Bayesian inference of causal effects with proxy measurements of the interference network

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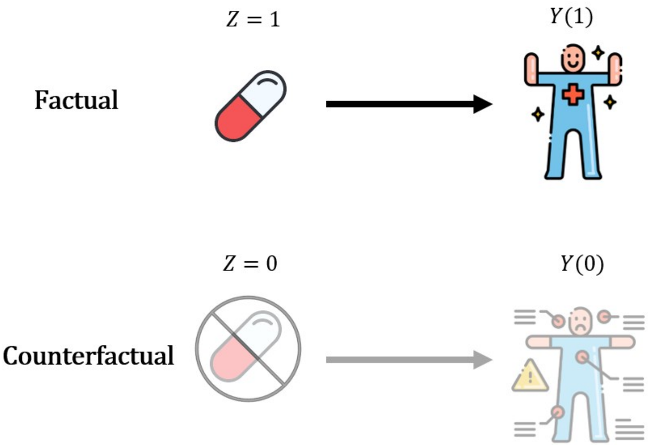
Joint work with Daniel Nevo

Outline

- Causal inference 101.
- Networked interference.
- Available proxies of the interference network.
- Inferential framework.

Causal Inference 101



Observation:

$$Y^{obs} = Y(1)$$

Causal effect:

$$Y(1) - Y(0)$$

Associations:

$$E[Y^{obs}|Z=1] - E[Y^{obs}|Z=0]$$

Interference

- Treatment of one unit affects the outcomes of other units.
- Units interact, resulting in the spread of treatment effects.
- Examples include
  1. Infectious diseases (Halloran et al. 1995; Hayek et al. 2022)
  2. Spread of addictive drugs (Buchanan et al. 2023)
  3. A/B testing in online platforms (Aral 2016)
  4. Social influence between friends (Christakis et al. 2007)

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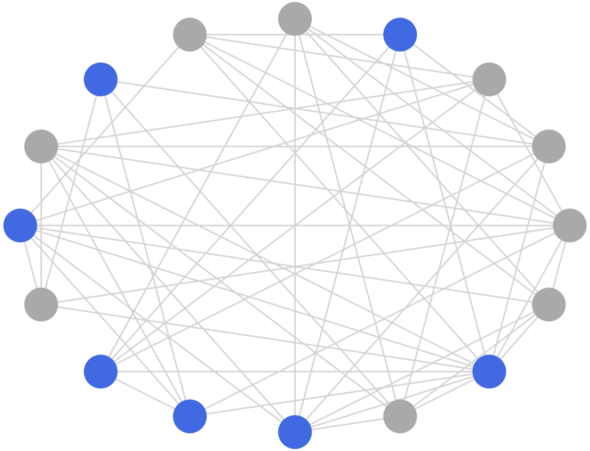
  4. Social influence between friends (Christakis et al. 2007)
- Encompass spillovers, peer effects, contamination, etc.
- Interference structure can be represented within a network.

Networked interference



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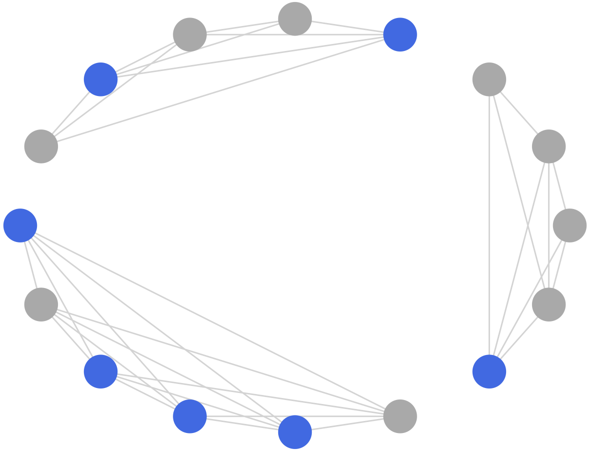

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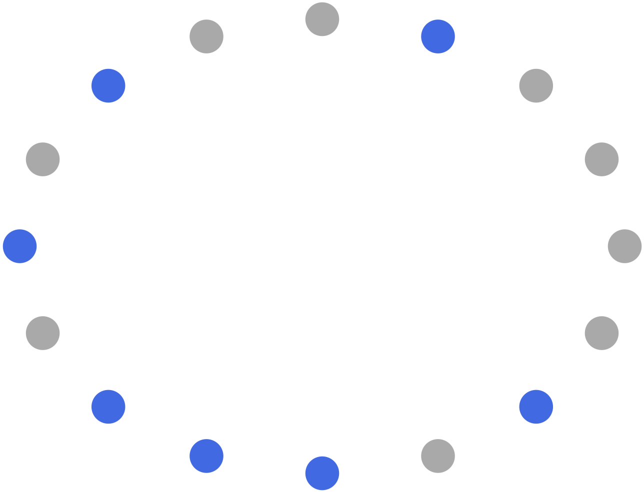

Formal settings

- Finite population of units $i \in \{1, \ldots, N\}$.
- $\boldsymbol{Z}$: population-level treatment vector with treatments' space $\mathcal{Z}$.
  - For example, for binary treatments $\mathcal{Z} \subseteq \{0, 1\}^N$.
- $Y_i(\boldsymbol{z})$: *potential* or *counterfactual* outcomes for each $\boldsymbol{z} \in \mathcal{Z}$.
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- $\boldsymbol{X} \subseteq \mathbb{R}^{N \times q}$: covariates matrix.
- $\boldsymbol{A}^*$: adjacency matrix of the true interference network.
    - For simplicity, undirected and unweighted.

Causal estimands

- Comparisons of population-level interventions:
  1. **Static.** $\mu(\boldsymbol{z}) = N^{-1} \sum_{i=1}^{N} \mathbb{E}[Y_i(\boldsymbol{z}) | \boldsymbol{A}^*]$
  2. **Dynamic.** $\mu(h) = N^{-1} \sum_{i=1}^{N} \mathbb{E}[Y_i(h) | \boldsymbol{A}^*]$, for function $h \equiv h(\boldsymbol{X}, \boldsymbol{A}^*)$.
  3. **Stochastic.** $\mu(\alpha_0) = N^{-1} \sum_{i=1}^{N} \sum_{\boldsymbol{z} \in \mathcal{Z}} \pi_{\alpha_0}(\boldsymbol{z}) \mathbb{E}[Y_i(\boldsymbol{z}) | \boldsymbol{A}^*]$, for stochastic policy $\pi_{\alpha_0}(\boldsymbol{z})$.

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Assumptions and estimands are expressed via $\boldsymbol{A}^*$. Can we accurately measure it?

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- Assume that we observe proxies $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_B)$ of $\boldsymbol{A}^*$.
  - Single or multiple imprecisely measured networks.
  - Multiple networks from varying sources.
  - Multi-layer networks.

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  - Single or multiple imprecisely measured networks.
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- **Our goal: estimate causal effects using the proxy measurements $\boldsymbol{A}$.**

Structural model

- $\boldsymbol{Z}_{-i}$ affects Y_i through values of summarizing function (Aronow et al. 2017).
- **Assumption 1.** *For any* $\boldsymbol{z}, \boldsymbol{z}' \in \mathcal{Z}$, *if* $z_i = z_i'$ *and* $f(\boldsymbol{z}_{-i}, \boldsymbol{A}_i^*) = f(\boldsymbol{z}'_{-i}, \boldsymbol{A}_i^*)$, *then* $Y_i(\boldsymbol{z}) = Y_i(\boldsymbol{z}')$.

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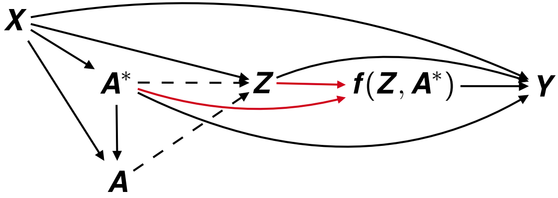
- Data generated by sequentially evaluating structural equations.
- Population-level directed acyclic graph (DAG):



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- $Y_i(\boldsymbol{z}) \perp\!\!\!\perp \boldsymbol{Z} | \boldsymbol{A}^*, \boldsymbol{X}$.
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  3. *Outcome model.* $p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X})$.
- Few limitations. The primary constraint is the computational burden.

Nonparameteric identification

- **Assumption 2** (Consistency). *If* $\boldsymbol{Z} = \boldsymbol{z}$, *then* $Y_i = Y_i(\boldsymbol{z})$, $\forall \boldsymbol{z} \in \mathcal{Z}$
- Causal estimands are identified by: $\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{A}^*] = \mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{A}^*, \boldsymbol{X}]$.

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- **Assumption 3.** *For some fixed functions* ϕ_1, ϕ_2, ϕ_3,

$$p(Y_i|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}) = p(Y_i|Z_i, f(\phi_1(\boldsymbol{Z}_{-i}, \boldsymbol{A}_i^*)), \boldsymbol{X}_i, \phi_2(\boldsymbol{X}_{-i}, \boldsymbol{A}_i^*), \phi_{3,i}(\boldsymbol{A}^*)), \ \forall i$$

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- yields

$$\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{A}^*] = \mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i|z_i, f(\phi_1), \boldsymbol{X}_i, \phi_2, \phi_{3,i}]$$

Bayesian inference

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- Posterior distribution:

$$
\begin{aligned}
p(\eta, \theta, \gamma, \boldsymbol{A}^* | \boldsymbol{O}) \propto\ & p(\boldsymbol{Y} | \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}, \eta) p(\eta) \\
& \cdot p(\boldsymbol{A} | \boldsymbol{A}^*, \boldsymbol{X}, \gamma) p(\gamma) \\
& \cdot p(\boldsymbol{A}^* | \boldsymbol{X}, \theta) p(\theta),
\end{aligned}
$$

- Mixed space. Discrete part ($\boldsymbol{A}^*$) consists of $\mathcal{O}(2^{N^2})$ terms!

Sampling from the posterior

- Metropolis-Hastings has slow convergence and poor mixing.
- Faster alternatives, like Hamiltonian Monte Carlo, often require continuity.
    - Extensions for mixed spaces doesn't scale well (e.g., Nishimura et al. 2020; Zhou 2020).

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- Bayesian modularization approach (Jacob et al. 2017). Posterior can be written as:

$$p(\eta, \theta, \gamma, \boldsymbol{A}^* | \boldsymbol{O}) \propto \underbrace{p(\eta | \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X})}_{\text{Outcome module}} \underbrace{p(\theta, \gamma, \boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A})}_{\text{Network module}} \underbrace{p(\boldsymbol{Y} | \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X})}_{\text{Feedback term}}$$

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- Sample from the “cut-posterior" instead:

$$p_{cut}(\eta, \theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O}) \propto p(\eta|\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}) p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{X}, \boldsymbol{A})$$

Sampling from the posterior

- Sequentially sampling from each module:
  1. **Network.** $p(\theta, \gamma, \boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A}) = p(\boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A}, \theta, \gamma) \sum_{\boldsymbol{A}^*} p(\theta, \gamma, \boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A}) \Rightarrow$ can be simplified!
  2. **Outcome.** $p(\eta | \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}) \propto p(\eta) p(\boldsymbol{Y} | \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}, \eta) \Rightarrow$ Bayesian regression.

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- In practice, this is performed via either:
  A. Sample (θ_m, γ_m) and $\boldsymbol{A}^*_m$. Sample η for each $\boldsymbol{A}^*_m$. Two options for drawing $\boldsymbol{A}^*_m$:
     i. Sample one network for each (θ_m, γ_m) (*"Three-stage"*).
     ii. Sample networks using the posterior mean $\mathbb{E}[\theta, \gamma | \cdot]$ (*"Two-stage"*).

Sampling from the posterior

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  B. Sample multiple $\boldsymbol{A}^*$. Estimate $\hat{\phi} = M^{-1} \sum_{m=1}^{M} \phi(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{A}^*_m)$. Sample η by replacing ϕ with $\hat{\phi}$ (*"Plug-in"*).

Numerical illustration

- Can be implemented in any probabilistic programming language.
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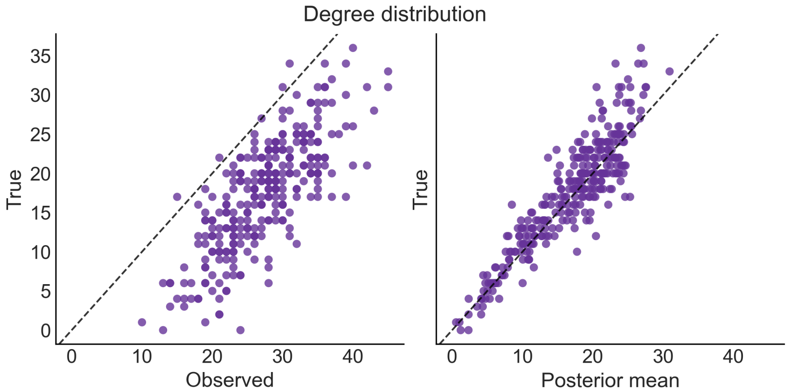

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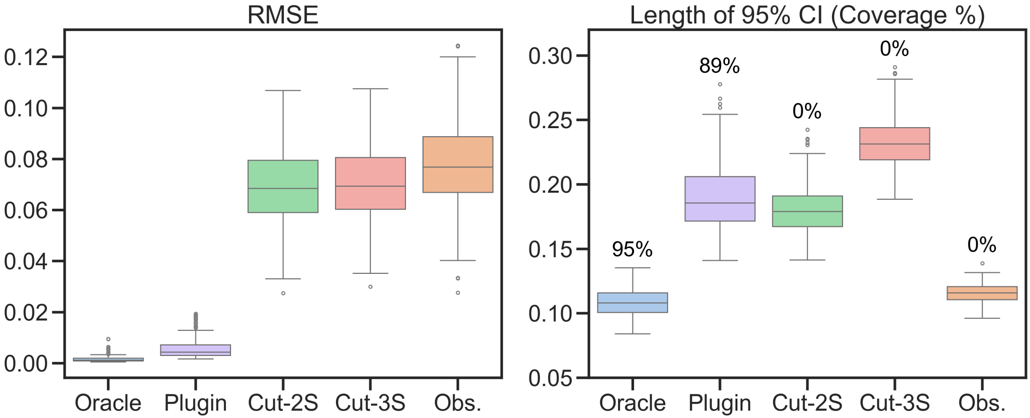

Thank You!

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Part I

Appendix

Postivity assumption

Assumption (positivity). $p(\boldsymbol{Z}_i = z, f(\boldsymbol{Z}_{-i}, \boldsymbol{A}_i^*) = c \mid \boldsymbol{X} = \boldsymbol{x}) > 0, \ \forall i, z \in \{0, 1\}, c \in \mathcal{C}, \boldsymbol{x}$.

Nonparameteric identification

$$
\begin{aligned}
\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{A}^*] &= \mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{A}^*, \boldsymbol{X}] \\
&= \mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{A}^*, \boldsymbol{X}] \\
&= \mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{A}^*, \boldsymbol{X}] \\
&= \mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i|z_i, f(\phi_1(\boldsymbol{z}_{-i}, \boldsymbol{A}_i^*)), \boldsymbol{X}_i, \phi_2(\boldsymbol{X}_{-i}, \boldsymbol{A}_i^*), \phi_{3,i}(\boldsymbol{A}^*)]
\end{aligned}
$$

Nonparameteric identification – stochastic estimand

$$
\begin{aligned}
N^{-1}\sum_{i=1}^{N}\sum_{\boldsymbol{z}\in\mathcal{Z}}\pi_{\alpha_0}(\boldsymbol{z})\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{A}^*] &= N^{-1}\sum_{\boldsymbol{z}\in\mathcal{Z}}\pi_{\alpha_0}(\boldsymbol{z})\sum_{i=1}^{N}\mathbb{E}[Y_i(\boldsymbol{z})|\boldsymbol{A}^*] \\
&= N^{-1}\sum_{\boldsymbol{z}\in\mathcal{Z}}\pi_{\alpha_0}(\boldsymbol{z})\sum_{i=1}^{N}\mathbb{E}_{\boldsymbol{X}}\mathbb{E}[Y_i|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{A}^*,\boldsymbol{X}],
\end{aligned}
$$

DAG factorization

$$p(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}, \boldsymbol{A}^*, \boldsymbol{X}) = p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X})p(\boldsymbol{Z}|\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{A}^*)p(\boldsymbol{A}|\boldsymbol{A}^*, \boldsymbol{X})p(\boldsymbol{A}^*|\boldsymbol{X})p(\boldsymbol{X}),$$

Example – random noise

We observe single $\boldsymbol{A}$ generated by measuring $\boldsymbol{A}^*$ with random error independently across edges with false positive and negative rates γ_0, γ_1, respectively. Namely, $\Pr(A_{ij} = 1 - k | A^*_{ij} = k) = \gamma_k,\ k = 0, 1$, and we can write

$$p(\boldsymbol{A}|\boldsymbol{A}^*, \boldsymbol{X}, \gamma) = \prod_{i>j} \sum_{k=0,1} \mathbb{I}\{A^*_{ij} = k\} \gamma_k^{A_{ij}} (1 - \gamma_k)^{1-A_{ij}} = \prod_{i>j} \xi_{ij}(A^*_{ij}, \gamma),$$

where $\xi_{ij}(k, \gamma) = \gamma_k^{A_{ij}} (1 - \gamma_k)^{1-A_{ij}},\ k = 0, 1$. Extending to heterogeneous random noise that depends on covariates is possible as well.

Example – Repeated noisy measurements

Researchers observe $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_B)$ networks each pertain to different noisy measurement of the true network (De Bacco et al. 2023; Redhead et al. 2023). A model can be assumed for each of the networks:

$$p(\boldsymbol{A}_b | \boldsymbol{A}^*, \boldsymbol{X}, \gamma_b) = \prod_{i>j} \xi_{b,ij}(A^*_{ij}, \gamma_b),$$

for some function $\xi_{b,ij}$. The joint distribution of $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_B$ can be assumed to be independent or dependent (e.g., via a hierarchical model).

Example – multi-layer networks

Researchers observe multi-layer collection of networks $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_B)$. Each $\boldsymbol{A}_b$ measures different social relationships. It is possible to model the observed multi-layer networks using LSM with shared latent positions:

$$p(\boldsymbol{A}_b | \boldsymbol{X}, \gamma_b, \boldsymbol{w}) = \prod_{i>j} \psi(\gamma_b - \|w_i - w_j\|)^{A_{b,ij}} (1 - \psi(\gamma_b - \|w_i - w_j\|))^{1 - A_{b,ij}},$$

for $\psi : \mathbb{R} \to [0, 1]$ and latent positions $w_i \in \mathbb{R}^d$. Covariates can also be included. Dependence between the networks can be modeled with multinomial (Salter-Townshend et al. 2017) or hierarchical (Sosa et al. 2022) models. The true network $\boldsymbol{A}^*$ can be viewed as a “consensus" or aggregate version of the observed networks. That can be achieved with a simple union (Salter-Townshend et al. 2017), or in a model-based approach (Sosa et al. 2022).

Feedback between modules

The posterior can be written as: $p(\eta, \theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O}) = p(\eta|\boldsymbol{O}, \theta, \gamma, \boldsymbol{A}^*)p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O})$. Thus

$$
\begin{aligned}
p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O}) &= \int_\eta p(\eta, \theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O}) d\eta \\
&\propto p(\theta)p(\gamma)p(\boldsymbol{A}|\boldsymbol{A}^*, \boldsymbol{X}, \gamma)p(\boldsymbol{A}^*|\boldsymbol{X}, \theta)p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}) \\
&\equiv p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{X}, \boldsymbol{A})p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X})
\end{aligned}
$$

where $p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}) = \int_\eta p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}, \eta)p(\eta)d\eta$. In addition,

$$
p(\eta|\boldsymbol{O}, \theta, \gamma, \boldsymbol{A}^*) = \frac{p(\eta, \theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O})}{p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{O})} \propto p(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}, \eta)p(\eta)
$$

Sampling from network module

Assume that both the network generation model $p(\boldsymbol{A}^*|\boldsymbol{X},\theta)$ and the observed network model $p(\boldsymbol{A}|\boldsymbol{A}^*,\boldsymbol{X},\gamma)$ are dyad independent, i.e., can be written as

$$p(\boldsymbol{A}^*|\boldsymbol{X},\theta) = \prod_{i>j} \nu_{ij}(\boldsymbol{A}^*_{ij},\theta), \quad p(\boldsymbol{A}|\boldsymbol{A}^*,\boldsymbol{X},\gamma) = \prod_{i>j} \xi_{ij}(\boldsymbol{A}^*_{ij},\gamma),$$

for some functions ν_{ij},ξ_{ij}. Thus,

$$\begin{aligned} p(\theta,\gamma|\boldsymbol{X},\boldsymbol{A}) &\propto p(\theta)p(\gamma)\sum_{\boldsymbol{A}^*}\prod_{i>j}\xi_{ij}(\boldsymbol{A}^*_{ij},\gamma)\nu_{ij}(\boldsymbol{A}^*_{ij},\theta) \\ &= p(\theta)p(\gamma)\prod_{i>j}\sum_{k=0,1}\xi_{ij}(k,\gamma)\nu_{ij}(k,\theta) \end{aligned}$$

Sampling networks

Generating samples of $\boldsymbol{A}^*$ is reduced to sampling edges independently from

$$
\begin{aligned}
p(\boldsymbol{A}^*|\boldsymbol{X}, \boldsymbol{A}, \theta, \gamma) &= \frac{p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{X}, \boldsymbol{A})}{\sum_{\boldsymbol{A}^*} p(\theta, \gamma, \boldsymbol{A}^*|\boldsymbol{X}, \boldsymbol{A})} \\
&= \prod_{i>j} \frac{\xi_{ij}(\boldsymbol{A}^*_{ij}, \gamma)\nu_{ij}(\boldsymbol{A}^*_{ij}, \theta)}{\sum_{k=0,1} \xi_{ij}(k, \gamma)\nu_{ij}(k, \theta)},
\end{aligned}
$$

Algorithm 1: Three-stage sampling

foreach $m = 1, \ldots, M$ **do**

  Sample $\theta_m, \gamma_m \sim p(\theta, \gamma | \boldsymbol{X}, \boldsymbol{A})$;

  Sample $\boldsymbol{A}^*_m \sim p(\boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A}, \theta_m, \gamma_m)$;

  foreach $\ell = 1, \ldots, L$ **do**

    Sample $\eta_\ell \sim p(\eta | \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}^*_m, \boldsymbol{X})$;

  end

end

Two-stage sampling

Algorithm 2: Two-stage sampling

foreach $m = 1, \ldots, M$ **do**

  Sample $\theta_m, \gamma_m \sim p(\theta, \gamma | \boldsymbol{X}, \boldsymbol{A})$;

end

Compute posterior means $\bar{\theta} = M^{-1} \sum_m \theta_m$ and similarly for $\bar{\gamma}$;

foreach $m = 1, \ldots, M$ **do**

  Sample $\boldsymbol{A}_m^* \sim p(\boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A}, \bar{\theta}, \bar{\gamma})$;

  foreach $\ell = 1, \ldots, L$ **do**

    Sample $\eta_\ell \sim p(\eta | \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}_m^*, \boldsymbol{X})$;

  end

end

Algorithm 3: Plug-in sampling

foreach $m = 1, \ldots, M$ **do**

  Sample $\theta_m, \gamma_m \sim p(\theta, \gamma | \boldsymbol{X}, \boldsymbol{A})$;

  Sample $\boldsymbol{A}^*_m \sim p(\boldsymbol{A}^* | \boldsymbol{X}, \boldsymbol{A}, \theta_m, \gamma_m)$;

end

Estimate the outcome models' summary statistics

$$\boldsymbol{A}^* \equiv \phi(\boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X}) = M^{-1} \sum_m \phi(\boldsymbol{Z}, \boldsymbol{A}^*_m, \boldsymbol{X}) ;$$

foreach $\ell = 1, \ldots, L$ **do**

  Sample $\eta_\ell \sim p(\eta | \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}^*, \boldsymbol{X})$;

end

Simulations DGP

- Outcome model $Y_i = \eta_0 + \eta_1 Z_i + \eta_2 \sum_{j \neq i} Z_j A^*_{ij} + \eta_3 X_i + \varepsilon_i$
- Network generation $logit(\Pr(A^*_{ij} = 1)) = \theta_0 + \theta_1 |X_i - X_j|$.
- Observed network from a random noise measurement error model.
- $N = 300$ for 300 iterations.

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