23-02 Tue

# MEAN VALUE THEOREM

Introduction :-

* let $y=f(x)$ is a continuous in ~~the~~ close interval $[a,b]$ these means that if $a<c<b$

$\lim_{x\to c} f(x) = f(c)$ and

$\lim_{x\to a+0} f(x) = f(a)$, $\lim_{x\to b-0} f(x) = f(b)$

* let $y=f(x)$ with differentiable in the closed interval ~~a,b~~ $[a,b]$ these means that $a<c<b$ the derivative of $f(x)$ at $x=c$ exists.

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \text{ exists}$$

NOTE

* If $f'(x)>0$ then $f(x)$ is increasing function
* If $f'(x)<0$ then $f(x)$ is decreasing function

* Rolle's theorem :-

* let $f(x)$ be a function such that
  (i) It is continuous in closed interval $[a,b]$
  (ii) It is differentiable in open interval $(a,b)$ and
  (iii) If $f(a)=f(b)$ then there exist atleast one point $c$ ~~such that~~ in open interval $(a,b)$ such that $f'(c)=0$

Q) Verify Rolle's theorem for $f(x) = (x-2)^3(x-3)^4$ in $[2,3]$

Q) let $f(x)$ be a function defined on $[a,b]$ such that
(i) It is continuous on $[a,b]$
(ii) It is differentiable on $(a,b)$

and (iii) $f(a) = f(b)$ then there exists atleast one point $c \in (a,b)$ st $f'(c) = 0$

(i) Since $f(x)$ is a polynomial every polynomial is continuous

$\therefore f(x)$ is continuous on $[-2,3]$

(ii) $f'(x) = \frac{d}{dx}\left((x+2)^3(x-3)^4\right)$

$$= \left((x+2)^3 \frac{d}{dx}(x-3)^4\right) + (x-3)^4 \frac{d}{dx}\left((x+2)^3\right)$$

$$= (x+2)^3\, 4(x-3)^3 + (x-3)^4\, 3(x+2)^2$$

$$= (x+2)^2(x-3)^3\left[4(x+2)+3(x-3)\right]$$

$$= (x+2)^2(x-3)^3\left[4x+8+3x-9\right]$$

$$f'(x) = (x+2)^2(x-3)^3\,(7x-1)$$

$f'(x)$ exists for every $x$

$f(x)$ is differentiable in $(-2,3)$

(iii) $f(a) = f(-2)$

$$= (-2+2)^3(-2-3)^4$$

$$= 0$$

$f(b) = f(3)$

$$= (3-2)^3(3-3)^4$$

$$= 0$$

$f(-2) = f(3)$

here $f(x)$ satisfied conditions of Rolle's theorem by Rolle's theorem

there exists $c \in (-2,3)$ st $f'(c)=0$

for verification

$$f'(c) = 0$$

$$(c+2)^2(c-3)^3(7c-1) = 0$$

$$(c+2)^2 = 0, \ (c-3)^3 = 0, \ 7c-1 = 0$$

$$c = -2, \ c = 3, \ 7c = 1$$

$$c = \frac{1}{7}$$

$$c = \frac{1}{7} \in (-2,3)$$

Hence Rolle's theorem verified.

Q) Verify rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2 = 0$ in $[-\sqrt{2}, \sqrt{2})$

S) let $f(x)$ be a function defined on $[a,b]$ such that

(i) It is continuous on $[a,b]$

(ii) It is differentiable on $(a,b)$

and (iii) $f(a) = f(b)$ then there exists atleast one point $c \in (a,b)$ st $f'(c)=0$

(i) Since $f(x)$ is a polynomial every polynomial is continous

$\therefore f(x)$ is continuous on $[-\sqrt{2}, \sqrt{2}]$

(ii) $f'(x) = \frac{d}{dx}(2x^3+x^2-4x-2)$

$$= 6x^2+2x-4$$

$$f'(x) = 6x^2+2x-4$$

$\therefore f'(x)$ exists for every $x$

$f(x)$ is differentiable on $[\sqrt{2}, \sqrt{2}]$

(iii) If $f(a) = f(b)$

$f(a) = f(-\sqrt{2})$

$= 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(\sqrt{2}) - 2$

$= \cancel{-6\sqrt{2}} + \cancel{2} + 4\sqrt{2} - \cancel{2} = 0$

$2(\cancel{\ })$

$-4\sqrt{2}$

If $f(b) = f(\sqrt{2})$

$= 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2$

$= 2(2^{3/2}) + \cancel{2} - \cancel{4\sqrt{2}} - \cancel{2} = 0$

$+4\sqrt{2}$

$f(-\sqrt{2}) = f(\sqrt{2})$

Here $f(x)$ satisfied condition of Rolle's theorem

by Rolle's theorem.

there exists $c \in (\sqrt{2}, \sqrt{2})$ st $f'(c) = 0$

$f'(c) = 0$

$24c^2$

$6c \quad -4c$

$6c^2 + 2c - 4 = 0$

$6c^2 + 6c - 4c - 4 = 0$

$6c(c+1) - 4(c+1) = 0$

$(6c-4)(c+1) = 0$

$6c - 4 = 0, \quad c+1 = 0$

$6c = 4, \quad c = -1$

$c = \cancel{\frac{4}{6}}\ \frac{2}{3}$

$c = \frac{2}{3} \in (-\sqrt{2}, \sqrt{2})$

Hence Rolle's theorem verified

(3) Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$

8) Rolle's theorem:- $f(x)$ is a function such that

1) → It is continuous $[a,b]$

2) It is differentiable on $(a,b)$

3) $f(a)=f(b)$

Then there exist $f'(c)=0 \in (a,b)$

(i) Since given function is polynamial, it is continuous

(ii) $\frac{d}{dx}(2x^3+x^2-4x-2)$

$$= 6x^2+2x-4$$

$$f(-\sqrt{3}) = 2(-\sqrt{3})^3+(-\sqrt{3})^2-4(\sqrt{3})-2$$

$$= -6\sqrt{3}+3+4\sqrt{3}-2$$

$$= 1-2\sqrt{3}$$

$$f(+\sqrt{3}) = 2\cdot(\sqrt{3})^3+(\sqrt{3})^2-4(\sqrt{3})-2$$

$$= 6\sqrt{3}+3-4\sqrt{3}-2$$

$$= 2\sqrt{3}+1$$

$$= 1+2\sqrt{3}$$

$$f(a) \neq f(b)$$

∴ Rolle's theorem is not verified

∴ The given polynamial doesnot satisfy Rolle's theorem.

NIL

④ Verify Rolle's theorem for $f(x) = e^{-x/2}(x)(x+3)$ in $[-3,0]$

Sol: let $f(x)$ be a function defined on $[a,b]$ such that
(i) It is continuous on $[a,b]$
(ii) It is differentiable on $(a,b)$
(iii) $f(a)=f(b)$ then there exists atleast one point $c\in(a,b)$ st $f'(c)=0$

(i) Since $f(x)$ $x(x+3)$ is a polynomial hence it is continuous and $e^{-x/2}$ is defined in $[-3,0]$ hence it is continuous $e^{-x/2}(x)(x-3)$ is continuous on $[-3,0]$

(ii) $f'(x) = \frac{d}{dx}\left(e^{-x/2}\,x(x+3)\right)$

$$= \frac{d}{dx}\left(e^{-x/2}(x^2+3x)\right)$$

$$= e^{-x/2}(2x+3) + (x^2+3x)\,e^{-x/2}\left(-\frac{1}{2}\right)$$

$= e^{-x/2}\left[2x+3-\frac{1}{2}x^2-\frac{3}{2}x\right]$

$= \frac{e^{-x/2}}{2}\left[4x+6-x^2-3x\right]$

$= \frac{e^{-x/2}}{2}\left[-x^2+x+6\right]$

$e^{-x/2}$ is defined in $(-3,0)$ it is not infinity at any values of $x$

$f(x)$ is differentiable on $(-3,0)$

(iii) $f(-3) = e^{(-3/2)}(-3)(-3+3) = 0$

$f(0) = e^{(0/2)}(0)(0+3) = 0$

$f(-3) = f(0)$

Hence, $f(x)$ satisfied three condition of rolle's theorem

Verification

$f'(c) = 0$

$\Rightarrow e^{-c/2}(2c+3) + (c^2+3c)\cdot -\frac{1}{2}e^{-c/2} = 0$

$e^{-c/2}\left[2c+3-\frac{(c^2+3c)}{2}\right] = 0$

$e^{-c/2}\left[4c+6-c^2-3c\right] = 0$

$e^{-c/2}\left[-c^2+c+6\right]$

$c^2-3c+2c-6 = 0$

$c(c-3)+2(c-3) = 0$

$(c+2)(c-3) = 0$

$c=-2, c=3$

$c=-2 \in [-3,0]$

Hence Rolles theorem is verified.

25-02 THU Verify rollel's theorem can be applied to the following functions. is the intervals

(i) $f(x) = \tan x$ $[0, \pi]$

(ii) $f(x) = 1/x^2$ $[-1, 1]$

(iii) $f(x) = x^3$ $[1, 3]$

Sol) (i) Since $\tan x$ at $x = \frac{\pi}{2}$ is not defined

It is not continuous on $[0, \pi]$

Given function $f(x)$ is not satisfied of the condition of Rollel's theorem

Hence Rolle's theorem is not applicable

(ii) Since $f(x) = \frac{1}{x^2}$ is not defined on at $x = 0$

so $f(x)$ is not continuous on $[-1, 1]$

Hence $f(x)$ is not satisified at one of the condition of Rolle's theorem.

Hence Rolle's theorem is not applicable

(iii) Since $f(x) = x^3$ is not defined on at $x = 0$

so $f(x)$ is not continuous on $[1, 3]$

Hence $f(x)$ is not satisified at one of the condition of Rolles theorem

Hence Rolle's theorem is not applicable

(Q) Verify rolle's theorem ~~where~~ $f(x) = (x-a)^m (x-b)^n$ where m,n are positive integers in $[a,b]$

(i) Since $f(x)$ is a polynomial

It is continuous on $[a,b]$

(ii) $f'(x) = (x-a)^m \frac{d}{dx}(x-b)^n + (x-b)^n \frac{d}{dx}(x-a)^m$

$= (x-a)^m n(x-b)^{n-1} + (x-b)^n m(x-a)^{m-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[n(x-a) + m(x-b)]$

$f'(x)$ is defined for any value of $x$.

$f(x)$ is differentiable on $(a,b)$

(iii) $f(a) = (a-a)^m (a-b)^n = 0$

$f(b) = (b-a)^m (b-b)^n = 0$

$f(a) = f(b)$

Hence $f(x)$ satisified all conditions of Rolle's theorem by Rolle's theorem exists atleast one point $c \in (a,b)$ st $f'(c) = 0$

For verification

$f'(c) = 0 \Rightarrow (c-a)^{m-1} (c-b)^{n-1}(n(c-a) + m(c-b)) = 0$

$\Downarrow$

$c=a$ or $c=b$ or $c = \frac{mb+na}{m+n}$

$a,b \notin (a,b)$

$c = \frac{mb+na}{m+n} \in (a,b)$ because c divides $(a,b)$ in the ratio $m:n$ internally.

(9) Verify rolle's theorem for the function

$f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$

Sol (i) Since $f(x)$ is a polynomial

it is continuous on $[0, \pi]$

(ii) $f'(x) = \cancel{\frac{\sin x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}\sin x}{(e^x)^2}}$

$= \sin x \cancel{e^x} + e^x \cos x$

$= \frac{e^x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(e^x)}{(e^x)^2}$

$= \frac{e^x \cos x - \sin x \, e^x}{(e^x)^2}$

$\cancel{f(x)} = e^x \left[\frac{\cos x - \sin x}{(e^x)^{\cancel{2}}}\right]$

$= \left[\frac{\cos x - \sin x}{e^x}\right] \in [0 \pi]$

$\therefore f(x)$ is defined for any value of $x$

$f(x)$ is differentiable on $[0, \pi]$

(iii) $f(a) = \frac{\sin 0}{e^0} = \frac{0}{1} = 0$

$f(b) = \frac{\sin \pi}{e^0} = \frac{0}{1} = 0$

$f(a) = f(b)$

Hence $f(x)$ satisified all conditions of rolle's theorem by rolle's theorem exists one point

$c \in (0 \pi)$ st $f'(c) = 0$

For verification

$c=0$, (or) $c=\pi$ (or)

$$c = \left[\frac{\cos c - \sin c}{e^c}\right] \in [0,\pi]$$

because $c$ divides $(0,\pi)$

(2) Verify Rolles theorem for $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$ $a>0$, $b>0$.

Sol (i) Since $f(x)$ is a polynomial

if it is continuous on $[0,\pi]$

(ii) $f'(x) = \frac{d}{dx}\log\left(\frac{x^2+ab}{x(a+b)}\right)$

$$= \frac{d}{dx}\left(\log(x^2+ab) - \log(x(a+b))\right)$$

$$= \frac{d}{dx}\log(x^2+ab) - \frac{d}{dx}\log(x(a+b))$$

$$= \frac{1}{x^2+ab}\cdot 2x - \frac{1}{x(a+b)}\cdot(a+b)$$

$$= \frac{2x}{x^2+ab} - \frac{(a+b)}{x(a+b)}$$

$$= \frac{2x}{x^2+ab} - \frac{1}{x}$$

$f'(x)$ is defined for any value $x$

$f(x)$ is differentiable on $[a,b]$

(iii) $f(a) = \log\left[\frac{a^2+ab}{a(a+b)}\right] = \log(1)$

$f(b) = \log\left[\frac{b^2+ab}{b(a+b)}\right] = \log(1)$

$f(a) = f(b)$

$\therefore$ Hence $f(x)$ satisfied all conditions of rolle's theorem by rolles theorem exists one point

$c \in (a,b)$ st $f'(c) = 0$

$$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} \in (a,b)$$

$$= \frac{2c^2 - c^2 - ab}{c^3 + abc} \in (a,b) \Rightarrow c^2 = ab \Rightarrow c = \sqrt{ab}$$

Hence Rolles theorem is verified.

(Q) Verify rolles theorem for $f(x) = x^{2/3} - 2x^{1/3}$ in $[0,8]$

Sol: (i) Since $f(x)$ is a polynomial

if it is continuous on $[0,8]$

(ii) $f'(x) = \frac{d}{dx}(x^{2/3} - 2x^{1/3})$

$$= \frac{d(x^{2/3})}{dx} - \frac{d(2x^{1/3})}{dx}$$

$$= \frac{2}{3}x^{\frac{2}{3}-1} - 2\frac{1}{3}x^{\frac{1}{3}-1}$$

$$= \frac{2}{3}x^{\frac{2-3}{3}} - \frac{2}{3}x^{\frac{1-3}{3}}$$

$$= \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3}$$

$$= \frac{2}{3}x^{-1/3}(1-x)$$

$f(x)$ is differentiable

$f(0) \Rightarrow 0 - 0 \Rightarrow 0$

$f(8) \Rightarrow (2^3)^{2/3} - 2 \cdot 8^{1/3}$

$$= (2^3)^{2/3} - 2(2^3)^{1/3}$$

$$= 4 - 4$$

$$= 0$$

$f'(c) = 0$

$\frac{2}{3}c^{-1/3}(1-c) = 0, \quad c = 1 \in (0,8)$

Hence rolles theorem is verified

(6) Verify the rolles theorem for $f(x) = e^x \sin x$ $(0,\pi)$

Sol (i) $f(x)$ is continuous function of $[0,\pi)$

(ii) $f'(x) = \frac{d(e^x \cos x)}{dx}$

$= e^x \cos x + \sin x \, e^x$

$= e^x (\cos x + \sin x)$

$f'(x)$ is continuous

$f(0) = 0 \quad | \; f(\pi) = 0$

$f(a) = f(b)$

$f'(c) = 0$

$e^c (\cos x + \sin c) = 0$

$\sin c = -\cos c$

$\sin c = -\sin\left(\frac{\pi}{2} - c\right)$

$\sin c + \sin\left(\frac{\pi}{2} - c\right) = 0$

$2\sin\left(\frac{\cancel{c} + \frac{\pi}{2} - \cancel{c}}{2}\right) \sin\left(\frac{c - \frac{\pi}{2} + c}{2}\right) = 0$

$2\sin\frac{\pi}{4} \sin\left(\frac{4c - \pi}{4}\right) = 0$

$\frac{4c - \pi}{4} = \pi$

$4c - \pi = 4\pi$

$4c = 5\pi$

$c = \frac{5\pi}{4} \in [0,\pi]$

Rolle's theorem is verified.

U2 FR1

let f(x) be a function such that it is continu-ous in [a, b] and differentiable in (a,b) then there exists atleast one point c in open interval a,b, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Lagrange's Mean Value theorem

Q Verify lagrange's Mean value theorem for $f(x) = x^3 - x^2 - 3x + 3$ in [0,4]

Sol (i) Since f(x) is a polynomial

Every polynomial is continuous

hence f(x) is continuous on [0,4]

(ii) $f'(x) = \frac{d}{dx}(x^3 - x^2 - 3x + 3)$

$= 3x^2 - 2x - 5$

f'(x) is defined

∴ f(x) is differentiable on (0,4)

hence f(x) satisfies all conditions of LMVT

by L.M.V.T there exists atleast one point c∈(0,4) st

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2 - 2c - 5 = \frac{f(4)-f(0)}{4-0}$$

For verification

$$3c^2 - 2c - 5 = \frac{(4)^3 - (4)^2 - 3(4) + \cancel{3} - \cancel{3}}{4}$$

$$= 16 - 4 - 5$$

$3c^2-2c-5 = 7$

$3c^2-2c-12 = 0.$

$$c = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-12)}}{2(3)}$$

$$c = \frac{2 \pm \sqrt{4+144}}{6}$$

$$c = \frac{2 \pm \sqrt{148}}{6}$$

$= \cancel{1 \pm 2\sqrt{37}} \; 3$

$$c = \frac{1 \pm \sqrt{37}}{3}$$

Hence L.M.V.T is verified

Q) Verify L.M.V.T for $f(x) = \log_e x$ in $[1, e]$

(i) $\log_e x$ is not defined $x \le 0$

It is defined for $x > 0$

Since $f(x)$ is continuous in $[1, e]$

(ii) $f'(x) = \frac{d}{dx} \log_e x$

$= \frac{1}{x}$

$\therefore f(x)$ is differentiable in $(1, e)$ because $1/x$ is defined in $1, e$,

$f(x)$ satisfies all condition L.M.V.T, by L.M.V.T there exists atleast one point $c$ belongs to $(1, e)$ such that $f'(c) = \frac{f(e)-f(1)}{e-1}$

For verification

$$\frac{1}{c} = \frac{1 - \log 1}{e}$$

f(x) belongs to 1, 2

(Q) Verify L.M.V.T for the $f(x) = x(x-2)(x-3)$ in $(0,4)$

Sol (i) Since f(x) is a polynomial
Every polynomial is continuous on [0,4]
Hence f(x) is continuous on [0,4]

(ii) $f'(x) = \frac{d}{dx}(x(x-2)(x-3))$

$= \cancel{(x(x-2)) \frac{d(x-3)}{dx} (x-3) \frac{d}{dx}}$

$= \frac{d}{dx}(x^2-2x)(x-3)$

$= \frac{d}{dx}(x^3-3x^2-2x^2+6x)$

$= \frac{d}{dx}(x^3-5x^2+6x)$

$= 3x^2-10x+6$

f(x) is defined

$\therefore$ f(x) is differentiable on (0,4)

Hence f(x) satisfies all conditions of L.M.V.T

by L.M.V.T there exists one point $c \in (0,4)$

st

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2-10c+6 = \frac{f(4)-f(0)}{4-0}$$

$$3c^2-10c+6 = \frac{4(4-2)(4-3)}{4}$$

$$= \frac{4(2)(1)}{4} = \frac{8}{4} = 2$$

$3x^2 - 10x + 4 = 0$ $\qquad 12x^2$

$\cancel{3x^2 - 6x - 2}$

$$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(4)}}{2(3)}$$

$$= \frac{10 \pm \sqrt{100-48}}{6}$$

$$\therefore \frac{10 \pm \sqrt{52}}{6} \in (0,4) \Rightarrow \frac{10 \pm \sqrt{13 \times 4}}{6} = \frac{10 \pm 2\sqrt{13}}{6}$$

$\therefore$ Hence L.M.V.T is verified $\quad = \frac{5 \pm \sqrt{13}}{3}\ c \in (0,4)$

(8) Find c of L.M.V.T for $f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$

Sol (i) Since $f(x)$ is a polynomial

Every polynomial is continuous on $[0,4]$

Hence $f(x)$ is continuous on $[0,4]$

(ii) $f'(x) = \frac{d}{dx}(x-1)(x-2)(x-3)$

$$= \frac{d}{dx}(x^2 - 2x - x + 2)(x-3)$$

$$= \frac{d}{dx}(x^2 - 3x + 2)(x-3)$$

$$= \frac{d}{dx}(x^3 - 3x^2 + 2x - 3x^2 + 9x + 6)$$

$$= \frac{d}{dx}(x^3 - 6x^2 + 11x + 6)$$

$$= 3x^2 - 12x + 11$$

$f'(x)$ is defined

$\therefore f(x)$ is differentiable on $(0,4)$

hence $f(x)$ satisfies all conditions of L.M.V.T

By L.M.V.T there exists one point $c \in (0,4)$

st

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2-12c+11 = \frac{(4+1)(4-2)(4-3)}{4}$$

$$3c^2-12c+11 = \frac{3(2)(1)}{4}$$

$$3c^2-12c+11 = \cancel{\frac{6}{4}}\ \frac{3}{2}$$

$$6c^2-24c+22 = 3$$

$$6c^2-24c+19 = 0$$

$$\frac{-(-24) \pm \sqrt{(-24)^2-4(6)(19)}}{2(6)}$$

$$\frac{24 \pm \sqrt{576-24(19)}}{12}$$

$$\frac{24 \pm \sqrt{576-456}}{12}$$

$$\frac{24 \pm \sqrt{120}}{12}$$

$$\Rightarrow \frac{24 \pm \sqrt{4 \times 30}}{12}$$

$$= \frac{24 + 2\sqrt{30}}{12}$$

$$\Rightarrow \frac{12+\sqrt{30}}{6}$$

27-02 SAT

Verify L.M.V.T for the function $f(x) = \cos x$ in $[0, \frac{\pi}{2}]$.

Sol (i) $f(x)$ is continuous because it is a polynomial

$\cos x$ is always defined

Hence $f(x)$ is continuous on $[0, \frac{\pi}{2}]$

(ii) $f'(x) = \frac{d \cos x}{dx}$

$= -\sin x$

$f(x)$ is defined

$\therefore f(x)$ is differentiable on $(0, 4)$.

Hence $f(x)$ satisifies all condition of L.M.V.T

$\therefore$ By L.MVT there exist one point $c \in (0, 4)$

S.T

$$f'(c) = \cancel{\sin c}\ \frac{f(b)-f(a)}{b-a}$$

$$-\sin c = \frac{\cos\frac{\pi}{2} - \cos 0}{\pi/2}$$

$+\left(\frac{2}{\pi}\right) = \not{} \sin c$

$c = \sin^{-1}\left[\frac{2}{\pi}\right] \in \left(0, \frac{\pi}{2}\right)$

Hence L.M.V.T is verified.

(Q) If $a<b$, prove that $\frac{b-a}{(1+b^2)} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{(1+a^2)}$ . using L.M.V.T. Deduce the following

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < \tan^{-1} 2 < \frac{\pi+2}{4}$

Sol) ~~Let f(x) = tan⁻¹x is continuous on [a,b]~~

Let $f(x) = \tan^{-1}x$ is $[a,b]$ $0<a<b<1$

(i) $f(x) = \tan^{-1}x$ is continuous on $[a,b]$

(ii) $f'(x) = \frac{d}{dx}(\tan^{-1}x)$

$= \frac{1}{1+x^2}$

$f'(x)$ exists at any value of $x$.

$\therefore f(x)$ is differentiable on $(a,b)$

Hence $f(x)$ satisified all condition of L.M.V.T.

By L.M.V.T there exists atleast one point

$c \in (a,b)$ st $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{f(b)-f(a)}{b-a} \rightarrow (1)$$

Since $c \in (a,b)$

$a<c<b$

$a^2<c^2<b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \to ③$$

From ①

$$\frac{1}{1+a^2} > \frac{f(b)-f(a)}{b-a} > \frac{1}{1+b^2}$$

$$\boxed{\frac{b-a}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a}\, b-a > \frac{b-a}{1+b^2}}$$

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

deduction

(i) Put $b = \frac{4}{3}$ $a=1$ in ③

$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\frac{4}{3} - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2}$$

$$\frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$$

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$$

(ii) Put $b=2$ , $a=1$ in ③

$$\frac{2-1}{1+(2)^2} < \tan^{-1}2 - \tan^{-1}(1) < \frac{2-1}{1+1^2}$$

$$\frac{1}{5} < \tan^{-1}2 - \tan^{-1}(1) < \frac{1}{2}$$

$$\frac{1}{5} < \frac{\pi}{4} - \tan^{-1}2 < \frac{1}{2} + \frac{\pi}{4}$$

$$\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$$

(7) P.T $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$

Sol: let $f(x) = \sin^{-1}(x)$ is $[a,b]$ $0<a<b<1$

(i) $f(x)$ is continuous function

(ii) $f'(x) = \frac{d}{dx}(\sin^{-1}x)$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$f'(x)$ exists at any value of $x$

$\therefore f(x)$ is differentiable on $(a,b)$

Hence $f(x)$ satisified all condition of L.M.V.T

By L.M.V.T then exists atleast one point $c \in (a,b)$

st $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\cancel{f(b)-f(}\sin^{-1}(b) - \sin^{-1}(a)}{b-a} \quad a<c<b$$

$$a<c<b$$

$$a^2<c^2<b^2$$

$$a^2-1 < c^2-1 < b^2-1$$

$$1-a^2 < 1-c^2 < 1-b^2$$

$$\frac{1}{1-a^2} < \frac{1}{1-c^2} < \frac{1}{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} > \frac{1}{\sqrt{1-c^2}} > \frac{1}{\sqrt{1-b^2}}$$

from (1)

$$\frac{1}{\sqrt{1-a^2}} > \frac{\sin^{-1}(b) - \sin^{-1}(a)}{b-a} > \frac{1}{\sqrt{1-b^2}}$$

$$\boxed{\frac{b-a}{\sqrt{1-a^2}} > \sin^{-1}(b) - \sin^{-1}(a) > \frac{b-a}{\sqrt{1-b^2}}}$$

$a=\frac{1}{2}$, $b=\frac{3}{5}$ we get (i) Result

(3) P.T $\frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3}-\frac{1}{8}$

Sol) $f(x)=\cos^{-1}x$.

$f(x)$ is continuous

$$f'(x)=\frac{d(\cos^{-1}x)}{dx}$$

$$=\frac{-1}{\sqrt{1-x^2}}$$

$f'(x)$ is ~~diff.~~ exists at any value of $x$.

$f(x)$ is differentiable on $(a,b)$

Hence $f(x)$ satisfied all conditions of L.M.V.T

By L.M.V.T there exists atleast one point $c\in(a,b)$

S.T
$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\frac{-1}{\sqrt{1-c^2}}=\frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a} \rightarrow (1)$$

$$a<c<b$$

$$a-1<c-1<b-1$$

$$a^2-1<c^2-1<b^2-1$$

$$1-a^2<1-c^2<1-b^2$$

$$\frac{1}{1-a^2}<\frac{1}{1-c^2}<\frac{1}{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}}>\frac{1}{\sqrt{1-c^2}}>\frac{1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}}>\frac{1}{\sqrt{1-c^2}}>\frac{1}{\sqrt{1-b^2}} \rightarrow (2)$$

- (1) in (2)

$$-\frac{1}{\sqrt{1-a^2}} < \frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a} < -\frac{1}{\sqrt{1-b^2}}$$

$$-\frac{(b-a)}{\sqrt{1-a^2}} < \cos^{-1}(b)-\cos^{-1}(a) < -\frac{(b-a)}{\sqrt{1-b^2}}$$

Taking $b=\frac{3}{5}$, $a=\frac{1}{2}$, we get result ②

i.e $\Rightarrow \frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3}-\frac{1}{8}$

x.01 unit

③ S.T 1) $1+x < e^x < 1+xe^x$

2) Calculate approximately $\sqrt[3]{245}$ by using L.M.V.T

3) Using mean value theorem p.T $\tan x > x$ in $0<x<\frac{\pi}{2}$

① sol Let $f(x)=e^x$ is continuous on $[0,x]$

(i) $f(x)=e^x$ is continuous on $[0,x]$

(ii) $f'(x)=e^x$

$f'(x)$ exists for every $x$.

so $f(x)$ is differentiable in $(0,x)$

Hence $f(x)$ satisifies all conditions of L.M.V.T

By L.M.V.T there exists atleast one point $c\in(0,3)$

S.T $f'(c)=\frac{f(x)-f(0)}{x-0}$

$$e^c=\frac{e^x-e^0}{x}$$

$$e^c=\frac{e^x-1}{x} \rightarrow ①$$

Since $c\in(0,x)$

$$0<c<x$$

$$e^0<e^c<e^x$$

$$1 < \frac{e^{x}-1}{x} < e^{x} \text{ from } (1)$$

$$x < e^{x}-1 < xe^{x}$$

$$\boxed{1+x < e^{x} < 1+xe^{x}}$$

Q

(ii) Let $f(x) = \sqrt[3]{x}$ in $[a,b]$

$a = 243$, $b = 245$

(i) $f(x)$ is continuous on $[243, 245]$;

(ii) $f'(x) = \frac{1}{3} x^{-2/3}$

$f(x)$ is differentiable on $(243, 245)$

$f(x)$ satisifies all conditions of L.M.V.T

So by L.M.V.T than exisits atleast one point

$c \in (243, 245)$

s.t $f'(c) = \frac{f(245) - f(243)}{245 - 243}$

$$\frac{1}{3} c^{-2/3} = \frac{(245)^{1/3} - (243)^{1/3}}{245 - 243}$$

$c = 244 \in (243, 245)$

$$\frac{1}{3}(244)^{-2/3} = \frac{(245)^{1/3} - (243)^{1/3}}{2}$$

$$\frac{2}{3}(244)^{-2/3} + (243)^{1/3} = (245)^{1/3}$$

$$\boxed{6.257324 = (245)^{1/3}}$$

(iii) Let $f(x) = \tan x$, $0 < \varepsilon < x < \frac{\pi}{2}$

(i) $f(x)$ is continuous in $[\varepsilon, x]$

(ii) $f'(x) = \sec^2 x$

$f'(x)$ = exist in $(\varepsilon, x)$

$f(x)$ is differentiable in $(\varepsilon, x)$

$f(x)$ satisifies all condition of L.M.V.T

By L.M.V.T there exists $c \in (\varepsilon, x)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(x) - f(\varepsilon)}{x - \varepsilon}$$

$$\sec^2 c = \frac{\tan x - \tan \varepsilon}{x - \varepsilon}$$

$$(\tan x - \tan \varepsilon) = (x - \varepsilon) \sec^2 c$$

If we ~~have~~ take $\varepsilon \to 0^+$

$$(\tan x - \tan 0) = (x - 0) \sec^2 c$$

$$\tan x = x \sec^2 c$$

$$\tan x > x \quad (1)$$

$$\boxed{\tan x > x}$$

3703 3π

**Cauchy Mean value theory :-**

If $f : [a,b] \to R$ & $g : [a,b] \to R$

are such that

(i) $f, g$ are continuous of $[a,b]$

(ii) $f, g$ are differentiable on $(a,b)$ and

(iii) $g'(x)$ not equal to zero, $g'(x) \neq 0 \ \forall x \in (a,b)$

then there exists atleast one point $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

(8) Find c of C.M.T by $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ in $[a,b]$ where $0<a<b$.

Sol (i) $f(x)$ & $g(x)$ are continuous on $[a,b]$ because $0<a<b$

(ii) $f'(x) = \frac{1}{2\sqrt{x}}$ ; $g'(x) = \frac{-1}{2}x^{-1/2-1}$

$$= \frac{-1}{2x^{3/2}}$$

These are not defined at $x=0$ but $0<a<b$

$f'(x)$ & $g'(x)$ exists in $(a,b)$

$f(x)$ and $g(x)$ are differentiable in $(a,b)$

(iii) $g'(x) = \frac{-1}{2x^{3/2}} \neq 0 \ \forall x \in (a,b)$

Hence f & g satisfies all conditions of C.M.V.T, by C.M.V.T there exists atleast one point $c \in (a,b)$ st.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\cancel{f'(c)}}{g'(c)} \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c^{3/2}}}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{b}\sqrt{a}}} = -\frac{c^{3/2}}{c^{1/2}}$$

$$\frac{\frac{(\sqrt{a}-\sqrt{b})}{(\sqrt{a}-\sqrt{b})}}{\sqrt{ab}} = \not{c}$$

$$\boxed{c = \sqrt{ab}}$$

Since $\sqrt{ab}$ is the geometric means

So $a < \sqrt{ab} < b$

$c = \sqrt{ab} \in (a,b)$

P(2) (i) Verify C.M.V.T for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$

P(3) (ii) Find $c$ of C.M.V.T on $[a,b]$ for $f(x) = e^x$ and $g(x) = e^{-x}$ $(a,b>0)$

P(4) (iii) Discuss the applicability of C.M.V.T $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ on $[a,b]$

(i) Given $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$

(i) $f(x)$ & $g(x)$ are continuous on $[1,2]$ ~~bec~~ because $f(x)$, $g(x)$ are polynomials.

(ii) $f'(x) = 2x$ ; $g'(x) = 3x^2$

are exists in $(1,2)$

$\therefore f(x), g(x)$ are differentiable in $(1,2)$

(iii) $g'(x) = 3x^2 \neq 0$ $\forall x \in (1,2)$

hence $f(x)$ and $g(x)$ satisfies all conditions of C.M.V.T, by C.M.V.T there exists atleast one point $c \in (1,2)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{4-1}{8-1} = \frac{2c}{3c^2}$$

$$\frac{3}{7} = \frac{2}{3c}$$

$$9c = 14$$

$$c = \frac{14}{9} = 1.5$$

$$\boxed{c = 1.5} \in (1,2)$$

Hence C.M.V.T are verified

(ii) Given $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$ ; $(a,b,>0)$

(i) $f(x)$ and $g(x)$ are continuous in $[a,b]$ because it is exponential functions

(ii) $f'(x) = e^x$ ; $g'(x) = -e^{-x}$ are exists in $(a,b)$

$\therefore$ $f(x)$ and $g(x)$ are differentiable in $(a,b)$

(iii) $g'(x) = -e^{-x} \neq 0 \ \forall x \in (a,b)$

Hence $f(x)$ and $g(x)$ satisifies all conditions of C.M.V.T, by C.M.V.T there exists atleast one point $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$e^b - e^a$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{\frac{1}{-e^c}}$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -(e^c)^2$$

$$\frac{e^b - e^a}{\frac{e^a - e^b}{e^a e^b}} = -(e^c)^2$$

$$\frac{\cancel{-}\cancel{(e^a - e^b)}}{\frac{\cancel{e^a - e^b}}{e^a e^b}} = \cancel{-}(e^c)^2$$

$$e^a e^b = e^{c^2}$$

$$e^{a+b} = e^{c^2}$$

$$\cancel{e}^{a+b} = \cancel{e}^{2c}$$

$$2c = a+b$$

$$c = \frac{a+b}{2} \in (a,b)$$

Hence C.M.V.T are verified

(iii) Given $f(x) = \frac{1}{x^2}$ , $g(x) = \frac{1}{x}$ on $[a,b]$

(i) $f(x)$ and $g(x)$ are continuous in $[a,b]$ because it is polynomial ~~fn~~

(ii) $f(x) = x^{-2}$ $\qquad$ $g(x) = x^{-1}$

$= -2x^{-3}$ $\qquad$ $= -1x^{-2}$ are exists in $(a,b)$

$\therefore$ f(x) and g(x) are differentiable in (a,b)

(iii) $g'(x) = -x^{-2} \neq 0 \quad \forall x \in (a,b)$

Hence f(x) and g(x) satisfies all conditions of C.M.V.T, by C.M.V.T there exists atleast one point $c \in (a,b)$ S.T

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}} = \frac{+2c^{-3}}{+c^{-2}}$$

$$\frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}} = \frac{2c^{-3}}{c^{-2}}$$

$$\frac{\frac{(a+b)\cancel{(a-b)}}{(ab)(ab)}}{\frac{\cancel{a-b}}{\cancel{ab}}} = \quad 2c^{-3} \times c^{2} = 2c^{-1}$$

$$\frac{(a+b)}{ab} = 2c^{-1}$$

$$2c^{-1} = \frac{a+b}{ab}$$

$$c^{-1} = \frac{(a+b)}{2ab}$$

$$c = \frac{2ab}{(a+b)} \in (a,b)$$

$\therefore$ Hence C.M.V.T are verified

(15) Verify C.M.V.T for $f(x) = \sin x$ & $g(x) = \cos x$ on $[0, \frac{\pi}{2}]$

(16) Verify C.M.V.T for $f(x)$ and $f(x)$ in $(1, e)$ given $f(x) = \log x$

(17) Verify generalised mean value theorem for $f(x) = e^x$, $g(x) = e^{-x}$ in $[3, 7]$

(18) If $f(x) = \log x$ and $g(x) = x^2$ in $[a, b]$ with $b > a > 1$ using C.M.V.T

Prove that $\frac{\log b - \log a}{b - a} = \frac{a+b}{2c^2}$

(15) Given $f(x) = \sin x$ ; $g(x) = \cos x$ on $[0, \frac{\pi}{2}]$

(i) $f(x)$ is continuous on $[0, \frac{\pi}{2}]$ because $f(x), g(x)$ are trignometric functions

(ii) $f'(x) = \cos x$ $\qquad g'(x) = -\sin x$

$f'(x)$ and $g'(x)$ exist in $(0, \frac{\pi}{2})$

$f(x)$ and $g(x)$ are differentiable in $(a, b)$

(iii) $g'(x) = -\sin x \neq 0 \quad \forall x \in (0, \frac{\pi}{2})$

hence $f(x)$ and $g(x)$ satisfies all conditions of C.M.V.T there exists atleast one point $c \in (0, \frac{\pi}{2})$ s.t

$$\frac{f(\frac{\pi}{2}) - f(0)}{g(\frac{\pi}{2}) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sin\frac{\pi}{2} - \sin 0}{\cos\frac{\pi}{2} - \cos 0} = \frac{-\sin c}{\cos c} \quad \frac{\cos c}{-\sin c}$$

$$\frac{1-0}{0-1} = \frac{-\sin c}{\cos c} \quad \frac{\cos c}{\sin c}$$

$1 = \cot c$

$\cancel{c=\theta}$

$\cot c = 1 \in (0, \frac{\pi}{2})$

$\boxed{c = \frac{\pi}{4}}$

Hence C.M.V.T is verified.

(16) Given ~~f(x) = sin x~~ ~~&~~ $f(x) = \log x$, $g(x) = f'(x) = \frac{1}{x}$ (1,e)

(i) f(x) and f(x) are continuous on [1,e] because f(x), g'(x) are polynomials

(ii) $f'(x) = \frac{1}{x}$ , $g'(x) = \frac{-1}{x^2} \in (1,e)$

f(x) and g(x) are differentiable on (1,e)

(iii) $g'(x) = \frac{-1}{x^2} \neq 0 \; \forall x \in (1,e)$

Hence f(x) and g(x) satisifies all conditions of C.M.V.T. By C.M.V.T there exists atleast on point $c \in (1,e)$

$$\frac{\log b - \log 1}{\frac{1}{e} - \frac{1}{1}} = \frac{\frac{1}{c}}{\frac{-1}{c^2}}$$

$$\frac{\log e - 0}{\frac{1-e}{e}} = \frac{\cancel{c}^2}{-\cancel{c}}$$

$$\frac{e}{1-e} = -c$$

$$c = \frac{-e}{1-e}$$

$c \in ($

Hence C.M.V.T is verified.

(9) Given $f(x) = e^{x}$, $g(x) = e^{-x}$ in $[3,7]$

(i) $f(x)$ and $g(x)$ is continuous on $[3,7]$ because $f(x)$ & $g(x)$ are exponential functions on $[3,7)$

(ii) $f'(x) = e^{x}$ , $g'(x) = -e^{-x}$ are exist in $(3,7)$

$f(x)$, $g(x)$ are differentiable on $(3,7)$

(iii) $g'(x) = -e^{-x} \neq 0 \;\forall\; x \in (3,7)$

Hence $f(x)$ & $g(x)$ satisfies all condition of C.M.V.T. By C.M.V.T there exists atleast one point $c \in (3,7)$ s.t

$$\frac{f(7) - f(3)}{g(7) - g(3)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^{7} - e^{3}}{e^{-7} - e^{-3}} = \frac{e^{c}}{-e^{-c}}$$

$$\frac{e^{7} - e^{3}}{\frac{1}{e^{7}} - \frac{1}{e^{3}}} = \frac{e^{c}}{\frac{1}{e^{-c}}}$$

$$\frac{e^{7} - e^{3}}{\frac{e^{3} - e^{7}}{e^{7}e^{3}}} = -(e^{c})^{2}$$

$$\frac{-\cancel{(e^{3} - e^{7})}}{\frac{\cancel{e^{3} - e^{7}}}{e^{7}e^{3}}} = \cancel{-}(e^{c})^{2}$$

$$e^{10} = e^{2c}$$

$$2c = 10$$

$\boxed{c=5} \in (3,7)$

Hence C.M.V.T is verified.

P8) Given $f(x) = \log x$ ; $g(x) = x^2$ in $[a,b]$

(i) $f(x)$ and $g(x)$ are continuous on $[a,b]$ bcoz $f(x)$ & $g(x)$ are polynomials

(ii) $f'(x) = \frac{1}{x}$ , $g'(x) = 2x \in (a,b)$

$f(x)$ and $g(x)$ are differentiable on $(a,b)$

(iii) $g'(x) = 2x \neq 0 \ \forall x \in (a,b)$

Hence $f(x)$ and $g(x)$ satisifies all conditions of C.M.V.T. By C.M.V.T then there exists atleast one point $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\log b - \log a}{b^2 - a^2} = \frac{1/c}{2c}$$

$$\frac{\log b - \log a}{(b+a)(b-a)} = \frac{1}{2c^2}$$

$$\frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2}$$

* Taylor's theorem:

If $f : [a,b] \to R$ is such that

(i) $f^{(n-1)}$ is continuous on $[a,b]$

(ii) $f^{(n-1)}$ is derivable on $(a,b)$

(iii) $f^{(n-1)}$ is derivable on $(a\ b)$

(or)

$f^{n}$ exists on $[a,b]$ and $p \in z^{+}$ then there exists a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!} \cdot f'(a) + \frac{(b-a)^2}{2!} f''(a) \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where $R_n = \dfrac{(b-a)^p (b-c)^{n-p} f^{n}(c)}{(n-1)!\ p}$

* Lagranges form of Remainder

Put $p = n$ we get $R_n = \dfrac{(b-a)^n \cancel{(b-c)} f^{n}(c)}{n!}$

* Cauchy's form of Remainder:

Put $p=1$, we get $R_n = \dfrac{(b-a)(b-c)^{n-1} f^{n}(c)}{(n-1)!}$

* Another form of Taylor's theorem

If $f : [a, a+\cancel{b}h] \to R$ such that

(i) $f^{(n-1)}$ is continuous on $[a, a+h]$

~~(ii) $f^{(n-1)}$ is differentiable~~

(ii) $f^{(n-1)}$ is derivable on $(a, a+h)$ and $p \in z^{+}$

Then there exists a real number $0<\theta<1$ s.t

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1} f^{(n-1)}(a)}{(n-1)!} + R_n$$

where $R_n = \dfrac{h^{n}(1-\theta)^{n-p} f^{n}(a+\theta h)}{p(n-1)!}$

* Lagranges form of Remainder :-

$$R_n = \frac{h^n f^n(a+\theta h)}{n!}$$

* Cauchy's form of Remainder :-

$$R_n = \frac{h^n(1-\theta)^{n-1} f^{(n)}(a+\theta h)}{(n-1)!}$$

* Maclaurin's theorem

If $f : [0,x] \to R$ such that

(i) $f^{(n-1)}$ is continuous on $[0,x]$

(ii) $f^{(n-1)}$ is derivable on $(0,x)$ and $p \in z^+$

then there exists a real number $\theta = (0,1)$ s.t

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)} f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x).$$

* Lagranges form of Remainder

Putting $p = n$ we get $R_n = \dfrac{x^n}{n!} f^n(\theta x)$

* Cauchy's form of Remainder

Put $p = 1$, we get $R_n = \dfrac{x^n(1-\theta)^{n-1} f^{(n)}(\theta x)}{(n-1)!}$

NOTE

1) Taylor series expansion of $f(x)$ about $x=a$ is

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots$$

2) Maclaurin's theorem series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

* obtain Taylor's series expansion of $\sin x$ in powers of $x - \frac{\pi}{4}$

Sol) The Taylor series expansion of $f(x)$ about $x=a$ is

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

put $a = \frac{\pi}{4}$

$$f(x) = f\left(\frac{\pi}{4}\right) + \frac{x-\frac{\pi}{4}}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots \rightarrow (1)$$

~~$f(x) = f\left(\frac{\pi}{4}\right) + x$~~

$f(x) = \sin x \Rightarrow f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

Substitute $f\left(\frac{\pi}{4}\right), f'\left(\frac{\pi}{4}\right), f''\left(\frac{\pi}{4}\right) \ldots$ in (1)

$$\sin x = \frac{1}{\sqrt{2}} + \frac{x-\frac{\pi}{4}}{1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{x-\pi}{4} - \frac{\left(x-\frac{\pi}{4}\right)^2}{2} + \cdots\right.$$

(2)(Q) Obtain taylor's series expansion of $e^x$ in powers of $x+1$

Sol The Taylor series expansion of $f(x)$ about $x=a$ is

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

put $a=-1$

$$f(x) = f(-1) + \frac{x+1}{1!} f'(-1) + \frac{(x+1)^2}{2!} f''(-1) + \cdots \rightarrow (1)$$

$$f(x) = e^x \Rightarrow f(-1) = e^{-1} = \frac{1}{e}$$

$$f'(x) = e^x \Rightarrow f'(-1) = e^{-1} = \frac{1}{e}$$

$$f''(x) = e^x \Rightarrow f''(-1) = e^{-1} = \frac{1}{e}$$

substituting $f(-1), f'(-1), f''(-1) \ldots$ in (1)

$$e^x = \frac{1}{e} + \frac{x+1}{1!}\frac{1}{e} + \frac{(x+1)^2}{2!}\frac{1}{e} + \cdots$$

$$= \frac{1}{e}\left[1 + \frac{x+1}{1!} + \frac{(x+1)^2}{2!} + \cdots\right]$$

(3)Q Find the Taylor series expansion of $\sin 2x$ about $x = \frac{\pi}{4}$

Sol The Taylor series expansion of $f(x)$ about $x=a$ is

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \rightarrow (1)$$

Put $a = \frac{\pi}{4}$

$$f(x) = f\left(\frac{\pi}{4}\right) + \frac{x-\frac{\pi}{4}}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots \rightarrow (2)$$

$f(x) = \sin 2x \Rightarrow f(\frac{\pi}{4}) = \sin 2\times\frac{\pi}{4} = 1$

$f'(x) = 2\cos 2x \Rightarrow f'(\frac{\pi}{4}) = 2\times\cos 2\times\frac{\pi}{4} = 0$

$f''(x) = 4\sin 2x \Rightarrow f''(\frac{\pi}{4}) = 4$

Sub above values in ①

$$\sin 2x = 1 + \frac{x-a}{1!}(0) + \frac{(x-a)^2}{2!}4 + \cdots +$$

$$\sin 2x = 1 + \frac{(x-a)^2}{2!}4 + \cdots$$

$$\sin 2x = 1 + \frac{4(x-a)^2}{2!} + \cdots$$

* Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval $[0,1]$

Sol (i) $f(x)$, $f'(x)$ are continuous in $[0,1]$

(ii) $f(x)$, $f'(x)$ are differentiable in $(0,1)$

$f(x)$ satisified conditions of Taylors theorem

by Taylors theorem with lagrang's form of remain-der

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(\theta x) \rightarrow ①$$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$$

$$f'(x) = \frac{5}{2}(1-x)^{5/2-1}(-1) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$$

$$f''(x) = \frac{-5}{2}\cdot\frac{3}{2}(1-x)^{3/2-1}(-1) = \frac{15}{4}(1-x)^{1/2}$$

$$\Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$$

Substituting $f(0)$, $f'(0)$, $f''(\theta x)$ in ①

$$(1-x)^{5/2} = 1+x\left(\frac{-5}{2}\right)+\frac{x^2}{2}\left(\frac{15}{4}\right)(1-\theta x)^{1/2}$$

put $x=1$

$$\theta = 1-\frac{5}{2}+\frac{1}{2}\left(\frac{15}{4}\right)(1-\theta)^{1/2}$$

$$\theta = \frac{8-20+15(1-\theta)^{1/2}}{8}$$

$$(1-\theta)^{1/2} = \frac{\cancel{12}}{\cancel{15}}\frac{4}{5}$$

$$1-\theta = \frac{16}{25}$$

$$\theta = 1-\frac{16}{25}$$

$$\theta = \frac{9}{25} = 0.36$$

$$\theta = 0.36 \in (0,1)$$

(a) Write Taylor series for $f(x) = (1-x)^{5/2}$ by using lagranges form of remainder upto 2 terms in $[0,1]$

Sol

$f(x) = (1-x)^{5/2}$ with

(i) $f(x), f'(x), f''(x)$ are continuous $[0,1]$

(ii) $f(x), f'(x), f''(x)$ are differentiable $(0,1)$

$f(x)$ satisified all conditions of Taylor th with lagranges form of remainder

by Taylor theorem

$$f(x) = f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(\theta x) \to 0$$

$$f(x) = (1-x)^{5/2} \to f(0) = 1$$

$$f'(x) = \frac{5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$$

$$f''(x) = +\frac{15}{4}(1-x)^{\frac{1}{2}} \Rightarrow f''(0) = \frac{15}{4}$$

$$f'''(x) = -\frac{15}{8}(1-x)^{-1/2} \Rightarrow f'''(\theta x) = -\frac{15}{8}(1-\theta x)^{-1/2}$$

substituting $f(0), f'(0), f''(0)$ & $f'''(\theta x)$ in ①

$$(1-x)^{5/2} = 1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{15}{48}(1-\theta x)^{-1/2}$$

(Q) Obtain maclaurin's series expansion of the following function

i) $e^x$, (ii) $\sin x$ (iii) $\cos x$ (iv) $\sin hx$ (v) $\cos hx$

Sol: Maclaurin's series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots \rightarrow ①$$

(i) $f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

Sub above values in ①

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots +$$

This expansion is verified for all $x$, R.H.S of expansion converges to $e^x$.

NIL

⑦ Obtain maclaurin's series expansion of $\log_e (1+x)$

⑧ Obtain the maclaurin's series exp for $f(x) = (1+x)^n$

⑨ S.T. $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{4x^3}{3!} + \cdots$

Sol $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, f(x) = \sin^{-1}(x) \rightarrow ①$

differentiating w.r.to $x$

$\sqrt{1-x^2}\, f'(x) + f(x) \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{(1-x^2)f'(x) - xf(x)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

$(1-x^2)f'(x) - xf(x) = 1 \rightarrow ②$

put $x=0$

$f'(0) - 0 = 1 \Rightarrow f'(0) = 1$

differentiating ② w.r.t $x$

$(1-x^2)f''(x) + f'(x)(-2x) - xf'(x) - f(x) = 0$

$(1-x^2)f''(x) - 3x f'(x) - f(x) = 0 \rightarrow ③$

put $x=0 \Rightarrow f''(0) - 0 - f(0) = 0$

$f''(0) = 0$

differentiating ③ w.r.t $x$

$(1-x^2) f'''(x) - 2xf''(x) - 3xf''(x) - 3f'(x) - f'(x) = 0$

$(1-x^2)f'''(x) - 5xf''(x) - 4f'(x) = 0$

put 42<47

$$f'''(0) - 0 \cdot 4f'(0) = 0$$

$$f'''(0) \cdot 4(1) = 0$$

$$f'''(0) = 4$$

$$\text{M.S.F} = \frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{4x^3}{3!} + \cdots$$

(9) S.T $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and hence deduce that

$$\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

Sol)

$\cdot$ ~~$f(x) = \log(1+e^x)$~~

let $f(x) = \log(1+e^x)$

M.S.F for $f(x)$ is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

$f(x) = \log(1+e^x) \Rightarrow f(0) = \log 2$

$f'(x) = \frac{e^x}{1+e^x} \Rightarrow f'(0) = \frac{1}{2}$

$$f''(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} \Rightarrow f''(0) = \frac{2-1}{4} = \frac{1}{4}$$

substituting above values in (1)

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \left(\frac{x^2}{2!}\right)\left(\frac{1}{4}\right) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$$

Differentiating above equation

$$\frac{1}{1+e^x}(1+e^x)' = 0 + \frac{1}{2} + \frac{2x}{8} + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

(a) Obtain the M.S.E for $f(x) = (1+x)^n$

Sol) let $f(x) = (1+x)^n$

M.S.E for $f(x)$ is $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} \ldots \rightarrow (1)$

$f(x) = (1+x)^n \Rightarrow f(0) = 1$

$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2} \Rightarrow f''(0) = n(n-1)$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \Rightarrow f'''(0) = n(n-1)(n-2)(n-3)$

sub the above value in (1)

$(1+x)^n = 1 + xn + \frac{x^2}{2!} n(n-1) + \frac{x^3}{3!} n(n-1)(n-2)(n-3) \ldots$

$f''(k) = n(n-1) \ldots [n-(k-1)][1+x]^{n-k}$

$f''(0) = n(n-1) \ldots [n-k+1]$

$f(x) = \sum_{k=0}^{\infty} f^k(0) = \frac{x^k}{k!}$

$= \sum_{k=0}^{\infty} (n(n-1) \ldots (n-k+1) \frac{x^k}{k!}$

This expansion is vaid in $-1 \leq x \leq 1$.

(a) Obtain the M.S.F for $f(x) = \log_e(1+x)$

Sol) Given $f(x) = \log_e(1+x)$

The maclaurin series expansion of $f(x)$

$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \rightarrow (1)$

$f(x) = \log(1+x) \Rightarrow f(0) = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(0) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = \frac{2}{1} = 2$

sub in (i)

$$\log(1+x) = x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \cdots +$$

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!}$$

This expansion is vaid only when

$-1 \leq x \leq 1$

(3) Obtain M.S.E for $f(x)$

(i) $\sin x$, (ii) $\cos x$, $\sin hx$, $\cos hx$.

The maclaurins series expansion for $f(x)$ is

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

(i) $f(x) = \sin x = f(0) = 0$

$f'(x) = \cos x = f'(0) = 1$

$f''(x) = -\sin x = f''(0) = 0$

$f'''(x) = -\cos x = f'''(0) = -1$

sub in (1)

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

(ii) $f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

sub above values in (1)

$\cancel{\cos x = 1 - \frac{x^2}{2!} + x^3}$

$\cos x = 1 - \frac{x^2}{2!} + \ldots$

(iii) $f(x) = \sinh x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = 1$

$f''(x) = \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = -1$

sub the above values in (1)

$\sinh x = 0 + 1(x) + 0 + 1\left(\frac{x^3}{3!}\right) + \ldots$

$\sinh x = x + \frac{x^3}{3!} + \ldots$

(iv) $f(x) = \cosh x \Rightarrow f(0) = 1$

$f'(x) = \sinh x \Rightarrow f'(0) = 0$

$f''(x) = \cosh x \Rightarrow f''(0) = 1$

$f'''(x) = \sinh x \Rightarrow f'''(0) = 0$

sub the above values in (1)

$\cosh x = 1 + \frac{x^2}{2!} + \ldots$

12-03 FRI

Expand $e^{x\sin x}$ in powers of $x$

Sol) The M.S.E of $f(x)$ is

$f(x) = f(0) + \frac{x}{1!} f'(x) + \frac{x^2}{2!} f''(x) + \ldots \rightarrow (1)$

$f(x) = e^{x\sin x} \Rightarrow f(0) = 1$

$f'(x) = e^{x\sin x} \frac{d}{dx}(x\sin x) = e^{x\sin x}(x\cos x + \sin x)$

$f'(0) = e^0 \sin 0 \, (0\cos 0 - i \sin 0)$

$= 0$

$f''(x) = f(x)(-x\sin x + \cos x + \cos x) + (x\cos x + \sin x) f'(x)$

$f''(0) = f(0)\{-0\sin 0 + \cos 0 + \cos 0) + (0+0) f'(0) = 2$

$= 2$

sub above values in ①

$e^{x\sin x} = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(2) + \cdots$

$= 1 + \frac{2x^2}{2!} + \cdots +$