

COURSE MATERIAL

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|---------------------------------|---|
| SUBJECT | LINEAR ALGEBRA & CALCULUS (MA20ABS101) |
| UNIT | I |
| COURSE | B. TECH |
| DEPARTMENT | COMMON TO ALL BRANCHES |
| SEMESTER | 1-1 |
| PREPARED BY (Faculty Name/s) | DEPARTMENT OF MATHEMATICS |
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1. COURSE OBJECTIVES

1. This course will illuminate the students in the concepts of calculus and linear algebra.
2. To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real-world problems and their applications.

2. PREREQUISITES

Students should have knowledge on

1. Basic Mathematics & Calculus

3. SYLLABUS**UNIT I**

Matrices:

Rank of a matrix by echelon form, normal form. Solving system of homogeneous and non-homogeneous linear equations. Eigen values and Eigenvectors and their properties, Cayley-Hamilton theorem (without proof), finding inverse and power of a matrix by Cayley-Hamilton theorem, Diagonalisation of a matrix.

4. COURSE OUTCOMES

After the completion of course, students will be able to

1. Develop the use of matrix algebra techniques that is needed by engineers for practical applications (L6)
2. Utilize mean value theorems to real life problems (L3)
3. Familiarize with functions of several variables which is useful in optimization (L3)
4. Apply multiple integrals to find the area and volumes for different functions. (L3)
5. Analyze the concepts of Beta and Gamma special function for different functions. (L4)

1. Co-PO / PSO Mapping

| BEEE | PO1 | PO2 | PO3 | PO4 | PO5 | PO6 | PO7 | PO8 | PO9 | P10 | PO11 | PO12 | PSO1 | PSO2 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|
| CO1 | 3 | 3 | | | | | | | | | | | | |
| CO2 | 3 | 3 | | | | | | | | | | | | |
| CO3 | 3 | 3 | | | | | | | | | | | | |
| CO4 | 3 | 3 | | | | | | | | | | | | |
| CO5 | 3 | 3 | | | | | | | | | | | | |

2. LESSON PLAN

| Lecture No. | Weeks | Topics to be covered | References |
|-------------|-------|---|------------|
| 1 | 1 | Introduction to Matrices and types of matrices | T1, R1 |
| 2 | | Rank of a matrix and Elementary transformations | T1, R1 |
| 3 | | Echelon form and problems | T1, R1 |
| 4 | | Normal form and problems | T1, R1 |
| 5 | 2 | System of Homogeneous and Non-Homogeneous equations | T1, R1 |
| 6 | | Problems | T1, R1 |
| 7 | | Problems | T1, R1 |
| 8 | | Eigen values and Eigen vectors | T1, R1 |
| 9 | 3 | Problems on Eigen values | T1, R1 |
| 10 | | Problems on Eigen values | T1, R1 |
| 11 | | Properties of eigen values | T1, R1 |
| 12 | | Properties of eigen values | T1, R1 |
| 13 | 4 | Cayley Hamilton theorem and problems | T1, R1 |
| 14 | | Problems | T1, R1 |
| 15 | | Diagonalization and problems | T1, R1 |
| 16 | | Diagonalization and problems | T1, R1 |

7. ACTIVITY BASED LEARNING

8. LECTURE NOTES

1.1.1 INTRODUCTION TO MATRICES

Matrix: A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between $[]$ (or) $()$ (or) $||$ is called a matrix of order $m \times n$.

$$\text{Eg: } \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n} \text{ where } 1 \leq i \leq m, 1 \leq j \leq n.$$

$m \times n$

1.1.2 Types of Matrices

1. **Square matrix:** A square matrix A of order $n \times n$ is sometimes called as a n -rowed matrix A (or)

simply a square matrix of order n

Eg: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2nd order matrix

2. **Rectangular matrix:** A matrix which is not a square matrix is called a rectangular matrix,

Eg: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2×3 matrix

3. **Row matrix:** A matrix of order $1 \times m$ is called a row matrix

Eg: $[1 \ 2 \ 3]_{1 \times 3}$

4. **Column matrix:** A matrix of order $n \times 1$ is called a column matrix

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3 \times 1}$

5. **Unit matrix:** if $A = [a_{ij}]_{n \times n}$ such that $a_{ij} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, then A is called a unit matrix.

$$\text{Eg: } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. **Zero matrix:** if $A = [a_{ij}]_{m \times n}$ such that $a_{ij} = 0 \forall i$ and j then A is called a zero matrix (or) null matrix

$$\text{Eg: } O_{3 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Principal Diagonal elements in a matrix:

$A = [a_{ij}]_{n \times n}$, the elements a_{ij} of A for which $i = j$, i.e. $(a_{11}, a_{22}, \dots, a_{nn})$ are called the principal diagonal elements of A

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

6. Diagonal matrix: A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If d_1, d_2, \dots, d_n are diagonal elements of a diagonal matrix A , then A is written as $A = \text{diag}(d_1, d_2, \dots, d_n)$

E.g.: $A = \text{diag}(3, 1, -2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

7. Scalar matrix: A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

E.g.: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8. Equal Matrices: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if

(i) A and B are of the same type (order) (ii) $a_{ij} = b_{ij}$ for every i & j

9. The transpose of a matrix: The matrix obtained from any given matrix A , by interchanging its rows and columns is called the transpose of A . It is denoted by A^1 (or) A^T .

If $A = [a_{ij}]_{m \times n}$ then the transpose of A is $A^1 = [b_{ji}]_{n \times m}$, where $b_{ji} = a_{ij}$ Also $(A^1)^1 = A$

Note: A^1 and B^1 be the transposes of A and B respectively, then

(i) $(A^1)^1 = A$

(ii) $(A+B)^1 = A^1 + B^1$

(iii) $(KA)^1 = KA^1$, K is a scalar

(iv) $(AB)^1 = B^1 A^1$

10. The conjugate of a matrix: The matrix obtained from any given matrix A , on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by \bar{A}

Note: if \bar{A} and \bar{B} be the conjugates of A and B respectively then,

(i) $\overline{(\bar{A})} = A$

(ii) $\overline{A + B} = \bar{A} + \bar{B}$

(iii) $\overline{(KA)} = \bar{K}\bar{A}$ (K is a any complex number)

(iv) $\overline{AB} = \bar{A}\bar{B}$

E.g.: if $A = \begin{bmatrix} 2 & -3i \\ 3+i & -3i \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} 2 & 3i \\ 3-i & 3i \end{bmatrix}$

11. The conjugate Transpose of a matrix:

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is

denoted by A^θ Thus $A^\theta = (\bar{A})^1$

i.e. the $(i,j)^{th}$ element of A^θ conjugate complex of the $(j, i)^{th}$ element of A.

E.g.: if $A = \begin{bmatrix} 2 & -3i & -i \\ 3+i & -3i & 1 \end{bmatrix}$ then $A^\theta = \begin{bmatrix} 2 & 3-i \\ 3i & 3i \\ i & 1 \end{bmatrix}$

14. (i) Upper Triangular matrix: A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix.

E.g.: $\begin{bmatrix} 2 & 3 & 8 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix.

(ii) Lower triangular matrix: A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e, $a_{ij}=0$ for $i < j$

E.g.: $\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 6 & 5 & 2 \end{bmatrix}$

(iii) Triangular matrix: A matrix is said to be triangular matrix it is either an upper Triangular matrix or a lower triangular matrix

15. Symmetric matrix: A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j thus, A is a symmetric matrix if $A^T = A$

E.g.: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

16. Skew – Symmetric: A square matrix $A = [a_{ij}]$ is said to be skew – symmetric if $a_{ij} = -a_{ji}$ for every i and j.

E.g.: $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Thus, A is a skew – symmetric iff $A = -A^1$ or $-A = A^1$

Note: Every diagonal element of a skew – symmetric matrix is necessarily zero.

$$\text{Since } a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$$

17. Multiplication of a matrix by a scalar.

Let 'A' be a matrix. The matrix obtained by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA.

$$\text{Thus: } A = [a_{ij}] \text{ then } KA = [ka_{ij}] = k[a_{ij}]$$

18. Sum of matrices:

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ be two matrices. The matrix $C = [c_{ij}]_{m \times n}$

where $c_{ij} = a_{ij} + b_{ij}$ is called the sum of the matrices A and B.

The sum of A and B is denoted by A+B.

$$\text{Thus } [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n} \text{ and } [a_{ij} + b_{ij}]_{m \times n} = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$$

19. The difference of two matrices: If A, B are two matrices of the same type then

A+ (-B) is taken as A – B

20. Matrix multiplication: Let $A = [a_{ij}]_{m \times n}$, $B = [b_{kj}]_{n \times p}$ then the matrix $C = [c_{ij}]_{m \times p}$ where c_{ij}

is called the product of the matrices A and B in that order and we write $C = AB$.

The matrix A is called the pre-factor & B is called the post – factor

Note: If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

21. Positive integral powers of a square matrix:

Let A be a square matrix. Then A^2 is defined $A.A$

Now, by associative law $A^3 = A^2.A = (AA)A = A(AA) = AA^2$

Similarly, we have $A^{m-1}A = A A^{m-1} = A^m$ where m is a positive integer

Note 1: Multiplication of matrices is distributive w.r.t. addition of matrices.

$$\text{i.e., } A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

Note 2: If A is a matrix of order $m \times n$ then $A I_n = I_n A = A$

22. Trace of A square matrix: Let $A = [a_{ij}]_{m \times n}$ the trace of the square matrix A is defined as $\sum_{i=1}^n a_{ii}$ and is denoted by 'tr A'

$$\text{Thus } \text{tr} A = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

$$\text{E.g.: } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ then } \text{tr} A = a+b+c$$

Properties: If A and B are square matrices of order n and λ is any scalar, then

$$(i) \quad \text{tr} (\lambda A) = \lambda \text{tr} A$$

$$(ii) \quad \text{tr} (A+B) = \text{tr} A + \text{tr} B$$

$$(iii) \quad \text{tr}(AB) = \text{tr}(BA)$$

23. Idempotent matrix: If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

24. Nilpotent Matrix: If A is a square matrix such that $A^m = 0$ where m is a +ve integer then A is called nilpotent matrix.

Note: If m is least positive integer such that $A^m = 0$ then A is called nilpotent of index m

25. Involuntary: If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

26. Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA^T = A^T A = I$

27. Minors and cofactors of a square matrix

Let $A = [a_{ij}]_{m \times n}$ be a square matrix when from A the elements of i^{th} row and j^{th} column is deleted the determinant of $(n-1)$ rowed matrix $[M_{ij}]$ is called the minor of a_{ij} of A and is denoted by $|M_{ij}|$

The signed minor $(-1)^{i+j} |M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij} .

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$|A| = a_{11} |M_{11}| + a_{12} |M_{12}| + a_{13} |M_{13}| \quad (\text{or}) \\ = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

E.g.: Find Determinant of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by using minors and co-factors.

$$\text{Sol: } A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1 \begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} \\ = 1(-12-12) - 1(-4-6) + 3(-4+6) \\ = -24+10+6 = -8$$

Similarly, we can find $\det A$ by using co-factors also.

Note:

1. If A is a square matrix of order n then, $|KA| = K^n|A|$ where k is a scalar.
2. If A is a square matrix of order n, then $|A| = |A^T|$
- 3: If A and B be two square matrices of the same order, then $|AB| = |A||B|$

28. Inverse of a Matrix: Let A be any square matrix, then a matrix B, if exists such that $AB = BA = I$ then B is called inverse of A and is denoted by A^{-1} .

Note 1: $(A^{-1})^{-1} = A$ 2: $I^{-1} = I$

29. Adjoint of a matrix:

Let A be a square matrix of order n. The transpose of the matrix got from A by replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by $\text{adj } A$.

Note:

1. For any scalar k, $\text{adj}(kA) = k^{n-1} \text{adj } A$
2. The necessary and sufficient condition for a square matrix to possess inverse is that $|A| \neq 0$
3. if $|A| \neq 0$ then $A^{-1} = \frac{\text{adj } A}{|A|}$

30. Singular and Non-singular Matrices:

A square matrix A is said to be singular if $|A| = 0$

If $|A| \neq 0$ then 'A' is said to be non-singular.

Note: 1. only non-singular matrices possess inverses.

2. The product of non-singular matrices is also non-singular.

3. If A, B are invertible matrices of the same order, then

$$(i). (AB)^{-1} = B^{-1}A^{-1}$$

$$(ii). (A^{-1})^{-1} = (A^{-1})^1$$

1.1.3 RANK OF A MATRIX:

Let A be m x n matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that r is the rank of A if

- (i) Every $(r+1)^{\text{th}}$ order minor of A is '0' (zero) &
- (ii) At least one r^{th} order minor of A which is not zero.

Note:

1. It is denoted by $\rho(A)$
2. Rank of a matrix is unique.
3. Every matrix will have a rank.
4. If A is a matrix of order $m \times n$,
Rank of $A \leq \min(m, n)$
5. If $\rho(A) = r$ then every minor of A of order $r+1$, or more is zero.
6. Rank of the Identity matrix I_n is n .
7. If A is a matrix of order n and A is non-singular then $\rho(A) = n$
8. The rank of a matrix is $\leq r$ if all minors of $(r+1)$ th order are zero.
9. The rank of a matrix is $\geq r$, if there is at least one minor of order 'r' which is not equal to zero.

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$$\rightarrow \det A = 1(48-40) - 2(36-28) + 3(30-28)$$

$$= 8 - 16 + 6 = -2 \neq 0$$

We have minor of order 3

$$\rho(A) = 3$$

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3×4

$$\text{Its Rank} \leq \min(3, 4) = 3$$

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

$$\text{Determinant of minor} = 1(-49) - 2(-56) + 3(35-48)$$

$$= -49 + 112 - 39 = 24 \neq 0.$$

Hence rank of the given matrix is '3'.

1.1.4 ELEMENTARY TRANSFORMATIONS ON A MATRIX:

- i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$
- (ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow k R_i$
- (iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \rightarrow R_j + k R_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$$C_i \leftrightarrow C_j, \quad C_i \rightarrow k C_i \quad C_j \rightarrow C_j + k C_i$$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A , then B is said to be equivalent to A and it is denoted as $B \sim A$.

Note: 1. If A and B are two equivalent matrices, then $\text{rank } A = \text{rank } B$.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

1.1.5 ECHELON FORM OF A MATRIX:

A matrix is said to be in Echelon form, if

- (i). Zero rows, if any exists, they should be below the non-zero row.
- (ii). the first non-zero entry in each non-zero row is equal to '1'.
- (iii). the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note:

- 1. The number of non-zero rows in echelon form of A is the rank of ' A '.
- 2. The rank of the transpose of a matrix is the same as that of original matrix.
- 3. The condition (ii) is optional.

E.g.:

$$1. \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a row echelon form.}$$

$$2. \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ is a row echelon form.}$$

PROBLEMS

1. Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A

$$R_1 \leftrightarrow R_3$$

$$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2/7, R_3 \rightarrow R_3/9$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon form of matrix A.

The rank of a matrix A = Number of non-zero rows = 2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'.

Sol: The given matrix is of the order 4x4

If its rank is 3 $\Rightarrow \det A = 0$

$$\Rightarrow \begin{vmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{vmatrix} = 0$$

By applying $R_2 \rightarrow 4R_2 - R_1, R_3 \rightarrow 4R_3 - kR_1, R_4 \rightarrow 4R_4 - 9R_1$

$$\sim \begin{vmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$$\sim 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 1[(8-4k)3 - 1(8-4k)(4k+27)] = 0$$

$$\Rightarrow (8-4k)(3-4k-27) = 0$$

$$\Rightarrow (8-4k)(-24-4k) = 0$$

$$\Rightarrow (2-k)(6+k) = 0$$

$$\Rightarrow k = 2 \text{ or } k = -6$$

3. Find the rank of the matrix by using Echelon form $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

By applying $R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 4R_1; R_4 \rightarrow R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

By applying $R_2 \rightarrow \frac{R_2}{-1}; R_3 \rightarrow \frac{R_3}{-1}; R_4 \rightarrow \frac{R_4}{-3}$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 - R_2; R_4 \rightarrow R_4 - R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore A$ is in Echelon form

Rank of A = Number of non-zero rows

$$\therefore \rho(A) = 2$$

4. Find the rank of the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

By applying $R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 3R_1; R_4 \rightarrow R_4 + R_1$

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 - R_2; R_4 \rightarrow R_4 + R_2$

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore A$ is in Echelon form

Rank of A = Number of non-zero rows

$$\therefore \rho(A) = 2$$

5. Find the rank of the matrix $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$

By applying $R_1 \rightarrow -R_1$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

By applying $R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

By applying $R_2 \rightarrow 8R_2$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 40 & 8 & 128 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

By applying $R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 8 & 4 & 31 \\ 0 & 40 & 8 & 128 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 - 5R_2$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 8 & 4 & 31 \\ 0 & 0 & -12 & -27 \end{bmatrix}$$

$\therefore A$ is in Echelon form

Rank of A = Number of non-zero rows

$$\therefore \rho(A) = 3$$

6. Find the rank of the matrix $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

By applying $R_2 \rightarrow R_2 + 2R_1; R_3 \rightarrow R_3 + R_1; R_4 \rightarrow R_4 + 3R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 - 2R_2; R_4 \rightarrow R_4 - 3R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore A$ is in Echelon form

Rank of A = Number of non-zero rows

$$\therefore \rho(A) = 2$$

1.2. NORMAL FORM:

Every $m \times n$ matrix of rank r can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (or) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (or) $\begin{bmatrix} I_r & 0 \end{bmatrix}$ by a finite number of elementary transformations, where I_r is the r -rowed unit matrix. Normal form another name is "canonical form".

1. By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

By applying $R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 / -2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

By applying $C_2 \rightarrow C_2 - 2C_1, C_3 \rightarrow C_3 - 3C_1, C_4 \rightarrow C_4 - 4C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

By applying $C_3 \rightarrow 3C_3 - 2C_2, C_4 \rightarrow 3C_4 - 5C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$$

By applying $C_2 \rightarrow C_2 / -3, C_4 \rightarrow C_4 / 18$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By applying $C_4 \leftrightarrow C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is in normal form $[I_3 \ 0]$

Hence Rank of A is '3'.

1.3. SYSTEM OF HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR EQUATIONS:

An equation of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ where x_1, x_2, \dots, x_n are unknowns and a_1, a_2, \dots, a_n, b are constants is called linear equation in n unknowns.

Definition: Consider the system of m linear equations in n unknowns x_1, x_2, \dots, x_n as given below:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n &= b_i \\ &\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

The number a_{ij} 's are known as coefficient and b_1, b_2, \dots, b_m are constants. An ordered n -tuple (x_1, x_2, \dots, x_n) satisfying all the equations simultaneously is called a solution of system.

HOMOGENEOUS SYSTEM:

If all $b_i = 0$ for $i = 1, 2, \dots, m$.

NON-HOMOGENEOUS SYSTEM:

If all $b_i \neq 0$ i.e. at least one $b_i \neq 0$.

MATRIX REPRESENTATION:

The above system of linear non-Homogeneous equations can be written in Matrix form as $AX=B$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

AUGMENTED MATRIX:

It is denoted by $[A/B]$ or $[A \ B]$ is obtained by Augmenting A by the column B .

$$\therefore [A/B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

By reducing $[A/B]$ into its row echelon form the existence and uniqueness of solution $AX = B$ exists.

NOTE:

Given a system, we do not know in general whether it has a solution or not. If there is at least one solution, then the system is said to be consistent. If does not have any solution then the system is inconsistent.

1. CONSISTENT: A system is said to be consistent if it has at least one solution

- If $\rho(AB) = \rho(A) = r = n$ (total number of unknowns) then the system is consistent and it has unique solution.
- If $\rho(AB) = \rho(A) = r < n$ (total number of unknowns) then the system is consistent and it has Infinitely many solutions.

NOTE: To obtain solutions set $(n - r)$ variables any arbitrary value & solve for remaining unknowns.

2. INCONSISTENT: If $\rho(AB) \neq \rho(A)$ then the system is inconsistent and it has no solution.

For Non-Homogeneous System, the system $AX = B$ is consistent i.e. it has a solution.

The system is inconsistent i.e., it has no solution.

NOTE: Find the rank A and rank $[A/B]$ by reducing the augmented matrix $[A/B]$ to Echelon form by elementary row operations. Then the matrix A will be reduced to Echelon form.

This procedure is illustrated through the following examples.

Example 1: Find whether the following equations are consistent, if so solve them

$$x + y + 2z = 4; 2x - y + 3z = 9; 3x - y - z = 2.$$

Solution: The given equations can be written in the matrix form as $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

$$\text{i.e. } AX = B$$

$$\text{The Augmented matrix } [A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A/B] = 3

Since the number of non-zero rows of matrix A are 3

Since the number of non-zero rows of matrix [A/B] is 3

$$\therefore \text{Rank of A} = \text{Rank of [A/B]}$$

$$\text{i.e. } \rho(A) = \rho(AB)$$

The given system is consistent. So, it has a solution.

Since Rank of A = Rank of [A/B] = Number of unknowns (n)

\therefore The given system has a unique solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$$\Rightarrow x + y + 2z = 4 \rightarrow (1)$$

$$-3y - z = 1 \rightarrow (2)$$

$$-17z = -34 \rightarrow (3)$$

$$\text{From (3) } z = 2$$

Substituting $z = 2$ in eq(2), we get $y = -1$

Substituting $z = 2$ & $y = -1$ in eq(1), we get $x = 1$.

$\therefore x = 1, y = -1, z = 2$ is the solution.

Example 2: Find whether the following system of equations is consistent. If so solve them.

$$x + 2y + 2z = 2; 3x - 2y - z = 5; 2x - 5y + 3z = -4; x + 4y + 6z = 0.$$

Solution: The given equations can be written in the matrix form as $AX = B$

$$\text{i.e., } \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

$$\text{The Augmented matrix } [A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 3R_1; R_3 \rightarrow R_3 - 2R_1; R_4 \rightarrow R_4 - R_1$

$$[A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

Applying $R_3 \rightarrow 8R_3 - 9R_2; R_4 \rightarrow 4R_4 + R_2$, we get

$$[A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3/55; R_4 \rightarrow R_4/9$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_4 \rightarrow R_4 - R_3$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Rank of $A = 3$ & Rank of $[A/B] = 3$

\therefore Rank of $A = \text{Rank of } [A/B]$

i.e. $\rho(A) = \rho(AB)$

The given system is consistent, so it has a solution.

Since Rank of $A = \text{Rank of } [A/B] = \text{Number of unknowns (n)}$

\therefore The given system has a unique solution.

We have
$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 2y + 2z = 2 \rightarrow (1)$$

$$-8y - 7z = -1 \rightarrow (2)$$

$$z = -1 \rightarrow (3)$$

From (3) $z = -1$

Substituting $z = -1$ in eq(2), we get $y = 1$

Substituting $z = -1$ & $y = 1$ in eq(1), we get $x = 2$.

$\therefore x = 2, y = 1, z = -1$ is the solution.

Example3: Show that the equations $x + y + z = 4$, $2x + 5y - 2z = 3$, $x + 7y - 7z = 5$ are not consistent.

Solution: The given equations can be written in the matrix form as $AX = B$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

We can see $\rho(AB) = 3$, since the number of non-zero rows is 3

Applying the same row operations on A, we get from above

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the number of non-zero rows is 2 so the rank of $A = 2$

Here we have $\rho(AB) \neq \rho(A)$.

\therefore The given system is not consistent.

Example 4: Discuss for what values of λ, μ the simultaneous equations $x + y + z = 6$;

$$x + 2y + 3z = 10; x + 2y + \lambda z = \mu \text{ have}$$

- i) no solution
- ii) a unique solution
- iii) an infinite number of solutions.

Solution: The given equations can be written in the matrix form as $AX = B$

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

We have the Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$ we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$$

Case i): Let $\lambda \neq 3$ then rank of $A = 3$ and rank of $[A/B] = 3$, So that they have same rank. Then the system of equations is consistent. Here the number of unknowns are 3 which is same as the rank of A . The system of equations will have a unique solution. This is true for any value of μ .

Thus if $\lambda \neq 3$ and μ has any value, the given system of equations will have a Unique solution.

Case ii): Suppose $\lambda = 3$ & $\mu \neq 10$, then we can see that rank of $A = 2$ and rank of $[A/B] = 3$.

Since the ranks of A and $[A/B]$ are not equal, we say that the system of equations have no solution (inconsistent).

Case iii): Suppose $\lambda = 3$ & $\mu = 10$. Then we have rank of $A = \text{rank of } [A/B] = 2$

\therefore The given system of equations will be consistent.

But here the number of unknowns $= 3 > \text{rank of } A$

Hence the system has infinitely many solutions.

HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS:**Definition:**

Consider the system of m homogeneous linear equations in n unknowns x_1, x_2, \dots, x_n as given below:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\dots$$

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = 0$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

MATRIX REPRESENTATION:

The above system of linear Homogeneous equations can be written in Matrix form as $AX = 0$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here A is called coefficient Matrix. It is clear that $x_1 = 0 = x_2 = x_3 = \dots = x_n$ is a solution.

This is called Trivial solution of $AX = 0$.

DEFINITIONS:

1. The system $AX = 0$ is always consistent since $X = 0$ is a Solution of $AX = 0$. This solution is called a trivial solution of the system. The Trivial solution is called zero solution.
2. Given $AX = 0$, we try to decide whether it has a solution $X \neq 0$. Such a solution, if exists, is called a non -Trivial solution.

Note: 1. If A is non-Singular matrix i.e. $(\det A \neq 0)$ then the linear system $AX = 0$ has only the zero solution.

2. The system $AX = 0$ possesses a non – zero solution $\Leftrightarrow A$ is singular matrix.

Working Rule for Finding the Solutions of The Equation $AX = 0$:

1. i) If $r = n$ (number of variables) \Rightarrow the system of equations has only Trivial solution (Zero solution)

ii) If $r < n \Rightarrow$ the system of equations has an infinite number of Non – Trivial solutions, we shall have $n - r$ linearly independent solutions.

To obtain infinite solutions, set $(n - r)$ variables any arbitrary value and solve for the remaining Unknowns.

2.If the number of equations is less than number of variables, the solution is always other than a Trivial Solution.

If the number of equations is equal to number of variables, the necessary and Sufficient condition for the solutions other than a Trivial solution is that the determinant of the coefficient matrix is zero.

Example 1: Solve completely the system of equations:

$$x + 2y + 3z = 0; 3x + 4y + 4z = 0; 7x + 10y + 12z = 0.$$

Solution: Taking $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get the system of equations as $AX = 0$

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 7R_1$, we get

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

This is in Echelon form. Number of non -zero rows is 3.

\therefore The rank of $A = 3$

\therefore No. of variables = 3

\therefore Number of non-zero solutions is $= n - r = 3 - 3 = 0$.

$\therefore x = 0, y = 0, z = 0$ is the only solution.

Example2: Solve completely the system of equations:

$$x + y + w = 0; y + z = 0; x + y + z + w = 0; x + y + 2z = 0.$$

Solution: The given equations can be written in the matrix form as $AX = 0$

$$\text{Where } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Consider } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_1, R_4 \rightarrow R_4 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

Applying $R_4 \rightarrow R_4 - 2R_3$, we get

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

\therefore Rank of $A = 4$ and Number of variables = 4

\therefore There is no non-zero solution

Hence $x = y = z = w = 0$ is the only solution.

Example 3: Find all the solutions of system of equations.

$$x + 2y - z = 0; 2x + y + z = 0; x - 4y + 5z = 0.$$

$$\text{Solution: Let } A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix} \text{ Then } \det A = 1(5+4) - 2(10-1) - 1(-8-1)$$

$$= 9 - 18 + 9 = 0$$

\therefore The rank of $A = 2 < 3$ (no. of variables)

Applying $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of $A = 2$

\therefore Number of non-zero solutions = $n - r = 3 - 2 = 1$.

From the above matrix, $-3y + 3z = 0$

$$\Rightarrow y = z$$

Let $z = k$. Then $y = k$

From the above matrix, $x + 2y - z = 0$

$$\Rightarrow x = z - 2y$$

$$\Rightarrow x = k - 2k = -k$$

\therefore The solutions are given by $x = -k, y = k, z = k$.

Example 4: Solve completely the system of equations:

$$x + y - 2z + 3w = 0; x - 2y + z - w = 0; 4x + y - 5z + 8w = 0; 5x - 7y + 2z - w = 0.$$

Solution: The given equations can be written in the matrix form as $AX = 0$

$$AX = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & -2 & 1 & -1 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 4R_1, R_4 \rightarrow R_4 - 5R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

Applying $R_4 \rightarrow R_4/4$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2, R_4 \rightarrow R_4 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form. We have

Rank of A = The number of non-zero rows in this echelon form = 2(r) and

number of unknowns = 4

Since $r < n$. The given system has infinite number of non-trivial solutions.

\therefore Number of independent solutions = $4 - 2 = 2$ Now we shall assign arbitrary values to 2 variables and the remaining 2 variables shall be found in terms of these. The given system is equivalent to

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the equations $x + y - 2z + 3w = 0$; $-3y + 3z - 4w = 0$.

Taking $z = k_1$ and $w = k_2$ We see that $x = k_1 - \frac{5}{3}k_2$, $y = k_1 - \frac{4}{3}k_2$, $z = k_1$, $w = k_2$ constitutes the general solution of the given system.

Example 5: show that the only real number λ for which the system

$x + 2y + 3z = \lambda x$; $3x + y + 2z = \lambda y$; $2x + 3y + z = \lambda z$ has non-zero solution is 6 and solve them when $\lambda = 6$.

Solution: The given equations can be written in the matrix form as $AX = 0$

Where $A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here the number of variables $L = n = 3$

The given system of equations possesses a non-zero (non-trivial) solution, if rank of $A < \text{no. of unknowns}$ i.e., Rank of $A < 3$

For this we must have $\det A = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ we get

$$(6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda) [(-2-\lambda)(-1-\lambda) + 1] = 0$$

$$\Rightarrow (6-\lambda) (\lambda^2 + 3\lambda + 3) = 0$$

$\Rightarrow \lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow 5R_2 + 3R_1$, $R_3 \rightarrow 5R_3 + 2R_1$, we get

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$, we get

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x + 2y + 3z = 0$$

$$\Rightarrow -19y + 19z = 0$$

$$\Rightarrow y = z$$

Since Rank of A < Number of unknowns. i.e., $r = 2 < n = 3$

\therefore The given system has infinite number of non-trivial solutions.

Set $n - r = 3 - 2 = 1$. Let $z = k \Rightarrow y = k$ and from the above equations

$$-5x + 2k + 3k = 0$$

$$\Rightarrow x = k$$

$\therefore x = k, y = k, z = k$ is the solution.

1.4.1. EIGEN VALUES & EIGEN VECTORS:

Definition: Characteristic vector of a matrix:

Let $A = [a_{ij}]$ be an $n \times n$ matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that $AX = \lambda X$.

Note: If $AX = \lambda X$ ($X \neq 0$), then we say ' λ ' is the Eigen value (or) characteristic root of 'A'.

E.g.: Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$ $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AX = 1 \cdot X$$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and Characteristic root of A is "1".

Note: We notice that an Eigen value of a square matrix A can be 0. But a zero vector cannot be an Eigen vector of A.

Method of finding the Eigen vectors of a matrix:

Let $A = [a_{ij}]$ be a $n \times n$ matrix. Let X be an Eigen vector of A corresponding to the Eigen value λ .

Then by definition $AX = \lambda X$.

$$AX = \lambda X$$

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0 \text{ ----- (1)}$$

This is a homogeneous system of n equations in n unknowns. Will have a non-zero solution X if and only $|A - \lambda I| = 0$

$(A - \lambda I)$ is called characteristic matrix of A and $|A - \lambda I|$ is a polynomial in λ of degree n and is called the characteristic polynomial of A

$\therefore |A - \lambda I| = 0$ is called the characteristic equation

Solving characteristic equation of A , we get the roots, $\lambda_1, \lambda_2, \dots, \lambda_n$ these are called the characteristic roots or Eigen values of the matrix.

Corresponding to each one of these n Eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ be a given square matrix

Characteristic matrix of A is $(A - \lambda I)$

$$(A - \lambda I) = \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A - \lambda I|$

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$ we solve the $|A - \lambda I| = 0$ we get n roots; these are called Eigen values or latent values or proper values.

Let each one of these Eigen values say λ their Eigen vector X
 corresponding the given value λ is obtained by solving Homogeneous system
 and determining the non-trivial solution

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

PROBLEMS

1. Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

The characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\begin{aligned} & \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0 \\ \Rightarrow & (8-\lambda)(2-\lambda) + 8 = 0 \\ \Rightarrow & 16 + \lambda^2 - 10\lambda + 8 = 0 \\ \Rightarrow & \lambda^2 - 10\lambda + 24 = 0 \\ \Rightarrow & (\lambda - 6)(\lambda - 4) = 0 \\ \Rightarrow & \lambda = 4, 6 \text{ are the eigen values of matrix A.} \end{aligned}$$

Consider the system $\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda=4$:

Put $\lambda=4$ in the above system, we get

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 - 4x_2 = 0 \text{ --- (1)}$$

$$\Rightarrow 2x_1 - 2x_2 = 0 \text{ --- (2)}$$

From equations (1) and (2) we have $x_1 = x_2 = K$

$$\text{Eigen vector } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a Eigen vector of matrix A corresponding to the Eigen value $\lambda=4$

Eigen vector corresponding to $\lambda=6$

Put $\lambda=6$ in the above system, we get

$$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 4x_2 = 0$$

$$\Rightarrow 2x_1 - 4x_2 = 0$$

from the above equations we have $x_1 = 2x_2$

Say $x_2 = k$ then $x_1 = 2k$

$$\text{Eigen vector } X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a Eigen vector of matrix A corresponding to the Eigen value $\lambda=6$

2. Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$$\text{Sol: Let } A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\text{i.e. } |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + 1[0 - (2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)^3 + (\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)(-\lambda^2 + 4\lambda - 3) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

$$\therefore \lambda = 1, 2, 3$$

The Eigen values of A are 1, 2, 3

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to $\lambda=1$:

$$\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_3 = 0$$

$$\Rightarrow x_2 = 0$$

$$\Rightarrow x_1 = -x_3$$

$$\Rightarrow \text{Let } x_3 = k \text{ then } x_1 = -k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigenvector to the corresponding Eigen value $\lambda=1$.

Eigen vector corresponding to $\lambda=2$:

$$\begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1 = 0$ and $x_3 = 0$ and we can take any arbitrary value for $x_2 = k$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda=3$:

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_3 = 0$$

$$\Rightarrow -x_2 = 0$$

$$x_2 = 0 \text{ and } x_1 = x_3 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Find the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\text{i.e. } |A - \lambda I| = \begin{vmatrix} 6 - \lambda & -2 & 2 \\ -2 & 3 - \lambda & -1 \\ 2 & -1 & 3 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (6 - \lambda)[(3 - \lambda)^2 - 1] + 2[-2(3 - \lambda) + 2] + 2[2 - 2(3 - \lambda)] = 0$$

$$\Rightarrow (6 - \lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 2, 2, 8$$

The Eigen values of A is 2, 2, 8

The Eigen vector of matrix A corresponding to $\lambda = 2$;

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

$$\Rightarrow (A - 2I)X = 0$$

$$\Rightarrow \begin{bmatrix} 6 - 2 & -2 & 2 \\ -2 & 3 - 2 & -1 \\ 2 & -1 & 3 - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

By observing that three equations are identical

Therefore, we have to take two arbitrary constants for any two variables

Let $x_1 = k_1, x_2 = k_2$

From third equation $2k_1 - k_2 + x_3 = 0$

$$x_3 = -2k_1 + k_2$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 + k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ -2k_1 \end{bmatrix} + \begin{bmatrix} 0 \\ k_2 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to $\lambda = 8$;

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

$$\Rightarrow (A - 8I)X = 0$$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

By solving above three equations we get $x_1 = 2, x_2 = -1, x_3 = 1$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

4. Find the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\text{i.e. } |A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8 - \lambda)[(7 - \lambda)(3 - \lambda) - 16] + 6[-6(3 - \lambda) + 8] + 2[24 - 2(7 - \lambda)] = 0$$

$$\Rightarrow (8 - \lambda)[\lambda^2 - 10\lambda + 5] + 6[6\lambda - 10] + 2[2\lambda - 10] = 0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda = 0, 3, 15$$

The Eigen values of A is 0, 3, 15

The Eigen vector of matrix A corresponding to $\lambda = 0$;

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

$$(A - 0I)X = 0$$

$$\Rightarrow \begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

By solving above any of the two equations we get $x_1 = 1, x_2 = 2, x_3 = 2$

$$\text{Eigen vector } X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to $\lambda = 3$;

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

$$(A - 3I)X = 0$$

$$\Rightarrow \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 = 0$$

By solving above any of the two equations we get $x_1 = -2, x_2 = -1, x_3 = 2$

$$\text{Eigen vector } X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to $\lambda = 15$;

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

$$(A - 15I)X = 0$$

$$\Rightarrow \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

By solving above any of the two equations we get $x_1 = 2, x_2 = -2, x_3 = 1$

$$\text{Eigen vector } X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{The Eigen vectors are } X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

1.4.2. PROPERTIES OF EIGEN VALUES:

1. The sum of the Eigen values of a matrix A is same as trace of the matrix A.

Proof: To prove this theorem, we consider third order square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned} &= |A - \lambda I| \\ &= \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} \\ &= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(a_{21}a_{12} + a_{13}a_{31} + \dots) + (a_{11}a_{23}a_{32} + a_{12}a_{23}a_{32} + \dots) \end{aligned}$$

We have

$$\begin{aligned} \text{Sum of the roots} &= \frac{-\lambda^2 \text{ coefficient}}{\lambda^3 \text{ coefficient}} \\ &= \frac{-(a_{11} + a_{22} + a_{33})}{-1} \\ &= a_{11} + a_{22} + a_{33} \\ &= \text{Trace of A} \end{aligned}$$

Hence the result.

2. The product of the Eigen values of a matrix A is equal to its determinant.

Proof: Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the Eigen values of $A_{n \times n}$. Then

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) = 0$$

$$\text{Put } \lambda = 0$$

$$\begin{aligned} |A| &= (-1)^n (-\lambda_1)(-\lambda_2) \dots (-\lambda_n) \\ &= (-1)^n (-1)^n (\lambda_1 \lambda_2 \dots \lambda_n) \\ &= (-1)^{2n} (\lambda_1 \lambda_2 \dots \lambda_n) \end{aligned}$$

$$|A| = \lambda_1 \lambda_2 \dots \lambda_n$$

Hence the result.

3. If λ is an Eigen value of A corresponding to the Eigen vector X then λ^n is the Eigen value of A^n corresponding to the Eigen vector X .

Proof:

Since λ is an Eigen value of A corresponding to the Eigen vector X , we have

$$AX = \lambda X \text{ ----- (1)}$$

Pre multiplying equation (1) by A

$$A(AX) = A(\lambda X)$$

$$(AA)X = \lambda (AX)$$

$$A^2X = \lambda (\lambda X)$$

$$A^2X = \lambda^2 X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen Vector.

Thus, the theorem is true for $n = 2$.

Let the result be true for $n = k$.

$$\text{Then } A^k X = \lambda^k X$$

Pre multiplying this by A

$$A(A^k X) = A(\lambda^k X)$$

$$A^{k+1} X = \lambda^k (AX)$$

$$= \lambda^k (\lambda X) \quad (\text{Since } AX = \lambda X)$$

$$= \lambda^{k+1} X$$

$$A^{k+1} X = \lambda^{k+1} X$$

Which implies that λ^{k+1} is Eigen value of A^{k+1} with X itself as the corresponding Eigen Vector.

Hence by the principle of mathematical induction, the theorem is true for all positive integers n .

4. A square matrix A and its transpose A^T have the same Eigen values.

Proof:

We have

$$(A - \lambda I)^T = A^T - \lambda I^T$$

$$= A^T - \lambda I$$

$$|(A - \lambda I)^T| = |A^T - \lambda I| \quad \text{or}$$

$$|A - \lambda I| = |A^T - \lambda I| \quad [\text{Since } |A^T| = |A|]$$

$$\therefore |A - \lambda I| = 0 \text{ if and only if } |A^T - \lambda I| = 0$$

i.e., λ is an Eigen value of A if and only if λ is an Eigen value of A^T .

Thus, the Eigen values of A and A^T are same.

5. If λ is an Eigen value of the matrix A corresponding to the Eigen vector X then $k + \lambda$ is an Eigen value of the matrix $A + kI$.

Proof:

Let λ be an Eigen value of the matrix A corresponding to the Eigen vector X then

$$AX = \lambda X \text{ ----- (1)}$$

$$\text{Now } (A + kI) X = AX + kIX$$

$$= \lambda X + kX \quad (\text{Since } AX = \lambda X)$$

$$= (\lambda + k)X$$

$$(A + kI) X = (\lambda + k)X \text{ -----(2)}$$

From Equation (2) we see that that the scalar $k + \lambda$ is an Eigen value of the matrix $A + kI$ and X is a corresponding Eigen vector.

6. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the Eigen values of a matrix A, then $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ are the Eigen values of the matrix KA, where k is a non-zero scalar.

Proof:

Let A be a square matrix of order n.

$$\begin{aligned} \text{Then } |kA - \lambda kI| &= |k(A - \lambda I)| \\ &= k^n |A - \lambda I| \end{aligned}$$

Since $k \neq 0$, therefore $|kA - \lambda kI| = 0$ if and only if $|A - \lambda I| = 0$

i.e., $k\lambda$ is an Eigen value of kA if and only if λ is an Eigen value of A.

Thus $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ are the Eigen Values of the Matrix kA if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the Eigen values of the matrix A.

7. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the Eigen Values of A, then $(\lambda_1 - k)(\lambda_2 - k) \dots (\lambda_n - k)$ are the Eigen Values of the matrix $(A - kI)$, where k is a non-zero scalar.

Proof:

Since $\lambda_1, \lambda_2, \dots, \lambda_n$ are the Eigen values of A, then the characteristic polynomial of A is

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) \text{ -----(1)}$$

Thus, the Characteristic polynomial of $(A - kI)$ is

$$|A - kI - \lambda I| = |A - (k + \lambda)I|$$

$$= [\lambda_1 - (\lambda + k)] [\lambda_2 - (\lambda + k)] \dots [\lambda_n - (\lambda + k)] \quad (\text{Since from (1)})$$

$$= [(\lambda_1 - k) - \lambda] [(\lambda_2 - k) - \lambda] \dots [(\lambda_n - k) - \lambda]$$

This shows that the Eigen values of $(A - kI)$ are $(\lambda_1 - k)(\lambda_2 - k) \dots (\lambda_n - k)$.

8. Prove that the Eigen values of A^{-1} are the reciprocals of the Eigen values of A.
(Or)

If λ is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X. Then λ^{-1} is an Eigen value of A^{-1} and corresponding Eigen vector X itself.

Proof:

Since A is non-singular and product of the Eigen values is equal to $|A|$, it follows that none of the Eigen values of A is 0.

If λ is an Eigen value of the non-singular matrix A and X is the corresponding Eigen vector,

$$\lambda \neq 0 \text{ and } AX = \lambda X$$

Pre multiplying this with A^{-1} , we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda(A^{-1}X)$$

$$(1) \quad X = \lambda(A^{-1}X)$$

$$\lambda^{-1}X = A^{-1}X$$

$$A^{-1}X = \lambda^{-1}X \quad \text{Since } \lambda \neq 0$$

Hence by the definition, it follows that λ^{-1} is an Eigen value of A^{-1} and X is the corresponding Eigen vector.

9. If λ is an Eigen value of a non-singular matrix A , then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix $\text{adj } A$.

Proof:

Since λ is an Eigen value of a non-singular matrix, therefore $\lambda \neq 0$.

Also λ is an Eigen value of A implies that there exist a non-zero vector X such that

$$AX = \lambda X \quad (1)$$

Pre multiplying (1) by $\text{adj } A$

$$(\text{adj } A)AX = (\text{adj } A)\lambda X$$

$$[(\text{adj } A)A]X = \lambda[(\text{adj } A)X]$$

$$|A|IX = \lambda(\text{adj } A)X \quad [\text{Since } (\text{adj } A)A = |A|I]$$

$$|A|X = \lambda(\text{adj } A)X$$

$$\frac{|A|}{\lambda}X = (\text{adj } A)X$$

$$(\text{adj } A)X = \frac{|A|}{\lambda}X$$

Since X is a non-zero vector, therefore from the relation It is clear that $\frac{|A|}{\lambda}$ is an Eigen value of the matrix $\text{adj } A$.

10. If λ is an Eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also an Eigen value.

Proof:

We know that if λ is an Eigen value of a matrix A , then $\frac{1}{\lambda}$ is an Eigen value of A^{-1} .

Since A is an orthogonal matrix, therefore

$$A^{-1} = A^T$$

$\frac{1}{\lambda}$ is an Eigen value of A^T .

But the matrices A and A^T have the same Eigen values. Since the determinants $|A - \lambda I|$ and $|A^T - \lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an Eigen value of A .

11. The Eigen values of a diagonal matrix is its diagonal elements.

Proof: Let

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

The characteristic equation of D is

$$|D - \lambda I| = 0$$

$$\begin{vmatrix} a_{11} - \lambda & 0 & 0 & \dots & 0 \\ 0 & a_{22} - \lambda & 0 & \dots & 0 \\ 0 & 0 & a_{33} - \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \dots (a_{nn} - \lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33} \dots a_{nn} \text{ are the Eigen values of } D.$$

12. The Eigen values of a triangular matrix are the diagonal elements.

Proof:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$ be the upper triangular matrix.

The Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \dots (a_{nn} - \lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33} \dots a_{nn}$$

Hence the Eigen values are the diagonal elements.

13. Eigen values of two similar matrices are same.

Proof:

Let A and B be two similar matrices so that

$$B = P^{-1}AP$$

To prove that A and $P^{-1}AP$ have the same Eigen values.

Let λ be Eigen value of B then,

$$|B - \lambda I| = 0$$

$$|P^{-1}AP - \lambda P^{-1}P| = 0$$

$$|P^{-1}P(A - \lambda I)| = 0$$

$$|P^{-1}||P| |A - \lambda I| = 0$$

$$|A - \lambda I| = 0 \quad \text{since } |P^{-1}||P| = 1$$

$$|B - \lambda I| = 0 \Rightarrow |A - \lambda I| = 0$$

Hence A and B have the same Eigen values.

14. If A and B are square matrices and if A is invertible then the matrices $A^{-1}B$ and BA^{-1} have the same Eigen values.

Proof:

Given A is invertible, then A^{-1} exists.

$$\text{Now } A^{-1}B = A^{-1}B I$$

$$= A^{-1}B (A^{-1}A)$$

$$= A^{-1}(BA^{-1})A$$

$$\Rightarrow A^{-1}B = A^{-1}(BA^{-1})A \quad \text{--- (1)}$$

By using the above property, matrices BA^{-1} and $A^{-1}(BA^{-1})A$ have the same Eigen values.

Now by (1) the matrices $A^{-1}B$ and BA^{-1} have the same Eigen values.

15. The Eigen values of a real symmetric matrix are always real.

Proof:

Let λ be an Eigen value of a real symmetric matrix A and let X be the corresponding Eigen vector. Then $AX = \lambda X$ -----(1)

Taking the Conjugate of (1)

$$\bar{A} \bar{X} = \bar{\lambda} \bar{X}$$

Taking the transpose on both sides

$$(\bar{A} \bar{X})^T = (\bar{\lambda} \bar{X})^T$$

$$(\bar{X})^T (\bar{A})^T = \bar{\lambda} (\bar{X})^T$$

Since A is symmetric, we have

$$\bar{A} = A \text{ and } A^T = A$$

$$\therefore (\bar{X})^T A = \bar{\lambda} (\bar{X})^T$$

Post multiplying by X, we get

$$(\bar{X})^T A X = \bar{\lambda} (\bar{X})^T X \text{ -----(2)}$$

Pre multiplying (1) with $(\bar{X})^T$, we get

$$(\bar{X})^T A X = (\bar{X})^T \lambda X \text{ -----(3)}$$

$$(2)-(3)$$

$$\Rightarrow (\lambda - \bar{\lambda}) (\bar{X})^T X = 0$$

$$\Rightarrow (\lambda - \bar{\lambda}) = 0 \quad \text{since } (\bar{X})^T X \neq 0$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \text{ is real}$$

Hence the result.

16. If λ is an Eigen value of A, then prove that the Eigen value of $B = a_0 A^2 + a_1 A + a_2 I$ is $a_0 \lambda^2 + a_1 \lambda + a_2$.

Proof:

If X be the Eigen vector corresponding to the Eigen value λ , then

$$A X = \lambda X \text{ -----(1)}$$

Pre multiplying by A on both sides of (1)

$$A(A X) = A(\lambda X)$$

$$(A A) X = \lambda (A X)$$

$$A^2 X = \lambda (A X) \quad (\text{Since } A X = \lambda X)$$

$$A^2 X = \lambda^2 X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen Vector.

We have

$$B = a_0 A^2 + a_1 A + a_2 I$$

$$B X = (a_0 A^2 + a_1 A + a_2 I) X$$

$$B X = a_0 A^2 X + a_1 A X + a_2 X$$

$$B X = a_0 \lambda^2 X + a_1 \lambda X + a_2 X$$

$$B X = (a_0 \lambda^2 + a_1 \lambda + a_2) X$$

This shows that $a_0 \lambda^2 + a_1 \lambda + a_2$ is an Eigen value of B and the corresponding Eigen Vector of B is X.

17. Prove that the two Eigen vectors corresponding to the two different Eigen Values are linearly independent.

Proof:

Let A be a square matrix. Let X_1 and X_2 be the two Eigen vectors of A corresponding to two distinct Eigen values λ_1 and λ_2 . Then

$$A X_1 = \lambda_1 X_1 \text{ And } A X_2 = \lambda_2 X_2 \text{ -----(1)}$$

Now we shall prove that the Eigen vectors X_1 and X_2 are linearly independent.

Let us assume that the X_1 and X_2 are linearly dependent.

Then for two scalars k_1 and k_2 not both zeroes such that $k_1 X_1 + k_2 X_2 = 0$ -----(2)

Multiplying both sides of (2) by A, we get

$$\begin{aligned}
 A(k_1 X_1 + k_2 X_2) &= A(0) \\
 A(k_1 X_1) + A(k_2 X_2) &= 0 \\
 k_1(A X_1) + k_2(A X_2) &= 0 \\
 k_1(\lambda_1 X_1) + k_2(\lambda_2 X_2) &= 0 \text{ ----- (3) (Since from (1))}
 \end{aligned}$$

$$\begin{aligned}
 (3) - \lambda_2 (1) &\Rightarrow k_1(\lambda_1 - \lambda_2)X_1 = 0 \\
 &\Rightarrow k_1 = 0 \quad \text{Since } \lambda_1 \neq \lambda_2 \text{ \& } X_1 \neq 0 \\
 \text{Similarly, } k_2 &= 0
 \end{aligned}$$

But this contradicts our assumption that k_1, k_2 are not zeroes. Hence our assumption that

X_1 & X_2 are Linearly independent is wrong.

Hence the statement is true.

5. Find the Eigen values and Eigen vectors of the matrix A and its inverse,

where $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of A is given by $|A - \lambda I| = 0$

$$\begin{aligned}
 \Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} &= 0 \\
 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) &= 0 \\
 \Rightarrow \lambda = 1, 2, 3
 \end{aligned}$$

Eigen values of the matrix are 1, 2, 3

Eigen vector corresponding to $\lambda=1$: $(A - \lambda I)X_1 = 0$

$$\Rightarrow \begin{bmatrix} 1-1 & 3 & 4 \\ 0 & 2-1 & 5 \\ 0 & 0 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2 + 4x_3 = 0$$

$$x_2 - 5x_3 = 0$$

$$2x_3 = 0$$

$$\therefore x_3 = 0, x_2 = 0 \text{ and let } x_1 = k, \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to $\lambda=2$: $(A - 2I)X_2 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0$$

$$\Rightarrow 5x_3 = 0, \text{ i.e., } x_3 = 0$$

$$-x_1 + 3x_2 = 0$$

$$x_1 = 3x_2$$

Let $x_2 = k$ then $x_1 = 3k$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to $\lambda=3$: $(A - 3I)X_3 = 0$

$$\Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 3x_2 + 4x_3 = 0$$

$$\Rightarrow -x_2 + 5x_3 = 0$$

Let $x_3 = k$ then $x_2 = 5k$

$$-2x_1 + 3(5k) + 4k = 0$$

$$x_1 = \frac{19}{2}k$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix} = \frac{k}{2} \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$$

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

Eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$

We know that Eigen vectors of A^{-1} are same as Eigen vectors of A .

6. Find the Eigen values of $3A^3 + 5A^2 - 6A + 2I$ where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 3, -2$$

Eigen values of the matrix are 1, 3, -2

We know that if λ is the eigen value of matrix A and $f(A)$ is a polynomial in A then the Eigen value of $f(A)$ is $f(\lambda)$

$$\text{Let } f(A) = 3A^3 + 5A^2 - 6A + 2I$$

Then Eigen values of $f(A)$ are $f(1)$, $f(3)$ and $f(-2)$

$$f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 4$$

$$f(3) = 3(3)^3 + 5(3)^2 - 6(3) + 2(1) = 110$$

$$f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2(1) = 10$$

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4, 110, 10

1.5. CAYLEY - HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation

1. Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation Hence

find A^{-1} .

Sol: The Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$C_2 \rightarrow C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

The characteristic equation is $\lambda^3 - \lambda^2 + \lambda - 1 = 0$

By Cayley - Hamilton theorem, we have $A^3 - A^2 + A - I = 0$ -----(1)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Matrix A satisfies Cayley-Hamilton theorem.

To find A^{-1} , Multiplying equation (1) with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = A^2 - A + I$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix

Where $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 7 - \lambda & 2 & -2 \\ -6 & -1 - \lambda & 2 \\ 6 & 2 & -1 - \lambda \end{vmatrix} = 0$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley - Hamilton theorem we have $A^3 - 5A^2 + 7A - 3I = 0 \dots (1)$

To find A^{-1} , Multiplying equation (1) with A^{-1} we get $A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$A^{-1} = \frac{1}{3} \left\{ \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - 5 \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

To find A^4 Multiply equation (1) with A, we get

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$A^4 = 5A^3 - 7A^2 + 3A$$

$$A^4 = \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

1.6. DIAGONALIZATION OF A MATRIX:

A square matrix is said to be diagonalizable if there exists an invertible matrix 'P' such that $P^{-1}AP = D$ is known as Diagonal matrix

Modal and Spectral matrices:

The matrix P in the above result which diagonalizable the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Calculation of powers of a matrix:

We can obtain the power of a matrix by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that

$$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$$

Similarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$(1)

To obtain A^n , Pre-multiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$

$$= (PP^{-1}) A^n (PP^{-1}) = A^n$$

$$\therefore A^n = PD^nP^{-1}$$

NOTE: The diagonal elements in the diagonal matrix are Eigen values of the given matrix.

PROBLEMS

1. Determine the modal matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & 3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } (\lambda - 5)(\lambda + 3)^2 = 0$$

Thus the Eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

when $\lambda=5$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

By solving above we get $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given Eigen value $\lambda=-3$ we can have two linearly independent Eigen vectors

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Modal matrix $P = [X_1, X_2, X_3]$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Now } |P| = 1(-1) - 2(2) + 3(-1) = -8 \neq 0$$

$\therefore P$ is invertible matrix so matrix A is diagonalizable.

$$P^{-1} = \frac{\text{adj}P}{|P|} = \frac{-1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{-1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{-1}{8} \begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Hence $P^{-1}AP$ is a diagonal matrix.

2. Find a matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form.

Hence calculate A^4 .

Sol: Characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 1, 2, 3$$

Thus, the eigen values of matrix are 1, 2, 3.

If x_1, x_2, x_3 be the components of an Eigen vector corresponding to the Eigen value λ , we have

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{i.e., } 0.x_1 + 0.x_2 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

By solving above equations $x_1 = 1, x_2 = -1, x_3 = 0$

Eigen vector is $[1, -1, 0]^T$

For $\lambda = 2, \lambda = 3$ we can obtain Eigen vector $[-2, 1, 2]^T$ and $[-1, 1, 2]^T$

Modal matrix $P = [X_1, X_2, X_3]$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj}P}{|P|} = \frac{-1}{2} \begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$\text{Now } P^{-1}AP = \frac{-1}{2} \begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \left\{ \frac{-1}{2} \begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix} \right\}$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

9. Practice Quiz

- If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a _____ [b]
 a) Idempotent b) Symmetric c) Skew – Symmetric d) Diagonal
- If A is a matrix of order $m \times n$, then rank of A is \leq _____ [c]
 a) Minimum of m b) minimum of n c) minimum of m, n d) none
- In Echelon form Rank of A = _____ [b]
 a) Number of non-zero columns only b) Number of non-zero rows only
 c) a & b d) none
- If $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$ then $\rho(A) =$ _____ [a]
 a) 1 b) 2 c) 3 d) Zero
- The square matrix A is said to be non-singular if _____ [c]
 a) $\rho(A) = n$ b) $\text{Det } A = 0$ c) $\text{Det } A \neq 0$ d) none
- The system of equation $AX = B$ has a unique solution if _____ [b]
 a) $\rho(A/B) = \rho(A) = r < n$ b) $\rho(A) = \rho(A/B) = r = n$
 c) $\rho(A/B) \neq \rho(A)$ d) $\rho(A) = \rho(A/B) = r > n$

7. If $\rho(A) = \rho(A/B) < n$ then given system $AX = B$ has _____ [c]
 a) No solution b) Unique solution
 c) Infinite number of solutions d) Zero solution
8. The augmented Matrix of a system is $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$ then the system is in consistent if _____ [c]
 a) $\lambda \neq 3, \mu \neq 10$ b) $\lambda = 3, \mu = 10$ c) $\lambda = 3, \mu \neq 10$ d) $\lambda \neq 3, \mu = 10$
9. If -2, 1, 3 are the eigen values of A then the eigen values of A^{-1} are _____ [c]
 a) -2, 1, 3 b) 2, -1, -3 c) -1/2, 1, 1/3 d) 1/2, -1, -1/3
10. The characteristic equation of A matrix is $\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$ then $A^3 =$ _____ [b]
 a) $11A^2 - 4A + I$ b) $11A^2 + 4A - I$ c) $-11A^2 - 4A + I$ d) $11A^2 + 4A + I$
11. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ then the eigen values of A^4 is _____ [c]
 a) 1, 5 b) -1, -5 c) 1, 625 d) -1, -625
12. If 2, 3, 4 are eigen values of A & $A^3 = PD^3P^{-1}$ then $D^3 =$ _____ [c]
 a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ c) $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ d) None
13. If λ is the eigen value of A, then eigen value of A^T is _____ [b]
 a) $1/\lambda$ b) λ c) 2λ d) $\lambda/2$
14. The characteristic roots of $A = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$ are _____ [a]
 a) 1, 5 b) 0, 2 c) -1, -5 d) -1, 5
15. The product of two orthogonal matrices is _____ [a]
 a) Orthogonal b) Unitary c) both a & b d) None

10. Assignments

| S.No | Question | BL | CO |
|------|--|----|-----|
| 1 | Convert the given matrix into Echelon form and hence find it's rank where $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ | L2 | CO1 |
| 2 | Convert the given matrix into Normal form and hence find it's rank where $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ | L2 | CO1 |
| 3 | Test for consistency and solve the following system of equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ | L4 | CO1 |
| 4 | Solve the following system of equations $x + y - 3z + 2w = 0$, $2x - y + 2z - 3w = 0$, $3x - 2y + z - 4w = 0$, $-4x + y - 3z + w = 0$ | L3 | CO1 |
| 5 | Using Cayley Hamilton theorem find A^{-1} and A^{-2} if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ | L3 | CO1 |
| 6 | Determine Eigen values and the corresponding Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ | L4 | CO1 |

11. Part A- Question & Answers

| S.No | Question& Answers | BL | CO |
|------|---|----|----|
| 1 | Define rank of a matrix. Sol: If A is non-null matrix, we say that 'r' is the rank of A if (i) every (r+1)th order minor of A is zero and (ii) there exists at least one rth order minor of A which is not zero. | L1 | 1 |
| 2 | Define an echelon form of a Matrix? Sol: A matrix is said to be in echelon form if it has the following properties (i) Zero rows, if any, are below any non-zero row. (ii) The first non-zero entry in each non-zero row is equal to 1. (iii) The number of zeros before the first non-zero element in a row is less than the Number of such zeros in the next row. | L1 | 1 |
| 3 | Convert the matrix to echelon form and find the rank of $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 5 \\ 2 & 1 & 2 \end{bmatrix}$ Sol: Applying $R_3 \leftrightarrow R_2$, we get $\sim \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 4 & 5 \end{bmatrix}$ by $R_2 \rightarrow 3R_2 - 2R_1$ $\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 4 & 5 \end{bmatrix}$ by $R_3 \rightarrow R_3 - 4R_2$, we get $\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -11 \end{bmatrix}$ this is in echelon form, here number of non-zero rows =3. Hence rank of given matrix=3. | L2 | 1 |
| 4 | Convert the matrix to echelon form find the rank of $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ Sol: Applying $R_3 \rightarrow R_3 + R_1$, we get $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ this is row echelon form and of number of non-zero rows is 3. Hence rank of A is 3. | L2 | 1 |
| 5 | Find the value of k such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2. Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ Given rank of A=2. $\therefore A =0$ | L1 | 1 |

| | | | |
|----|---|----|---|
| | $\Rightarrow 1(10k-42)-2(20-21)+3(12-3k)=0. \Rightarrow K=4.$ | | |
| 6 | Write condition for the system $AX=B$ is consistent. Sol: System $AX=B$ is consistent if rank of A = rank of $[A/B]$ | L1 | 1 |
| 7 | Write condition for the system $AX=0$ has non-trivial solutions. Sol: If the rank of $A=r < n$ (number of unknowns) then the system has non-trivial solutions. | L1 | 1 |
| 8 | Find the eigen values of $A=\begin{bmatrix} 8 & 9 \\ 9 & 5 \end{bmatrix}$ Sol: The characteristic equation of A is $\begin{vmatrix} 8-\lambda & 9 \\ 9 & 5-\lambda \end{vmatrix} = 0.$ $\Rightarrow (8-\lambda)(5-\lambda) - 81 = 0$ $\Rightarrow \lambda^2 - 13\lambda - 41 = 0$ \therefore Eigen values are $\lambda = \frac{13 \pm \sqrt{333}}{2}$ | L1 | 1 |
| 9 | Find the sum of the squares of the eigen values of $A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ Sol: A is a triangular matrix. Therefore, the eigen values of A are 3, 2 and 5. The sum of squares of the eigen values of $A = 3^2 + 2^2 + 5^2 = 9 + 4 + 25 = 38.$ | L1 | 1 |
| 10 | State Cayley-Hamilton theorem. Sol: Every square matrix satisfies its own characteristic equation | L1 | 1 |

12. Part B- Questions

| S.No | Question | BL | CO |
|------|---|----|----|
| 1 | (a) Convert the matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ into Echelon form and hence find its rank. (b) Prove that the Eigen values of Hermitian matrix are real. | L2 | 1 |
| 2 | (a) Convert the following matrix into its normal form and hence find its rank $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (b) Prove that sum of Eigen values of a matrix is its trace and product of Eigen values of matrix is its trace. | L2 | 1 |
| 3 | Test for Consistency and solve $5x+3y+7z = 4$, $3x+26y+2z = 9$, $7x+2y+10z = 5$. | L4 | 1 |

| | | | |
|----|--|----|---|
| 4 | Determine Eigen values and Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ | L4 | 1 |
| 5 | Convert the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to diagonal form. | L2 | 1 |
| 6 | Determine the eigen values and corresponding Eigen Vectors of the matrix $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. | L4 | 1 |
| 7 | Determine the characteristic roots and characteristic vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ | L4 | 1 |
| 8 | Find the rank of matrix $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ by converting into Echelon Form | L2 | 1 |
| 9 | Investigate the values of λ and μ so that the equations $2x+3y+5z = 9$, $7x+3y-2z = 8$, $2x+3y+\lambda z = \mu$, have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. | L5 | 1 |
| 10 | Show that the only real number λ for which the system $x + 2y + 3z = \lambda x$; $3x + y + 2z = \lambda y$; $2x + 3y + z = \lambda z$ has non zero solution is 6 and solve them when $\lambda = 6$. | L3 | 1 |

13. Supportive Online Certification Courses

- Advanced Engineering Mathematics by Prof. P. N. Agarwal
IIT Roorkee – 12 weeks

14. Real Time Applications

| S. No | Application | CO |
|-------|---|----|
| 1 | Matrices are also useful in electrical circuits and quantum mechanics and also used to solve AC network equations in electric circuits. | 1 |

15. Contents Beyond the Syllabus

- Model population growth using an age transition matrix and an age distribution vector.
- Determination of Natural Frequencies and Mode Shapes

16. Prescribed Text Books & Reference Books

Text Book

- B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
- Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.

References:

1. R. K. Jain and S. R. K. Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002.
2. B. V. Ramana, Higher Engineering Mathematics, McGraw Hill Education

