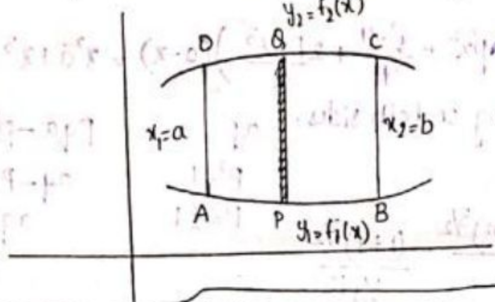
UNIT-04

Multiple Integrals :-

Double Integrals :-

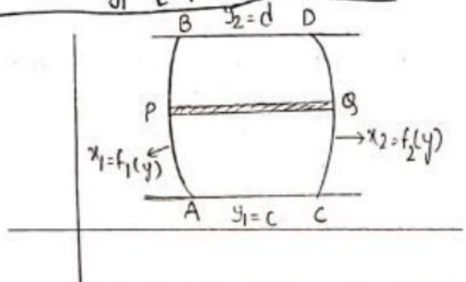
1. If x_1, x_2 are constants and y_1, y_2 are functions of 'x'
$x_1 = a$, $x_2 = b$, $y_1 = f_1(x)$, $y_2 = f_2(x)$. We take a,b & c,d curves. Take a vertical strip (y strip) PQ such that strip lies between $x=a$ & $x=b$. First integrate w.r.t 'y' then total Expression, we get interms of 'x' then finally integrating w.r.t 'x' we get the required solution.



$$\boxed{\therefore \iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\left[\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy\right]dx}$$

2. If y_1, y_2 are constants and x_1, x_2 are functions of 'y'. similarly

$$\boxed{\therefore \iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy}$$



3. If x_1, x_2 & y_1, y_2 are constants. then the order of integration is not importa

$$\boxed{\therefore \iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx}$$

(or)

$$\boxed{\therefore \iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy}$$

Q. Evaluate $\int_0^1 \int_0^{1-y} x^2+y^2 \,dx\,dy$

Ans. Given, $\int_0^1 \left[\int_0^{1-y} (x^2+y^2)\,dx\right]dy = \int_0^1 \left(\frac{x^3}{3} + y^2 x\right)_0^{1-y} dy = \int_0^1 \left(\frac{(1-y)^3}{3} + y^2(1-y)\right)dy - (0)$

$$= \frac{1}{3}\left[\frac{(1-y)^4}{-4}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^1$$

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$$= \frac{1}{3}\left[0-\left(\frac{-1}{4}\right)\right] + \left[\frac{1}{3}-0\right] - \left[\frac{1}{4}-0\right]$$

$$= \frac{1}{12}+\frac{1}{3}-\frac{1}{4} = \frac{1+4-3}{12} = \frac{5-3}{12} = \frac{2}{12} = \frac{1}{6}$$

Q. Evaluate $I = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dx\cdot dy$.

An. Given, $\int_0^1 \left[\int_0^{1-x} (x^2+y^2)dy\right] dx = \int_0^1 (x^2y)_0^{1-x} + \left(\frac{y^3}{3}\right)_0^{1-x} dx$

$$= \int_0^1 \left[x^2(1-x)\right] + \frac{(1-x)^3}{3}\,dx$$

$$= \int_0^1 -x^3+x^2+\frac{(1-x)^3}{3}\,dx$$

$$= \left(\frac{-x^4}{4}\right)_0^1 + \left(\frac{x^3}{3}\right)_0^1 + \frac{1}{3}\left(\frac{(1-x)^4}{-4}\right)_0^1$$

$$= \frac{-1}{4}+\frac{1}{3}+\frac{1}{3}\left[0-\left(\frac{-1}{4}\right)\right] = \frac{-1}{4}+\frac{1}{3}+\frac{1}{12} = \frac{1}{6}$$

Q. Evaluate $\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\;dx\cdot dy$

An. Given, $\int_{x=0}^{1}\left[\int_{y=x^2}^{2-x} xy\,dy\right]dx = \int_{x=0}^{1} x\left(\frac{y^2}{2}\right)_{x^2}^{2-x} dx = \int_{x=0}^{1} x\cdot\left(\frac{(2-x)^2}{2}-\frac{x^4}{2}\right)dx$

$$= \int_{x=0}^{1} \frac{x}{2}\left(4+x^2-4x-x^4\right)dx = \frac{1}{2}\int_{x=0}^{1}(4x+x^3-4x^2-x^5)dx$$

$$= \frac{1}{2}\left[\frac{4x^2}{2}+\frac{x^4}{4}-4\frac{x^3}{3}-\frac{x^6}{6}\right]_0^1 = \frac{1}{2}\left[2+\frac{1}{4}-\frac{4}{3}-\frac{1}{6}\right] = \frac{1}{2}\left[\frac{9}{12}\right]$$

$$= \frac{3}{8}$$

Q. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\;dx\cdot dy$ $\left[\because \int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

An. Given, $\int_0^a\left[\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-y^2-x^2}\,dx\right]dy = \int_0^a\left[\int_0^{\sqrt{a^2-y^2}}\sqrt{\left(\sqrt{a^2-y^2}\right)^2-x^2}\,dx\right]dy$

$$= \int_0^a \left[\frac{x}{2}\sqrt{\left(\sqrt{a^2-y^2}\right)^2-x^2}\right] + \left[\frac{\left(\sqrt{a^2-y^2}\right)^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left(0+\left[\frac{a^2-y^2}{2}\sin^{-1}(1)-0\right]\right)dy = \int_0^a \frac{\pi}{4}(a^2-y^2)\,dy$$

$$= \frac{\pi}{4}\left[a^2y-\frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3-\frac{a^3}{3}\right] = \frac{\pi}{4}a^3\left[1-\frac{1}{3}\right]$$

$$= \frac{2\pi}{12}a^3 = \frac{\pi a^3}{6}$$

Q. Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dx\cdot dy$.

Ans. Given, $\int_0^1\left[\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\,dy\right]dx = \int_0^1 \left[\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\cdot\frac{1}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}dx$

$= \int_0^1 \tan^{-1}(1)\frac{1}{\sqrt{1+x^2}}\,dx = \frac{\pi}{4}\int_0^1\frac{1}{\sqrt{1+x^2}}\,dx$

$= \frac{\pi}{4}\left(\sinh^{-1}x\right)_0^1 = \frac{\pi}{4}\left(\sinh^{-1}(1)\right)$

Q. Evaluate the following integrals:

i) $\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\cdot dy$ ii) $\int_0^4\int_0^{x^2} e^{y/x}\,dx\cdot dy$

$\left[\because \int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx\right]$

iii) $\int_0^1\int_0^{x^2} e^{y/x}\,dy\cdot dx$ iv) $\int_0^2\int_0^x e^{x+y}\,dy\cdot dx$

i)
Ans. Given, $\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\cdot dy = \left(\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx\right)\left(\int_0^1\frac{1}{\sqrt{1-y^2}}\,dy\right)$

$= (\sin^{-1}x)_0^1\,(\sin y)_0^1 = \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$

ii)
Ans. Given, $\int_0^4\left[\int_0^{x^2} e^{y/x}\,dy\right]dx = \int_0^4\left[\frac{e^{y/x}}{1/x}\right]_0^{x^2}dx = \int_0^4 x(e^x-1)\,dx = \int_0^4 xe^x\,dx - \int_0^4 x\,dx$

$= \left[e^x(x-1)\right]_0^4 - \left[\frac{x^2}{2}\right]_0^4 = \left[e^4(3)-e^0(-1)\right] - \left[\frac{16}{2}\right] = 3e^4+1-8$
$= 3e^4-7$

iii)
Ans. Given,
$\int_0^1\int_0^{x^2} e^{y/x}\,dy\cdot dx = \int_0^1\left(\frac{e^{y/x}}{1/x}\right]_0^{x^2}dx = \int_0^1 x(e^x-1)\,dx$

$= \int_0^1 xe^x\,dx - \int_0^1 x\,dx = \left[e^x(x-1)\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$

$= \left[0-e^0(0-1)\right] - \left[\frac{1}{2}-0\right] = 1-\frac{1}{2} = \frac{1}{2}$

iv)
Ans. Given, $\int_0^2\int_0^x e^{x+y}\,dy\cdot dx = \int_0^2 e^x\left[\int_0^x e^y\,dy\right]dx = \int_0^2 e^x\left[e^y\right]_0^x dx = \int_0^2 e^x\left[e^x-e^0\right]dx$

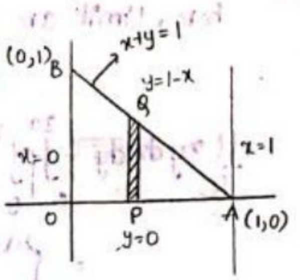
$= \int_0^2 e^{2x}\,dx - \int_0^2 e^x\,dx = \left[\frac{e^{2x}}{2}\right]_0^2 - \left[e^x\right]_0^2$

$= \left[\frac{e^4}{2}-\frac{1}{2}\right] - \left[e^2-e^0\right]$

$= \frac{e^4}{2}-\frac{1}{2}-e^2+1 = \frac{e^4}{2}-e^2+\frac{1}{2}$

Q. Evaluate $\iint_R$ over the region in positive Quadrant for which $x+y \leq 1$

$$\iint_R (x^2+y^2)\,dx.dy.$$



First Method:-

Ans. Here, we can take 'PQ' strip which enter at the point 'P' where $y=0$ and emerges at 'Q' where $y=1-x$. such strip lies between $x=0$ and $x=1$ then the limits are

$$y \to 0 \text{ to } 1-x \ \& \ x \to 0 \text{ to } 1$$

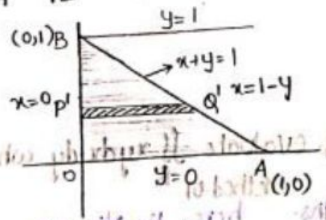
$$\iint_R (x^2+y^2)dx.dy = \int_0^1 \left[\int_0^{1-x} (x^2+y^2)dy\right]dx = \int_0^1 \left(x^2y+\frac{y^3}{3}\right)_0^{1-x} dx = \int_0^1 \left(x^2(1-x)+\frac{(1-x)^3}{3}\right)dx$$

$$= \int_0^1 x^2-x^3+\frac{(1-x)^3}{3}\,dx = \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{1}{3}\left(\frac{(1-x)^4}{-4}\right)\right]_0^1$$

$$= \frac{1}{3}-\frac{1}{4}+\frac{1}{12} = \frac{1}{6}$$

Second Method:-

Here, limits are $y \to 0$ to 1 & $x \to 0$ to $1-y$



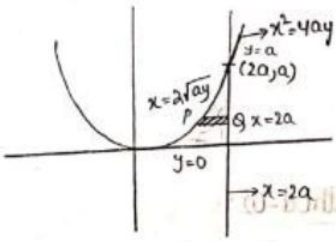
$$\iint_R (x^2+y^2)dxdy = \int_0^1\left[\int_0^{1-y}(x^2+y^2)dx\right]dy = \int_0^1 \left[\frac{x^3}{3}+y^2x\right]_0^{1-y} dy$$

$$= \int_0^1 \frac{(1-y)^3}{3}+y^2(1-y)\,dy$$

$$= \int_0^1 y^2-y^3+\left(\frac{(1-y)^3}{3}\right)dy = \left[\frac{y^3}{3}-\frac{y^4}{4}+\left(\frac{(1-y)^4}{-4(3)}\right)\right]_0^1$$

$$= \frac{1}{3}-\frac{1}{4}+\left(0-\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\right) = \frac{1}{3}-\frac{1}{4}+\frac{1}{12} = \frac{1}{6}$$

Q. Evaluate integration over 'R' $\iint_R xy\,dx.dy$ where R is region bounded by x-axis ordinates $x=2a$ and the curve $x^2=4ay$



Method-01

Ans. Here, we take 'PQ' strip which enter at the point 'P' where $x=2\sqrt{ay}$ and emerges at Q where $x=2a$. such strip lies between $y=0$ and $y=a$ then limits are

$$y \to 0 \text{ to } a \quad x \to 2\sqrt{ay} \text{ to } 2a$$

$$\iint_R xy\,dx.dy = \int_0^a\left[\int_{2\sqrt{ay}}^{2a} xy\,dx\right]dy = \int_0^a \left[y\left(\frac{x^2}{2}\right)\right]_{\sqrt{4ay}}^{2a} dy = \int_0^a y\left(\frac{4a^2}{2}-\frac{4ay}{2}\right)dy$$

$$= 2\int_0^a a^2y-ay^2\,dy = 2\left[a^2\frac{y^2}{2}-a\frac{y^3}{3}\right]_0^a$$

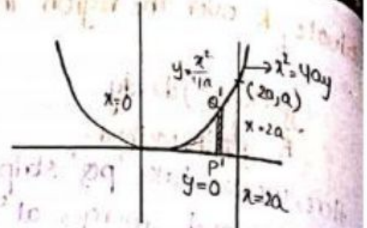
$$= 2\left[a^2\left(\frac{a^2}{2}\right)-a\left(\frac{a^3}{3}\right)\right] = 2\left[\frac{a^4}{2}-\frac{a^4}{3}\right] = 2\left[\frac{3a^4-2a^4}{6}\right]$$

$$= \frac{a^4}{3}.$$

Method-02

here, limits are $y \to 0$ to $\frac{x^2}{4a}$

$x \to 0$ to $2a$



$$\iint_R xy\,dx\cdot dy = \int_0^{2a}\left[\int_0^{x^2/4a} xy\cdot dy\right]dx$$

$$= \int_0^{2a}\left[x\,\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_0^{2a} x\left(\frac{x^4}{16a^2(2)}\right)dx$$

$$= \int_0^{2a} \frac{1}{32a^2}\,x^5\,dx = \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{32a^2}\left[\frac{2^6a^6}{6}\right] = \frac{16(4)a^6}{32(6)a^2}$$

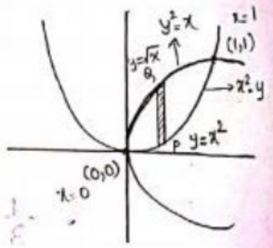
$$\therefore \iint_R (xy)\,dx\cdot dy = \frac{a^4}{3}$$

Q. Evaluate $\iint_R xy\,dx\cdot dy$ where 'R' is the region bounded by curves $y^2=x$ & $x^2=y$

Method-01

Ans. here, limits are $y \to x^2$ to $\sqrt{x}$

$x \to 0$ to 1



$$\iint_R xy\,dx\cdot dy = \int_0^1\left[\int_{x^2}^{\sqrt{x}} xy\,dy\right]dx$$

$$= \int_0^1\left[x\,\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = \int_0^1\left[-x\left(\frac{x^4}{2}\right)+x\left(\frac{x}{2}\right)\right]dx$$

$$= \int_0^1 -\frac{x^5}{2}+\frac{x^2}{2}\,dx = \frac{1}{2}\left[-\frac{x^6}{6}+\frac{x^3}{3}\right]_0^1$$

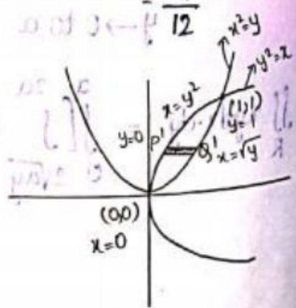
$$= \frac{1}{2}\left[-\frac{1}{6}+\frac{1}{3}\right] = \frac{1}{2}\left[\frac{-1+2}{6}\right]$$

$$= \frac{1}{12}$$

Method-02:-

here, limits are $x \to y^2$ to $\sqrt{y}$

$y \to 0$ to 1



$$\iint_R xy\,dx\cdot dy = \int_0^1\left[\int_{y^2}^{\sqrt{y}} xy\,dx\right]dy = \int_0^1\left(\frac{x^2}{2}y\right)_{y^2}^{\sqrt{y}} dy$$

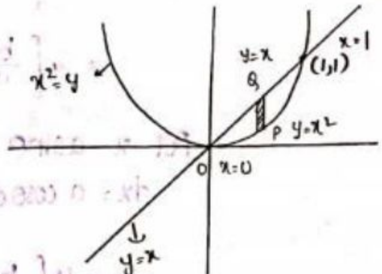
$$= \int_0^1\left[\frac{y^2}{2}-\frac{y^5}{2}\right]dy$$

$$= \left[\frac{y^3}{6}-\frac{y^6}{12}\right]_0^1 = \frac{1}{6}-\frac{1}{12} = \frac{2-1}{12} = \frac{1}{12}$$

Q. Evaluate $\int$ integration over 'R' $\iint_R xy(x+y)\,dx\,dy$ where 'R' is the region bounded by the curves $y=x$ and $x^2=y$

Ans. Method-01 :-

Here, we can take 'pq' strip which enter at the point 'p' where $y=x^2$ and emerges at 'q' where $y=x$. such strip lies between $x=0$ and $x=1$ then limits are



$$y \to x^2 \text{ to } x$$
$$x \to 0 \text{ to } 1$$

$$\therefore \iint_R xy(x+y)\,dx\cdot dy = \int_0^1\left[\int_{x^2}^{x} x^2y+xy^2\,dy\right]dx = \int_0^1\left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_{x^2}^{x}dx$$

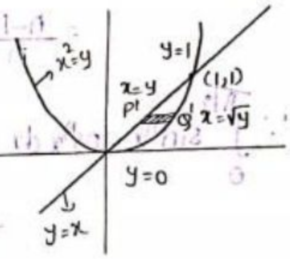
$$=\int_0^1\left[\frac{x^4}{2}+\frac{x^4}{3}-\left[\frac{x^6}{2}-\frac{x^7}{3}\right]\right]dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1+\frac{1}{3}\left[\frac{x^5}{5}\right]_0^1-\frac{1}{2}\left[\frac{x^7}{7}\right]_0^1-\frac{1}{3}\left[\frac{x^8}{8}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{1}{5}\right]+\frac{1}{3}\left[\frac{1}{5}\right]-\frac{1}{2}\left[\frac{1}{7}\right]-\frac{1}{3}\left[\frac{1}{8}\right]$$

$$=\frac{1}{10}+\frac{1}{15}-\frac{1}{14}-\frac{1}{24} = \frac{3+2}{30}-\left[\frac{12+7}{168}\right] = \frac{5}{30}-\frac{19}{168} = \frac{3}{56}$$

Method-02:-

here limits are $y \to 0$ to 1
$x \to y$ to $\sqrt{y}$



$$\int_0^1\left[\int_y^{\sqrt{y}} x^2y+xy^2\,dx\right]dy = \int_0^1\left[\frac{x^3}{3}y+\frac{x^2}{2}y^2\right]_y^{\sqrt{y}}dy$$

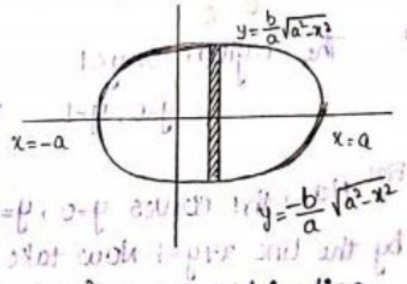
$$=\int_0^1\left[\frac{y^{3/2}}{3}y+\frac{y^3}{2}-\frac{y^4}{3}-\frac{y^4}{2}\right]dy$$

$$=\int_0^1\left[\frac{y^{5/2}}{3}+\frac{y^3}{2}-\frac{5y^4}{6}\right]dy = \int_0^1\frac{y^{5/2}}{3}+\frac{y^3}{2}-\frac{5y^4}{6}\,dy = \left(\frac{y^{7/2}}{7/2}\right)_0^1\frac{1}{3}+\frac{1}{2}\left(\frac{y^4}{4}\right)_0^1-\frac{5}{6}\left(\frac{y^5}{5}\right)_0^1$$

$$=\frac{3}{56}$$

Q. Evaluate $\iint_R (x+y)^2\,dx\cdot dy$ where R is the region bounded by the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Ans. here, $x \to -a$ to a
$y \to -\frac{b}{a}\sqrt{a^2-x^2}$ to $\frac{b}{a}\sqrt{a^2-x^2}$



$$\iint_R (x^2+y^2+2xy)\,dx\cdot dy = \int_{-a}^{a}\left[\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}(x+y)^2\,dy\right]dx$$

here in $(x+y)^2$ $2xy$ is odd function and x^2+y^2 is even

$$=2\int_{-a}^{a}\left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2+y^2\,dy\right]dx$$

$$=2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx$$

$$= 2\int_{-a}^{a} x^2\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\,dx$$

$$= 4\left[\frac{b}{a}\int_0^a x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2}\,dx\right]$$

Put $x = a\sin\theta$ $\qquad x\to 0$ then $\theta\to 0$

$dx = a\cos\theta\, d\theta$ $\qquad x\to a$ then $\theta\to \pi/2$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^4\sin^2\theta\cos^2\theta\, d\theta + \frac{b^3}{a^3(3)}\int_0^{\pi/2}\cos^4\theta\, a^4\, d\theta\right]$$

$$= 4\left[ba^3\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{ab^3}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta\right]$$

$$= 4ba^3\left[\frac{1}{4}\int_0^{\pi/2}\sin^2\theta\, d\theta\right] + \frac{4b^3a}{3}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 4ba^3\left[\frac{1}{4}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\right] + \frac{4b^3a}{3}\left[\frac{3}{16}\pi\right] = \frac{ba^3\pi}{4} + \frac{b^3a\pi}{4}$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

$$\because \int_0^{\pi/2}\sin^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\dots\frac{1}{2}\cdot\frac{\pi}{2} \text{ if } n \text{ is even}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\dots\frac{2}{3} \text{ if } n \text{ is odd}$$

$$\because \int_0^{\pi/2}\sin^m x\cdot\cos^n x\, dx = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdots\cdots\frac{1}{m+1} \text{ if } n \text{ is odd}$$

$$= \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdots\cdots\int_0^{\pi/2}\sin^m x\, dx \text{ if } n \text{ is even}$$

CHANGE OF ORDER OF INTEGRATION :-

It means change the limits of integration i.e $\int_{x_1}^{x_2}\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(xy)\,dx\cdot dy$ can be changed to $\int_{y_1}^{y_2}\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(xy)\,dy\cdot dx$

Q. Evaluate $\int_0^1\int_0^{1-y}(x^2+y^2)\,dx\,dy$ by using change of order of integration.

Ans. The region, $0<y<1 \qquad 0<x<1-y$

$y=0,\ y=1 \qquad x=0,\ x=1-y$

$x+y=1$

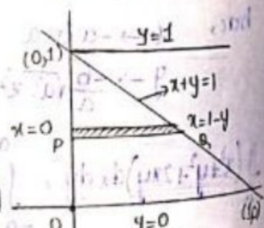


Fig-1

In Fig1. Here, the curves $y=0$, $y=1$, $x=0$, ~~x+y=1~~ are bounded by the line $x+y=1$. Now take a strip PQ which enter at the point P where $x=0$ & emerges at Q where $x=1-y$, such strip lies between the lines $y=0$ & $y=1$

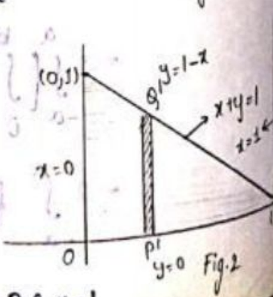


Fig.2

Instead of horizontal strip we should draw the vertical strip (P'Q'), such strip ~~emerges~~ enter at P' where $y=0$ and emerges at Q' where $y=1-x$ & lies between lines $x=0$ & $x=1$

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By the change of order of integration,

New limits are $y \to 0$ to $1-x$

$x \to 0$ to 1

$$I = \int_0^1 \int_0^{1-y} (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_0^{1-x} (x^2+y^2)\,dy \right] dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx = \int_0^1 x^2 - x^3 + \frac{(1-x)^3}{3}\, dx$$

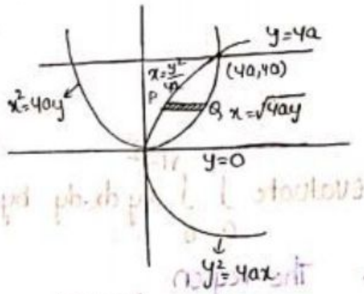
$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{-4(3)} \right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

Q. evaluate $\int_0^{4a} \int_{y^2/4a}^{\sqrt{4ay}} dx.dy$ by using change of order of integration.

Ans. the region, $0 < y < 4a$ $\quad \frac{y^2}{4a} < x < \sqrt{4ay}$

$y=0$, $y=4a$ $\quad x = \frac{y^2}{4a}$ $\quad x = \sqrt{4ay}$

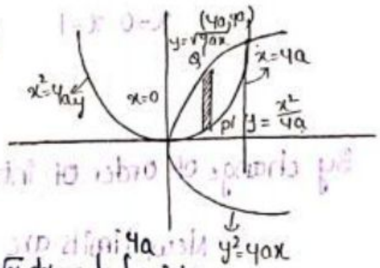
$y^2 = 4ax$ $\quad x^2 = 4ay$



By change of order of integration,

New limits are $y \to \frac{x^2}{4a}$ to $\sqrt{4ax}$

$x \to 0$ to $4a$



$$\int_0^{4a} \left[\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy \right] dx = \int_0^{4a} [y]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx = \int_0^{4a} 2\sqrt{a}\,\sqrt{x}\,dx - \frac{1}{4a}\int_0^{4a} x^2 dx$$

$$= 2\sqrt{a} \int_0^{4a} x^{1/2} dx - \frac{1}{4a}\int_0^{4a} x^2 dx = 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a}$$

$$= 2\sqrt{a} \left[\frac{2}{3} (4a)^{3/2} \right] - \frac{1}{4a} \left[\frac{64a^3}{3} \right] = \frac{4\sqrt{a}}{3} [4\sqrt{4}\,\underset{a\sqrt{a}}{a^{3/2}}] - \frac{1}{4a} \left[\frac{64a^3}{3} \right]$$

$$= \frac{32}{3} a^2 - \frac{16a^2}{3} = \frac{32a^2 - 16a^2}{3} = \frac{16a^2}{3}$$

In fig, the curves $x^2 = 4ay$ & $y^2 = 4ax$ are bounded by the lines $y=0$ & $y=4a$ the strip 'pq' enter at P where $x = \frac{y^2}{4a}$ & Emerges at Q where $x = \sqrt{4ay}$ such strip lies between $y=0$ to $y=4a$

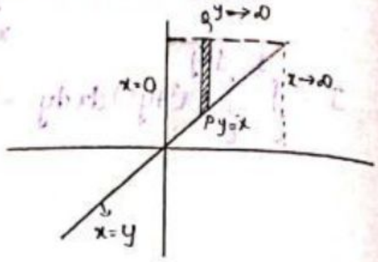
By change of order of integration we can take vertical strip instead of horizontal strip. such strip Enter at P' where $y = \frac{x^2}{4a}$ & Emerges at Q' where $y = \sqrt{4ax}$ such strip lies between $x=0$ & $x=4a$.

Q. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ by using change of order of integration

Ans. The region, $0<x<\infty$ $\quad x<y<\infty$

$x=0,\ x=\infty \qquad x=y,\ y=\infty$

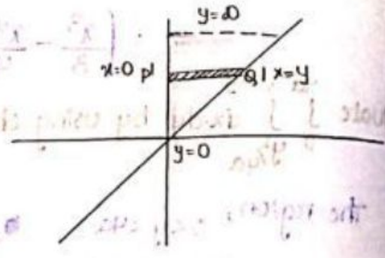
$x\to\infty \qquad y\to\infty$



By change of order of integration,

New limits are $x\to 0$ to y

$y\to 0$ to ∞



$$I=\int_0^\infty\left[\int_0^y \frac{e^{-y}}{y}dx\right]dy=\int_0^\infty\left[\frac{e^{-y}}{y}x\right]_0^y dy$$

$$=\int_0^\infty \frac{e^{-y}}{y}(y)\,dy$$

$$=\int_0^\infty e^{-y}dy=\left[\frac{e^{-y}}{-1}\right]_0^\infty=[0-(-1)]=1.$$

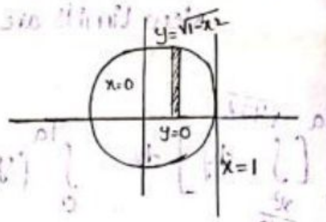
Q. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dx.dy$ by using change of order of integration

Ans. The region $\quad 0<x<1 \qquad 0<y<\sqrt{1-x^2}$

$x=0\ \ x=1 \qquad y=0 \quad y=\sqrt{1-x^2}$

$y^2=1-x^2$

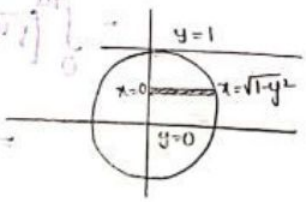
$x^2+y^2=1$



By change of order of integration,

New limits are $x\to 0$ to $\sqrt{1-y^2}$

$y\to 0$ to 1



$$I=\int_0^1\left[\int_0^{\sqrt{1-y^2}} xy\,dx\right]dy=\int_0^1\left[y\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}}dy$$

$$=\int_0^1 \frac{y(1-y^2)}{2}dy=\frac{1}{2}\left[\int_0^1 y\,dy-\int_0^1 y^3dy\right]$$

$$=\frac{1}{2}\left[\frac{y^2}{2}\right]_0^1-\frac{1}{2}\left[\frac{y^4}{4}\right]_0^1=\frac{1}{4}-\frac{1}{8}$$

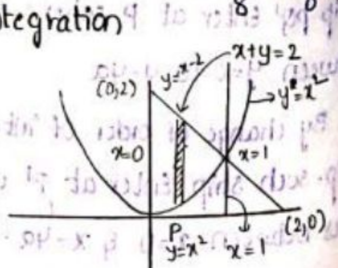
$$=\frac{2-1}{8}=\frac{1}{8}$$

Q. Evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dx.dy$ by using change of order of integration

Ans. The region, $0<x<1 \qquad x^2<y<2-x$

$x=0,\ x=1 \qquad y=x^2,\ y=2-x$

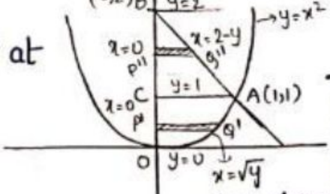
$x+y=2$



The given region 'R' can be divided into R_1 & R_2 regions (R_1 OACO, R_2 ABCA)

In R_1 region, take a strip $p'q'$ which enter at p' where $x=0$ & emerges at q' where $x=\sqrt{y}$

Such strip lies between $y=0$ & $y=1$



Similarly in R_2, Take a strip $p''q''$ which Enter at p'' where $x=0$ Emerges at q'' where $x=$

Such strip lies b/n $y=1$ & $y=2$

For the region R_1 OACO, limits are

$y \to 0$ to 1

$x \to 0$ to $\sqrt{y}$

For the region R_2 ABCA, limits are

$y \to 1$ to 2

$x \to 0$ to $2-y$

$$I_1 = \int_0^1 \left[\int_0^{\sqrt{y}} xy\,dx\right] dy = \int_0^1 \left[\frac{x^2}{2}y\right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{y^2}{2}dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$I_1 = \frac{1}{2}\left[\frac{1}{3}\right] = \frac{1}{6}$$

$$I_2 = \int_1^2 \left[\int_0^{2-y} xy\,dx\right] dy$$

$$= \int_1^2 \left[\frac{x^2}{2}y\right]_0^{2-y} dy = \int_1^2 \frac{(2-y)^2 y}{2}dy$$

$$= \frac{1}{2}\int_1^2 (y^2+4-4y)y\,dy$$

$$= \frac{1}{2}\int_1^2 y^3+4y-4y^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4}+\frac{4y^2}{2}-\frac{4y^3}{3}\right]_1^2$$

$$I_2 = \frac{1}{2}\left[\frac{1}{4}(16-1)+2[4-1]-\frac{4}{3}[8-1]\right]$$

$$I_2 = \frac{1}{2}\left[\frac{15}{4}+6-\frac{28}{3}\right] = \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}$$

Finally,

$$\therefore \iint_R (xy)\,dx\,dy = I_1 + I_2$$

$$= \frac{1}{6}+\frac{5}{24}$$

$$= \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}$$

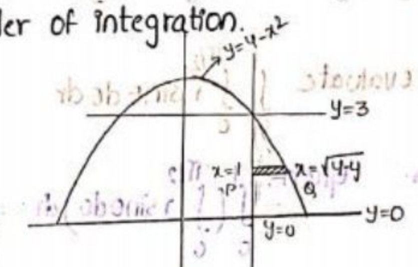
Q. Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ by using change of order of integration.

Ans. The region, $0<y<3 \quad 1<x<\sqrt{4-y}$

$y=0 \quad y=3 \qquad x=1$

$x=\sqrt{4-y}$

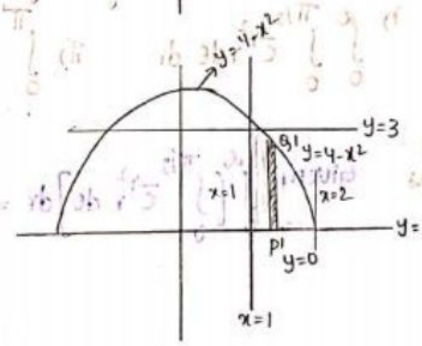
$x^2+y=4$



By change of order of integration, $y=4-x^2$

New limits are $y \to 0$ to $4-x^2$

$x \to 1$ to 2



$$\int_1^2\left[\int_0^{4-x^2}(x+y)dy\right]dx = \int_1^2 \left(xy+\frac{y^2}{2}\right)_0^{4-x^2} dx$$

$$= \int_1^2 \left[x(4-x^2)+\frac{(4-x^2)^2}{2}\right]dx$$

$$= \int_1^2 \left[4x-x^3+\frac{1}{2}[x^4+16-8x^2]\right]dx$$

$$= \int_1^2 4x-x^3\,dx + \frac{1}{2}\int_1^2 x^4+16-8x^2\,dx = \left[4\frac{x^2}{2}-\frac{x^4}{4}\right]_1^2 + \frac{1}{2}\left[\frac{x^5}{5}+16x-\frac{8x^3}{3}\right]_1^2$$

$$= 2(4-1)-\frac{1}{4}(16-1)+\frac{1}{2}\left[\frac{1}{5}(32-1)+16(2-1)-\frac{8}{3}(8-1)\right]$$

$$= 6-\frac{15}{4}+\frac{1}{2}\left[\frac{31}{5}+16-\frac{56}{3}\right] = 6-\frac{15}{4}+\frac{1}{2}\left[\frac{53}{15}\right] = 6-\frac{15}{4}+\frac{53}{30}$$

$$= \frac{241}{60}$$

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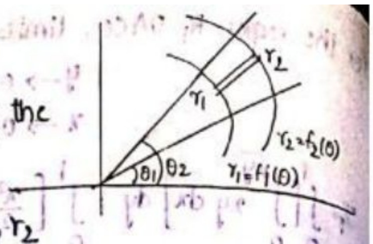
Double Integrals in polar co-ordinates :-



Q. Evaluate $\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta=\theta_1$ to $\theta=\theta_2$ and the curves $r_1=f_1(\theta)$, $r_2=f_2(\theta)$

We first integrate w.r.t 'r' between the limits r_1 & r_2 [Treating 'θ' as a constant]. The resulting expression in terms of 'θ' then integrate w.r.t 'θ' from θ_1 to θ_2.

$$\boxed{\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} f(r,\theta)\,dr\,d\theta = \int_{\theta_1}^{\theta_2}\left[\int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r,\theta)\,dr\right]d\theta}$$

Q. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Ans. Given, $\int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta = \int_0^{\pi}\frac{a^2\sin^2\theta}{2}\,d\theta$

$$= \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$\left[\because n\int_0^a f(x)\,dx = \int_0^{an} f(x)\,dx\right]$

$$= a^2\int_0^{\pi/2}\sin^2\theta\,d\theta = a^2\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi a^2}{4}$$

Q. Evaluate $\int_0^1\int_0^{\pi/2} r\sin\theta\,d\theta\,dr$

Ans. Given, $\int_0^1\left(\int_0^{\pi/2} r\sin\theta\,d\theta\right)dr = -\int_0^1 \left[r\cos\theta\right]_0^{\pi/2} dr = \int_0^1 +r\,dr = +\left[\frac{r^2}{2}\right]_0^1 = +\frac{1}{2}$

Q. evaluate the following integrals

i) $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$ ii) $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$ iii) $\int_0^{\pi/2}\int_0^{\infty}\frac{r\,dr\,d\theta}{(r^2+a^2)^2}$ iv) $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

i)
Ans. Given, $\int_0^{\infty}\left[\int_0^{\pi/2} e^{-r^2} r\,d\theta\right]dr = \int_0^{\infty}\left[e^{-r^2} r\,\theta\right]_0^{\pi/2} dr = \int_0^{\infty}\frac{\pi}{2}e^{-r^2}(r)\,dr = \frac{\pi}{2}\int_0^{\infty} r\,e^{-r^2}\,dr$

$r^2=t$, $2r\,dr=dt$, $r\to 0$ $t\to 0$, $r\to\infty$ $t\to\infty$

$$= \frac{\pi}{2}\int_0^{\infty} e^{-t}\frac{1}{2}\,dt = \frac{\pi}{4}\int_0^{\infty} e^{-t}\,dt$$

$$= \frac{\pi}{4}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} = \frac{-\pi}{4}[0-1] = \frac{\pi}{4}$$

ii)
Ans. Given, $\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \int_0^{\pi}\frac{a^2}{2}(1+\cos\theta)^2\,d\theta$

$$= \frac{a^2}{2}\int_0^{\pi}(1+\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left\{\left[\theta\right]_0^{\pi} + \frac{1}{2}\left(\frac{\pi}{2}\right) + \left[2(\sin\theta)\right]_0^{\pi}\right\}$$

$$= \frac{a^2}{2}\left[(\pi-0) + \frac{\pi}{4} + 2(0-0)\right] = \frac{a^2}{2}\left(\pi + \frac{\pi}{4}\right)$$

$$= \frac{\pi a^2}{2} + \frac{\pi a^2}{4} = \frac{3\pi a^2}{4}$$

iii) Given, $\int_0^{\pi/2} \frac{1}{2}\left[\int_0^{\infty} \frac{2r\,dr}{(r^2+a^2)^2}\right] d\theta = \int_0^{\pi/2} \frac{1}{2}\left[\frac{-1}{r^2+a^2}\right]_0^{\infty} d\theta$ $\qquad \int \frac{f'(x)}{f(x)}$

-Ans. $\qquad \int \frac{1}{x^2}dx = \frac{-1}{x}$

$$= \int_0^{\pi/2} \frac{1}{2}\left(0 - \left(\frac{-1}{a^2}\right)\right) d\theta = \int_0^{\pi/2} \frac{1}{2a^2} d\theta$$

$$= \frac{1}{2a^2}[\theta]_0^{\pi/2} = \frac{1}{2a^2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4a^2}.$$

iv) Given, $\int_0^{\pi/4} \frac{-1}{2}\left[\int_0^{a\sin\theta} \frac{-2r\,dr}{\sqrt{a^2-r^2}}\right] d\theta = \int_0^{\pi/4} \frac{-1}{2}\left[2\sqrt{a^2-r^2}\right]_0^{a\sin\theta} d\theta$

-Ans.

$$= -\int_0^{\pi/4} \left(a\sqrt{1-\sin^2\theta} - a\right) d\theta$$

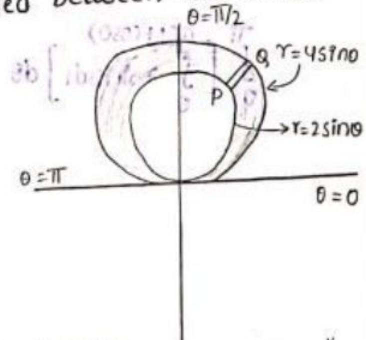
$$= -\int_0^{\pi/4} a(\cos\theta - 1)\, d\theta = -a\left[\sin\theta - \theta\right]_0^{\pi/4}$$

$$= -a\left[\left(\sin\frac{\pi}{4} - \sin 0\right) - \left(\frac{\pi}{4} - 0\right)\right]$$

$$= -a\left[\left(\frac{1}{\sqrt{2}} - 0\right) - \frac{\pi}{2}\right] = -a\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right].$$

Q. Evaluate integration $I = \iint r^3\, dr.d\theta$ over the region included between the circles $r = 2\sin\theta$, $r = 4\sin\theta$.



Ans. Here limits are $r \to 2\sin\theta$ to $4\sin\theta$
$\theta \to 0$ to π

$$I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\, dr.d\theta = \int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3\, dr\right] d\theta$$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \int_0^{\pi} \frac{4^4\sin^4\theta}{4} - \frac{2^4\sin^4\theta}{4}\, d\theta$$

$$= \int_0^{\pi} \frac{\sin^4\theta\,(2^8 - 2^4)}{4} d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= \frac{240}{2}\int_0^{\pi/2} \sin^4\theta\, d\theta = 120\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{45\pi}{2}$$

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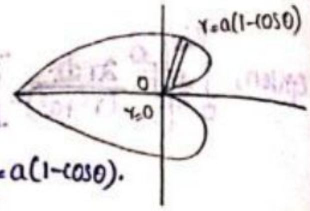
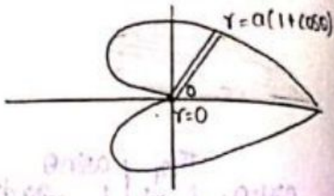
Q. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the caridiod $r=a(1-\cos\theta)$, $r=a(1+\cos\theta)$ above the initial line



Ans. The cardiod $r=a(1-\cos\theta)$ is symetrical about the initial line and it passes through the pole 'O' when $\theta=0$. The region of integration is above the initial line is covered by radial strip whose ends are $r=0$ & $r=a(1-\cos\theta)$.
The strip starting from $\theta=0$ to $\theta=\pi$



then, $$\int_0^{\pi}\left[\int_0^{a(1-\cos\theta)} r\sin\theta\,dr\right]d\theta = \int_0^{\pi}\sin\theta\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)}d\theta$$

$$=\int_0^{\pi}\sin\theta\,\frac{a^2(1-\cos\theta)^2}{2}\,d\theta$$

$$=\frac{a^2}{2}\int_0^{\pi}\sin\theta(1+\cos^2\theta-2\cos\theta)\,d\theta = \frac{a^2}{2}\left[\int_0^{\pi}\sin\theta\,d\theta+\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta - \int_0^{\pi}2\sin\theta\cos\theta\,d\theta\right]$$

$$=\frac{a^2}{2}\left[(-\cos\theta)_0^{\pi}\right]+a^2\frac{1}{3}\int_0^{\pi/2}\sin\theta\,d\theta-\frac{a^2}{2}\int_0^{\pi}\sin2\theta\,d\theta$$

$$=-\frac{a^2}{2}(1-1)+\frac{a^2}{3}\left[-\cos\theta\right]_0^{\pi/2}-\frac{a^2}{2}\left[\frac{-\cos2\theta}{2}\right]_0^{\pi} = -\frac{a^2}{3}\left[0-1\right]+\frac{a^2}{4}\left[\cos2\pi-\cos0\right]$$

$$=\frac{a^2}{3}+\frac{a^2}{4}[1-1]=\frac{a^2}{3}$$

For cardiod $r=a(1+\cos\theta)$,

Limits are $\theta=0$ to $\theta=\pi$

$r=0$ to $r=a(1+\cos\theta)$

$$\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r\sin\theta\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}\sin\theta\,d\theta = \int_0^{\pi}\sin\theta\left[\frac{a^2(1+\cos^2\theta+2\cos\theta)}{2}\right]d\theta$$

$$=\frac{a^2}{2}\left[\int_0^{\pi}\sin\theta\,d\theta+\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta+\int_0^{\pi}2\sin\theta\cos\theta\,d\theta\right]$$

Wrong (x)

$$=a^2\left[(-\cos\theta)_0^{\pi/2}+\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta+\int_0^{\pi/2}\sin2\theta\,d\theta\right]$$

$$=a^2\left[-(0-1)+\frac{1}{3}\int_0^{\pi/2}\sin\theta\,d\theta+\left[\frac{-\cos2\theta}{2}\right]_0^{\pi/2}\right]$$

$$=a^2\left[1+\frac{1}{3}\left[-\cos\theta\right]_0^{\pi/2}+\left(\frac{-1}{2}\right)\left[\cos\pi-\cos0\right]\right]$$

$$=a^2\left[1+\frac{-1}{3}(0-1)+\left(\frac{-1}{2}\right)(1-1)\right]=a^2\left[1+\frac{1}{3}\right]=\frac{4a^2}{3}$$

$$=\int_0^{\pi}\sin\theta(1-\cos\theta)^2\left(\frac{a^2}{2}\right)d\theta$$

$f(x)=1-\cos\theta$

$f'(x)=\sin\theta$

$\left(\because \int f'(x)\,(f(x))^n=\frac{(f(x))^{n+1}}{n+1}\right)$

$$=\frac{a^2}{2}\int_0^{\pi}\sin\theta(1-\cos\theta)^2\,d\theta$$

$$=\frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}=\frac{a^2}{6}\left[(1-\cos\theta)^3\right]_0^{\pi}$$

$$=\frac{a^2}{6}\left[(1-\cos\pi)^3-(1-\cos0)^3\right]$$

$$=\frac{a^2}{6}\left[(1+1)^3-(1-1)^3\right]=\frac{a^2}{6}\left[2^3\right]=\frac{8a^2}{6}$$

$$=\frac{4a^2}{3}$$

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For cardiod $r = a(1+\cos\theta)$ $\quad \theta \to 0$ to π

$r \to 0$ to $a(1+\cos\theta)$

$$\int_0^\pi \left[\int_0^{a(1+\cos\theta)} r\sin\theta\, dr\right] d\theta = \int_0^\pi \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \int_0^\pi \sin\theta\left(\frac{a^2}{2}\right)(1+\cos^2\theta+2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\int_0^\pi \sin\theta\, d\theta + \int_0^\pi \sin\theta\cos^2\theta\, d\theta + \int_0^\pi \sin2\theta\, d\theta\right]$$

$$= a^2\left[\int_0^{\pi/2} \sin\theta\, d\theta + \int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta + \int_0^{\pi/2} \sin2\theta\, d\theta\right]$$

$$= a^2\left[(-\cos\theta)_0^{\pi/2} + \frac{1}{3}\int_0^{\pi/2}\sin\theta\, d\theta + \left[\frac{-\cos2\theta}{2}\right]_0^{\pi/2}\right]$$

$$= a^2\left[-(0-1) + \frac{1}{3}[0-1] - \frac{1}{2}[\cos\pi - \cos0]\right] = a^2\left[1+\frac{1}{3}-\frac{1}{2}\left(\frac{-1}{-1}\right)\right]$$

$$= a^2\left[1+\frac{1}{3}\right.$$

$$= \int_0^\pi \sin\theta\left(\frac{a^2}{2}\right)(1+\cos\theta)^2 d\theta \qquad f(x) = 1+\cos\theta, \quad f'(x) = -\sin\theta$$

$$= -\frac{a^2}{2}\int_0^\pi (-\sin\theta)(1+\cos\theta)^2 d\theta = -\frac{a^2}{2}\left[\frac{(1+\cos\theta)^3}{3}\right]_0^\pi = \frac{-a^2}{2(3)}\left[(1+\cos\pi)^3 - (1+\cos0)^3\right]$$

$$= \frac{-a^2}{6}\left[(1-1)^3 - (1+1)^3\right] = \frac{-a^2}{6}[-8] = \frac{4a^2}{3}$$

Change of Variables in double Integrals :-

Sometimes to Evaluate double or Triple integrals with a pleasant form may not be simple to Evaluate by choice of an appropriate co-ordinate system, the given integral can be transformed into a simple integral involving new variables

Transformation of co-ordinates :-

Let, $x = f(u,v)$ & $y = g(u,v)$ be a relation between old variables 'x' & 'y' with new variables 'u' & 'v' of the new co-ordinate system.

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(f(u,v), g(u,v))\,du\,dv\,|J|$$

$$= \iint_R f(r,\theta)\, r\, dr\, d\theta$$

where, $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = J\left(\frac{x,y}{u,v}\right)$

RESULT :-

→ Change of variables from cartesian co-ordinates to polar co-ordinates,

$x = r\cos\theta$, $y = \sin\theta(r)$ and $dx\,dy = r\,dr\,d\theta$.

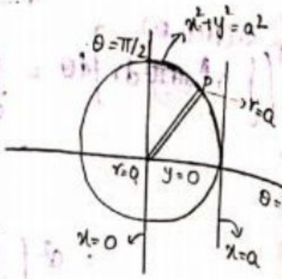
→ Change of variables from cartesian co-ordinates to spherical polar coordinates in triple integrals $\quad x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and $dx\,dy\,dz = r^2\sin\theta\, d\theta\, d\phi\, dr$

* $\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r,\theta)\, r\, dr\, d\theta$

* $\iint_R f(x,y,z)\,dx\,dy\,dz = \iint_R f(r,\theta,\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$.

Q. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx.dy$ by using change of variables.



Ans. put, $x=r\cos\theta$ $y=r\sin\theta$ $dxdy = r dr d\theta$

$$\int_0^a \int_0^{\sqrt{a^2-\text{[redacted]}x^2}} (r\sin\theta)\sqrt{r^2(\sin^2\theta+\cos^2\theta)}\; r dr d\theta$$

The region, $0<x<a$ $\quad 0<y<\sqrt{a^2-x^2}$

$x=0,\ x=a$ $\quad y=0$ $\quad x^2+y^2=a^2$ i.e which is a circle bounded by the lines $x=0; x=a, y=0$.

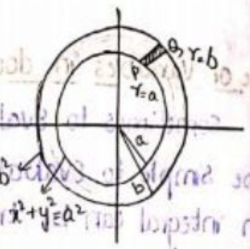
New limits are, $\theta \to 0$ to $\pi/2$

By considering a strip, $r\to 0$ to a

$$\int_0^{\pi/2}\int_0^a r^3 \sin\theta\, dr d\theta = \left(\int_0^{\pi/2} \text{[redacted]} \sin\theta\, d\theta\right)\left(\int_0^a r^3 dr\right)$$

$$= (-\cos\theta)_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^a = -(0-1)\left(\frac{a^4}{4}\right) = \frac{a^4}{4}$$

Q. Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2} dxdy$. by changing into polar co-ordinates, over the annular's region between the circles $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(b>a)$



Ans. Here limits are, $\theta\to 0$ to 2π

$r \to a$ to b

put $x=r\cos\theta$, $y=r\sin\theta$, $dxdy = r d\theta dr$

$$\int_a^b\int_0^{2\pi} \frac{r^4(\sin^2\theta\cos^2\theta)}{r^2}(rdrd\theta) = \int_0^{2\pi}\int_a^b r^3(\sin^2\theta\cos^2\theta)\, dr d\theta$$

$$= \left(\int_a^b r^3 dr\right)\left(\int_0^{2\pi}\sin^2\theta\cos^2\theta\, d\theta\right)$$

$$= \left(\frac{r^4}{4}\right)_a^b \left[\text{[redacted]}\int_0^{2\pi}\sin^2\theta(1-\sin^2\theta)\, d\theta\right]$$

$$= \left(\frac{b^4}{4}-\frac{a^4}{4}\right)\left[\int_0^{2\pi}\sin^2\theta\, d\theta - \int_0^{2\pi}\sin^4\theta\, d\theta\right]$$

$$= (b^4-a^4)\left[\int_0^{\pi/2}\sin^2\theta\, d\theta - \int_0^{\pi/2}\sin^4\theta\, d\theta\right]$$

$$= (b^4-a^4)\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{(b^4-a^4)\pi}{4}\left(1-\frac{3}{4}\right)$$

$$= \frac{\pi(b^4-a^4)}{16}$$

Q. Evaluate the following integrals

i) $\int_0^a\int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dx\, dy$ $\quad$ ii) $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy$ $\quad$ iii) $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dx\, dy$

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i) Given, $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx.dy$

Ans.

The region, $0<x<a$ $\quad 0<y<\sqrt{a^2-x^2}$

$x=0,\ x=a$ $\quad y=0\ \ y^2+x^2=a^2$

As in above question, new limits are, $\theta \to 0$ to $\pi/2$

$r \to 0$ to a

Put, $x = r\cos\theta$, $y = r\sin\theta$ $\quad dx.dy = r\,dr\,d\theta$

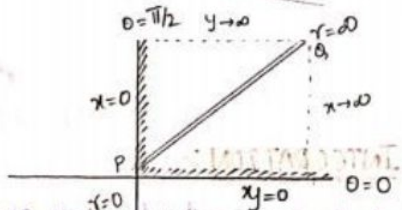
$$\int_0^a \int_0^{\pi/2} e^{-r^2(\cos^2\theta+\sin^2\theta)} \cdot r\, dr\, d\theta$$

$$\left[= \left(\int_0^a r\,dr\right)\left(\int_0^{\pi/2} e^{-a^2}\, d\theta\right)\right.$$

$$\left.= \left(\frac{r^2}{2}\right)_0^a \left(e^{-a^2}\left(\frac{\pi}{2}-0\right)\right) = \frac{a^2}{2}\left(\frac{\pi}{2}e^{-a^2}\right) = \frac{\pi a^2 e^{-a^2}}{4}\right]$$

ii) Given, $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx.dy$

Ans.



The region $0<x<\infty$ $\quad 0<y<\infty$

New limits are, $\theta \to 0$ to $\pi/2$, $r \to 0$ to ∞

Put $x = r\cos\theta$, $y = r\sin\theta$, $dx dy = r\, d\theta\, dr$

$$\int_0^{\pi/2}\left[\int_0^\infty e^{-r^2}\, r\,dr\right] d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^\infty e^{-t}\,\frac{dt}{2}\right] d\theta$$

Put, $r^2 = t$, $2r\,dr = dt$; if $r \to 0$ then $t \to 0$, $r \to \infty$ then $t \to \infty$

$$= \int_0^{\pi/2} \frac{1}{2}\,\frac{(e^{-t})_0^\infty}{-1}\, d\theta$$

$$= \int_0^{\pi/2} \frac{-1}{2}(0-1)\,d\theta = \int_0^{\pi/2} \frac{-1}{2}(-1)\,d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$

i)

Ans.

$$\int_0^{\pi/2}\left[\int_0^a e^{-r^2}(r)\,dr\right] d\theta = \int_0^{\pi/2} \frac{1}{2}\left[\int_0^{a^2} e^{-t}\,dt\right] d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{e^{-t}}{-1}\right)_0^{a^2} d\theta = \frac{1}{2}\int_0^{\pi/2} -(e^{-a^2}-1)\,d\theta$$

Put $r^2 = t$

$2r\,dr = dt$

$r \to 0$ then $t \to 0$

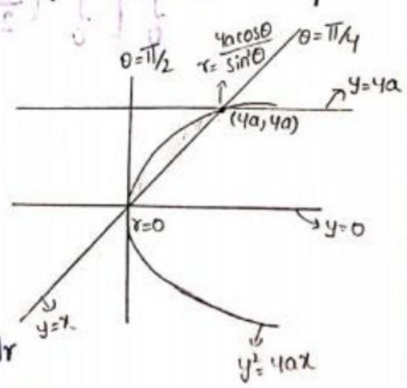
$r \to a$ then $t \to a^2$

$$= \frac{(1-e^{-a^2})}{2}\int_0^{\pi/2} d\theta = \frac{(1-e^{-a^2})}{2}[\theta]_0^{\pi/2}$$

$$= \frac{(1-e^{-a^2})}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi(1-e^{-a^2})}{4}$$

iii) Given, $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\,dy$

Ans.



The region, $0<y<4a$ $\quad \frac{y^2}{4a}<x<y$

$y=0,\ y=4a$ $\quad x=y,\ x=\frac{y^2}{4a}$

$y^2 = 4ax$

Put $x = r\cos\theta$, $y = r\sin\theta$

$dx dy = r\,d\theta \cdot dr$

New limits are, $\theta \to \pi/4$ to $\pi/2$

$r \to 0$ to $\frac{4a\cos\theta}{\sin^2\theta}$ $\quad r^2\sin^2\theta = 4a\, r\cos\theta \Rightarrow r = \frac{4a\cos\theta}{\sin\theta}$

$$\therefore \int_{\pi/4}^{\pi/2}\int_{0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}\, r\,dr\,d\theta = \left[\int_{\pi/4}^{\pi/2}\cos^2\theta\,d\theta - \int_{\pi/4}^{\pi/2}\sin^2\theta\,d\theta\right]\left[\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} r\,dr\right]$$

$$= \int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\,d\theta\left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}} = \int_{\pi/4}^{\pi/2}\frac{16a^2}{2}\frac{\cos^2\theta}{\sin^4\theta}(\cos^2\theta-\sin^2\theta)\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}(\cot^4\theta-\cot^2\theta)\,d\theta =$$

TRIPLE INTEGRATION :-

$\longrightarrow$ if x_1, x_2 are constants, y_1, y_2 are functions of x and z_1, z_2 are functions of xy Here first integrate w.r.t z, then the Expression in terms of x & y. Now we integrate w.r.t y, then total expression in terms of x. Finally integrate w.r.t x we get required solution.

$$\therefore \iiint_V f(x,y,z)\,dv = \int_{x_1}^{x_2}\left[\int_{y_1=f_1(x)}^{y_2=f_2(x)}\left[\int_{z_1=g_1(x,y)}^{z_2=g_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx.$$

Q. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx.$ $\left[\because \int\frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)\right]$

Ans. Given, $\int_0^1\left(\int_0^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{(\sqrt{1-x^2-y^2})^2-z^2}}\,dz\right]dy\right]dx$

$$= \int_0^1\left[\int_0^{\sqrt{1-x^2}}\left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}}dy\right]dx \qquad \left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \int_0^1\left[\int_0^{\sqrt{1-x^2}}\left(\frac{\pi}{2}-0\right)dy\right]dx = \int_0^1\frac{\pi}{2}(y)_0^{\sqrt{1-x^2}}\,dx = \int_0^1\frac{\pi}{2}\sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2}\int_0^1\sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}(0)+\frac{1}{2}\sin^{-1}(1)-0-\frac{1}{2}\sin^{-1}(0)\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)\right] = \frac{\pi^2}{8}$$

Q. Evaluate $\int_0^1\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Ans. Given, $\int_0^1\left[\int_0^x\left[\int_0^{x+y}(e^{x+y})e^z dz\right]dy\right]dx = \int_0^1\left[\int_0^x e^{x+y}(e^z)_0^{x+y}dy\right]dx$

$= \int_0^1\left[\int_0^x e^{x+y}\left[e^{x+y}-e^0\right]dy\right]dx$

$= \int_0^1\left[\int_0^x e^{2(x+y)}-e^{x+y}\,dy\right]dx = \int_0^1\left[\int_0^x e^{2x}e^{2y}dy - \int_0^x e^x e^y dy\right]dx$

$= \int_0^1 e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x - e^x\left[e^y\right]_0^x dx$

$= \int_0^1 e^{2x}\left[\frac{e^{2x}}{2}-\frac{1}{2}\right]-e^x\left[e^x-1\right]dx = \frac{1}{2}\int_0^1(e^{4x}-e^{2x})dx - \int_0^1(e^{2x}-e^x)dx$

$= \frac{1}{2}\left[\frac{e^{4x}}{4}-\frac{e^{2x}}{2}\right]_0^1 - \left[\frac{e^{2x}}{2}-e^x\right]_0^1$

$= \frac{1}{2}\left[\frac{e^4}{4}-\frac{e^2}{2}-\frac{1}{4}+\frac{1}{2}\right]-\left[\frac{e^2}{2}-e^1-\frac{1}{2}+1\right]$

$= \frac{1}{2}\left[\frac{e^4}{4}-\frac{e^2}{2}+\frac{1}{4}\right]-\frac{e^2}{2}+e^1-\frac{1}{2} = \frac{e^4}{8}-\frac{e^2}{4}-\frac{e^2}{2}+e+\frac{1}{8}-\frac{1}{2}$

$= \frac{e^4}{8}-\frac{3e^2}{4}+e-\frac{3}{8} = \frac{e^4-6e^2+8e-3}{8}$

Q. Evaluate the following integrals.

i) $\int_{-1}^1\int_0^z\int_{x-z}^{x+z} x+y+z\;dz\,dy\,dx$ ii) $\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,dz$ iii) $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

i) Ans. Given $\int_{-1}^1\left[\int_0^z\left[\int_{x-z}^{x+z}(x+y+z)dy\right]dx\right]dz = \int_{-1}^1\left[\int_0^z\left[x(y)_{x-z}^{x+z}+\left(\frac{y^2}{2}\right)_{x-z}^{x+z}+z(y)_{x-z}^{x+z}\right]dx\right]dz$

$= \int_{-1}^1\left[\int_0^z x(x+z-x+z)+\frac{4z^2}{2}+z(2z)\,dx\right]dz$

$= \int_{-1}^1\left[\int_0^z 2xz+2z^2+2z^2dx\right]dz$ $\cancel{= \int_{-1}^1 2\left[\frac{x^3}{3}+z^2x+z\right]}$

$= \int_{-1}^1\left[2z\left(\frac{x^2}{2}\right)_0^z+2z^2(x)_0^z+2z^2(x)_0^z\right]dz$

$= 2\int_{-1}^{+1} z\left(\frac{z^2}{2}\right)+z^2(z)+z^2(z)\,dz = 2\left[\frac{1}{2}\left[\frac{z^4}{4}\right]_{-1}^1+\left[\frac{z^4}{4}\right]_{-1}^1 + \left[\frac{z^4}{4}\right]_{-1}^1\right]$

$= 2\left[\frac{1}{2}\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{4}\right)\right] = 2(0) = 0$

ii)
Ans. Given, $\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,dz = \left(\int_0^1 x\,dx\right)\left(\int_1^2 y\,dy\right)\left(\int_2^3 z\,dz\right) = \left(\frac{x^2}{2}\right)_0^1\left(\frac{y^2}{2}\right)_1^2\left(\frac{z^2}{2}\right)_2^3$

$$= \frac{1}{2}(1-0)\cdot\frac{1}{2}(4-1)\frac{1}{2}(9-4) = \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) = \frac{15}{8}$$

iii)
Ans. Given, $\int_0^{\log 2}\left[\int_0^x\left[\int_0^{x+\log y} e^{x+y}\, e^z\,dz\right]dy\right]dx = \int_0^{\log 2}\left[\int_0^x\left[e^{x+y}(e^z)_0^{x+\log y}\right]dy\right]dx$

$$= \int_0^{\log 2}\left[\int_0^x e^{x+y}\left(e^{x+\log y}-1\right)dy\right]dx$$

$$= \int_0^{\log 2}\left[\int_0^x e^{2x}\, e^{y+\log y} - e^x\cdot e^y\,dy\right]dx$$

$$= \int_0^{\log 2}\left[\int_0^x e^{2x}\,(e^y)\,y\,dy - \int_0^x e^x\cdot e^y\,dy\right]dx$$

$$= \int_0^{\log 2}\left[e^{2x}\left(e^y(y-1)\right)_0^x - e^x\left(e^y\right)_0^x\right]dx = \int_0^{\log 2}\left[e^{2x}\left(e^x(x-1)-e^0(-1)\right) - e^x\left(e^x-1\right)\right]dx$$

$$= \int_0^{\log 2} e^{2x}\left(xe^x-e^x+1\right)-e^{2x}+e^x\,dx = \int_0^{\log 2} xe^{3x}-e^{3x}+\cancel{e^{2x}}-\cancel{e^{2x}}+e^x\,dx$$

$$= \int_0^{\log 2} \underset{u}{x}\,\underset{v}{e^{3x}}\,dx - \int_0^{\log 2} e^{3x} + \int_0^{\log 2} e^x\,dx$$

$$= \left[x\left[\frac{e^{3x}}{3}\right]-\int 1\cdot\frac{e^{3x}}{3}\,dx\right]_0^{\log 2} - \left[\frac{e^{3x}}{3}\right]_0^{\log 2} + \left(e^x\right)_0^{\log 2}$$

$$= \left[\frac{xe^{3x}}{3}-\frac{e^{3x}}{9}\right]_0^{\log 2} - \left[\frac{e^{3\log 2}}{3}-\frac{1}{3}\right] + \left(e^{\log 2}-1\right)$$

$$= \left[\frac{\log 2\; e^{\log 8}}{3}-\frac{e^{\log 8}}{9}-0+\frac{1}{9}\right] - \frac{8}{3}+\frac{1}{3}+(2-1)$$

$$= \log 2\left(\frac{8}{3}\right)-\frac{8}{9}+\frac{1}{9}-\frac{7}{3}+1 = \log 2\left(\frac{8}{3}\right)-\frac{7}{9}-\frac{7}{3}+1 = \frac{8}{3}\log 2-\frac{28}{9}+1$$

$$= \frac{8}{3}\log 2-\frac{19}{9}$$

Q. Evaluate $\iiint_V xy^2z\,dxdydz$ taken through the +ve quadrant of the sphere $x^2+y^2+z^2=a^2$

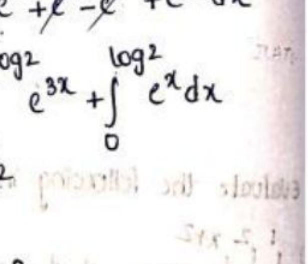
Ans. Given, $x^2+y^2+z^2=a^2 \Rightarrow z=\pm\sqrt{a^2-x^2-y^2}$

Put $z=0 \Rightarrow x^2+y^2=a^2$

$y=\sqrt{a^2-x^2}$ $z\to 0$ to $\sqrt{a^2-x^2-y^2}$

hence limits are, $x\to 0$ to a

$y\to 0$ to $\sqrt{a^2-x^2}$



$$\iiint_V xy^2z = \int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\left[\int_0^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz\right]dy\right]dx$$

$$= \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} xy^2 \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy \right) dx$$

$$= \frac{1}{2}\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} xy^2 (a^2-x^2-y^2)\, dy \right] dx$$

$$= \frac{1}{2}\int_0^a x \left[\int_0^{\sqrt{a^2-x^2}} a^2y^2 - x^2y^2 - y^4\, dy \right] dx$$

$$= \frac{1}{2}\int_0^a x \left[a^2\frac{y^3}{3} - x^2\frac{y^3}{3} - \frac{y^5}{5} \right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x \left[\frac{a^2(a^2-x^2)^{3/2}}{3} - \frac{x^2(a^2-x^2)^{3/2}}{3} - \frac{(a^2-x^2)^{5/2}}{5} \right] dx$$

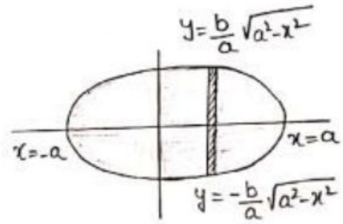
$$= \frac{1}{2}\int_0^a x \left[\frac{(a^2-x^2)^{3/2}}{3}(a^2-x^2) - \frac{(a^2-x^2)^{5/2}}{5} \right] dx$$

$$= \frac{1}{2}\int_0^a x \left[\frac{2(a^2-x^2)^{5/2}}{15} \right] dx = \frac{1}{15}\int_0^a x (a^2-x^2)^{5/2} dx = \frac{-1}{30}\int_0^a (-2x)(a^2-x^2)^{5/2} dx$$

$$= \frac{-1}{30}\left[\frac{(a^2-x^2)^{7/2}}{7/2} \right]_0^a \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{-1}{30}\left[0 - \frac{(a^2)^{7/2}}{7/2} \right] = \frac{+2}{30(7)} a^7 = \frac{+a^7}{105}$$

Q. Find the volume of the $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ elliposied



Ans. Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$

$$z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Put $z = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$x \to -a$ to a

$y \to \frac{-b}{a}\sqrt{a^2-x^2}$ to $\frac{b}{a}\sqrt{a^2-x^2}$

$z \to -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ to $+c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

Volume $(v) = \iiint_V dv$

$$= \int_{-a}^{a} \left(\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \left[\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{+c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz \right] dy \right] dx = 8\int_0^a \left(\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left(\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz \right) dy \right) dx$$

$$= 8\int_0^a \left(\int_0^{b/a\sqrt{a^2-x^2}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy \right) dx \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad f(x) = \text{Even};\ = 0 \text{ if } f(x) = \text{odd} \right]$$

$$= 8\int_0^a \left(\int_0^{b/a\sqrt{a^2-x^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy \right) dx$$

$$= 8\int_0^a \frac{c}{b}\left(\int_0^{b/a\sqrt{a^2-x^2}} \sqrt{\frac{b^2}{a^2}(a^2-x^2) - y^2}\, dy \right) dx$$

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$$= \frac{8c}{b}\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left(\sqrt{\left(\sqrt{\frac{b^2}{a^2}(a^2-x^2)}\right)^2 - y^2}\, dy \right) dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{\left(\sqrt{\frac{b^2}{a^2}(a^2-x^2)}\right)^2 - y^2} + \frac{\frac{b^2}{a^2}(a^2-x^2)}{2} \sin^{-1}\left(\frac{y}{\sqrt{\frac{b^2}{a^2}(a^2-x^2)}}\right) \right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= \frac{8c}{b}\int_0^a \frac{\frac{b^2}{a^2}(a^2-x^2)}{2}\left(\frac{\pi}{2}\right) dx = \frac{2cb\pi}{a^2}\int_0^a (a^2-x^2)\,dx$$

$$= \frac{2cb\pi}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{2cb\pi}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] \quad = 2cba\pi\left[1-\frac{1}{3}\right] = 2abc\pi\left[\frac{2}{3}\right]$$

$$\therefore V = \frac{4\pi}{3}abc$$

1) Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\, dy \right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int_{x=0}^{1} \left[\sin^{-1}(y) \right]_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int_{x=0}^{1} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \left(\frac{\pi}{2} - 0 \right) \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2} \left[\sin^{-1}(x) \right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1}(1) - \sin^{-1}(0) \right]$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{4}$$

*2) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\; dx\,dy$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\; dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2) - x^2}\; dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}\; dx \right] dy$$

w.k.T, $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$= \int_{y=0}^{a} \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \frac{a^2-y^2}{2} \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right) \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_{y=0}^{a} \left[0 + \frac{a^2-y^2}{2} \sin^{-1}\left(\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right) - 0 \right] dy \quad (\because \sin^{-1}(0) = 0)$$

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$$= \int_{y=0}^{a} \frac{a^2-y^2}{2} \cdot \sin^{-1}(1)\, dy$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a$$

$$= \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} - 0 \right] = \frac{\pi}{4} \left(\frac{3a^3 - a^3}{3} \right)$$

$$= \frac{\pi}{\cancel{4}_2} \left(\frac{\cancel{2}^1 a^3}{3} \right) = \frac{\pi a^3}{6} \;//$$

*3) $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy \right] dx$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2}\, dy \right] dx$$

we have $\displaystyle\int \frac{1}{a^2+y^2}\, dy = \frac{1}{a} \tan^{-1}(y/a)$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \left\{ \tan^{-1}\left(\frac{\cancel{\sqrt{1+x^2}}}{\cancel{\sqrt{1+x^2}}} \right) - \tan^{-1}(0) \right\} \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4} - 0 \right) \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

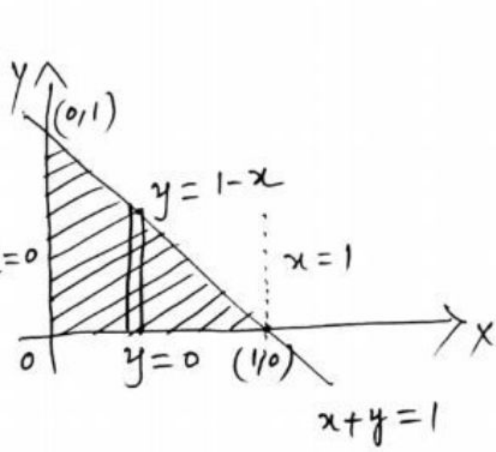
$$= \frac{\pi}{4} \left[\sinh^{-1}(x) \right]_0^1$$

$$= \frac{\pi}{4} \left[\sinh^{-1}(1) - \sinh^{-1}(0) \right]$$

$$= \frac{\pi}{4} \sinh^{-1}(1)$$

$\left(\because \sinh 0 = \frac{e^0 - e^{-0}}{2} = 0 \Rightarrow \sinh^{-1}(0) = 0 \right)$

*4) Evaluate $\iint\limits_R (x^2+y^2)\,dx\,dy$ in the positive quadrant for which $x+y \le 1$



Sol: The line $x+y=1$ intersect the coordinate axes at $(1,0)$ & $(0,1)$. shaded area is the region of integration.

Now $$\iint\limits_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{1-y} (x^2+y^2)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} (x^2+y^2)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1}\left[x^2(1-x) + \frac{1}{3}(1-x)^3 - 0\right]dx$$

$$= \int_{x=0}^{1}\left[x^2 - x^3 + \frac{1}{3}(1-x)^3\right]dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\,\frac{(1-x)^4}{-4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0 - 0 + 0 + \frac{1}{12}(1-0)^4$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$3\,\underline{|\,3, 4, 12}$
$\quad 1, 4, 4$

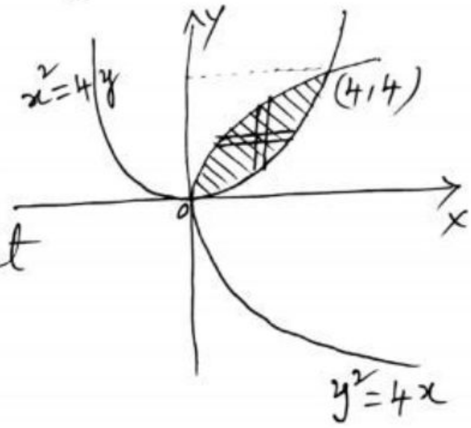
$$= \frac{4-3+1}{12} = \frac{2}{12} = \boxed{\frac{1}{6}}$$

*5) Evaluate i) $\iint_R y\,dx\,dy$

ii) $\iint_R y^2\,dx\,dy$

where R is the region bounded by the parrabolas

$y^2 = 4x$ & $x^2 = 4y$



Sol: Given curves $y^2 = 4x$ & $x^2 = 4y$
$\Rightarrow y = x^2/4$

solving these two, we get

$$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x$$

$$\Rightarrow \frac{x^4}{16} - 4x = 0$$

$$\Rightarrow x^4 - 64x = 0$$

$$\Rightarrow x(x^3 - 64) = 0$$

$$x = 0 \text{ (or) } x^3 - 64 = 0$$

$$\Rightarrow x^3 = 64 = 4^3$$

$$\Rightarrow x = 4$$

when $x = 0$, $y = 0$ & when $x = 4$, $y = \sqrt{16} = 4$

$\therefore$ Given two curves intersecting at the points $(0,0)$ & $(4,4)$. Shaded area is the region of integration.

$$\text{i) } \iint_R y\,dx\,dy = \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dx\,dy = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{2\sqrt{y}} y\,dx\,dy$$

$$= \int_{x=0}^{4} \left(\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dy \right) dx$$

$$= \int_{x=0}^{4} \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2} \int_{x=0}^{4} \left(4x - \frac{x^4}{16} \right) dx$$

$$= \frac{1}{2} \left(4 \cdot \frac{x^2}{2} - \frac{1}{16} \cdot \frac{x^5}{5} \right)_0^4$$

$$= \frac{1}{2} \left(2 \times 16 - \frac{1}{16} \times \frac{4^5}{5} - 0 \right)$$

$$= \frac{1}{2} \left(32 - \frac{64}{5} \right)$$

$$= \frac{1}{2} \left(\frac{160 - 64}{5} \right)$$

$$= \frac{1}{2} \left(\frac{96}{5} \right) = \boxed{\frac{48}{5}}$$

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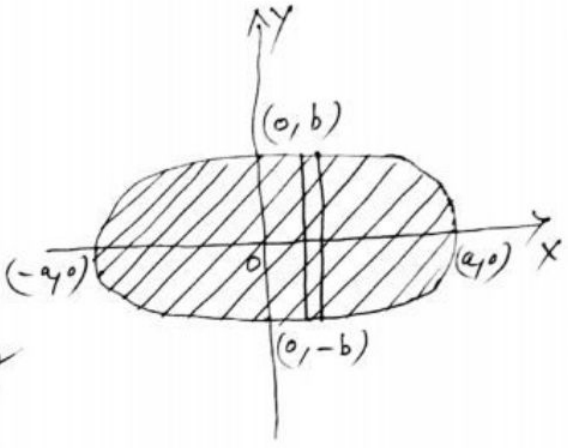
Find $\iint_R (x+y)^2\,dx\,dy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Sol: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow (1)$

put $y=0$, we get

$$\frac{x^2}{a^2}=1 \Rightarrow x^2=a^2$$
$$\Rightarrow x=\pm a$$

From (1) $\Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$$\Rightarrow y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$
$$\Rightarrow y=\pm\sqrt{\frac{b^2}{a^2}(a^2-x^2)}$$
$$=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Shaded area is the region of integration.

$\therefore\ x \rightarrow -a$ to a

$y \rightarrow -\frac{b}{a}\sqrt{a^2-x^2}$ to $+\frac{b}{a}\sqrt{a^2-x^2}$

Now $\iint_R (x+y)^2\,dx\,dy = \iint_R (x^2+y^2+2xy)\,dx\,dy$

$$=\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2)\,dx\,dy+\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}2xy\,dx\,dy$$

we have $\int_{x=-a}^{a} f(x)\,dx = \begin{cases} 2\int_{x=0}^{a} f(x)\,dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd}\end{cases}$

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$$= 2\times 2 \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy + 0$$

since x^2+y^2 is an even function

xy is an odd function

$$= 4\int_{x=0}^{a}\left[\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\right]dx$$

$$= 4\int_{x=0}^{a}\left[x^2y + \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\int_{x=0}^{a}\left[x^2\cdot\frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3 - 0\right]dx$$

$$= 4\int_0^{a}\left[\frac{b}{a}\,x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}\,(a^2-x^2)^{\frac{3}{2}}\right]dx$$

put $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

Limits: when $x=0$, $\theta = \sin^{-1}(0) = 0$

when $x = a$, $\theta = \sin^{-1}(1) = \pi/2$

$$= 4\int_{\theta=0}^{\pi/2}\left[\frac{b}{\cancel{a}}\cdot \cancel{a}^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta} + \frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]\cdot a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left[ab\sin^2\theta\cdot a\sqrt{1-\sin^2\theta} + \frac{b^3}{3a^3}(a^2)^{\frac{3}{2}}(1-\sin^2\theta)^{3/2}\right]\cdot a\cos\theta\, d\theta$$

we have $\sin^2\theta + \cos^2\theta = 1$

$$= 4\int_0^{\pi/2}\left[a^2b\sin^2\theta\sqrt{\cos^2\theta} + \frac{b^3}{3\cancel{a^3}}\cdot \cancel{a^3}(\cos^{\cancel{2}}\theta)^{3/\cancel{2}}\right]\cdot a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left(a^3b\sin^2\theta\cdot\cos^2\theta + \frac{ab^3}{3}\cdot\cos^4\theta\right)d\theta$$

$$= 4\int_0^{\pi/2}\left[a^3 b\,(1-\cos^2\theta)\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right]d\theta$$

$$= 4\int_0^{\pi/2}\left[a^3 b\,(\cos^2\theta - \cos^4\theta) + \frac{ab^3}{3}\cos^4\theta\right]d\theta$$

W.K.T, $\int_0^{\pi/2}\cos^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2}$, if n is even

$$= 4\left[a^3 b\left(\frac{2-1}{2}\cdot\frac{\pi}{2} - \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right) + \frac{ab^3}{3}\left(\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right)\right]$$

$$= 4\left[a^3 b\,\frac{1}{2}\cdot\frac{\pi}{2}\left(1-\frac{3}{4}\right) + \frac{ab^3}{\cancel{3}}\cdot\frac{\pi}{2}\cdot\frac{\cancel{3}}{4}\cdot\frac{1}{2}\right]$$

$$= 4\left[\frac{\pi a^3 b}{4}\cdot\left(\frac{1}{4}\right) + \frac{\pi a b^3}{16}\right]$$

$$= \frac{\pi a^3 b}{4} + \frac{\pi a b^3}{4}$$

$$= \frac{\pi}{4}ab\,(a^2+b^2)$$

Example 17 : Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

Solution : Consider $\iint\limits_R xy\,dx\,dy = \int\left(\int x\,dx\right)y\,dy$ over R.

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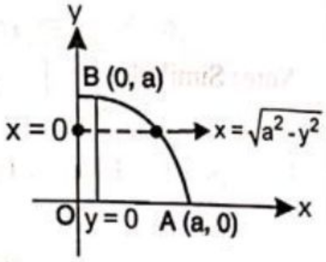
$x^2+y^2=a^2$ is a circle with centre at (0,0) and radius a units. The given region R of integration is bounded by $OABO$. Let us fix y. For a fixed y, to be in the region, we have to vary x from 0 to $\sqrt{a^2-y^2}$. However, we will be within the region only if we vary y from 0 to a.

Hence the given integral

$$= \int_{y=0}^{a}\left(\int_{x=0}^{\sqrt{(a^2-y^2)}} x\,dx\right) y\,dy = \int_{y=0}^{a}\left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} y\,dy$$

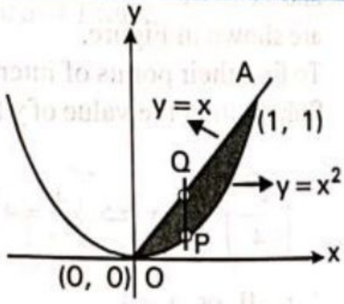


$$= \int_{y=0}^{a}\left[\frac{a^2-y^2}{2}\right] y\,dy = \int_{y=0}^{a}\left(\frac{a^2y-y^3}{2}\right)dy$$

$$= \frac{1}{2}\left(\frac{a^2y^2}{2}-\frac{y^4}{4}\right)_{y=0}^{a} = \frac{1}{2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right)=\frac{a^4}{8}.$$

Example 18 : Evaluate $\iint xy(x+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$.

Solution : $y=x^2$ is a parabola through (0,0) symmertic about y-axis. $y=x$ is a straight line through (0, 0) with slope 1. Let us find their points of intersection.



Solving $y=x^2$, $y=x$ we get $x^2=x \Rightarrow x=0,1$. Hence $y=0,1$. The points of intersection of the curves are (0, 0), (1, 1).

Hence the region is as in figure.

Consider $\iint_R xy(x+y)dy\,dx$. For the evaluation of the integral, we have to fix x first. For a fixed x, y varies from x^2 to x. Then to be in the region, we can vary x from 0 to 1. Hence the given integral is equal to

$$\int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy(xy+y)\,dy\right]dx = \int_{x=0}^{1}\left[\int_{y=x^2}^{x}(x^2y+xy^2)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left(x^2\frac{y^2}{2}+\frac{xy^3}{3}\right)_{x^2}^{x}dx = \int_{x=0}^{1}\left[\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right]dx$$

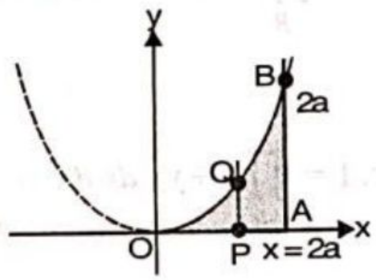
$$= \int_{x=0}^{1}\left(\frac{5}{6}x^4-\frac{x^6}{2}-\frac{x^7}{3}\right)dx = \left(\frac{5}{6}.\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{9}{168}=\frac{3}{56}.$$

Example 19 : Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x- axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$. **[JNTU (A) June 2010 (Set No. 3)]**

Solution : Let us draw the parabola $x^2 = 4ay$, the line $x = 2a$ and identify the region R of integration. It is as in figure. The integral $\iint_R xy dx\,dy$ is same as $\iint_R xy\,dy\,dx$.

Let us consider a fixed x (Draw a line $x = k$ in the region). Now for this fixed x, y varies from 0 to $x^2/4a$. To be in the region, we have to vary x from 0 to $2a$.



Hence the given integral $= \int_{x=0}^{2a} \int_{y=0}^{x^2/4a} xy\,dy\,dx$

$$= \int_{x=0}^{2a} \left[\int_{y=0}^{x^2/4a} y\,dy \right] x\,dx$$

$$= \int_{x=0}^{2a} \left[\frac{y^2}{2} \right]_{y=0}^{x^2/4a} x\,dx$$

$$= \int_{x=0}^{2a} \frac{x^4}{32a^2} x\,dx = \frac{1}{32a^2} \int_{x=0}^{2a} x^5\,dx$$

$$= \frac{1}{32a^2} \left(\frac{x^6}{6} \right)_{x=0}^{2a} = \frac{64\,a^6}{32.\,a^2.6} = \frac{a^4}{3}$$

Example 27 : Evaluate $\iint_R xy\, dx\, dy$ where R is the region bounded by the line $x+2y=2$, lying in the first quadrant. **[JNTU (A) June 2009 (Set No. 1)]**

Solution : The region R is bounded by the lines $y=0, y=\frac{1}{2}(2-x), x=0$ and $x=2$.

Hence $\iint_R xy\, dx\, dy = \int_{x=0}^{2}\int_{y=0}^{\frac{1}{2}(2-x)} xy\, dy\, dx$

$$= \int_{x=0}^{2} x.\left(\frac{y^2}{2}\right)_0^{\frac{1}{2}(2-x)} dx = \frac{1}{2}\int_0^2 \frac{x}{4}.(2-x)^2\, dx$$

$$= \frac{1}{8}\int_0^2 (4x-4x^2+x^3)\, dx = \frac{1}{8}\left(4.\frac{x^2}{2}-4\frac{x^3}{3}+\frac{x^4}{4}\right)_0^2$$

$$= \frac{1}{8}\left(8-\frac{32}{3}+4\right) = \frac{1}{8}\left(12-\frac{32}{3}\right) = \frac{1}{24}(4) = \frac{1}{6}$$

Double integrals in polar co-ordinates :—

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\, dr\, d\theta$ over the region bounded by the lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$. First we integrate w.r.t. r between the limits $r=r_1$ and $r=r_2$ keeping θ fixed. The resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Geometrically, AB and CD are the curves $r_1 = f_1(\theta)$ & $r_2 = f_2(\theta)$ bounded by the straight lines $\theta=\theta_1$ & $\theta=\theta_2$.

So that ABCD is the region of integration.

$$\therefore \iint_R f(r,\theta)\, dr\, d\theta = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, dr\, d\theta = \int_{r=r_1}^{r_2}\int_{\theta=f_1(r)}^{f_2(r)} f(r,\theta)\, d\theta\, dr$$

Ex: 1) Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Sol: $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$

$= \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$

$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$

$= \frac{a^2}{4}\left[\pi - 0\right] = \frac{a^2\pi}{4}$ //

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2) Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

Sol: $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r\,dr\right]d\theta$

$= \int_0^{\pi}\left[\left\{\frac{r^2}{2}\right\}_0^{a(1+\cos\theta)}\right]d\theta$

$= \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\left(2\cos^2\theta/2\right)^2 d\theta$

$= 2a^2\int_0^{\pi}\cos^4\theta/2\,d\theta$

put $\theta/2 = t$

$d\theta = 2dt$

$= 2a^2\int_0^{\pi/2}\cos^4 t\,(2dt)$

$= 4a^2\;\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\pi/2$

$= \cancel{4}a^2\cdot\frac{3}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{4}$ //

*3) Evaluate $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

<u>Sol</u>: $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \int_0^{\pi/4}\left[\int_0^{a\sin\theta} \frac{r\,dr}{\sqrt{a^2-r^2}}\right] d\theta$.

$$= -\frac{1}{2}\int_0^{\pi/4} 2\left[\sqrt{a^2-r^2}\right]_0^{a\sin\theta} d\theta$$

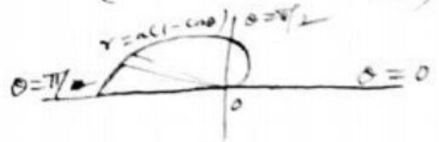
$$= -\int_0^{\pi/4}\left[\sqrt{a^2-a^2\sin^2\theta} - \sqrt{a^2}\right] d\theta$$

$$= (-a)\int_0^{\pi/4} (\cos\theta - 1)\, d\theta$$

$$= (-a)\left[\sin\theta - \theta\right]_0^{\pi/4}$$

$$= (-a)\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = a\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$ //

4) Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.



Sol: The cardioid $r = a(1-\cos\theta)$ is symmetrical about the initial line and it passes through the pole o when $\theta = 0$

The region of integration R above the initial line is covered by radial strips whose ends are $r=0$ and $r = a(1-\cos\theta)$, the strips starting from $\theta=0$ and ending at $\theta = \pi$

$$\therefore \iint_R r\sin\theta \, dr\, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1-\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \, \frac{a^2(1-\cos\theta)^2}{2} \, d\theta$$

$$= \cancel{\frac{a^2}{2} \int_0^{\pi} \sin\theta \cdot \left(2\sin^2\theta/2\right)^2 d\theta}$$

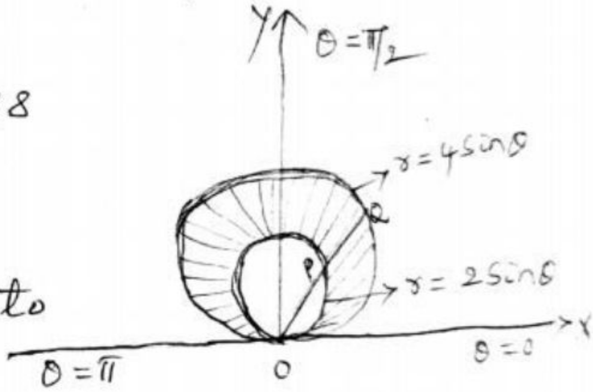
$$= \frac{a^2}{2} \int_0^{\pi} \sin\theta \, (1-\cos\theta)^2 \, d\theta.$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}$$

$$= \frac{a^2}{6}\left(2^3 - 0\right) = \frac{4a^2}{3} \,//$$

5) Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol: The region of integration R is the shaded area.



Here r varies from $r = 2\sin\theta$ to $r = 4\sin\theta$ and to cover the whole region θ varies from 0 to π.

$$\therefore \iint r^3\, dr\, d\theta = \int_{\theta=0}^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta.$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

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$$= \int_{\theta=0}^{\pi} \frac{1}{4}\left(256 \sin^4\theta - 16\sin^4\theta\right) d\theta$$

$$\begin{array}{r} 256 \\ 16 \\ \hline 240 \end{array}$$

$$= 60 \int_{\theta=0}^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \times 2 \int_{0}^{\pi/2} \sin^4\theta \, d\theta$$

$$= 120 \left[\frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2} \right]$$

$$= \cancel{120}^{\cancel{30}\,15} \times \frac{3}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2}$$

$$= \frac{45\pi}{2} \;//$$

Ans: $4a^2/3$

* Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the cardioid $r = a(1+\cos\theta)$ above the initial line.

change of variables in double integral :—

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables x, y with the new variables u, v of the new coordinate system.

Then $$\iint_R F(x,y)\, dx\, dy = \iint_R F(f,g)\, |J|\, du\, dv \quad \to (1)$$

where $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

which is called the Jacobian of the coordinate transformation.

1) change of variables from cartesian to polar coordinates:

In this case, we have $u = r$ & $v = \theta$

$$x = r\cos\theta \;/\; y = r\sin\theta$$

Now $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Hence eqn (1) becomes

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

This corresponds to $\iint_R F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\, r\,dr\,d\theta$

*Ex: 1) Evaluate the following integral by transforming into polar coordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$

Sol: $0 \le x \le a, \quad 0 \le y \le \sqrt{a^2-x^2}$

$x=0, x=a \; ; \quad y=0, \; y=\sqrt{a^2-x^2}$

$\Rightarrow x^2+y^2=a^2$

put $x = r\cos\theta, \quad y = r\sin\theta$

we have $x^2+y^2=r^2$ & $dx\,dy = r\,dr\,d\theta$

The limits for r: 0 to a and for θ: 0 to $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^a (r\sin\theta)\cdot r\,(r\,dr\,d\theta)$$

$$= \int_0^{\pi/2}\int_0^a r^3 \sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\int_0^a r^3\,dr\right] \sin\theta\,d\theta$$

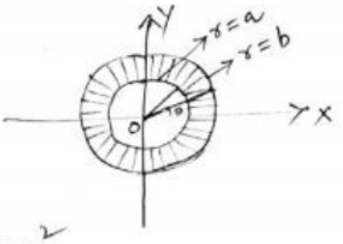
$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a \sin\theta\,d\theta$$

$$= \frac{a^4}{4}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{4}(0+1) = \frac{a^4}{4}$$

2) By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annulus region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$

Sol: change to polar coordinates by putting



$x = r\cos\theta$, $y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta$

Now $x^2+y^2=a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\Rightarrow r^2 = a^2 \Rightarrow r = a$

$x^2+y^2=b^2 \Rightarrow r^2=b^2 \Rightarrow r=b$

and θ varies from 0 to 2π

$$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta\cdot r^2\sin^2\theta}{r^2}\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta\left[\frac{r^4}{4}\right]_a^b d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} \cos^2\theta\sin^2\theta\,(b^4-a^4)\,d\theta$$

$$= \frac{b^4-a^4}{4}\int_0^{2\pi} \frac{\sin^2 2\theta}{4}\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

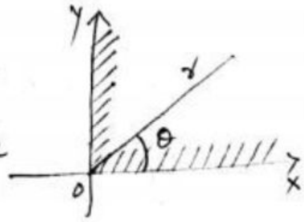
$$= \frac{b^4-a^4}{32}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{32}\left[2\pi - 0\right]$$

$$= \frac{\pi(b^4-a^4)}{16}$$

3) Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

Sol: Given $x \to 0$ to ∞, $y \to 0$ to ∞.



Therefore the region of integration is the first quadrant of the xy-plane.

changing to polar coordinates, by putting

$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx\,dy = r\,dr\,d\theta$

In the region of integration r varies from 0 to ∞ and θ varies from 0 to $\pi/2$.

$$\therefore \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

put $r^2 = t$

$2r\,dr = dt$

$$= \int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} (0+1)\,d\theta$$

$$= \frac{1}{2}\,[\theta]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Change of order of Integration :—

The change of order of integration is nothing but changes the limits of integration.

For instance, to interchanged the order of integration is

$$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx = \int_{y=a}^{b}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$

we first sketch the region of integration followed by taking up of horizontal strip (instead of vertical strip). Thus the new limits are

$y = a$ to b and $x = f_1(y)$ to $f_2(y)$ after changing order of integration.

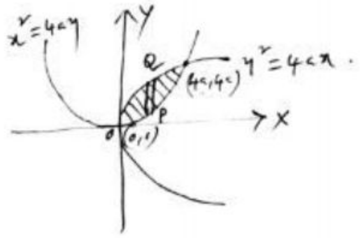
Ex: 1) change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol: Given $0 \le x \le 4a$ & $\frac{x^2}{4a} \le y \le 2\sqrt{ax}$

$$\frac{x^2}{4a} = y \quad , \quad y = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 4ay \quad , \quad y^2 = 4ax .$$

The region of integration is the shaded region in figure.



changing the order of integration, we must fix y first.

For a fixed y, x varies from $y^2/4a$ to $2\sqrt{ay}$ and then y varies from 0 to 4a.

In this case the vertical strip slides as a horizontal strip.

Hence the integral is equal to

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right] dy$$

$$= \int_{y=0}^{4a} \left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left[\frac{2\sqrt{a}\, y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16\times \cancel{4}\times a^3}{\cancel{12}_{3}\, a}$$

$$= \frac{32a^2 - 16a^2}{3} = \frac{16a^2}{3}$$

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*2) Change the order of integration and evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$$

Sol: Given limits are

$$0 \le x \le 1 \quad \& \quad x^2 \le y \le 2-x$$

x varies from 0 to 1. $\quad y = x^2 \to (1), \quad y = 2-x \Rightarrow x+y=2 \to (2)$

solving (1) & (2), we get

$$x + x^2 = 2$$
$$\Rightarrow x^2 + x - 2 = 0 \qquad 2x - 1 = -2$$
$$x^2 + 2x - x - 2 = 0 \qquad 2 - 1 = 1$$
$$x(x+2) - 1(x+2) = 0$$
$$(x-1)(x+2) = 0$$
$$x = 1 \text{ or } x = -2$$

If $x = 1, \; y = 1$

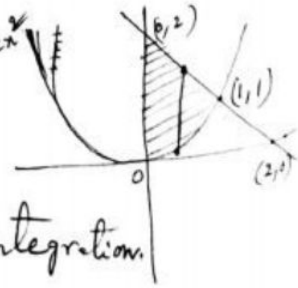
$x = -2, \; y = 4$

Hence the points of intersection of the curves are $(1,1)$ & $(-2,4)$.

The line $x+y=2$ passes through $(0,2), (2,0)$.

we shall draw the curves $y = x^2$ & $y = 2-x$

The shaded region in the figure is the region of integration.



Suppose we change the order of integration.

In this case the vertical strip slides as a horizontal strip.

In the changed order we have to take two horizontal strips since during sliding one edge of the strip remains on $x=0$ but the

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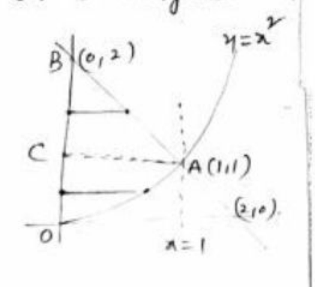
other edge of the strip doesn't remain on a single curve.



$\therefore$ The region is

Area OAB = Area OAC + Area CAB

We shall fix y first.

For the region OACO, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1.

For the region CABC, x varies from 0 to $2-y$ and y varies from 1 to 2

Hence $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy + \int_{y=1}^{2} \left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

$$= \int_{y=0}^{1} \left(\frac{y^2}{2}\right) dy + \int_{y=1}^{2} \frac{(2-y)^2 y}{2}\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2\,dy + \frac{1}{2}\int_{y=1}^{2} (4y + y^3 - 4y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$\begin{array}{r} 3 \\ 28\times 4 \\ \hline 112 \end{array}$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{120 - 112 - 3}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\cdot\frac{5}{12} = \frac{4+5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}$$

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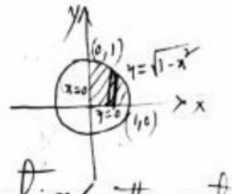
* 3) By changing the order of integration, evaluate
$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$$

Sol: Given $0 \le x \le 1$ & $0 \le y \le \sqrt{1-x^2}$

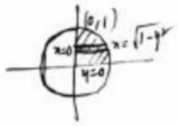
$$y = 0, \; y = \sqrt{1-x^2}$$
$$\rightarrow x^2 + y^2 = 1$$

using these limits we can draw the region of integration.



For changing the order of integration, the vertical strip slides as a horizontal strip.

we shall fix y first, x varies from 0 to $\sqrt{1-y^2}$ and y varies from 0 to 1.



$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 \, dx \, dy$$

$$= \int_{y=0}^{1} \left[x y^2 \right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \int_{y=0}^{1} \left(\sqrt{1-y^2} \, y^2 \right) dy$$

put $y = \sin\theta$

$dy = \cos\theta \, d\theta$

$$= \int_0^{\pi/2} \cos\theta \cdot \sin^2\theta \, (\cos\theta \, d\theta)$$

$$= \int_0^{\pi/2} \cos^2\theta \, \sin^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \, (1 - \cos^2\theta) \, d\theta.$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{4} \left(1 - \frac{3}{4} \right) = \frac{\pi}{16}$$

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4) change the order of integration and evaluate $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$

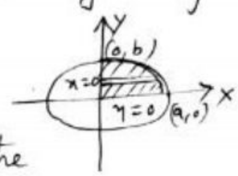
Sol: Given $0 \le y \le b$ & $0 \le x \le \frac{a}{b}\sqrt{b^2-y^2}$

$y=0$ & $y=b$, $x=0$ & $x = \frac{a}{b}\sqrt{b^2-y^2}$

$\Rightarrow b^2x^2 = a^2b^2 - a^2y^2$

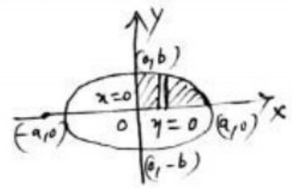
$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

using these limits we can draw the region of integration.



on changing the order of integration, the horizontal strip slides as a vertical strip.

we shall fix x first, y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$ and x varies from 0 to a.



$$\therefore \int_{y=0}^{b} \int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_{x=0}^{a} \left[\frac{xy^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \left[\frac{x}{2}(a^2-x^2)\frac{b^2}{a^2}\right] dx$$

$$= \frac{b^2}{2a^2}\int_{x=0}^{a} (a^2x - x^3)\,dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{4}\right) = \frac{a^2b^2}{8} \,//$$

5) Evaluate the integral by changing the order of integration $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ Ans: $\frac{241}{60}$

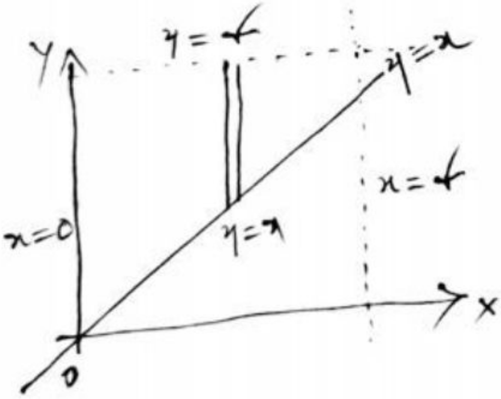
Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dx\,dy$, by changing the order of integration

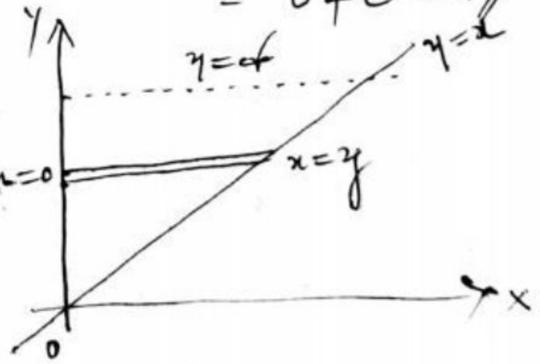
$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}\left[x\right]_0^{y} dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}(y-0)\,dy = \left[-e^{-y}\right]_0^{\infty}$$

$$= 0 + e^{0} = 1$$





Triple Integrals:

Let $f(x, y, z)$ be a function defined over a 3-dimensional finite region V.

The triple integral of f over the volume V and is represented by $\iiint_V f\,dv$ or $\int_V f\,dv$

If the region V is bounded by the surfaces $x = x_1$, $x = x_2$; $y = y_1$, $y = y_2$; $z = z_1$, $z = z_2$ then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dx\,dy\,dz$$

Evaluation of Triple integrals :—

Case i): If x_1, x_2 ; y_1, y_2 ; z_1, z_2 are all constants, then the order of integration is not important provided the limits of integration are changed accordingly.

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dz\,dy\,dx = \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x, y, z)\,dx\,dz\,dy$$

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\,dx\,dy\,dz$$

Case ii): If z_1, z_2 are functions of x & y and y_1, y_2 are functions of x while x_1 & x_2 are constants. Then the integration must be performed first w.r.t z then w.r.t. y and finally w.r.t. x.

$$\iiint_V f(x, y, z)\,dv = \int_{x=a}^{b} \boxed{\int_{y=f_1(x)}^{y=g_1(x)} \boxed{\int_{z=f_2(x, y)}^{z=g_2(x, y)} f(x, y, z)\,dz}\,dy}\,dx$$

Ex: 1) Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2z \, dz\, dy\, dx$

Sol: $\int_0^2 \int_1^3 \int_1^2 xy^2z \, dz\, dy\, dx = \int_0^2 \int_1^3 \left[\frac{xy^2z^2}{2}\right]_1^2 dy\, dx$

$= \int_0^2 \int_1^3 \left(\frac{4xy^2}{2} - \frac{xy^2}{2}\right) dy\, dx$

$= \int_0^2 \int_1^3 \frac{3xy^2}{2}\, dy\, dx$

$= \frac{3}{2} \int_0^2 \left[\frac{xy^3}{3}\right]_1^3 dx$

$= \frac{1}{2} \int_0^2 (27x - x)\, dx$

$= \frac{1}{2} \left[\frac{26x^2}{2}\right]_0^2$

$= \frac{13}{2}(4) = 26$ //

*2) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$

Sol: $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz = \int_0^a \int_0^x e^{x+y}\left[\int_0^{x+y} e^z\, dz\right] dx\, dy$

$= \int_0^a \int_0^x e^{x+y} \left[e^z\right]_0^{x+y} dx\, dy$

$= \int_0^a \int_0^x e^{x+y} \left(e^{x+y} - 1\right) dx\, dy$

$= \int_0^a \int_0^x \left[e^{2(x+y)} - e^{x+y}\right] dx\, dy$

$= \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx.$

$= \int_0^a \left\{\left(\frac{e^{4x}}{2} - e^{2x}\right) - \left(\frac{e^{2x}}{2} - e^x\right)\right\} dx$

$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x\right]_0^a$

$= \left(\frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^a\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$

$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$ //

*3) Evaluate the triple integral $\int_0^1 \int_y^1 \int_0^{1-x} x \, dz\, dx\, dy$

Sol: $$\int_0^1 \int_y^1 \int_0^{1-x} x \, dz\, dx\, dy = \int_0^1 \int_y^1 x\,[z]_0^{1-x} \, dx\, dy$$

$$= \int_0^1 \int_y^1 (x - x^2)\, dx\, dy.$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_y^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{3} - \frac{y^2}{2} + \frac{y^3}{3}\right) dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right) dy$$

$$= \left[\frac{y}{6} - \frac{y^3}{6} + \frac{y^4}{12}\right]_0^1$$

$$= \cancel{\frac{1}{6}} - \cancel{\frac{1}{6}} + \frac{1}{12} = \frac{1}{12}$$

*4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$

Sol: $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right] dx$$

$$= \frac{1}{8}\int_0^1 \left(2x - 2x^3 - 2x^3 + 2x^5 - \overset{x - x^5 + 2x^3}{\cancel{x + x^2}}\right) dx$$

$$= \frac{1}{8}\int_0^1 (x^5 - 2x^3 + x)\, dx$$

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$$= \frac{1}{8}\left[\frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{8}\left(\frac{1}{6} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\right) = \frac{1}{48} \,//$$

*5) Evaluate $\int_0^{\pi/2}\int_x^{\pi/2}\int_0^{xy} \cos(z/x)\, dz\, dy\, dx$

Sol: $\int_0^{\pi/2}\int_x^{\pi/2}\int_0^{xy} \cos(z/x)\, dz\, dy\, dx = \int_0^{\pi/2}\int_x^{\pi/2}\left[\frac{\sin(z/x)}{1/x}\right]_0^{xy} dy\, dx$

$$= \int_0^{\pi/2}\int_x^{\pi/2} (x \sin y)\, dy\, dx$$

$$= \int_0^{\pi/2} \left[-x\cos y\right]_x^{\pi/2} dx$$

$$= \int_0^{\pi/2} (0 + x\cos x)\, dx$$

$$= \left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx$$

$$= \frac{\pi}{2} + \left[\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + 0 - 1$$

$$= \frac{\pi}{2} - 1 \,//$$

6) Evaluate $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

Sol: $\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\, dx dy dz = \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy + \frac{y^2}{2} + yz\right]_{x-z}^{x+z} dx dz$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} \left\{\left[x(x+z) + \frac{(x+z)^2}{2} + (x+z)z\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + (x-z)z\right]\right\} dx dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} \left\{\cancel{x^2} + xz + \frac{1}{2}(\cancel{x^2} + \cancel{z^2} + 2xz) + xz + z^2 - \cancel{x^2} + xz - \frac{1}{2}(\cancel{x^2} + \cancel{z^2} - 2xz) - \cancel{xz} + z^2\right\} dx dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} (4xz + 2z^2)\, dx dz$$

$$= \int_{z=-1}^{1} \left(2x^2 z + 2xz^2\right)_0^{z} dz$$

$$= \int_{z=-1}^{1} \left(2z^3 + 2z^3\right) dz$$

$$= \left[\frac{4z^4}{4}\right]_{-1}^{1}$$

$$= 1 - 1 = 0$$

7) Evaluate $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r\,dr\,d\theta\,dz$

<u>Sol</u>: $$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \int_{z=0}^{(a^2-r^2)/a} r\,dr\,d\theta\,dz = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} r\left[z\right]_0^{\frac{a^2-r^2}{a}} dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} r\left(\frac{a^2-r^2}{a}\right) dr\,d\theta$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \left(a^2 r - r^3\right) dr\,d\theta$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi/2} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi/2} \left(\frac{a^4 \sin^2\theta}{2} - \frac{a^4 \sin^4\theta}{4}\right) d\theta$$

$$= \frac{a^3}{4} \int_{\theta=0}^{\pi/2} \left(2\sin^2\theta - \sin^4\theta\right) d\theta$$

$$= \frac{a^3}{4}\left[\cancel{2}\cdot\frac{2-1}{\cancel{2}}\cdot\frac{\pi}{2} - \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{a^3}{4}\left(\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{\pi a^3}{8}\left(1 - \frac{3}{8}\right)$$

$$= \frac{\pi a^3}{8}\cdot\left(\frac{5}{8}\right) = \frac{5\pi a^3}{64}$$

8) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

<u>Sol</u>: $$\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - (z)^2}}\,dz\right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right) - 0\right] dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \left[y\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= \frac{\pi}{2}\left[0 + \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi^2}{8}$$

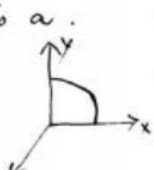
*9) Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2 = a^2$

Sol: Given sphere is $x^2+y^2+z^2 = a^2 \Rightarrow z = \sqrt{a^2-x^2-y^2}$

The projection of the sphere on the xy-plane is the circle $x^2+y^2 = a^2$

So, this circle is covered as y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.

$$\therefore \iiint xyz\,dx\,dy\,dz = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\,dx\,dy\,dz$$



$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= \frac{1}{2}\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,(a^2-x^2-y^2)\, dx\,dy$$

$$= \frac{1}{2}\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\, dx\,dy$$

$$= \frac{1}{2}\int_{x=0}^{a} \left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_{y=0}^{\sqrt{a^2-x^2}}$$

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$$= \frac{1}{2}\int_{x=0}^{a}\left[\frac{a^2x}{2}(a^2-x^2) - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\frac{a^4x - a^2x^3}{2} - \frac{a^2x^3 - x^5}{2} - \frac{a^4x + x^5 - 2a^2x^3}{4}\right]dx$$

$$= \frac{1}{8}\int_{x=0}^{a}\left[2a^4x - 2a^2x^3 - \cancel{2a^2x^3} + 2x^5 - a^4x - x^5 + \cancel{2a^2x^3}\right]dx$$

$$= \frac{1}{8}\int_{x=0}^{a}\left[a^4x - 2a^2x^3 + x^5\right]dx$$

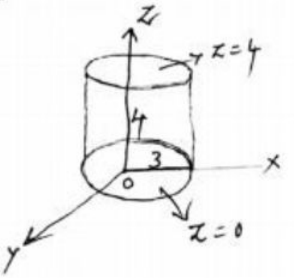
$$= \frac{1}{8}\left[\frac{a^4x^2}{2} - \frac{2a^2x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[\cancel{\frac{a^6}{2}} - \cancel{\frac{2a^6}{4_2}} + \frac{a^6}{6}\right]$$

$$= \frac{a^6}{48}\ //$$

*

10) Find the volume of the solid enclosed by $x^2+y^2=9$, $z=0$ and $z=4$ by using triple integral.



Sol: Given $x^2+y^2=9 \rightarrow (1)$, $z=0$ & $z=4$

$\Rightarrow y = \pm\sqrt{9-x^2}$

y varies from $-\sqrt{9-x^2}$ to $+\sqrt{9-x^2}$

put $y=0$ in (1) $\Rightarrow x = \pm 3$

x varies from -3 to $+3$.

$$\therefore \text{volume of the solid} = \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=0}^{4} dz\,dy\,dx$$

$$= \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\,dy\,dx$$

$$= 4\int_{x=-3}^{3}\left(\sqrt{9-x^2}+\sqrt{9-x^2}\right)dx$$

$$= 8\int_{x=-3}^{3}\sqrt{3^2-x^2}\,dx$$

$$= 8\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}(x/3)\right]_{-3}^{3}$$

$$= 8\left[0 + \frac{9}{2}\left(\frac{\pi}{2}\right) - 0 - \frac{9}{2}\left(-\frac{\pi}{2}\right)\right]$$

$$= 8\left(\frac{18\pi}{4}\right) = 36\pi$$

11) Evaluate $\int_{0}^{\pi/4}\int_{0}^{\log\sec z}\int_{-\infty}^{2y} e^{x}\,dx\,dy\,dz$

Sol:
$$\int_{z=0}^{\pi/4}\int_{y=0}^{\log\sec z}\int_{x=-\infty}^{2y} e^{x}\,dx\,dy\,dz = \int_{z=0}^{\pi/4}\int_{y=0}^{\log\sec z}\left[e^{x}\right]_{-\infty}^{2y} dy\,dz$$

$$= \int_{z=0}^{\pi/4}\int_{y=0}^{\log\sec z}\left(e^{2y}-0\right)dy\,dz$$

$$= \int_{z=0}^{\pi/4}\left[\frac{e^{2y}}{2}\right]_{0}^{\log\sec z} dz$$

$$= \frac{1}{2}\int_{z=0}^{\pi/4}\left(e^{2\log\sec z}-1\right)dz$$

$$= \frac{1}{2}\int_{z=0}^{\pi/4}\left(\sec^{2}z-1\right)dz$$

$$= \frac{1}{2}\left[\tan z - z\right]_{0}^{\pi/4}$$

$$= \frac{1}{2}\left(1-\frac{\pi}{4}\right) = \frac{4-\pi}{8}$$

12) Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ taken over the volume bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ and $z=0$, $z=1$

Sol:
$$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(x+y+z)\,dx\,dy\,dz = \int_{x=0}^{1}\int_{y=0}^{1}\left[xz+yz+\frac{z^{2}}{2}\right]_{0}^{1} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left(x+y+\frac{1}{2}\right)dy\,dx$$

$$= \int_{x=0}^{1}\left(xy+\frac{y^{2}}{2}+\frac{y}{2}\right)_{0}^{1} dx$$

$$= \int_{x=0}^{1}\left(x+\frac{1}{2}+\frac{1}{2}\right)dx$$

$$= \left(\frac{x^{2}}{2}+\frac{x}{2}+\frac{x}{2}\right)_{0}^{1}$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 3/2$$

13) Evaluate $\iiint_{V} (x+y+z)\,dx\,dy\,dz$, where the domain V is bounded by the plane $x+y+z=a$ $(a>0)$ and the coordinate planes. Ans: $\frac{a^{4}}{8}$

Sol:
$$\iiint (x+y+z)\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[(x+y)\,z + \frac{z^2}{2}\right]_0^{a-x-y} dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[(x+y)(a-x-y) + \frac{(a-x-y)^2}{2}\right] dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[x(a-x-y) + (a-x)y - y^2 + \frac{(a-x-y)^2}{2}\right] dx\,dy$$

$$= \int_{x=0}^{a}\left[\frac{x(a-x-y)^2}{-2} + (a-x)\frac{y^2}{2} - \frac{y^3}{3} + \frac{(a-x-y)^3}{-6}\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a}\left[0 + \frac{(a-x)^3}{2} - \frac{(a-x)^3}{3} + 0 + \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{6}\right] dx$$

$$= \left[\frac{(a-x)^4}{-8} + \frac{(a-x)^4}{12} + \frac{(a-x)^4}{-24} + \frac{1}{2}\left(\frac{a^2x^2}{2} + \frac{x^4}{4} - \frac{2ax^3}{3}\right)\right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2} + \frac{a^4}{4} - \frac{2a^4}{3}\right) + \frac{a^4}{8} - \frac{a^4}{12} + \frac{a^4}{24}$$

$$= \frac{3a^4}{8} - \frac{a^4}{3} + \frac{a^4}{8} - \frac{a^4}{12} + \frac{a^4}{24}$$

$$= \frac{(9-8+3-2+1)a^4}{24} = \frac{(13-10)a^4}{24} = \frac{a^4}{8}$$

*14) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$ Ans: $\frac{8abc}{3}(a^2+b^2+c^2)$

15) Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} dx\,dy\,dz$ Ans: $1/6$

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2014 1) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy = \frac{1}{4}\left(e^2 - 8e + 13\right)$

2) Evaluate the triple integral $\iiint xy^2z \, dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Sol: The eqns of the sphere in the first octant are

$$x^2+y^2+z^2=a^2, \quad x \geq 0, y \geq 0, z \geq 0$$

$$\underline{\text{Ans}}: \frac{a^7}{105}.$$

$$= \frac{8c}{b}\int_{x=0}^{a}\left[0 + \frac{p^2}{2}\sin^{-1}(1) - 0\right]dx$$

$$= \frac{4c}{\cancel{b}}\int_{x=0}^{a}\frac{\pi}{2}\cdot b^{\cancel{2}}\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc \int_{x=0}^{a}\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left(x - \frac{x^3}{3a^2}\right)_0^a$$

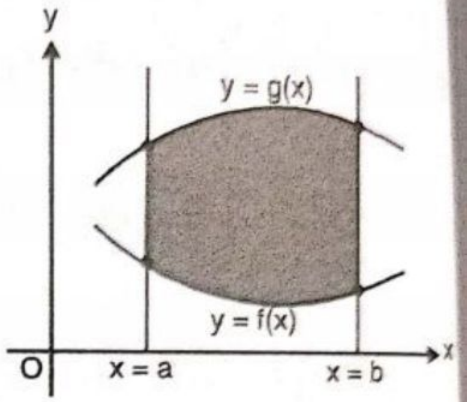
$$= 2\pi bc\left(a - \frac{a^3}{3a^2} - 0\right)$$

$$= 2\pi bc \, \frac{2a^{\cancel{3}}}{3\cancel{a^2}} = \frac{4\pi}{3}abc$$

(3) Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz = \frac{8}{3}\log 2 - \frac{19}{9}$

Applications of Multiple Integrals:

8.7 AREA ENCLOSED BY A PLANE CURVE



Consider the area enclosed by the curves $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ in the xy plane.

The area of the region R bounded by the given curves is given by

$$\iint_R dx\,dy \quad \text{or} \quad \iint_R dy\,dx = \int_{x=a}^{b} \int_{y=f(x)}^{g(x)} dy\,dx$$

If the region is represented through polar coordinates, then the area is given by $\iint_R r\,dr\,d\theta$.

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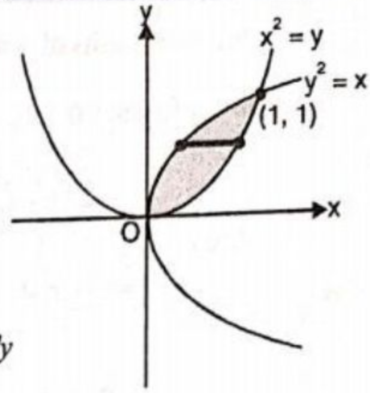
SOLVED EXAMPLES

Example 1 : Find the area enclosed by the parabolas $x^2 = y$ and $y^2 = x$.

Solution : Given curvers are $x^2 = y$ ... (1) and $y^2 = x$ ... (2)



To find their points of intersection, solve (1) and (2).

Squaring on both sides of (1), $x^4 = y^2 = x$, using (2)

$\Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1$

$\therefore$ Given parabolas intersect at the points O(0, 0) and P(1, 1).

$$\therefore \text{ The required area} = \iint_R dx\, dy = \int_{y=0}^{1}\left(\int_{x=y^2}^{\sqrt{y}} dx\right) dy = \int_{y-0}^{1}\left(\sqrt{y} - y^2\right) dy$$

$$= \left(\frac{2}{3}y^{3/2} - \frac{y^3}{3}\right)_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq.units.}$$

Example 2 : Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solution : Given curves are $y^2 = 4ax$ ... (1) and $x^2 = 4ay$ ... (2)

To find their points of intersection, solve (1) and (2).

Squaring (2), we get

$x^4 = 16a^2y^2 = 16a^2(4ax)$, using (1)

$\therefore x^4 = 64a^3x \Rightarrow x[x^3 - (4a)^3] = 0 \Rightarrow x = 0, 4a$

When $x = 0, y = 0$ and when $x = 4a, y = 4a$.

Hence the two parabolas intersect at O(0, 0) and P(4a, 4a).

$\therefore$ Area, $A = \iint_R dx\, dy$

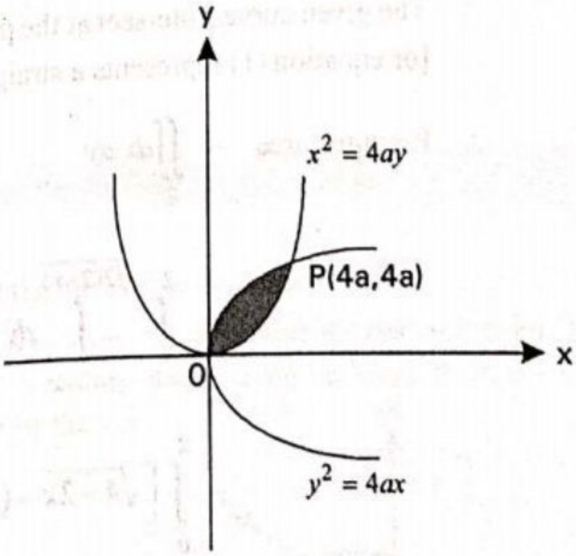
The region R can be covered by varying x from the upper curve $x = y^2/4a$ to the lower curve $x = 2\sqrt{ay}$, while y varies from 0 to $4a$.

$$\text{Thus A} = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy$$

$$= \int_{y=0}^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{a}\sqrt{y}} dy$$

$$= \int_0^{4a}\left(2\sqrt{a}\sqrt{y} - \frac{y^2}{4a}\right) dy$$

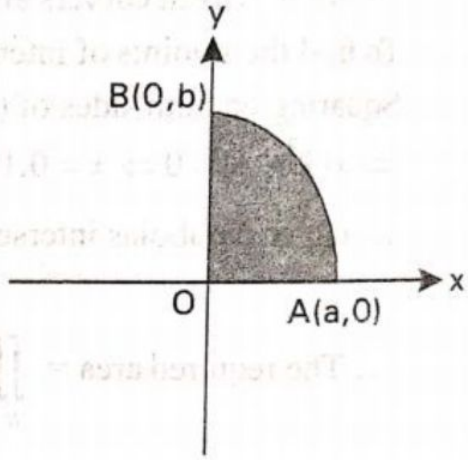
$$= \left[\frac{2\sqrt{a}\,y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{16a^2}{3}$$



Example 3 : Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Limits of y are : $0 \to b\cdot\sqrt{1-\frac{x^2}{a^2}}=\frac{b}{a}\sqrt{a^2-x^2}$

Limits of x are : $0 \to a$



$$\therefore \text{Area} = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx = \int_{x=0}^{a}[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)\right] = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4} \text{ sq.units.}$$

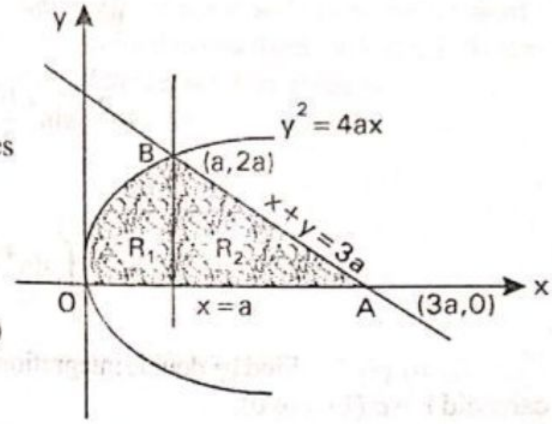
Example 5 : Using double integration determine the area of the region bounded by the curves $y^2=4ax, x+y=3a$ and $y=0$.

Solution : Given curves are

$y^2=4ax$ ...(1)

$x+y=3a$ ...(2)

$y=0$ ...(3)



To find the points of intersection of the two curves $y^2=4ax$ and $x+y=3a$, solve (1) and (2).

Substituting the value of y from eqn. (2) in eqn. (1), we get

$(3a-x)^2=4ax$

i.e. $9a^2+x^2-6ax=4ax$ or $x^2-10ax+9a^2=0$

or $(x-a)(x-9a)=0$

$\therefore x=a, x=9a$

Substituting $x=a$ in (2), we get

$y=2a$

$\therefore$ The curves (1) and (2) intersect at the point B $(a, 2a)$.

Similarly the curves (2) and (3) meet at the point A $(3a, 0)$.

$$\text{Hence required area, A} = \iint_R dx\,dy = \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{4ax}} dy\,dx + \int_{x=a}^{3a}\int_{y=0}^{3a-x} dy\,dx$$

$$= \int_{x=0}^{a} (y)_0^{\sqrt{4ax}}dx + \int_{a}^{3a}(y)_0^{3a-x}dx = 2\sqrt{a}\int_0^a \sqrt{x}dx + \int_a^{3a}(3a-x)dx$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left(x^{3/2}\right)_0^a + \left[\frac{(3a-x)^2}{-2}\right]_a^{3a} = \frac{4\sqrt{a}}{3}(a^{3/2}) - \frac{1}{2}[0-4a^2]$$

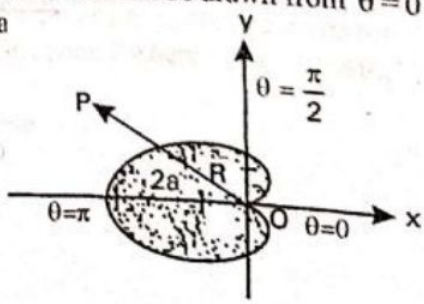
$$\therefore A = \frac{4a^2}{3}+2a^2 = \frac{10a^2}{3} \text{ sq. units.}$$

Example 6 : Using double integral, find the area of the cardioid $r=a(1-\cos\theta)$.

Solution :

The cardioid $r=a(1-\cos\theta)$ is symmetrical about the initial line *i.e.* about $\theta=0$.

To determine the polar limits of integration, imagine a radius vector through the region R from O which emerges at the point P where $r=a(1-\cos\theta)$. Such radii vectors can be drawn from $\theta=0$ to $\theta=\pi$. The region R is made into two equal parts by the x-a



$$\text{Hence required area} = 2\iint_R r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

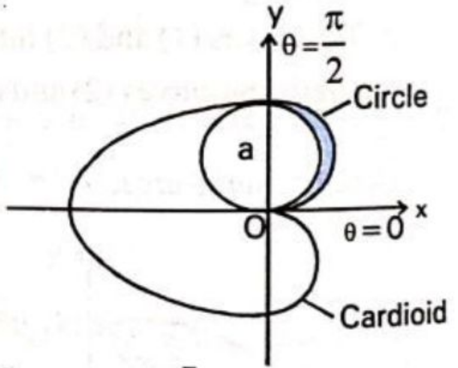
$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta = a^2\int_0^{\pi}(1-\cos\theta)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\sin^4\frac{\theta}{2}d\theta = 4a^2\int_0^{\pi/2}\sin^4\phi\cdot 2d\phi \quad \left[\text{Putting } \frac{\theta}{2}=\phi\right]$$

$$= 8a^2\int_0^{\pi/2}\sin^4\phi\, d\phi = 8a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

Example 7 : Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Solution : Here the outer curve is the circle $r = a\sin\theta$ and the inner curve is the cardioid $r = a(1-\cos\theta)$ in the shaded region.



The two curves meet at $\theta = 0$ and $\theta = \pi/2$.

Hence the required area

$$= \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta = \int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi/2}[\sin^2\theta-(1-\cos\theta)^2]\, d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left[-1+(\sin^2\theta-\cos^2\theta)+2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[-1-\cos 2\theta+2\cos\theta]d\theta = \frac{a^2}{2}\left[-\theta-\frac{\sin 2\theta}{2}+2\sin\theta\right]_0^{\pi/2} = \frac{a^2}{2}\left(-\frac{\pi}{2}+2\right) = a^2\left(1-\frac{\pi}{4}\right).$$

8.9 VOLUME AS A TRIPLE INTEGRAL

Suppose a three dimensional solid is cut into elemental rectangular parallelopipeds by drawing planes parallel to the coordinate planes. The volume of an elemental parallelopiped δV is $\delta x\ \delta y\ \delta z$.

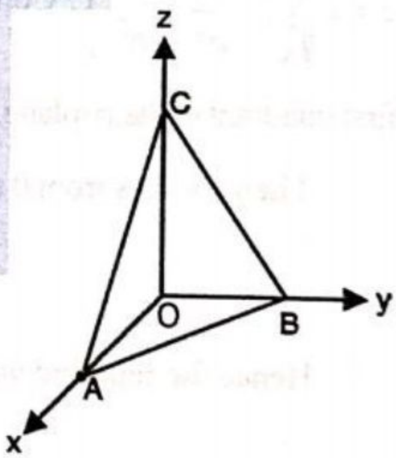
Hence the total volume of the solid is $\iiint_V dv = \iiint_V dx\, dy\, dz$ where the integration is carried over the entire volume.

SOLVED EXAMPLES

Example 1 : Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

(or) Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes by triple integral.

[JNTU (H) June 2009 (Set No.3)]

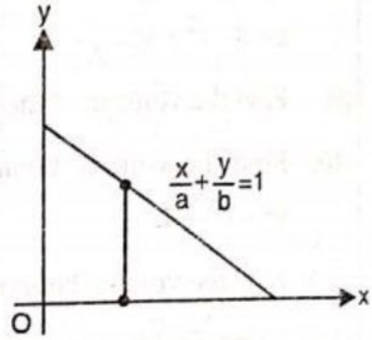


Solution : The required volume $= \iiint_V dx dy dz$

On the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\ z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Hence for a fixed (x,y) on the xy plane within the ΔOAB, z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ within the solid. Then for a fixed x within the ΔOAB, y varies from 0 to $b\left(1-\frac{x}{a}\right)$. Then x varies from 0 to a.

$\therefore$ The required volume of the tetrahedron



$$= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

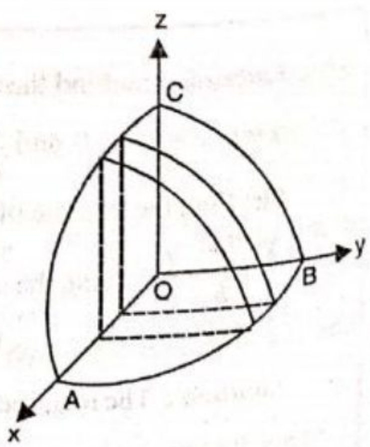
$$= \int_{x=0}^{a}\left[c\,y\left(1-\frac{x}{a}\right)-\frac{c}{b}\frac{y^2}{2}\right]_{y=0}^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \int_{x=0}^{a}\left[cb\left(1-\frac{x}{a}\right)^2-\frac{cb}{2}\left(1-\frac{x}{a}\right)^2\right]dx = \left[\frac{cb}{2}\left(1-\frac{x}{a}\right)^3 \cdot \frac{1}{3\left(-\frac{1}{a}\right)}\right]_{x=0}^{a} = \frac{abc}{6}\,c.\text{units}.$$

Example 2 : Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. **[JNTU 1998]**

(or) Find the volume of the greatest rectangular parallelopied that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. **[JNTU 1999, (H) June, Dec. 2010, 2011 (Set No. 4)]**

Solution : The solid figure $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut into 8 equal pieces by the three coordinate planes. Hence the volume of the solid is equal to 8 times the volume of the solid bounded by $x=0$, $y=0, z=0$ and the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.



For a fixed point (x, y) on the xy plane, z varies from $z=0$ to $z=c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$. Consider the quadrant of the ellipse in the first quadrant of the xy plane. For a fixed x, y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

Then x varies from 0 to a.

Hence the required volume $= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}} dz\,dy\,dx$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx \qquad \dots (1)$$

Write $1-\frac{x^2}{a^2} = \frac{p^2}{b^2}$

$\therefore$ The required volume

$$= 8 \int_{x=0}^{a} \int_{y=0}^{p} \frac{c}{b}\sqrt{p^2 - y^2}\, dy\, dx = 8\frac{c}{b} \int_{x=0}^{a} \left[\int_{y=0}^{p} \sqrt{(p^2 - y^2)} dy \right] dx \qquad \dots (2)$$

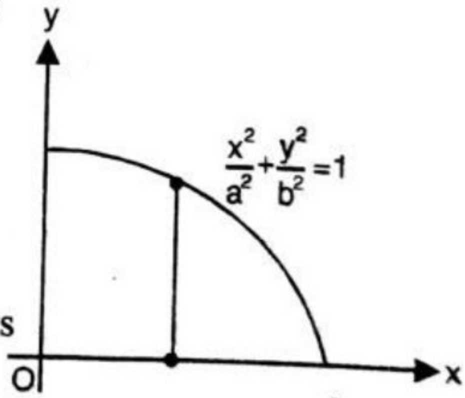
But $\int_{y=0}^{p} \sqrt{p^2 - y^2} dy = \int_{0}^{\pi/2} p\cos\theta . p\cos\theta d\theta$ $\left[\begin{array}{l} Put\ y = p\sin\theta \Rightarrow dy = p\cos\theta\ d\theta \\ if\ y = 0, \theta = 0 \text{ and } if\ y = p, \theta = \pi/2 \end{array} \right.$

$$= p^2 \int_{\theta=0}^{\pi/2} \cos^2\theta d\theta = p^2 . \frac{\pi}{4} = \frac{\pi}{4} b^2 \left(1 - \frac{x^2}{a^2} \right) \qquad \dots (3)$$

Using (3) in (2), the required volume

$$= \frac{8c}{b} . \frac{\pi}{4} b^2 \int_{x=0}^{a} \left(1 - \frac{x^2}{a^2} \right) dx = 2\pi bc \left[x - \frac{x^3}{3a^2} \right]_0^a$$

$$= 2\pi bc \left[a - \frac{a}{3} \right] = 2\pi bc . \frac{2a}{3} = \frac{4\pi}{3} abc \text{ cubic units}$$



Note : Putting $a = b = c$, we obtain volume of the sphere $x^2 + y^2 + z^2 = a^2$ as $\frac{4\pi a^3}{3}$.

Example 4 : Find the volume bounded by the xy plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.

Solution : Given surfaces are

$$z=0 \quad \text{...(1)}$$

$$x^2+y^2=1 \quad \text{...(2)}$$

and $x+y+z=3 \Rightarrow z=3-x-y$ ...(3)

The projection of the volume on the xy-plane is the region R enclosed by the circle $x^2+y^2=1$.

$\therefore$ The required volume can be covered as follows :

z : From 0 to $3-x-y$

y : From $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

x : From -1 to 1

Thus the volume bounded by the given surfaces,

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{3-x-y} dz\, dy\, dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [z]_0^{3-x-y}\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3-x-y)dy\, dx = \int_{-1}^{1}\left[3y-xy-\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\left\{3\cdot 2\sqrt{1-x^2}-x\cdot 2\sqrt{1-x^2}-\frac{1}{2}(0)\right\}dx$$

$$= \int_{-1}^{1}[6\cdot\sqrt{1-x^2}-2x\sqrt{1-x^2}\,]dx = 6\cdot 2\int_{0}^{1}\sqrt{1-x^2}\,dx-2(0)$$

[$\because$ Since integrand is odd function in second integral]

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$$= 12\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = 12\left[\frac{1}{2}\sin^{-1}(1)\right]$$

$$= 6\cdot\frac{\pi}{2} = 3\pi \text{ cubic units.}$$

Example 5 : Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$.

Solution : The region of integration is given by

z : From 0 to $1-x-y$

y : From 0 to $1-x$

x : From 0 to 1

$$\therefore \iiint_V (x+y+z)\,dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (x+y+1-x-y)^2\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} dy\,dx = \frac{1}{2}\int_0^1 [y]_0^{1-x}\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\left[\frac{(1-x)^2}{-2}\right]_0^1$$

$$= -\frac{1}{4}(0-1) = \frac{1}{4}.$$

Exampl...

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