Gamma function :-

The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the Gamma function and is denoted by $\Gamma(n)$.

$\therefore \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ converges only for $n > 0$

Gamma function is also called Eulerian integral of the second kind.

Properties of Gamma function :-

1. Show that $\Gamma(n+1) = n\Gamma(n)$

Sol: By the definition, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \longrightarrow (1)$$

put $n = n+1$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n} dx$$

$$= \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1} dx$$

$$= 0 + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\Gamma(n) \quad (\because (1))$$

$$\boxed{\therefore\ \Gamma(n+1) = n\,\Gamma(n)\ ,\quad n>0}$$

(2) Show that $\Gamma(n) = (n-1)\,\Gamma(n-1)\ ,\quad n>1$

Sol: By the definition, we have

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx \longrightarrow (1)$$

$$= \left[x^{n-1}(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x})\,(n-1)\,x^{n-2}\, dx$$

$$= 0 + (n-1)\int_0^\infty e^{-x}\, x^{(n-1)-1}\, dx.$$

$$= (n-1)\,\Gamma(n-1) \qquad (\because (1))$$

$$\boxed{\therefore\ \Gamma(n) = (n-1)\,\Gamma(n-1)\ ,\quad n>1}$$

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(3) If n is a non-negative integer, then S.T $\Gamma(n+1) = n!$

Sol: We know that,

$$\Gamma(n+1) = n\,\Gamma(n) \quad \& \quad \Gamma(n) = (n-1)\,\Gamma(n-1) \rightarrow (1)$$

$$\therefore \Gamma(n+1) = n\,\Gamma(n)$$

$$= n(n-1)\,\Gamma(n-1) \quad (\because (1))$$

$$= n(n-1)(n-2)\,\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)\,\Gamma(n-3)$$

$$= n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1\,\Gamma(1)$$

$$= n! \quad (\because \Gamma(1) = 1)$$

Thus $\Gamma(n+1) = n!$, $n = 0, 1, 2, \ldots\ldots$

Note: If n is a positive fraction, then we can write

$$\Gamma(n) = (n-1)(n-2)\cdots(n-r)\,\Gamma(n-r), \text{ where } (n-r) > 0$$

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Ex: 1) $\Gamma(7/2) = (7/2 - 1)\,\Gamma(7/2 - 1)$

$$= 5/2\,\Gamma(5/2) = 5/2 \cdot (5/2 - 1)\,\Gamma(5/2 - 1)$$

$$= 5/2 \cdot 3/2 \cdot \Gamma(3/2)$$

$$= 5/2 \cdot 3/2 \cdot (3/2 - 1)\,\Gamma(3/2 - 1)$$

$$= 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)$$

similarly $\Gamma(8/3) = 5/3 \cdot 2/3 \cdot \Gamma(2/3)$.

Note: we know that, $\Gamma(n+1) = n\,\Gamma(n)$

$$\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \quad (n \neq 0, -1, -2, -3, \dots)$$

1. $\Gamma(n)$ is defined when $n > 0$
2. $\Gamma(n)$ is defined when 'n' is a negative fraction.
3. But $\Gamma(n)$ is not defined when $n = 0$ and 'n' is a negative integer.

i.e., $\Gamma(0)$, $\Gamma(-1)$, $\Gamma(-2)$, $\Gamma(-3)$, ... are all not defined.

*Ex: 1) P.T $\Gamma(1/2) = \sqrt{\pi}$

Sol: By the definition, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

put $n = 1/2$

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} dx$$

put $x = t^2$

$dx = 2t\,dt$

$$= \int_0^\infty e^{-t^2} t^{-1} \cdot 2t\,dt$$

$$= 2\int_0^\infty e^{-t^2} dt$$

Now $\Gamma(1/2) \cdot \Gamma(1/2) = \left[2\int_0^\infty e^{-u^2} du\right]\left[2\int_0^\infty e^{-v^2} dv\right]$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(u^2+v^2)}\, du\, dv$$

This double integral in the first quadrant is evaluated by changing cartesian coordinates into polar coordinates.

Let $u = r\cos\theta$, $v = r\sin\theta$.

$du = -r\sin\theta\, d\theta$, $dv = r\cos\theta\, d\theta$

$\Rightarrow du\, dv = r\, dr\, d\theta$

$$\therefore [\Gamma(1/2)]^2 = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

put $r^2 = t$

$2r\, dr = dt$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\, d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} (0+1)\, d\theta$$

$$= 2\, [\theta]_0^{\pi/2} = 2 \cdot \left(\frac{\pi}{2}\right) = \pi$$

$$\Rightarrow [\Gamma(1/2)]^2 = \pi$$

$$\boxed{\therefore \Gamma(1/2) = \sqrt{\pi}}$$

Another form of Gamma function:

- prove that $\Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2n-1}\, dx$, $n > 0$

Sol: By the definition, we have

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

put $x = y^2$

$dx = 2y\, dy$

$$\Gamma(n) = \int_0^\infty e^{-y^2} (y^2)^{n-1}\, 2y\, dy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$= 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

$$\therefore \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx, \quad n > 0$$

Ex: compute i) $\Gamma(9/2)$ ii) $\Gamma(-1/2)$ iii) $\Gamma(-7/2)$

Sol: i) we know that $\Gamma(n+1) = n\Gamma(n)$

$$\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

put $n = 7/2$, we get

$$\Gamma(9/2) = 7/2 \cdot \Gamma(7/2)$$

$$= 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)$$

$$= \frac{105}{16}\sqrt{\pi}$$

ii) $$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

put $n = -1/2$

$$\Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{-1/2} = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

iii) put $n = -7/2$

$$\Gamma(-7/2) = \frac{\Gamma(-7/2+1)}{-7/2}$$

$$= \frac{\Gamma(-5/2)}{-7/2} = -2/7 \cdot \frac{\Gamma(-5/2+1)}{-5/2}$$

$$= -2/7 \cdot -2/5\, \Gamma(-3/2)$$

$$= \frac{4}{35}\, \frac{\Gamma(-3/2+1)}{-3/2}$$

$$= \frac{-8}{105}\, \Gamma(-1/2) = \frac{-8}{105} \times (-2\sqrt{\pi})$$

$$= \frac{16}{105}\sqrt{\pi}$$

Applications of Gamma function :-

Ex: 1) Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol: By the def, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

put $n = \frac{1}{2}$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x} x^{-1/2} dx$$

put $x = t^2$

$dx = 2t\,dt$

$$\therefore \Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t^2} (t^2)^{-1/2} 2t\,dt$$

$$= 2\int_0^\infty e^{-t^2} dt$$

$$(\text{or}) \int_0^\infty e^{-x^2} dx = \frac{\Gamma(\frac{1}{2})}{2}$$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{ //}$$

(OR)

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2} \text{ //}$$

2) P.T $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$

Sol: $\int_{-\infty}^\infty e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx$ ($\because e^{-x^2}$ is an even function)

$$= 2\left(\frac{\sqrt{\pi}}{2}\right)$$

$$= \sqrt{\pi} \text{ //}$$

3) show that $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx, \; n > 0$

Sol: By the def, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

put $x = \log \frac{1}{y}$

$\Rightarrow x = -\log y \Rightarrow y = e^{-x}$

$dx = -\frac{1}{y} dy$

Limits: If $x = 0$ then $y = e^{-0} = 1$

$x = \infty$ then $y = e^{-\infty} = 0$

$$\therefore \Gamma(n) = \int_1^0 \cancel{y} \left(\log \frac{1}{y}\right)^{n-1} \left(-\frac{1}{\cancel{y}}\right) dy$$

$$= -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} dy$$

$$= \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

$$= \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$$

4) Evaluate i) $\int_0^\infty e^{-2x} x^6 dx$ ii) $\int_0^\infty e^{-4x} x^{3/2} dx$

Sol: i) put $2x = y \Rightarrow 2dx = dy$

$$\therefore \int_0^\infty e^{-2x} x^6 dx = \int_0^\infty e^{-y} \left(\frac{y}{2}\right)^6 \frac{dy}{2}$$

$$= \frac{1}{2^7} \int_0^\infty e^{-y} y^{7-1} dy$$

$$= \frac{1}{2^7} \Gamma(7)$$

$$= \frac{1}{2^7} \Gamma(6+1) = \frac{6!}{2^7} = \frac{\overset{15}{\cancel{\overset{30}{\cancel{120}}}} \times \overset{3}{\cancel{6}}}{\underset{2}{\cancel{8}} \times \underset{8}{\cancel{16}}}$$

$$= \frac{45}{8}$$

ii) put $4x = y \Rightarrow 4dx = dy$

$$\therefore \int_0^\infty e^{-4x} x^{3/2} dx = \int_0^\infty e^{-y} \left(\frac{y}{4}\right)^{3/2} \frac{dy}{4}$$

$$= \frac{1}{4} \int_0^\infty e^{-y} \frac{y^{3/2}}{8} dy$$

$$= \frac{1}{32}\int_0^\infty e^{-y}\, y^{5/2 - 1}\, dy$$

$$= \frac{1}{32}\Gamma(5/2) = \frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)$$

$$= \frac{3}{128}\sqrt{\pi}$$

5. Evaluate i) $\int_0^\infty x^2 e^{-x^2} dx$ ii) $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$

Sol: i) put $x^2 = y \Rightarrow 2x\,dx = dy$

$$\therefore \int_0^\infty x^2 e^{-x^2} dx = \int_0^\infty e^{-y}\, y \cdot \frac{1}{2\sqrt{y}}\, dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{1/2}\, dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{\frac{3}{2}-1}\, dy$$

$$= \frac{1}{2}\Gamma(3/2) = \frac{1}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{\sqrt{\pi}}{4}$$

ii) put $x^2 = y \Rightarrow 2x\,dx = dy$

$$\therefore \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \int_0^\infty \sqrt{y^{1/2}}\, e^{-y}\cdot\frac{1}{2\sqrt{y}}\, dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{\frac{1}{4}-\frac{1}{2}}\, dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{-1/4}\, dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{\frac{3}{4}-1}\, dy$$

$$= \frac{1}{2}\Gamma(3/4)$$

*6. Evaluate $\int_0^1 x^4 \left(\log\frac{1}{x}\right)^3 dx$

Sol: put $\log\frac{1}{x} = t \Rightarrow \frac{1}{x} = e^{t}$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t}\, dt$

Limits: when $x = 0$, $t = \infty$ and when $x = 1$, $t = 0$

$$\therefore \int_0^1 x^4 \left(\log\frac{1}{x}\right)^3 dx = \int_{+\infty}^0 e^{-4t}\, t^3 \left(-e^{-t}\,dt\right)$$

$$= \int_0^\infty e^{-5t}\, t^3\, dt$$

put $5t = u$

$5\,dt = du$

$$= \int_0^\infty e^{-u} \left(\frac{u}{5}\right)^3 \frac{du}{5}$$

$$= \frac{1}{5^4} \int_0^\infty e^{-u}\, u^{4-1}\, du$$

$$= \frac{1}{625}\,\Gamma(4) = \frac{3!}{625} = \frac{6}{625}$$

7. Evaluate $\int_0^1 \frac{dx}{\sqrt{(-\log x)}}$

Sol: put $-\log x = t \Rightarrow x = e^{-t}$

$dx = -e^{-t}\,dt$

Limits: when $x = 0$, $t = \infty$ and when $x = 1$, $t = 0$

$$\therefore \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_\infty^0 \frac{1}{\sqrt{t}} \left(-e^{-t}\,dt\right)$$

$$= \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$$= \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt$$

$$= \Gamma(1/2) = \sqrt{\pi}$$

8. prove that $\int_0^1 x^{n-1} \left(\log\frac{1}{x}\right)^{m-1} dx = \frac{\Gamma(m)}{n^m}$, $m > 0, n > 0$

Sol: put $\log\frac{1}{x} = t \Rightarrow \frac{1}{x} = e^t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t}\,dt$

Limits: when $x = 0$, $t = \infty$ and when $x = 1$, $t = 0$

$$\therefore \int_0^1 x^{n-1} \left(\log\frac{1}{x}\right)^{m-1} dx = \int_\infty^0 \left(e^{-t}\right)^{n-1} (t)^{m-1} \left(-e^{-t}\,dt\right)$$

$$= \int_0^{\infty} e^{-nt+t} \cdot t^{m-1} e^{-t}\, dt$$

$$= \int_0^{\infty} e^{-nt}\, t^{m-1}\, dt$$

put $nt = u$

$n\,dt = du$

$$= \int_0^{\infty} e^{-u} \left(\frac{u}{n}\right)^{m-1} \frac{du}{n}$$

$$= \frac{1}{n^{m-1+1}} \int_0^{\infty} e^{-u}\, u^{m-1}\, du$$

$$= \frac{\Gamma(m)}{n^{m}}$$

9. prove that $\int_0^{\infty} e^{-y^{1/m}}\, dy = m\Gamma(m)$

Sol: put $y^{1/m} = x \Rightarrow y = x^{m}$

$dy = m x^{m-1}\, dx$

$$\therefore \int_0^{\infty} e^{-y^{1/m}}\, dy = \int_0^{\infty} e^{-x} (m x^{m-1}\, dx)$$

$$= m \int_0^{\infty} e^{-x} x^{m-1}\, dx$$

$$= m\Gamma(m)$$

*10. P.T $\int_0^{1} x^{m} (\log x)^{n}\, dx = \frac{(-1)^{n}\, n!}{(m+1)^{n+1}}$, where n is a +ve integer and $m > -1$

Sol: put $\log x = -t \Rightarrow x = e^{-t}$

$dx = -e^{-t}\, dt$

Limits: when $x = 0$, $t = \infty$ and when $x = 1$, $t = 0$

$$\therefore \int_0^{1} x^{m} (\log x)^{n}\, dx = \int_{\infty}^{0} (e^{-t})^{m} (-t)^{n} (-e^{-t}\, dt)$$

$$= (-1)^{n} \int_0^{\infty} e^{-(m+1)t}\, t^{n}\, dt$$

put $(m+1)t = y$

$(m+1)\,dt = dy$

$$= (-1)^{n} \int_0^{\infty} e^{-y} \left(\frac{y}{m+1}\right)^{n} \frac{dy}{m+1}$$

$$= (-1)^n \cdot \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n \, dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^{(n+1)-1} \, dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}$$

*11. Show that $\int_0^\infty x^n e^{-a^2x^2} dx = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right) \quad (n > -1)$

and hence deduce that $\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{1}{2}\sqrt{\pi/2}$

Sol: put $a^2x^2 = y \Rightarrow x = \sqrt{y}/a$

$2a^2x \, dx = dy$

$$\therefore \int_0^\infty x^n e^{-a^2x^2} dx = \int_0^\infty \left(\frac{\sqrt{y}}{a}\right)^n e^{-y} \frac{dy}{2a^{\cancel{2}} \frac{\sqrt{y}}{\cancel{a}}}$$

$$= \frac{1}{2a^{n+1}} \int_0^\infty e^{-y} y^{\frac{n}{2} - \frac{1}{2}} \, dy$$

$$= \frac{1}{2a^{n+1}} \int_0^\infty e^{-y} y^{\left(\frac{n-1}{2}+1\right)-1} \, dy$$

$$= \frac{1}{2a^{n+1}} \Gamma\left(\frac{n-1}{2}+1\right)$$

$$= \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$$

$$\therefore \int_0^\infty x^n e^{-a^2x^2} dx = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right) \longrightarrow (1)$$

Deduction: putting $n = 0$ and $a^2 = i$ in (1), we get

$$\int_0^\infty e^{-ix^2} dx = \frac{1}{2(i)^{1/2}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{1}{(\cos\pi/2 + i\sin\pi/2)^{1/2}} \cdot \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2}\left(\cos\pi/2 + i\sin\pi/2\right)^{-1/2}$$

i.e., $$\int_0^\infty (\cos x^2 - i\sin x^2)\,dx = \frac{\sqrt{\pi}}{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} - i\,\frac{\sqrt{\pi}}{2\sqrt{2}}$$

Equating real and imaginary parts, we get

$$\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\pi/2} \quad \& \quad \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\pi/2}$$

$$\therefore \int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\pi/2}$$

*12. Show that $\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8}$

Sol: put $x^2 = y \Rightarrow x = \sqrt{y}$

$2x\,dx = dy$

$$\therefore \int_0^\infty x^4 e^{-x^2}\,dx = \int_0^\infty (\sqrt{y})^4 e^{-y}\,\frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{2-\frac{1}{2}}\,dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{3/2}\,dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{5/2-1}\,dy$$

$$= \frac{1}{2}\Gamma(5/2) \qquad \left(\because \Gamma(3/2+1) = \frac{3}{2}\Gamma(3/2)\right)$$

$$= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{3\sqrt{\pi}}{8} \;//$$

*13. Evaluate $\int_0^\infty e^{-2x} x^{5/2}\,dx$

Sol: put $\cancel{x}$ $2x = y \Rightarrow 2\,dx = dy$

$$\therefore \int_0^\infty e^{-2x} x^{5/2}\,dx = \int_0^\infty e^{-y}\left(\frac{y}{2}\right)^{5/2}\frac{dy}{2}$$

$$= \frac{1}{2^{7/2}}\int_0^\infty e^{-y}\, y^{7/2-1}\,dy$$

$$= \frac{1}{2^{7/2}}\Gamma(7/2)$$

$$= \frac{1}{2^{7/2}}\cdot 5/2\cdot 3/2\cdot 1/2\cdot\sqrt{\pi}$$

$$= \frac{15}{2^{7/2+3}}\sqrt{\pi}$$

$$= \frac{15}{2^{13/2}}\sqrt{\pi} \;//$$

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14. P.T $\int_0^\infty \cos(b x^{1/n})\, dx = \frac{1}{b^n}\,\Gamma(n+1)\cdot \cos\frac{n\pi}{2}$

Sol: put $x^{1/n} = y \Rightarrow x = y^n$

$dx = n y^{n-1}\, dy$

Also when $x = 0$, $y = 0$ and when $x \to \infty$, $y \to \infty$

$$\therefore \int_0^\infty \cos(b x^{1/n})\, dx = \int_0^\infty \cos by \; n y^{n-1}\, dy$$

$$= n \int_0^\infty y^{n-1} \cos by \, dy$$

$$= \text{R.P. of } \left\{ n \int_0^\infty y^{n-1} e^{-iby}\, dy \right\}$$

put $iby = t$

$ib\, dy = dt$

$$= \text{R.P. of } \left\{ n \int_0^\infty e^{-t} \left(\frac{t}{ib}\right)^{n-1} \frac{dt}{ib} \right\}$$

$$= \text{R.P. of } \left\{ \frac{n}{(ib)^n} \int_0^\infty e^{-t}\, t^{n-1}\, dt \right\}$$

$$= \text{R.P. of } \left\{ \frac{n}{(ib)^n}\, \Gamma(n) \right\}$$

$$= \text{R.P. of } \left\{ \frac{\Gamma(n+1)}{b^n}\, (i)^{-n} \right\}$$

$$= \text{R.P. of } \left\{ \frac{\Gamma(n+1)}{b^n} \left(\cos \pi/2 + i \sin \pi/2\right)^{-n} \right\}$$

$$= \text{R.P. of } \left\{ \frac{\Gamma(n+1)}{b^n} \left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right) \right\}$$

$$= \frac{\Gamma(n+1)}{b^n} \cos\frac{n\pi}{2}$$

15) S.T $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$

Beta Function :—

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ is called the Beta function and is denoted by $B(m, n)$.

Thus, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ converges for $m>0, n>0$.

Beta function is also called Eulerian integral of the first kind.

Properties of Beta function:

1. **Symmetry of Beta function:** $\beta(m,n) = \beta(n,m)$

Sol: By the definition, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $1-x = y \Rightarrow dx = -dy$

Limits: when $x=0$, $y=1$ and when $x=1$, $y=0$

$$\therefore \beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \beta(n,m)$$

Hence, $\beta(m,n) = \beta(n,m)$

2. **Beta function in terms of trigonometric functions:**

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Sol: By the definition, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta \Rightarrow dx = \sin 2\theta\, d\theta$.

Limits: when $x=0$, $\theta=0$ and when $x=1$, $\theta=\pi/2$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\sin 2\theta\, d\theta$$

$$= \int_0^{\pi/2} (\sin^{2m-2}\theta)(\cos^{2n-2}\theta)\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Note: $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$.

3. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Sol: By the definition, we have

$$\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^{m+1-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$$

$$= \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \beta(m,n)$$

Hence, $\beta(m,n) = \underline{\underline{\beta(m+1,n) + \beta(m,n+1)}}$

4. If m and n are positive integers, then

$$\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

Note: 1. put $m=1$ in $\beta(m,n) = \dfrac{(m-1)!\,(n-1)!}{(m+n-1)!}$

we have $\beta(1,n) = \dfrac{(n-1)!}{n!} = \dfrac{\cancel{(n-1)!}}{\cancel{(n-1)!}\,n} = \dfrac{1}{n}$

2. Similarly, putting $n=1$, we get $\beta(m,1) = \dfrac{1}{m}$

Other forms of Beta function :—

* 1. Show that $\beta(m,n) = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol: By the definition, we have

$\boxed{\text{put } x = \dfrac{y}{1+y}}$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \dfrac{1}{1+y} \Rightarrow dx = \dfrac{-dy}{(1+y)^2}$

Limits: when $x=0$, $y=\infty$ and when $x=1$, $y=0$

$$\therefore \beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{-dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1+2}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\,dy$$

$$= \int_0^\infty y^{n-1}\,\frac{1}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Since Beta function is symmetrical in m and n, we have

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Hence, $\beta(m,n) = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

2. Show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol: We know that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \to (1)$$

Take the second integral $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

put $x = \frac{1}{y} \Rightarrow dx = \frac{-1}{y^2}\,dy$

Limits: when $x=1$, $y=1$ and when $x=\infty$, $y=0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{(1/y)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy$$

$$= \int_0^1 \frac{y^{m+n}}{(1+y)^{m+n}\,y^{m+1}}\,dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Hence (1) becomes

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

3. Show that $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$

Sol: $$R.H.S = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx$$

$$= \frac{a^m b^n}{b^{m+n}} \int_0^\infty \frac{x^{m-1}}{\left(\frac{a}{b}x+1\right)^{m+n}}\,dx$$

put $\frac{ax}{b} = t \Rightarrow dx = \frac{b}{a}\,dt$ and $x = \frac{bt}{a}$

$$\therefore \quad \frac{a^{m} b^{n}}{b^{m+n}} \int_0^{\infty} \frac{x^{m-1}}{\left(\frac{ax}{b}+1\right)^{m+n}} dx = \frac{a^{m} b^{n}}{b^{m+n}} \int_0^{\infty} \frac{\frac{b^{m-1}}{a^{m-1}} \cdot t^{m-1}}{(t+1)^{m+n}} \left(\frac{b}{a}\right) dt$$ (10)

$$= \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \beta(m, n) = L.H.S$$

$\therefore$ L.H.S = R.H.S

$$\therefore \beta(m, n) = a^{m} b^{n} \int_0^{\infty} \frac{x^{m-1}}{(ax+b)^{m+n}} dx$$

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$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

5. show that $\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\beta(m,n)$

Sol: We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \frac{t-b}{a-b} \Rightarrow dx = \frac{dt}{a-b}$

$$\therefore \beta(m,n) = \int_b^a \frac{(t-b)^{m-1}}{(a-b)^{m-1}}\left[1-\frac{t-b}{a-b}\right]^{n-1} \cdot \frac{dt}{a-b}$$

$$= \int_b^a \frac{(t-b)^{m-1}}{(a-b)^{m-1+n-1+1}}(a-t)^{n-1}\,dt$$

$$= \frac{1}{(a-b)^{m+n-1}}\int_b^a (t-b)^{m-1}(a-t)^{n-1}\,dt$$

$$= \frac{1}{(a-b)^{m+n-1}}\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx$$

$$\therefore \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\beta(m,n)$$

Ex: 1) Express the following integrals in terms of Beta function

i) $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ ii) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Sol: i) put $x^2 = y \Rightarrow 2x\,dx = dy$

Limits: when $x=0$, $y=0$ and when $x=1$, $y=1$.

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-y}}\frac{dy}{2}$$

$$= \frac{1}{2}\int_0^1 (1-y)^{-1/2}\,dy$$

$$= \frac{1}{2}\int_0^1 (y-0)^{1-1}(1-y)^{\frac{1}{2}-1}\,dy$$

$$= \frac{1}{2}(1-0)^{1+\frac{1}{2}-1}\beta(1,\tfrac{1}{2}) = \frac{1}{2}\beta(1,\tfrac{1}{2})$$

ii) put $x^2 = 9y \Rightarrow 2x\,dx = 9\,dy$

Limits: when $x=0$, $y=0$ and when $x=3$, $y=1$

$$\therefore \int_0^3 \frac{dx}{\sqrt{9-x^2}} = \int_0^1 \frac{1}{\sqrt{9-9y}} \cdot \frac{9}{2} \cdot \frac{1}{3\sqrt{y}}\,dy$$

$$= \frac{1}{2}\int_0^1 (1-y)^{-1/2}\, y^{-1/2}\,dy$$

$$= \frac{1}{2}\int_0^1 (y-0)^{1/2-1} (1-y)^{1/2-1}\,dy$$

$$= \frac{1}{2}(1-0)^{\frac{1}{2}+\frac{1}{2}-1}\,\beta\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)$$

2) Evaluate $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}}$ in terms of β function.

Sol: put $x^5 = y \Rightarrow 5x^4\,dx = dy$

$$\therefore \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}} = \int_0^1 \frac{x^2}{x^4}\cdot\frac{1}{\sqrt{1-x^5}}\cdot x^4\,dx$$

$$= \int_0^1 x^{-2}\,\frac{1}{\sqrt{1-x^5}}\,x^4\,dx$$

$$= \int_0^1 y^{-2/5}(1-y)^{-1/2}\,\frac{dy}{5}$$

$$= \frac{1}{5}\int_0^1 (y-0)^{3/5-1}(1-y)^{1/2-1}\,dy$$

$$= \frac{1}{5}(1-0)^{3/5+1/2-1}\,\beta(3/5, 1/2)$$

$$= \frac{1}{5}\beta(3/5, 1/2)$$

3) show that $\int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n+1}\beta(m+1, n+1)$

Sol: put $x = \frac{t-a}{b-a} \Rightarrow dx = \frac{1}{b-a}\,dt$

Limits: when $x=a$, $t=$

we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \frac{t-a}{b-a} \Rightarrow dx = \frac{1}{b-a}\,dt$

Limits: when $x=0$, $t=a$ and when $x=1$, $t=b$

$$\beta(m,n) = \int_a^b \frac{(t-a)^{m-1}}{(b-a)^{m-1}} \left(1 - \frac{t-a}{b-a}\right)^{n-1} \cdot \frac{1}{b-a}\, dt$$

$$= \int_a^b \frac{(t-a)^{m-1}}{(b-a)^{m-1+n-1+1}} (b-t)^{n-1}\, dt$$

$$\beta(m,n) = \frac{1}{(b-a)^{m+n-1}} \int_a^b (t-a)^{m-1}(b-t)^{n-1}\, dt$$

Replace m by $m+1$ and n by $n+1$, we get

$$\beta(m+1,n+1) = \frac{1}{(b-a)^{m+n+1}} \int_a^b (t-a)^m (b-t)^n\, dt$$

$$= \frac{1}{(b-a)^{m+n+1}} \int_a^b (x-a)^m (b-x)^n\, dx$$

$$\therefore \int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\, \beta(m+1,n+1)$$

Relation between Beta and Gamma functions:-

* statement: $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ where $m>0, n>0$

Aliter: By definition,

$$\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} \, dx$$

put $x = t^2 \Rightarrow dx = 2t\,dt$

$$\Gamma(m) = \int_0^{\infty} e^{-t^2} \cdot t^{2m-2} \cdot 2t\,dt$$

$$= 2\int_0^{\infty} e^{-t^2} t^{2m-1} \, dt$$

Then $\Gamma(m) \cdot \Gamma(n) = \left[2\int_0^{\infty} e^{-x^2} x^{2m-1} \, dx\right] \times \left[2\int_0^{\infty} e^{-y^2} y^{2n-1} \, dy\right]$

$$= 4\int_0^{\infty}\int_0^{\infty} x^{2m-1} \cdot y^{2n-1} \cdot e^{-(x^2+y^2)} \, dx\,dy$$

Introduce polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$

and $dx\,dy = r\,dr\,d\theta$

As x, y vary in the first quadrant (i.e., $0 < x < \infty$, $0 < y < \infty$)

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r varies from 0 to ∞ and θ from 0 to $\pi/2$.

$$\therefore \Gamma(m)\cdot\Gamma(n) = 4\int_0^{\pi/2}\int_0^{\infty} (r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}\cdot e^{-r^2}\cdot r\,dr\,d\theta$$

$$= 4\left[\int_0^{\pi/2} \sin^{2n-1}\theta\cdot\cos^{2m-1}\theta\,d\theta\right]\times\left[\int_0^{\infty} e^{-r^2} r^{2(m+n-1)}\,r\,dr\right]$$

$$= 4\left[\frac{1}{2}\beta(m,n)\right]\left[\frac{1}{2}\int_0^{\infty} e^{-t}\,t^{m+n-1}\,dt\right] \quad (\because r^2=t)$$

$$= \beta(m,n)\int_0^{\infty} e^{-x}\,x^{m+n-1}\,dx$$

$$= \beta(m,n)\,\Gamma(m+n)$$

Hence, $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

Prob: Show that $\Gamma(1/2) = \sqrt{\pi}$

Sol: We know that $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Taking $m = n = \frac{1}{2}$, we have

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \frac{[\Gamma(1/2)]^2}{\Gamma(1)} = \underbrace{[\Gamma(\tfrac{1}{2})]^2}_{\to (1)} \qquad (\because \Gamma(1)=1)$$

But $$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

put $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

__Limits:__ when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \pi/2$

$$\therefore\ B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2} (1-\sin^2\theta)^{-1/2} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot \sin\theta\cos\theta\, d\theta$$

$$= 2\,[\theta]_0^{\pi/2} = 2\left(\tfrac{\pi}{2}\right) = \pi$$

$$(1) \Rightarrow \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi.$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

(OR)

We know that $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

put $n = \frac{1}{2}$, we get

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1-\tfrac{1}{2}\right) = \frac{\pi}{\sin \pi/2}$$

$$\Rightarrow \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi \Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

*Ex: 1) P.T $\displaystyle\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta = \frac{\pi}{32}$

Sol: We know that $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$ → (1)

put $2m-1 = 2$ & $2n-1 = 4$

$\Rightarrow m = 3/2$ $\qquad \Rightarrow n = 5/2$

Then (1) becomes

$$\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta = \frac{1}{2}\beta(3/2, 5/2)$$

$$= \frac{1}{2}\cdot\frac{\Gamma(3/2)\cdot\Gamma(5/2)}{\Gamma\left(\frac{3}{2}+\frac{5}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\frac{1}{2}\Gamma(1/2)\,\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{\Gamma(4)}$$

$$= \frac{3\pi}{16\cdot 3!} = \frac{\pi}{32}$$

2) Evaluate i) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ ii) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Sol: i) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$

We know that, $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n) \longrightarrow (1)$

put $2m-1 = -\frac{1}{2}$ & $2n-1 = \frac{1}{2}$ in (1), we get.

$\Rightarrow 2m = \frac{1}{2}$ $\quad \Rightarrow 2n = 3/2$

$\Rightarrow m = \frac{1}{4}$ $\quad \Rightarrow n = 3/4$

$$(1) \Rightarrow \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4}+\frac{3}{4})}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{2 \times \frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}} //$$

ii) put $x^2 = \sin\theta \Rightarrow x = \sqrt{\sin\theta}$

$$\Rightarrow dx = \frac{1}{2\sqrt{\sin\theta}} \cdot \cos\theta\, d\theta$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{2\sqrt{\sin\theta}} \cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta$$

$(\because 2m-1 = -1/2$, $m = \frac{1}{4}$, $2n-1 = 0$, $n = 1/2)$

$$= \frac{1}{2} \cdot \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \frac{\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})}$$

$$= \frac{1}{4} \frac{\Gamma(1/4) \cdot \sqrt{\pi}}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

3) P.T $\int_0^1 (1-x^n)^{1/n}\, dx = \frac{1}{n} \frac{[\Gamma(\frac{1}{n})]^2}{2\Gamma(\frac{2}{n})}$

Sol: put $x^n = t \Rightarrow x = t^{1/n}$

$\Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1}\, dt$

$\therefore \int_0^1 (1-x^n)^{\frac{1}{n}} dx = \int_0^1 (1-t)^{\frac{1}{n}} \cdot \frac{1}{n} t^{\frac{1}{n}-1} dt$

$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{(\frac{1}{n}+1)-1} dt$

$= \frac{1}{n} \beta\left(\frac{1}{n}, \frac{n+1}{n}\right)$

$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\, \Gamma(\frac{1}{n}+1)}{\Gamma(\frac{1}{n}+\frac{1}{n}+1)}$

$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\, \frac{1}{n}\Gamma(\frac{1}{n})}{\Gamma(\frac{2}{n}+1)}$

$= \frac{1}{n^2} \frac{[\Gamma(\frac{1}{n})]^2}{\frac{2}{n}\Gamma(\frac{2}{n})}$

$= \frac{1}{n} \frac{[\Gamma(\frac{1}{n})]^2}{2\Gamma(\frac{2}{n})}$

4) P.T $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(1/n)}{\Gamma(\frac{1}{n}+\frac{1}{2})}$

Sol: Let $x^n = \sin^2\theta \Rightarrow x = \sin^{2/n}\theta$

$dx = \frac{2}{n} \sin^{\frac{2}{n}-1}\theta \cos\theta\, d\theta$.

Limits: when $x=0$, $\theta = 0$ & when $x=1$, $\theta = \pi/2$

$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{2}{n} \sin^{\frac{2}{n}-1}\theta \cos\theta\, d\theta$

$= \frac{2}{n} \int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta\, d\theta$. $\quad (\because \frac{2}{n}-1 = 2m-1$, $2m = \frac{2}{n}$, $m = \frac{1}{n})$

$= \frac{2}{n} \beta\left(\frac{1}{n}, \frac{1}{2}\right) \cdot \frac{1}{2}$

$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\, \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n}+\frac{1}{2})} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{n}+\frac{1}{2})}$

*5) show that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ and deduce that $\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{n+2}{2}\right)}$

Sol:

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\cdot\beta(m,n) \quad \rightarrow (1)$$

i) put $2m-1 = n$ & $2n-1 = 0$ in (1)

$\Rightarrow m = \frac{n+1}{2}$ $\quad \Rightarrow n = 1/2$

$$(1) \Rightarrow \int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{n+2}{2}\right)}$$

ii) put $2m-1 = 0$ & $2n-1 = n$ in (1)

$\Rightarrow m = \frac{1}{2}$ $\quad \Rightarrow n = \frac{n+1}{2}$

$$(1) \Rightarrow \int_0^{\pi/2} \cos^n\theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}\beta\left(\frac{1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n+1}{2}\right)}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{n+2}{2}\right)}$$

Hence, $\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\cdot\sqrt{\pi}}{2\Gamma\left(\frac{n+2}{2}\right)}$

6) ~~put $x = \sqrt{\tan\theta}$~~ Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using $\beta - \Gamma$ functions.

Sol: put $x = \sqrt{\tan\theta} \Rightarrow dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta\, d\theta$

Limits: when $x = 0$, $\theta = 0$ and when $x = \infty$, $\theta = \tan^{-1}(\infty) = \pi/2$.

$$\therefore 4\int_0^\infty \frac{x^2}{1+x^4}\,dx = 4\int_0^{\pi/2} \frac{\tan\theta}{1+\tan^2\theta} \cdot \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan\theta}{\sec^2\theta} \cdot \frac{\sec^2\theta}{\sqrt{\tan\theta}}\, d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta\, d\theta$$

$$= 2 \cdot \frac{1}{2} \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \beta\left(3/4, \frac{1}{4}\right)$$

$$= \frac{\Gamma(3/4) \cdot \Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}+\frac{1}{4})}$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{\Gamma(1)}$$

$$= \frac{\pi}{\sin\frac{\pi}{4}} \qquad \left(\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

$$= \frac{\pi}{1/\sqrt{2}} = \underline{\underline{\sqrt{2}\,\pi}}$$

7) Evaluate $\int_0^2 (8-x^3)^{1/3}\,dx$ using $\beta - \Gamma$ function

Hint: put $x^3 = 8y$; Ans: $\int_0^2 (8-x^3)^{1/3} = \frac{2}{3} \frac{[\Gamma(1/3)]^2}{\Gamma(2/3)}$

8) Evaluate $\int_0^1 x^3\sqrt{1-x}\,dx$ using $\beta - \Gamma$ functions

Ans: $\frac{32}{315}$

9) Evaluate i) $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta\, d\theta$ ii) $\int_0^{\pi/2} \sin^6\theta \cos^7\theta\, d\theta = \frac{2^4}{77 \times 39} = \frac{16}{3003}$

Ans: $\frac{64}{1989}$ iii) $\int_0^1 x^7(1-x)^5\,dx = \frac{1}{10296}$

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10) P.T $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

Sol: Let $I_1 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta$$

We know that $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m, n)$ $\rightarrow (1)$

put $2m-1 = -\frac{1}{2}$ & $2n-1 = 0$ in (1), we get

$\Rightarrow 2m = \frac{1}{2}$ $\quad\Rightarrow n = \frac{1}{2}$

$\Rightarrow m = \frac{1}{4}$

$$\therefore \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\, \Gamma(\frac{1}{4})}{\Gamma(3/4)}$$

Let $I_2 = \int_0^{\pi/2} \sin^{1/2}\theta \, d\theta$

put $2m-1 = \frac{1}{2}$ & $2n-1 = 0$ in (1), we get

$\Rightarrow m = 3/4$ $\quad\Rightarrow n = \frac{1}{2}$

$$\therefore \int_0^{\pi/2} \sin^{1/2}\theta \, d\theta = \frac{1}{2}\beta\left(3/4, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\, \Gamma(1/2)}{\Gamma(3/4+\frac{1}{2})}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\, \Gamma(3/4)}{\Gamma(5/4)}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\, \Gamma(3/4)}{\frac{1}{4}\Gamma(1/4)} \qquad (\because \Gamma(5/4) = \Gamma(\tfrac{1}{4}+1))$$

$$= 2 \frac{\sqrt{\pi}\, \Gamma(3/4)}{\Gamma(1/4)}$$

$$\therefore \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \frac{1}{2} \frac{\sqrt{\pi}\, \cancel{\Gamma(\frac{1}{4})}}{\cancel{\Gamma(3/4)}} \times 2 \frac{\sqrt{\pi}\, \cancel{\Gamma(3/4)}}{\cancel{\Gamma(\frac{1}{4})}}$$

$$= \pi$$

11) P.T $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = 0$ using $\beta-\Gamma$ functions.

Sol: $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8-x^{14}}{(1+x)^{24}}dx$

$= \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$

$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{24}}dx$

$= \beta(9,15) - \beta(15,9) \quad \left(\because \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\right)$

$= \beta(9,15) - \beta(9,15) \quad (\because \beta(m,n) = \beta(n,m))$

$= \underline{\underline{0}}$

12) P.T $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\cdots\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Sol: Let $K = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\cdots\cdots\Gamma\left(\frac{n-1}{n}\right) \longrightarrow (1)$

writing (1) in reverse order, we get

$K = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{3}{n}\right)\cdots\cdots\Gamma\left(1-\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right) \longrightarrow (2)$

multiplying (1) & (2), we get

$K^2 = \Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\cdots\cdots\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)$

W.K.T, $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$\therefore K^2 = \frac{\pi}{\sin\frac{\pi}{n}} \cdot \frac{\pi}{\sin\frac{2\pi}{n}} \cdots\cdots \frac{\pi}{\sin\frac{(n-1)\pi}{n}}$

$= \frac{\pi^{n-1}}{n/2^{n-1}} \quad \left(\because \sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdots\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}, n>1\right)$

$$= \frac{(2\pi)^{n-1}}{n}$$

$$\therefore \quad K = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}} \quad //$$

13) Express the integral $\int_0^{\infty} \frac{x^c}{c^x}\,dx$ $(c>1)$ in terms of Gamma function.

Sol: we have $c = e^{\log c} \Rightarrow c^x = e^{x\log c}$

$$\therefore \int_0^{\infty} \frac{x^c}{c^x}\,dx = \int_0^{\infty} \frac{x^c}{e^{x\log c}}\,dx$$

$$= \int_0^{\infty} e^{-\log c \cdot x}\, x^c\,dx$$

$$\cancel{= \int_0^{\infty} e^{-\log c \cdot x}\, x^{(c+1)-1}\,dx}$$

put $x\log c = t \Rightarrow x = \frac{t}{\log c}$

$\Rightarrow \log c\,dx = dt$

$\Rightarrow dx = \frac{dt}{\log c}$

$$= \int_0^{\infty} e^{-t} \left(\frac{t}{\log c}\right)^c \frac{dt}{\log c}$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^{\infty} e^{-t}\, t^{(c+1)-1}\,dt$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}} \quad // \qquad (\because \text{by the def.})$$

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* S.T $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Sol: W.K.T, $\beta(m,n) = \int_0^\infty \dfrac{x^{n-1}}{(1+x)^{m+n}}\,dx \rightarrow (1)$

we have $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} \rightarrow (2)$

From eqns (1) & (2), we have

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

put $m+n=1 \Rightarrow m = 1-n$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \Gamma(n)\cdot\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \quad (\because \Gamma(1)=1) \rightarrow (3)$$

W.K.T, $\int_0^\infty \dfrac{x^{2m}}{1+x^{2n}}\,dx = \dfrac{\pi}{2n}\operatorname{cosec}\left(\dfrac{2m+1}{2n}\right)\pi$

where $m>0$, $n>0$ & $n>m$

Let $x^{2n} = t$ & $\frac{2m+1}{2n} = s$

$\Rightarrow x = t^{\frac{1}{2n}}$

$\Rightarrow dx = \frac{1}{2n} \cdot t^{\frac{1}{2n}-1} dt$

$$\Rightarrow \int_0^{\infty} \frac{(t^{\frac{1}{2n}})^{2m}}{1+t} \cdot \frac{1}{2n} \cdot t^{\frac{1}{2n}-1} dt = \frac{\pi}{2n} \operatorname{cosec} s\pi$$

$$\Rightarrow \cancel{\frac{1}{2n}} \int_0^{\infty} \frac{t^{\frac{2m}{2n}}}{1+t} \cdot t^{\frac{1}{2n}-1} dt = \frac{\pi}{\cancel{2n}} \operatorname{cosec} s\pi$$

$$\Rightarrow \int_0^{\infty} \frac{t^{\frac{2m}{2n}+\frac{1}{2n}-1}}{1+t} dt = \pi \operatorname{cosec} s\pi$$

$$\Rightarrow \int_0^{\infty} \frac{t^{\left(\frac{2m+1}{2n}\right)-1}}{1+t} dt = \frac{\pi}{\sin s\pi}$$

$$\Rightarrow \int_0^{\infty} \frac{t^{s-1}}{1+t} dt = \frac{\pi}{\sin s\pi}$$

$$(\text{or}) \quad \int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$$

From (3) $\Rightarrow$ $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Find i) $\int_0^1 x^3 \sqrt{1-x}\, dx$ ii) $\int_0^1 x^7 (1-x)^5 dx$

Sol: $\int_0^1 x^3 (1-x)^{1/2} dx = \int_0^1 x^{4-1} (1-x)^{\left(\frac{1}{2}+1\right)-1} dx$

$$= \int_0^1 x^{4-1} (1-x)^{\frac{3}{2}-1} dx$$

we have $\int_0^1 x^{m-1} (1-x)^{n-1} dx = \beta(m, n)$

$$= \beta\left(4, \frac{3}{2}\right)$$

we have $\beta(m, n) = \dfrac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

$$= \frac{\Gamma(4)\, \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(4+\frac{3}{2}\right)}$$

$$= \frac{\Gamma(3+1) \cdot \Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{11}{2}\right)}$$

we have $\Gamma(n+1) = n!$ & $\Gamma(n+1) = n\,\Gamma(n)$

$$= \frac{3!\ \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}+1\right)}$$

$$= \frac{\cancel{3}6 \cdot \frac{1}{\cancel{2}} \cdot \sqrt{\pi}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{\cancel{3}}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)}$$

$$= \frac{32 \sqrt{\pi}}{315 \sqrt{\pi}} = \frac{32}{315} //$$

3×5

315

ii) $\int_0^1 x^7 (1-x)^5 dx = \int_0^1 x^{8-1} (1-x)^{6-1}$

$$= \beta(8,6)$$

$$= \frac{\Gamma(8) \cdot \Gamma(6)}{\Gamma(8+6)}$$

$$= \frac{\Gamma(7+1) \cdot \Gamma(5+1)}{\Gamma(13+1)}$$

$$= \frac{7! \times 5!}{13!}$$

$$= \frac{\cancel{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \; \cancel{7!} \times \cancel{120}^{12}}{13 \times \cancel{12} \times 11 \times \cancel{10} \times 9 \times 8 \times \cancel{7!}}$$

$$= \frac{1}{10296}$$

Example 2 : Show that

(*i*) $B(m+1, n) = \dfrac{m}{m+n} B(m,n) \quad (m > 0, n > 0)$ **[JNTU 2004S (Set No. 1)]**

(*ii*) $B(m, n+1) = \dfrac{n}{m+n} B(m, n) \quad (m > 0, n > 0)$

(*iii*) $B(m, n) = B(m+1, n) + B(m, n+1) \quad (m > 0, n > 0)$

Solution : We have $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ ... (1)

(*i*) $$B(m+1, n) = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} = \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} \quad [\because \Gamma(n) = (n-1)\Gamma(n-1)]$$

$$= \frac{m}{m+n} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{m}{m+n} B(m, n) \text{ [by (1)]} \quad \text{... (2)}$$

(*ii*) $$B(m, n+1) = \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{\Gamma(m) \cdot n\Gamma(n)}{(m+n)\Gamma(m+n)} = \frac{n}{m+n} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{n}{m+n} B(m, n) \text{ [by (1)]} \quad \text{... (3)}$$

(*iii*) (2) + (3) gives

$$B(m+1, n) + B(m, n+1) = \frac{m}{m+n} B(m, n) + \frac{n}{m+n} B(m, n)$$

$$= B(m,n)\left[\frac{m}{m+n} + \frac{n}{m+n}\right] = B\ (m,\ n)$$

Example 5 : Evaluate

(i) $\int_0^1 x^5 (1-x)^3 \, dx$

(ii) $\int_0^1 x^4 (1-x)^2 \, dx$ **[JNTU (H) Nov. 2009 (Set No. 4), (A) June 2016]**

(iii) $\int_0^2 x \left(8-x^3\right)^{1/3} dx$ **[JNTU (H) Nov. 2009, (A) Nov. 2011 (Set No. 2)]**

(iv) $\int_0^1 x^{5/2} (1-x^2)^{3/2} \, dx$ **[JNTU (K) Nov. 2009 (Set No. 4)]**

Solution : (i) $\int_0^1 x^5 (1-x)^3 \, dx = \int_0^1 x^{6-1} (1-x)^{4-1} \, dx = B(6,4) = \dfrac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}$

$$= \frac{5!\,3!}{9!} = \frac{5! \times 6}{9 \times 8 \times 7 \times 6 \times 5!} \quad [\because \ \Gamma(n+1) = n!]$$

$$= \frac{1}{9 \times 8 \times 7} = \frac{1}{504}$$

(ii) $\int_0^1 x^4 (1-x)^2 \, dx = \int_0^1 x^{5-1} (1-x)^{3-1} \, dx$

$$= B(5,3) = \frac{\Gamma(5)\,\Gamma(3)}{\Gamma(8)} = \frac{4!\,2!}{7!} = \frac{2}{7 \times 6 \times 5} = \frac{1}{105}$$

(iii) Let $x^3 = 8y \Rightarrow x = 2y^{1/3}$ so that $dx = \frac{2}{3} y^{-2/3}\, dy$.

When $x = 0$; $y = 0$; when $x = 2$; $y = 1$.

$$\therefore \int_0^2 x\left(8-x^3\right)^{1/3} dx = \int_0^1 2y^{1/3}\,(8-8y)^{1/3} \cdot \frac{2}{3} y^{-2/3}\, dy$$

$$= \frac{8}{3}\int_0^1 y^{-1/3}\,(1-y)^{1/3}\, dy = \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}\,(1-y)^{\frac{4}{3}-1}\, dy$$

$$= \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{8}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} = \frac{8}{3}\cdot\frac{\Gamma\left(\frac{2}{3}\right)\left(\frac{4}{3}-1\right)\Gamma\left(\frac{4}{3}-1\right)}{\Gamma(2)}$$

$$[\because \Gamma(n) = (n-1)\Gamma(n-1)]$$

$$= \frac{8}{3}\cdot\frac{1}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right) = \frac{8}{9}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$\therefore \int_0^2 x\left(8-x^3\right)^{1/3} dx = \frac{8}{9}\,\frac{\pi}{\sin(\pi/3)} = \frac{16\pi}{9\sqrt{3}} \quad \left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

(iv) Put $x^2 = y$ so that $2x\,dx = dy$ or $dx = \dfrac{dy}{2\sqrt{y}}$

$$\therefore I = \int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx = \int_0^1 y^{5/4}(1-y)^{3/2}\,\frac{dy}{2\sqrt{y}} = \frac{1}{2}\int_0^1 y^{3/4}(1-y)^{3/2}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{7}{4}-1}(1-y)^{\frac{5}{2}-1}\,dy = \frac{1}{2}B\left(\frac{7}{4},\frac{5}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{2}\right)} = \frac{1}{2}\cdot\frac{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{17}{4}\right)}$$

$$= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{\frac{3}{2}\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\frac{13}{4}\cdot\frac{9}{4}\cdot\frac{5}{4}\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{8}{65}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}$$

Example 12 : Evaluate

(i) $\int_0^\infty 3^{-4x^2}\,dx.$ **[JNTU (A) Nov. 2009 (Set No 4)]**

(ii) $\int_0^\infty a^{-bx^2}\,dx$ **[JNTU Nov. 2008 (Set No 4)]**

(iii) $\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx$ **[JNTU Nov. 2008, June 2009S, (A) Nov. 2011 (Set No. 1)]**

(iv) $\int_0^1 x^2\left(\log\frac{1}{x}\right)^3 dx$ **[JNTU Nov. 2008S (Set No 4)]**

Solution : (i) Since $3 = e^{\log 3}$ $\therefore$ $3^{-4x^2} = e^{-4x^2\log 3}$ $\therefore \int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty e^{-4x^2\log 3}\,dx$

Put $2x\sqrt{\log 3} = y$ so that $dx = \dfrac{dy}{2\sqrt{\log 3}}$

$$\therefore \quad \int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty e^{-y^2}\frac{dy}{2\sqrt{\log 3}} = \frac{1}{2\sqrt{\log 3}}\int_0^\infty e^{-y^2}\,dy$$

$$= \frac{1}{2\sqrt{\log 3}}\cdot\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}} = \sqrt{\frac{\pi}{16\log 3}}$$

(ii) $\int_0^\infty a^{-bx^2}\,dx = \int_0^\infty e^{-bx^2\log a}\,dx$

Put $(b\log a)\,x^2 = t$ so that $dx = \dfrac{dt}{2x\,b\log a} = \dfrac{dt}{2\sqrt{t}\cdot\sqrt{b\log a}}$

$$\therefore \quad \int_0^\infty a^{-bx^2}\,dx = \frac{1}{2\sqrt{b\log a}}\int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,dt = \frac{1}{2\sqrt{b\log a}}\cdot\int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{2\sqrt{b\log a}}\Gamma\left(\frac{1}{2}\right), \text{ by definition of } \Gamma \text{ function.}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{b\log a}} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

(iii) Put $\log\dfrac{1}{x} = t$ *i.e.,* $\dfrac{1}{x} = e^t$ or $x = e^{-t}$ $\therefore dx = -e^{-t}\,dt$

Also when $x = 1, t = 0$ and when $x \to 0,\ t \to \infty$

$$\therefore \int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx = \int_\infty^0 e^{-4t}t^3\cdot(-e^{-t}dt) = \int_0^\infty e^{-5t}t^3\,dt$$

Put $5t = u$ so that $dt = \frac{du}{5}$

$$\therefore \int_0^1 x^4 \left(\log\frac{1}{x}\right)^3 dx = \int_0^\infty e^{-u}\left(\frac{u}{5}\right)^3 \frac{du}{5} = \frac{1}{625}\int_0^\infty e^{-u}u^3 du$$

$$= \frac{1}{625}\int_0^\infty e^{-u}u^{4-1}du = \frac{1}{625}\cdot\Gamma(4) = \frac{3!}{625} = \frac{6}{625}$$

(iv) Proceeding as in (*iii*), we get

$$\int_0^1 x^2 \log\left(\frac{1}{x}\right)^3 dx = \int_0^\infty e^{-3t}.t^3 dt$$

$$= \frac{1}{27}\int_0^\infty e^{-u}.u^3 du \quad [\text{Putting } 3t = u]$$

$$= \frac{1}{27}\Gamma(4) = \frac{6}{27} = \frac{2}{9}$$

Example 29 : Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$ using $B-\Gamma$ functions.

Solution : $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx = \int_0^\infty \frac{x^4+x^9}{(1+x)^{15}}$

$$= \int_0^\infty \frac{x^4}{(1+x)^{15}}dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}dx$$

$$= B(5,10) + B(10,5) \quad \left[\because\ B(m.n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\right]$$

$$= 2\,B(5,10)\ [\because\ B(m,n) = B(n,m)]$$

$$= 2.\frac{\Gamma(5)\Gamma(10)}{\Gamma(15)} = \frac{2.4!.9!}{14!} = \frac{2\times 24}{14\times 13\times 12\times 11\times 10} = \frac{1}{5005}$$

Example 39 : Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of Gamma function and evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$. **[JNTU (A) June 2015]**

Solution : Put $x^n = t$ *i.e.,* $x = t^{1/n}$ so that $dx = \frac{1}{n}t^{\frac{1}{n}-1}\,dt$

Also when $x = 0$, $t = 0$ and when $x = 1$, $t = 1$

$$\therefore \quad \int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 t^{\frac{m}{n}}(1-t)^p \cdot \frac{1}{n}t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^p\,dt = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{(p+1)-1}\,dt$$

$$= \frac{1}{n}B\left(\frac{m+1}{n}, p+1\right) \qquad \text{... (1)}$$

$$= \frac{1}{n}\cdot\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)} \qquad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

Comparing $\int_0^1 x^5(1-x^3)^{10}\,dx$ with $\int_0^1 x^m(1-x^n)^p\,dx$, we have $m = 5$, $n = 3$, $p = 10$

$\therefore$ From (1), we have

$$\int_0^1 x^m (1-x^n)^p \, dx = \frac{1}{3} B\left(\frac{5+1}{3}, 10+1\right) = \frac{1}{3} B(2, 11)$$

$$= \frac{1}{3} \frac{\Gamma(2)\Gamma(11)}{\Gamma(13)} = \frac{1}{3} \cdot \frac{1!\,10!}{12!} = \frac{1}{3\times12\times11} = \frac{1}{396}$$

Example 57 : Evaluate $\int_0^a x^4\sqrt{a^2-x^2}\,dx$ **[JNTU (H) Nov. 2009 (Set No. 4)]**

Solution : Put $x = a\sqrt{y}$ so that $dx = \frac{a}{2\sqrt{y}}dy$

$$\therefore \int_0^a x^4\sqrt{a^2-x^2}\,dx = \int_0^1 (a\sqrt{y})^4\sqrt{a^2-a^2y}\,.\frac{a}{2\sqrt{y}}dy = \frac{a^6}{2}\int_0^1 y^{3/2}(1-y)^{1/2}dy$$

$$= \frac{a^6}{2}\int_0^1 y^{\frac{5}{2}-1}(1-y)^{\frac{3}{2}-1}dy = \frac{a^6}{2}\mathrm{B}\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= \frac{a^6}{2}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}$$

$$= \frac{a^6}{2}\cdot\frac{\frac{3}{2}\left[\Gamma\left(\frac{3}{2}\right)\right]^2}{\Gamma(4)} = \frac{3a^6}{4}\cdot\frac{1}{3!}\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]^2$$

$$= \frac{\pi a^6}{32} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 58 : Prove that $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

[JNTU 2003S, (K) June 2012 (Set No. 2)]

Solution : Let $I_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}$ **[JNTU April 2006 (Set No. 4)]**

Put $x^2 = \sin\theta$ *i.e.*, $x = \sqrt{\sin\theta}$ so that $dx = \dfrac{1}{2\sqrt{\sin\theta}}\cos\theta\, d\theta$

When $x = 0,\ \theta = 0$; when $x = 1,\ \theta = \pi/2$

$$\therefore \quad I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta\, d\theta}{2\sqrt{\sin\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right) \qquad \left[\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2} B(m, n)\right]$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{3}{4}\right)}{\left(\frac{5}{4}-1\right)\Gamma\left(\frac{5}{4}-1\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \qquad \text{... (1)}$$

Let $I_2 = \displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^2 = \tan\phi$ so that $dx = \dfrac{\sec^2\phi}{2\sqrt{\tan\phi}}\, d\phi$

$$\therefore I_2 = \int_0^{\pi/4} \frac{\sec^2\phi}{2\sqrt{1+\tan^2\phi}\,\sqrt{\tan\phi}}\, d\phi$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\phi}{\sec\phi\sqrt{\tan\phi}}\, d\phi = \frac{1}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{\sin\phi\cos\phi}}$$

$$= \frac{\sqrt{2}}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{2\sin\phi\cos\phi}} = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\phi}{\sqrt{\sin 2\phi}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{dt}{2\sqrt{\sin t}} \quad \text{(Putting } 2\phi = t\text{)}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-\frac{1}{2}} t\, dt = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-\frac{1}{2}} t \cos^0 t\, dt$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \ldots (2)$$

Hence $\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = I_1 \times I_2$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \times \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \quad \text{[by (1) \& (2)]}$$

$$= \frac{\pi}{4\sqrt{2}}$$

Example 59 : Show that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$

[JNTU 2003S, April 2006, (A) Nov. 2010 (Set No. 2)]

Solution : Consider $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$ *i.e.* $x = \sin^{1/2}\theta$ so that $dx = \frac{1}{2}\sin^{-\frac{1}{2}}\theta \cdot \cos\theta\, d\theta$

$$\therefore \int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{2}\sin^{-\frac{1}{2}}\theta \cdot \cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right) \qquad \left[\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2} B(m,n)\right]$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} \qquad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{3}{4}\right)}{\left(\frac{5}{4}-1\right)\Gamma\left(\frac{5}{4}-1\right)} \quad [\because \Gamma(n)=(n-1)\Gamma(n-1)]$$

$$= \sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(1/4)} \qquad \ldots (1)$$

Now consider $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$ *i.e.*, $x = \sin^{\frac{1}{2}}\theta$ so that $dx = \frac{1}{2}\sin^{-\frac{1}{2}}\theta\cos\theta\, d\theta$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{1}{2}\frac{\sin^{-\frac{1}{2}}\theta\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta\cos^0\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \left[\because B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma(3/4)} \qquad \ldots (2)$$

$$\text{Hence} \int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma(1/4)} \times \frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$= \frac{\pi}{4}, \text{ using (1) and (2)}$$

5. Evaluate (*i*) $\int_0^{\infty} x^3\, 3^{-x}\, dx$

(*ii*) $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^{\infty} x^2 e^{-x^4}\, dx = \frac{\pi}{4\sqrt{2}}$ **[Hint. Put** $z = \sin^2 \theta$**]**

6. Prove that (*i*) $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^{\infty} x^2 e^{-x^4}\, dx = \frac{\pi}{4\sqrt{2}}$ **[Hint. Put** $x^2 = y$**]**

(*ii*) $\int_0^{\infty} x^2 e^{-x^4}\, dx \times \int_0^{\infty} e^{-x^4}\, dx = \frac{\pi\sqrt{2}}{16}$ **[Hint. Put** $x^4 = y$**]**

Example 51 : Prove that $2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\Gamma(2n)\sqrt{\pi}$ **[JNTU (K) Jan 2012 (Set No. 4)]**

(or) Show that $\Gamma\left(\frac{1}{2}\right)\Gamma(2n)=2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$ **[JNTU 2008S, (A) Nov. 2010 (Set No.3)]**

(or) Prove that $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

[JNTU Nov. 2008, (H) Nov. 2009, (K) June 2012 (Set No. 4)]

Solution : By definition, we have $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

or $\int_0^1 x^{n-1}(1-x)^{m-1}\,dx=B(n,m)=\dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$ ... (1)

Put $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta$

Now (1) becomes $\int_0^{\pi/2}\sin^{2n-2}\theta(1-\sin^2\theta)^{m-1}(2\sin\theta\cos\theta)d\theta=\dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$

or $\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta=\dfrac{\Gamma(n)\Gamma(m)}{2\Gamma(m+n)}$ ... (2)

Putting $m=\frac{1}{2}$ in (2), we get

$$\int_0^{\pi/2}\sin^{2n-1}\theta\,d\theta=\frac{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(n+\frac{1}{2}\right)}=\frac{\sqrt{\pi}}{2}\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} \quad \text{... (3)}$$

Now putting $m=n$ in (2), we get

$$\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{[\Gamma(n)]^2}{2\Gamma(2n)}$$

or $$\frac{[\Gamma(n)]^2}{2\Gamma(2n)}=\frac{1}{2^{2n-1}}\int_0^{\pi/2}(2\sin\theta\cos\theta)^{2n-1}d\theta=\frac{1}{2^{2n-1}}\int_0^{\pi/2}\sin^{2n-1}2\theta\,d\theta$$

$$=\frac{1}{2^{2n-1}}\times\frac{1}{2}\int_0^{\pi}\sin^{2n-1}\phi\,d\phi \text{ (Putting } 2\theta=\phi)$$

$$=\frac{1}{2^{2n}}\times 2\int_0^{\pi/2}\sin^{2n-1}\phi\,d\phi=\frac{1}{2^{2n-1}}\times\frac{\sqrt{\pi}}{2}\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}\text{, by (3)}$$

$$\therefore\quad 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\Gamma(2n)\sqrt{\pi}$$

Example 32 : Show that

(*i*) $B\left(m, \frac{1}{2}\right) = 2^{2m-1} B(m, m)$ **[JNTU Nov. 2008, (K) June 2012 (Set No. 4)]**

(*ii*) $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$ **[JNTU (K) June 2012 (Set No. 4)]**

(or) $\Gamma\left(\frac{1}{2}\right)\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$ **[JNTU Aug. 2007S (Set No. 1)]**

(or) $2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \Gamma(2n)\sqrt{\pi}$ **[JNTU (K) Jan. 2012 (Set No. 4)]**

Solution : (*i*) We know that $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ ... (1)

Putting $n = \frac{1}{2}$ in (1), we get $B\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$ ... (2)

Again putting $n = m$ in (1), we get

$$B(m, m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi\left(\frac{d\phi}{2}\right), \text{putting } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi = \frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

or $2^{2m-1}B(m, m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = B\left(m, \frac{1}{2}\right)$ [by (2)]

Hence $B\left(m, \frac{1}{2}\right) = 2^{2m-1} B(m, m)$

ii) From (*i*), we have $B\left(m, \frac{1}{2}\right) = 2^{2m-1} B(m, m)$

Writing the above result in terms of Gamma function, we have

$$\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)}$$

or $$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}\frac{\Gamma(m)}{\Gamma(2m)}$$ or $$\frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}\frac{\Gamma(m)}{\Gamma(2m)}$$

or $$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$$