UNIT-1

Laplace Transforms

The knowledge of Laplace Transforms is an essential Part of mathematics required by engineers.

The Laplace transform is an essential tool for solving the linear differential Equations with given initial values of an unknown function and its derivatives without the necessity of first finding the general solution (CF+PI) and then evaluating from it the particular solution satisfying the given conditions. This technique is used to solve some partial differential Equations as well.

By using this Laplace transform we can directly find the solution of differential equation without finding the general solution i.e., GS = CF+PI.

Definition :-

Let $f(t)$ be a given function defined for all $t \geq 0$ then the Laplace transform of $f(t)$

is denoted by $L\{f(t)\}$ or $\bar{f}(s)$ and is defined by

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \longrightarrow (1)$$

→ The symbol L transform $f(t)$ into $\bar{f}(s)$ is called the Laplace ~~operator~~ transform operator

→ The Relation (1) can also be written as

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

in such case $f(t)$ is said to be inverse laplace transform of $\bar{f}(s)$.

→ The symbol L^{-1} transforms $\bar{f}(s)$ into $f(t)$ is called inverse Laplace transform operator.

Sufficient Conditions for the Existance of the Laplace transform of a function :-

→ The function $f(t)$ must satisfy the following conditions for the Existance of the Laplace transform

1. The function $f(t)$ must be piece-wise continous (or) sectionally continous in any limited interval $0 < a \le t \le b$

2. The function $f(t)$ is of the Exponential order.

Piece-wise Continous function :-

A function $f(t)$ is said to be piece-wise continous over the closed interval $[a,b]$ if it is defined on that interval and is such that the interval can be divided into finite no. of sub intervals in each of which $f(t)$ is continous.

Ex :- 1) $f(t) = \sin t$, if $0 < t < \pi$
$\qquad 0$ if $t > \pi$

2) $$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 1 & \text{if } 2 < t < 4 \\ 2 & \text{if } 4 < t < 6 \\ 3 & \text{if } t > 6 \end{cases}$$

Function of Exponential order :-

A function $f(t)$ is said to be of Exponential order a if $\lim_{t \to \infty} e^{-at} f(t)$ = a finite quantity.

Linearity Property :-

i) $L\{c\,f(t)\} = c\,L\{f(t)\}$
where c is a constant

ii) If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$ then

$$L\{a f(t) + b g(t)\} = L\{a f(t)\} + L\{b g(t)\}$$
$$= a L\{f(t)\} + b L\{g(t)\}$$
$$= a \bar{f}(s) + b \bar{g}(s)$$

$$\boxed{\therefore L\{a f(t) + b g(t)\} = a \bar{f}(s) + b \bar{g}(s)}$$

Laplace transforms of some standard functions :-

1. $L\{1\} = \frac{1}{s}$

Sol:- we know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$L\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt$$
$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty$$
$$= \frac{e^{-s\infty}}{-s} + \frac{e^{-s(0)}}{s}$$
$$= 0 + \frac{1}{s}$$
$$= \frac{1}{s}$$

2. $L\{t\} = \frac{1}{s^2}$

Sol:- $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$L\{t\} = \int_0^\infty e^{-st} \cdot t\, dt$$
$$= \left[t \frac{e^{-st}}{-s} - \int 1 \cdot \int e^{-st} dt\right]_0^\infty$$
$$= \left[-t \frac{e^{-st}}{s} - \int \frac{e^{-st}}{-s} dt\right]_0^\infty$$
$$= \left[-t \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$
$$= 0 - \left\{0 - \frac{1}{s^2}\right\} = \frac{1}{s^2}$$

3. $L\{e^{at}\} = \frac{1}{s-a}$ where $s-a>0$ i.e., $s>a$

Sol:- By the definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$
$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$
$$= \int_0^\infty e^{(-s+a)t}\, dt$$
$$= \left[\frac{e^{(-s+a)t}}{(-s+a)}\right]_0^\infty$$
$$= 0 - \left\{\frac{e^0}{(-s+a)}\right\} = 0 - \left\{-\frac{1}{(s-a)}\right\}$$
$$= \frac{1}{s-a}$$

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4. $L\{e^{-at}\} = \frac{1}{s+a}$, where $s+a>0$

Sol:- $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt$

$= \int_0^\infty e^{-(s+a)t}\, dt$

$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$

$= 0 - \left\{\frac{e^0}{-(s+a)}\right\}$

$= \frac{1}{s+a}$

Note:-

i) $\sinh x = \frac{e^x - e^{-x}}{2}$ ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

iii) $\sinh(at) = \frac{e^{at} - e^{-at}}{2}$ iv) $\cosh at = \frac{e^{at} + e^{-at}}{2}$

5. $L\{\sinh at\} = \frac{a}{s^2-a^2}$

Sol:- we know that $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$

$= \frac{1}{2} L\{e^{at} - e^{-at}\}$

$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{(s-a)(s+a)}\right]$

$= \frac{1}{\cancel{2}} \frac{\cancel{2}a}{s^2-a^2} = \frac{a}{s^2-a^2}$

6. $L\{\cosh at\} = \frac{s}{s^2-a^2}$

Sol:- W.K.T $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$

$= \frac{1}{2} L\{e^{at} + e^{-at}\}$

$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$

$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$

7. $L\{\sin at\} = \frac{a}{s^2+a^2}$ and $L\{\cos at\} = \frac{s}{s^2+a^2}$

Sol:- W.K.T $L\{e^{at}\} = \frac{1}{s-a}$

$\Rightarrow L\{e^{iat}\} = \frac{1}{s-ia} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}$

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$\therefore L\{e^{iat}\} = \frac{s+ia}{s^2+a^2}$

$L\{\cos at + i \sin at\} = \frac{s+ia}{s^2+a^2}$

$L\{\cos at\} + i\, L\{\sin at\} = \frac{s}{s^2+a^2} + \frac{i\,a}{s^2+a^2}$

comparing real and imaginary parts

$L\{\cos at\} = \frac{s}{s^2+a^2}$ and $L\{\sin at\} = \frac{a}{s^2+a^2}$

8. $L\{t^n\} = \frac{n!}{s^{n+1}}$ where n is +ve integer

Sol:- By the definition of Laplace transform

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$L\{t^n\} = \int_0^\infty t^n e^{-st}\, dt$

$= \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n\, t^{n-1} \frac{e^{-st}}{-s}\, dt$

$= 0 - 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\, dt$

$\therefore L\{t^n\} = \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\, dt$

$= \frac{n}{s} L\{t^{n-1}\}$

lly $L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$

$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$

$L\{t^{n-3}\} = \frac{n-3}{s} L\{t^{n-4}\}$

$\vdots$

$L\{t^{n-(n-1)}\} = \frac{n-(n-1)}{s} L\{t^{n-n}\}$

$\therefore L\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots\cdots \frac{n-(n-1)}{s} L\{t^{n-n}\}$

$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots\cdots \frac{1}{s} L\{1\}$

$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots\cdots \frac{1}{s} \cdot L\{1\}$

$= \frac{1}{s} \frac{2}{s} \frac{3}{s} \cdots\cdots \frac{n-2}{s} \frac{n-1}{s} \frac{n}{s} \cdot L\{1\}$

$= \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$

Problems:-

1. Find the Laplace transform of $e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$.

Sol:- Given

$f(t) = e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

Taking Laplace transform on both sides

$L\{f(t)\} = L\{e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\}$

$= L\{e^{3t}\} - 2L\{e^{-2t}\} + L\{\sin 2t\} + L\{\cos 3t\} + L\{\sinh 3t\} - 2L\{\cosh 4t\} + 9L\{1\}$

$= \frac{1}{s-3} - 2\left(\frac{1}{s+2}\right) + \frac{2}{s^2+2^2} + \frac{s}{s^2+3^2} + \frac{3}{s^2-3^2} - 2\left(\frac{s}{s^2-4^2}\right) + 9\left(\frac{1}{s}\right)$

2. Find the Laplace Transform of $e^{2t}+4t^3-2\sin 3t+3\cos 3t$.

Sol:- Given

$f(t) = e^{2t}+4t^3-2\sin 3t+3\cos 3t$

$L\{f(t)\} = L\{e^{2t}+4t^3-2\sin 3t+3\cos 3t\}$

$= L\{e^{2t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3L\{\cos 3t\}$

$= \frac{1}{s-2} + 4\left[\frac{3!}{s^{3+1}}\right] - 2\left(\frac{3}{s^2+3^2}\right) + 3\left(\frac{s}{s^2+3^2}\right)$

3. Find the Laplace transform of $\sin(\omega t+\alpha)$

Sol:- $f(t) = \sin(\omega t+\alpha)$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$L\{f(t)\} = L\{\sin(\omega t+\alpha)\}$

$= L\{\sin\omega t\cos\alpha + \cos\omega t\sin\alpha\}$

$= \cos\alpha\, L\{\sin\omega t\} + \sin\alpha\, L\{\cos\omega t\}$

$= \cos\alpha\left[\frac{\omega}{s^2+\omega^2}\right] + \sin\alpha\left[\frac{s}{s^2+\omega^2}\right]$

4. Find the Laplace Transform of $\cos(\omega t+\alpha)$

Sol:- $f(t) = \cos(\omega t+\alpha) = \cos\omega t\cos\alpha - \sin\omega t\sin\alpha$

$L\{\cos(\omega t+\alpha)\} = L\{\cos\omega t\cos\alpha - \sin\omega t\sin\alpha\}$

$= L\{\cos\omega t\cos\alpha\} - L\{\sin\omega t\sin\alpha\}$

$= \cos\alpha\, L\{\cos\omega t\} - \sin\alpha\, L\{\sin\omega t\}$

$= \cos\alpha \frac{s}{s^2+\omega^2} - \sin\alpha \frac{\omega}{s^2+\omega^2}$

5. Find the L.T of $\cos 5t \cdot \cos 3t$

$2\cos A\cos B = \cos(A+B)+\cos(A-B)$

Sol:- $f(t) = \cos 5t\cos 3t$

$= \frac{1}{2}[2\cos 5t\cos 3t]$

$= \frac{1}{2}[\cos(5t+3t) + \cos(5t-3t)]$

$= \frac{1}{2}[\cos 8t + \cos 2t]$

$\therefore L\{f(t)\} = \frac{1}{2}L[\cos 8t + \cos 2t]$

$= \frac{1}{2}[L\{\cos 8t\} + L\{\cos 2t\}]$

$= \frac{1}{2}\left[\frac{s}{s^2+8^2} + \frac{s}{s^2+2^2}\right\}$

6. Find the L.T of $f(t) = \begin{cases} \cos t & 0<t<\pi \\ \sin t & t>\pi \end{cases}$

Note:- 1) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

2) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

Sol:- By the definition of Laplace Transform

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$= \int_0^\pi e^{-st} f(t)\, dt + \int_\pi^\infty e^{-st} f(t)\, dt$

$$= \int_0^{\pi} e^{-st}\cos t\,dt + \int_{\pi}^{\infty} e^{-st}\sin t\,dt$$

$$= \left[\frac{e^{-st}}{(-s)^2+1^2}\left[-s\cos t + \sin t\right]\right]_0^{\pi} + \left[\frac{e^{-st}}{(-s)^2+(1)^2}\left[-s\sin t - \cos t\right]\right]_{\pi}^{\infty}$$

$$= \left[\frac{e^{-s\pi}}{s^2+1}\left[-s\cos\pi + \sin\pi\right]\right] - \left[\frac{1}{s^2+1}\left[-s\cos 0 + \sin 0\right]\right]$$

$$+ 0 - \left[\frac{e^{-s\pi}}{s^2+1}\left[-s\sin\pi - \cos\pi\right]\right]$$

$(\because \cos n\pi = (-1)^n)$

$$= \cancel{\left[\frac{e^{-s\pi}}{s^2+1}(s\right.}$$

$$= \frac{s e^{-s\pi}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-s\pi}}{s^2+1}$$

$$= \frac{s e^{-s\pi} + s - e^{-s\pi}}{s^2+1}$$

$$= \frac{e^{-s\pi}(s-1)+s}{s^2+1}$$

$=$

7. Find the Laplace transform of $f(t) = \begin{cases} 1 & \text{if } 0<t<2 \\ 2 & \text{if } 2<t<4 \\ 3 & \text{if } 4<t<6 \\ 0 & \text{if } t>6 \end{cases}$

Sol:- W.K.T $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$

$$= \int_0^{2} e^{-st} f(t)\,dt + \int_2^{4} e^{-st} f(t)\,dt + \int_4^{6} e^{-st} f(t)\,dt + \int_6^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{2} e^{-st}\cdot 1\,dt + \int_2^{4} e^{-st}\, 2\,dt + \int_4^{6} e^{-st}\, 3\,dt + \int_6^{\infty} e^{-st}\, 0\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^2 + 2\left(\frac{e^{-st}}{-s}\right)_2^4 + 3\left(\frac{e^{-st}}{-s}\right)_4^6 + 0$$

$$= \left[\frac{e^{-2s}}{-s} - \frac{e^{0}}{-s}\right] + 2\left[\frac{e^{-4s}}{-s} - \frac{e^{-2s}}{-s}\right] + 3\left[\frac{e^{-6s}}{-s} - \frac{e^{-4s}}{s}\right]$$

$$= \frac{e^{-2s}}{-s} + \frac{1}{s} + \frac{2e^{-4s}}{-s} + \frac{2e^{-2s}}{s} + \frac{3e^{-6s}}{-s} + \frac{3e^{-4s}}{s}$$

$$= -\frac{1}{s}\left[e^{-2s} - 1 + 2e^{-4s} - 2e^{-2s} + 3e^{-6s} - 3e^{-4s}\right]$$

$$= -\frac{1}{s}\left[-e^{-2s} - 1 - e^{-4s} + 3e^{-6s}\right]$$

$$= \frac{e^{-2s} + 1 + e^{-4s} - 3e^{-6s}}{s}$$

$=$

8. Find the Laplace transform of $f(t) = \begin{cases} t/\pi & 0<t<\pi \\ 1 & t>\pi \end{cases}$

Sol:-

9. Verify whether the function $f(t) = t^3$ is exponential order and find its transform.

Sol:- Given $f(t) = t^3$

The function $f(t)$ is said to be an exponential order if it satisfies the condition

$$\lim_{t\to\infty} e^{-at} f(t) = \text{a finite value}$$

Now

$$\lim_{t\to\infty} e^{-at} f(t) = \lim_{t\to\infty} e^{-at} t^3$$

$$= \lim_{t\to\infty} \frac{t^3}{e^{at}} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

By using L Hospital rule

$$= \lim_{t\to\infty} \frac{3t^2}{a e^{at}} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{t\to\infty} \frac{6t}{a^2 e^{at}} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{t\to\infty} \frac{6}{a^3 e^{at}}$$

$$= \frac{6}{a^3 e^{\infty}} = \frac{6}{\infty} = 0$$

$\therefore f(t) = t^3$ is of exponential order.

The Laplace transform of t^3 is

$$L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{3!}{s^4}$$

10. P.T the function $f(t) = t^2$ is of exponential order 3.

Sol:- By the definition of exponential order

$$\lim_{t\to\infty} e^{-at} f(t) = \lim_{t\to\infty} e^{-3t} t^2$$

$$= \lim_{t\to\infty} \frac{t^2}{e^{+3t}}$$

$$= \lim_{t\to\infty} \frac{2t}{+3e^{3t}} \quad \text{(L Hospital rule)}$$

$$= \lim_{t\to\infty} \frac{2}{9e^{3t}}$$

$$= \frac{2}{\infty} = 0 \quad \text{(a finite quantity)}$$

Hence t^2 is an exponential order 3.

First Translation or first shifting Theorem :-

Statement :- If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

$L\{e^{-at} f(t)\} = \bar{f}(s+a)$

Proof :- By the Definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

$$L\{e^{at} f(t)\} = \cancel{\int_0^\infty e^{-st} f(t)\,dt}$$

$$= \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

Put $s-a=u$

$\therefore L\{e^{at} f(t)\} = \int_0^\infty e^{-ut} f(t)\, dt = \bar{f}(u)$ (By def $L\{f(t)\}=\bar{f}(s)$)

$= \bar{f}(s-a)$

Illy $L\{e^{-at} f(t)\} = \bar{f}(s+a)$

Based on above theorem

1. $L\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

2. $L\{e^{-at} t^n\} = \dfrac{n!}{(s+a)^{n+1}}$

3. $L\{e^{at} \sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

4. $L\{e^{-at} \sin bt\} = \dfrac{b}{(s+a)^2+b^2}$

5. $L\{e^{at} \sinh bt\} = \dfrac{b}{(s-a)^2-b^2}$

6. $L\{e^{-at} \sinh bt\} = \dfrac{b}{(s+a)^2-b^2}$

7. $L\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

8. $L\{e^{-at} \cos bt\} = \dfrac{s+a}{(s+a)^2+b^2}$

9. $L\{e^{at} \cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}$

10. $L\{e^{-at} \cosh bt\} = \dfrac{s+a}{(s+a)^2-b^2}$

Note:- by first shifting theorem $s \to s-a$ when e^{at} occurs
$s \to s+a$ when e^{-at} occurs

Problems :-

1. Find $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

Sol:- $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = L\{2e^{-3t}\cos 5t - 3e^{-3t}\sin 5t\}$

$= 2L\{e^{-3t}\cos 5t\} - 3L\{e^{-3t}\sin 5t\}$

$= 2\left(\dfrac{s+3}{(s+3)^2+25}\right) - 3\left(\dfrac{5}{(s+3)^2+25}\right)$

$= 2\left[\dfrac{s+3}{(s+3)^2+25}\right] - 3\left[\dfrac{5}{(s+3)^2+25}\right]$

$= \dfrac{2s+6-15}{(s+3)^2+25} = \dfrac{2s-9}{s^2+6s+9+25} = \dfrac{2s-9}{s^2+6s+34}$

2. Find $L\{e^{-t}(3\sin 2t - 5\cosh 2t)\}$

Sol:- $L\{e^{-t}(3\sin 2t - 5\cosh 2t)\}$

$= L\{3e^{-t}\sin 2t - 5e^{-t}\cosh 2t\}$

$$= 3L\{e^{t}\sin 2t\} - 5L\{e^{-t}\cosh 2t\}$$

$$= 3\left[\frac{2}{(s+1)^2+4}\right] - 5\left[\frac{s+1}{(s+1)^2-4}\right]$$

$$= 3\left[\frac{2}{s^2+2s+1+4}\right] - 5\left[\frac{s+1}{s^2+2s+1-4}\right]$$

$$= \frac{6}{s^2+2s+5} - \frac{5s+5}{s^2+2s-3}$$

$\left(\because L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}\right)$

$\left(\because L\{e^{-at}\cosh bt\} = \frac{s+a}{(s+a)^2-b^2}\right)$

3. $L\{e^{3t}\sin^2 t\}$

Sol:- $L\{e^{3t}\sin^2 t\} = L\{e^{3t}\,\frac{1-\cos 2t}{2}\}$

$$= \frac{1}{2}L\{e^{3t}(1-\cos 2t)\}$$

$$= \frac{1}{2}L\{e^{3t} - e^{3t}\cos 2t\}$$

$$= \frac{1}{2}\left[L\{e^{3t}\} - L\{e^{3t}\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-3} - \frac{(s-3)}{(s-3)^2+2^2}\right]$$

$\left(\because L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}\right)$

$$= \frac{1}{2}\left[\frac{1}{s-3}\right] - \frac{1}{2}\left[\frac{s-3}{s^2+9-6s+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-3}\right] - \frac{1}{2}\left[\frac{s-3}{s^2-6s+13}\right]$$

4. $L\{e^{t}\cos^2 t\}$

Sol:- $L\{e^{-t}\cos^2 t\} = L\{e^{-t}\,\frac{(1+\cos 2t)}{2}\}$

$$= \frac{1}{2}L\{e^{-t} + e^{-t}\cos 2t\}$$

$$= \frac{1}{2}\left[L\{e^{-t}\} + L\{e^{-t}\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{(s+1)^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{s^2+2s+1+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{s^2+2s+5}\right]$$

5. Show that i) $L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

ii) $L\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$

Sol:- we have $L\{t\} = \frac{1}{s^2}$

$$L\{e^{at}t\} = \frac{1}{(s-a)^2}$$

$$L\{t\,e^{iat}\} = \frac{1}{(s-ia)^2} \times \frac{(s+ia)^2}{(s+ia)^2} = \frac{(s+ia)^2}{(s^2+a^2)^2}$$

$\left((a-b)^2(a+b)^2 = (a^2-b^2)^2\right)$

$$L\{t\,e^{iat}\} = \frac{s^2-a^2+2isa}{(s^2+a^2)^2}$$

$L\{t(\cos at + i\sin at)\} = \dfrac{s^2-a^2+i2as}{(s^2+a^2)^2}$

Equating real and imaginary parts

$L\{t\cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$ $\quad L\{t\sin at\} = \dfrac{2as}{(s^2+a^2)^2}$

Gamma function :- The Integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is called the Gamma function. It is denoted by $\Gamma(n)$.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \quad \text{where } n>0$$

Properties of Gamma-function:-

1. $\Gamma(n+1) = n!$ where n is +ve integer
2. $\Gamma(n+1) = n\Gamma(n)$ where $n>0$ (integers (or) fractional parts also)
3. $\Gamma(1) = 1$
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
5. $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$
6. Γ function is not defined for negative values and for zero, i.e., $\Gamma(0), \Gamma(-1)\, \Gamma(-2) \ldots\ldots$ are not defined
7. $\Gamma(n) = (n-1)\Gamma(n-1)$

P.T ~~solve~~ $L\{t^n\} = \dfrac{n!}{s^{n+1}}$ by using Gamma function. also P.T $L\{t^n\} = \dfrac{n\Gamma(n)}{s^{n+1}}$ where $n>0$

Sol:- By def of Laplace transform

$$L\{t^n\} = \int_0^\infty e^{-st} t^n\,dt$$

put $st = x \Rightarrow t = \frac{x}{s}$

$s\,dt = dx \Rightarrow dt = \frac{dx}{s}$

when $t=0 \Rightarrow x=0$

$t=\infty \Rightarrow x=\infty$

$$\therefore L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \left(\frac{dx}{s}\right)$$

$$= \int_0^\infty e^{-x} \cdot \frac{x^n}{s^{n+1}}\,dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\,dx \qquad \left(\because \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx,\ \Gamma(n+1) = \int_0^\infty e^{-x}x^n dx\right)$$

$$\boxed{L\{t^n\} = \frac{1}{s^{n+1}}\,\Gamma(n+1)} \quad \longrightarrow (1)$$

$$= \frac{1}{s^{n+1}}\, n! = \frac{n!}{s^{n+1}}$$

also from (1)

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n\Gamma(n)}{s^{n+1}} \quad \text{where } n>0$$

2. Find L.T of $\left[\sqrt{t}+\frac{1}{\sqrt{t}}\right]^3$

Sol:- $L\left\{\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{(\sqrt{t})^3+\left(\frac{1}{\sqrt{t}}\right)^3+3(\sqrt{t})^2\frac{1}{\sqrt{t}}+3\sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2\right\}$

$= L\{t^{3/2}\}+L\{t^{-3/2}\}+3L\{t^{1/2}\}+3L\{t^{-1/2}\}$

W.K.T $L\{t^n\} = \cancel{\frac{\Gamma(n+1)}{s^{n+1}}}\ n\frac{\Gamma(n)}{s^{n+1}}$

$L\left\{\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right\} = \frac{\frac{3}{2}\Gamma(\frac{3}{2})}{s^{3/2+1}} - \frac{\frac{3}{2}\Gamma(-\frac{3}{2})}{s^{-\frac{3}{2}+1}}$

$\frac{3\,\frac{1}{2}\Gamma(\frac{1}{2})}{s^{1/2+1}} + \frac{3(-\frac{1}{2})\Gamma(-\frac{1}{2})}{s^{-1/2+1}}$

$= \frac{\frac{3}{2}\,\frac{1}{2}\Gamma(\frac{1}{2})}{s^{5/2}} - \frac{\frac{3}{2}(-\frac{5}{2})\Gamma(-\frac{5}{2})}{s^{-1/2}} + \frac{\frac{3}{2}\Gamma(\frac{1}{2})}{s^{3/2}}$

Sol:- $L\left\{\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{(\sqrt{t})^3+\left(\frac{1}{\sqrt{t}}\right)^3+3(\sqrt{t})^2\frac{1}{\sqrt{t}}+3\sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2\right\}$

$= L\{t^{3/2}\}+L\{t^{-3/2}\}+3L\{t^{1/2}\}+3L\{t^{-1/2}\}$

W.K.T $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$L\left\{\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right\} = \frac{\Gamma(\frac{3}{2}+1)}{s^{3/2+1}} + \frac{\Gamma(-\frac{3}{2}+1)}{s^{-3/2+1}} + 3\frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} + 3\frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}}$

$= \frac{\frac{3}{2}\Gamma(\frac{3}{2})}{s^{5/2}} + \frac{\Gamma(-\frac{1}{2})}{s^{-1/2}} + \frac{3\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} + \frac{3\Gamma(\frac{1}{2})}{s^{1/2}}$

$= \frac{\frac{3}{2}\,\frac{1}{2}\Gamma(\frac{1}{2})}{s^{5/2}} + \frac{\Gamma(-\frac{1}{2})}{s^{-1/2}} + \frac{\frac{3}{2}\Gamma(\frac{1}{2})}{s^{3/2}} + 3\frac{\Gamma(\frac{1}{2})}{s^{1/2}}$

$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - 2\sqrt{\pi s} + \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + 3\sqrt{\frac{\pi}{s}}$

$=$

3. Find $L\{t^{7/2}e^{3t}\}$

(By Gamma function $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$L\{e^{at}t^n\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$

Sol:- $L\{e^{3t}t^{7/2}\} = \cancel{L\{t^{7/2}\}_{s\to s-3}}$

$= \frac{\Gamma(\frac{7}{2}+1)}{(s-3)^{7/2+1}}$

$= \frac{\frac{7}{2}\cdot\Gamma(\frac{7}{2})}{(s-3)^{7/2+1}} = \frac{\frac{7}{2}\cdot\frac{5}{2}\Gamma(\frac{5}{2})}{(s-3)^{9/2}}$

$= \frac{\frac{7}{2}\cdot\frac{5}{2}\,\frac{3}{2}\Gamma(\frac{3}{2})}{(s-3)^{9/2}} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\frac{1}{2}\Gamma(\frac{1}{2})}{(s-3)^{9/2}}$

$= \frac{105\sqrt{\pi}}{16(s-3)^{9/2}}$

4. using the equation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

S.T $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} [e^{-1/4s}]$

Sol:- Given $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\sin\sqrt{t} = \sqrt{t} - \frac{1}{3!}(\sqrt{t})^3 + \frac{1}{5!}(\sqrt{t})^5 - \frac{1}{7!}(\sqrt{t})^7 + \cdots$$

$$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!} L\{t^{5/2}\} - \frac{1}{7!} L\{t^{7/2}\}$$

$$= \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} - \frac{1}{3!}\frac{\Gamma(\frac{3}{2}+1)}{s^{3/2+1}} + \frac{1}{5!}\frac{\Gamma(\frac{5}{2}+1)}{s^{5/2+1}} - \frac{1}{7!}\frac{\Gamma(\frac{7}{2}+1)}{s^{7/2+1}}$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{7/2}} - \frac{1}{7!}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{9/2}}$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}} - \frac{\cancel{3}}{\cancel{24}\,8}\frac{\sqrt{\pi}}{s^{5/2}} + \frac{1}{64}\frac{\sqrt{\pi}}{s^{7/2}} - \frac{\sqrt{\pi}}{768\, s^{9/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$$\left(\because e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right.$$
$$\left. e^{-1/4s} = 1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots \right)$$

UNIT STEP FUNCTION :-

The unit step function at $t=a$ is defined as

$$u(t-a) \text{ (or) } H(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$$

This function is also known as Heaviside unit function

Laplace Transform of unit step function :-

we know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left(\frac{e^{-st}}{-s}\right)_a^\infty = 0 - \left(\frac{e^{-sa}}{-s}\right) = \frac{e^{-sa}}{s}$$

$$\therefore L\{u(t-a)\} = \frac{e^{-sa}}{s}$$

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Second Translation or second shifting Theorem :-

statement:-

If $L\{f(t)\} = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$

then $L\{g(t)\} = e^{-as}\bar{f}(s)$

Proof:-

By the definition of laplace transform

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)dt + \int_a^\infty e^{-st} g(t)dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = u$
$dt = du$

If $t = a \Rightarrow u = 0$
$t = \infty \Rightarrow u = \infty$

$$\therefore L\{g(t)\} = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-su-sa} f(u)\,du$$

$$= \int_0^\infty e^{-su}.e^{-sa} f(u)\,du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)du$$

$$= \cancel{e^{-sa}} \int_0^\infty e^{-st} \cancel{f(t)dt \text{ (replacing u by t)}}$$

$$= e^{-sa}\int_0^\infty e^{-st} f(t)dt \quad \text{(replacing u by t)}$$

$$= \cancel{e^{-sa}\, L\{f(t)\}}$$

$$= e^{-sa}\, L\{f(t)\}$$

$$= e^{-sa}\,\bar{f}(s)$$

$$\therefore \boxed{L\{g(t)\} = e^{-sa}\bar{f}(s)}$$

Another form of second shifting theorem :-

statement:- If $L\{f(t)\} = \bar{f}(s)$ and $a > 0$

then $L\{f(t-a) \cdot H(t-a)\} = e^{-as}\bar{f}(s)$

where $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$ and $H(t)$ is called Heavyside unit step function.

Proof :- By the definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a)\,dt$$

Put $t-a = u \Rightarrow t = u+a$
$dt = du$

If $t=0$ $u = -a$
$t = \infty$ $u = \infty$

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$\therefore L\{f(t-a)H(t-a)\} = \int_{-a}^{\infty} e^{-s(u+a)} f(u) H(u)\, du$

$= \int_{-a}^{0} e^{-s(u+a)} f(u) H(u)\, du + \int_{0}^{\infty} e^{-s(u+a)} f(u)\, du$

$= \int_{-a}^{0} e^{-s(u+a)} f(u) (0)\, du + \int_{0}^{\infty} e^{-s(u+a)} f(u)\, 1\, du$

$= \int_{0}^{\infty} e^{-s(u+a)} f(u)\, du$

$= \int_{0}^{\infty} e^{-su} e^{-sa} f(u)\, du$

$= e^{-sa} \int_{0}^{\infty} e^{-su} f(u)\, du$

$= e^{-sa} \int_{0}^{\infty} e^{-st} f(t)\, dt$

$= e^{-sa} L\{f(t)\}$

(we can replace H by u because H or u is unit step function)

$= e^{-sa} \bar{f}(s)$

$\therefore \boxed{L\{f(t-a)\, u(t-a)\} = e^{-as} \bar{f}(s)}$

Problems :-

1. Find L.T of $g(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{if } t > \frac{2\pi}{3} \\ 0 & \text{if } t < \frac{2\pi}{3} \end{cases}$

Sol :- By second shifting theorem

we know that $g(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$

$\therefore$ here $f(t-a) = \cos\left(t - \frac{2\pi}{3}\right)$

$\Rightarrow a = \frac{2\pi}{3}$ and let $f(t) = \cos t$

~~Now: $L\{e^{-at}$~~

Now $L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1}$

By second shifting theorem we have

$L\{g(t)\} = e^{-as} \bar{f}(s)$

$= e^{-as} L\{f(t)\}$

$= e^{-as} \frac{s}{s^2+1}$

$= e^{-\frac{2\pi}{3} \cdot s} \frac{s}{s^2+1}$

$= \frac{s\, e^{-\frac{2\pi}{3} s}}{s^2+1}$

2. Find the Laplace transform of the following functions

i) $g(t) = \begin{cases} \cos(t - \pi/3) & \text{if } t > \pi/3 \\ 0 & \text{if } t < \pi/3 \end{cases}$

ii) $g(t) = \begin{cases} \sin(t - \pi/3) & \text{if } t > \pi/3 \\ 0 & \text{if } t < \pi/3 \end{cases}$

Sol:- ii) By second shifting theorem

w.k.T

$$g(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

$\therefore f(t-a) = \sin(t - \pi/3)$

$\therefore a = \pi/3$

and let $f(t) = \sin t$

$L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

By second shifting theorem

$L\{g(t)\} = e^{-as} \bar{f}(s)$

$= e^{-as} L\{f(t)\}$

$= e^{-as} \frac{1}{s^2+1}$

$= e^{-\pi/3 s} \frac{1}{s^2+1}$

3. Find the laplace transform of $3\cos 4(t-2) \cdot u(t-2)$

Sol:- w.k.T By the another form of second shifting theorem

$L\{f(t-a)\, H(t-a)\} = e^{-as} L\{f(t)\}$

$= e^{-as} \bar{f}(s)$

on comparing we have $f(t-a) = 3\cos 4(t-2)$

$u(t-a) = u(t-2)$

here $a=2$ let $f(t) = 3\cos 4t$

$L\{f(t)\} = 3L\{\cos 4t\} = 3 \frac{s}{s^2+16}$

Now by another form of second shifting theorem

$L\{3\cos 4(t-2) \cdot u(t-2)\} = e^{-as} \bar{f}(s)$

$= e^{-2s} L\{f(t)\}$

$= e^{-2s} \cdot \frac{3s}{s^2+16}$

$= \frac{3s e^{-2s}}{s^2+16}$

4. Find the L.T of i) $L\{(t-2)^3 u(t-2)\}$ ii) $e^{-3t} u(t-2)$

Sol:- i) on comparing with $L\{f(t-a)\, H(t-a)\}$

$f(t-a) = (t-2)^3$, $H(t-a) = H(t-2)$

here $a=2$ and let $f(t) = t^3$

$L\{f(t)\} = L\{t^3\} = \frac{6}{s^4}$

By second shifting theorem

$L\{(t-2)^3 u(t-2)\} = e^{-as} \bar{f}(s)$

$= e^{-as} L\{f(t)\}$

$= e^{-2s} \frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$

ii) $e^{-3t} u(t-2)$

$$e^{-3t} u(t-2) = e^{-3t+6-6} u(t-2)$$
$$= e^{-3(t-2)-6} u(t-2)$$
$$= e^{-3(t-2)} e^{-6} u(t-2)$$

$$L\{e^{-3t} u(t-2)\} = e^{-6} L\{e^{-3(t-2)} u(t-2)\}$$

By second shifting theorem (another form)

$f(t-a) = e^{-3(t-2)}$

$H(t-a) = u(t-2)$

here $a = 2$

Let $f(t) = e^{-3t}$

$L\{f(t)\} = L\{e^{-3t}\} = \frac{1}{s+3}$

$$\therefore L\{e^{-3t} u(t-2)\} = e^{-6} L\{e^{-3(t-2)} u(t-2)\}$$
$$= e^{-6} \cdot e^{-as} \bar{f}(s)$$
$$= e^{-6} \cdot e^{-2s} L\{f(t)\}$$
$$= e^{-6} \cdot e^{-2s} \frac{1}{s+3}$$
$$= \frac{e^{-2(s+3)}}{s+3}$$

Change of scale property :-

statement :-

If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$

Proof :- By the def of L.T

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s)$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

Put $at = x \Rightarrow t = \frac{x}{a}$

$a\,dt = dx \Rightarrow dt = \frac{dx}{a}$

also if $t = 0$ then $x = 0$

if $t = \infty$ then $x = \infty$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-s\frac{x}{a}} f(x) \frac{1}{a} dx$$
$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}x} f(x)\, dx$$

(replacing x by t)

$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}t} f(t)\, dt = \frac{1}{a} L\{f(t)\}_{s \to \frac{s}{a}}$$
$$= \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$$

Note :-

→ If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(\frac{t}{a})\} = a\, \bar{f}(as)$

→ $L\{f(at)\} = \frac{1}{a} L\{f(t)\}_{s \to \frac{s}{a}}$

→ $L\{f(\frac{t}{a})\} = a\, [L\{f(t)\}]_{s \to as}$

1. If $L\{f(t)\} = \frac{9s^2-12s+15}{(s-1)^3}$ then find $L\{f(3t)\}$ using change of scale property.

Sol :- Given $L\{f(t)\} = \frac{9s^2-12s+15}{(s-1)^3} = \bar{f}(s)$

w.k.T

$L\{f(at)\} = \frac{1}{a} L\{f(t)\}_{s \to s/a}$

$L\{f(3t)\} = \frac{1}{3}\left[\frac{9s^2-12s+15}{(s-1)^3}\right]_{s \to s/3}$

$= \frac{1}{3}\left[\frac{9\frac{s^2}{9} - 12\frac{s}{3} + 15}{(\frac{s}{3}-1)^3}\right]$

$= \frac{1}{3}\left[\frac{s^2-4s+15}{\frac{(s-3)^3}{27}}\right]$

$= \frac{1}{3}\left[\frac{s^2-4s+15}{(s-3)^3}\right] \times 27$

$= \frac{9[s^2-4s+15]}{(s-3)^3}$

$=$

2. Find $L\{e^{-3t}\sinh 3t\}$ using change of scale property.

Sol :- By the change of scale property

$L\{f(at)\} = \frac{1}{a} L\{f(t)\}_{s \to s/a}$

$L\{\sinh 3t\} = \frac{1}{3}\left[L\{\sinh t\}\right]_{s \to s/3}$

$= \frac{1}{3}\left[\frac{1}{s^2-1}\right]_{s \to s/3}$

$= \frac{1}{3}\left[\frac{1}{(\frac{s}{3})^2-1}\right]$

$= \frac{1}{3}\left[\frac{1}{\frac{s^2}{9}-1}\right]$

$= \frac{1}{3}\left[\frac{1}{\frac{s^2-9}{9}}\right]$

$= \frac{1}{3} 9\left[\frac{1}{s^2-9}\right]$

$= \frac{3}{s^2-9}$

$L\{e^{-3t}\sinh 3t\} = \frac{3}{(s+3)^2-9} = \frac{3}{s^2+9+6s-9} = \frac{3}{s^2+6s}$

~~(By first shifting~~

$L\{e$ ~~$\sinh bt = \frac{b}{(s+a)^2-b^2}$~~

$=$

3. If $L\{f(t)\} = \frac{1}{s}e^{-1/s}$ ~~Prove that~~ then find $L\{e^{-t}f(3t)\}$

Sol :- Given $L\{f(t)\} = \frac{1}{s}e^{-1/s} = \bar{f}(s)$

By change of scale property

$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$L\{f(3t)\} = \frac{1}{3}\bar{f}\left(\frac{s}{3}\right) e^{-3/s}$$
$$= \frac{1}{\cancel{3}} \frac{\cancel{3}}{s} e^{-3/s}$$
$$= \frac{e^{-3/s}}{s}$$

Now $L\{e^{-t} f(3t)\} = \dfrac{e^{-3/(s+1)}}{(s+1)}$ (By 1st shifting theorem)

4. If $L\{f(t)\} = \dfrac{20-4s}{s^2-4s+20}$ then find $L\{f(3t)\}$ and $L\{e^{-t} f(2t)\}$

Sol:- Given $L\{f(t)\} = \dfrac{20-4s}{s^2-4s+20} = \bar{f}(s)$

By change of scale property

$L\{f(at)\} = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$

$$L\{f(3t)\} = \frac{1}{3}\bar{f}\left(\frac{s}{3}\right)$$
$$= \frac{20-4\left(\frac{s}{3}\right)}{\left(\frac{s}{3}\right)^2-4\left(\frac{s}{3}\right)+20} \cdot \frac{1}{3}$$
$$= \frac{1}{3} \cdot \frac{\frac{60-4s}{3}}{\frac{s^2}{9}-\frac{4s}{3}+20}$$
$$= \frac{1}{3} \cdot \frac{\frac{60-4s}{3}}{\frac{s^2-12s+180}{9}}$$
$$= \frac{1}{\cancel{3}} \frac{60-4s}{\cancel{3}} \times \frac{\cancel{9}}{s^2-12s+180} = \frac{60-4s}{s^2-12s+180}$$

ii) $L\{e^{-t} f(2t)\}$

$$L\{f(2t)\} = \frac{1}{2}\bar{f}\left(\frac{s}{2}\right)$$
$$= \frac{1}{2} \frac{20-\cancel{4}^{2}\frac{s}{\cancel{2}}}{\left(\frac{s}{2}\right)^2-\frac{4s}{2}+20}$$
$$= \frac{1}{2} \frac{20-2s}{\frac{s^2}{4}-2s+20} = \frac{1}{2} \frac{20-2s}{\frac{s^2-8s+80}{4}}$$
$$= \frac{1}{\cancel{2}} \frac{20-2s}{s^2-8s+80} \times \cancel{4}^{2}$$
$$= \frac{40-4s}{s^2-8s+80}$$

Now $L\{e^{-t} f(2t)\} = \dfrac{40-4(s+1)}{s^2-8(s+1)+80}$

Laplace transform of derivatives

Statement:- If $f(t)$ is continous and of exponential order and $f'(t)$ is piecewise continous then the Laplace transform of $f'(t)$ is given by

$L\{f'(t)\} = s\bar{f}(s) - f(0)$ where $\bar{f}(s) = L\{f(t)\}$

Proof:- By the definition $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Now

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d[f(t)]$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty s e^{-st} f(t)\,dt$$

$$= 0 - 1.f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L\{f(t)\}$$

$$= s\,L\{f(t)\} - f(0)$$

$$= s\bar{f}(s) - f(0)$$

Theorem:-

Prove that $L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$

Proof:-

$L\{f'(t)\} = s\bar{f}(s) - f(0)$

$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$

$L\{f''(t)\} = s\,L\{f'(t)\} - f'(0)$

$= s.[s\bar{f}(s) - f(0)] - f'(0)]$

$= s^2\bar{f}(s) - sf(0) - f'(0)$

$L\{f'''(t)\} = s\,L\{f''(t)\} - f''(0)\}$

$= s\,[s^2\bar{f}(s) - s\,f(0) - f'(0)] - f''(0)$

$= s^3\bar{f}(s) - s^2f(0) - sf'(0) - f''(0)$

$\vdots$

$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots f^{n-1}(0)$

1. Using the theorem of transformation of derivatives find the L.T of the following functions.

i) e^{at} ii) $\cos at$ iii) $t\sin at$

Sol:- Given

i) $f(t) = e^{at}$

$\Rightarrow f(0) = 1$

$f'(t) = e^{at} \cdot a$

Applying L.T on both sides

$L\{f'(t)\} = L\{a e^{at}\}$

$s L\{f(t)\} - f(0) = a L\{e^{at}\}$

$s L\{e^{at}\} - 1 = a L\{e^{at}\}$

$s L\{e^{at}\} - a L\{e^{at}\} = 1$

$L\{e^{at}\}(s-a) = 1$

$L\{e^{at}\} = \frac{1}{s-a}$

ii) $f(t) = \cos at$

$\Rightarrow f(0) = 1$

$f'(t) = (-\sin at) \cdot a = -a \sin at$

~~$L\{f'(t)\} = L\{a(-\sin at\}$~~ $\Rightarrow f'(0) = 0$

$f''(t) = -a \cos at \cdot a$

$= -a^2 \cos at$

Applying laplace transform on both sides

$L\{f''(t)\} = -L\{a^2 \cos at\}$

$\Rightarrow s^2 L\{f(t)\} - s f(0) - f'(0) = -L\{a^2 \cos at\}$

$\Rightarrow s^2 L\{\cos at\} - s(1) - 0 = -L\{a^2 \cos at\}$

$\Rightarrow s^2 L\{\cos at\} + a^2 L\{\cos at\} = s$

$\Rightarrow L\{\cos at\}(s^2+a^2) = s$

$\Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$

iii) $f(t) = t \sin at$

$\Rightarrow f(0) = 0$

$f'(t) = t \cos at (a) + \sin at (1)$

$= at \cos at + \sin at$

$\Rightarrow f'(0) = 0$

$f''(t) = at(-\sin at)(a) + \cos at(a) + \cos at(a)$

$= -a^2 t \sin at + a \cos at + a \cos at$

$= -a^2 t \sin at + 2a \cos at$

$L\{f''(t)\} = -a^2 L\{t \sin at\} + 2a L\{\cos at\}$

$s^2 L\{f(t)\} - s f(0) - f'(0) = -a^2 L\{t \sin at\} + 2a L\{\cos at\}$

$s^2 L\{t \sin at\} - s f(0) - f'(0) = -a^2 L\{t \sin at\} + 2a L\{\cos at\}$ $\left(\begin{array}{l} f'(0)=0 \\ f(0)=0 \end{array} \right.$

$s^2 L\{t \sin at\} + a^2 L\{t \sin at\} = 2a \frac{s}{s^2+a^2}$

$L\{t \sin at\}(s^2+a^2) = \frac{2as}{s^2+a^2}$

$L\{t \sin at\} = \frac{2as}{(s^2+a^2)^2}$ //

Q) If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$ then find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Sol:- Let $f(t) = \sin\sqrt{t} \Rightarrow f(0) = 0$

$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

$f(t) = \sin\sqrt{t}$
$f'(t) = \cos\sqrt{t} \cdot \frac{1}{2} t^{1/2 - 1}$
$= \frac{\cos\sqrt{t}}{2\sqrt{t}}$

$L\{f'(t)\} = s\bar{f}(s) - f(0)$

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = sL\{\sin\sqrt{t}\} - 0$

$\frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{s\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}$

$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-1/4s} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

Laplace Transform of Integrals

Theorem:- If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\bar{f}(s)$

Proof:- Let $g(t) = \int_0^t f(t)dt \Rightarrow g(0) = \int_0^0 f(0)dt = 0$

diff w.r.to 't'

$g'(t) = \frac{d}{dt}\int_0^t f(t)dt$

$\Rightarrow g'(t) = f(t)$

$L\{g'(t)\} = L\{f(t)\}$

$sL\{g(t)\} - g(0) = L\{f(t)\}$

$s L\left\{\int_0^t f(t)dt\right\} - 0 = \bar{f}(s)$

$L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\bar{f}(s)$

Note:-

If $L\{f(t)\} = \bar{f}(s)$ then

i) $L\left\{\int_0^t \int_0^t f(t)\,dt\,dt\right\} = \frac{1}{s^2}\bar{f}(s)$

ii) $L\left\{\int_0^t \int_0^t \int_0^t f(t)\,dt\,dt\,dt\right\} = \frac{1}{s^3}\bar{f}(s)$

iii) $L\{\underbrace{\int_0^t \int_0^t \cdots}_{n\text{ times}} f(t)\underbrace{dt\,dt \cdots dt}_{n\text{ times}}\} = \frac{1}{s^n}\bar{f}(s)$

Find $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$

Here $f(t) = e^{-t}\cos t$

$L\{f(t)\} = L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2} = \bar{f}(s)$

$\therefore L\left\{\int_0^t e^{-t}\cos t\,dt\right\} = \frac{1}{s}\bar{f}(s)$

$= \frac{1}{s}\left[\frac{s+1}{s^2+2s+2}\right]$

Multiplication by t^n :-

If $L\{f(t)\} = \bar{f}(s)$ then $L\{t \cdot f(t)\} = (-1)\frac{d}{ds}\bar{f}(s)$

then for t^n

$$L\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

Proof :-

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

Diff w.r.to s

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)dt$$

$$= \int_0^\infty e^{-st}(-t) f(t)dt$$

$$= -\int_0^\infty e^{-st} t \cdot f(t)dt$$

$$= (-1)\int_0^\infty e^{-st}\, t f(t)dt$$

$$= (-1)\, L\{t\, f(t)\}$$

$$\Rightarrow (-1)\frac{d}{ds}\bar{f}(s) = L\{t f(t)\}$$

By the method of mathematical Induction

it is true for $n=1$

Let us assume that the theorem is true for $n=k$

$$L\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k}\bar{f}(s)$$

$$\int_0^\infty e^{-st}\{t^k f(t)\}\,dt = (-1)^k \frac{d^k}{ds^k}\bar{f}(s)$$

diff w.r.to s

$$\frac{d}{ds}\int_0^\infty e^{-st} t^k f(t)\,dt = (-1)^k \frac{d}{ds}\frac{d^k}{ds^k}\bar{f}(s)$$

$$\int_0^\infty e^{-st}(-t)t^k f(t)\,dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$$

$$-\int_0^\infty e^{-st} t^{k+1} f(t)\,dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$$

$$\int_0^\infty e^{-st} t^{k+1} f(t)\,dt = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$$

$$L\{t^{k+1} f(t)\} = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$$

$\therefore$ The statement is true for $n = k+1$

Hence by Principle of mathematical Induction the theorem is true for all values of n

$$\therefore L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

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1. Evaluate $L\{t \sin at\}$

Sol:- $L\{t \sin at\}$ is in form of $L\{f(t)\}$

here $f(t) = \sin at$

$L\{f(t)\} = L\{\sin at\} = \frac{a}{s^2+a^2} = \bar{f}(s)$

$L\{t\, f(t)\} = (-1)\frac{d}{ds}\bar{f}(s)$

$= (-1)\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$

$= (-1)\left[\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2}\right] = \frac{2as}{(s^2+a^2)^2}$

2. Find $L\{t \sin 3t \cos 2t\}$

Sol:- Here $f(t) = \sin 3t \cos 2t$

$= \frac{1}{2}\, 2\sin 3t \cos 2t$

$= \frac{1}{2}[\sin 5t + \sin t]$

$(2\sin A \cos B = \sin(A+B) + \sin(A-B))$

$L\{f(t)\} = \frac{1}{2}[L\{\sin 5t\} + L\{\sin t\}]$

$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$

Now $L\{t\, f(t)\} = (-1)\frac{d}{ds}\bar{f}(s)$

$= (-1)\frac{d}{ds}\left[\frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]\right]$

$= (-1)\frac{1}{2}\left[5\frac{d}{ds}\left(\frac{1}{s^2+25}\right) + \frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$

$= -\frac{5}{2}\left[\frac{(s^2+25)(0) - 1(2s)}{(s^2+25)^2}\right] - \frac{1}{2}\left[\frac{(s^2+1)(0) - 2s}{(s^2+1)^2}\right]$

$= -\frac{5}{2}\left[\frac{-2s}{(s^2+25)^2}\right] - \frac{1}{2}\left[\frac{-2s}{(s^2+1)^2}\right]$

$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$

3. Find $L\{t^2 \sin 2t\}$

Sol:- here $f(t) = \sin 2t$

$L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4} = \bar{f}(s)$

W.K.T $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]$

$= \frac{d}{ds}\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right] = \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$

$= \frac{(s^2+4)^2(-4) + 4s\,2(s^2+4)(2s)}{(s^2+4)^4}$

$= \frac{-4(s^2+4)^2 + 16s^2(s^2+4)}{(s^2+4)^4}$

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$$\cancel{= s^2+4\left[-4+16s^2\right]}$$

$$= \frac{s^2+4\left[-4(s^2+4)+16s^2\right]}{(s^2+4)^4}$$

$$= \frac{-4s^2-16+16s^2}{(s^2+4)^3} = \frac{-16+12s^2}{(s^2+4)^3}$$

$$= 4\left[\frac{3s^2-4}{(s^2+4)^3}\right]$$

4. Find $L\left\{\int_0^t t e^{-t} \sin 2t \, dt\right\}$

Sol:- By using Laplace transform of integrals

$$L\left\{\int_0^t t e^{-t} \sin 2t \, dt\right\} = \frac{1}{s} L\left\{t e^{-t} \sin 2t\right\}$$

$$L\left\{e^{-t} \sin 2t\right\} = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

Now $L\left\{t e^{-t} \sin 2t\right\} = (-1)\frac{d}{ds} L\left\{e^{-t}\sin 2t\right\}$

$$= (-1)\frac{d}{ds}\left[\frac{2}{s^2+2s+5}\right]$$

$$= (-1)\left[\frac{s^2+2s+5(0) - 2(2s+2)}{(s^2+2s+5)^2}\right]$$

$$= (-1)\left[\frac{-4s-4}{(s^2+2s+5)^2}\right]$$

$$= \frac{4s+4}{(s^2+2s+5)^2} = \frac{4(s+1)}{(s^2+2s+5)^2}$$

$$\therefore L\left\{\int_0^t t e^{-t} \sin 2t \, dt\right\} = \frac{1}{s} L\left\{t e^{-t} \sin 2t\right\}$$

$$= \frac{1}{s}\left[\frac{4(s+1)}{(s^2+2s+5)^2}\right]$$

5. Find L.T of $t \cos at$

6. Find $L\{t e^{-t} \cosh t\}$

7. Find L.T of $t^2 e^{-2t} \cos 2t$

Sol:- Let $f(t) = e^{-2t}\cos 2t$

$$L\{f(t)\} = L\{e^{-2t}\cos 2t\} = \frac{s+2}{(s+2)^2+4} = \frac{s+2}{s^2+4s+8}$$

$$L\{t^2 e^{-2t}\cos 2t\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s+2}{s^2+4s+8}\right]$$

$$= (1)\frac{d}{ds}\left[\frac{(s^2+4s+8)(1) - (s+2)(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+4s+8 - 2s^2 - 8}{(s^2+4s+8)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2+4s}{(s^2+4s+8)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2+4s}{(s^2+4s+8)^2}\right]$$

$$= \frac{(s^2+4s+8)^2(-2s+4) - (-s^2+4s)\, 2(s^2+4s+8)(2s+4)}{(s^2+4s+8)^{4}}$$

$$= \frac{\cancel{(s^2+4s+8)}\left[(s^2+4s+8)(-2s+4) - (-s^2+4s)\,2(2s+4)\right]}{(s^2+4s+8)^{3}}$$

$$= \frac{-2s^3-4s^2-16s+4s^2+16s+32-(-4s^3+16s^2-8s^2+32s)}{(s^2+4s+8)^3}$$

$$= \frac{-2s^3-\cancel{4s^2}-\cancel{16s}+\cancel{4s^2}+\cancel{16s}+32+4s^3-16s^2+8s^2-32s}{(s^2+4s+8)^3}$$

$$= \frac{2s^3-8s^2-32s+32}{(s^2+4s+8)^3} = \frac{2(s^3-4s^2-16s+16)}{(s^2+4s+8)^3}$$

Division by t :-

Theorem :- If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

Proof :- W.K.T

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

Now $\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty L\{f(t)\}\,ds$

$$= \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\,ds\,dt$$

$$= \int_{t=0}^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt$$

$$= \int_{t=0}^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$= \int_{t=0}^\infty \left[\frac{e^{-\infty}-e^{-st}}{-t}\right]\cdot f(t)\,dt$$

$$= \int_{t=0}^\infty \frac{0-e^{-st}}{-t} f(t)\,dt$$

$$= \int_{t=0}^\infty \frac{e^{-st}}{t} f(t)\,dt$$

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$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Hence $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds$

1. Find $L\left\{\frac{\sin t}{t}\right\}$

Sol:- $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{\sin t\}\, ds$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$(90+\theta)$ $(180-\theta)$ $\sin$	$(90-\theta)$ All $(360+\theta)$
Tan $(270-\theta)$ $(180+\theta)$	Cos $(360-\theta)$ $(270+\theta)$

$(\because \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2})$

(or)

Put $\frac{\pi}{2} - \tan^{-1} s = x$

$\frac{\pi}{2} - x = \tan^{-1} s$

$\tan(\frac{\pi}{2} - x) = s$

$\cot x = s$

$x = \cot^{-1} s$

2. Find $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

Sol:- $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty L\{e^{-at} - e^{-bt}\}\, ds$

$$= \int_s^\infty L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log \frac{s+a}{s+b} - \log\left(\frac{s+a}{s+b}\right)$$

$$= \lim_{s\to\infty} \log \frac{\cancel{s}(1+\frac{a}{s})}{\cancel{s}(1+\frac{b}{s})} - \log\left(\frac{s+a}{s+b}\right)$$

$$= \lim_{s\to\infty} \log \left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right] - \log\left(\frac{s+a}{s+b}\right) \qquad (\because \frac{a}{\infty} = \frac{b}{\infty} = 0)$$

$$= \log(1) - \log\left(\frac{s+a}{s+b}\right)$$

$$= 0 - [\log(s+a) - \log(s+b)\}$$

$$= \log(s+b) - \log(s+a)$$

$$= \log\left[\frac{s+b}{s+a}\right]$$

3. Find $L\left\{\frac{\sin 3t \cos t}{t}\right\}$

Sol:- $L\left\{\frac{\sin 3t \cos t}{t}\right\} = \int_s^\infty L\{\sin 3t \cos t\}\, dt$

$L\{\sin 3t \cos t\} = \frac{1}{2} L\{2 \sin 3t \cos t\}$

$$= \frac{1}{2} L\{\sin 4t + \sin 2t\}$$

$$= \frac{1}{2}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right]$$

$\therefore L\left\{\frac{\sin 3t \cos t}{2}\right\} = \int_s^\infty \frac{1}{2}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] ds$

$= \int_s^\infty \left(\frac{2}{s^2+16} + \frac{1}{s^2+4}\right) ds$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$= \int_s^\infty \left(\frac{2}{s^2+4^2} + \frac{1}{s^2+2^2}\right) ds$

$= \left[2\,\frac{1}{4}\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \left(\frac{1}{2}\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2}\right) - \left[\frac{1}{2}\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]$

$= \left(\frac{\pi}{4} + \frac{\pi}{4}\right) - \frac{1}{2}\left(\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right)$

$= \frac{\pi}{2} - \frac{1}{2}\left(\tan^{-1}\left(\frac{s}{4}\right)\right) - \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)$

4. Find $L\left\{\frac{1-e^t}{t}\right\}$

5. Find $L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\}$

Sol:- $L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\} = \frac{1}{s} L\left\{\frac{e^t \sin t}{t}\right\}$

$= \frac{1}{s} L\left\{e^t \frac{\sin t}{t}\right\}$

$= \frac{1}{s}\left[L\left\{\frac{\sin t}{t}\right\}\right]_{s \to s-1}$

$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,ds$

$= \int_s^\infty \frac{1}{s^2+1}\,ds$

$= [\tan^{-1}(s)]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$

$\therefore L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\} = \frac{1}{s}\left[\cot^{-1}s\right]_{s \to s-1}$

$= \frac{1}{s}\left[\cot^{-1}(s-1)\right]$

(or)

Sol:- $L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\} = \frac{1}{s} L\left\{\frac{e^t \sin t}{t}\right\}$

$= \frac{1}{s} L\left\{e^t \frac{\sin t}{t}\right\}$

$L\left\{e^t \frac{\sin t}{t}\right\} = \int_s^\infty L\{e^t \sin t\}\,ds$

$= \int_s^\infty \frac{1}{(s-1)^2+1}\,ds$

$= [\tan^{-1}(s-1)]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}(s-1)$

$= \cot^{-1}(s-1)$

$\therefore L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}(s-1)$

Evaluation of Integrals by Laplace Transforms :-

Sometimes Evaluation of improper integrals i.e., integrals having lower limit 0 and upper limit ∞ can be done easily by using Laplace transform technique

1. Using Laplace transform Evaluate $\int_0^\infty t e^{-t} \sin t\, dt$

Sol :- We know that $\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$

$\therefore \int_0^\infty t e^{-t} \sin t\, dt = L\{t \sin t\}$ where $s=1$

$= (-1)\frac{d}{ds} L\{\sin t\}$

$= (-1)\frac{d}{ds}\frac{1}{s^2+1}$

$= -\left[\frac{(s^2+1)0 - 1(2s)}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$

$= \frac{2s}{s^4+2s^2+1}$

$(\because s=1)$

$= \frac{2}{1+2+1}$

$= \frac{2}{4} = \frac{1}{2}$

2. Using L.T Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt$

Sol :- W.K.T $\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$

$\therefore \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\}$ where $s=0$

$= \int_s^\infty L\{e^{-t}-e^{-2t}\}\,ds$

$= \int_s^\infty \frac{1}{s+1} - \frac{1}{s+2}\,ds$

$= \left[\log(s+1) - \log(s+2)\right]_s^\infty$

$= \left[\log\left(\frac{s+1}{s+2}\right)\right]_s^\infty$

$= \lim_{s\to\infty} \log\left(\frac{s+1}{s+2}\right) - \log\left(\frac{s+1}{s+2}\right)$

$= \lim_{s\to\infty} \log\left(\frac{\cancel{s}(1+\frac{1}{s})}{\cancel{s}(1+\frac{2}{s})}\right) - \log\left(\frac{s+1}{s+2}\right)$

$= \log 1 - \log\left(\frac{s+1}{s+2}\right)$

$= -\left[\log(s+1) - \log(s+2)\right]$

$= \log(s+2) - \log(s+1) = \log\left(\frac{s+2}{s+1}\right)$

where $s=0$

$= \log 2$

3. Evaluate $\int_0^\infty e^{-4t}\left[\frac{2\sin t - 3\sinh t}{t}\right]dt$

Sol :- W.K.T $\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$

$\int_0^\infty e^{-4t}\left[\frac{2\sin t - 3\sinh t}{t}\right] = L\left\{\frac{2\sin t - 3\sinh t}{t}\right\}$

$= \int_s^\infty L\{2\sin t - 3\sinh t\}\,ds$

$$= \int_s^\infty \left[2\frac{1}{s^2+1} - 3\frac{1}{s^2-1}\right]ds \qquad \left[\because \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\left(\frac{x-a}{x+a}\right)\right|\right]$$

$$= \int_s^\infty \left(\frac{2}{s^2+1} - \frac{3}{s^2-1}\right)ds$$

$$= 2\left[\tan^{-1}(s)\right]_s^\infty - 3\left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$= 2\left[\tan^{-1}(\infty) - \tan^{-1}(s)\right] - \frac{3}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$= 2\left[\frac{\pi}{2} - \tan^{-1}(s)\right] - \frac{3}{2}\left[\lim_{s\to\infty}\log\left(\frac{s-1}{s+1}\right) - \log\left(\frac{s-1}{s+1}\right)\right]$$

$$= 2\left[\frac{\pi}{2} - \tan^{-1}(s)\right] - \frac{3}{2}\left[\lim_{s\to\infty}\log\left[\frac{s(1-\frac{1}{s})}{s(1+\frac{1}{s})}\right] - \log\left(\frac{s-1}{s+1}\right)\right]$$

$$= 2\left[\frac{\pi}{2} - \tan^{-1}(s)\right] - \frac{3}{2}\left[\log(1) - \log\left(\frac{s-1}{s+1}\right)\right]$$

where $s = 4$

$$= 2\left(\frac{\pi}{2} - \tan^{-1}(4)\right] - \frac{3}{2}\left[-\log\left[\frac{3}{5}\right]\right]$$

$=$

4. $\int_0^\infty t^2 e^{-4t}\sin 2t\, dt$

Laplace transforms of periodic functions

Periodic function:- A function $f(x)$ is said to be periodic P iff $f(x+P) = f(x)$ where P is least +ve integer.

Ex:- 1. $\sin x$ and $\cos x$ are periodic functions with period 2π

2. $\tan x$ and $\cot x$ are periodic functions with period π

3. $|\sin x|$ and $|\cos x|$ are periodic functions with period 2π

Note:- If P is a period of $f(x)$ then the period of $f(ax+b)$ is $\frac{P}{a}$

Theorem:- If $f(t)$ is a periodic function with period T then $L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$

Proof:- By the def of L.T

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

$$= \int_0^T e^{-st}f(t)dt + \int_T^{2T} e^{-st}f(t)dt + \int_{2T}^{3T} e^{-st}f(t)dt + \cdots\cdots$$

Put $t = u+T$, $dt = du$; If $t=T$ then $u=0$, $t=2T$ then $u=T$

Put $t = u+2T$, $dt = du$; If $t=2T \Rightarrow u=0$, $t=3T \Rightarrow u=T$

$$= \int_0^T e^{-st}f(t)\,dt + \int_0^T e^{-s(u+T)}f(u+T)\,du + \int_0^T e^{-s(u+2T)}f(u+2T)du + \cdots$$

$$= \int_0^T e^{-st}f(t)dt + \int_0^T e^{-su}e^{-sT}f(u+T)\,du + \int_0^T e^{-su}e^{-2Ts}f(u+2T)du + \cdots$$

$$= \int_0^T e^{-st}f(t)dt + e^{-sT}\int_0^T e^{-su}f(u+T)du + e^{-2Ts}\int_0^T e^{-su}f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2Ts}\int_0^{T} e^{-su} f(u)\,du \ldots$$

$[\because f(t)$ is periodic with period T
i.e., $f(t) = f(t+T) = f(t+2T) \ldots]$

replacing u by t

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \ldots$$

$$= [1 + e^{-sT} + e^{-2sT} + \ldots]\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Hence $\boxed{L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$

Hence the theorem is proved

1. Find $L\{f(t)\}$ whose $f(t)$ is a periodic function of period 2π and it is given by
$f(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$

Sol:- W.K.T $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-s(2\pi)}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s\pi}} \left[\int_0^{\pi} e^{-st} f(t)\,dt + \int_{\pi}^{2\pi} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} (0)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1^2}[(-s)\sin t - \cos t]\right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1}(0-(-\cos\pi)) - \frac{e^0}{s^2+1}(0-\cos 0)\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1}(1) - \frac{e^0}{s^2+1}(-1)\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1}\right] = \frac{e^{-s\pi}+1}{(s^2+1)(1-e^{-2\pi s})} //$$

2. Find $L\{f(t)\}$ whose $f(t) = \begin{cases} t & 0<t<b \\ 2b-t & b<t<2b \end{cases}$
$2b$ being the period of $f(t)$.

Sol:- Given $f(t)$ is a periodic function with period $2b$
i.e., $T = 2b$

W.K.T $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-s(2b)}} \int_0^{2b} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s(2b)}} \left[\int_0^b e^{-st}\, t\, dt + \int_b^{2b} e^{-st}(2b-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^b + \left[(2b-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_b^{2b} \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left[\left(b\frac{e^{-sb}}{-s} - \frac{e^{-sb}}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right] + \left[\left(0 + \frac{e^{-2bs}}{s^2} \right) - \frac{be^{-sb}}{-s} + \frac{e^{-sb}}{s^2} \right] \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ -\cancel{\frac{be^{-sb}}{s}} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2} + \frac{e^{-2sb}}{s^2} + \cancel{\frac{be^{-sb}}{s}} - \frac{e^{-sb}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{1}{s^2} - \frac{2e^{-sb}}{s^2} + \frac{e^{-2bs}}{s^2} \right]$$

3. H.W If $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases}$ is a periodic function with period 2 Find its L.T

Inverse Laplace Transforms

The Inverse Laplace Transforms are useful for solving differential Equations without finding general solutions and arbitrary constants

We know that

$$L\{f(t)\} = \bar{f}(s)$$

$$\Rightarrow f(t) = L^{-1}\{\bar{f}(s)\}$$

Where $f(t)$ is called inverse Laplace transform of $\bar{f}(s)$ and the symbol L^{-1} stands for the Inverse Laplace transform operator.

Standard formulae :-

1) $L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\{\frac{1}{s}\} = 1$

2) $L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$

3) $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$

4) $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$

5) $L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L\{\frac{1}{s^2-a^2}\} = \frac{\sinh at}{a}$

6) $L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$

7) $L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\{\frac{1}{s-a}\} = e^{at}$

8) $L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\{\frac{1}{s+a}\} = e^{-at}$

Note:- **Linearity property**

$L^{-1}\{c_1 \bar{f}(s) + c_2 \bar{g}(s)\} = c_1 L^{-1}\{\bar{f}(s)\} + c_2 L^{-1}\{\bar{g}(s)\}$

where c_1 and c_2 are constants and $\bar{f}(s)$, $\bar{g}(s)$ are Laplace transforms of $f(t)$ and $g(t)$ respectively.

1) Find $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$

Sol:- $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right\}$

$= L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

$= L^{-1}\{\frac{1}{s}\} - 3L^{-1}\{\frac{1}{s^2}\} + 4L^{-1}\{\frac{1}{s^3}\}$

$= 1 - 3\frac{t}{1!} + 4\frac{t^2}{2!}$

$= 1 - 3t + \cancel{4}^{2}\frac{t^2}{\cancel{2}}$

$= 1 - 3t + 2t^2$

2) Find $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$

Sol:- $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\} = L^{-1}\left\{\frac{3(s^4+4-4s^2)}{2s^5}\right\}$

$= L^{-1}\left\{\frac{3s^4}{2s^5} + \frac{12}{2s^5} - \frac{12s^2}{2s^5}\right\}$

$= L^{-1}\left\{\frac{3}{2s} + \frac{6}{s^5} - \frac{6}{s^3}\right\}$

$= L^{-1}\{\frac{3}{2s}\} + L^{-1}\{\frac{6}{s^5}\} - L^{-1}\{\frac{6}{s^3}\}$

$= \frac{3}{2}L^{-1}\{\frac{1}{s}\} + 6L^{-1}\{\frac{1}{s^5}\} - 6L^{-1}\{\frac{1}{s^3}\}$

$= \frac{3}{2}(1) + 6\left(\frac{t^4}{4!}\right) - 6\left(\frac{t^2}{2!}\right)$

$= \frac{3}{2} + \frac{6t^4}{24} - \frac{6t^2}{2}$

$= \frac{3}{2} + \frac{t^4}{4} - 3t^2$

Inverse Laplace Transform by using Partial fractions:-

1. Find the inverse L.T of $\frac{4}{(s+1)(s+2)}$

Sol:- Let $\bar{f}(s) = \frac{4}{(s+1)(s+2)}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\}$

$= 4L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} \rightarrow (1)$

$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \rightarrow (2)$

$\frac{1}{\cancel{(s+1)(s+2)}} = \frac{A(s+2)+B(s+1)}{\cancel{(s+1)(s+2)}}$

$1 = A(s+2) + B(s+1)$

$As + 2A + Bs + B = 1$

Comparing coefficients

$$A+B=0$$
$$2A+B=1$$
$$-A=-1$$
$$A=1$$

$$A+B=0$$
$$1+B=0 \Rightarrow B=-1$$

from (2)

$$\therefore \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

from (1)

$$L^{-1}\{\bar{f}(s)\} = 4\,L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\}$$
$$= 4\left[L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}\right]$$
$$= 4\left[e^{-t} - e^{-2t}\right]$$

2. Find $L^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$

Sol:- Given $L^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$s^2+s-2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3)$$
$$s^2+s-2 = A[s^2-2s+3s-6] + B(s^2-2s) + C(s^2+3s)$$
$$s^2+s-2 = As^2 + As - 6A + Bs^2 - 2Bs + Cs^2 + 3Cs$$

Comparing coefficients

$$A+B+C = 1 \rightarrow (1)$$
$$A-2B+3C = 1 \rightarrow (2)$$
$$-6A = -2 \rightarrow (3)$$
$$\Rightarrow A = \frac{1}{3}$$

Solving (1) and (2)

$$(1)\times 3 \Rightarrow 3A+3B+3C = 3$$
$$(2) \Rightarrow A-2B+3C = 1$$
$$2A+5B = 2$$

$$\frac{2}{3}+5B = 2$$
$$5B = 2-\frac{2}{3} = \frac{4}{3}$$
$$B = 4/15$$

from (1)

$$\frac{1}{3}+\frac{4}{15}+C = 1$$
$$\frac{5+4}{15}+C = 1$$
$$\frac{3}{5}+C = 1$$
$$C = 1-\frac{3}{5} = \frac{2}{5}$$

$$\therefore L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\} = L^{-1}\left\{\frac{1}{3s} + \frac{4}{15(s+3)} + \frac{2}{5(s-2)}\right\}$$
$$= \frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{15}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{s-2}\right\}$$
$$= \frac{1}{3}(1) + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$
$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

3. $L^{-1}\left\{\frac{s^2+2s-4}{(s^2+9)(s-5)}\right\}$

Sol:- $\frac{s^2+2s-4}{(s^2+9)(s-5)} = \frac{As+B}{(s^2+9)} + \frac{C}{(s-5)}$

$s^2+2s-4 = (As+B)(s-5) + C(s^2+9)$

$(s^2+2s-4) = As^2+Bs-5As-5B+Cs^2+9C$

Comparing coefficients

$A+C=1 \rightarrow (1)$

$B-5A=2 \rightarrow (2)$

$9C-5B=-4 \rightarrow (3)$

solve (1) and (2)

(1) $\times 5 \Rightarrow$ $\cancel{5A}+5C=5$

(2) $\Rightarrow$ $\cancel{-5A}+B=2$

$B+5C=7 \rightarrow (4)$

from (3) & (4)

(3) $\Rightarrow$ $9C-\cancel{5B}=-4$

(4) $\times 5 \Rightarrow 25C+\cancel{5B}=35$

$34C = 31$

$C = 31/34$

from (1)

$A+C=1$

$A+\frac{31}{34}=1$

$A=1-\frac{31}{34}=\frac{3}{34}$

from (2)

$B-5A=2$

$B-5\left(\frac{3}{34}\right)=2$

$B-\frac{15}{34}=2$

$B=2+\frac{15}{34}=\frac{83}{34}$

$\therefore L^{-1}\left\{\frac{s^2+2s-4}{(s^2+9)(s-5)}\right\} = L^{-1}\left\{\frac{As+B}{s^2+9}+\frac{C}{s-5}\right\}$

$= L^{-1}\left\{\frac{\frac{3}{34}s+\frac{83}{84}}{s^2+9}\right\} + L^{-1}\left\{\frac{\frac{31}{34}}{s-5}\right\}$

$= \frac{3}{34}L^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{83}{84}\left\{\frac{1}{s^2+9}\right\} + \frac{31}{34}L^{-1}\left\{\frac{1}{s-5}\right\}$

$= \frac{3}{34}L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{83}{84}\left\{\frac{1}{s^2+3^2}\right\} + \frac{31}{34}L^{-1}\left\{\frac{1}{s-5}\right\}$

$= \frac{3}{34}(\cos 3t) + \frac{83}{84}\left(\frac{\sin 3t}{3}\right) + \frac{31}{84}(e^{5t})$

$=$

4. Find $L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$

Sol:- $\frac{s^2}{(s^2+4)(s^2+9)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+9)}$

$s^2 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$

$s^2 = As^3+Bs^2+9As+9B+Cs^3+Ds^2+4Cs+4D$

$A+C=0 \rightarrow (1)$

$B+D=1 \rightarrow (2)$

$9A+4C=0 \rightarrow (3)$

$9B+4D=0 \rightarrow (4)$

solving (1) & (3)

$(1)\times 9 \Rightarrow 9A+9C=0$

$\cancel{9A}+4C=0$

$\underset{(-)}{} \quad \underset{(-)}{}$

$5C=0 \Rightarrow C=0$

solving (2) & (4)

$(2)\times 9 \Rightarrow 9B+9D=9$

$\cancel{9B}+4D=0$

$\underset{(-)}{} \quad \underset{(-)}{}$

$5D=9$

$D=9/5$

from (1)

$A+C=0 \Rightarrow A=0$

from (2)

$B+D=1 \Rightarrow B+\frac{9}{5}=1 \Rightarrow B=1-\frac{9}{5}=-\frac{4}{5}$

$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\} = L^{-1}\left\{\frac{(0)s+\left(-\frac{4}{5}\right)}{s^2+4}\right\} + L^{-1}\left\{\frac{(0)s+\left(\frac{9}{5}\right)}{s^2+9}\right\}$

$= -\frac{4}{5}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} + \frac{9}{5}L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$

$= -\frac{\cancel{4}^{2}}{5}\frac{\sin 2t}{\cancel{2}_{1}} + \frac{\cancel{9}^{3}}{5}\frac{\sin 3t}{\cancel{3}_{1}}$

$= -\frac{2}{5}\sin 2t + \frac{3}{5}\sin 3t$

5. Find $L^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$ [(Hint):- $s^2+4s+5=(s+2)^2+1$]

First shifting theorem :-

If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\{\bar{f}(s-a)\}=e^{at}f(t)$

Ily $L^{-1}\{\bar{f}(s+a)\}=e^{-at}f(t)$.

Problems :-

1. Find $L^{-1}\left\{\frac{s+2}{(s-2)^3}\right\}$ using first shifting theorem.

Sol :- $\frac{s+2}{(s-2)^3} = \frac{s+2-2+2}{(s-2)^3} = \frac{(s-2)+4}{(s-2)^3}$

$L^{-1}\left\{\frac{s+2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{(s-2)+4}{(s-2)^3}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^3}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$

$$= e^{2t}\, L^{-1}\left\{\frac{s}{s^3}\right\} + 4e^{2t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{2t}\, L^{-1}\left\{\frac{1}{s^2}\right\} + 4e^{2t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{2t}\frac{t}{1!} + 4e^{2t}\frac{t^2}{2!}$$

$$= te^{2t} + 2e^{2t}t^2 = e^{2t}\left[t + 2t^2\right]$$

2. Find $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol:- $\frac{s}{(s+3)^2} = \frac{s+3-3}{(s+3)^2} = \frac{s+3}{(s+3)^2} - \frac{3}{(s+3)^2}$

$$L^{-1}\left\{\frac{s}{(s+3)^2}\right\} = L^{-1}\left\{\frac{s+3}{(s+3)^2}\right\} - 3L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{s^2}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\}e^{-3t}$$

$$= e^{-3t}L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\}e^{-3t}$$

$$= e^{-3t}(1) - 3e^{-3t}\frac{t}{1!}$$

$$= e^{-3t}\left[1 - 3t\right]$$

3. Find $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$ using first shifting theorem.

Sol:- $s^4 + 4a^4 = (s^2)^2 + (2a^2)^2$

$= (s^2+2a^2)^2 - 2s^2 2a^2$

$= (s^2+2a^2)^2 - (2as)^2$

$= (s^2+2a^2+2as)(s^2+2a^2-2as)$

Now $\frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$

$s = \frac{As+B}{s^2+2a^2+2as} + \frac{Cs+D}{s^2+2a^2-2as}$

$s = (As+B)(s^2+2a^2-2as) + (Cs+D)(s^2+2a^2+2as)$

$s = As^3 + Bs^2 + 2a^2As + 2a^2B - 2as\,As - 2as\,B$
$+ Cs^3 + Ds^2 + 2a^2Cs + 2a^2D + 2asCs + 2asD$

Comparing coefficients

$A + C = 0 \rightarrow (1) \Rightarrow A = -C$

$~~B + D = 0 \rightarrow (2)~~$

$-2aA + B + 2aC + D = 0 \rightarrow (2)$

sub $A = -C$

we have

$2aC + B + 2aC + D = 0$

$\Rightarrow 4aC + B + D = 0 \rightarrow (3)$

$2Aa^2 - 2aB + 2a^2C + 2aD = 1 \rightarrow (4)$

sub $A = -C$

$-2Ca^2 - 2aB + 2Ca^2 + 2aD = 1 \Rightarrow -2Ba + 2Da = 1 \rightarrow (5)$

Comparing Coefficients of s^3

$A+C=0 \rightarrow (1)$

we have $A=-C$

Comparing Coefficients of s^2

$-2Aa+B+2aC+D=0$ $(\because A=-C)$

$2aC+B+2aC+D=0$

$4aC+B+D=0 \rightarrow (2)$

Comparing Coefficients of s

$2Aa^2-2aB+2a^2C+2aD=1$

sub $A=-C$

$-2\cancel{C}a^2-2aB+2\cancel{C}a^2+2aD=1$

$-2Ba+2Da=1 \rightarrow (3)$

Comparing Constants

$2Ba^2+2Da^2=0$

$a^2(2B+2D)=0$

$2a^2(B+D)=0$

$B+D=0 \rightarrow (4)$

$\Rightarrow B=-D$

frm (3)

$2Da+2Da=1$

$4Da=1$

$D=1/4a$

$\therefore B=-1/4a$

frm (2)

$4aC+B+D=0$ $(\because B=-D)$

$\Rightarrow 4aC=0$

$\Rightarrow C=0$

frm (1)

$A+C=0$

$\Rightarrow A=-C \Rightarrow A=0$

$$\therefore \frac{s}{s^4+4a^4} = \frac{-1/4a}{s^2+2a^2+2as} + \frac{1/4a}{s^2+2a^2-2as}$$

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{-1}{4a} L^{-1}\left\{\frac{1}{s^2+2a^2+2as}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2+2a^2-2as}\right\}$$

$$= -\frac{1}{4a} L^{-1}\left\{\frac{1}{s^2+2as+a^2+a^2}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2-2as+a^2+a^2}\right\}$$

$$= -\frac{1}{4a} L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$= -\frac{1}{4a} e^{-at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} + \frac{1}{4a} e^{at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{-1}{4a} e^{-at} \frac{\sin at}{a} + \frac{1}{4a} e^{at} \frac{\sin at}{a}$$

$$= \frac{1}{4a} \frac{\sin at}{a}\left[e^{at}-e^{-at}\right]$$

$$= \frac{\sin at}{2a^2}\left[\frac{e^{at}-e^{-at}}{2}\right] \quad \left(\because \frac{e^x-e^{-x}}{2}=\sinh x\right)$$

$$= \frac{\sin at}{2a^2}\left[\sinh at\right] //$$

4. Find $L^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$

Sol:- $L^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = L^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\}$

$= L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$

$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2e^{-2t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= e^{-2t}\cos t - 2e^{-2t}\frac{\sin t}{1}$

$= e^{-2t}[\cos t - 2\sin t]$

5. Find $L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\}$

Sol:- $\frac{s+3}{s^2-10s+29} = \frac{s+3}{s^2-10s+25+4}$

$= \frac{s+3}{(s-5)^2+4}$

$L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\} = L^{-1}\left\{\frac{s+3}{(s-5)^2+4}\right\}$

$= L^{-1}\left\{\frac{s+3-5+5}{(s-5)^2+4}\right\}$

$= L^{-1}\left\{\frac{(s-5)+8}{(s-5)^2+4}\right\}$

$= L^{-1}\left\{\frac{(s-5)}{(s-5)^2+4}\right\} + 8L^{-1}\left\{\frac{1}{(s-5)^2+4}\right\}$

$= e^{5t}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 8e^{5t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$

$= e^{5t}\cos 2t + \cancel{8}^{4}e^{5t}\frac{\sin 2t}{\cancel{2}}$

$= e^{5t}[\cos 2t + 4\sin 2t]$

H.W

Find $L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$

Second shifting theorem:-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{e^{-as}\bar{f}(s)\} = G(t)$

where $G(t) = \begin{cases} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$

Problems:-

1) Find $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$ using second shifting theorem.

Sol:- $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} \rightarrow (1)$

~~= sint~~ +

Now $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

here $a=-\pi$ $\quad \bar{f}(s) = \frac{1}{s^2+1}$

$\therefore L^{-1}\{\bar{f}(s)\} = \frac{1}{s^2+1} = \sin t = f(t)$

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$\therefore f(t) = \sin t$

$f(t-a) = \sin(t-a)$ and we know that

$G(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$

hence

$G(t) = \begin{cases} \cancel{f(t-\pi)}\ \sin(t-\pi) & \cancel{\text{if } t > \pi} \\ 0 & \cancel{\text{if } t < \pi} \end{cases}$

$\therefore$ from ①

$$L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$$

$$= \sin t + G(t)$$

where $G(t) = \begin{cases} \sin(t-\pi) & \text{if } t > \pi \\ 0 & \text{if } t < \pi \end{cases}$

2. Find $L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\}$

Sol:- $\cancel{L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\}}$

$\cancel{= e^{-4t} L^{-1}\left\{\frac{e^4 \cdot e^{-3s}}{s^{5/2}}\right\}}$

$$L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^4 L^{-1}\left\{e^{-3s} \cdot \frac{1}{(s+4)^{5/2}}\right\}$$

Sol:- Let $\bar{f}(s) = \frac{1}{(s+4)^{5/2}}$ $\quad \left(\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}\right)$

$$L^{-1}\{\bar{f}(s)\} = e^{-4t} L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$= e^{-4t} \frac{t^{3/2}}{\Gamma(\frac{3}{2}+1)}$$

$$= e^{-4t} \frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})} = e^{-4t}\frac{4}{3}\frac{t^{3/2}}{\sqrt{\pi}} = f(t)$$

Now $L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^4 L^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\} \rightarrow$ ①

~~here a=3~~

~~Consider~~ $\cancel{L^{-1}\left\{e^{-3s}\frac{1}{(s+4)^{5/2}}\right\} = e^4 \{ e^{-4(t-3)}}$

Consider $L^{-1}\left\{e^{-3s}\frac{1}{(s+4)^{5/2}}\right\}$

here $a = 3$ $\quad \bar{f}(s) = \frac{1}{(s+4)^{5/2}}$

w.k.T $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\} = e^{-4t}\frac{4}{3}\frac{t^{3/2}}{\sqrt{\pi}} = f(t)$

$\therefore$ from ①

$$L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = \begin{cases} e^4\, e^{-4(t-3)}\,\frac{4}{3}\frac{(t-3)^{3/2}}{\sqrt{\pi}} & \text{if } t > 3 \\ 0 & \text{if } t < 3 \end{cases}$$

Note:- If $f(t) = L^{-1}\{\bar{f}(s)\}$ then $L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)\,H(t-a)$
(another form)
($H(t-a)$ is unit step function)

3. Find $L^{-1}\left\{\frac{3+5s}{s^2e^{2s}}\right\}$

Let $\bar{f}(s) = \frac{3+5s}{s^2} = \frac{3}{s^2}+\frac{5}{s}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{3}{s^2}+\frac{5}{s}\right\} = 3\left[L^{-1}\left\{\frac{1}{s^2}\right\}\right]+5L^{-1}\left\{\frac{1}{s}\right\}$

$= 3t+5 = f(t)$

Now $L^{-1}\left\{\frac{3+5s}{s^2e^{2s}}\right\} = L^{-1}\left\{e^{-2s}\,\frac{3+5s}{s^2}\right\}$ here $a=2$

$= f(t-a)\,H(t-a)$

~~$= 3(t-2)\,H(t-2)$~~

$= [3(t-2)+5]\,H(t-2)$

$= (3t-6+5)\,H(t-2)$

$= (3t-1)\,H(t-2)$

(or)

$L^{-1}\left\{\frac{3+5s}{s^2e^{2s}}\right\} = L^{-1}\left\{e^{-2s}\,\frac{3+5s}{s^2}\right\}$

$= \begin{cases} 3(t-3)+5 & \text{if } t>3 \\ 0 & \text{if } t<3 \end{cases}$

$= \begin{cases} 3t-1 & \text{if } t>3 \\ 0 & \text{if } t<3 \end{cases}$

The problems ① & ② can also be expressed in the another form of 2nd shifting theorem

Problem 1:- $f(t) = \sin t$
$f(t-a) = \sin(t-a)$ here $a=-\pi$
$f(t-\pi) = \sin(t-\pi)$

$L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = f(t-a)\,H(t-a)+\sin t$
$= f(t-\pi)\,H(t-\pi) = \sin(t-\pi)\,H(t-\pi)+\sin t$

Problem 2:- $f(t) = e^{-4t}\,\frac{4}{3}\,\frac{t^{3/2}}{\sqrt{\pi}}$

Now $L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^4\cdot f(t-a)\,H(t-a)$ here $a=3$

$= e^4\cdot e^{-4(t-3)}\,\frac{4}{3}\,\frac{(t-3)^{3/2}}{\sqrt{\pi}}\,H(t-3)$

H.W Find $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Change of scale property:-

If $f(t) = L^{-1}\{\bar{f}(s)\}$ then $L^{-1}\{\bar{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right)$

Problems:-

1. If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$ then find $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol:- we have $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t = f(t)$ say

writing as for s

$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2}\,\frac{1}{a}\,\frac{t}{a}\sin\frac{t}{a}$ (by c.s.p)

$= \frac{t}{2a^2}\sin\frac{t}{a}$

Put $a=2$

$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{t}{8}\sin\frac{t}{2}$

$\Rightarrow L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{t}{2}\sin\frac{t}{2}$

2. If $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ find $L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}$

Sol:- we have $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}} = f(t)$

writing ks for s

$$L^{-1}\left\{\frac{e^{-1/ks}}{(ks)^{1/2}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t/k}}$$

$$L^{-1}\left\{\frac{e^{-1/ks}}{\sqrt{ks}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t/k}} \quad \text{(By C.S.P)}$$

$$\left(= \frac{1}{k}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}\cdot\sqrt{k}\right)$$

$$\left(= \frac{1}{\sqrt{k}}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}\right)$$

$$\Rightarrow \cancel{\frac{1}{\sqrt{k}}}\, L^{-1}\left\{\frac{e^{-1/ks}}{\sqrt{s}}\right\} = \cancel{\frac{1}{\sqrt{k}}}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}$$

Put $k = \frac{1}{a}$

we get $L^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$

Inverse Laplace transform of Derivatives :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}^{(n)}(s)\} = (-1)^n t^n f(t)$

where $\bar{f}^{(n)}(s) = \frac{d^n}{ds^n}[\bar{f}(s)]$

(not necessary)

Note:- W.K.T $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

$t^n f(t) = (-1)^n L^{-1}\left[\frac{d^n}{ds^n}\bar{f}(s)\right]$

$(-1)^n t^n f(t) = L^{-1}\{\bar{f}^{(n)}(s)\}$

1. Find $L^{-1}\left\{\log\left(\frac{1+s}{s^2}\right)\right\}$

Sol:- $L^{-1}\left\{\log\left(\frac{1+s}{s^2}\right)\right\}$

This is of the form $L^{-1}\{\bar{f}(s)\}$

where $\bar{f}(s) = \log\left(\frac{1+s}{s^2}\right) = \log(1+s) - \log(s^2)$

$\bar{f}^{(1)}(s) = \frac{1}{1+s} - \frac{1}{s^2}(2s) = \frac{1}{1+s} - \frac{2}{s}$

$L^{-1}\{\bar{f}^{(1)}(s)\} = L^{-1}\left\{\frac{1}{1+s} - \frac{2}{s}\right\}$

$(-1)\, t f(t) = L^{-1}\left\{\frac{1}{1+s}\right\} - 2L^{-1}\left\{\frac{1}{s}\right\}$

$-t f(t) = e^{-t} - 2(1)$

$f(t) = \frac{e^{-t} - 2}{-t}$

$\therefore L^{-1}\{\bar{f}(s)\} = \frac{2 - e^{t}}{t}$

$L^{-1}\left\{\log\left(\frac{1+s}{s^2}\right)\right\} = \frac{2 - e^{-t}}{t}$

2. Find $L^{-1}\left\{\log\left(\frac{s^2+4}{s^2+9}\right)\right\}$

Sol:- $\bar{f}(s) = \log\left(\frac{s^2+4}{s^2+9}\right) = \log(s^2+4) - \log(s^2+9)$

$\bar{f}'(s) = \frac{1}{s^2+4}(2s) - \frac{1}{s^2+9}(2s)$

$= \frac{2s}{s^2+4} - \frac{2s}{s^2+9}$

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{2s}{s^2+4} - \frac{2s}{s^2+9}\right\}$$

$$(-1)\, t f(t) = 2L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 2\, L^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

$$= 2\cos 2t - 2\cos 3t$$

$$f(t) = \frac{2\cos 2t - 2\cos 3t}{-t}$$

$$= \frac{2\cos 3t - 2\cos 2t}{t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{2\cos 3t - 2\cos 2t}{t}$$

$$L^{-1}\left\{\log\left(\frac{s^2+4}{s^2+9}\right)\right\} = \frac{2\cos 3t - 2\cos 2t}{t}$$

3. Find $L^{-1}\left\{\log\left(\frac{s+3}{s+2}\right)\right\}$

4. Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol:- Let $\bar{f}(s) = \frac{1}{s^2+a^2}$

$$L^{-1}\{\bar{f}(s)\} = \frac{1}{a}\sin at = f(t)$$

~~(-1) t f(t)~~

$$\bar{f}^{(1)}(s) = \frac{(s^2+a^2)(0) - 1(2s)}{(s^2+a^2)^2} = \frac{-2s}{(s^2+a^2)^2}$$

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\}$$

$$(-1)\, t f(t) = L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\}$$

$$(-1)\, t \frac{1}{a}\sin at = L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

5. Find $L^{-1}\left\{\frac{s}{(s^2-25)^2}\right\}$

Sol:- Let $\bar{f}(s) = \frac{1}{s^2-25} = \frac{1}{s^2-5^2}$

$$L^{-1}\{\bar{f}(s)\} = \frac{\sinh 5t}{5} = f(t)$$

$$\bar{f}'(s) = \frac{-2s}{(s^2-5^2)^2} = \frac{-2s}{(s^2-25)^2}$$

$$L^{-1}\{\bar{f}^{(1)}(s)\} = L^{-1}\left\{\frac{-2s}{(s^2-25)^2}\right\}$$

$$(-1)\, t f(t) = L^{-1}\left\{\frac{-2s}{(s^2-25)^2}\right\}$$

$$(-1)\, t \frac{\sinh 5t}{5} = L^{-1}\left\{\frac{-2s}{(s^2-25)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2-25)^2}\right\} = \frac{t}{10}\sinh 5t$$

6. Find inverse laplace transform of $\log\left(1+\frac{16}{s^2}\right)$

Sol:- $\bar{f}(s) = \log\left(1+\frac{16}{s^2}\right) = \log\left(\frac{s^2+16}{s^2}\right)$

$$= \log(s^2+16) - \log(s^2)$$

$$= \log(s^2+16) - 2\log s$$

$$\bar{f}^{(1)}(s) = \frac{1}{s^2+16}(2s) - 2\left(\frac{1}{s}\right)$$

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$$\bar{f}^{(1)}(s) = \frac{2s}{s^2+16} - \frac{2}{s}$$

$$L^{-1}\{\bar{f}^{(1)}(s)\} = 2L^{-1}\left\{\frac{s}{s^2+4^2}\right\} - 2L^{-1}\left\{\frac{1}{s}\right\}$$

$$(-1)t f(t) = 2\cos 4t - 2$$

$$f(t) = \frac{2(\cos 4t - 1)}{-t} = \frac{2(1-\cos 4t)}{t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{2(1-\cos 4t)}{t}$$

$$L^{-1}\left\{\log\left(1+\frac{16}{s^2}\right)\right\} = \frac{2}{t}(1-\cos 4t)$$

7. Find $L^{-1}\{\cot^{-1}(s)\}$

Sol :- $\bar{f}(s) = \cot^{-1}(s)$

$$\bar{f}^{(1)}(s) = -\frac{1}{1+s^2}$$

$$L^{-1}\{\bar{f}^{(1)}(s)\} = L^{-1}\left\{\frac{-1}{1+s^2}\right\}$$

$$(-1)t f(t) = (-1)L^{-1}\left\{\frac{1}{1+s^2}\right\}$$

$$t f(t) = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{\sin t}{t}$$

$$\Rightarrow L^{-1}\{\cot^{-1}(s)\} = \frac{\sin t}{t} \quad //$$

Inverse laplace transform of Integrals :-

If $f(t) = L^{-1}\{\bar{f}(s)\}$ then $L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t}$

Note:- 1. W.K.T $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

$$\Rightarrow \frac{f(t)}{t} = L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\}$$

2. $L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t} = \frac{L^{-1}\{\bar{f}(s)\}}{t}$

$$\Rightarrow t\,L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = L^{-1}\{\bar{f}(s)\}$$

1. Find $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$ using Inverse L.T of integrals

Sol:- $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} = L^{-1}\left\{\frac{s+1}{((s+1)^2+1)^2}\right\}$

$$= e^{-t} L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$= e^{-t}\, t\, \frac{1}{2} L^{-1}\left\{\int_s^\infty \frac{2s}{(s^2+1)^2}\,ds\right\} = \frac{te^{-t}}{2} L^{-1}\left\{\int_s^\infty \frac{d}{ds}\left(\frac{-1}{s^2+1}\right)ds\right\}$$

$$= \frac{te^{-t}}{2} L^{-1}\left[\frac{-1}{s^2+1}\right]_s^\infty$$

$$= \frac{te^{-t}}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\frac{t}{2}\sin t$$

2. Find $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$ by Inverse L.T of integrals.

Sol:- $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = t\, L^{-1}\left\{\frac{1}{2}\int_s^\infty \frac{2s}{(s^2-a^2)^2}\,ds\right\}$

$= \frac{t}{2} L^{-1}\left\{\int_s^\infty \frac{d}{ds}\left(\frac{-1}{s^2-a^2}\right) ds\right\}$

$= \frac{t}{2} L^{-1}\left\{\left(\frac{-1}{s^2-a^2}\right)_s^\infty\right\}$

$= \frac{t}{2} L^{-1}\left\{0-\left(\frac{-1}{s^2-a^2}\right)\right\}$

$= \frac{t}{2} L^{-1}\left\{\frac{1}{s^2-a^2}\right\}$

$= \frac{t}{2}\,\frac{1}{a}\sinh at$

$= \frac{t}{2a}\sinh at$

3. Find $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$

Sol:- $L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$

$= e^{-2t}t.L^{-1}\left\{\frac{s}{s^2+1}\right\}$

$= e^{-2t}t\frac{1}{2}\, L^{-1}\left\{\int_s^\infty\frac{2s}{s^2+1}ds\right\}$

$= \frac{e^{-2t}}{2}t.L^{-1}\left\{\int_s^\infty \frac{d}{ds}\left(\frac{-1}{s^2+1}\right)ds\right\}$

$= \frac{te^{-2t}}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{te^{-2t}}{2}\sin t$

4. $L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

Sol:- $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t\, L^{-1}\left\{\int_s^\infty \frac{1}{(s+1)^2}\,ds\right\}$

$= t\, L^{-1}\left\{\int_s^\infty \frac{d}{ds}\left(\frac{-1}{s+1}\right) ds\right\}$

$= t\, L^{-1}\left\{\left(\frac{-1}{s+1}\right)_s^\infty\right\}$

$= t\, L^{-1}\left\{\frac{1}{s+1}\right\} = t e^{-t}$ //

~~$= te^{-t} L^{-1}\left\{\frac{1}{s}\right\}$~~

Multiplication by the Powers of s :-

If $f(t) = L^{-1}\{\bar{f}(s)\}$ and $f(0)=0$ then $L^{-1}\{s\bar{f}(s)\} = f'(t)$

Note:- In General

$L^{-1}\{s^n \bar{f}(s)\} = f^n(t)$

$L^{-1}\{s\bar{f}(s)\} = f'(t) = \{L^{-1}\bar{f}(s)\}'$

$L^{-1}\{s^2\bar{f}(s)\} = f''(t) = [L^{-1}\{f(s)\}]''$

Problems :-

1. Find $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol:- $L^{-1}\left\{\frac{s}{(s+3)^2}\right\} = L^{-1}\left\{s\cdot\frac{1}{(s+3)^2}\right\}$

This is of the form $L^{-1}\{s.\bar{f}(s)\}$

$\therefore L^{-1}\left\{s.\frac{1}{(s+3)^2}\right\} = \left[L^{-1}\left\{\frac{1}{(s+3)^2}\right\}\right]'$

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$$= \left[e^{-3t} \; L^{-1}\left\{\frac{1}{s^2}\right\}\right]'$$

$$= \left(e^{-3t} t\right)'$$

$$= e^{-3t}(1) + t\, e^{-3t}(-3)$$

$$= e^{-3t} - 3te^{-3t} = e^{-3t}[1-3t]$$

2. Find $L^{-1}\left\{s \cdot \log\left(\frac{s-1}{s+1}\right)\right\}$

Sol :- $L^{-1}\left\{s \log\left(\frac{s-1}{s+1}\right)\right\}$

This is of form $L^{-1}\{s \cdot \bar{f}(s)\}$

$$\therefore L^{-1}\left\{s \log\left(\frac{s-1}{s+1}\right)\right\} = \left[L^{-1}\left\{\log\left(\frac{s-1}{s+1}\right)\right\}\right]' \rightarrow (1)$$

To find $L^{-1}\left\{\log\left(\frac{s-1}{s+1}\right)\right\}$

$$\bar{f}(s) = \log(s-1) - \log(s+1)$$

$$\bar{f}'(s) = \frac{1}{s-1} - \frac{1}{s+1}$$

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$(-1)\, t\, f(t) = e^{t} - e^{-t}$$

$$f(t) = \frac{e^{t} - e^{-t}}{-t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{e^{t} - e^{-t}}{t}$$

$$L^{-1}\left\{\log\left(\frac{s-1}{s+1}\right)\right\} = \frac{e^{t} - e^{-t}}{-t} = \frac{2}{2} \frac{e^{t} - e^{-t}}{-2t} = \frac{-2}{t} \frac{e^{t} - e^{-t}}{2}$$

$$= \frac{-2}{t} \sinh t$$

$\therefore$ from (1)

$$L^{-1}\left\{s \log\left(\frac{s-1}{s+1}\right)\right\} = \left\{\frac{\cancel{e^{t} - e^{-t}}}{\cancel{t}}\right\} = \left[\frac{-2}{t} \sinh t\right]'$$

$$= -2\left\{\frac{1}{t} \cosh t + \sinh t\left(-\frac{1}{t^2}\right)\right\}$$

$$= -2\left[\frac{\cosh t}{t} - \frac{\sinh t}{t^2}\right\}$$

3. Find $L^{-1}\left\{\frac{s-3}{s^2+4s+13}\right\}$

Sol :- $\cancel{L^{-1}\left\{\frac{s-3}{s^2+4s+13}\right\} = L^{-1}\left\{\frac{s-3}{}\right.}$

$$\text{Let } \bar{f}(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+3^2}$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{s^2+3^2}\right\} = e^{-2t} \frac{\sin 3t}{3} = f(t)$$

Clearly $f(0) = 0$

$$L^{-1}\left\{\frac{s-3}{s^2+4s+13}\right\} = L^{-1}\left\{\frac{s}{s^2+4s+13}\right\} - 3\, L^{-1}\left\{\frac{1}{s^2+4s+13}\right\}$$

$$= L^{-1}\{s \bar{f}(s)\} - 3\, L^{-1}\{\bar{f}(s)\}$$

$$= f'(t) - 3 f(t)$$

$$= \frac{d}{dt}\left\{e^{-2t} \frac{\sin 3t}{3}\right\} - \cancel{3} e^{-2t} \frac{\sin 3t}{\cancel{3}}$$

$$= \frac{1}{3}\left[e^{-2t} 3\cos 3t + (-2e^{-2t})\sin 3t\right] - e^{-2t} \sin 3t$$

$$= \frac{1}{3}\left[3e^{-2t}\cos 3t - 2e^{-2t}\sin 3t\right] - e^{-2t}\sin 3t$$

Division by s :-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$

Note:- $L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t\int_0^t f(t)\,dt\,dt$

IIIly $L^{-1}\left\{\frac{\bar{f}(s)}{s^n}\right\} = \int_0^t\int_0^t \dots \int_0^t f(t)\,dt\,dt \dots dt$

1. Find $L^{-1}\left\{\frac{1}{s(s^2+1)(s^2-1)}\right\}$

Sol:- $L^{-1}\left\{\frac{1}{s(s^2+1)(s^2-1)}\right\} = L^{-1}\left\{\frac{\frac{1}{(s^2+1)(s^2-1)}}{s}\right\}$

where $\bar{f}(s) = \frac{1}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1}$

$1 = (As+B)(s^2-1) + (Cs+D)(s^2+1)$

$1 = As^3 + Bs^2 - As - B + Cs^3 + Ds^2 + Cs + D$

$A + C = 0 \rightarrow (1)$

$B + D = 0 \rightarrow (2)$

$-A + C = 0 \rightarrow (3)$

$-B + D = 1 \rightarrow (4)$

Solving (1) & (3)

$A + C = 0$

$-A + C = 0$

$2C = 0$

$C = 0$

$\therefore A = 0$

Solving (2) & (4)

$B + D = 0$

$-B + D = 1$

$2D = 1$

$D = 1/2$

$B = -1/2$

$\therefore \frac{1}{(s^2+1)(s^2-1)} = \frac{-1/2}{s^2+1} + \frac{1/2}{s^2-1}$

Now $L^{-1}\left\{\frac{1}{(s^2+1)(s^2-1)}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-1}\right\}$

$= -\frac{1}{2}\sin t + \frac{1}{2}\sinh t$

$= \frac{1}{2}[\sinh t - \sin t] = f(t)$

$\therefore L^{-1}\left\{\frac{\frac{1}{(s^2+1)(s^2-1)}}{s}\right\} = \int_0^t f(t)\,dt$

$= \int_0^t \frac{1}{2}[\sinh t - \sin t]\,dt$

$= \frac{1}{2}[\cosh t + \cos t]_0^t$

$= \frac{1}{2}\{[\cosh t + \cos t] - [\cosh(0) + \cos 0]\}$

$= \frac{1}{2}[\cosh t + \cos t - 2]$

$=$

2. $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

Sol:- $L^{-1}\left\{\frac{\frac{1}{s^2+9}}{s}\right\}$

where $\bar{f}(s) = \frac{1}{s^2+9}$ $\quad L^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{\sin 3t}{3} = f(t)$

$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t \frac{\sin 3t}{3}\,dt$

$= \frac{1}{3}\int_0^t \sin 3t\,dt$

$= \frac{1}{3}\left[\frac{-\cos 3t}{3}\right]_0^t$

$= \frac{1}{9}[-\cos 3t + \cos 0]$

$= \frac{1}{9}[1 - \cos 3t]$

3. Find $L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$

Sol:- $L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = L^{-1}\left\{\frac{4(s-1)+9}{(s-1)^2[(s-1)+3]}\right\}$

$= e^t L^{-1}\left\{\frac{4s+9}{s^2(s+3)}\right\}$

$= e^t L^{-1}\left\{\frac{4}{s(s+3)}\right\} + 9 L^{-1}\left\{\frac{1}{s^2(s+3)}\right\} \rightarrow (1)$

$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

$1 = As + 3A + Bs$

$A + B = 0 \Rightarrow A = -B$

$3A = 1 \Rightarrow A = 1/3 \quad \therefore B = -1/3$

$\frac{1}{s(s+3)} = \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$

$L^{-1}\left\{\frac{1}{s(s+3)}\right\} = \frac{1}{3}\left[L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\}\right]$

$= \frac{1}{3}[1 - e^{-3t}] \rightarrow (2)$

Now $L^{-1}\left\{\frac{1}{s^2(s+3)}\right\} = L^{-1}\left\{\frac{\frac{1}{s(s+3)}}{s}\right\}$

$= \frac{1}{3}\int_0^t (1 - e^{-3t})\,dt$

$= \frac{1}{3}\left[t - \frac{e^{-3t}}{-3}\right]_0^t$

$= \frac{1}{3}\left[t + \frac{e^{-3t}}{3}\right]_0^t$

$= \frac{1}{3}\left[t + \frac{e^{-3t}}{3} - \frac{1}{3}\right]$

$= \frac{1}{3}\left[t + \frac{1}{3}(e^{-3t} - 1)\right] \rightarrow (3)$

from (1) & (2) and (3) we have

$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = e^t\left[\frac{4}{3}(1 - e^{-3t}) + \frac{9}{3}\left(t + \frac{1}{3}(e^{-3t} - 1)\right)\right]$

$= e^t\left[\frac{4}{3} - \frac{4}{3}e^{-3t} + 3t + e^{-3t} - 1\right]$

$= e^t\left[\frac{1}{3} + 3t - \frac{1}{3}e^{-3t}\right]$

4. Find $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

Sol:- $L^{-1}\left\{\frac{1}{s((s+1)^2+1)}\right\} = L^{-1}\left\{\frac{\frac{1}{(s+1)^2+1}}{s}\right\}$

Let $\bar{f}(s) = \frac{1}{(s+1)^2+1} \Rightarrow L^{-1}\{\bar{f}(s)\} = e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= e^{-t}\sin t = f(t)$

$\therefore L^{-1}\left\{\frac{\frac{1}{(s+1)^2+1}}{s}\right\} = \int_0^t f(t)\,dt = \int_0^t e^{-t}\sin t\,dt$

$= \left[\frac{e^{-t}}{1^2+1^2}(-\sin t - \cos t)\right]_0^t$

$= \left[\frac{e^{-t}}{2}(-\sin t - \cos t)\right] - \left[\frac{1}{2}(-1)\right]$

$= \left[\frac{e^{-t}}{2}(-\sin t - \cos t) + \frac{1}{2}\right]$

$= \left[-\frac{e^{-t}}{2}(\sin t + \cos t) + \frac{1}{2}\right]$

$= \frac{1}{2}\left[1 - e^{-t}(\sin t + \cos t)\right]$

5. Find $L^{-1}\left\{\frac{s+1}{s^2(s^2+1)}\right\}$

Sol :- Let $\bar{f}(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= \cos t + \sin t = f(t)$

Now $L^{-1}\left\{\frac{\frac{s+1}{s^2+1}}{s}\right\} = \int_0^t f(t)\,dt = \int_0^t \cos t + \sin t \, dt$

$= (\sin t - \cos t)_0^t$

$= (\sin t - \cos t) - (\sin 0 - \cos 0)$

$= 1 + (\sin t - \cos t)$

$L^{-1}\left\{\frac{\frac{s+1}{s^2+1}}{s^2}\right\} = \int_0^t (1 + \sin t - \cos t)\,dt$

$= (t - \cos t - \sin t)_0^t$

$= (t - \cos t - \sin t) - (0 - \cos 0 - \sin 0)$

$= 1 + t - \cos t - \sin t$

Convolution :-

Convolution is used for obtaining inverse L.T of a Product of two transformations

Convolution theorem :-

Let $f(t)$ and $g(t)$ be the two functions defined for $t > 0$.

If $f(t) = L^{-1}\{\bar{f}(s)\}$ and $g(t) = L^{-1}\{\bar{g}(s)\}$ then $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$

$= f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

where $*$ represents the Convolution operator.

1. By using Convolution ~~find~~ theorem find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Sol :- $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\}$

This is in the form $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$

~~By~~ where $\bar{f}(s) = \frac{1}{s^2+a^2}$ $\quad \bar{g}(s) = \frac{1}{s^2+a^2}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$ $\quad L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$

$f(t) = \frac{\sin at}{a}$ $\quad g(t) = \frac{\sin at}{a}$

$\Rightarrow f(u) = \frac{1}{a}\sin au$ $\quad g(t-u) = \frac{1}{a}\sin a(t-u)$

By Convolution theorem

$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t)$

$= \int_0^t f(u)\, g(t-u)\,du$

$= \int_0^t \frac{1}{a}\sin au \,\frac{1}{a}\sin a(t-u)\,du$

$= \frac{1}{a^2}\int_0^t \sin au \,\sin(at-au)\,du$

$= \frac{1}{2a^2}\int_0^t 2\sin au\,\sin(at-au)\,du$

$= \frac{1}{2a^2}\int_0^t \cos(au-at+au) - \cos(au+at-au)\,du$

$= \frac{1}{2a^2}\int_0^t \cos(2au-at) - \cos at\,du$

$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - \cos at\, u\right]_0^t$

$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \left[\frac{\sin(-at)}{2a} - \cos 0\right]\right]$

$\cancel{= \frac{\sin at}{4a^3}} -$

$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \left[-\frac{\sin at}{2a} - 0\right]\right]$

$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at + \frac{\sin at}{2a}\right]$

$= \frac{1}{2a^2}\left[\frac{2\sin at}{2a}\right] - \frac{1}{2a^2}[t\cos at]$

$= \frac{2\sin at}{4a^3} - \frac{t\cos at}{2a^2}$

2. Find $L^{-1}\left\{\frac{1}{(s^2+5^2)^2}\right\}$ using Convolution theorem.

Sol :- $L^{-1}\left\{\frac{1}{(s^2+5^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+5^2}\cdot\frac{1}{s^2+5^2}\right\}$

here $\bar{f}(s) = \frac{1}{s^2+5^2}$ $\quad \bar{g}(s) = \frac{1}{s^2+5^2}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+5^2}\right\}$ $\quad L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+5^2}\right\}$

$f(t) = \frac{\sin 5t}{5}$ $\quad g(t) = \frac{\sin 5t}{5}$

$f(u) = \frac{1}{5}\sin 5u$ $\quad g(t-u) = \frac{1}{5}\sin 5(t-u)$

By Convolution theorem

$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t) = \int_0^t f(u)\, g(t-u)\,du$

$= \int_0^t \frac{1}{5}\sin 5u \cdot \frac{1}{5}\sin 5(t-u)\,du$

$= \frac{1}{2\times 25}\int_0^t 2\sin 5u\,\sin 5(t-u)\,du$

$= \frac{1}{2\times 25}\int_0^t \cos(5u-5t+5u) - \cos(\cancel{5u}+5t-\cancel{5u})\,du$

$= \frac{1}{2\times 25}\int_0^t \cos(10u-5t) - \cos(5t)\,du$

$$= \frac{1}{50}\left[\frac{\sin(10u-5t)}{10} - \cos(5t)\cdot u\right]_0^t$$

$$= \frac{1}{50}\left[\frac{\sin 5t}{10} - t\cos 5t - \left(-\frac{\sin 5t}{10} - 0\right)\right]$$

$$= \frac{1}{50}\left[\frac{\cancel{2}\sin 5t}{\cancel{10}5} - t\cos 5t\right]$$

$$= \frac{1}{50}\left[\frac{\sin 5t}{5} - t\cos 5t\right]$$

$$= \frac{\sin 5t}{250} - \frac{t\cos 5t}{50}$$

3. Find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ by using convolution theorem.

Sol:- $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right\} = L^{-1}\{\underbrace{\frac{1}{s^2}}_{\bar{g}(s)}\}\cdot L^{-1}\{\underbrace{\frac{1}{(s+1)^2}}_{\bar{f}(s)}\}$

$$= t \cdot e^{-t}t$$

$$= g(t) * f(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{-u} u\, (t-u)\, du$$

$$= t\int_0^t u e^{-u} du - \int_0^t u^2 e^{-u} du$$

$$= t\left[u\frac{e^{-u}}{-1} - e^{-u}\right]_0^t - \left[-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\right]_0^t$$

$$= t\left[-te^{-t} - e^{-t} - (0+1)\right] - \left[-t^2e^{-t} - 2te^{-t} - 2e^{-t} - (-2)\right]$$

$$= -t^2\cancel{e^{-t}} - te^{-t} + t + t^2\cancel{e^{-t}} + 2te^{-t} + 2e^{-t} - 2$$

$$= te^{-t} + 2e^{-t} + t - 2$$

4. Find $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$ using convolution theorem.

Sol:- $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\left\{\frac{1}{(s+a)}\cdot\frac{1}{(s+b)}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$

where $\bar{f}(s) = \frac{1}{s+a}$ $\quad \bar{g}(s) = \frac{1}{s+b}$

$L^{-1}\{\bar{f}(s)\} = e^{-at} = f(t)$ $\quad L^{-1}\{\bar{g}(s)\} = e^{-bt} = g(t)$

$f(u) = e^{-au}$ and $g(t-u) = e^{-b(t-u)}$

By convolution theorem

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{-au}\, e^{-b(t-u)}\, du$$

$$= e^{-bt}\int_0^t e^{-au} e^{bu}\, du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\, du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t = e^{-bt}\left[\frac{e^{(b-a)t}}{b-a} - \frac{1}{b-a}\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right] \,//$$

5. Find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$ using convolution theorem.

Sol:- $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$

$\bar{f}(s) = \frac{s}{s^2+a^2}$ $\quad\quad$ $\bar{g}(s) = \frac{s}{s^2+b^2}$

$L^{-1}\{\bar{f}(s)\} = \cos at = f(t)$ $\quad\quad$ $L^{-1}\{\bar{g}(s)\} = \cos bt = g(t)$

$f(u) = \cos au$ $\quad\quad$ $g(t-u) = \cos b(t-u)$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos au \; \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t 2\cos au \; \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu)\, du$$

$$= \frac{1}{2}\int_0^t \cos[(a-b)u+bt] + \cos[(a+b)u-bt]\, du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin[at+bt-bt]}{a+b} - \left(\frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

6. Find $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$ using convolution theorem.

Sol:- $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2^2}\right\}$

here let $\bar{f}(s) = \frac{1}{s^2+2^2}$ $\quad\quad$ $\bar{g}(s) = \frac{1}{s}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$ $\quad\quad$ $L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$

$= \frac{1}{2}\sin 2t$ $\quad\quad$ $g(t-u) = 1$

$= f(t)$

$f(u) = \frac{1}{2}\sin 2u$

By convolution theorem

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2^2}\right\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \frac{1}{2}\sin 2u \cdot 1\, du$$

$$= \frac{1}{2}\int_0^t \sin 2u\, du = \frac{1}{2}\left[\frac{-\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{4}\left[-\cos 2t - (-\cos 0)\right]$$

$$= \frac{1}{4}\left[1 - \cos 2t\right] //$$

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Applications to ordinary differential Equations :-

Solution of ordinary differential Equations with Constant Coefficients

The solutions of ODE Can be done easily by using Laplace transform method without finding the general solution by the following Procedure.

Step 1:- Take the Laplace transform of bothsides of the given D.E.

Step 2:- Use the formula

i) $L\{y'(t)\} = s\bar{y}(s) - y(0) = sL\{y\} - y(0)$

ii) $L\{y''(t)\} = s^2\bar{y}(s) - sy(0) - y'(0) = s^2L\{y\} - sy(0) - y'(0)$

iii) $L\{y'''(t)\} = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0)$
$= s^3L\{y\} - s^2y(0) - sy'(0) - y''(0)$

Step 3:- Replace $y(0), y'(0), y''(0)$ with the given initial Conditions

Step 4:- Transpose terms to the right hand side and divide by the Coefficient of $L\{y\}$ or $\bar{y}$

Step 5:- Resolve the function of 's' obtained in the above step into partial fractions

Step 6:- Take the inverse L.T on bothsides this gives y as a function of t which is the required solution of the given Equation.

Problems :-

1. Solve $(D^2+4D+5)y = 5$ given that $y(0)=0$ & $y'(0)=0$

Sol:- Given $(D^2+4D+5)y = 5$

$\Rightarrow D^2y + 4Dy + 5y = 5$

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5$

$y'' + 4y' + 5y = 5$

Applying L.T on both sides

$L\{y'' + 4y' + 5y\} = L\{5\}$

$L\{y''\} + 4L\{y'\} + 5L\{y\} = L\{5\}$

$L\{y''\} + 4L\{y'\} + 5L\{y\} = 5 \cdot \frac{1}{s}$

$\{s^2L\{y\} - sy(0) - y'(0)\} + 4\{sL\{y\} - y(0)\} + 5L\{y\} = \frac{5}{s}$

$s^2L\{y\} - s(0) - 0 + 4sL\{y\} - 4(0) + 5L\{y\} = \frac{5}{s}$

$L\{y\}(s^2+4s+5) = \frac{5}{s}$

$L\{y\} = \frac{5}{s(s^2+4s+5)} \rightarrow (1)$

Now $\frac{1}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$

$1 = As^2 + 4As + 5A + Bs^2 + Cs$

$A+B=0 \rightarrow (1)$

$4A+C=0 \rightarrow (2)$

$5A=1 \rightarrow (3) \Rightarrow A = 1/5$

$4A+C=0 \Rightarrow \frac{4}{5} + C = 0 \Rightarrow C = -4/5$

$A+B=0 \Rightarrow B = -A \Rightarrow B = -1/5$

$$L\{y\} = 5\left[\frac{1/5}{s} - \frac{\frac{1}{5}s + 4/5}{s^2+4s+5}\right]$$

$$L\{y\} = 5\left[\frac{1}{5s} - \frac{s+4}{5(s^2+4s+5)}\right]$$

Apply Inverse L.T on both sides

$$y = 5\,L^{-1}\left\{\frac{1}{5s} - \frac{s+4}{5(s^2+4s+5)}\right\}$$

$$y = \frac{\cancel{5}}{\cancel{5}} L^{-1}\left\{\frac{1}{s}\right\} - \frac{\cancel{5}}{\cancel{5}} L^{-1}\left\{\frac{s+4}{s^2+4s+5}\right\}$$

$$y = 1 - L^{-1}\left\{\frac{(s+2)+2}{(s+2)^2+1}\right\}$$

$$y = 1 - e^{-2t} L^{-1}\left\{\frac{s+2}{s^2+1}\right\}$$

$$y = 1 - e^{-2t}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} + 2L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$y = 1 - e^{-2t}\left[\cos t + 2\sin t\right]$$

2. Solve $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 8x = e^{2t}$, $x(0)=2$ & $x'(0)=2$

Sol:- $$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 8x = e^{2t}$$

$$x'' - 4x' + 8x = e^{2t}$$

$$L\{x'' - 4x' + 8x\} = L\{e^{2t}\}$$

$$L\{x''\} - 4L\{x'\} + 8L\{x\} = L\{e^{2t}\}$$

$$\Rightarrow s^2L\{x\} - sx(0) - x'(0) - 4[sL\{x\} - x(0)] + 8L\{x\} = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 4s + 8)L\{x\} - s(2) - 2 + 4(2) = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 4s + 8)L\{x\} = \frac{1}{s-2} + 2s - 6$$

$$= \frac{1+(2s-6)(s-2)}{s-2}$$

$$\Rightarrow L\{x\} = \frac{1+(2s-6)(s-2)}{(s-2)(s^2-4s+8)} = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-6}{s^2-4s+8}$$

~~Now~~ $\frac{1}{(s-2)(s^2-4s+8)}$

Apply Inverse L.T on both sides

$$x = L^{-1}\left\{\frac{1}{(s-2)(s^2-4s+8)}\right\} + L^{-1}\left\{\frac{2s-6}{s^2-4s+8}\right\} \rightarrow 1$$

$$= \cancel{L^{-1}\left\{\frac{1}{(s-2)(s-2)^2+4}\right\} + L^{-1}\{}$$

$$\text{Now } L^{-1}\left\{\frac{1}{(s-2)(s^2-4s+8)}\right\} = L^{-1}\left\{\frac{1}{(s-2)(s-2)^2+4}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{\frac{1}{s^2+4}}{s}\right\} \qquad F(s) = \frac{1}{s^2+4}$$

$$= e^{2t}\int_0^t f(t)\,dt \qquad f(t) = L^{-1}\{F(s)\} = \frac{\sin 2t}{2}$$

$$= e^{2t}\int_0^t \frac{\sin 2t}{2}\,dt$$

$$= \frac{e^{2t}}{2}\left(-\frac{\cos 2t}{2}\right)_0^t$$

$$= \frac{e^{2t}}{4}[1-\cos 2t] \rightarrow 2$$

$$L^{-1}\left\{\frac{2s-6}{s^2-4s+8}\right\} = L^{-1}\left\{\frac{2s-6}{(s-2)^2+4}\right\}$$

$$= L^{-1}\left\{\frac{2(s-2)-2}{(s-2)^2+4}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{2s-2}{s^2+4}\right\}$$

$$= e^{2t}\left[2L^{-1}\left\{\frac{s}{s^2+4}\right\} - 2L^{-1}\left\{\frac{1}{s^2+4}\right\}\right]$$

$$= e^{2t}\left[2\cos 2t - \cancel{2}\frac{\sin 2t}{\cancel{2}}\right]$$

$$= e^{2t}[2\cos 2t - \sin 2t] \rightarrow (3)$$

Sub (2) & (3) in Eqn (1)

$$x = \frac{e^{2t}}{4}[1-\cos 2t] + e^{2t}[2\cos 2t - \sin 2t]$$

3. solve $(D^2+1)y = 6\cos 2t$ given that $y=3$ $dy=1$ when $t=0$

sol:- Given $y(0)=3$ $y'(0)=1$

$$y'' + y = 6\cos 2t$$

$$L\{y''\} + L\{y\} = 6\,L\{\cos 2t\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = 6\frac{s}{s^2+4}$$

$$(s^2+1)L\{y\} - s(3) - 1 = \frac{6s}{s^2+4}$$

$$(s^2+1)L\{y\} = \frac{6s}{s^2+4} + 3s + 1$$

$$L\{y\} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1} \rightarrow (1)$$

But $\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$

$$s = As^3 + Bs^2 + As + B + Cs^3 + Ds^2 + 4Cs + 4D$$

$A + C = 0 \rightarrow (1)$

$B + D = 0 \rightarrow (2)$

$A + 4C = 1 \rightarrow (3)$

$B + 4D = 0 \rightarrow (4)$

solving (1) & (2)

$$\begin{array}{l} \cancel{A} + C = 0 \\ \underline{\cancel{A} + 4C = 1} \\ -3C = -1 \end{array} \Rightarrow C = 1/3 \quad \therefore A = -1/3$$

solving (2) & (4)

$$\begin{array}{l} \cancel{B} + D = 0 \\ \underline{\cancel{B} + 4D = 0} \\ -3D = 0 \end{array} \Rightarrow D = 0 \quad \therefore B = 0$$

$$\therefore \frac{s}{(s^2+4)(s^2+1)} = \frac{-1}{3(s^2+4)} + \frac{1}{3(s^2+1)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$\therefore L\{y\} = \frac{\overset{2}{\cancel{6}}}{\cancel{3}}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] + 3\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$y = 2L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\} + 3L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2\sin t - \sin 2t + 3\cos t + \sin t$$

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{-s}{3(s^2+4)} + \frac{s}{3(s^2+1)}$$

$$= \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

from (1)

$$L\{y\} = \cancel{6}^{2}\frac{1}{\cancel{3}}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right] + 3\left[\frac{s}{s^2+1}\right] + \frac{1}{s^2+1}$$

$$y = 2L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2\cos t - 2\cos 2t + 3\cos t + \sin t$$

$$= 5\cos t - 2\cos 2t + \sin t$$

4. solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 3te^{-t}$ Given $x(0)=4$, $\frac{dx}{dt}=0$ at $t=0$

Sol:-

$$x'' + 2x' + x = 3te^{-t}$$

$$L\{x''\} + 2L\{x'\} + L\{x\} = 3L\{te^{-t}\}$$

$$L\{x''\} + 2L\{x'\} + L\{x\} = 3L\{te^{-t}\}$$

$$s^2L\{x\} - sx(0) - x'(0) + 2[sL\{x\} - x(0)] + L\{x\} = 3L\{te^{-t}\}$$

$$s^2L\{x\} - 4s - 0 + 2sL\{x\} - 8 + L\{x\} = 3\frac{1}{(s+1)^2}$$ (By 1st shifting theorem)

$$L\{x\}(s^2+2s+1) = \frac{3}{(s+1)^2} + 4s + 8$$

$$L\{x\}(s+1)^2 = \frac{3}{(s+1)^2} + 4s + 8$$

$$L\{x\} = \frac{3}{(s+1)^4} + \frac{4s}{(s+1)^2} + \frac{8}{(s+1)^2}$$

$$x = L^{-1}\left\{\frac{3}{(s+1)^4}\right\} + 4L^{-1}\left\{\frac{s}{(s+1)^2}\right\} + 8L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{(s+1)^4}\right\} + 4L^{-1}\left\{\frac{(s+1)-1}{(s+1)^2}\right\} + 8L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= 3e^{-t}L^{-1}\left\{\frac{1}{s^4}\right\} + 4e^{-t}L^{-1}\left\{\frac{s-1}{s^2}\right\} + 8e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 3e^{-t}L^{-1}\left\{\frac{1}{s^4}\right\} + 4e^{-t}\left[L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\}\right] + 8e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 3e^{-t}L^{-1}\left\{\frac{1}{s^4}\right\} + 4e^{-t}(1-t) + 8e^{-t}(t)$$

$$= 3e^{-t}\frac{t^3}{3!} + 4e^{-t}(1-t) + 8e^{-t}(t)$$

$$= 3e^{-t}\frac{t^3}{2} + 4e^{-t} - 4te^{-t} + 8te^{-t}$$

$$= e^{-t}\frac{t^3}{2} + 4e^{-t} + 4te^{-t}$$

$$= e^{-t}\left[\frac{t^3}{2} + 4t + 4\right]$$

5. solve $y''' + 2y'' - y' - 2y = 0$ Given $y(0)=y'(0)=0$ and $y''(0)=6$.

Sol:-

$$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = 0$$

$$\cancel{s^3L\{y\} - s^2y''}$$

$$s^3L\{y\} - s^2y(0) - sy'(0) - y''(0) + 2[s^2L\{y\} - sy(0) - y'(0)]$$

$$- sL\{y\} - y(0) - 2L\{y\} = 0$$

$$L\{y\}(s^3 + 2s^2 - s - 2) - 6 = 0$$

$$L\{y\} = \frac{6}{(s^3+2s^2-s-2)} = \frac{6}{(s+1)(s-1)(s+2)}$$

$$L\{y\} = \frac{6}{(s+1)(s-1)(s+2)} \rightarrow ①$$

Now $\frac{1}{(s+1)(s-1)(s+2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+2}$

$$1 = A(s-1)(s+2) + B(s+1)(s+2) + C(s+1)(s-1)$$
$$= A(s^2+s-2) + B(s^2+3s+2) + C(s^2-1)$$
$$= As^2+As-2A+Bs^2+3Bs+2B+Cs^2-C$$

$A+B+C=0 \rightarrow ①$

$A+3B=0 \rightarrow ②$

$-2A+2B-C=1 \rightarrow ③$

Solving ① & ③

$A+B+\cancel{C}=0$

$-2A+2B-\cancel{C}=1$

$-A+3B=1 \rightarrow ④$

Solving ② & ④

$\cancel{A}+3B=0$

$-\cancel{A}+3B=1$

$6B=1$

$B=1/6$

$\therefore A+3B=0$

$A+3(\frac{1}{6})=0$

$A=-1/2$

$-\frac{1}{2}+\frac{1}{6}+C=0$

$\frac{-3+1}{6}+C=0$

$\frac{-2}{6}+C=0$

$C=1/3$

$$\therefore \frac{1}{(s+1)(s-1)(s+2)} = \frac{-1}{2(s+1)} + \frac{1}{6(s-1)} + \frac{1}{3(s+2)}$$

from ①

$$L\{y\} = 6\left\{-\frac{1}{2(s+1)} + \frac{1}{6(s-1)} + \frac{1}{3(s+2)}\right\}$$
$$= \frac{-3}{s+1} + \frac{1}{s-1} + \frac{2}{s+2}$$

$$y = L^{-1}\left\{-\frac{3}{s+1} + \frac{1}{s-1} + \frac{2}{s+2}\right\}$$
$$y = -3L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} + 2L^{-1}\left\{\frac{1}{s+2}\right\}$$
$$y = -3e^{-t} + e^{t} + 2e^{-2t}$$

6. solve $4y'' + \pi^2 y = 0$, $y(0)=2$ $y'(0)=0$

Sol :- $4y'' + \pi^2 y = 0$

$$4L\{y''\} + \pi^2 L\{y\} = 0$$
$$4[s^2 L\{y\} - sy(0) - y'(0)] + \pi^2 L\{y\} = 0$$
$$(4s^2+\pi^2)L\{y\} - 8s = 0$$
$$L\{y\} = \frac{8s}{4s^2+\pi^2} = \frac{8s}{4(s^2+(\frac{\pi}{2})^2)} = \frac{2s}{s^2+(\frac{\pi}{2})^2}$$
$$y = 2L^{-1}\left\{\frac{s}{s^2+(\frac{\pi}{2})^2}\right\} = 2L^{-1}\left\{\frac{s}{s^2+(\frac{\pi}{2})^2}\right\}$$

$$\boxed{\Rightarrow y = 2\cos\frac{\pi}{2}t}$$