BLG202E Homework № 4

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1 Problem 1

1.1 Newton's Divided Difference Method

$$p_n(x) = \sum_{j=0}^{n} \left(f[x_0, x_1, ..., x_j] \prod_{i=0}^{j-1} (x - x_i) \right)$$

The equation that is given above is generalized Newton's Divided Difference Interpolation formula. We need a quadratic interpolation, so $n = 2$. Our data points are,

- $x_0 = 0.00, y_0 = 1.20$
- $x_1 = -0.60, y_1 = 1.04$
- $x_2 = -1.20, y_2 = 0.00$

If we apply Newton's Divided Difference Method to our data points:

For j=0:

$$c_0 = f(x_0) \rightarrow \mathbf{c_0 = 1.20}$$

For j=1:

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \rightarrow c_1 = \frac{f(-0.60) - f(0.00)}{-0.60 - 0.00} = \frac{\mathbf{4}}{\mathbf{15}} \approx \mathbf{0.26}$$

$$c_1(x - x_0) = 0.26(x)$$

For j=2:

$$c_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} \rightarrow c_2 = \frac{\frac{f(-1.20)-f(-0.60)}{-1.20+0.60} - \frac{f(-0.60)-f(0)}{-0.60-0}}{-1.20 - 0}$$

$$c_2 = \frac{\frac{1.04}{0.60} - \frac{0.16}{0.60}}{-1.20} \approx \textbf{-1.22}$$

$$c_2(x - x_0)(x - x_1) = -1.76(x)(x + 0.60)$$

If we sum all results, we get our interpolation equation:

$$p(x) = 1.20 + 0.26(x) - 1.22(x)(x + 0.60)$$

Interpolated value of point x=-1.04:

$$p(1.04) = 1.20 + 0.26(-1.04) - 1.22(1.04)(-1.04 + 0.60) \approx \textbf{0.371}$$

In dataset f(-1.04)=0.60, therefore absolute value of the relative error is

$$\frac{|0.60 - 0.371|}{|0.60|} \approx \textbf{0.38166}$$

2 Deriving Formulas Using Taylor Series

With the help of Taylor Series, we can determine derivation any point of the function without differentiation with a small truncation error.

2.0.1 Second Order Centered Difference Formula

We are trying to find derivative of $x = x_0$ by using $x = x_0 + h$ and $x = x_0 - h$. By Taylor Formula:

$$f(x_0 + h) = f(x_0) + hf^{'}(x_0) + \frac{h^2}{2}f^{''}(x_0) + \frac{h^3}{6}f^{'''}(x_{x_0})$$
$$f(x_0 - h) = f(x_0) - hf^{'}(x_0) + \frac{h^2}{2}f^{''}(x_0) - \frac{h^3}{6}f^{'''}(x_{x_0})$$

If we subtract second from the first and take the derivative to left:

$$f^{'}(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \frac{h^2}{6}f^{'''}(x_0)$$

Truncation error which is second order:

$$\varepsilon(h) = \frac{h^2}{6}f^{'''}(x_0)$$

2.0.2 Fourth Order Centered Difference Formula

We are trying to find derivative of $x = x_0$ by using $x = x_0 + h, x = x_0 - h, x = x_0 + 2h, x = x_0 - 2h$.
By Taylor Formula:
For $\mathbf{x = x_0 + h, x = x_0 - h}$:

$$f(x_0 + h) = f(x_0) + hf^{'}(x_0) + \frac{h^2}{2}f^{''}(x_0) + \frac{h^3}{6}f^{'''}(x_0) + \frac{h^4}{24}f^{iv}(x_0) + \frac{h^5}{120}f^{v}(x_0) + \frac{h^6}{720}f^{vi}(x_0)$$
$$f(x_0 - h) = f(x_0) - hf^{'}(x_0) + \frac{h^2}{2}f^{''}(x_0) - \frac{h^3}{6}f^{'''}(x_0) + \frac{h^4}{24}f^{iv}(x_0) - \frac{h^5}{120}f^{v}(x_0) + \frac{h^6}{720}f^{vi}(x_0)$$

If we subtract second from the first:

$$f(x_0 + h) - f(x_0 - h) = 2hf^{'}(x_0) + \frac{2h^3}{6}f^{'''}(x_0) + \frac{2h^5}{120}f^{v}(x_0)$$

$$1) f^{'}(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \frac{h^2}{6}f^{'''}(x_0) - \frac{h^4}{120}f^{v}(x_0)$$

For $\mathbf{x = x_0 + 2h, x = x_0 - 2h}$:

$$f(x_0+2h) = f(x_0)+2hf^{'}(x_0)+\frac{4h^2}{2}f^{''}(x_0)+\frac{8h^3}{6}f^{'''}(x_0)+\frac{16h^4}{24}f^{iv}(x_0)+\frac{32h^5}{120}f^{v}(x_0)+\frac{64h^6}{720}f^{vi}(x_0)$$

$$f(x_0-2h)=f(x_0)-2hf^{'}(x_0)+\frac{4h^2}{2}f^{''}(x_0)-\frac{8h^3}{6}f^{'''}(x_0)+\frac{16h^4}{24}f^{iv}(x_0)-\frac{32h^5}{120}f^{v}(x_0)+\frac{64h^6}{720}f^{vi}(x_0)$$

If we subtract second from the first:

$$f(x_0+2h)-f(x_0-2h)=4hf^{'}(x_0)+\frac{16h^3}{6}f^{'''}(x_0)+\frac{64h^5}{120}f^{v}(x_0)$$

$$2)f^{'}(x_0)=\frac{f(x_0+2h)-f(x_0-2h)}{4h}-\frac{4h^2}{6}f^{'''}(x_0)-\frac{16h^4}{120}f^{v}(x_0)$$

If we subtract 4 times of first equation from the second one to cancel out third order derivative.

$$-3f^{'}(x_0)=\frac{f(x_0+2h)-f(x_0-2h)}{4h}-\frac{4f(x_0+h)-4f(x_0-h)}{2h}-\frac{12h^4}{120}f^{v}(x_0)$$

$$f^{'}(x_0)=\frac{1}{12h}\Big(f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)\Big)+\frac{h^4}{30}f^{v}(x_0)$$

Truncation error which is fourth order:

$$\varepsilon(h)=\frac{h^4}{30}f^{v}(x_0)$$

2.0.3 Sixth Order Centered Difference Formula

We are trying to find derivative of $x=x_0$ by using $x=x_0+h, x=x_0-h, x=x_0+2h, x=x_0+3h, x=x_0-3h$. By Taylor Formula:

For $\mathbf{x=x_0+h, x=x_0-h}$:

$$f(x_0+h)=f(x_0)+hf^{'}(x_0)+\frac{h^2}{2}f^{''}(x_0)+\frac{h^3}{6}f^{'''}(x_0)+\frac{h^4}{24}f^{iv}(x_0)+\frac{h^5}{120}f^{v}(x_0)$$
$$+\frac{h^6}{720}f^{vi}(x_0)+\frac{h^7}{5040}f^{vii}(x_0)+\frac{h^8}{40320}f^{viii}(x_0)$$
$$f(x_0-h)=f(x_0)-hf^{'}(x_0)+\frac{h^2}{2}f^{''}(x_0)-\frac{h^3}{6}f^{'''}(x_0)+\frac{h^4}{24}f^{iv}(x_0)-\frac{h^5}{120}f^{v}(x_0)$$
$$+\frac{h^6}{720}f^{vi}(x_0)-\frac{h^7}{5040}f^{vii}(x_0)+\frac{h^8}{40320}f^{viii}(x_0)$$

If we subtract second from the first:

$$f(x_0+h)-f(x_0-h)=2hf^{'}(x_0)+\frac{h^3}{3}f^{'''}(x_0)+\frac{h^5}{60}f^{v}(x_0)+\frac{h^7}{2520}f^{vii}(x_0)$$

$$1)f^{'}(x_0)=\frac{f(x_0+h)-f(x_0-h)}{2h}-\frac{h^2}{6}f^{'''}(x_0)-\frac{h^4}{120}f^{v}(x_0)-\frac{h^6}{5040}f^{vii}(x_0)$$

For $\mathbf{x=x_0+2h, x=x_0-2h}$:

$$f(x_0+2h)=f(x_0)+2hf^{'}(x_0)+\frac{4h^2}{2}f^{''}(x_0)+\frac{8h^3}{6}f^{'''}(x_0)+\frac{16h^4}{24}f^{iv}(x_0)+\frac{32h^5}{120}f^{v}(x_0)$$
$$+\frac{64h^6}{720}f^{vi}(x_0)+\frac{128h^7}{5040}f^{vii}(x_0)+\frac{256h^8}{40320}f^{viii}(x_0)$$
$$f(x_0-2h)=f(x_0)-2hf^{'}(x_0)+\frac{4h^2}{2}f^{''}(x_0)-\frac{8h^3}{6}f^{'''}(x_0)+\frac{16h^4}{24}f^{iv}(x_0)-\frac{32h^5}{120}f^{v}(x_0)$$
$$+\frac{64h^6}{720}f^{vi}(x_0)-\frac{128h^7}{5040}f^{vii}(x_0)+\frac{256h^8}{40320}f^{viii}(x_0)$$

If we subtract second from the first:

$$f(x_0+2h)-f(x_0-2h)=4hf^{'}(x_0)+\frac{8h^3}{3}f^{'''}(x_0)+\frac{32h^5}{60}f^{v}(x_0)+\frac{128h^7}{2520}f^{vii}(x_0)$$

$$2)f^{'}(x_0)=\frac{f(x_0+2h)-f(x_0-2h)}{4h}-\frac{2h^2}{3}f^{'''}(x_0)-\frac{8h^4}{120}f^{v}(x_0)-\frac{32h^6}{2520}f^{vii}(x_0)$$

For $\mathbf{x = x_0 + 3h, x = x_0 - 3h}$: For $\mathbf{x = x_0 + h, x = x_0 - h}$:

$$f(x_0+3h) = f(x_0) + 3hf^{'}(x_0) + \frac{9h^2}{2}f^{''}(x_0) + \frac{27h^3}{6}f^{'''}(x_0) + \frac{81h^4}{24}f^{iv}(x_0) + \frac{243h^5}{120}f^{v}(x_0)$$
$$+\frac{729h^6}{720}f^{vi}(x_0) + \frac{2187h^7}{5040}f^{vii}(x_0) + \frac{6561h^8}{40320}f^{viii}(x_0)$$
$$f(x_0-3h) = f(x_0) - 3hf^{'}(x_0) + \frac{9h^2}{2}f^{''}(x_0) - \frac{27h^3}{6}f^{'''}(x_0) + \frac{81h^4}{24}f^{iv}(x_0) - \frac{243h^5}{120}f^{v}(x_0)$$
$$+\frac{729h^6}{720}f^{vi}(x_0) - \frac{2187h^7}{5040}f^{vii}(x_0) + \frac{6561h^8}{40320}f^{viii}(x_0)$$

If we subtract second from the first:

$$f(x_0+3h) - f(x_0-3h) = 6hf^{'}(x_0) + \frac{27h^3}{3}f^{'''}(x_0) + \frac{243h^5}{60}f^{v}(x_0) + \frac{2187h^7}{2520}f^{vii}(x_0)$$

$$3)f^{'}(x_0) = \frac{f(x_0+3h) - f(x_0-3h)}{6h} - \frac{27h^2}{18}f^{'''}(x_0) - \frac{243h^4}{360}f^{v}(x_0) - \frac{2187h^6}{15120}f^{vii}(x_0)$$

We have found three centered second order approximations. These are given below. I will increase the order by doing algebraic operations.

$$1)f^{'}(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{6}f^{'''}(x_0) - \frac{h^4}{120}f^{v}(x_0) - \frac{h^6}{5040}f^{vii}(x_0)$$
$$2)f^{'}(x_0) = \frac{f(x_0+2h) - f(x_0-2h)}{4h} - \frac{2h^2}{3}f^{'''}(x_0) - \frac{8h^4}{120}f^{v}(x_0) - \frac{32h^6}{2520}f^{vii}(x_0)$$
$$3)f^{'}(x_0) = \frac{f(x_0+3h) - f(x_0-3h)}{6h} - \frac{27h^2}{18}f^{'''}(x_0) - \frac{243h^4}{360}f^{v}(x_0) - \frac{2187h^6}{15120}f^{vii}(x_0)$$

If I subtract 4 times equation 1 from equation 2:

$$a){-3}f^{'}(x_0) = \frac{3f(x_0+2h) - 3f(x_0-2h) + 24f(x_0+h) - 24f(x_0-h)}{12h} - \frac{36h^4}{360}f^{v}(x_0) - \frac{180h^6}{15120}f^{vii}(x_0)$$

If I subtract 9 times equation 1 from equation 3:

$$b){-8}f^{'}(x_0) = \frac{2f(x_0+3h) - 2f(x_0-3h) - 54f(x_0+h) + 54f(x_0-h)}{12h} - \frac{216h^4}{360}f^{v}(x_0) - \frac{2160h^6}{15120}f^{vii}(x_0)$$

We have found two centered fourth order approximations. Then, we will cancel fourth order and we will get sixth order approximation. If I subtract 6 times equation a from equation b:

$$10f^{'}(x_0) = \frac{2\Big(f(x_0+3h) - f(x_0-3h)\Big) + 18\Big(f(x_0-2h) - f(x_0+2h)\Big) + 90\Big(f(x_0+h) - f(x_0-h)\Big)}{12h} - \frac{1080h^6}{15120}f^{vii}(x_0)$$

$$f^{'}(x_0) = \frac{2\Big(f(x_0+3h) - f(x_0-3h)\Big) + 18\Big(f(x_0-2h) - f(x_0+2h)\Big) + 90\Big(f(x_0+h) - f(x_0-h)\Big)}{120h} - \frac{108h^6}{15120}f^{vii}(x_0)$$

Truncation error which is sixth order:

$$\varepsilon(h) = \frac{108h^6}{15120}f^{vii}(x_0)$$

2.1 Python Implementation

Listing 1: Python Implementation of Derivation Algorithm

```python
import math
import matplotlib.pyplot as plt
import numpy as np
import sympy as sym
from sympy import Symbol, Derivative
#Function which returns theoretical and real absolute error
def errorComparasion(f,diff,x0,k,order,h):
    #Finds approximate value of f' using centered difference formula
    res= derivative(f,x0,k,order,h)
    #Calculates h value
    h = math.pow(h,-k)
    #Calculates absolute error
    absolute_error=abs(diff(1,x0)-res)
    #Finds corresponding theoretical error values
    if order==2:
        theoretical = (h**2/6)*diff(3,x0)
    elif order==4:
        theoretical =  (h**4/30)*diff(5,x0)
    elif order==6:
        theoretical =  (h**6/15120)*diff(7,x0)
    #Returns absolute and theoretical errors
    return absolute_error, theoretical
#Function which calculates derivatives with using centered difference
def derivative(f,x0,k,order,h):
    #Calculates h value
    h = math.pow(h,-k)
    #Returns corresponding centered difference formulas for the first derivative
    if order==2:#Returns corresponding
        res= (f(x0+h)-f(x0-h))/(2*h)
        return res
    elif order==4:
        res = (1/(12*h))*( f(x0-2*h)-8*f(x0-h)+8*f(x0+h)-f(x0+2*h))
        return res
    elif order==6:
        res =(2*( f(x0+3*h)-f(x0-3*h))+18*(f(x0-2*h)-f(x0+2*h))+90*(f(x0+h)-f(x0-h)))/(120*h)
        return res
    else:
        #Prints invalid input error
        print("Order can be only 2, 4 or 6")
        return
def f(x):
```

```python
    return math.exp(x)*math.cos(2*x)
#Function which returns higher order derivatives of f function7
def diff(order,x0):
    x = sym.Symbol('x')
    function= sym.exp(x)*sym.cos(2*x)
    for i in range(order):
        function =  Derivative(function, x)
        function.doit()
    return sym.N(function.subs(x, x0))
#Function which prints theoretical and real absolute errors of calculations
def Compare(f,diff,h,k_0,k_1, x_0):
    for order in [2,4,6]:
        print("Order of "+str(order)+":")
        for k in range (k_0,k_1+1):
            print("Step size: "+str(h)+"^"+str(k))
            absolute, theoretical=errorComparasion(f,diff,x_0,k,order,h)
            print("Absolute error:" +str(absolute)+" | "+"Theoretical error:" +str(theoretical))
def Graph(f,diff,h,k_0,k_1, x_0):
    second=[] #list for second order approximations
    fourth=[] #list for fourth order approximations
    sixth=[] #list for sixth order approximations
    h_array=[] #list for keeping h values
    #Fills h array with negative powers of 10 up to -10th power
    for k in range(k_0,k_1+1):
        h1 = math.pow(h,-k)
        h_array.append(h1)
    #Fills lists with approximations for every power of 10
    for order in [2,4,6]:
        for k in range (k_0,k_1+1):
            absolute_error, theoretical =errorComparasion(f,diff,x_0,k,order,h)
            if(order==2):
                second.append(absolute_error)
            if(order==4):
                fourth.append(absolute_error)
            if(order==6):
                sixth.append(absolute_error)
    plt.figure()
    plt.xlabel("h")
    plt.ylabel("Absolute Error")
    plt.loglog( h_array,second,"--",label="second order")
    plt.loglog(h_array, fourth,"--",label="fourth order")
    plt.loglog( h_array,sixth,"--",label="sixth order")
    plt.legend()
```

```
Graph(f,diff,10,1,10, 1)
Compare(f,diff,10,1,10, 1)
```

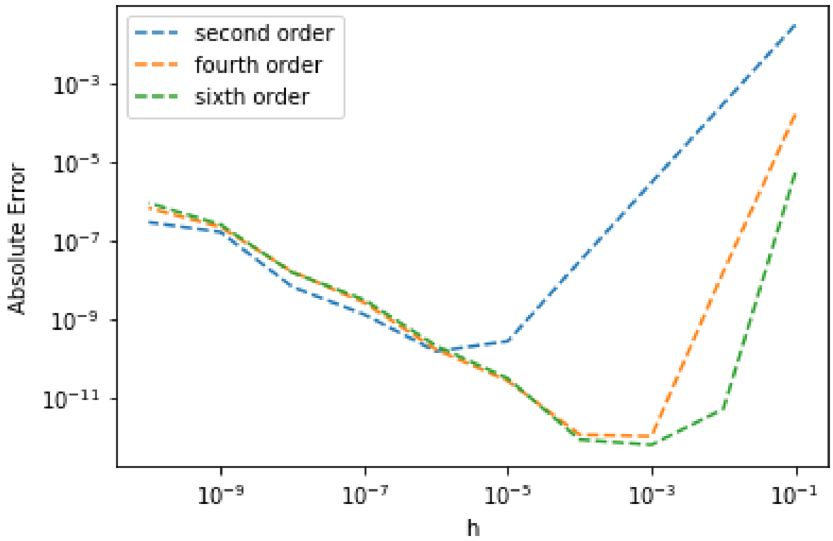

Figure 1: Errors-h & Order Relations of$e^x cos(2x)$

Observations show that higher order formulas are more reliable and they gives more closer results. In the graphic, we can observe that sixth order approximations are in below. But for lower h values, absolute error is getting bigger. Because of round-off errors, computer cannot perform operations smoothly.

In the table which is given in next page, for smaller k values our the theoretical error is very close to real absolute error. But increase in k values causes larger mismatches between theory and real results.

In essence, differentiation operations with using Taylor series give us accurate results, if we choose optimal h values. In my opinion, it is simple and reliable technique.

Absolute Error- Theoretical Findings			
Order	**k**	**Absolute Error**	**Theoretical Error**
2	1	0.0290173150437871	0.0289778359422243
4		0.000155637768665784	0.000158487446007179
6		5.09898860734381e-6	-4.76154062133788e-8
2	2	0.000289782321460486	0.000289778359422243
4		1.58458925980653e-8	1.58487446007179e-8
6		5.14965847742133e-12	-4.76154062133788e-14
2	3	2.89778441153743e-6	2.89778359422243e-6
4		1.04449782156735e-12	1.58487446007179e-12
6		6.35047570085590e-13	-4.76154062133788e-20
2	4	2.89773307571295e-8	2.89778359422243e-8
4		1.15463194561016e-12	1.58487446007179e-16
6		8.59756710269721e-13	-4.76154062133788e-26
2	*5*	2.71405120599866e-10	2.89778359422243e-10
4		2.65050204006911e-11	1.58487446007179e-20
6		3.05755420981768e-11	-4.76154062133788e-32
2	6	1.49279699712679e-10	2.89778359422243e-12
4		1.67784008908711e-10	1.58487446007179e-24
6		2.08491002240407e-10	-4.76154062133788e-38
2	7	1.29401023230002e-9	2.89778359422243e-14
4		2.58927101981499e-9	1.58487446007179e-28
6		3.07036707170028e-9	-4.76154062133788e-44
2	8	6.47755094007607e-9	2.89778359422243e-16
4		1.57294106628569e-8	1.58487446007179e-32
6		1.49892604994761e-8	-4.76154062133788e-50
2	9	1.60055902753697e-7	2.89778359422243e-18
4		2.15567053984955e-7	1.58487446007179e-36
6		2.48873743835532e-7	-4.76154062133788e-56
2	10	2.84033307096365e-7	2.89778359422243e-20
4		6.54107648045965e-7	1.58487446007179e-40
6		8.76152253859175e-7	-4.76154062133788e-62