Probability

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$$p(x|y) = \cancel{p(x) \cdot p(y)} \; \frac{p(x,y)}{p(y)}$$

if x & y are independent.

$$p(x|y) = \frac{p(x)\,p(y)}{p(y)} = p(y)$$

Theorem of total probability.

$$p(x) = \sum_y p(x|y) \cdot p(y) \quad \{\text{discrete case}$$

$$p(x) = \int p(x|y) \cdot p(y)\, dy \quad \text{continuous case}$$

Bayes rule.

$$p(x|y) = \frac{p(y|x) \cdot p(x)}{p(y)} = \frac{p(y|x) \cdot p(x)}{\sum_{x'} p(y|x') \cdot p(x')}$$

(Discrete)

$$p(x|y) = \frac{p(y|x) \cdot p(x)}{p(y)} = \frac{p(y|x) \cdot p(x)}{\int p(y|x') \cdot p(x')\, dx'}$$

(continuous)

~~Function : 1) Initialization~~
~~2) Predict, Sigma points~~

$p(y)$ does not depend on x. Thus, the factor $p(y)^{-1}$ is often written as a 'normalizer' ~~equation~~ variable denoted by (η)

$\Rightarrow p(x|y) = \eta\,(p(y|x) \cdot p(x))$

* We simply use normalizer 'η' to indicate that the final result has to be normalized to 1.

✓ Expectation of random variable X is given by:

$$E(X) = \sum_x x \cdot p(x).$$

$$E(X) = \int x \cdot p(x) \cdot dx.$$

$$E(aX+b) = a \cdot E[X] + b.$$

$$Cov[X] = E[X - E[X]]^2 = E[X^2] - E[X]^2$$

✓ for a discrete random variable, entropy is given by:

$$H(P) = E[-\log_2 p(x)] = -\sum_x p(x) \log_2 p(x)$$

✓ Conditioning Bayes rule:

$$p(x|y,z) = \frac{p(y|x,z)\, p(x|z)}{p(y|z)}$$

$$p(x,y|z) = p(x|z) \cdot p(y|z)$$

$$p(x|z) = p(x|z,y)$$

$$p(y|z) = p(y|z,x)$$

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$$p(x, y|z) = p(x|z) \cdot p(y|z) \not\Rightarrow p(x,y) = p(x) \cdot p(y)$$

✓ belief over state variable x_t.

$$\Rightarrow bel(x_t) = p(x_t | z_{1:t}, u_{1:t})$$

W.K.T

$$\underbrace{p(a|b)}_{\text{posterior}} = \frac{\overbrace{p(b|a)}^{\text{likelihood}} \cdot \overbrace{p(a)}^{\text{prior}}}{\underbrace{p(b)}_{\text{Normalizing constant}}}$$

calculate a posterior before incorporating z_t, just after executing the control u_t. hence posterior is denoted as:

$$\overline{bel}(x_t) = p(x_t | z_{1:t-1}, u_{1:t})$$

↳ predicts the state at time t based on the previous state posterior, before incorporating the measurement at time t.

Calculating $bel(x_t)$ from $\overline{bel}(x_t)$ is called measurement update.

Algorithm for Bayes filter

Algorithm Bayes_filter $(bel(x_{t-1}), u_t, z_t)$:

for all x_t do:

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$$

$$bel(x_t) = \eta\, p(z_t \mid x_t) \cdot \overline{bel}(x_t)$$

endfor

return $bel(x_t)$.

Derivation of Bayes filter

$$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) = p(x_t \mid z_t, z_{1:t-1}, u_{1:t})$$

$$= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})}$$

$$= \eta\, \underbrace{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})}_{\text{observation model}} \cdot p(x_t \mid z_{1:t-1}, u_{1:t}) \quad (*)$$

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t).$$

↳ If we (hypothetically) knew the state x_t and were interested in predicting the measurement z_t, no past measurement or control i/p's provide us information.

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$$p(x_t \mid z_{1:t}, u_{1:t}) = \eta \cdot p(z_t \mid x_t) \cdot p(x_t \mid z_{1:t-1}, u_{1:t})$$
~~(correction model)~~ ——— (a)

hence,

$$bel(x_t) = \eta\, p(z_t \mid x_t) \cdot \overline{bel}(x_t)$$

Next, expand term $\overline{bel}(x_t)$.

$$\overline{bel}(x_t) = \overbrace{p(x_t \mid z_{1:t-1}, u_{1:t})}^{\text{motion model}}$$

$$= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \cdot p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \cdot dx_{t-1}$$ ——— (b).

Markov Assumption for Motion model

W.K.T $\{ p(A) = \int p(A \mid B') \cdot p(B')\, dB'$.

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$ ——— (c)

→ This implies if we know x_{t-1}, past measurements & controls convey no information regarding x_t. This gives above equation

Here we retain the control i/p u_t, since it does not kill the state x_{t-1}.

$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t) \cdot p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$$
$$= \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx_{t-1}$$
(refer above motion model)

u_t

$bel(x_t) = p(x_t | z_t, z_{1:t-1}, u_{1:t})$

$bel(x_t) = \eta \, p(z_t | x_t) \cdot \overline{bel}(x_t)$

where,

$$\overline{bel}(x_t) = \int \underbrace{p(x_t | x_{t-1}, u_t)}_{\text{transition model}} \cdot bel(x_{t-1}) \, dx_{t-1}$$

Motion model $\longrightarrow$ prediction step.

Observation model $\longrightarrow$ update step

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Observation model

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

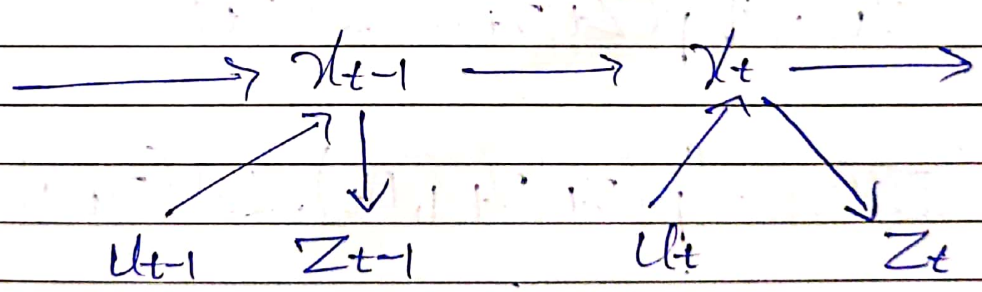
where, $z_{1:t} = \{ z_1, \dots z_t \}$

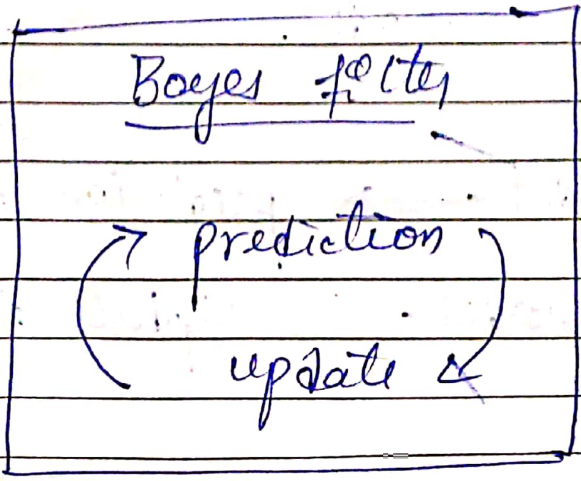
$$z_t = [z_t^1, \dots z_t^k]$$

$$\therefore p(z_t \mid x_t) = p(z_t^1, \dots, z_t^k \mid x_t)$$

$$= \prod_{k=1}^{K} p(z_t^k \mid x_t)$$

Summary of Bayes filter





Particle filter

Given the previous set of particles X_{t-1}, u_t & Z_t, we perform each of the following steps:-

1) Apply the forward motion model for all the particles. This means that, for each i, we sample x_t^i from $p(x_t | x_{t-1}^i, u_t)$

2) Update the weights for each particles, which means we set $w^i = p(Z_t | x_t^i)$ for each i.

3) Normalize the weights, set $w_i = \frac{w_i}{\sum_i w_i}$ for each i.

4) Perform the resampling step. We resample each particle according to the weights w^i.

5) Reset each particle's weight to $\frac{1}{n}$ & return the set of particles.

$X_t = \{ x_t^{[1]}, x_t^{[2]}, \ldots \; x_t^{[M]} \}$

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Particle filter Pseudocode (GPS)

Previous belief — sensor measurement

Particle filter (X_{t-1}, u_t, Z_t)

↳ Actuation command (Yaw rate, Velocity)

Initialization ← $X_t = X_t = \phi$

we estimate our position from sensor (GPS or ~~[illegible]~~ i/p), later we refine this estimate to localise our vehicle.

for $m = 1$ to M:

Prediction ← Sample $x_t^{[m]} \sim p(x_t | u_t, x_{t-1}^{[m]})$

we add control i/p's (Yaw rate, velocity) to all the particles.

update ←
$$\begin{cases} w_t^{[m]} = p(Z_t | x_t^{[m]}) \\ \bar{X}_t = \bar{X}_t + \langle x_t^{[m]}, w_t^{[m]} \rangle \end{cases}$$

we update our particle weights using map landmark positions and feature ~~the~~ measurements (lidar & radar fused data)

end for

Resampling ←
for $m = 1$ to M:
draw i with probab $\propto w_t^{[i]}$
add $x_t^{[i]}$ to X_t
end for.

return X_t

return new particle set. Now we have refined estimate of the vehicles position based on i/p evidence

'f' → prob. density funcⁿ (target distributⁿ)

The distributⁿ that correspond the density 'g' is called proposal distributⁿ

density 'g' must be such that $f(x) > 0$
$\Rightarrow g(x) > 0$.

In particular, for any interval $A \subseteq \text{range}(X)$, the emperical count of particles that fall into 'A', converges to the integral of 'g' under 'A':

$$\frac{1}{M} \sum_{m=1}^{M} I(x^{[m]} \in A) \longrightarrow \int_A g(x) \cdot dx.$$

To offset this difference betⁿ f and g, particles $x^{[m]}$ are weighted by quotient

$$w^{[m]} = \frac{f(x^{[m]})}{g(x^{[m]})}.$$

Mathematical derivation of particle filter

Think of particles as samples of state sequences.

$$x_{0:t}^{[m]} = x_0^{[m]}, x_1^{[m]}, \ldots x_t^{[m]}$$

This particle filter calculates the posterior over all sequences:

$$bel(x_{0:t}) = p(x_{0:t} | u_{1:t}, z_{1:t})$$

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Instead of belief $bel(x_t) = p(x_t | u_{1:t}, z_{1:t})$

$$bel(x_{0:t}) = p(x_{0:t} | u_{1:t}, z_{1:t})$$
(target distribution)

$$= p(x_{0:t} | z_{1:t}, u_{1:t})$$

$$= \eta \, p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) \cdot p(x_{0:t} | z_{1:t-1}, u_{1:t})$$

(Bayes) — from (A).

$$= \eta \, p(z_t | x_t) \cdot p(x_{0:t} | z_{1:t-1}, u_{1:t})$$

(Markov) — from (a).

$$= \eta \, p(z_t | x_t) \cdot p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) \cdot p(x_{0:t-1}, z_{1:t-1}, u_{1:t})$$

— from (b)

$$= \eta \, p(z_t | x_t) \cdot p(x_t | x_{t-1}, u_t) \cdot p(x_{0:t-1} | z_{1:t-1}, u_{1:t-1})$$

— from (c).

Assuming that our I particle set is obtained by sampling the prior $p(x_0)$.

Let us assume that the particle set at time $t-1$ is distributed according to $bel(x_{0:t-1})$.

For the m^{th} particle $x_{0:t-1}^{[m]}$ in this set, the sample $x_t^{[m]}$ generated from the proposal
(prediction) distribution (q)

$$p(x_t \mid x_{t-1}, u_t) \cdot bel(x_{0:t-1})$$
$$= p(x_t \mid x_{t-1}, u_t)\, p(x_{0:t-1} \mid z_{0:t-1}, u_{0:t-1})$$
(proposal distribution)

with

$$w_t^{[m]} = \frac{\text{target distribution}}{\text{proposal distribution}}$$

$$= \eta\, p(z_t \mid x_t) \quad \text{—— (d)}$$

The constant 'η' plays no role since the resampling takes place with probabilities proportional to importance weights.

By resampling particles with probability proportional to $w_t^{[m]}$, The resulting particles are indeed distributed according to the product of proposal & the importance weight $w_t^{[m]}$

$$\eta \, w_t^{[m]} \, p(x_t \mid x_{t-1}, u_t) \cdot p(x_{0:t-1} \mid z_{0:t-1}, u_{0:t-1})$$

$$= bel(x_{0:t})$$

(e)

Note: Constant factor 'η' here differs.

If $x_{0:t}^{[m]}$ {particle set} is distributed according to $bel(x_{0:t})$, then the state sample $x_t^{[m]}$ is distributed according to $bel(x_t)$.

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Bayes filter framework

$p(x_t \mid u_{t-1}, x_{t-1})$ → Action model

Given,

1) prior prob. of state system $p(x)$

2) likelihood sensor model. $p(z \mid x)$

3) stream of observations 'z' & control data 'u':

$data := \{u_1, z_2, \ldots, u_{t-1}, z_t\}$

find :- Estimate of the state x at time t.

- posterior also called belief:

$Bel(x_t) = p(x_t \mid u_1, z_2 \ldots u_{t-1}, z_t)$

Sensor independence

$$p(Z_t \mid x_{0:t}, Z_{1:t-1}, u_{1:t}) = p(Z_t \mid x_t)$$

$$p(x_t \mid x_{1:t-1}, Z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Markov rules

Bayes rule

likelihood. prior

$$p(x|z) = \frac{p(z|x) \cdot p(x)}{p(z)}$$ normalizing constant.

$$= \eta \, p(z|x) \cdot p(x) \propto p(z|x) \cdot p(x)$$

Baye's filters

$$Bel(x_t) = p(x_t \mid u_1, z_2, \ldots, u_{t-1}, z_t)$$

(Bayes)

$$= \eta \, p(z_t \mid x_t, u_1, z_2 \ldots u_{t-1}) \cdot p(x_t \mid u_1, z_2, \ldots, u_{t-1})$$

(sensor ind)

$$= \eta \, p(z_t \mid x_t) \, p(x_t \mid u_1, z_2 \ldots, u_{t-1})$$

(total probability of prior)

$$= \eta \, p(z_t \mid x_t) \int p(x_t \mid u_1, z_2 \ldots, u_{t-1}, x_{t-1}) \cdot p(x_{t-1} \mid u_1, z_2 \ldots, u_{t-1}) \, dx_{t-1}$$

{ what is the probab. of going from x_{t-1} to x_t

$\{p(a|b,c) = \int p(a|d) \cdot p(d|b,c)\, d\cancel{b}\}$

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markov rule

$$= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1}) \cdot p(x_{t-1}|u_1, z_2 \ldots, u_{t-1})\, dx_{t-1}$$

$$\therefore Bel(x_t) = \eta\, p(z_t|x_t) \underbrace{\int p(x_t|u_{t-1}, x_{t-1}) \cdot Bel(x_{t-1})\, dx_{t-1}}$$

Recursive.

prediction Before taking measurement (like in KF)