Unscented Kalman Filter (UKF)

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✓ Motion models.

✓ In EKF we used constant velocity (CV) model.

✓ in UKF we use constant turn rate velocity (CTRV) model

* CV model predict turning rate incorrectly.

✓ CTRV model state vector.

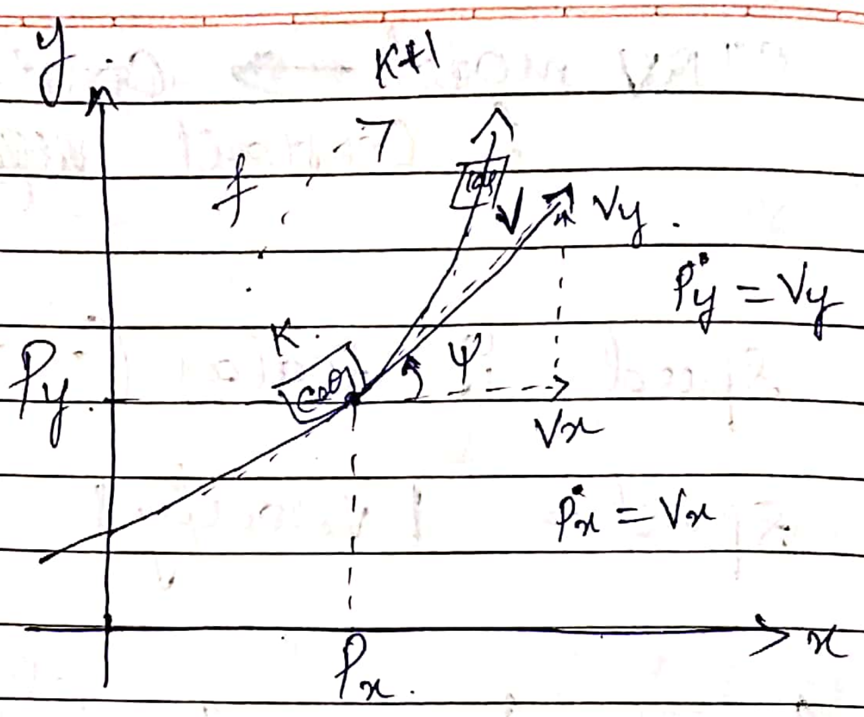
→ speed (v) → magnitude of velocity

→ Yaw angle (ψ) or orientation

→ Rate of change of Yaw ($\dot{\psi}$) (Yaw rate)

$$x = \begin{pmatrix} P_x \\ P_y \\ v \\ \psi \\ \dot{\psi} \end{pmatrix}$$

CV model predict turning rate incorrectly hence we go for CTRV model



* State Vector (x)

$$x_k = \begin{pmatrix} P_x \\ P_y \\ V \\ \psi \\ \dot{\psi} \end{pmatrix}$$

* Process model

$$x_{k+1} = f(x_k, v_k)$$

* Changle rate of State ($\dot{x}$)

$$\dot{x} = \begin{pmatrix} \dot{P_x} \\ \dot{P_y} \\ \dot{v} \\ \dot{\psi} \\ \ddot{\psi} \end{pmatrix}$$

<u>Differential Equation</u>

$$\dot{x} = g(x)$$

$$\begin{bmatrix} \dot{P_x} = \cos(\psi) \cdot v \\ \dot{P_y} = \sin(\psi) \cdot v \\ \dot{v} = 0 \\ \dot{\psi} = \dot{\psi} \\ \ddot{\psi} = 0. \end{bmatrix}$$

$\dot{P_x} = \frac{d}{dx}(P_x) = V_x$

V_y.

$\dot{V} = \frac{d}{dx}(\dot{v}) = 0.$

$\hookrightarrow$ constant velocity

$\dot{\psi} = \frac{d}{dx}(\psi)$

$\ddot{\psi} = \frac{d}{dx}(\dot{\psi}) = 0$

$\hookrightarrow$ constant yaw rate

$\cos(\psi(t))dt \rightarrow \cos(\psi_k + \dot{\psi}_k(t - t_k))\,dt$.

Here we can explicitly express yaw angle as a function of t, using the assumption of constant yaw rate.

Time difference

$$\Delta t = t_{k+1} - t_k \quad (0.1\ \text{sec})$$

Integral

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} \begin{bmatrix} \dot{p}_x(t) \\ \dot{p}_y(t) \\ \dot{v}(t) \\ \dot{\psi}(t) \\ \ddot{\psi}(t) \end{bmatrix} dt$$

$$\Rightarrow x_{k+1} = x_k + \begin{bmatrix} \int_{t_k}^{t_{k+1}} v(t)\cdot\cos(\psi(t))\cdot dt \\ \int_{t_k}^{t_{k+1}} v(t)\cdot\sin(\psi(t))\,dt \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix}$$

{0 -- we have assumed constant velocity & yaw rate for our CTRV model}.

{ constant velocity }

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$$x_{k+1} = x_k + \begin{bmatrix} V_k \int_{t_k}^{t_{k+1}} \cos\left(\psi_k + \dot{\psi}_k \cdot (t - t_k)\right) dt \\ V_k \int_{t_k}^{t_{k+1}} \sin\left(\psi_k + \dot{\psi}_k \cdot (t - t_k)\right) dt \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix}$$

$$\Rightarrow x_{k+1} = x_k + \begin{bmatrix} \frac{V_k}{\dot{\psi}_k}\left(\sin(\psi_k + \dot{\psi}_k \Delta t) - \sin(\psi_k)\right) \\ \frac{V_k}{\dot{\psi}_k}\left(-\cos(\psi_k + \dot{\psi}_k \Delta t) + \cos(\psi_k)\right) \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix}$$

$\Delta t = t_{k+1} - t_k$
(0.1 sec)

~~$\dot{\psi}$ (Yaw rate is 0 means~~ vehicle is driving on straight road

(Process noise)

* Noise vector

$$V_K = \begin{bmatrix} V_{a,k} \\ V_{\ddot{\psi},k} \end{bmatrix}$$

* Process model

$$x_{k+1} = f(x_k, V_k)$$

$V_{a,k}$ = longitudinal acceleration noise

$$V_{a,k} \sim N(0, \sigma_a^2)$$

$V_{\ddot{\psi},k}$ = Yaw acceleration noise

$$V_{\ddot{\psi},k} \sim \underbrace{N(0, \sigma_{\ddot{\psi}}^2)}$$

It is normal distributed white noise with 0 mean & variance (σ^2) of $\ddot{\psi}$

> It is normal distributed white noise with 0 mean & variance of acceleration

* Process model

$$x_{k+1} = x_k + \begin{bmatrix} \frac{v_k}{\dot{\psi}_k}\left(\sin(\psi_k + \dot{\psi}_k \Delta t) - \sin(\psi_k)\right) \\ \frac{v_k}{\dot{\psi}_k}\left(-\cos(\psi_k + \dot{\psi}_k \Delta t) + \cos(\psi_k)\right) \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}}_{\nu_k}$$

✓ Influence Yaw acceleration on Yaw rate ($\dot{\psi}$)

$$e = \Delta t \cdot \nu_{\ddot{\psi},k}$$

✓ Influence of $\nu_{a,k}$ & $\nu_{\ddot{\psi},k}$ on velocity (v)

$$c = \Delta t \cdot \nu_{a,k}$$

✓ Influence of $\nu_{a,k}$ & $\nu_{\ddot{\psi},k}$ on Yaw angle (ψ)

$$d = \frac{1}{2}(\Delta t)^2 \cdot \nu_{\ddot{\psi},k}$$

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* acceleration offset if the car were driving perfectly straight ('a)

$$a = \frac{1}{2} (\Delta t)^2 \cdot \cos(\varphi_k) \cdot V_{a,k}$$

y acceleration offset if the car were driving perfectly straight (b)

$$b = \frac{1}{2} (\Delta t)^2 \cdot \sin(\varphi_k) \cdot V_{a,k}$$

* Till now we learnt CTRV model eqn's

* In EKF we used Jacobian matrix to linearize the non-linear function.

* Here in UKF, no need to linearize non-linear funcn's, instead UKF takes representative points from a Gaussian distributn.
These points will be plugged into the non-linear eqn's.

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* prediction — Generate sigma points
  predict sigma points
  predict mean & covariance

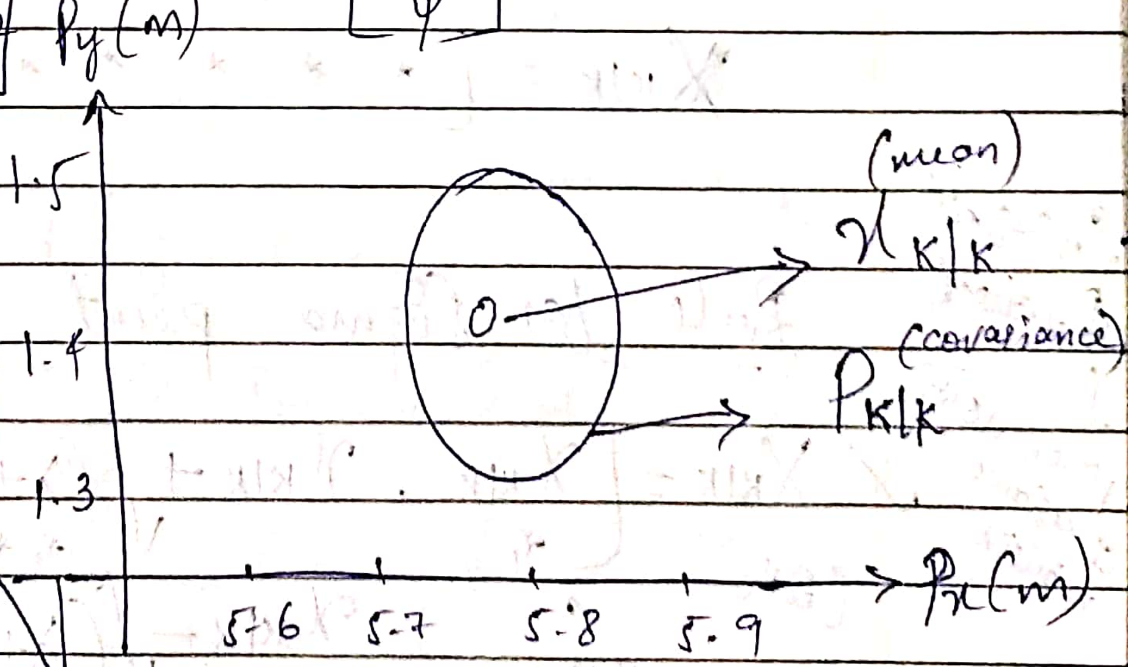
update — predict measurement
  update state

<u>Map</u> → <u>Generate sigma points</u>

<u>State vector</u> → $x = \begin{bmatrix} P_x \\ P_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$

State Dimension
$n_x = 5$
($\because x$ has 5 var's)

$P_{k|k} = \begin{bmatrix} 0.0043m^2 & -0.0013m^2 \\ -0.0013m^2 & 0.0077m^2 \end{bmatrix}$



Let,

$x_{k|k} = \begin{bmatrix} 5.7441m \\ 1.3800m \end{bmatrix}$

$P_{k|k} = \begin{bmatrix} 0.0065m & 0m \\ -0.019m & 0.085m \end{bmatrix}$

* Sigma points serve as a representative of whole distribution, Next we calculate the mean & covariance of these group of sigma points.

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No. of Sigma points

$n_\sigma = 2n_x + 1 = 11$

∴ matrix of sigma point size
$= (n_x, 2n_x+1)$
$= (5, 11)$
i.e. (5×11)

* * * * *

*

* * * * *

mean

Let :- $x = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$ $n_x = 2$

$\therefore n_\sigma = 2n_x + 1 = 5$

matrix with sigma points

$\mathcal{X}_{k|k} = [\ * \quad * \quad * \quad * \quad *\ ]$

Rule for sigma point matrix

"without Q matrix" ↓ process noise covariance matrix

$$X_{k|k} = \left[\underset{*_1}{x_{k|k}} \ . \ \underset{*_2\ *_3}{x_{k|k} + \sqrt{(\lambda + n_x) P_{k|k}}} \ . \ \underset{*_4\ .\ *_5}{x_{k|k} - \sqrt{(\lambda + n_x) P_{k|k}}} \right]$$

(2×5)

mean σ point

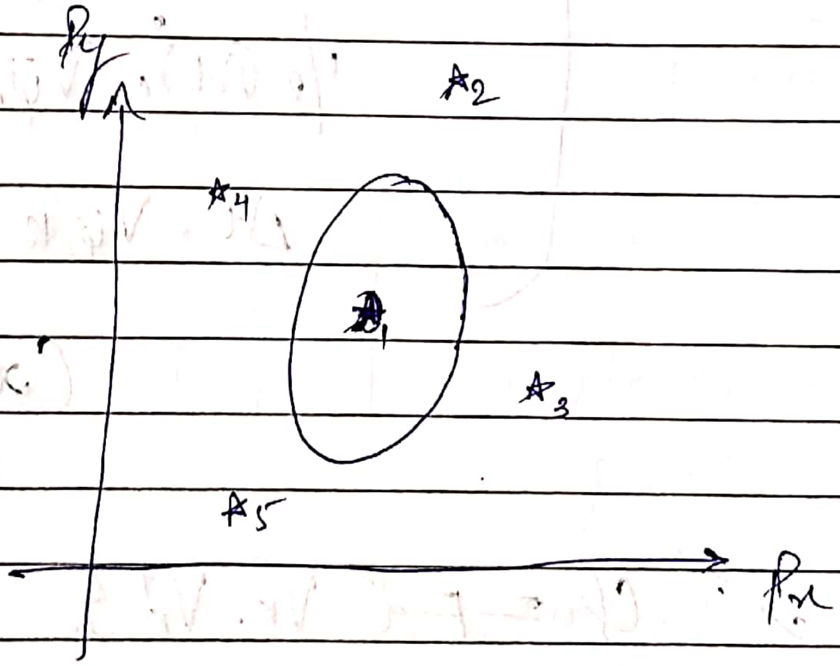
* $\lambda \rightarrow$ decides how far we need to spread sigma points

✓ λ = Design parameter (Scaling factor)

$\lambda = 3 - n_x = 3 - 2 = 1$

✓ $P_{K|K} = A^T A \Rightarrow \sqrt{P_{K|K}} = A$

$$A = \sqrt{P_{K|K}} = \begin{bmatrix} 0.00656m & 0m \\ -0.0191m & 0.0855m \end{bmatrix}$$



* ∴ Sigma point matrix size

$= (n_x, 2.n_x + 1)$
$= (2, 5)$
ie 2×5

UKF Augmentation

* Process noise

(Given) $V_K = \begin{bmatrix} V_{a,K} \\ V_{\ddot{\psi},K} \end{bmatrix}$ $\begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$ $\begin{matrix} m/s^2 \\ rad/s^2 \end{matrix}$

$$V_K = \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \cos(\psi_K) \cdot V_{a,K} \\ \frac{1}{2}(\Delta t)^2 \sin(\psi_K) \cdot V_{a,K} \\ \Delta t \cdot V_{a,K} \\ \frac{1}{2}(\Delta t)^2 \cdot V_{\ddot{\psi},K} \\ \Delta t \cdot V_{\ddot{\psi},K} \end{bmatrix}$$

(refer process model)

✓ $Q = E\{V_K \cdot V_K^T\}$

↓

process noise covariance vector (matrix)

(illy in KF)

→ Representation of process noise covariance matrix (Q) with sigma points is Augmented state.

$$\therefore Q = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_{\ddot{\psi}}^2 \end{bmatrix}$$

✓ Representation of process noise covariance matrix (Q) with sigma points-

Augmented state

$$\underset{(7\times1)}{x_{a,k}} = \begin{bmatrix} P_x \\ P_y \\ V \\ \psi \\ \dot{\psi} \\ V_a \\ V_{\ddot{\psi}} \end{bmatrix}$$

$Q \left\{ V_a, V_{\ddot{\psi}} \right.$

Augmented state Diamension

$n_a = 7$

$\therefore$ No. of sigma points

$n_\sigma = 2n_a + 1 = 15$

* * * * * * * *

* * * * * * * *

Augmented Covariance matrix

$$\underset{(7\times7)}{P_{a,k|k}} = \begin{bmatrix} P_{k|k} & 0 \\ 0 & Q \end{bmatrix}$$

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(Generate)

Calculate Augmented Sigma points (including $\ddot{\varphi}$)

$$X_{a,k|k} = \left[x_{a,k|k} \quad x_{a,k|k} + \sqrt{(\lambda+n_a) P_{a,k|k}} \quad x_{a,k|k} - \sqrt{(\lambda+n_a) P_{a,k|k}} \right]$$

(7×15)

$(n_a, 2n_a+1)$

with scaling factor, $\lambda = 3 - n_a$

given $\begin{cases} V_a = \text{longitudinal acceleration noise } (m/s^2) \\ V_{\ddot{\varphi}} = \text{Yaw acceleration noise } (m/s^2) \end{cases}$

* for Augmented state, each sigma point has 7 parameters
{total " " = 15}
* for Predicted state each sigma point has 5 parameters

Predict Sigma Points

$$\underset{(7\times15)}{\mathcal{X}_{a,k|k}} = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

$$x_{k+1} = f(x_k, \nu_k)$$

ν_k → process noise

(process model)

$$\underset{(5\times15)}{\mathcal{X}_{k+1|k}} = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

Now, we put every sigma points to process model

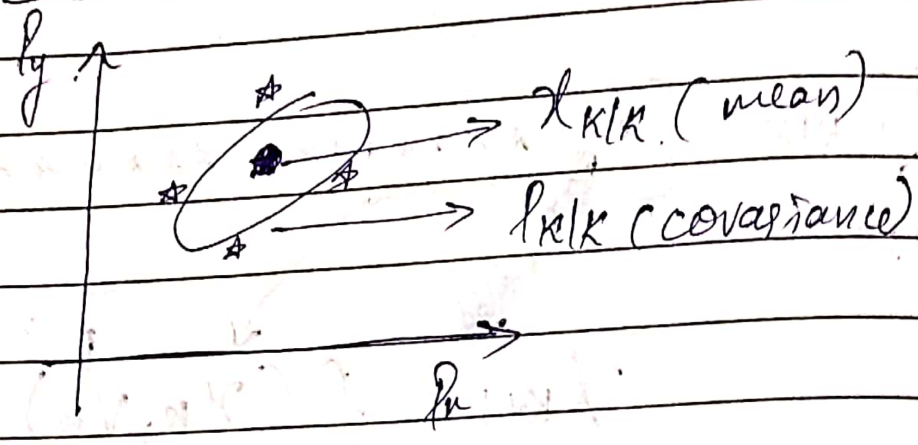
Sigma point in Augmented state $(\mathcal{X}_{a,k|k})$ (7x15) =

$$\begin{bmatrix} p_x & \cdots \\ p_y & \cdots \\ v & \cdots \\ \psi & \cdots \\ \dot{\psi} & \cdots \\ \nu_a & \cdots \\ \nu_{\ddot{\psi}} & \cdots \end{bmatrix}_{(7\times15)}$$

Sigma point in Predicted state $(\mathcal{X}_{k+1|k})$ (5x15) =

$$\begin{bmatrix} p_x & \cdots \\ p_y & \cdots \\ v & \cdots \\ \psi & \cdots \\ \dot{\psi} & \cdots \end{bmatrix}_{(5\times15)}$$

Predict mean & Covariance

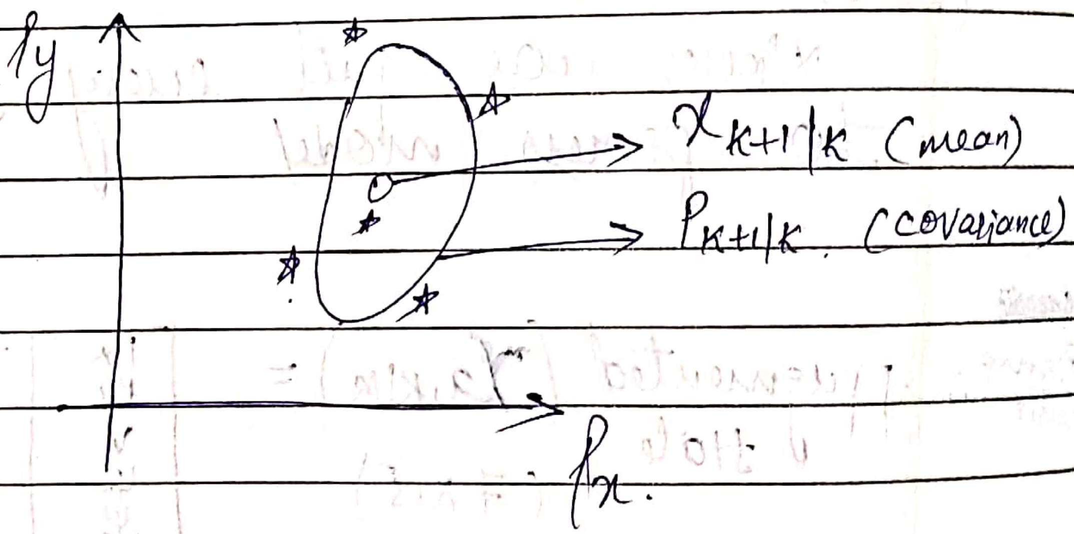


x - Px
y - Py
Vx
Vy

process model..

$x_{k+1} = f(x_k, v_k)$

↳ process noise



* λ → decides how far we need to spread sigma points.

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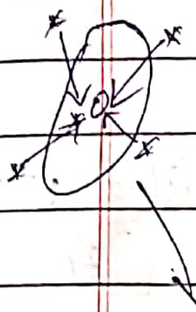
✓ Predicted mean (state vector @ $k+1$)

i = column

$$x_{k+1|k} = \sum_{i=0}^{n_a} w_i \mathcal{X}_{k+1|k,i}$$
(5×1)

→ i → each column

→ weights → $w_i = \dfrac{\lambda}{\lambda + n_a}$, $i = 0$



$$w_i = \frac{1}{2(\lambda + n_a)}, \quad i = 1 \ldots n_a$$

∫ weights invert the spreading of sigma points (i.e inverse of what λ does)

✓ Predicted covariance (state covariance mat. @ $k+1$) i.e $P_{k+1|k}$

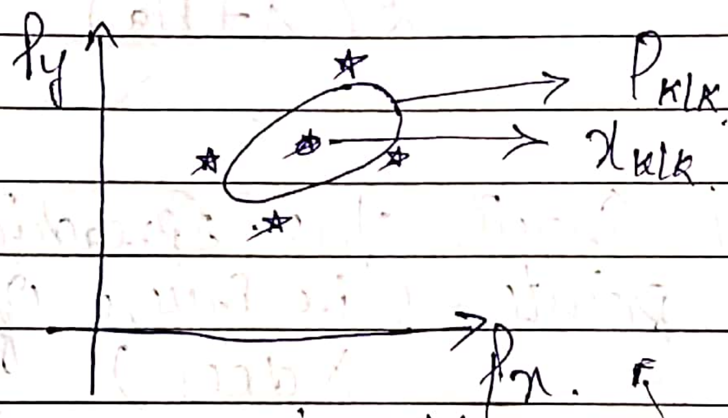
$$P_{k+1|k} = \sum_{i=0}^{2n_a} w_i \left(\mathcal{X}_{k+1|k,i} - x_{k+1|k}\right)\left(\mathcal{X}_{k+1|k,i} - x_{k+1|k}\right)^T$$
(5×5)

Sigma point Generation

$$\mathcal{X}_{a,k|k} = \left[x_{a,k|k} \quad x_{a,k|k} + \sqrt{(\lambda + n_a) P_{a,k|k}} \quad x_{a,k|k} - \sqrt{(\lambda + n_a) P_{a,k|k}} \right]$$

① predict sig. pts in terms of $\rho, \varphi, \dot{\rho}$
+
② predict mean & cov. in terms of $\rho, \varphi, \dot{\rho}$

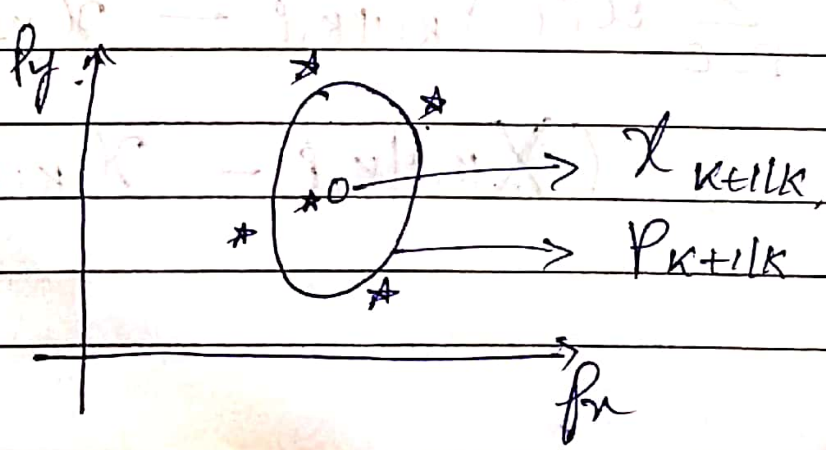
Predict Measurement



Predicted state mean & covariance

(pass these sigma points in this model)

Process model
$$x_{k+1} = f(x_k, \nu_k)$$



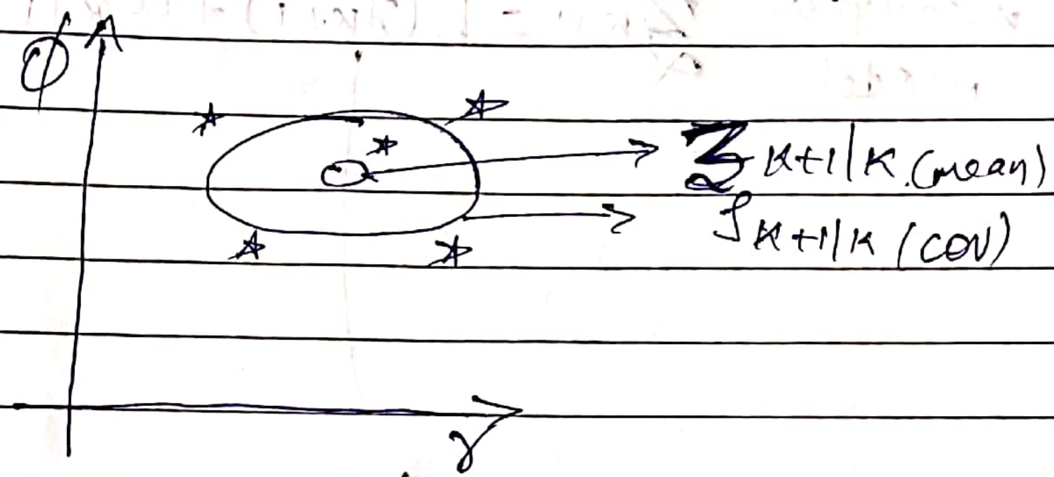
(pass these sigma points in model) → measurement model.

(∵ we later consider R)

→ Measurement model

$$Z_{k+1} = h(x_{k+1}) + w_{k+1}$$

predicted measurement mean & Covariance



w_{k+1} = measurement noise

$h(x_{k+1})$ → is a non-linear function

⇒ Here, we are not again generating sigma points but we we are using same sigma " which we've generated earlier (not augmented sigma points ∵ here we're considering w_{k+1} ie measurement noise). the Calculate the mean (state vector @ k+1) & covariance (state covar-matrix @ k+1).

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predicted sigma points $\mathcal{X}_{k+1|k} = [\ast \; \ast \ast \ast \ast \ast \ast \ast \ast \;\; \ast \ast \ast \ast \ast \ast \ast]$ (5×15)

State Vector

$$x_{k+1|k} = \begin{bmatrix} P_x \\ P_y \\ V \\ \psi \\ \dot{\psi} \end{bmatrix}$$

measurement model $z_{k+1} = h(x_{k+1}) + \overset{0}{\cancel{w_{k+1}}}$

measurement sigma points $\mathcal{Z}_{k+1|k} = [\ast \; \ast \ast \ast \ast \ast \ast \ast \;\; \ast \ast \ast \ast \ast \ast \ast]$ (3×15)

measurement Vector

$$z_{k+1|k} = \begin{bmatrix} \rho \\ \varphi \\ \dot{\rho} \end{bmatrix}$$

where, $\rho = \sqrt{P_x^2 + P_y^2}$

$$\varphi = \arctan\left(\frac{P_y}{P_x}\right)$$

$$\dot{\rho} = \frac{P_x \cdot \cos(\psi) \cdot V + P_y \sin(\psi) \cdot V}{\sqrt{P_x^2 + P_y^2}}$$

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(2) ✓ Predicted measurement mean

$$z_{k+1|k} = \sum_{i=1}^{n_\sigma} w_i Z_{k+1|k,i}$$

(1) ✓ Measurement model

$$z_{k+1|k} = h(x_{k+1}) + \overset{0}{w_{k+1}}$$

$\rightarrow$ measurement noise

(3) ✓ Predicted measurement Covariance

$$S_{k+1|k} = \sum_{i=0}^{2n_\sigma} w_i \left(Z_{k+1|k,i} - z_{k+1|k}\right)\left(Z_{k+1|k,i} - z_{k+1|k}\right)^T + R$$

✓ Measurement noise Covariance

$$R = E\{w_k \, w_k^T\}$$

$$R = \begin{bmatrix} \sigma_\rho^2 & 0 & 0 \\ 0 & \sigma_\varphi^2 & 0 \\ 0 & 0 & \sigma_{\dot\rho}^2 \end{bmatrix}$$

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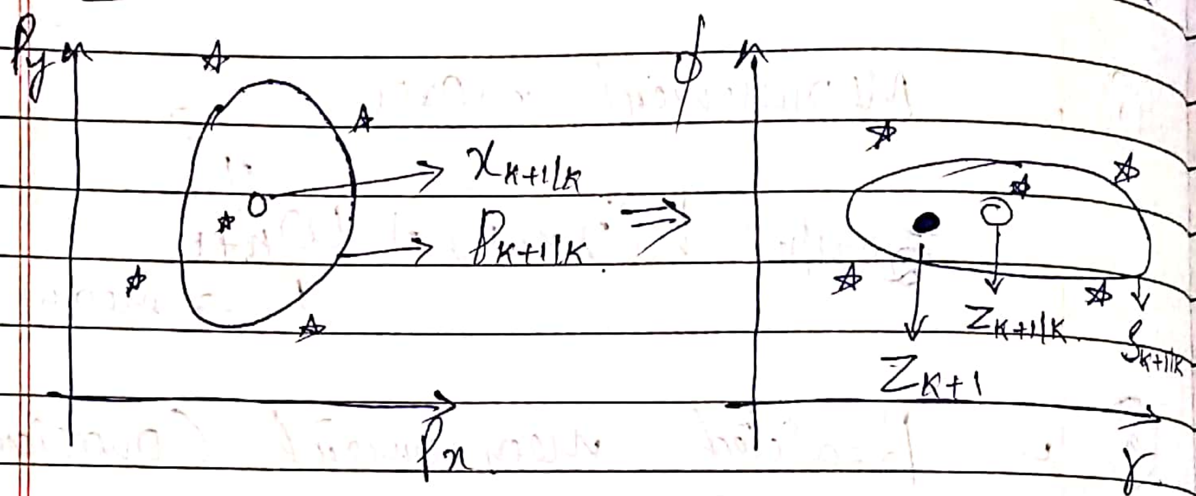
UKF update state



(2) Kalman Gain

$$K_{k+1|k} = T_{k+1|k} S_{k+1|k}^{-1}$$

(3) State update

$$x_{k+1|k+1} = x_{k+1|k} + K_{k+1|k} (z_{k+1} - z_{k+1|k})$$

(5×1). $\begin{pmatrix} P_x \\ P_y \\ v \\ \psi \\ \dot{\psi} \end{pmatrix}$

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(4) Covariance matrix update

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k} S_{k+1|k} K^T_{k+1|k}$$
(5×5)

(1) Cross-correlation betn sigma points in state-space and measurement space

$$T_{k+1|k} = \sum_{i=0}^{2n_\sigma} w_i \left(\mathcal{X}_{k+1|k,i} - x_{k+1|k} \right) \left(Z_{k+1|k,i} - z_{k+1|k} \right)^T$$

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Predict sigma points (cont...)

$$x = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$$

If $\dot{\psi}_k$ is not zero i.e. {vehicle is turning}

$$x_{k+1} = x_k + \begin{bmatrix} \frac{v_k}{\dot{\psi}_k}\left(\sin(\psi_k + \dot{\psi}_k \Delta t) - \sin(\psi_k)\right) \\ \frac{v_k}{\dot{\psi}_k}\left(-\cos(\psi_k + \dot{\psi}_k \Delta t) + \cos(\psi_k)\right) \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \cos(\psi_k) v_{a,k} \\ \frac{1}{2}(\Delta t)^2 \sin(\psi_k) v_{a,k} \\ \Delta t \cdot v_{a,k} \\ \frac{1}{2}(\Delta t)^2 v_{\ddot{\psi},k} \\ \Delta t \cdot v_{\ddot{\psi},k} \end{bmatrix}$$

If $\dot{\psi}_k$ is zero i.e. {vehicle is moving straightly}

$$x_{k+1} = x_k + \begin{bmatrix} v_k \cos(\psi_k) \cdot \Delta t \\ v_k \sin(\psi_k) \cdot \Delta t \\ 0 \\ \dot{\psi}_k \Delta t \quad (0) \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \cos(\psi_k) v_{a,k} \\ \frac{1}{2}(\Delta t)^2 \sin(\psi_k) v_{a,k} \\ \Delta t \cdot v_{a,k} \\ \frac{1}{2}(\Delta t)^2 v_{\ddot{\psi},k} \\ \Delta t \cdot v_{\ddot{\psi},k} \end{bmatrix}$$

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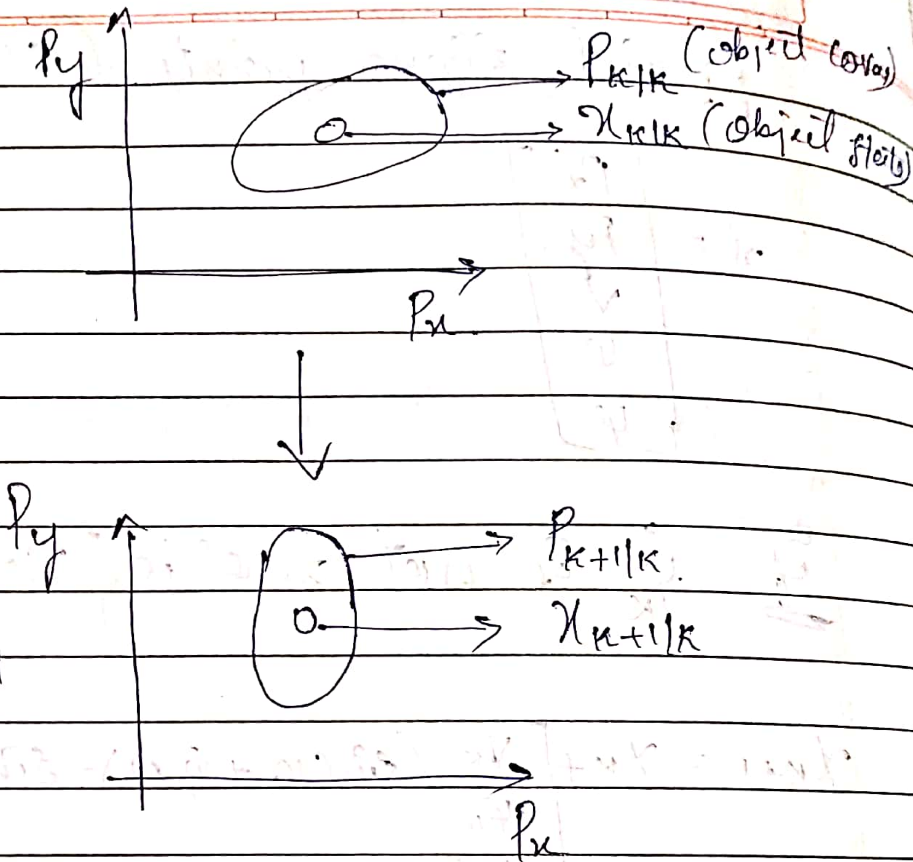
@ K

@ K+1



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✓ What does this elipse mean

→ It visualises the uncertainity distribution of our.

✓ All points on this elipse has same probability density.

✓ If the uncertainty is normally distributed then the shape will be elipse.

✓ It is the visualization of covariance matrix P.