Nuclear Moments of Inertia

By T. H. R. SKYRME
Atomic Energy Research Establishment, Harwell, Berks.

Communicated by B. H. Flowers; *MS. received* 14*th March* 1957

Abstract. A formula proposed by the author for the calculation of the moment of inertia of a nuclear system is discussed. A simple calculation using the model of a deformed oscillator well for the mean nuclear potential gives the result $M\Sigma z^2$ for a prolate distribution, approximately half the rigid moment; it is suggested that this result incorporates most of the effects of the residual interactions between nucleons. The problem of other potential shapes and the velocity-dependence of the potential is discussed qualitatively.

§ 1. Introduction

Many nuclei are now known to exhibit bands of energy levels with the spectra

$$E_I = (\hbar^2/2\mathscr{I})I(I+1) \qquad \ldots\ldots(1)$$

where I is the angular momentum of the level and $\mathscr{I}$ is an empirical constant (for each band) called the moment of inertia by analogy with the spectrum of energy levels of a rigid rotating top. These are especially well developed in the nuclei with masses between 150 and 190 and greater than 225, nuclei which are also considered to be markedly deformed into an ellipsoidal shape on the basis of their large electric quadrupole moments (and quadrupole transition matrix elements). The experimental information has been collated by various authors, Bohr and Mottelson (1955), Moszkowski (1956), Blin-Stoyle (1956). The present paper offers some theoretical comments on the magnitude of $\mathscr{I}$.

A qualitative explanation of the deformation of these nuclei was given by Rainwater (1950), who showed that on the basis of the independent particle model of the nucleus the state of lowest energy would generally be deformed due to the polarizing action of those (outermost) nucleons which were in incompletely filled shells. Calculations have been made by various authors, for example Nilsson (1955), showing that indeed large deformations would be expected in the mass-regions mentioned due to the overlapping and mixing of a large number of single-particle levels.

The self-consistency of this deformed potential was discussed by Bohr and Mottelson (1953) in terms of a hydrodynamic model in which the potential was ascribed to a 'core' of nucleons whose motion was assumed to be that of an incompressible, irrotational fluid, having a surface tension tending to oppose the polarizing action of the individual nucleons coupled to the potential. In the 'strong-coupling' limit of this model, the system would execute small vibrations about an equilibrium state characterized by a deformed core and the

lowest states of the system would be rotational ones with a moment of inertia given by

$$\mathscr{I}_{\text{irrot}} = \mathscr{I}_{\text{rig}}\epsilon^2 \qquad \ldots\ldots(2)$$

where ϵ is the ratio of the difference of axes to the mean radius of the equilibrium ellipsoidal shape.

The deformation ϵ could be inferred directly from the quadrupole data, and it was found that the observed moments of inertia determined by the level spacing were appreciably greater than given by the formula (2), presumably due to the impropriety of describing the nucleon motion as irrotational fluid flow.

An alternative method of discussing the problem was suggested by Inglis (1954), who introduced the idea of independent particle motion in a deformed potential well rotated with a fixed impressed angular velocity Ω. In a frame of reference fixed with respect to the potential the rotation introduces as a perturbation a Coriolis force. It can then be argued that the reduction in energy caused by this perturbation should be equal to the increase in energy for free rotation of this system, and the moment of inertia is found by equating this energy change to $\frac{1}{2}\mathscr{I}\Omega^2$; this leads to the cranking model formula

$$\mathscr{I}_{\text{cr}} = 2\sum_i |\langle 0|J_x|i\rangle|^2/(E_i - E_0). \qquad \ldots\ldots(3)$$

Some of the consequences of this formula have been discussed by Inglis and by Bohr and Mottelson (1955). In particular they have shown that, used in conjunction with a deformed oscillator well, the formula leads to the rigid moment of inertia for the equilibrium configuration; they have also shown that this value may be markedly reduced by the effects of residual interactions between the nucleons.

A number of authors, for example Villars (to be published), have considered the general problem of exhibiting the rotational motion by a suitable transformation of the many-particle Hamiltonian describing the nucleus, but this fundamental approach will not be considered further in this paper. A more phenomenological approach has been used by Yoccoz (1956), applying a variational principle to a wave function of the type suggested by Hill and Wheeler.

The method followed here is that suggested by Skyrme (1957), and is closely related to the method of Yoccoz in this problem. In the following section the basic formula in this approach is derived; in § 3 it is applied to a wave function of shell-model type; next a simplified formula is deduced, making use of the fact that the interaction forces are predominantly of short range. In § 5 this is used in conjunction with a deformed oscillator well and in the following sections some discussion is given of the problems presented by a more realistic form of potential. The results are discussed in § 8.

§2. Formula for the Moment of Inertia

It will be supposed at first that the nuclear system has a set of energy levels E_I, which are given exactly by the formula (1) for a rigid rotator, and corresponding eigenfunctions $\psi_M{}^I(x)$, where the arguments x stand for all the coordinates of the A constituent nucleons. According to the qualitative picture developed by Bohr and Mottelson (1953), these levels can be described phenomenologically

in terms of the rotational motion of a top; if θ, ϕ and ψ, or Ω for brevity, are the Eulerian coordinate angles of this top, the wave functions of this model are the well-known transformation matrices $D_{KM}^{I}(\Omega)$, the eigenfunctions of the Hamiltonian

$$(1/2\mathscr{I})\mathbf{K}^2 \qquad \ldots\ldots(4)$$

representing the rotational energy, where $\mathbf{K}$ is the angular momentum operator corresponding to the coordinates Ω.

According to the general procedure suggested by Skyrme (1957) an enlarged state function $\psi(x, \Omega)$ is constructed according to the definition

$$\psi(x, \Omega) = \sum_{IM} A_I \psi_M^I(x) D_{KM}^{*I}(\Omega). \qquad \ldots\ldots(5)$$

Here K is the intrinsic spin postulated for the set of levels considered and the A_I are coefficients, arbitrary for the moment. Conversely from this equation

$$A_I \psi_M^I(x) = \int \psi(x, \Omega) D_{KM}^{I}(\Omega)\, d\Omega \qquad \ldots\ldots(6)$$

which exhibits the wave function in a general form that has been suggested by Hill and Wheeler.

Since it has been assumed that the levels E_I are given by the eigenvalues of (4), it follows from the definition (5) that $\psi(x, \Omega)$ must satisfy the enlarged Schrodinger equation

$$(H - (1/2\mathscr{I})\mathbf{K}^2)\psi(x, \Omega) = E_0 \psi(x, \Omega) \qquad \ldots\ldots(7)$$

where H is the Hamiltonian of the nuclear system, the sum of kinetic energy T and potential energy V. It is also clear from the definition that $\Psi(x, \Omega)$ is invariant under simultaneous rotations of the coordinates x and of the Eulerian angles Ω, so that

$$\mathbf{K}\Psi(x, \Omega) = \mathbf{J}\Psi(x, \Omega) \qquad \ldots\ldots(8)$$

where $\mathbf{J}$ is the total angular momentum vector of the nucleonic system; in other words $\Psi(x, \Omega)$ can be obtained from $\Psi(x, 0)$ simply by a rotation of coordinates, and since $\mathbf{K}^2$ is equivalent to $\mathbf{J}^2$, all mention of Ω can be eliminated from (7). Hereafter $\Psi(x, 0)$ will simply be written as $\Psi(x)$, which must satisfy

$$(H - (1/2\mathscr{I})\mathbf{J}^2)\Psi(x) = E_0 \Psi(x). \qquad \ldots\ldots(9)$$

In this equation both the function Ψ and the coefficient $(1/2\mathscr{I})$ are unknown. Now it was originally supposed that the energy levels E_I were given exactly by the formula (1); this was unrealistic and the coefficient $(1/2\mathscr{I})$ will be so chosen as to minimize the difference between the actual energy levels and those given by the model formula. One of the simplest ways of doing this is to impose the variational condition

$$\delta \int \Psi^*(x)[H - (1/2\mathscr{I})\mathbf{J}^2 - E_0]^2 \Psi(x)\, dx = 0 \qquad \ldots\ldots(10)$$

on variations both of the state-function and of the parameters $\mathscr{I}$ and E_0. Variation of the state-function Ψ leads back to equation (9), and variation of the parameters leads, after the elimination of E_0, to a formula for the moment of inertia

$$1/2\mathscr{I} = [\langle H\mathbf{J}^2\rangle - \langle H\rangle\langle \mathbf{J}^2\rangle]/[\langle \mathbf{J}^4\rangle - \langle \mathbf{J}^2\rangle^2] \qquad \ldots\ldots(11)$$

where the bracket symbols $\langle\ \rangle$ denote the expectation value with respect to the function Ψ determined by (9), or some approximation to it.

§3. Self-consistent Potential

Further progress in this problem depends upon making some approximation to the function $\Psi(x)$. The qualitative success of the single-particle model of deformed nuclei suggests that $\Psi(x)$, for a suitable choice of A_{I} in equation (5), might be taken as a determinant of single-particle wave functions $\phi_n(x_i)$. It would then be most consistent to determine the best possible set of ϕ_n from the variational condition (10); however, this leads to excessive algebraic complexity and the (more usual) variational condition will be used in which the integrand is not squared, that is

$$\delta\langle H-(1/2\mathscr{I})\mathbf{J}^2-E_0\rangle=0 \qquad \ldots\ldots(12)$$

(this also leads to equation (9) for arbitrary variations of $\Psi(x)$).

The set of ϕ_n determined by this condition (12) are eigenstates of a single-particle Hamiltonian

$$h=t+u-(1/2\mathscr{I})g \qquad \ldots\ldots(13)$$

where t is the kinetic energy of a single nucleon, u is a single-particle non-local potential, the Hartree–Fock potential derived from V and ϕ_n, and g is a similar single-particle operator derived from $\mathbf{J}^2$.† The total determinantal state Ψ is then an eigenstate of the Hamiltonian H' formed by summing the single-particle Hamiltonians h_i,

$$H'=T+U-(1/2\mathscr{I})G \qquad \ldots\ldots(14)$$

and

$$H=H'+(V-U)+(1/2\mathscr{I})G. \qquad \ldots\ldots(15)$$

This last expression is now substituted for H in the formula (11). H' may be dropped since Ψ is supposed now to be an eigenstate of it, and when the terms in $(1/2\mathscr{I})$ are collected together the result is to give

$$(1/2\mathscr{I})=[\langle(V-U)\mathbf{J}^2\rangle-\langle(V-U)\rangle\langle\mathbf{J}^2\rangle]/[\langle(\mathbf{J}^2-G)\mathbf{J}^2\rangle-\langle(\mathbf{J}^2-G)\rangle\langle\mathbf{J}^2\rangle]. \ldots(16)$$

Now the operators V and $\mathbf{J}^2$ may, with respect to the set of functions ϕ_n, be divided into an expectation value, a 'one-particle part' giving rise to excitations of one nucleon from the ground state, a 'two-particle part' and so on. Obviously the expectation value does not contribute to the formula (12) and the one-particle parts are cancelled exactly by U and G respectively; so if the operator Q is defined to be zero acting upon the ground state and all states derived from it by excitation of only one nucleon, and unity otherwise, the formula may be written more concisely as

$$(1/2\mathscr{I})=\langle VQ\mathbf{J}^2\rangle/\langle\mathbf{J}^2Q\mathbf{J}^2\rangle. \qquad \ldots\ldots(17)$$

These expectation values may now easily be written in terms of matrix elements. The matrix element between the states ϕ_m and ϕ_n of the single-particle angular momentum operator $\mathbf{j}$ will be written as $(n|\mathbf{j}|m)$; then

$$\langle\mathbf{J}^2Q\mathbf{J}^2\rangle=2\sum_{m}\sum_{n}\sum_{m'}\sum_{n'}\sum_{r}\sum_{s}(m|j_r|m')(n|j_r|n')[(m'|j_s|m)(n'|j_s|n)-(m'|j_s|n)(n'|j_s|m)] \qquad \ldots\ldots(18)$$

where the summation of m and n extends over all occupied states and that of m'

† With the notation used below in equation (18) and (19), the matrix elements of u and g given by

$$(a\mid u\mid b)=\sum_{n}(an-na\mid v\mid bn)$$

$$(a\mid g\mid b)=(a\mid j^2\mid b)-2\sum_{n}\sum_{r}(a\mid j_r\mid n)(n\mid j_r\mid b)$$

and n' over all unoccupied ones. It will be assumed hereafter that V is a sum of two-nucleon interactions

$$\sum_{i<j} v_{ij};$$

then in a similar way

$$\langle VQ\mathbf{J}^2\rangle = 2\sum_{m}\sum_{n}\sum_{m'}\sum_{n'}\sum_{r}(m, n-n, m|v|m'n')(m'|j_r|m)(n'|j_r|n) \quad \ldots\ldots(19)$$

where the matrix element of v is the difference of the direct and exchange matrix elements.

In the above discussion the question of centre-of-mass motion has been ignored. The wave functions introduced in § 2 should be those of the internal motion of the system, whereas the approximate determinantal ones involve some motion of the mass-centre. Then it would be more consistent with the general philosophy used here to replace the total angular momentum $\mathbf{J}$ in formulae (11) and (17) by the internal angular momentum. This has not however been considered in any detail because of the complexity of the modified matrix elements.

§ 4. A Simplified Formula

The formulae found in the preceding sections could be evaluated numerically for particular assumed trial wave functions, but it is difficult to draw general qualitative conclusions from them. For this it would be desirable to have a form which depends principally upon the mean self-consistent potential rather than upon the internucleon potential V. This can be done in the following way.

By the substitution of (15) into the fundamental relation (11), two formally different identities may be obtained

$$(1/2\mathscr{I})\langle\mathbf{J}^2(\mathbf{J}^2-G-\langle\mathbf{J}^2-G\rangle)\rangle = \langle\mathbf{J}^2(V-U-\langle V-U\rangle)\rangle \quad \ldots\ldots(20)$$

$$(1/2\mathscr{I})\langle J_i(\mathbf{J}^2-G-\langle\mathbf{J}^2-G\rangle)J_i\rangle = \langle J_i(V-U-\langle V-U\rangle)J_i\rangle + \langle J_i[H', J_1]\rangle \quad \ldots\ldots(21)$$

of which the former is the same as equation (16). In the latter equation the coefficient of $(1/2\mathscr{I})$ is not very different from (18) being explicitly

$$\sum_{r}\sum_{s}\sum_{m}\sum_{n}\sum_{m'}\sum_{n'}(m|j_i|m')(n'|j_i|n)[(m'|j_s|m)(n|j_s|n') - (m'|j_s|n')(n|j_s|m)] \quad \ldots\ldots(22)$$

with the same direct term. The first term on the right is equal to

$$\sum_{i}\sum_{m}\sum_{n}\sum_{m'}\sum_{n'}(m|j_i|m')(n'|j_i|n)(m'n|V|mn'-n'm) \quad \ldots\ldots(23)$$

and is approximately equal to the negative of (19), for the following reasons:

(*a*) in the heavy deformed nuclei the orbital angular momenta are much larger than the spins, so that $\mathbf{j}$ may be replaced by $\mathbf{l}$; then, if the states are taken to have real orbital wave functions (which is not a significant restriction),

$$(n'|l_i|n) = -(n|l_i|n'), \quad \ldots\ldots(24)$$

(*b*) the 'direct' matrix elements of V are the same in the two cases; the exchange terms are different, but will be small for short range forces.

A useful formula can be obtained then by assuming this last equality and by retaining only the similar direct terms in (18) and (22). Some further examination of these simplifications is made in Appendix A. Within the framework of the present analysis J_z is a good quantum number, equal to K, so that only J_x and J_y contribute to the matrix elements and in a symmetrical way; for example

$$\sum_{m}\sum_{m'}(m|j_1|m')(m'|j_3|m) = \delta_{13}(1-\delta_{13})\langle J_x{}^2\rangle \quad \ldots\ldots(25)$$

and the simplified formula is

$$\mathscr{I} = 2\langle J_x^2\rangle^2/\langle J_x(H' - \langle H'\rangle)J_x\rangle. \qquad \ldots\ldots(26)$$

In a similar notation the formula given by the cranking model is

$$\mathscr{I}_{\mathrm{cr}} = 2\langle J_x(H' - \langle H'\rangle)^{-1}J_x\rangle \qquad \ldots\ldots(27)$$

which will always be greater than (26) unless all the excitation energies involved are equal. The evaluation of (26) is discussed in the following sections.

§ 5. The Oscillator Model

The simplest model and the only one for which explicit calculations can easily be made is that of articles moving in a deformed oscillator well, such as has been used by Inglis (1954, 1955, 1956), Bohr and Mottelson (1955) and other authors. With this assumption

$$H' = -(1/2M)\nabla^2 + \tfrac{1}{2}M\omega^2[(x^2+y^2)e^{2\epsilon/3} + z^2e^{-4\epsilon/3}] \qquad \ldots\ldots(28)$$

where the parameter ϵ is a convenient measure of the deformation, related to the parameter B introduced by Bohr and Mottelson (1953) by

$$B = (16\pi/45)^{1/2}(e^{2\epsilon/3} - e^{-\epsilon/3}) \sim 1{\cdot}06\epsilon. \qquad \ldots\ldots(29)$$

The individual particle wave functions are oscillator wave functions with deformation $\frac{1}{2}\epsilon$, and can be labelled by the oscillator quantum numbers l, m, n corresponding to the x, y, z directions.

The deformation will be assumed to be prolate with cylindrical symmetry, so that

$$\sum(l+\tfrac{1}{2}) = \sum(m+\tfrac{1}{2}) < \sum(n+\tfrac{1}{2}) = C, \quad \text{say.} \qquad \ldots\ldots(30)$$

The assumed potential (28) can only be self-consistent if the density distribution has approximately the same deformation; since the wave functions are distorted only by $\frac{1}{2}\epsilon$, the asymmetry of the configuration must make up the difference, the expectation values of y^2 and z^2 will have the correct ratio if

$$\sum(m+\tfrac{1}{2}) = Ce^{-\epsilon}. \qquad \ldots\ldots(31)$$

This result may also be obtained by minimizing $\langle H'\rangle$ with respect to variations of ϵ, but that argument is dubious (see also § 7).

Acting upon the state $|l,m,n\rangle$ the operator j_x can transform it into any of the four states $|l,m\pm1,n\pm1\rangle$ that are unoccupied, and the value of $\langle J_x^2\rangle$ is found by summing the squares of all the allowed transition matrix elements (cf. the formula (25)). This summation can easily be performed by summing over all, allowed and forbidden, matrix elements and then subtracting the forbidden ones. The result, as has been found by Bohr and Mottelson and by Inglis, is

$$\begin{aligned}\langle J_x^2\rangle &= \cosh^2(\tfrac{1}{2}\epsilon)\sum(n-m) + \sinh^2(\tfrac{1}{2}\epsilon)\sum(m+n+1)\\ &= C\sinh\epsilon \qquad \ldots\ldots(32)\end{aligned}$$

with the help of the condition (31).

The denominator of (26) can be evaluated in just the same way, leading to the result

$$\mathscr{I} = (C/\omega)e^{2\epsilon/3} = M\sum z^2 \qquad \ldots\ldots(33)$$

so that the ratio to the rigid moment $M\sum(y^2+z^2)$ is $1/(1+e^{-2\epsilon})$, a little greater than $\frac{1}{2}$ for the moderate deformations observed.

§ 6. General Estimate

The oscillator model is only a rough approximation for heavy nuclei, so it is important to investigate the generality of the preceding result and in particular how sensitive it is to the well shape and to the velocity dependence of the mean potential (Blin-Stoyle 1956).

The denominator of formula (26) can reasonably be estimated, since it can be written in terms of commutators as†

$$\langle \tfrac{1}{2}[J_x, [U, J_x]] \rangle. \qquad \ldots\ldots(34)$$

As a model of a distorted velocity-dependent potential it will be assumed that U can be taken in the form

$$U = U_0(r') + \mathbf{p}U_1(r')\mathbf{p} \qquad \ldots\ldots(35)$$

where

$$r'^2 = (x^2 + y^2)e^{2\epsilon/3} + z^2 e^{-4\epsilon/3} \qquad \ldots\ldots(36)$$

so that the effective mass is given by

$$1/2M^* = 1/2M + U_1(r'). \qquad \ldots\ldots(37)$$

It will also be assumed, consistently, that the density ρ and 'square-of-momentum' density τ are also functions only of r'. Then the expectation value (34) can be expressed in terms of an integral over r',

$$\tfrac{4}{3}\sinh^2\epsilon(\tfrac{1}{2}\langle[J_x[U, J_x]]\rangle) = -\frac{2}{15}\sinh^2\epsilon \int \left[\frac{dU_0}{dr'}\frac{d\rho}{dr'} + \frac{dU_1}{dr'}\frac{d\tau}{dr'}\right] r'^2\, dV. \quad \ldots\ldots(38)$$

It can be verified that for the oscillator potential, $U_0 = \frac{1}{2}M\omega^2 r'^2$, this gives the same result as that calculated directly in § 5, which is

$$2\omega C e^{-2\epsilon/3}\sinh^2\epsilon \sim \omega^2 \mathscr{I}_{\text{rig}}\sinh^2\epsilon \qquad \ldots\ldots(39)$$

proportional to A, the total number of nucleons, when the oscillator frequency is adjusted to give constant nuclear density. For a more realistic square potential shape, the integral should be proportional to $A^{4/3}$, at least for large A; an estimate can be made by using the Thomas–Fermi approximation relating ρ and τ to the potential, when (38) can be transformed into

$$\frac{4}{27}\sinh^2\epsilon \int \frac{\tau}{2M^*}\left(\frac{r'}{\rho}\frac{d\rho}{dr'}\right)^2 dV \qquad \ldots\ldots(40)$$

as shown in Appendix B, and of which a numerical estimate is

$$A^{4/3}(7{\cdot}2\,\text{MeV})\sinh^2\epsilon \qquad \ldots\ldots(41)$$

whereas the coefficient of $\sinh^2\epsilon$ in (39), for the oscillator potential, has the numerical value A (22 MeV). Thus for the heavy nuclei with A around 200 the estimate (41) has about twice the value given by the oscillator model.

The estimation of $\langle J_x^2 \rangle$ is more difficult, as this depends upon the configuration of minimum energy for a given deformation. One possible line of approach is to follow the suggestion made by Nilsson (1955) that in the case of large deformations perturbation theory may be applied to the deformed oscillator model. Then the configuration of the oscillator model is retained, but $\langle J_x^2 \rangle$ is altered by the first-order change in the wave functions.

† Here the reaction term $(1/2\mathscr{I})G$ in H' has been neglected, or it may be supposed that U is modified to include it.

If U_{pert} is the correction applied to the oscillator potential, then in first order the change in $\langle J_x^2 \rangle$ is

$$2\langle J_x^2 (E_0 - H')^{-1} U_{\text{pert}} \rangle. \qquad \ldots\ldots(42)$$

The perturbation used by Nilsson, proportional to l^2 depresses states of high angular momentum and makes this correction positive. Its numerical value fluctuates with the configuration considered, and suggests increases of 25–50% in $\langle J_x^2 \rangle$.

Apart from the crudity of the method of estimation by perturbation theory it is not clear that Nilsson's model is sufficiently realistic to give a reliable value for $\langle J_x^2 \rangle$. In the next section a different approach is attempted, to bring in explicitly the condition of optimum configuration.

§ 7. Condition of Least Energy

The first requirement is a knowledge of the single-particle wave functions in a well with a given distortion, such as (35). The approximation suggested by Moszkowski (1955) will be used, according to which the single-particle wave functions are approximated by those of a spherical system, distorted by some amount η to be fixed by a variational calculation.

The states ϕ_n are functions of the distorted coordinates $x' = xe^{\eta/3}$, $z' = ze^{-2\eta/3}$, so that $d\phi_n/dx = e^{\eta/3}(d\phi_n/dx')$, and so on. The expectation values are worked out by expressing all quantities in terms of the distorted coordinates. For the determination of η only terms of the first and second order in η and ϵ will be retained; thus, dropping the primes from the distorted coordinates, the expectation value of the kinetic energy is

$$\frac{1}{2M}\left\langle \sum(p_x^2 + p_y^2 + p_z^2) + \frac{2\eta}{3}\sum(p_x^2 + p_y^2 - 2p_z^2) + \frac{2\eta^2}{9}\sum(p_x^2 + p_y^2 + 4p_z^2)\right\rangle. \qquad \ldots\ldots(43)$$

The linear term in η is cancelled exactly, according to the virial theorem, by a similar term arising in first order from the potential energy. The remaining quadratic terms are

$$\tfrac{1}{9}\eta^2 Y + \tfrac{1}{9}(\epsilon - \eta)^2 X \qquad \ldots\ldots(44)$$

where

$$Y = \tfrac{1}{2}\langle (1/2M^*)\sum(p_x^2 + p_y^2 + 4p_z^2)\rangle \sim \langle (1/2M^*)\sum(p_x^2 + p_y^2 + p_z^2)\rangle \qquad \ldots\ldots(45)$$

to this order of approximation in the deformation, and where X is the integral defined in equation (38).

Therefore the optimum distortion is given by

$$\eta/\epsilon = (1 + Y/X)^{-1}. \qquad \ldots\ldots(46)$$

For the oscillator case $X = Y$ and $\eta = \frac{1}{2}\epsilon$, as we already know. For the realistic case X is about doubled in value, and so will be Y for a reduced mass around $\frac{1}{2}M$, so their ratio will not differ much from unity.

The second requirement is the fixing of the distortion ϵ. As was mentioned earlier it is not *a priori* satisfactory to do this simply by minimizing with respect to ϵ, because it is possible that the potential varies in other ways with ϵ. It is, however, correct to assert that the energy, calculated as the expectation value of the many-body Hamiltonian, must be stationary for small uniform distortions of the determinantal wave function, since the latter was defined to be the best. In particular then the condition

$$[H, D] = [H, \sum(2zp_z - xp_x - yp_y)] = 0 \qquad \ldots\ldots(47)$$

must be satisfied in the equilibrium state.

The potential energy will be taken in the phenomenological form used by Skyrme (1956),

$$V=\sum_{i<j}\delta(x_i-x_j)(t_0+t_1(p_i-p_j)^2). \qquad \ldots\ldots(48)$$

The commutator with D is easy to evaluate in this form, because the delta-function commutes with the distortion, leading to the result

$$i[H,D]=\langle(1/M^*)\sum(2p_z^2-p_x^2-p_y^2)\rangle+B\int[2(d\rho/dz)^2-(d\rho/dx)^2-(d\rho/dy)^2]\,dV \qquad \ldots\ldots(49)$$

where B is proportional to t_1 (see reference quoted for the precise definition).

The second term in this result is just the hydrodynamical estimate of the potential energy of distortion, and is a relatively small surface effect. The contribution of the first term may be divided into two parts, one arising from the asymmetry of the configuration and the other, a volume effect, arising from the uniform distribution of the system. Since the latter is much larger, for heavy nuclei, than the surface effect, the vanishing of the whole must be due to a balance between the asymmetry of the configuration and the volume distortion effect, modified slightly by the effects of the last term. If that is neglected, the use of distorted wave functions gives the condition

$$\langle(1/M^*)\sum(2p_z^2-p_x^2-p_y^2)\rangle=(8\eta/3)Y \qquad \ldots\ldots(50)$$

on the configuration of the equilibrium state.

Now for oscillator wave functions, in a configuration of minimum energy

$$(1/M)\langle\sum(2p_z^2-p_x^2-p_y^2)\rangle=\omega\langle\mathbf{J}^2\rangle. \qquad \ldots\ldots(51)$$

(Both are proportional to $\sum(2n-m-l)$, see § 5.) Also both these quantities depend only upon the particles in the last partly filled shell, and mainly upon those of higher angular momentum. It is plausible to suppose that the corresponding states in a more realistic well might yield a similar relation

$$\langle(1/M^*)\sum(2p_z^2-p_x^2-p_y^2)\rangle=\omega^*\langle\mathbf{J}^2\rangle \qquad \ldots\ldots(52)$$

for some suitable effective oscillator frequency ω^*. Then from equations (46), (50) and (52), to order ϵ,

$$\langle\mathbf{J}^2\rangle=(8\epsilon/3)(XY/\omega^*)/(X+Y). \qquad \ldots\ldots(53)$$

The moment of inertia is obtained by substituting this result and (38) into the formula (26), giving

$$\mathscr{I}=\frac{2}{3}\frac{Y}{\omega^{*2}}\frac{4XY}{(X+Y)^2}. \qquad \ldots\ldots(54)$$

Since X and Y are nearly equal the last factor is very close to unity, and will be dropped. Y is the total kinetic energy evaluated with the reduced mass,

$$Y=A\frac{3}{5}\frac{k_F^2}{2M_1} \qquad \ldots\ldots(55)$$

where k_F is the maximum momentum of nucleons and M_1 is a mass close to the value of M^* inside the nucleus. ω^* can be estimated by comparison of the kinetic energies for the topmost nucleons

$$\omega^*k_FR\sim\langle(1/M^*)\mathbf{p}^2\rangle=k_F^2/M_2 \qquad \ldots\ldots(56)$$

where M_2 is another 'effective mass' appropriate to a particle at the top of the momentum distribution. Finally then the ratio of the moment of inertia to the rigid one is

$$\mathscr{I}/\mathscr{I}_{\text{rig}}=\tfrac{1}{2}(M_2^2/MM_1) \qquad \ldots\ldots(57)$$

for small deformations. M_2 will certainly lie between M and M_1 so that the factor (M_2^2/MM_1) will be between $\frac{1}{2}$ and 2, if M_1 is about $\frac{1}{2}M$; a closer estimate

does not appear possible without a detailed model of the velocity dependence and of surface shape.

§ 8. Discussion

The first formula (11) derived in § 2 should really be regarded as an identity satisfied by the true state, but since it is in fact stationary for small variations from that state it can be used as an estimator of $\mathscr{I}$ on the basis of an approximate wave function. This latter was then assumed to be the determinantal wave function satisfying (12), and the original formula could then be simplified to (17). Up to this point no assumption is made about the nature of the nuclear interactions, except that they are such as to make the determinantal wave function a reasonable approximation.

In § 4 a much simpler formula (26) was obtained, involving only the mean self-consistent potential, by making some approximations which involved in an essential way the short-range nature of the interactions. This agrees with the prediction of the cranking model only when the excitation energies involved are all equal; applied to the particular case of the deformed oscillator well this is precisely the case in which the irrotational formula is reproduced, a point of only academic interest since there is then no equilibrium deformation. In general, application to this model, with the self-consistent equilibrium deformation, gives a moment of inertia equal to one-half of the rigid value, for small deformations, whereas the cranking model gives the rigid value in these circumstances.

This result does not necessarily imply any contradiction between the two methods of calculating the moment of inertia, for, as has been emphasized by Bohr and Mottelson (1955), residual interactions between nucleons modify considerably the result obtained with the cranking model. On the other hand the original formula (11), being associated with a variational principle, should be insensitive to errors in the wave function so that, in so far as the reduction to (26) is justified, the latter should incorporate most of these residual effects.

The question then arises whether this factor of one-half is a realistic explanation of the tendency of the observed moments to cluster around this value (Blin-Stoyle 1956, Moszkowski 1956). The present analysis shows that there are various effects operating both to raise and to lower this value. In the first place the oscillator model gives, for deformations with $\epsilon = 0{\cdot}3$–$0{\cdot}4$, a rather larger ratio of about 0·65 (§ 5). The effects of the shape of the potential well and of its velocity dependence cannot easily be estimated, but the calculations made in the last two sections indicate that there may be some reduction arising from the velocity dependence, as suggested by Blin-Stoyle; since, however, the rotational phenomena are determined mainly by surface interactions, the effective mass relevant here should be greater than the mean value for all the nucleons. There is a third effect tending to reduce the influence of the velocity dependence this arises from the fact that the finite range of the nucleon interaction, of which one manifestation is the effective mass, also introduces a correction to the simplified formula (26) tending to increase the moment of inertia (see Appendix A).

Acknowledgment

The author is indebted to Professor R. E. Peierls for several discussions about the work of Yoccoz and Peierls, which has close contact with the analysis in this paper through equation (6), and which provided the stimulus for the present investigation.

APPENDIX A

The main contribution to $\langle J_x^2 \rangle$ comes from transitions within the same shell, i.e. for the oscillator model the first term of (32), which arises by summing $(m+1)n$ over those occupied states from which a transition can be made. The most important effect of the exchange term in (18) is to remove those terms in the sum with $m=n$, introducing therefore a correction of the relative order

$$(\sum'(m+1)^2n^2)/[\sum'(m+1)n]^2. \qquad \ldots\ldots(A\,1)$$

This must, however, be reduced by a factor $\frac{1}{4}$ because exchange can take place only between particles with the same spin and charge, and by a further factor $\frac{1}{2}$ because the exchange term in (22) is of smaller order, and it is the average of (18) and (22) which occurs in the formula (26). The resultant correction is unlikely to exceed 5% in the heavy nuclei where several states contribute to the sums in (A 1).

The errors arising from the finite range of the potential may be investigated with the help of the potential (48). It will be assumed that the condition (*a*) of § 4 is satisfied so that (24) holds. Then a direct evaluation of the expectation value $2\langle VQJ_x^2 \rangle$ can be made for comparison with the formula (38). The t_0 term gives immediately

$$2t_0 \int (\sum' \phi_n l_x \phi_n)^2 \, dV \qquad \ldots\ldots(A\,2)$$

where $\sum'$ denotes summation over those states from which a transition can be made, or

$$\begin{aligned}\sum' \phi_n l_x \phi_n &= \sum \phi_n l_x \phi_n - \sum\sum \phi_n \phi_m \int \phi_m l_x \phi_n \, dV \\ &= \tfrac{1}{2} l_x \sum \phi_n{}^2 = \tfrac{1}{2} l_x \rho\end{aligned}$$

since the last term vanishes by the condition (24). This leads to a verification of the formula (38) with $U_0 = t_0\rho$, as was to be expected.

The t_1 term can be analysed in a similar way, but this time the condition (24) is not sufficient to express the result only in terms of the mean potential. When the expressions derived have been simplified as far as possible the result takes the form

$$2\langle VQJ_x^2 \rangle = \tfrac{1}{2}[l_x, U][l_x, \rho] + t_1[\sum'(\phi_n l_x \phi_n - \phi_n l_x \phi_n)]^2 \qquad \ldots\ldots(A\,3)$$

of which the first term is positive, leading to (38), and the second residual term is negative (for j_x is an imaginary operator). This second term represents the change in strength of the interaction due to the change of momentum produced by the operator j_x. This momentum change will, for a large system, be of the order of spacing between states in momentum and so small compared with the momenta at the top of the Fermi distribution.

APPENDIX B

Suppose that the potential energy of the system can be written in the form

$$\langle V \rangle = \int W(\rho, \tau) \, dV \qquad \ldots\ldots(B\,1)$$

then the resultant single-particle potential is of the form (35) with

$$U_0 = \partial W/\partial \rho, \qquad U_1 = \partial W/\partial \tau. \qquad \ldots\ldots(B\,2)$$

The integrand of (38) can be transformed as follows:

$$-\frac{d\rho}{dr}\frac{dU_0}{dr}-\frac{d\tau}{dr}\frac{dU_1}{dr}=-\frac{d\rho}{dr}\left\{\frac{d}{dr}\left[\frac{\partial W}{\partial\rho}+\frac{d\tau}{d\rho}\frac{\partial W}{\partial\tau}\right]\right\}+\frac{d\rho}{dr}\frac{\partial W}{\partial\tau}\frac{d}{dr}\frac{d\tau}{d\rho}$$

$$=\frac{d\rho}{dr}\left(\frac{1}{2M}+\frac{\partial W}{\partial\tau}\right)\frac{d}{dr}\frac{d\tau}{d\rho} \qquad \ldots\ldots(B\,3)$$

with the help of the Thomas–Fermi relation

$$(1/2M)(d\tau/d\rho)+\partial W/\partial\rho+(\partial W/\partial\tau)(d\tau/d\rho)=\text{const.}$$

The result (B 3) with $\tau\propto\rho^{5/3}$ leads immediately to (40). The numerical result (41) was obtained using the density distribution

$$\rho/\rho_0=[1+\exp\{(r-R)/0{\cdot}5\}]^{-1} \qquad \ldots\ldots(B\,4)$$

and an effective mass $1/2M^*=(1/2M)(1+\rho/\rho_0)$. Closely similar results were obtained by the use of the figures adopted by Skyrme (1956) and by a direct estimate of (38) in terms of density and potential gradients.

References

Blin-Stoyle, R. J., 1956, *Nuclear Phys.*, **2,** 169.

Bohr, A., and Mottelson, B. R., 1953, *K. Danske Vidensk. Selsk., Mat.-fys. Medd.*, **27,** no. 16; 1955, *Ibid.* **30,** no. 1

Inglis, D., 1954, *Phys Rev.* **96,** 1059; 1955, *Ibid.*, **97,** 701; 1956, *Ibid*, **103,** 1786.

Moszkowski, S A., 1955, *Phys. Rev.*, **99,** 803; 1956, *Ibid.*, **103,** 1328.

Nilsson, S. G., 1955, *K. Danske Vidensk. Selsk, Mat.-fys. Medd.*, **29,** no 16.

Rainwater, J., 1950, *Phys. Rev.*, **79,** 432

Skyrme, T. H R., 1956, *Phil. Mag.*, **1,** 1043; 1957, *Proc. Roy. Soc.* A, **239,** 399.

Yoccoz, J., 1956, *J. Phys. Radium*, **7,** 517.