

1 EXTENSIVE STUDY OF THE POSITIVE AND NEGATIVE PARITY
2 WOBBLING STATES FOR AN ODD-MASS TRIAXIAL NUCLEUS II:
3 CLASSICAL TRAJECTORIES

4 R. POENARU^{1,2,a}, A. A. RADUTA^{2,3,b}

5 ¹Doctoral School of Physics, University of Bucharest, Bucharest, Romania

6 *E-mail*^a: robert.poenaru@drd.unibuc.ro

7 ²Department of Theoretical Physics - Horia Hulubei National Institute for Physics and Nuclear
8 Engineering, Măgurele-Bucharest, Romania

9 *E-mail*^b: raduta@nipne.ro (corresponding author)

10 ³Academy of Romanian Scientists, Bucharest, Romania

11 Received: April 20, 2021 (RJP v2.0 r2018a)

12 *Abstract.* **To be implemented...**

13 *Key words:* Triaxial Nuclei, Wobbling Motion, Angular Momentum, Energy Ellipsoid.

1. INTRODUCTION

14 Collective phenomena in deformed nuclei such as the *wobbling motion* have
15 been drawing a lot of attention lately, mainly due to its elusive character, but also
16 due to the real experimental and theoretical challenges it implies. Considered as a
17 clear fingerprint of nuclear triaxiality, wobbling motion (w.m.) has been predicted
18 theoretically by Bohr and Mottelson more than 40 years ago [7], when they were
19 discussing the excited spectra of even-even nuclei using a triaxial rigid rotor with
20 three different moments of inertia (MOI).

21 W.m. can be viewed as the quantum analogue for the motion of the asymmetric
22 top, whose rotation around the axis with the largest moment of inertia (MOI) is
23 energetically the most favored. A uniform rotation about this axis will have the lowest
24 energy for a given angular momentum (spin). As the energy increases, this axis
25 will start to precess, due to the anisotropy of the three moments of inertia, with a
26 harmonic type of oscillation about the space-fixed angular momentum vector, giving
27 rise to a family of wobbling bands, each characterized by a wobbling phonon number
28 n_w . The resulting quantal spectrum will be a sequence of $\Delta I = 2\hbar$ rotational
29 bands, with an alternating signature number ($\alpha = \pm \frac{1}{2}$ in odd- A nuclei and $\alpha = 0, 1$
30 in even- A nuclei) for each wobbling excitation.

31 Although Bohr and Mottelson made predictions for these excitations in even-
32 even nuclei, the first experimental evidence of this nuclear behavior has been identified
33 in an odd-mass nucleus: the $A = 163$ isotope of Lu, where a single one-phonon

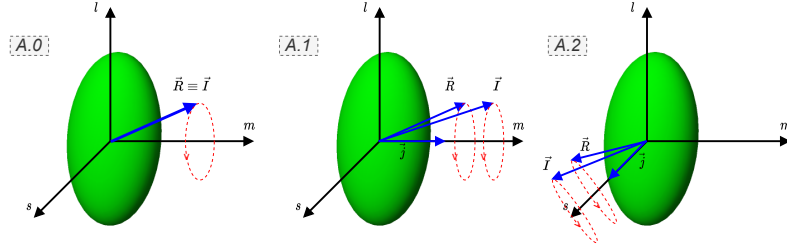


Fig. 1 – A.0: The geometry for the angular momentum of a simple wobbler. A.1: coupling geometry for a longitudinal wobbler (LW). A.2: coupling geometry for a transverse wobbler (TW). The short- s , long- l , and medium- m axes are defined in the body-fixed frame. The vectors \vec{R} , \vec{j} , and \vec{I} represent the set of angular momenta of the core, odd particle, and the total nuclear system, respectively.

wobbling band was measured initially [22], followed by two additional wobbling bands discovered one year later [23, 24]. Other experimental evidence came quickly after that and now the following nuclei are considered as wobblers: ^{105}Pd [31], ^{127}Xe [29], ^{133}La [26], ^{135}Pr [27, 28], $^{161,163,165,167}\text{Lu}$ [8, 10, 11, 23, 24], ^{167}Ta [9], $^{183,187}\text{Au}$ [32, 33]. Regarding the wobbling motion for the even-even nuclei the ^{112}Ru ($Z = 44$) nucleus has three wobbling bands [35], two of them being the excited one- and two-wobbling phonon bands. Another nucleus is ^{114}Pd [36], with two excited bands of wobbling character, similar to ^{112}Ru . The even-even nucleus ^{130}Ba ($Z = 56$) [37–39] was also confirmed very recently to exhibit wobbling behavior.

Compared to the wobbling mode described in [7], which has a purely collective form, in the case of odd- A nuclei, it turns out that the wobbling mode appears due to the coupling of a valence nucleon (the so-called $\pi(i_{13/2})$ intruder) to a triaxial core, driving the nucleus up to large deformation ($\epsilon \approx 0.4$) [25] and giving rise to excited states of the deformed nuclear system, each belonging to a particular wobbling band.

Frauendorf et al. [40] showed that in the case of odd- A nuclei, depending on the coupling between the triaxial core (with a.m. \vec{R}) and the single-particle (“valence” nucleon with a.m. \vec{j}), there can be two wobbling regimes: transverse (TW) and longitudinal (LW). The triaxial core is viewed as a Liquid Drop, such that the main rotation is along the intermediate m -axis (since this one has the largest MOI). When the odd particle aligns its angular momentum along the m -axis, then the system is said to achieve a *longitudinal wobbling* character (LW). If the odd-particle aligns its a.m. with an axis that is perpendicular to the m -axis (i.e., either the long l - or short s -axis of the triaxial rotor), then the system achieves a so-called *transverse wobbling* character. For a better understanding of the wobbling regimes in terms of angular momentum alignment, Figure 1 depicts three particular cases, namely a simple wobbler - inset A.0 (the case firstly developed by Bohr and Mottelson [7]), a longitudinal wobbler - inset A.1, and a transverse wobbler - inset A.2.

The current work represents the second piece of a two-part series of papers that

focuses on the description of the wobbling properties in odd-mass nuclei. Starting from an existing formalism concerning the interpretation of the wobbling structure of ^{163}Lu [19, 69], that initial framework (which will be further denoted by $\mathbb{W}1$) is extended with a proper description of the states with positive and negative parity, by adopting the concept of *Parity Partner Bands*. In the newly developed approach (denoted hereafter by $\mathbb{W}2$), a single particle (the odd $i_{j=13/2}$ proton) couples to the triaxial core, generating a sequence of four triaxial strongly deformed bands (called TSD_1, TSD_2, TSD_3 and TSD_4), with a total of 63 rotational states in all the bands. Previously in $\mathbb{W}1$, two different particle-core couplings were considered: one that consisted in the $i_{j=13/2}$ proton+core for the bands $TSD_{1,2,3}$ and one $h_{j=9/2}$ proton+core for TSD_4 band, which resulted in two separate fitting procedures required to obtain the energy spectrum of this isotope. Within $\mathbb{W}2$, as per the first part of this series of papers (denoted throughout the paper with **I** [20]), a single fitting procedure was required to find the excitation energies of ^{163}Lu , since only one proton was considered to align its angular momentum with that of the triaxial core. A successful description of the wobbling spectrum of ^{163}Lu was made in **I**, together with calculation of other relevant quantities (e.g. rotational frequencies) that put $\mathbb{W}2$ to the test in terms of correctness. The obtained agreement with the experimental data was impressive.

In this second part (hereafter denoted by **II**), attention is given to the properties of the classical energy function of ^{163}Lu , and the geometrical interpretation of the total angular momentum. The classical expression of the energy function, which can be obtained via the Time-Dependent Variational Principle (TDVE) applied in **I** allows the study of the wobbling stability, and also provides an insight into the classical features of nucleonic motion within the angular momentum space. Its expression signifies the initial quantal Hamiltonian of the deformed system but is brought to a dequantized form with the help of a set of coordinates that describe the dynamics. By expressing the angular momentum vector I of the triaxial nucleus and the energy function E as surfaces in a three-dimensional space, it is possible to obtain the trajectories of the rotating system by intersecting the two shapes. This aspect will be analyzed in detail later on.

The structure of this work is as follows. In Section 2 a brief overview of the key characteristics for the $\mathbb{W}2$ approach that emerged from **I** is made. In Section 3, the prerequisites for obtaining a classical expression of the energy function are formulated. Section 4 is devoted to the numerical results concerning the wobbling stability of ^{163}Lu . Wobbling stability is studied in terms of contour plots of the aforementioned energy function. The nuclear trajectories (i.e., the intersection curves of the energy surface with angular momentum surface) of the system are graphically represented for given values of angular momentum and energy. Several regimes of rotational motion emerge from this analysis. Discussion of the results is also made in Section 4. The conclusions of this current work are given in Section 5.

2. NEW FORMALISM FOR THE DESCRIPTION OF WOBBLING STATES

The $\mathbb{W}2$ formalism which emerged in **I** [20] consists in a re-interpretation of the four wobbling bands from ^{163}Lu . Namely, the bands TSD_2 and TSD_4 are *parity partner bands*: $\Delta I = 2$ sequences with identical spins but opposite parity ($\pi_2 = +1$ and $\pi_4 = -1$). Arguments for this came from the analysis of the wave function of the system. The function is an admixture of states with both positive and negative parity since the initial Hamiltonian is symmetric to rotations by a specific amount (D_2 implies invariance to rotations by π). A complete description of the properties of the wave function and the Hamiltonian concerning the parity property is made in **I**. In terms of wobbling excitations, both TSD_2 and TSD_4 are considered to be ground-state bands (zero-wobbling-phonon), obtained by coupling the $j_1 = 13/2$ proton (with parity $\pi_{j_1} = +1$) to a triaxial core of odd spins $R_2^+ = 1^+, 3^+, 5^+, \dots$ for TSD_2 and $R_2^- = 1^-, 3^-, 5^-, \dots$ for TSD_4 . The band TSD_1 is also regarded as a ground state band, but here the proton couples to a core of even spin states $R_1 = 0^+, 2^+, 4^+, \dots$, and TSD_3 is indeed an excited wobbling band (one-wobbling-phonon) that is built on top of TSD_2 via the action of a phonon operator. The coupling schemes for $\mathbb{W}2$ are described in **I** (denoted by C'_1 , C'_2 , and C'_3). For a clearer picture, Appendix A contains a diagram with all three mechanisms (see Fig. 10).

It is worth noting that this interpretation of the wobbling structure of ^{163}Lu contrasts the previously known band configuration [40, 42, 73] where the bands TSD_2 and TSD_3 were regarded as one- and two-wobbling phonon excitations built on the yrast TSD_1 band. However, it was recently shown [19, 59, 69] that TSD_1 and TSD_2 behave as signature partner bands, both being ground states with the favored (unfavored) $\alpha_f = +\frac{1}{2}$ ($\alpha_u = -\frac{1}{2}$) band as TSD_1 (TSD_2). This aspect, together with the fact that TSD_2 and TSD_4 are parity partners comprise the main ideas behind the $\mathbb{W}2$ formalism adopted in **I** and **II**. The workflow involved in $\mathbb{W}2$ is drawn in Fig. 9, and for the sake of completeness, the initial $\mathbb{W}1$ approach is also sketched in Fig. 8 from Appendix A.

3. THEORETICAL FORMALISM

In this section, a description of the framework used for obtaining the wobbling spectrum of ^{163}Lu is made. As stated in the previous section, the system is described with a similar Hamiltonian used in $\mathbb{W}1$, namely the Hamiltonian for the triaxial PRM.

$$H = H_{\text{core}} + H_{\text{s.p.}} \quad (1)$$

The Hamiltonian from Eq. 1 describes a system in which an odd j particle interacts with a triaxial even-even core i.e., the odd nucleon is moving in a quadrupole deformed mean-field that is generated by the core. As such, the first term in the

Hamiltonian H_{core} describes the motion of a triaxial core, while the second term $H_{\text{s.p.}}$ represents the single-particle potential characterizing the valence proton.

Indeed, the core Hamiltonian is given by:

$$H_{\text{core}} = \sum_{i=1,2,3} \frac{1}{2\mathcal{I}_i} (I_i - j_i)^2, \quad (2)$$

where the core angular momentum is $\vec{R} = \vec{I} - \vec{j}$ and the terms \mathcal{I}_i represent the moments of inertia for a triaxial ellipsoid, along the principal axes. These three moments of inertia will be considered as free parameters in the present calculations, but, compared to the work W1, a unique set of MOIs will be attributed to the four bands, since the triaxial core will create an alignment with a unique particle, that is j_1 . Because of this, there is no option for their nature (i.e., rigid or hydrodynamic).

The single-particle Hamiltonian from Eq. 1 is derived from the well-known Nilsson potential [88, 89]:

$$h(\beta_2, \gamma) = C \left\{ \cos \gamma Y_{20}(\theta, \varphi) + \frac{\sin \gamma}{\sqrt{2}} [Y_{22}(\theta, \varphi) + Y_{2-2}(\theta, \varphi)] \right\}, \quad (3)$$

where the coupling parameter C causes the level splitting in the deformed field and it is proportional to the quadrupole deformation β_2 . The potential h from Eq. 3 is written in terms of the quadrupole deformation and triaxiality parameter that play the role of deformation parameters within a triaxial system (β_2, γ) . Its expression using the coupling parameter C is widely used when working with a particle-rotor-model [90–92]. In the present this case, the change $h(\beta_2, \gamma) \rightarrow H_{\text{s.p.}}$ is done by applying the Wigner-Eckart theorem for the single- j particle, and the following expression for $H_{\text{s.p.}}$ will be obtained:

$$H_{\text{s.p.}} = \frac{V}{j(j+1)} \left[\cos \gamma (3j_3^2 - \vec{j}^2) - \sqrt{3} \sin \gamma (j_1^2 - j_2^2) \right] + \epsilon_j. \quad (4)$$

This term describes the motion of an odd particle with angular momentum j in a mean-field generated by a triaxial core, with a potential strength V characterized by the quadrupole deformation ($V \propto \beta_2$). In fact, the single-particle potential strength V will be considered as the fourth free parameter within the calculations and its behavior will dictate the coupling of the j particle with all four TSD bands. The term ϵ_j from Eq. 4 represents the single-particle energy that corresponds to the odd j proton from the i -orbital. One should not mix up the j_1 proton notation used throughout the paper with the components of the single-particle angular momentum from Eq. 4.

Regarding the triaxial deformation γ which enters in Eq. 4, its value will be considered as another free parameter of the current problem. In other words, having V and γ as free parameters means that the system will be described by its deformation

parameters which will be obtained through a fitting procedure, keeping an agreement with the experimental data regarding the excitation energies of the rotational states belonging to $TSD_{1,2,3,4}$.

From Eqs. 2 and 4, the free parameter set can be obtained, hereafter denoted by \mathcal{P} . It comprises three moments of inertia, the single-particle potential strength, and the triaxial deformation. As such, \mathcal{P} can be written as:

$$\mathcal{P} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma] . \quad (5)$$

Solving the problem of W2 is equivalent to finding the eigenvalues of H given in Eq. 1. In a similar approach as in W1, the eigenvalues of interest are obtained on the base of a semi-classical approach. Thus, the first step is to perform a de-quantization procedure on H through a TDVE [58, 66, 68]:

$$\delta \int_0^t \langle \Psi_{IjM} | H - i \frac{\partial}{\partial t'} | \Psi_{IjM} \rangle dt' = 0 . \quad (6)$$

Working within a semi-classical approach allows one to keep close contact with the system's dynamics in terms of equations of motion for the generalized coordinates. The trial function from Eq. 6 is carefully chosen as a product of two basis states comprising the states with total angular momentum I and j , respectively:

$$|\Psi_{IjM}\rangle = \mathbf{N} e^{z\hat{I}_-} e^{s\hat{j}_-} |IMI\rangle |jj\rangle , \quad (7)$$

where the operators \hat{I}_- and \hat{j}_- denote the lowering operators for the intrinsic angular momenta \vec{I} and \vec{j} , respectively, and \mathbf{N} plays the role of the normalization constant. One must remark the fact that the states $|IMI\rangle$ and $|jj\rangle$ from Eq. 7 are extremal states for the operators (\hat{I}^2, \hat{I}_3) and (\hat{j}^2, \hat{j}_3) , respectively, and they correspond to the maximally allowed states for a given set of angular momenta I and j . As an observation, the trial function is an admixture of components of definite K , which is consistent with the fact that for a triaxial nucleus, K is not a good quantum number.

The variables z and s from Eq. 7 are complex functions of time, and they play the role of classical coordinates in the phase spaces that describe the motion of the core and the odd particle:

$$z = \rho e^{i\varphi} , \quad s = f e^{i\psi} . \quad (8)$$

In order to obtain a set of classical equations in a Hamilton Canonical form, a new pair of variables are introduced:

$$r = \frac{2I}{1+\rho^2} , \quad t = \frac{2j}{1+f^2} , \quad (9)$$

where $r \in [0, 2I]$ and $t \in [0, 2j]$. Thus the equations of motion acquire the form:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial r} &= \dot{\varphi}; \quad \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r}, \\ \frac{\partial \mathcal{H}}{\partial t} &= \dot{\psi}; \quad \frac{\partial \mathcal{H}}{\partial \psi} = -\dot{t}.\end{aligned}\quad (10)$$

The function \mathcal{H} denotes the average of the Hamiltonian operator H (Eq. 1) with the trial function $|\Psi_{IjM}\rangle$ given in Eq. 7, and it plays the role of classical energy:

$$\mathcal{H}(\varphi, r; \psi, t) = \langle \Psi_{IjM} | H | \Psi_{IjM} \rangle, \quad (11)$$

Starting from the equations of motion given in Eq. 10, one can observe that the function \mathcal{H} is a constant of motion, that is $\dot{\mathcal{H}} \equiv 0$. This equation will define a surface, a so-called equi-energy surface $\mathcal{H} = \text{const}$. It is worth mentioning the fact that such equality holds since the entire set of equations of motion emerged from a variational principle. The sign of the Hessian associated to this classical function will indicate its stationary points. Among them, some are minima. The critical points which are of interest for the present study are those obtained when the following ordering for the three moments of inertia holds: $\mathcal{I}_1 > \mathcal{I}_2 > \mathcal{I}_3$. There is no restriction on γ .

With a linearization procedure for the equations of motion around the minimum point of \mathcal{H} , a dispersion equation will be obtained:

$$\Omega^4 + B\Omega^2 + C = 0. \quad (12)$$

The above equation describes a harmonic type of motion for the nuclear system, with the solutions to this algebraic equation as the *wobbling frequencies* Ω . The terms B and C are functions of total angular momentum I , single-particle a.m. j , inertial parameters $A_k = 1/(2\mathcal{I}_k)$, $k = 1, 2, 3$, single-particle potential strength V , and triaxiality parameter γ . The B term from Eq. 12 has the expression [69]:

$$-B = [(2I-1)(A_3 - A_1) + 2jA_1][(2I-1)(A_2 - A_1) + 2jA_1] + 8A_2A_3Ij + T_B^1 T_B^2, \quad (13)$$

where the terms T_B^1 and T_B^2 are defined as:

$$\begin{aligned}T_B^1 &= \left[(2j-1)(A_3 - A_1) + 2IA_1 + V \frac{2j-1}{j(j+1)} \sqrt{3}(\sqrt{3} \cos \gamma + \sin \gamma) \right], \\ T_B^2 &= \left[(2j-1)(A_2 - A_1) + 2IA_1 + V \frac{2j-1}{j(j+1)} 2\sqrt{3} \sin \gamma \right].\end{aligned}\quad (14)$$

Accordingly, the C term from Eq. 12 has the expression [69]:

$$\begin{aligned}C &= \{[(2I-1)(A_3 - A_1) + 2jA_1] T_C^1 - 4IjA_3^2\} \\ &\quad \times \{[(2I-1)(A_2 - A_1) + 2jA_1] T_C^2 - 4IjA_2^2\},\end{aligned}\quad (15)$$

where the terms T_C^1 and T_C^2 are defined as:

$$\begin{aligned} T_C^1 &= \left[(2j-1)(A_3 - A_1) + 2IA_1 + V \frac{2j-1}{j(j+1)} \sqrt{3}(\sqrt{3} \cos \gamma + \sin \gamma) \right], \\ T_C^2 &= \left[(2j-1)(A_2 - A_1) + 2IA_1 + V \frac{2j-1}{j(j+1)} 2\sqrt{3} \sin \gamma \right]. \end{aligned} \quad (16)$$

170 It can be seen that the terms which enter in B and C , namely (T_B^1, T_B^2) from
171 Eq. 14 and (T_C^1, T_C^2) from Eq. 16 correspond to the quadrupole deformation that
172 causes the single-particle to move in the mean-field of the triaxial core. The terms
173 also define the triaxiality that the nucleus achieves once the odd proton couples to the
174 triaxial core, driving the system up to a large (and stable) deformation.

Going back to Eq. 12, under the restrictions for the MOIs defined above, the dispersion equation admits two real and positive solutions (hereafter denoted with Ω_1^I and Ω_2^I , where $\Omega_1^I < \Omega_2^I$) defined for $j_1 = i_{13/2}$, given by:

$$\Omega_{1,2}^I = \sqrt{\frac{1}{2} (-B \mp (B^2 - 4C)^{1/2})}. \quad (17)$$

These two solutions are interpreted as *wobbling frequencies* associated with the motion of the core, and the motion of the odd-particle respectively. As such, each wobbling frequency has an associated wobbling-phonon number:

$$\Omega_1^I \rightarrow n_{w_1}; \quad \Omega_2^I \rightarrow n_{w_2}. \quad (18)$$

Now the analytical expressions for the four TSD bands in ^{163}Lu are readily obtained:

$$\begin{aligned} E_{\text{TSD1}}^I &= \epsilon_j + \mathcal{H}_{\min}^{(I,j)} + \mathcal{F}_{00}^I, \quad I = 13/2, 17/2, 21/2 \dots \\ E_{\text{TSD2}}^I &= \epsilon_j^1 + \mathcal{H}_{\min}^{(I,j)} + \mathcal{F}_{00}^I, \quad I = 27/2, 31/2, 35/2 \dots \\ E_{\text{TSD3}}^I &= \epsilon_j + \mathcal{H}_{\min}^{(I-1,j)} + \mathcal{F}_{10}^{I-1}, \quad I = 33/2, 37/2, 41/2 \dots \\ E_{\text{TSD4}}^I &= \epsilon_j^2 + \mathcal{H}_{\min}^{(I,j)} + \mathcal{F}_{00}^I, \quad I = 47/2, 51/2, 55/2 \dots, \end{aligned} \quad (19)$$

where $\mathcal{F}_{n_{w_1} n_{w_2}}^I$ is a function of the wobbling frequencies:

$$\mathcal{F}_{n_{w_1} n_{w_2}}^I = \Omega_1^I \left(n_{w_1} + \frac{1}{2} \right) + \Omega_2^I \left(n_{w_2} + \frac{1}{2} \right), \quad (20)$$

and $\mathcal{H}_{\min}^{(I,j)}$ is the classical energy evaluated in its minimal point. For the present case, its analytical expression is given by the following equation:

$$\mathcal{H}_{\min}^{(I,j)} = (A_2 + A_3) \frac{I+j}{2} + A_1(I-j)^2 - V \frac{2j-1}{j+1} \sin \left(\gamma + \frac{\pi}{6} \right). \quad (21)$$

A few aspects regarding the energy spectrum defined in Eq. 19 are worth mentioning. To each band, there is a specific energy ϵ_j associated with the single-particle state. In this case, the odd-proton $j_1 = 13/2$ from the i -orbital is the one that couples to the triaxial core. However, for the bands TSD_2 and TSD_4 , a different renormalization of ϵ_j is considered, since TSD_2 is the unfavored signature partner of TSD_1 , and TSD_4 is the negative parity partner of TSD_2 within the band structure. These quantities will shift the overall energy states belonging to the two bands, each by a different amount. As a result, both ϵ_j^1 and ϵ_j^2 will be adjusted throughout the numerical calculations such that the energy spectrum is best reproduced. Another aspect concerns the band TSD_3 ; since this is the only excited wobbling band within the family, its configuration is built on top of TSD_2 , with the action of a single phonon ($n_{w1} = 1$) operator. Consequently, an energy state I belonging to TSD_3 is obtained from a state $I - 1$ from TSD_2 .

3.1. ENERGY FUNCTION - GEOMETRICAL INTERPRETATION

The analytical expression for the average of H with the trial function describing the system was previously calculated in W1. Indeed, the energy function \mathcal{H} was given in terms of the phase space coordinates $(r, \varphi; t, \psi)$ as follows [69]:

$$\begin{aligned} \mathcal{H} = & \frac{I}{2}(A_1 + A_2) + A_3 I^2 + \frac{2I-1}{2I} r(2I-r) \mathcal{A}_\varphi + \frac{j}{2}(A_1 + A_2) + A_3 j^2 + \frac{2j-1}{2j} t(2j-t) \mathcal{A}_\psi \\ & - 2\sqrt{r(2I-r)t(2j-t)} \mathcal{A}_{\varphi\psi} + A_3 [r(2j-t) + t(2I-r)] - 2A_3 Ij + V \frac{2j-1}{j+1} \mathcal{A}_\gamma, \end{aligned} \quad (22)$$

with:

$$\begin{aligned} \mathcal{A}_\varphi(\varphi) &= (A_1 \cos^2 \varphi + A_2 \sin^2 \varphi - A_3), \\ \mathcal{A}_{\varphi\psi}(\varphi, \psi) &= (A_1 \cos \varphi \cos \psi + A_2 \sin \varphi \sin \psi), \\ \mathcal{A}_\psi(\psi) &= (A_1 \cos^2 \psi + A_2 \sin^2 \psi - A_3), \\ \mathcal{A}_\gamma(t, \psi) &= \left[\cos \gamma - \frac{t(2j-t)}{2j^2} \sqrt{3}(\sqrt{3} \cos \gamma + \sin \gamma \cos 2\psi) \right]. \end{aligned} \quad (23)$$

It is instructive to check the dependence of the energy function on the angular momentum components, e.g., the coordinates $x_k \stackrel{\text{not.}}{=} I_k$, $k = 1, 2, 3$, where the quantization axis is chosen as the 3-axis. By expressing the angular momentum coordinates $x_{1,2,3}$ in terms of the polar angles (θ, φ) and a radius I , one obtains:

$$x_1 = I \sin \theta \cos \varphi, \quad x_2 = I \sin \theta \sin \varphi, \quad x_3 = I \cos \theta. \quad (24)$$

Using this coordinates system and evaluating the energy function around its

minimum point $p_0 = (0, I; 0, j)$, the following expression for \mathcal{H} will be obtained:

$$\mathcal{H}|_{p_0} = I \left(I - \frac{1}{2} \right) \sin^2 \theta \cdot \mathcal{A}_\varphi(\varphi) - 2A_1 I j \sin \theta + T_{\text{core}} + T_{\text{s.p.}} . \quad (25)$$

The last two terms in this equation are independent on the polar angles (θ, φ) and they have the following form:

$$\begin{aligned} T_{\text{core}} &= \frac{I}{2} (A_1 + A_2) + A_3 I^2 , \\ T_{\text{s.p.}} &= \frac{j}{2} (A_2 + A_3) + A_1 j^2 - V \frac{2j-1}{j+1} \sin \left(\gamma + \frac{\pi}{6} \right) . \end{aligned} \quad (26)$$

188 The classical equations of motion admit two constants of motion: the total en-
189 ergy (E) and the total angular momentum (I). Consequently, by finding the intersec-
190 tion line(s) between the energy surface E and the surface of the angular momentum
191 (i.e., a sphere of radius $r = I$), the system's trajectory at that particular energy and
192 spin is obtained. Such representations will be made in the following section.

By changing Eq. 25 from polar coordinates into Cartesian coordinates, the energy surface E will become:

$$\begin{aligned} E &= \left(1 - \frac{1}{2I} \right) A_1 x_1^2 + \left(1 - \frac{1}{2I} \right) A_2 x_2^2 + \left[\left(1 - \frac{1}{2I} \right) A_3 + A_1 \frac{j}{I} \right] x_3^2 - \\ &- I \left(I - \frac{1}{2} \right) A_3 - 2A_1 I j + T_{\text{rot}} + T_{\text{sp}} . \end{aligned} \quad (27)$$

Indeed, one can notice that the three coordinates x_k appear as a squared sum. If some notations are made for the terms appearing next to the coordinates and the coordinate-free terms, one arrives at the following expression for the energy surface:

$$E = \mathcal{S}_1 x_1^2 + \mathcal{S}_2 x_2^2 + \mathcal{S}_3 x_3^2 + C_0^{\text{rot+sp}} . \quad (28)$$

193 From Eq. 28, it is now clear that the energy surface will be an ellipsoid, while
194 the free-term $C_0^{\text{rot+sp}}$ will produce an overall shift of the entire surface. The shift is in
195 fact caused by the core+particle coupling.

For a total angular momentum \vec{I} , the vector generates a sphere of radius $r = I$ described by the equation:

$$I^2 = x_1^2 + x_2^2 + x_3^2 . \quad (29)$$

196 The trajectories obtained through the intersection of Eqs. 27 and 29 will show
197 a classical visualization of the wobbling character for a triaxial nucleus.

4. NUMERICAL RESULTS

198 Numerical results.

Table 1

The parameter set \mathcal{P} that was determined by a fitting procedure of the excitation energies for ^{163}Lu , provided via calculations from I.

$\mathcal{I}_1 [\hbar^2/\text{MeV}]$	$\mathcal{I}_2 [\hbar^2/\text{MeV}]$	$\mathcal{I}_3 [\hbar^2/\text{MeV}]$	$\gamma [\text{deg.}]$	$V [\text{MeV}]$
72	15	7	22	2.1

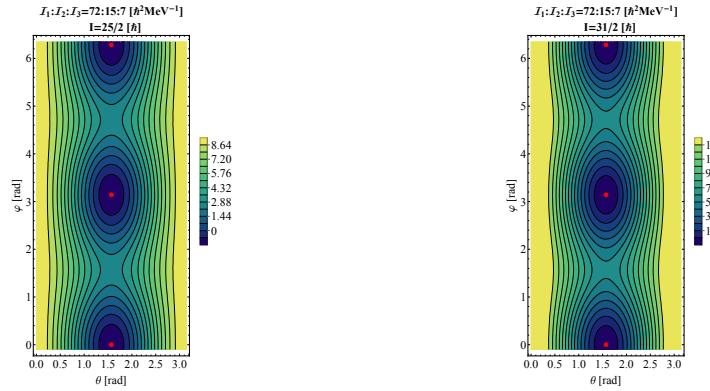


Fig. 2 – Contour plots with the energy function \mathcal{H} given by Eq. 25 for a state in TSD_1 (left) and a state from TSD_2 (right). Calculations were performed with the numerical parameters given in Table 1. The minimum points for \mathcal{H} are marked by red dots, and they represent the regions in space where the nucleus has a stable wobbling character. The darker *islands* also indicate a stable motion of the triaxial nucleus.

4.1. STABILITY OF THE WOBBLING REGION

The expression for the classical energy function, which plays a crucial role in analyzing the nucleus's stability for a given rotational state, was presented in the previous section, through Eq. 25. This will be used within the present numerical calculations to pinpoint the regions in space where the minimal points of \mathcal{H} do exist. A special interest is devoted to the low-lying states from each of the four bands. Namely, for each band, a spin-state close to the band-head is chosen, then using the parameter set \mathcal{P} , a graphical representation in the (θ, φ) -coordinate space is realized, and in each case, the extremal points with minimum character are identified. These graphical representations are shown in Figures 2 and 3.

The four contour plots shown in Figures 2 and 3 have many similarities, suggesting common collective properties, but also differences which are caused by the fact that the minima have different depths. A common feature consists in that the equi-energy curves surround a sole minimum for low values in energy, but as the energy increases, the trajectories go around all minima, the lack of localization indicating unstable wobbling motion. The unstable regions might also relate to phase transitions, where the nucleus can undergo a major change in its rotational char-

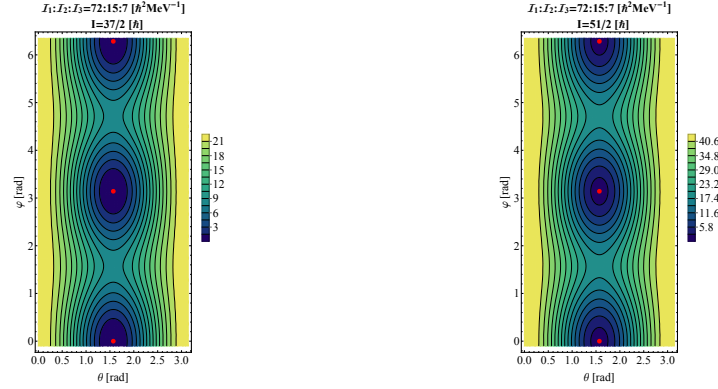


Fig. 3 – Contour plots with the energy function \mathcal{H} given by Eq. 25 for a state in TSD_3 (left) and a state from TSD_4 (right). Calculations were performed with the numerical parameters given in Table 1. The minimum points for \mathcal{H} are marked by red dots, and they represent the regions in space where the nucleus has a stable wobbling character. The darker *islands* also indicate a stable motion of the triaxial nucleus.

acter. This aspect will also be discussed in the next subsection, devoted to the 3-dimensional representation of the energy ellipsoid and the classical trajectories of the triaxial system.

Regarding the minimum points (marked by red dots on the contour plots), their position remains unchanged for all four bands and any rotational state I , as long as the MOI order stays the same. Remarkable is the fact that only with the obtained set of parameters (i.e., the current MOI ordering) it was possible to define contours with stable motion (marked by the darker regions). Indeed, if the two ratios $\mathcal{I}_1/\mathcal{I}_2$ and $\mathcal{I}_2/\mathcal{I}_3$ would have been smaller, a larger unstable region would prevail (with islands of maximal character), constraining thus the stable wobbling motion. This could indicate the fact that the single-particle term $T_{s.p.}$ from \mathcal{H} is sensitive to larger triaxiality, and only for certain values will the system achieve a stable motion characterized by large deformation (see Eq. 26).

An additional step consists in the analysis of the energy function, more precisely to see its evolution in one of the minimum points with respect to the angular momentum I . As it was already observed from the contour plots shown in Figures 2-3, the depth of the minima differs from one spin state to another, so it would be useful to have a quantitative view on that change. By fixing \mathcal{H} in one of its critical points (e.g., the minimum $p_{\min}(\theta, \varphi) = (\frac{\pi}{2}, 0)$), the angular momentum I was varied within a large interval, and the evolution of \mathcal{H} was evaluated. Graphical representation is shown in Figure 4.

As it can be seen from Figure 4, the classical energy \mathcal{H} is an increasing function of angular momentum, which is to be expected, since the wobbling energies of the four bands increase with respect to the increase in spin. The negative values of \mathcal{H}

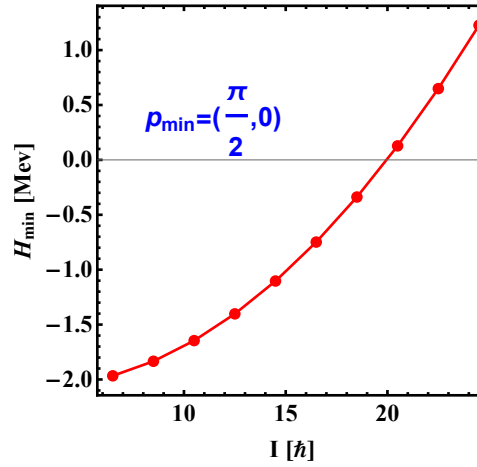


Fig. 4 – The change in the minimum depth of \mathcal{H} , evaluated in the one of its critical points $p_{\min}(\theta, \varphi) = (\frac{\pi}{2}, 0)$, using the parameters given in Table 1.

for low-lying wobbling states do not indicate that the nucleus has negative energy states since *the rest* of the nucleus' energy is also given by the single-particle energy ϵ_j terms and the phononic $\mathcal{F}_{n_{w_1} n_{w_2}}^I$ terms.

Another useful insight would be the study of the classical energy function \mathcal{H} as a function of the polar angles (θ, φ) . This can be achieved by choosing a minimum point, keeping one of the polar coordinates fixed, and then let the other one vary across its corresponding interval. For ^{163}Lu , such a graphical representation was done for the point $p_{\min} = (\frac{\pi}{2}, 0)$ (that is the bottom-most red dot from each of the four contour plots depicted in Figures 2-3). Results can be seen in Figure 5.

4.2. CLASSICAL TRAJECTORIES - 3D REPRESENTATION

The final step of the present work is to obtain an insight into the classical features of ^{163}Lu concerning the total angular momentum and its rotational motion. As already mentioned, the trajectories are given by the intersection curves of the energy ellipsoid E given in Eq. 27 with the angular momentum sphere I given in Eq. 29. In the 3-dimensional space generated by the three components of the angular momentum vector \vec{I} , these intersection curves characterize the motion of the system, as each curve will be oriented along with one of the three axes x_k , $k = 1, 2, 3$, suggesting a rotational motion (the precession of the total a.m.) around a particular direction preferred by the system.

As such, the dependence of the classical trajectories on the angular momenta as well as on energies must be analyzed in W2. Indeed, when the model Hamiltonian is diagonalized for a given I , a set of $2I + 1$ energies are obtained. Therefore, it

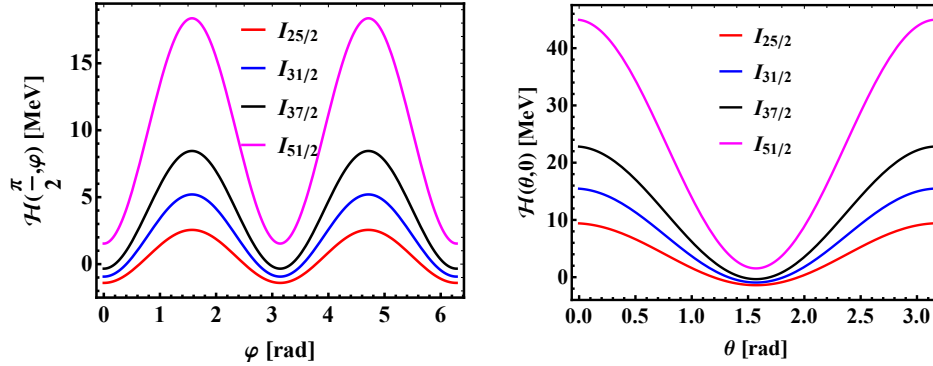


Fig. 5 – The energy function \mathcal{H} , evaluated in one of its minimum points, as a function of the polar coordinates. One coordinate is fixed while the other one is varied within its interval of existence. For $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. The chosen minimum is $p_{\min} = (\frac{\pi}{2}, 0)$. Each spin state corresponds to one of the four triaxial bands of ^{163}Lu (with $I_{25/2} \in TSD_1$, $I_{31/2} \in TSD_2$, $I_{37/2} \in TSD_3$, and $I_{51/2} \in TSD_4$).

is justified to study the evolution of trajectories when the energy of the nucleus is increasing. The curves are represented as the manifold given by the intersection of the two constants of motion, that is E and I . An example of such trajectories are depicted in Figures 6-7.

Each row from the Figures 6-7 represents a rotational state within a band. A low-lying spin state was chosen from each band in particular as an example. The left inset of each row represents the real excitation energy for the state I at which the energy ellipsoid is evaluated. It can be seen that two distinct (but symmetric) trajectories are observed along the 1-axis, for all four states. This suggests that the states of the triaxial nucleus are obtained from the rotation of the angular momentum along x_1 . Indeed, for low energies, the rotation is more pronounced along the x_1 - and $-x_1$ -axes. As the energy of the nucleus increases, the two trajectories approach each other, which results in a tilted rotation axis corresponding to both curves. The tilted axis implies that the rotation axis is being misaligned, the rotational axis moving away from its *equilibrium point*, marking the tilted-axis-rotation. Note that this picture is fully consistent with the one described by Lawrie et al. [98]. Further increase in energy will result in the two trajectories intersect with each other. The point where the intersection between the two orbits occurs is marked in the middle inset from each figure. Consequently, the intersection of these two orbits marks an unstable motion within the system. Finally, when the energy increases even more, beyond this *critical point*, one arrives again in a two-trajectories regime but with a different rotation axis, lying closer to the x_3 axis. This case is shown in the right inset within each figure, where the axis x_3 is marked by red color, signaling the change in the rotational mode of the nucleus. However, it is worth noting that such energies are

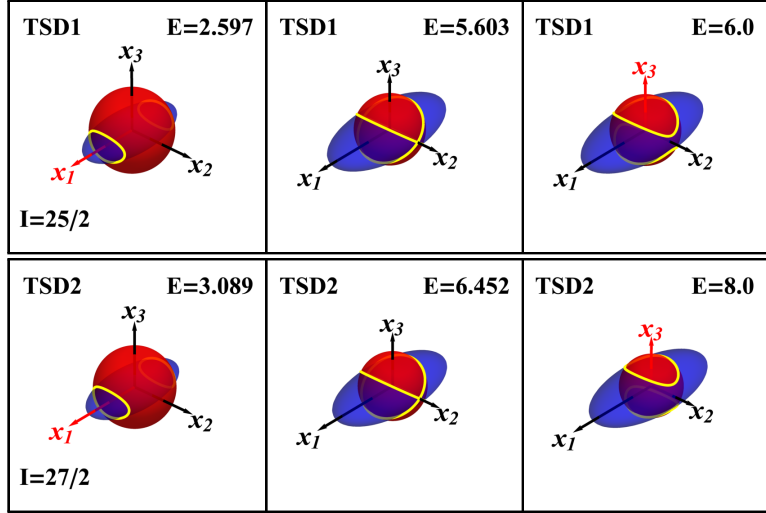


Fig. 6 – The nuclear trajectories of the system, evaluated for two spin states belonging to TSD_1 and TSD_2 . Intersection lines marked by yellow color represents the actual orbits. Axis colored in red represents the direction along which the system rotates (it precesses). The left-most inset corresponds to the real excitation energy for that particular spin state I .

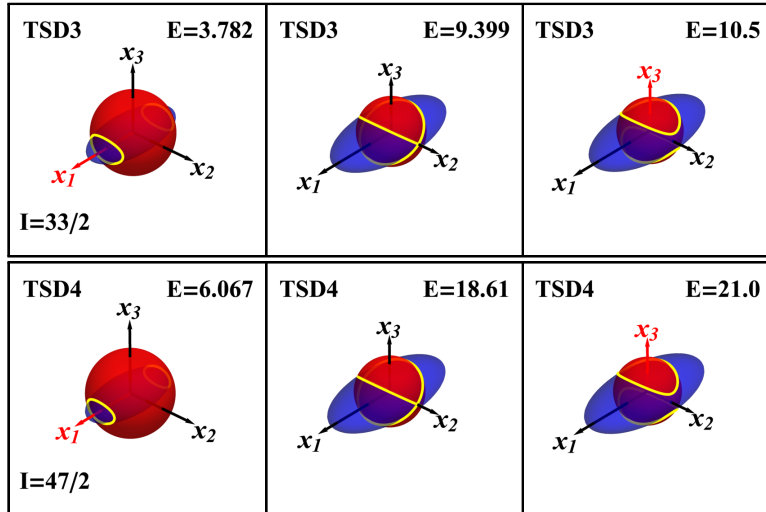


Fig. 7 – The nuclear trajectories of the system, evaluated for two spin states belonging to TSD_3 and TSD_4 . Intersection lines marked by yellow color represents the actual orbits. Axis colored in red represents the direction along which the system rotates (it precesses). The left-most inset corresponds to the real excitation energy for that particular spin state I .

way too large for such a phase transition to occur naturally in ^{163}Lu . For example, in the case of $I_{25/2} \in TSD_1$, the energy at which ^{163}Lu undergoes a phase transition with regards to the rotational mode is close to 5.6 MeV (middle inset for TSD_1 from Figure 2), but the real excitation energy which corresponds to this state is half that value (left inset for TSD_1 from Figure 6). Nevertheless, it is a remarkable fact that with the current model, a phase transition between rotational modes in a triaxial nucleus can be identified. A proper microscopic formalism based on this approach might also provide a more detailed picture with regards to the allowed trajectories for the system.

5. CONCLUSIONS & OUTLOOK

Concluding the present work, this newly developed formalism proves to be a successful tool for accurately describing the wobbling spectrum of ^{163}Lu and also for providing an insight into the rotational motion of the nuclear system with respect to its total spin.

Acknowledgments. This work was supported by UEFISCU, through the project **PCE-16/2021**.

A. APPENDIX - WORKFLOW DIAGRAMS

The two models described throughout the paper, namely the formalisms W1 and W2 are schematically represented. The W1 model corresponds to the work given in Refs. [19, 69], and the W2 corresponds to the formalism developed in this two-part series of papers. For W1, the diagram is shown in Figure 8, while for the newly developed approach W2, the diagram is shown in Figure 9. The coupling scheme for W2 is also represented in Figure 10.

REFERENCES

1. Peter Möller, Ragnar Bengtsson, B Gillis Carlsson, Peter Olivius, and Takatoshi Ichikawa. *Physical review letters*, 97(16):162502, 2006.
2. Peter Möller, Arnold J Sierk, Takatoshi Ichikawa, Akira Iwamoto, Ragnar Bengtsson, Henrik Uhrenholt, and Sven Åberg. *Physical Review C*, 79(6):064304, 2009.
3. I Hamamoto and H Sagawa. *Physics Letters B*, 201(4):415–419, 1988.
4. R Bengtsson, H Frisk, FR May, and JA Pinston. *Nuclear Physics A*, 415(2):189–214, 1984.
5. J Stachel, N Kaffrell, E Grosse, H Emling, H Folger, R Kulesa, and D Schwalm. *Nuclear Physics A*, 383(3):429–467, 1982.
6. Stefan Frauendorf and Jie Meng. *Nuclear Physics A*, 617(2):131–147, 1997.
7. Aage Niels Bohr and Ben R Mottelson. *Nuclear Structure (In 2 Volumes)*.

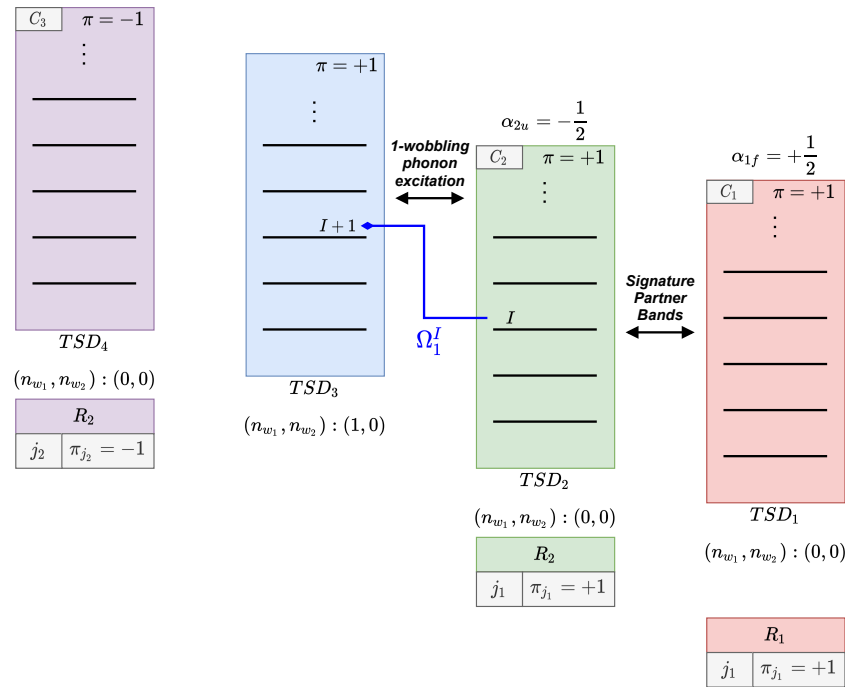


Fig. 8 – Schematic representation of the band structure adopted for ^{163}Lu in the W1 model. For each band, the wobbling phonon numbers are shown. The main features and linking properties between bands are represented with arrows. The bottom part shows the coupling scheme (the core and the valence nucleon) for each wobbling band. The blue arrow marks the activation of TSD_3 states via the phonon operator.

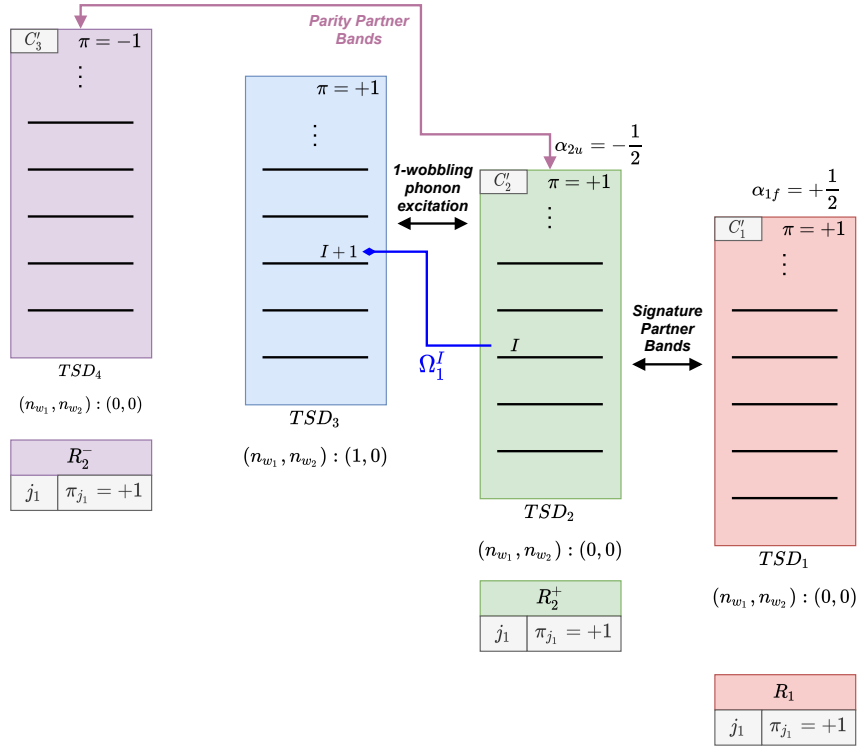


Fig. 9 – Schematic representation of the band structure adopted for ^{163}Lu in the w_2 model. For each band, the wobbling phonon numbers are shown. The main features and linking properties between bands are represented with arrows. The bottom part shows the coupling scheme (the core and the valence nucleon) for each wobbling band. The blue arrow marks the activation of TSD_3 states via the phonon operator.

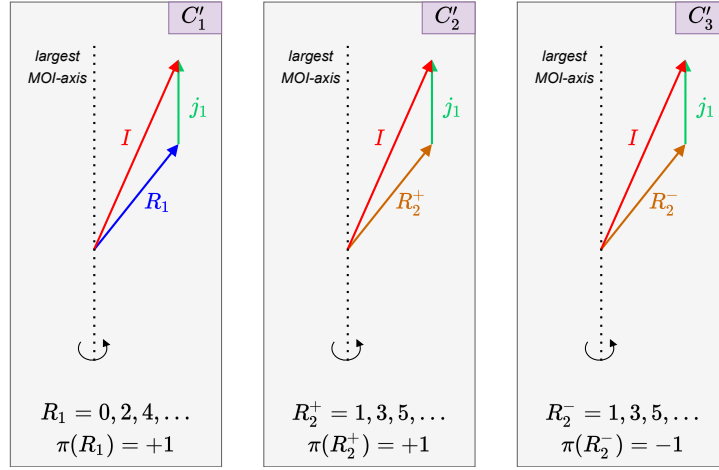


Fig. 10 – A schematic representation with the three coupling schemes that characterize the W_2 model. The same odd particle ($j_1 = i_{13/2}$ proton) is coupled with two positive cores with even (odd) integer spin sequences for TSD_1 (TSD_2), and one negative core in the case of TSD_4 with odd integer spin sequence. The total spin of the system precesses around the axis with the largest MOI, as it is the case for a triaxial rotor.

- 314 8. P Bringel, GB Hagemann, H Hübel, A Al-Khatib, P Bednarczyk, A Bürger, D Curien, G Gan-
- 315 gopadhyay, B Herskind, DR Jensen, et al. *The European Physical Journal A-Hadrons and Nuclei*,
- 316 24(2):167–172, 2005.
- 317 9. DJ Hartley, RVF Janssens, LL Riedinger, MA Riley, A Aguilar, MP Carpenter, CJ Chiara,
- 318 P Chowdhury, IG Darby, U Garg, et al. *Physical Review C*, 80(4):041304, 2009.
- 319 10. H Amro, WC Ma, GB Hagemann, RM Diamond, J Domscheit, P Fallon, A Görgen, B Herskind,
- 320 H Hübel, DR Jensen, et al. *Physics Letters B*, 553(3-4):197–203, 2003.
- 321 11. G Schönwaßer, H Hübel, GB Hagemann, P Bednarczyk, G Benzoni, A Bracco, P Bringel, R Chap-
- 322 man, D Curien, J Domscheit, et al. *Physics Letters B*, 552(1-2):9–16, 2003.
- 323 12. BW Xiong and YY Wang. *Atomic Data and Nuclear Data Tables*, 125:193–225, 2019.
- 324 13. VI Dimitrov, S Frauendorf, and F Döna. *Physical review letters*, 84(25):5732, 2000.
- 325 14. T Koike, K Starosta, CJ Chiara, DB Fossan, and DR LaFosse. *Physical Review C*, 67(4):044319,
- 326 2003.
- 327 15. J Meng, J Peng, SQ Zhang, and S-G Zhou. *Physical Review C*, 73(3):037303, 2006.
- 328 16. AA Raduta, Al H Raduta, and CM Petrache. *Journal of Physics G: Nuclear and Particle Physics*,
- 329 43(9):095107, 2016.
- 330 17. CM Petrache, BF Lv, A Astier, E Dupont, YK Wang, SQ Zhang, PW Zhao, ZX Ren, J Meng,
- 331 PT Greenlees, et al. *Physical Review C*, 97(4):041304, 2018.
- 332 18. BF Lv, CM Petrache, QB Chen, J Meng, A Astier, E Dupont, P Greenlees, H Badran, T Calverley,
- 333 DM Cox, et al. *Physical Review C*, 100(2):024314, 2019.
- 334 19. AA Raduta, R Poenaru, and CM Raduta. *Journal of Physics G: Nuclear and Particle Physics*,
- 335 47(2):025101, 2020.
- 336 20. R Poenaru and AA Raduta, *Romanian Journal of Physics*, Paper submitted, 2021.
- 337 21. Gudrun B Hagemann and Ikuko Hamamoto. *Nuclear Physics News*, 13(3):20–24, 2003.

22. SW Ødegård, GB Hagemann, DR Jensen, M Bergström, B Herskind, G Sletten, S Törmänen, JN Wilson, PO Tjøm, I Hamamoto, et al. *Physical review letters*, 86(26):5866, 2001.
23. DR Jensen, GB Hagemann, I Hamamoto, SW Ødegård, B Herskind, G Sletten, JN Wilson, K Spohr, H Hübel, P Bringel, et al. *Physical review letters*, 89(14):142503, 2002.
24. D Ringkøbing Jensen, GB Hagemann, I Hamamoto, SW Ødegård, M Bergström, B Herskind, G Sletten, S Törmänen, JN Wilson, PO Tjøm, et al. *Nuclear Physics A*, 703(1-2):3–44, 2002.
25. H Schnack-Petersen, Ragnar Bengtsson, RA Bark, P Bosetti, A Brockstedt, H Carlsson, LP Ekström, GB Hagemann, B Herskind, F Ingebretsen, et al. *Nuclear Physics A*, 594(2):175–202, 1995.
26. S Biswas, R Palit, S Frauendorf, U Garg, W Li, GH Bhat, JA Sheikh, J Sethi, S Saha, Purnima Singh, et al. *The European Physical Journal A*, 55(9):1–7, 2019.
27. James Till Matta. Transverse wobbling in 135 pr. In *Exotic Nuclear Excitations: The Transverse Wobbling Mode in 135 Pr*, pages 77–93. Springer, 2017.
28. N Sensharma, U Garg, S Zhu, AD Ayangeakaa, S Frauendorf, W Li, GH Bhat, JA Sheikh, MP Carpenter, QB Chen, et al. *Physics Letters B*, 792:170–174, 2019.
29. S Chakraborty, HP Sharma, SS Tiwary, C Majumder, AK Gupta, P Banerjee, S Ganguly, S Rai, S Kumar, A Kumar, et al. *Physics Letters B*, 811:135854, 2020.
30. C M Petrache. *LNL Annual Report*, 2018.
31. J Timár, QB Chen, B Kruzsicz, D Sohler, I Kuti, SQ Zhang, J Meng, P Joshi, R Wadsworth, K Starosta, et al. *Physical review letters*, 122(6):062501, 2019.
32. S Nandi, G Mukherjee, QB Chen, S Frauendorf, R Banik, Soumik Bhattacharya, Shabir Dar, S Bhattacharyya, C Bhattacharya, S Chatterjee, et al. *Physical Review Letters*, 125(13):132501, 2020.
33. N Sensharma, U Garg, QB Chen, S Frauendorf, DP Burdette, JL Cozzi, KB Howard, S Zhu, MP Carpenter, P Copp, et al. *Physical review letters*, 124(5):052501, 2020.
34. S Guo, XH Zhou, CM Petrache, EA Lawrie, S Mthembu, YD Fang, HY Wu, HL Wang, HY Meng, GS Li, et al. *arXiv preprint arXiv:2011.14354*, 2020.
35. JH Hamilton, SJ Zhu, YX Luo, AV Ramayya, S Frauendorf, JO Rasmussen, JK Hwang, SH Liu, GM Ter-Akopian, AV Daniel, et al. *Nuclear Physics A*, 834(1-4):28c–31c, 2010.
36. YX Luo, JH Hamilton, AV Ramaya, JK Hwang, SH Liu, JO Rasmussen, S Frauendorf, GM Ter-Akopian, AV Daniel, Yu Ts Oganessian, et al. Triaxial and triaxial softness in neutron rich ru and pd nuclei. In *Exotic Nuclei: EXON-2012*, pages 215–224. World Scientific, 2013.
37. CM Petrache, PM Walker, S Guo, QB Chen, S Frauendorf, YX Liu, RA Wyss, D Mengoni, YH Qiang, A Astier, et al. *Physics Letters B*, 795:241–247, 2019.
38. YK Wang, FQ Chen, and PW Zhao. *Physics Letters B*, 802:135246, 2020.
39. QB Chen, S Frauendorf, and CM Petrache. *Physical Review C*, 100(6):061301, 2019.
40. S Frauendorf and F Döna. *Physical Review C*, 89(1):014322, 2014.
41. Ikuko Hamamoto. *Physical Review C*, 65(4):044305, 2002.
42. Kosai Tanabe and Kazuko Sugawara-Tanabe. *Physical Review C*, 73(3):034305, 2006.
43. Shi Wen-Xian and Chen Qi-Bo. *Chinese Physics C*, 39(5):054105, 2015.
44. AS Davydov and GF Filippov. *Nuclear Physics*, 8:237–249, 1958.
45. Yoshifumi R Shimizu and Masayuki Matsuzaki. *Nuclear Physics A*, 588(3):559–596, 1995.
46. Masayuki Matsuzaki, Yoshifumi R Shimizu, and Kenichi Matsuyanagi. *Physical Review C*, 65(4):041303, 2002.
47. Masayuki Matsuzaki, Yoshifumi R Shimizu, and Kenichi Matsuyanagi. *The European Physical Journal A-Hadrons and Nuclei*, 20(1):189–190, 2003.

- 384 48. Masayuki Matsuzaki and Shin-Ichi Ohtsubo. *Physical Review C*, 69(6):064317, 2004.
- 385 49. Masayuki Matsuzaki, Yoshifumi R Shimizu, and Kenichi Matsuyanagi. *Physical Review C*,
386 69(3):034325, 2004.
- 387 50. Yoshifumi R Shimizu, Masayuki Matsuzaki, and Kenichi Matsuyanagi. *Physical Review C*,
388 72(1):014306, 2005.
- 389 51. Yoshifumi R Shimizu, Takuya Shoji, and Masayuki Matsuzaki. *Physical Review C*, 77(2):024319,
390 2008.
- 391 52. Takuya Shoji and Yoshifumi R Shimizu. *Progress of theoretical physics*, 121(2):319–355, 2009.
- 392 53. QB Chen, SQ Zhang, PW Zhao, and J Meng. *Physical Review C*, 90(4):044306, 2014.
- 393 54. QB Chen, SQ Zhang, J Meng, et al. *Physical Review C*, 94(5):054308, 2016.
- 394 55. S Mukhopadhyay, D Almeded, U Garg, S Frauendorf, T Li, PV Madhusudhana Rao, X Wang,
395 SS Ghugre, MP Carpenter, S Gros, et al. *Physical review letters*, 99(17):172501, 2007.
- 396 56. Bin Qi, SQ Zhang, J Meng, SY Wang, and S Frauendorf. *Physics Letters B*, 675(2):175–180, 2009.
- 397 57. Makito Oi, Ahmad Ansari, Takatoshi Horibata, and Naoki Onishi. *Physics Letters B*, 480(1-2):53–
398 60, 2000.
- 399 58. AA Raduta, R Poenaru, and L Gr Ixaru. *Physical Review C*, 96(5):054320, 2017.
- 400 59. AA Raduta, CM Raduta, and R Poenaru. *Journal of Physics G: Nuclear and Particle Physics*,
401 48(1):015106, 2020.
- 402 60. K Tanabe and K Sugawara-Tanabe. *Physics Letters B*, 34(7):575–578, 1971.
- 403 61. Kosai Tanabe and Kazuko Sugawara-Tanabe. *Physical Review C*, 77(6):064318, 2008.
- 404 62. Mitsuhiro Shimada, Yudai Fujioka, Shingo Tagami, and Yoshifumi R Shimizu. *Physical Review*
405 *C*, 97(2):024318, 2018.
- 406 63. Kenji Hara and Yang Sun. *International Journal of Modern Physics E*, 4(04):637–785, 1995.
- 407 64. PW Zhao, P Ring, and J Meng. *Physical Review C*, 94(4):041301, 2016.
- 408 65. M Konieczka, Markus Kortelainen, and W Satuła. *Physical Review C*, 97(3):034310, 2018.
- 409 66. AA Raduta, R Budaca, and CM Raduta. *Physical Review C*, 76(6):064309, 2007.
- 410 67. AA Raduta, R Poenaru, and Al H Raduta. *Journal of Physics G: Nuclear and Particle Physics*,
411 45(10):105104, 2018.
- 412 68. R Budaca. *Physical Review C*, 97(2):024302, 2018.
- 413 69. AA Raduta, R Poenaru, and CM Raduta. *Physical Review C*, 101(1):014302, 2020.
- 414 70. Tord Bengtsson. *Nuclear Physics A*, 512(1):124–148, 1990.
- 415 71. A Grgen, RM Clark, M Cromaz, P Fallon, GB Hagemann, H Hbel, IY Lee, AO Macchiavelli,
416 G Sletten, D Ward, et al. *Physical Review C*, 69(3):031301, 2004.
- 417 72. GB Hagemann. *Acta Physica Polonica B*, 36(4):1043, 2005.
- 418 73. DR Jensen, GB Hagemann, I Hamamoto, B Herskind, G Sletten, JN Wilson, SW Ødegård,
419 K Spohr, H Hbel, P Bringel, et al. *The European Physical Journal A-Hadrons and Nuclei*,
420 19(2):173–185, 2004.
- 421 74. Kosai Tanabe and Kazuko Sugawara-Tanabe. *Physical Review C*, 95(6):064315, 2017.
- 422 75. S Frauendorf. *Physical Review C*, 97(6):069801, 2018.
- 423 76. Kosai Tanabe and Kazuko Sugawara-Tanabe. *Physical Review C*, 97(6):069802, 2018.
- 424 77. Yang Sun, Shuxian Wen, et al. *Physical Review C*, 50(5):2351, 1994.
- 425 78. AM Khalaf, Hayam Yassin, and Eman R Abo Elyazeed. *Journal: Journal Of Advances In Physics*,
426 11(1), 2015.
- 427 79. VS Uma and Alpna Goel. *The European Physical Journal Plus*, 130(6):1–6, 2015.
- 428 80. HM Mittal and Anshul Dadwal. In *Proceedings of the DAE-BRNS Symp. on Nucl. Phys*, volume 61,
429 page 134, 2016.

- 430 81. Ikuko Hamamoto and Ben Mottelson. *Physics Letters B*, 127(5):281–285, 1983.
- 431 82. Ikuko Hamamoto. *Physics Letters B*, 193(4):399–404, 1987.
- 432 83. Ikuko Hamamoto. *Physica Scripta*, 91(2):023004, 2016.
- 433 84. S Torilov, S Thummerer, W Von Oertzen, Tz Kokalova, G De Angelis, HG Bohlen, A Tumino,
434 M Axiotis, E Farnea, N Marginean, et al. *The European Physical Journal A-Hadrons and Nuclei*,
435 19(3):307–317, 2004.
- 436 85. ME Debray, MA Cardona, D Hojman, AJ Kreiner, M Davidson, J Davidson, H Somacal, G Lev-
437 inton, DR Napoli, S Lenzi, et al. *Physical Review C*, 62(2):024304, 2000.
- 438 86. AA Radutaa and CM Radutab. *arXiv preprint arXiv:0903.0076*, 2009.
- 439 87. AA Raduta and CM Raduta. *Annals of the University of Craiova, Physics*, 21(1):28–53, 2011.
- 440 88. J Meyer-ter Vehn. *Nuclear Physics A*, 249(1):111–140, 1975.
- 441 89. SY Wang, SQ Zhang, B Qi, J Peng, JM Yao, J Meng, et al. *Physical Review C*, 77(3):034314,
442 2008.
- 443 90. J Peng, J Meng, and SQ Zhang. *Physical Review C*, 68(4):044324, 2003.
- 444 91. T Koike, K Starosta, and I Hamamoto. *Physical review letters*, 93(17):172502, 2004.
- 445 92. SY Wang, SQ Zhang, B Qi, and J Meng. *Physical Review C*, 75(2):024309, 2007.
- 446 93. CW Reich and Balraj Singh. *Nuclear Data Sheets*, 111(5):1211–1469, 2010.
- 447 94. RR Chasman. *Physics Letters B*, 96(1-2):7–10, 1980.
- 448 95. AA Raduta, CM Raduta, and Amand Faessler. *Physics Letters B*, 635(2-3):80–84, 2006.
- 449 96. AA Raduta, Al H Raduta, and CM Raduta. *Physical Review C*, 74(4):044312, 2006.
- 450 97. Wang Shou-Yu, Qi Bin, and Zhang Shuang-Quan. *Chinese Physics Letters*, 26(5):052102, 2009.
- 451 98. EA Lawrie, O Shirinda, and CM Petrache. *Physical Review C*, 101(3):034306, 2020.