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NUCLEI
Theory

Anharmonicity in Nuclear Wobbling Motion*

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Abstract—An unexpected strong anharmonicity was observed in the wobbling spectrum in ^{163}Lu. In an attempt to understand what causes the deviation from the original wobbling model by Bohr and Mottelson, an analysis is presented using several different approaches, such as exact diagonalization, a semiclassical model to deal with anharmonic wobbling motion, and a microscopic method based on the self-consistent cranking calculation.

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1. INTRODUCTION

In the presence of triaxial deformation, a nucleus can wobble at very high spin ($I \gg 1$) [1]. With a macroscopic rotor Hamiltonian,

$$H = \frac{I_1^2}{2\mathfrak{I}_1} + \frac{I_2^2}{2\mathfrak{I}_2} + \frac{I_3^2}{2\mathfrak{I}_3}, \qquad (1)$$

it is possible to demonstrate how nuclear wobbling motion emerges. Here, $\mathfrak{I}_i$ denotes the moment of inertia around the ith axis, which can be assumed to be $\mathfrak{I}_1 > \mathfrak{I}_2 > \mathfrak{I}_3$ without a loss of generality. The Hamiltonian is invariant with respect to discrete operations, $R_i = e^{-i\pi I_i}$, which correspond to a representation of the D_2 symmetry. This symmetry means a presence of triaxiality in the relevant system. Following Bohr and Mottelson [1], let us choose the first axis to be the quantization axis, although the corresponding quantum number (let us write it as K) is not a good quantum number owing to a violation of axial symmetry in the Hamiltonian. Because the angular momentum operators are defined in the intrinsic frame, the total angular momentum I, which can be classically expressed as $\sqrt{I_1^2 + I_2^2 + I_3^2}$, is a conserved quantity. In this way, the eigenstate of the Hamiltonian can be expressed as

$$|\Psi_w^I\rangle = \sum_K C_K^I |IK;\alpha\rangle, \qquad (2)$$

where α denotes other quantum numbers, such as the signature.

The high-spin condition, $I \gg 1$, can be considered as a semiclassical limit. Under this limit, a fluctuation ΔK can be controlled to be small despite the fact that K is not a good quantum number. In this case, the components of the wave function can be well localized around the $|II\rangle$, which implies $K \simeq I$, or $|C_K^I|^2 \simeq 1$. This semiclassical state is energetically favored because the moment of inertia takes the maximum value for the first axis. The meaning of this approximation is that rotational motion of the triaxial rotor is nearly one-dimensional. Let us call this approximation the "near-1D rotation" assumption. In short, the essence of the Bohr–Mottelson model for the wobbling motion relies on two assumptions: (i) $I \gg 1$ (the high-spin assumption) and (ii) $K \simeq I$ (the near-1D rotation assumption).

The Hamiltonian can be decomposed into two parts, $H = H_0 + H_w$. The first term, H_0, represents the (1D) rotational energy around the first axis, that is, $H_0 = \dfrac{I_1^2 + I_2^2 + I_3^2}{2\mathfrak{I}_1} = \dfrac{I(I+1)}{2\mathfrak{I}_1}$, while the second term is responsible for the orientational fluctuation around the first axis,

$$H_w = \frac{1}{2}\left(\frac{1}{\mathfrak{I}_2} - \frac{1}{\mathfrak{I}_1}\right) I_2^2 + \frac{1}{2}\left(\frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}_1}\right) I_3^2. \qquad (3)$$

Note that the first term is reduced to a C number, while the second remains a Q number.

For convenience, new operators are introduced here: $a_\pm = \dfrac{1}{\sqrt{2I}}\left(I_2 \pm iI_3\right)$. They correspond to the raising and lowering operators of angular momentum. The commutation relation is given as $[a_-, a_+] = I_1/I$. After the introduction of these operators, the Hamiltonian is to contain the so-called dangerous terms, such as a_-a_- and a_+a_+. These terms can be interpreted as the pairing correlations in the sense

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Numerical results at $I = 40\hbar$ (left) and $41\hbar$ (right) (energies are shown relative to the ground state; units for angular momentum is $\hbar$)

State ID	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
r	+1	−1	+1	−1	+1	−1	+1	−1	−1	+1	−1	+1	−1	+1	−1	+1
E/E_0	1.00	2.97	4.87	6.71	8.48	10.18	11.81	13.36	1.00	2.97	4.88	6.72	8.50	10.20	11.84	13.41
$I - \langle\hat{K}\rangle$	0.13	1.39	2.69	4.01	5.37	6.78	8.25	9.79	0.13	1.39	2.69	4.01	5.36	6.77	8.22	9.75
ΔK	0.54	0.95	1.46	2.03	2.62	3.26	3.92	4.64	0.54	0.94	1.46	2.02	2.62	3.25	3.91	4.61

that the particle number conservation is violated by these terms. As is widely known, these dangerous terms can be eliminated from the Hamiltonian using the Bogolyubov transformation. From a formal point of view, the transformation is equivalent to the canonical transformation, which keeps the commutation relation. The transformation reads $c_+ = xa_+ - ya_-$, with $x^2 - y^2 = 1$. The resultant wobbling Hamiltonian becomes

$$H_w = \hbar\omega_w \left(c_+ c_- + \frac{I_1}{2I} \right). \tag{4}$$

So far, everything is obtained in an exact manner, and no approximation is introduced.

Now, using the second assumption (the near-1D rotation assumption), we can treat the operators $c_\pm$ as bosonic creation–annihilation operators, that is,

$$[c_-, c_+] = I_1/I \simeq 1. \tag{5}$$

Then, the Hamiltonian turns out to have a simple harmonic form

$$H = \frac{I(I+1)}{2\mathfrak{I}_1} + \hbar\omega_w \left(n + \frac{1}{2} \right), \tag{6}$$

where n denotes the wobbling phonon number and the harmonic (wobbling) energy is expressed as

$$\hbar\omega_w = \frac{I}{\mathfrak{I}_1} \sqrt{\frac{(\mathfrak{I}_1 - \mathfrak{I}_2)(\mathfrak{I}_1 - \mathfrak{I}_3)}{\mathfrak{I}_2 \mathfrak{I}_3}}. \tag{7}$$

Let us hereafter call this wobbling phonon a "wobblon" for the sake of simplicity.

Subsequent to the first experimental report on evidence of one-wobblon excitation [2], evidence of the two-wobblon band was reported for the triaxial strongly deformed (TSD) state in ^{163}Lu [3]. The deformation parameters (β, γ) for this state are estimated to be $(0.4, -20^\circ)$ with the cranked Nilsson calculation [the Ultimate cranker (UC)] [4], where $\gamma = \arctan \dfrac{\sqrt{2}Q_2}{Q_0}$ (Q_i denote quadrupole moments). The moments of inertia for the 0-, 1-, and 2-wobblon bands were found to be quite similar, and linking transitions were observed to be present between these bands. In addition, the measured transition strengths were quite consistent with the calculations of the particle-plus-rotor model, which is based on the wobbling triaxial rotor interacting with the aligned $i_{13/2}$ proton [5]. From these facts, it was suggested that these rotational bands are a manifestation of the wobbling motion.

However, several unexpected features were also noticed. They are still unaccounted for by any theories (though some suggestions were presented by the author [6, 7]). The first mystery is that no wobbling bands are observed so far in the neighboring Hf isotopes, although the UC calculation suggests the existence of the TSD states. The second feature is the I dependence of the wobbling excitation energy. According to Eq. (7), the excitation energy $\hbar\omega_w$ should increase linearly in the total angular momentum I if the deformation is fixed, but the experimental data show a linear decline of the excitation energy. This result is inconsistent with the Bohr–Mottelson model. The third and most serious problem is a strong anharmonicity seen in the wobblon spectrum. The ratio of the excitation energies between $n = 2 \to n = 1$ and $n = 1 \to n = 0$ is almost 1 : 2.

The goal of this work is to understand why and how the anharmonicity occurs in the observed wobbling spectra. To this aim, firstly, we solve the problem of the triaxial Hamiltonian exactly through diagonalization in order to check how much the anharmonic ingredients are contained in the original Hamiltonian. Then, using the semiclassical approximation, a new anharmonic model is constructed to investigate the anharmonicity in a more explicit manner. Finally, with microscopic and self-consistent many-body calculations, we show how much one can explain the experimental data for even–even Hf systems, which are expected to show the wobbling motion.

2. THE EXACT SOLUTION

The harmonic spectrum derived within the Bohr–Mottelson model, that is, Eq. (6), is primarily a consequence of the assumption of the near-1D rotation. This means that, without any assumptions,

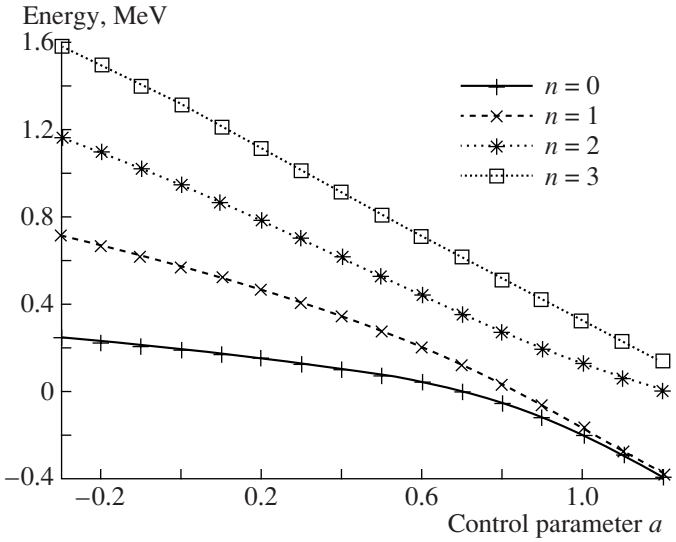

Fig. 1. Wobbling spectrum as a function of the control parameter a. The rigid-body moment of inertia is employed with $(\beta, \gamma) = (0.4, -20^\circ)$.

the original triaxial-rotor Hamiltonian may show the anharmonicity like the one seen in the experiment. It is thus interesting to examine how much the original triaxial-rotor Hamiltonian contains the anharmonic ingredients. For this purpose, all the assumptions, that is, $K \simeq I \gg 1$, are removed, and the equation is solved by means of exact diagonalization.

With the wave function of the form of Eq. (2), the Hamiltonian is expressed in a matrix form. The diagonal matrix elements are expressed as

$$\langle IK|H|IK\rangle = \frac{I(I+1)}{2\mathfrak{J}_1} + \frac{1}{4}\left(\frac{1}{\mathfrak{J}_2}+\frac{1}{\mathfrak{J}_3}-\frac{2}{\mathfrak{J}_1}\right)\left\{I(I+1)-K^2\right\}, \tag{8}$$

and the nonvanishing off-diagonal matrix elements are only given by

$$\langle IK \pm 2|H|IK\rangle = \frac{1}{8}\left(\frac{1}{\mathfrak{J}_3}-\frac{1}{\mathfrak{J}_2}\right) \times \sqrt{(I \pm K)(I \pm K - 1)(I \mp K + 1)(I \mp K + 2)}. \tag{9}$$

There is a reason why nonvanishing matrix elements follow the above type; that is, only the transitions between K and $K \pm 2$ are allowed for the given Hamiltonian. The reason is the D_2 symmetry or a combination of the signature symmetries [1]. As a result, the summation with respect to K in Eq. (2) needs to be taken only for even K when the signature is positive and only for odd K when the signature is negative. Taking care of this property, the diagonalization is performed for two high-spin cases ($I = 40\hbar$ and $41\hbar$) with the TSD shape ($\beta = 0.4, \gamma = -20^\circ$). The results are shown in the table. It indicates that the obtained

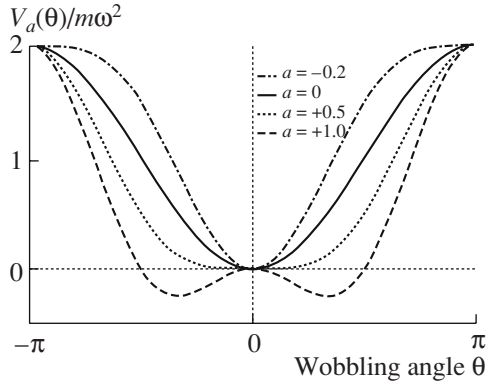

Fig. 2. The wobbling potential energy with strong anharmonicity, $V_a(\theta)$, for different control parameter a.

spectrum shows a fairly good harmonicity at low-lying excited states, against the experimental data showing the strong anharmonicity. Therefore, it is reasonable to conclude that the observed bands are something beyond the scope of the original Hamiltonian.

3. ANHARMONIC WOBBLING MODEL

Coming back to the exact form of the wobbling Hamiltonian (4), we introduce a dynamical variable θ in such a way that

$$\cos\theta = \frac{\langle I_1\rangle}{I}. \tag{10}$$

Semiclassically, this angle means a wobbling angle measured from the quantization axis. After a little

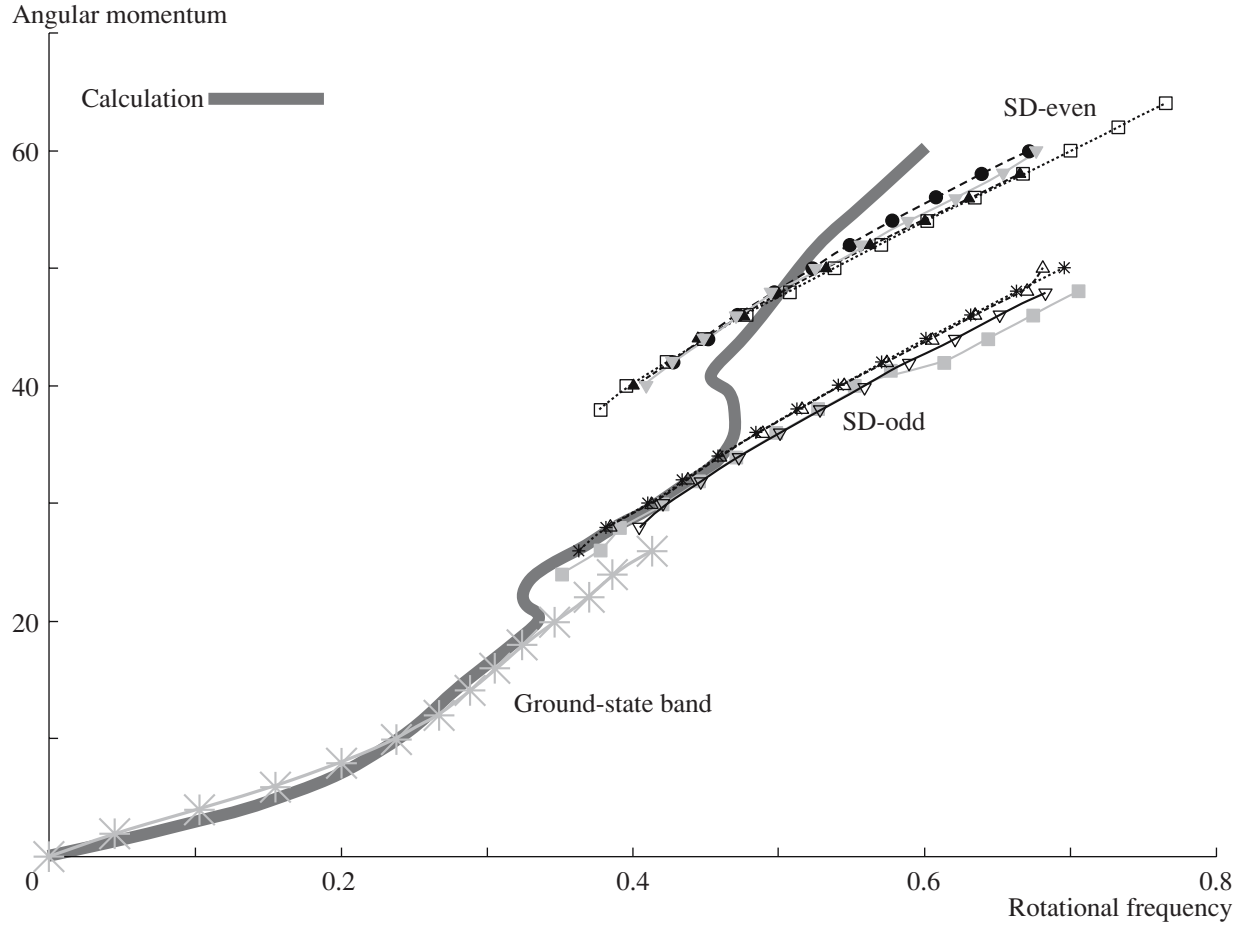

Fig. 3. So-called backbending plot. The ordinate (abscissa) corresponds to the total angular momentum (rotational frequency). The first backbend is due to the neutron quasiparticle excitation, while the second one is due to the simultaneous alignments in protons and neutrons.

algebra using the assumptions ($K \simeq I \gg 1$), the energy expectation value can be written as a function of θ:

$$E^I(\theta) \simeq C(1 - \cos\theta), \qquad (11)$$

where the proportional coefficient C is given as $C = \frac{I^2}{8}\left(\frac{1}{\mathfrak{J}_2} + \frac{1}{\mathfrak{J}_3} - \frac{2}{\mathfrak{J}_1}\right)$. When θ is small ($\theta \ll 1$), the above energy function is of a harmonic form, which is consistent with the original wobbling model ($K \simeq I$). Therefore, the dynamical motion associated with the above energy function can be considered as an extended wobbling motion, or anharmonic wobbling. The corresponding dynamics corresponds to a quantum fluctuation around the semiclassical minimum ($\theta = 0^\circ$) of the energy surface $E^I(\theta)$. A proper treatment of the relevant dynamics can be handled through the requantization with respect to the dynamical variable θ, regarding the energy function as an effective potential in the problem. The quantized oscillatory motion is a generalized wobblon. The mass of the wobblon is a parameter, but it can be determined from the experimental data (through the formula $m = C/\Delta E^2$, where the excitation energy from the 0- to 1-wobblon state is denoted as ΔE). For the details of the formulation, please see [6].

As shown in Fig. 1, the strong anharmonicity was not obtained again for this effective potential (see the spectrum for $a = 0.0$). To bring the strong anharmonicity, the potential needs a modification. The author proposed the following form, as one of the possibilities,

$$V_a(\theta) \propto 1 - \cos\theta - a\left(1 - \cos^2\theta\right), \qquad (12)$$

where a is a new parameter to enhance the anharmonicity. When $a = 1.1$, which gives rise to a tilted-rotation minimum (see Fig. 2), the extended wobbling model can reproduce the experimental data. The origin of this extra term should be microscopic degrees of freedom, which is not taken into account

in the original Hamiltonian. However, this speculation needs an examination by means of microscopic calculations [8].

4. MICROSCOPIC CALCULATION WITH THE CRANKED HFB

Experimentalists pursue a different path to tackle the problem. Recalling that the original model is for a triaxial rotor, the Lu isotopes, which are currently studied intensively, may not be appropriate for studying the typical features of the wobbling motion. The system is basically a system consisting of a triaxial rotor and a valence (proton) particle, which may disturb the wobbling motion of the rotor through the interactions, such as the Coriolis and recoiling ones.

Hartley et al. [9] executed an experimental search for a possible TSD band in even–even Hf isotopes. In ^{174}Hf, many regular high-spin bands were discovered by this experimental group, but they failed to identify the linking transitions to known bands. Consequently, the spin–parity assignment was not made for these new bands. The transitional quadrupole moments (Q_t) were measured for these bands and compared with the calculations of the UC, which shows multiple minima. It was found that none of these minima can reproduce the large Q_t values of the experimental data, so they concluded that the existence of the TSD bands was not confirmed in their experiment.

Walker et al. [10] found a new high-spin band ($40\hbar \lesssim I \lesssim 60\hbar$) in ^{175}Hf, which has similar characteristics to the new bands found in ^{174}Hf. The linking transition of this new band was identified. They performed the total-routhian-surface (TRS) analysis and suggested that the new band corresponds to a near-axial shape $(\beta, \gamma) = (0.35, -14^\circ)$, unlike the prediction by UC. They argued that the discrepancy may come from the difference in the nuclear potential: the TRS uses the Woods–Saxon potential, while the UC uses the Nilsson potential.

The self-consistent cranking calculations are performed for ^{174}Hf by the author with the pairing + quadrupole–quadrupole two-body interaction, and a similar result to the TRS calculation is obtained. The result is illustrated through the backbending plot in Fig. 3. As a reference, the ground-state band is compared between the experiment and the calculation, giving very good agreement. Although the new bands can "float" owing to the failure in the spin–parity assignment, good agreement is obtained by a proper shifting toward the calculated curve representing the neutron quasiparticle excitations. The state with the simultaneous alignments in protons and neutrons does not match the experimental data well.

5. CONCLUSIONS

The strong anharmonicity observed in the wobbling spectrum was analyzed by the exact diagonalization, the anharmonic wobbling model, and the microscopic many-body calculations. The original Hamiltonian is found not to contain such a strong anharmonic ingredient. The anharmonic wobbling model developed newly by the author implies that an emergence of tilted rotation may be important. But the microscopic calculations and new experimental data suggest a different explanation for the problem, which is based on the near-axial shape. The question of how the strong anharmonicity appears in the wobbling spectra is still a deep mystery requiring further investigations.

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