THE COLLECTIVE MODEL OF NUCLEI[1]

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I. SURVEY

1. Introduction and Outline

A short time only after the introduction of the *j-j*-coupling shell model in 1949 (1, 2), collective (cooperative) effects were brought back into the picture of the nucleus, in order to cope with the model's main difficulty: the explanation of the giant quadrupole moments (3). Since then, the systematic study of collective effects went almost parallel with the further development of the shell model. As a result, a synthesis of the two aspects of nuclear dynamics, single particle, and collective behaviour, has become one of today's most interesting problems of nuclear structure.

Collective aspects had for a long time dominated the thinking in nuclear physics, ever since in 1937 Bohr and Kalckar presented the liquid drop picture, and showed the implication of this and associated concepts for nuclear reactions, in particular. It is not in this form that the "collective model" has re-entered the scene in 1951. In fact, the original picture of collective motion attributable to very strongly interacting nucleons had to be considerably revised, since it has become clear that in moderate energy nuclear reactions the redistribution of excitation energy is a much slower process than one had previously assumed (4, 5, 6). This, together with the wealth of experimental evidence so neatly summed up in the shell model, is a clear indication that the idea of individual nuclear "orbits" (single particle quantum states) is a sensible concept up to excitation energies of 10 to 15 Mev. (For surveys of the shell model, see Refs. 7 to 11.)

Thus, if the "collective aspect" is revived today, it is not in the sense of questioning again all the results which point toward independent particle motion. The collective phenomena considered here are of a much more subtle type and, most essentially, they are intimately connected with the independent particle aspect of nuclear structure (12).

This independent particle aspect finds traditionally its expression in the Hartree-Fock equation, in which the single particle states of all nucleons co-operate to define a "self-consistent" one particle potential. In this non-linearity of the Hartree-Fock problem, we have already a typical collective effect: a truly self-consistent solution of these equations leads, indeed, to a potential, whose size, depth, shape, and deformation[3] depends on the nature

[1] The survey of literature pertaining to this review was completed on March 1st, (1957).

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[3] We use the word "shape" to describe the radial dependence, the word "deformation" to describe the angular dependence of the well.

of the occupied states. Unfortunately, this collective aspect of the method cannot easily be exploited to display the collective dynamics. Most of the purely theoretical approaches to the problem of collective motion deal with this difficulty and with ways to overcome it.

The collective and unified models have been presented in various publications: condensed reviews by Bohr (13), by Bohr & Mottelson (14), and by Alder, *et al.* (15), a detailed account in the monograph by Bohr & Mottelson (16), and, with different emphasis, in the paper by Hill & Wheeler (12). This makes it possible here to emphasize more fully some developments which have not yet been adequately reviewed, and deal with the problem of a more detailed understanding of the fundamental assumptions of the model, of the role of single particle motion in the model, and of the outstanding problems still to be solved.

In the next two sections we shall give a short survey, presenting the unified model and the nature of the evidence supporting it.

Part II of this review is purely theoretical, and deals mainly with attempts to formulate collective motion in terms of particle concepts. The purpose of such theoretical attempts is twofold: first, they give insight into the nature and justification of the simplifying assumptions made in the construction of actual models; secondly, they are the starting point for new techniques of approach to the problem of finding self-consistent solutions of the Schroedinger equation. The achievement so far accomplished by these purely theoretical considerations is somewhat limited. But they have, for instance, helped to clear up the nature of the coupling of collective rotation and intrinsic motion, they have contributed to our understanding of the Inglis model, etc. They are presented here also as a temptation to stimulate further research along the lines sketched in that part.

In part III various models constructed to display collective aspects in nuclear structure are presented; an attempt is made to describe and to assess the nature of the assumptions involved in the construction of the model. Finally, definite statements made by such models are compared with the experimental evidence.

2. The Nature of the Unified Model

Only very recently have there been any attempts to calculate a really self-consistent nuclear potential (17). Such attempts are discouraged by the unreliability of the approximation. The original Hartree-Fock approximation makes sense only for rather unrealistic types of two-body potentials [Bethe (18)]; more refined methods, expressing the energy in terms of the two-body scattering matrix [Brueckner & Levinson (19); Eden & Francis (20); Bethe & Goldstone (21)] have been used with a certain success by Brueckner, Eden & Francis (22), but are in a too preliminary stage to permit wholesale application. In addition, such methods were primarily designed to handle the problem of nuclear saturation (that is, determination of energy), whereas here the attention is focused on other properties: electric and magnetic moments, transition probabilities, moments of inertia. As Eden & Francis

have shown in an outline of a general "theory of models" (20), the correct evaluations of such quantities as mentioned above is a much more subtle problem than minimizing the energy. For the special case of magnetic moments, this difficulty has been illustrated by Bell, Eden & Skyrme (23). In this report we cannot but adopt a very elementary point of view and essentially ignore that problem; but this should not mean that we minimize its relevance for the questions to be discussed here.

Even apart from the questions of reliability, there is the formidable technical problem of carrying out a numerical Hartree-Fock calculation. As a consequence, all the thinking has been dominated by ready-made isotropic single particle potentials of given shape; their isotropy had at least the advantage of allowing the use of angular momentum eigenstates. The large quadrupole moments in the region $150 \lesssim A \lesssim 190$ and $A \gtrsim 225$ have finally forced us to abandon this picture, and to admit nonspherical (deformed) potentials (3).

The introduction of deformed wells as a better approximation to self-consistency (from an energetic point of view) leads to states that cannot be *eigenstates* of the total angular momentum. But a similar situation has always been tacitly tolerated with respect to the total linear momentum. The total linear and angular momenta are collective dynamical variables, which cannot be properly taken care of in a self-consistent independent particle approximation; and it is certainly a step forward to recognize that nothing is gained by trying to sacrifice the flexibility of the Hartree-Fock method in an attempt to deal exclusively with angular momentum eigenstates. Overstating the case, we may say that both total linear and angular momentum are collective variables which have little to do with the internal structure, and are better discarded in a first approximation to the latter.

For the case of the angular momentum, Bohr, in a beautiful analysis (24), has shown how to interpret these states that do not conserve it: the angular momentum of the Hartree-Fock solution is an "intrinsic" angular momentum $\mathbf{J}$; this latter differs from the (conserved) total angular momentum $\mathbf{I}$ by a quantity $\mathbf{R} = (\mathbf{I} - \mathbf{J})$ representing the angular momentum of the precession of the nonspherical intrinsic structure about the axis $\mathbf{I}$. As this precession leaves the intrinsic structure relatively unaffected (see, however, part III), there exist, under suitable conditions, states of identical intrinsic structure, differing by the value of $\mathbf{R}$ only. These states form the rotational spectrum of excitations.

Although this point of view strongly appeals to our intuition, its mathematical formulation presents difficult problems. The simplest approach is based on the assumption that a few nucleons only are responsible for the actual deformation of the well; these nucleons are then treated as individuals, and the remaining part of the nucleus is treated as a "soft" core, the dynamics of which is expressed in terms of a few suitable collective deformation parameters. The dynamic coupling between core and particles is introduced by the assumption that the density distribution $\rho(r, \theta, \varphi)$ of the core defines the well for the extra nucleons. Although this is a conceptually clean

scheme for constructing a unified model, it is subject to serious objections: (*a*) In order to obtain a workable "model" in the above mentioned way, the properties of the core, as defined here, should be either uniform or smoothly varying functions of the number of particles (N_c, Z_c) it contains. There is, however, every indication that if the number of extra nucleons is kept small (in the extreme limit: zero or two for even, one for odd nuclei), then the properties of the core show strong shell effects. It results that no purely collective model will reproduce the properties of such a core. (*b*) On the other hand, theoretical considerations, especially the results of the Inglis model (see part III), show that a relatively uniform behaviour may be expected for closed shell cores. But a division of the nucleus into a collectively described closed shell core and extra nucleons is hardly desirable either: First, the deformation of the core is then no longer a measure of the actual deformation of the nucleus. Second, since the number of extra nucleons may now be large, their direct interaction contributes significantly to the self-consistent potential, and they may be mainly responsible for the actual values of moments of inertia and deformation; just the most interesting collective aspects would then result from particles described individually. (*c*) To achieve self-consistency, one clearly needs a description which is symmetric in the particles, since they all contribute in a symmetric way to the definition of the single particle well. The only symmetric description of this kind which leads to a "model" is based on the method of Hill & Wheeler which is discussed in part II.2. This method gives a simple and straightforward way to superimpose the collective aspects upon the independent particle description of the nucleus, with the result that all quantities (inertial parameters, deformation potentials, collective flow pattern, electromagnetic moments) are determined by the underlying particle dynamics. A more ambitious and fundamental approach, which also describes all particles in a symmetric manner, is the method of redundant variables, presented in part II.4. This method is, however, still in its infancy.

3. Achievements of the Unified Model

The outstanding success of the unified point of view is of course the prediction and subsequent actual identification of rotational levels, together with an explanation of the rather well defined regions of their occurrence in the periodic table and the correlation between moments of inertia and intrinsic quadrupole moments. We now give a very short survey of this and other results of interest.

a) *Rotational levels.*—The number of rotational spectra, or of spectra with at least three levels fitting into a rotational band, is very considerable now (15, 25, 26). What characterizes these spectra so strikingly, even from a purely empirical point of view, is the accuracy with which they fit a one- (or two-, for $K=\frac{1}{2}$ bands) parameter formula. In adding a second (third) parameter in the form of a rotation-vibration coupling term, the levels can be reproduced with a really outstanding accuracy (see the examples given in Fig. 1). The one- (two-) parameter formulae are

$$E_I = \frac{\hbar^2}{2\Im}\{I(I+1) + a\delta_{K,1/2}(-1)^{I+1/2}(I+1/2)\} \qquad 1.$$

$\Im$ being the effective moment of inertia and a the decoupling parameter (16, 27). K is the quantum number of the projection of the total angular momentum on the body fixed symmetry axis, and characterizes a rotational band:

$$I = 0, 2, 4, \cdots; \qquad K = 0\text{: even nuclei}$$
$$I = I_0, I_0+1, I_0+2, \cdots; \qquad K = I_0\text{: odd nuclei}$$

Occasionally more than one band is observed in a nucleus, (W^{183}). All levels of a band have the parity of the ground state member.

The rotation vibration interaction adds a term

$$-\text{ const } I^2(I+1)^2 \qquad 2.$$

to the energy levels. The experimental value of the (positive) constant agrees in order of magnitude with what one expects on the basis of vibrational excitation energies. In Figure 1, we give an example, which illustrates three different types of rotational spectra.

The purity of the rotational level spacings leads us to interpret the dynamics in terms of a very simple coupling scheme: All particle motions are first coupled to the intrinsic symmetry axis, which in turn precesses, with an interpretation of adiabatic slowness, about the axis of the total angular momentum **I**.

b) *Quadrupole moments and E2-transitions.*—The giant quadrupole moments were, as already mentioned, the primary argument for the introduction of the collective viewpoint. Although the striking parallelism between magnitude of the intrinsic electric quadrupole moments Q_0 and the rotational moments of inertia $\Im$ is qualitatively understood, we are still far from having a quantitative theory of the latter; this has in fact proven to be one of the most interesting problems to be solved yet, since the value of $\Im$ depends so sensitively on the degree of residual internucleon coupling (See part III.2).

For strongly deformed nuclei, careful distinction must be made between intrinsic and "spectroscopic" quadrupole moments, the former measuring the deformation in the body fixed, the latter in the space fixed (laboratory) system. $E2$-transition probabilities B_{E2} are directly expressible in terms of the intrinsic moments Q_0, and the consistency between values of Q_0 from Q and B_{E2} measurements is reasonably good (25). This again indicates the basic correctness of the simple coupling scheme for large deformations.

For nuclei not displaying rotational levels, the situation may be rather complex. Generally, $E2$-transitions appear to be strongly enhanced. It has been shown however that already a very small "contamination" of the rigorous shell picture with collective effects ("weak coupling") will enhance the $E2$-transition rates, without otherwise affecting the intrinsic dynamics, which is still dominated by single particle level order and residual inter-

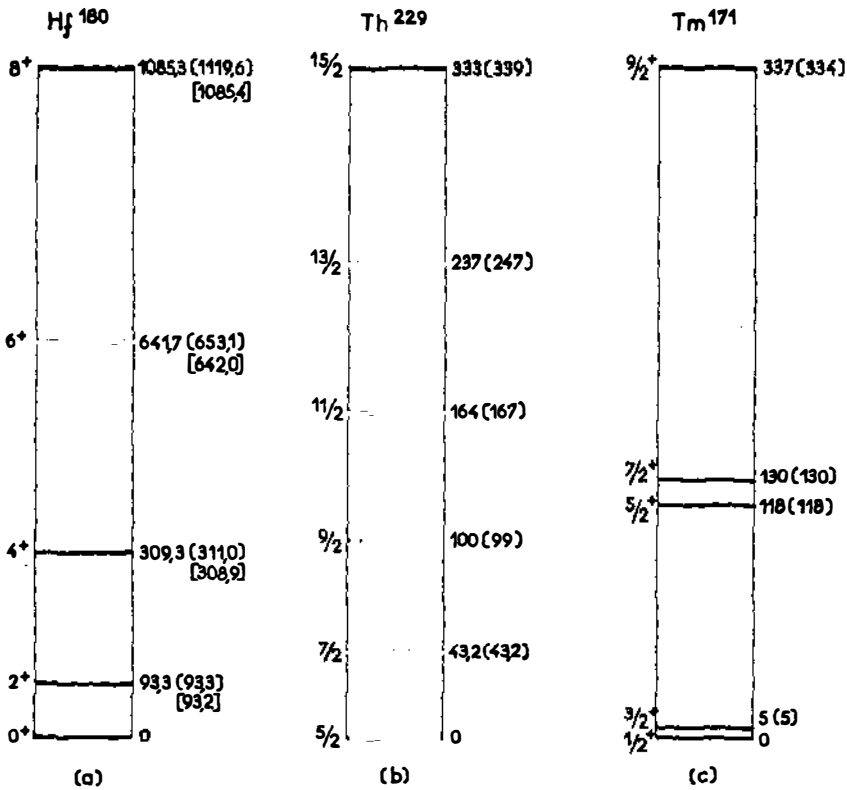

FIG. 1.

(*a*) Spectrum of an even nucleus:

$$E_I = (\hbar^2/2\Im)I(I+1); \qquad I = 0, 2, 4, \cdots$$

Odd values of I are suppressed by the symmetry condition on the collective wave function. This example is discussed in detail in Refs. (13) and (15). To the left are spins and parities; to the right observed energies, calculated energies without, ( ), and with, [ ], rotation-vibration interaction. (*b*) Spectrum of an odd nucleus with $K \neq \frac{1}{2}$:

$$E_I = (\hbar^2/2\Im)\{I(I+1) - I_0(I_0+1)\}; \qquad I = I_0, I_0+1, \cdots$$

This example is due to Goldin *et al.* (31), and found in the α-decay of U^{233}. (*c*) Spectrum of an odd nucleus with $K = \frac{1}{2}$:

$$E_I = (\hbar^2/2\Im)\{I(I+1) + a(-1)^{I+1/2}(I+1/2)\}$$

This example is due to Johansson (32); the experimentally adjusted value of a is $a = -0.86$; the levels are found by β-decay of Er^{171}.

particle couplings. This is so, at least, for odd nuclei. In many even nuclei, the frequently observed situations with $E_2/$[redacted] together with 2^+ for E_1, $2^+(4^+)$ for E_2 and the inhibition of the cross-over transition $E_2(2^+) \rightarrow E_0(0^+)$

seems to warrant a purely collective description in term of a harmonic oscillation. The situation is however not quite clarified yet, as no simple model covers all the facts in a quantitative way.

c) *Magnetic moments.*—The simple coupling scheme for strongly deformed nuclei again reduces the expression for μ to a two parameter formula (3 parameters for $K=\frac{1}{2}$):

$$\mu = \frac{1}{I+1}(g_\Omega I^2 + g_c I) \qquad 3.$$

and an analysis consists in an interpretation of g_c and g_Ω, the g-factors for core precession and intrinsic motion, respectively [with $(g_\Omega - g_c)^2$ supplied by $M1$-transition probabilities, see (16)]. Naively one would then expect g_c to be $=Z/A\sim 0.4$. Actually the experimental values deviate considerably from 0.4 (25), (15), even in cases where other evidence supports the coupling scheme.

One should, however, not necessarily consider this as a weakness of the theory. As may be seen in part II, all theoretical ("fundamental") approaches to the problem of collective rotation have stopped short of a complete decoupling of collective and intrinsic motion. In simple terms, this means that they have not found yet the "right" body fixed coordinate system: If we require the intrinsic motion, described in the body fixed system, to be decoupled from the rotation of that system, then that system is not defined by the kinematics of the nucleus alone, but is determined by the intrinsic dynamics too. The "right" intrinsic coordinate axes then depend not only on the distribution of nucleons, but on their momenta as well. In such a situation it cannot even be asserted that g_c should be equal to Z/A so that, until further notice, there is no question of a discrepancy. Investigations of these problems are only in their beginning (28).

d) *Fission.*—Space does not permit us to enter into this subject, but we wish to recall the work of Hill & Wheeler (12), Bohr (29), and Wheeler (30), in which the fission process is analyzed from the point of view of the unified model.

II. GENERAL THEORY

1. Introductory Comments

We have pointed out the collective aspects in the self-consistent independent particle approximation; but these features appear there merely in the form of collective (variational) parameters: size, shape, deformation (anisotropy), of the "self-consistent" potential well. It is particularly evident in the case of the well deformation that one would much prefer to consider these quantities as dynamical variables rather than as mere parameters; this would enable us to describe, through their time variation, the precession in space of the nonspherical potential well, and thus allow us to deal with eigenstates of the total angular momentum. This is clearly also a prerequisite for the possibility of describing collective (rotational) excitations,

that is, a class of states which differ, roughly speaking, mainly by the precession velocity of the well, but whose intrinsic structures are as similar as they can be under the circumstances.

There are essentially three methods with which this problem of describing the collective dynamics may be attacked. All these methods are more ambitious than a mere "collective model"; they aim at an understanding of the interdependence of collective and individual particle dynamics, and the results of these more fundamental approaches have a deeper interest than a ready-made model. a) In the first method [Hill & Wheeler (12), Griffin & Wheeler (33), and Peierls & Yoccoz (34, 35)], the starting point is an independent particle wave function, $\psi(x_1, \cdots, x_A; \alpha)$, depending on one or several variational parameters α, for instance, the deformation of the single particle well, or the orientation of a nonspherical well in space. One then constructs the wave function

$$\phi(x_1, \cdots x_A) = \int d\alpha \chi(\alpha) \psi(x_1, \cdots x_A; \alpha) \qquad 4.$$

and

$$\langle H \rangle = \int d\tau \phi^* H \phi = \int\int d\alpha d\alpha' \chi^*(\alpha) \chi(\alpha') h(\alpha; \alpha')$$
$$\langle N \rangle = \int d\tau \phi^* \phi = \int\int d\alpha d\alpha' \chi^*(\alpha) \chi(\alpha') n(\alpha; \alpha') \qquad 5.$$

with an undetermined distribution amplitude $\chi(\alpha)$ of α. The variational problem $\delta(\langle H \rangle - \epsilon \langle N \rangle) = 0$ then leads to a "Schroedinger equation" for $\chi(\alpha)$, and an *eigen*value spectrum that obviously belongs to one single type of intrinsic structure. Centre of mass motion, collective rotations and vibrations may be described this way; in the two first cases the proper choice of the amplitudes, χ, makes ϕ *eigenstates* of total linear and total angular momentum, respectively, whereas the associated ψ do not have this property. b) In a second method one tries to isolate appropriate collective aspects by carrying out a (canonical) transformation of variables, thus replacing the original particle position coordinates by new variables, some of which directly represent the collective modes in question. A familiar example of this is the separation of the centre of mass; another example will be the separation of the total angular momentum. Total linear and total angular momentum are clearly the two most obvious collective variables for any system, and one might expect a general rule for their separation. Indeed, we shall show that, irrespective of the system, the kinetic energy of any system of N particles of mass m may be written as (36):

$$T = \frac{1}{2mN} P^2 + \frac{1}{2} \sum_{A,B} Q_{AB}(\xi)(L_A - L_{A'})(L_B - L_{B'}) + \frac{1}{2} \sum_{\sigma,\tau} D_{\sigma\tau}(\xi) \pi_\sigma \pi_\tau, \qquad 6.$$

P being the total linear momentum, L_A a component of the total angular momentum; ξ_σ, π_σ being "intrinsic" variables and their conjugate momenta,

and finally $L_A'(\xi_\sigma, \pi_\sigma)$ an intrinsic angular momentum, and $Q_{AB}(\xi)$ a reciprocal moment of inertia tensor. In contrast to P, L cannot be completely decoupled from the intrinsic motion; only in certain special cases is this approximately possible, making use of the specific properties of the system at hand.

A drawback of this method lies in the fact that it is not possible to identify the $(3N\text{-}6)$ variables, ξ_σ, with particle position coordinates; thus the description of the intrinsic motion in terms of "independent" particle states seems impossible by this method. Ways to overcome, at least partially, this difficulty will be indicated in the appropriate place. c) The third method was developed mainly with the idea to avoid to a certain extent the above mentioned difficulty. In order to be able to use "shell model" wave functions, the particle coordinates should be kept as independent variables. Now the introduction of collective dynamical variables may be viewed as a transformation $x_i \to x_i'$, α_s $(s=1, \cdots, f)$, x_i being the original position coordinates, α_s the collective variables and the x_i' transformed position coordinates which, however, are subject to f conditions of constraint $F_s(x_i') \equiv 0$ $(s=1, \cdots, f)$. The present method consists then of lifting this constraint temporarily, and to consider the F_s as dynamical variables alongside the x_i (method of redundant variables). Redundant variables without compensating subsidiary conditions have been introduced by various authors: (13, 37, 38). The reversible transformation, F_s, $x_i \rightleftarrows \alpha_s$, x_i', can then be completed to a full canonical transformation: G_s, $p_i \rightleftarrows \pi_s$, p_i', G_s being the momenta conjugate to the F_s: $[F_s, G_t] = -i\hbar\delta_{s,t}$. The original Hamiltonian, in the original variables, is independent of the F_s and G_s, and therefore commutes with both; this property is preserved also if H is written in terms of the new variables: $H(p_i; X_i) = \tilde{H}(p_i', \pi_s; x_i', \alpha_s)$. It follows that any function $\tilde{\psi}(x_i', \alpha_s)$ is an *eigen*function of $\tilde{H}$ if, in terms of the old variables, it takes the form

$$U(F_s)\psi_n(x_i) \qquad 7.$$

ψ_n being an eigenfunction of H, and $U(F)$ an arbitrary function. Thus, all one seems to get so far is a spurious degeneracy of *eigen*states introduced by the arbitrariness of the amplitudes U. However, for example, assume now that the structure of $\tilde{H}$ is of type

$$\tilde{H} = H_1(\pi_s; \alpha_s) + H_2(p_i', x_i') + H_3(\alpha_s, x_i')$$

with H_3 small, a situation which clearly suggests in lowest order of approximation the use of a wave function

$$\tilde{\psi}^{(0)}(\alpha_s, x_i') = V(\alpha_s)\varphi(x_i') \qquad 7a.$$

This function does not reduce to type 7, if rewritten in terms of the old variables, and is therefore strictly speaking inadmissible; on the other hand, nothing prevents us from using 7a as a zero order approximation, to be improved by perturbation or variational methods. In this way, a completely new type of approximation technique may be formulated.

2. The Method of Continuous Variational Parameters

a) Illustration of the method. Centre of mass motion.—It is well known that in the Hartree-Fock approximation, the wave functions are not *eigen*functions of the total momentum. The expectation value of the Hamiltonian H contains therefore a spurious centre of mass energy $\langle(\sum p_i)^2\rangle/2Am$; in addition, there appear "ghost" levels, corresponding to an excited state of the centre of mass motion in the space fixed potential well (39).

Now, if $\psi(x_i' \cdots, X_A)$ is such a wave function, it is easily seen that

$$\phi_P(x_1, \cdots x_A) = \int d\xi \exp(-iP\cdot\xi)\psi(x_1+\xi, \cdots, x_A+\xi) \qquad 8.$$

is an *eigen*function of the total linear momentum P:

$$P_{op}\phi_P = -i\hbar\sum_i \partial\phi_P/\partial x_i = P\phi_P$$

ϕ_P is thus obtained from ψ by projecting out the component with total linear momentum P. We now discuss the improvement which is obtained by using ϕ_P instead of ψ in calculating expectation values: (*a*) Since all functions $\psi(x+\xi)$ give the same expectation value of the energy $\langle H\rangle = \int d\tau\psi^* H\psi$, this expectation value is infinitely degenerate with respect to the continuous displacement parameter ξ. As is well known in such cases [the classic example being the exchange energy obtained by (anti)symmetrization of a degenerate two particle wave function], a proper linear combination $\int d\xi\chi(\xi)\psi(x+\xi)$ of such degenerate states will lower the energy. In our case, we know that with $\chi(\xi) = \exp(-iP\cdot\xi)$ we construct *eigen*states of P, and therefore with $P=0$ we get the lowest possible energy by eliminating the spurious centre of mass kinetic energy. (*b*) In connection with this last point, it is also seen that such a procedure eliminates all the ghost states: $\phi_{\text{ghost}} \equiv 0$ (34, 39, 41). (*c*) Choosing $\chi(\xi) = \exp(-iP\cdot\xi)$. ($P \neq 0$) we may compare the P dependence of

$$E_P \simeq \frac{\langle H\rangle_P}{\langle N\rangle_P} = \frac{\int d\tau\phi_P^* H\phi_P}{\int d\tau\phi_P^*\phi_P} \qquad 9.$$

with the known exact dependence of the energy on P:

$$E_P = E_{P=0} + \frac{P^2}{2Am}$$

In this way we get an estimate of the adequacy of the independent particle wave function ψ to determine a collective inertial parameter, here the total mass.

We carry out this comparison in some detail, since it is characteristic for the general method. We have, in Eq. 9:

$$\langle N_P \rangle = \text{const.} \int d\xi \exp(-iP\cdot\xi) \int d\tau \psi^*(x)\psi(x+\xi)$$
$$\langle H \rangle_P = \text{const.} \int d\xi \exp(-iP\cdot\xi) \int d\tau \psi^*(x) H \psi(x+\xi) \qquad 10.$$

Now the τ-integrals are 1 and $\overline{H}$, respectively, for $\xi=0$, and drop off to zero for large values of ξ. For small ξ, one gets by expansion

$$1 - \frac{1}{2}\xi^2 \int d\tau \psi^*(x)(\sum p_i)^2 \psi(x) = 1 - \frac{1}{2}\xi^2 \overline{P^2}$$

and

$$\overline{H} - \frac{1}{2}\xi^2 \int d\tau \psi^*(x) H (\sum p_i)^2 \psi(x) = 1 - \frac{1}{2}\xi^2 \overline{HP^2}$$

This suggests that the ξ-dependence of these integrals may be approximated by the Gaussians[4]

$$\exp(-\tfrac{1}{2}\xi^2 \overline{P^2})$$

and

$$\exp(-\tfrac{1}{2}\xi^2 \overline{P^2})(\overline{H} - \tfrac{1}{2}\xi^2(\overline{HP^2} - \overline{H}\,\overline{P^2}) + \cdots)$$

respectively. Clearly then, if in Eq. 10, P is assumed to be sufficiently small, $\exp(-iP\cdot\xi)$ is a slowly varying function and may be expanded; it follows by Eq. 9, that

$$E_P = \left(\overline{H} - \frac{\overline{HP^2} - \overline{H}\,\overline{P^2}}{2\overline{P^2}}\right) + \frac{1}{2} P^2 \left(\frac{\overline{HP^2} - \overline{H}\,\overline{P^2}}{(\overline{P^2})^2}\right) + \cdots \qquad 11.$$

defining a total mass

$$\frac{1}{M_{\text{tot}}} = (\overline{HP^2} - \overline{H}\,\overline{P^2})/(\overline{P^2})^2 \qquad 12.$$

The interpretation of the second term in Eq. 11, is simple and illuminating. Taking the coefficient of $\frac{1}{2}P^2$ in the last term as a "definition" of the total mass (Eq. 12), we have

$$E_{P=0} = \overline{H} - \frac{\overline{P^2}}{2M_{\text{tot}}}$$

which shows that the method indeed cuts out the spurious centre of mass fluctuation energy.

b) *Collective rotations and moments of inertia.*—This method acquires a new dimension in connection with the application to rotational levels. A self consistent independent particle approximation will generally yield a nonspherical single particle well with arbitrary, but fixed orientation in space, and hence states which are not *eigenstates* of the total angular

[4] For nonantisymmetrized wave functions the Gaussian dependence follows in the limit of large N.

momentum. Here again, we find a degeneracy of the expectation values of H with respect to the orientation in space; again, a proper linear combination of degenerate states will yield a better approximation. The particular linear combinations one must choose are, of course, just those which make the new trial functions ϕ eigenfunctions of the total angular momentum. Assuming a self consistent well with a symmetry axis Z', we start with the functions $\psi_K(x')$, K being the value of $I_{Z'}$, and construct eigenfunctions ϕ_{IM} of I^2 and I_Z;

$$\phi_{IM}(x) = \int d\Omega \mathfrak{D}^{I*}{}_{MK}(\Omega)\psi_K(x'(\Omega)) \qquad 13.^{5}$$

Ω gives the orientation of the body fixed coordinate system (X') in space, in terms of Euler angles θ, ψ, φ; $d\Omega = \sin\theta d\theta d\psi d\varphi$ and the $\mathfrak{D}^{I*}{}_{MK}$ are the eigenfunctions of I^2, I_Z, $I_{Z'}$, to eigenvalues $I(I+1)$, M and K, respectively. If follows that

$$\langle H\rangle_I = \sum_M \int d\tau \phi_{IM}{}^* H \phi_{IM} = \text{const.} \int d\Omega \mathfrak{D}^{I*}{}_{KK}(\Omega) \int d\tau \psi_K{}^*(x) H \psi_K(x'(\Omega))$$

$$\langle N\rangle_I = \text{const.} \int d\Omega \mathfrak{D}^{I*}{}_{KK}(\Omega) \int d\tau \psi_K{}^*(x)\psi_K(x'(\Omega))$$

The same arguments as in the centre of mass problem are now applied. If the self-consistent well is sufficiently anisotropic, the overlap between $\psi(x)$ and $\psi(x'(\Omega))$ will decrease rapidly with increasing angle θ between the Z- and the Z'-axes. Let us follow up the case $K=0$ (ground states of even-even nuclei). We have then, tentatively:

$$\int d\tau \psi_0{}^*(x)\psi_0(x'(\Omega)) \cong \exp\left[-\tfrac{1}{2}\theta^2(\textstyle\sum \mathbf{j})^2\right]$$

In addition, we make use of the small angle expansion

$$\mathfrak{D}^{I}{}_{00}(\Omega) = P_I(\cos\theta) = 1 - \tfrac{1}{4}I(I+1)\theta^2 + \cdots$$

Then, in straight analogy with the centre of mass problem, we get:

$$E_I = \frac{\langle H\rangle_I}{\langle N\rangle_I} = E_{I=0} + \frac{I(I+1)}{2\Im_{\text{eff}}} \qquad 14.$$

$$\frac{1}{\Im_{\text{eff}}} \approx \frac{\overline{HJ^2} - \overline{H}\,\overline{J^2}}{(\overline{J^2})^2} \qquad 15.$$

$\mathbf{j}$ being $(\sum \mathbf{j})$.

This method makes it particularly apparent how the appearance of a rotational spectrum is tied to a sufficiently large deviation from spherical symmetry. Only then may we expect that the small angle expansion of P_I (cos θ)

[5] We use here a definition of the eigenfunctions $\mathfrak{D}^{j}{}_{mm'}$ of the symmetric top as given by Edmonds (41): $\mathfrak{D}^{j}{}_{mm'}(\psi,\theta,\varphi) = e^{-im\psi}\langle jm|e^{-i\theta J_x}|jm'\rangle e^{-im'\varphi}$. Functions of type (13) have previously been used by Redlich & Wigner (42) in the calculation of β-decay matrix elements for deformed nuclei.

is reasonable, and that the series may be terminated at the $I\ (I+1)$ term. Evaluation of the higher terms in principle allows to check the accuracy of the approximation.

Some preliminary numerical results have been obtained by Yoccoz, and agree reasonably well with experimental values. A result similar to Eq. 15 has been derived by Skyrme (40) based on a variational method closely related to the one presented here.

c) Electric quadrupole moments.—The improved wave functions ϕ_{IM}, (Eq. 13), may be used to calculate other ground state properties of the nucleus. As an example we give here the expression for the spectroscopic electric quadrupole moment, represented by the tensor operator $Q_{2m}(x)$. Q is the expectation value of Q_{20} for a state I, $M=I$. A straightforward calculation, using Eq. 13 and the properties of the $\mathfrak{D}^I_{KM}$, gives

$$Q = \int d\tau \phi_{IM}{}^* Q_{20} \phi_{IM} \Big/ \int d\tau \phi_{IM}{}^* \phi_{IM}\big|_{M=I}$$

$$= \sum_m C_m(I,K) \frac{\int d\Omega \mathfrak{D}^I_{K+m,K}(\Omega) \int d\tau \psi_K{}^*(x'(\Omega)) Q_{2m}(x) \psi_K(x)}{\int d\Omega \mathfrak{D}^I_{K,K}(\Omega) \int d\tau \psi_K{}^*(x'(\Omega)) \psi_K(x)} \qquad 16.$$

the $C_m(I, K)$ being a product of Wigner-coefficients. For $m=0$, $C_0\ (I, K)$ is the well-known projection factor of the strong-coupling theory (see part III.3d).

$$C_0(I,K) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)}$$

For strongly deformed states, only the term $m=0$ in (Eq. 16) contributes substantially, since $\mathfrak{D}^I_{K+M,K} \propto (\sin \theta/2)^m$. The result, Eq. 16, has therefore some resemblance to the result of the Bohr-Mottelson model in the strong coupling limit:

$$Q = C_0(IK) Q_0(K); \qquad Q_0(K) = \int d\tau \psi_K{}^*(x) Q_{20}(x) \psi_K(x) + \text{corr. terms.}$$

It is more general in the sense that the validity of Eq. 16 is not restricted to large deformations. In the limit of zero deformation, Eq. 16 properly reduces to $Q=Q_0(I)$.

No calculations using Eq. 16 seem to exist yet, and the same holds for magnetic moments.

Transition probabilities may also be calculated, provided the transitions occur within a given "rotational band" (fixed value of K). It must be remembered that wave functions belonging to same IM, but different K-values, are not orthogonal, in contrast to the corresponding wave functions in the Bohr-Mottelson model.

3. The Method of Canonical Transformations[6]

a) Systems with rotational levels.—Any closed system of interacting particles has two obvious collective constants of motion: the total linear and the total angular momentum. We want here to answer the question: is it possible to write the Hamiltonian of the system in such a way as to display its dependence on these two constants of motion? This can indeed be done by a suitable transformation of variables. For the case of the angular momentum, it is necessary to define an "intrinsic" coordinate system (that is, a system defined by the relative positions of the particles in space) so that a common rotation of all can be considered as a rotation of this intrinsic frame (36, 43). The Euler angles of this intrinsic frame are then the collective variables, and the total angular momentum components their conjugate. The three Euler angles $\theta_s = (\psi, \theta, \varphi)$ define an orthogonal transformation $R_{A\alpha}(\theta_s)$ from space fixed components $x_{i\alpha}$ to body-fixed components x'_{iA}:

$$x_{i\alpha} = X_\alpha + R_{A\alpha}(\theta) x'_{iA} \tag{17.}$$

In Eq. 17 we have isolated also the center of mass X_α. The x'_{iA} cannot be independent variables, but are subject to the conditions α): $\sum_i x'_{iA} \equiv 0$ (Definition of the centre of mass) β): three other, not yet specified conditions which will define the body-fixed frame: $F_s(x'_{iA}) \equiv 0$. We therefore write $x'_{iA}(\xi_\sigma)$ in terms of $(3N\text{-}6)$ independent, but unspecified, variables ξ_σ. The condition β:

$$F_s(x'_{iA}) = F_s(R_{A\alpha}(\theta)(x_{i\alpha} - X_\alpha)) \equiv 0 \tag{18.}$$

may be used to define the θ_s as functions of the $x_{i\alpha}$. From Eq. 17 it follows that the momentum is

$$p_{i\alpha} = \frac{1}{N} P_\alpha + \sum_\sigma \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} \pi_\sigma + \sum_s \frac{\partial \theta_s}{\partial x_{i\alpha}} \Pi_s$$

π_σ and Π_s being the momenta conjugate to ξ_σ and θ_s. Π_s may be expressed in terms of the total angular momentum components L_{AB} (with respect to the body-fixed axes)

$$\Pi_s = \sum_{A<B} \left(\frac{\partial R_{A\gamma}}{\partial \theta_s} R_{B\gamma} \right) L_{AB} \tag{19.}$$

and the intrinsic momentum split into two terms:

$$\sum_\sigma \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} \pi_\sigma = R_{A\alpha} p'_{iA} - \sum_{A<B} \left(\frac{\partial R_{A\gamma}}{\partial x_{i\alpha}} R_{B\gamma} \right) L'_{AB}$$

L'_{AB} being the intrinsic angular momentum

$$L'_{AB} = \sum_i (x'_{iA} p'_{iB} - x'_{iB} p'_{iA})$$

[6] There exist methods which are not so easily labelled; we mention here only Tomonaga's method of "approximately canonical" transformations (44, 45, 46), or a combination of canonical transformation and variational methods (47).

and

$$p'_{iA} \equiv \sum_{\sigma\tau} \frac{\partial x'_{iA}}{\partial \xi_\sigma} c_{\sigma\tau}^{-1} \pi_\tau \qquad c_{\sigma\tau} = \sum_{i\alpha} \left(\frac{\partial x_{i\alpha}}{\partial \xi_\sigma} \frac{\partial x_{i\alpha}}{\partial \xi_\tau} \right)$$

It is thus possible to write the momentum $p_{i\alpha}$ in the form

$$p_{i\alpha} = \frac{1}{N} P_\alpha + R_{A\alpha} p'_{iA} + \sum_{A<B} \left(\frac{\partial R_{A\gamma}}{\partial x_{i\alpha}} R_{B\gamma} \right) (L_{AB} - L'_{AB}) \qquad 20.$$

It is now easily seen that the three terms of Eq. 20 are "orthogonal" to each other, that is, do not contribute any cross terms to the kinetic energy, which now simply becomes

$$T = \frac{P^2}{2mN} + \sum_{iA} \frac{p'^2_{iA}}{2m} + \frac{1}{2} \sum_{A,B} Q_{AB}(\xi)(L\hat{A} - L'\hat{A})(L\hat{B} - L'\hat{B}) \qquad 21.^7$$

where

$$mQ_{AB} = \sum_{i\alpha} M_{i\alpha,\hat{A}} M_{i\alpha,\hat{B}}; \qquad M_{i\alpha,\hat{A}} = \left(\frac{\partial R_{B\gamma}}{\partial x_{i\alpha}} R_{C\gamma} \right) \qquad 21a.$$

This result is completely general. We recall that neither x'_{iA} nor the p'_{iA} are independent variables, but we have with good purpose introduced a notation suggestive of an approximation one is tempted to try at this place, if the number of particles is sufficiently large: to consider at least some of the x'_{iA} and p'_{iA} in fact as independent variables, in which case they become conjugate pairs, of course. This approximation is the basis for the Hamiltonians as used in (37, 38).

As an illustration of Eq. 21 and 21a we use as definition of the body-fixed axes:

$$\sum_i x'_{iA} x'_{iB} \equiv 0 \qquad (A \neq B) \qquad 22.$$

This relation has to hold identically in the independent variables ξ_σ. With Eq. 18 and 21a, we get:

$$M_{i\alpha,AB} = (x'_{iA} R_{B\alpha} + x'_{iB} R_{A\alpha})$$

I_A being $\sum_i (x'_{iA})^2$. The inertia tensor Q_{AB} becomes diagonal, with *eigenvalues*

$$Q_{AA} = (I_B + I_C)/(I_B - I_C)^2 \qquad 23.$$

This value is the equivalent of Bohr's result derived from irrotational liquid drop motion. (See part III.3d.)

Now let us consider what is gained by these results, both from the point of view of insight, and from the point of view of useful application: (*a*): Expression 21 for the kinetic energy shows to which extent the collective rotational motion may be decoupled from the intrinsic motion. The coupling terms express the fact that the intrinsic motion is described in a non-inertial frame and account for the centrifugal and Coriolis-forces acting on the in-

[7] Whenever convenient, we use the notation $L_{BA} = L\hat{c}$ for axial vector components.

trinsic degrees of freedom. No purely kinematic argument for the choice of the body-fixed frame will remove these coupling terms. We must conclude from this that the choice (19) of the body-fixed frame is as good as any other, at least for nuclei (for molecules one is able to do considerably better, due to the quasi-rigidity of the system).[8] (*b*): The effective moments of inertia, $\Im_{eff}$, which we shall define, for a system with rotational levels, by

$$E_I = E_0 + \frac{I(I+1)}{2\Im_{eff}},$$

cannot be found without solving the problem of the intrinsic motion to some extent at least. There is absolutely no reason to assume that $\Im_{eff}$ is directly related to the eigenvalues of the inertia tensor Q_{AB}, which are determined by the rather arbitrary choice of the body-fixed system. (*c*): To handle the intrinsic dynamics, clearly some "model" has to be made up. Such models will be discussed in part III. (*d*): In view of the observed shell structure in nuclei, it is often found convenient to single out some "valence nucleons" whose motion in the collective field created by the others (the "core"-nucleons) one is particularly interested in. It is then possible to carry out the transformation (Eq. 17, 18, 19) for the core-nucleons only, and to describe the motion of the extra particles in the body-fixed frame defined by the others. Calling x_r and p_r positions and momenta of the extra particles, we obtain an unsymmetric Hamiltonian:

$$H = \frac{P_c^2}{2mN_c} + H_{intr}(\pi_\sigma, \xi_\sigma) + \sum_{r,A}\left(\frac{p_{rA}^2}{2m} + \sum_{i=1}^{N_c} V(x_{rA} - x'_{iA}(\xi))\right) + \sum_{r,s} V_{r,s} + \frac{1}{2}\sum_{A,B} Q_{AB}\left(L_{\hat{A}} - L'_{\hat{A}} - \sum_r l_{r,\hat{A}}\right)\left(L_{\hat{B}} - L'_{\hat{B}} - \sum_r l_{r,\hat{B}}\right) \qquad 24.$$

In Eq. 24, the total angular momentum now includes the angular momentum of the valence nucleons; but it is still given by the operator (19). The form (Eq. 24) makes it of course impractical to include the exchange effects between the two groups of particles, but it is the adequate starting point for any approximation obtained by taking some kind of average over the intrinsic motion; in particular shall we be able to get a better understanding of the original Bohr model on the basis of Eq. 24.

N.B. to include the nucleon spin in Eq. 24, it is sufficient to make the substitution: $L_{\hat{A}} - L'_{\hat{A}} - \sum_r l_{r,\hat{A}} \to L_{\hat{A}} - J'_{\hat{A}} - \sum_r j_{r,\hat{A}}$ with $J'_{\hat{A}} = L'_A + \frac{1}{2}\sum_i \sigma_{i,\hat{A}}$; $j_{r,\hat{A}} = l_{r,\hat{A}} + \frac{1}{2}\sigma_{r,\hat{A}}$. $L_{\hat{A}}$ remains the same operator, but its interpretation changes to that of the total angular momentum of the total system. (*e*): It must thus be realized that a form like Eq. 21 or Eq. 24 will in no way tell directly whether rotational levels occur in a system. These expressions only furnish suitable starting points for approximations, or suggest models, in

[8] Equation 21a may of course be applied to molecules as well. Due to the existence of equilibrium positions x_i^0: $x_i' = x_i'^0 + \delta x_i'$, there exists a more appropriate condition to define the body-fixed frame: $\sum_i (x_{iA}^0\ \delta x'_{iB} - x_{iB}^0 \delta x'_{iA}) \equiv 0$, from which the rigid body moments of inertia follow (36, 48).

which the programme of decoupling the total angular momentum from the intrinsic motion can be carried one step further; but in each case such a programme will have to take into account the intrinsic dynamics, or a model thereof. These questions will therefore be discussed in part III.

b) *Small amplitude collective vibrations.*—The transformation of the previous section (part II.3a) was appropriate for a situation, in which the mass distribution in the nucleus, say given by the tensor

$$I_{\alpha\beta} = \sum_i (x_{i\alpha} - X_\alpha)(x_{i\beta} - X_\beta) \qquad 25.$$

is sufficiently anisotropic to define a nonspherical shape; this implies in particular that the eigenvalues I_A of $I_{\alpha\beta}$ show sufficiently small fluctuations:

$$\langle I_A^2\rangle - \langle I_A\rangle^2 \ll \langle (I_A - I)^2\rangle; \qquad (I = \sum I_A/3)$$

If this condition is not satisfied, then it is better to build up a formalism describing the fluctuations of $I_{\alpha\beta}$ about an isotropic average $I_0\delta_{\alpha\beta}$. The best way to do this is to think of the actual distribution $I_{\alpha\beta}$ as arising from a fictitious isotropic distribution $I_0\delta_{\alpha\beta}$ by action of a collective displacement field $\delta_\alpha(x) = w_{\alpha\beta}x_\beta$. The actual positions $x_{i\alpha}$ of the nucleons may then be viewed as the result of this displacement upon some "intrinsic" positions $x'_{i\alpha}$; that is

$$x_{i\alpha} = X_\alpha + (\delta_{\alpha\beta} + w_{\alpha\beta})x'_{i\beta}(\xi) \qquad 26.$$

X_α being again the center of mass.

The quantity $\dot{w}_{x\beta}x'_\beta$ is then clearly a collective velocity field; in analogy to the original formulation of Bohr's model we assume this field to be irrotational and incompressible; this is equivalent to $w_{\alpha\beta} = w_{\beta\alpha}$; $\sum_\alpha w_{\alpha\alpha} = 0$; $w_{\alpha\beta}$ is thus a symmetric, traceless tensor, and may be written in terms of five independent parameters λ_μ:

$$w_{\alpha\beta} = \sum_{\mu=-2}^{+2} \lambda_\mu C_{\alpha\beta}{}^\mu \qquad 26a.$$

the λ_μ thus chosen are complex and supposed to transform like the spherical harmonics $Y_{2\mu}{}^*$; it follows then that $\lambda_\mu{}^* = (-1)^\mu\lambda_{-\mu}$; the $C_{\alpha\beta}{}^\mu$ are normalized to $\sum_{\alpha\beta} C^*{}_{\alpha\beta}{}^\nu C_{\alpha\beta}{}^\mu = \delta_{\mu\nu}$. In order to introduce the λ_μ as dynamical variables, we must impose five restrictive conditions on the intrinsic positions (plus three others for the elimination of the center of mass). These are

$$\sum_i x'_{i\alpha} \equiv 0 \qquad 27a. \qquad\qquad \sum_i x'_{i\alpha}x'_{i\beta} \equiv I_0\delta_{\alpha\beta} \qquad 27b.$$

The meaning of this is clearly, as announced above, that the actual positions arise through a collective, five parametric displacement from an isotropic "intrinsic" distribution of positions.

Writing the momentum $p_{i\alpha}$ in the new variables, we have again

$$p_{i\alpha} = \frac{1}{N}P_\alpha + \sum_\sigma \frac{\partial\xi_\sigma}{\partial x_{i\alpha}}\pi_\sigma + \sum_\mu \frac{\partial\lambda_\mu}{\partial x_{i\alpha}}p_\mu$$

From here on all developments must be based on the assumed smallness of the λ_μ, in order to avoid extremely complicated expressions. A step analogous to the procedure in part II.3a, shows that

$$\sum_\sigma \frac{\partial \xi_\sigma}{\partial x_{i\alpha}} \pi_\sigma = p'_{i\alpha} + O(\lambda_\mu)$$

$p'_{i\alpha}$ being therein defined by $p'_{i\alpha} = (\partial x_{i\alpha}/\partial \xi_\sigma) c_{\sigma\tau}^{-1} \pi_\tau$, $c_{\sigma\tau}$ being the same matrix as there. The reason for introducing $p'_{i\alpha}$ is of course again its orthogonality to the two other terms in $p_{i\alpha}$. $\partial \lambda_\mu / \partial x_{i\alpha}$ is determined by the defining Eq. 26, 27b, which give, to order λ:

$$\partial \lambda_\mu / \partial x_{i\alpha} = C^*_{\alpha\beta}{}^\mu x'_{i\beta} / I_0$$

The kinetic energy is therefore approximately:

$$T = \frac{P^2}{2mN} + \sum_{i\alpha} \frac{p'^2_{i\alpha}}{2m} + \frac{1}{2mI_0} \sum_\mu p_\mu^* p_\mu \qquad 28.$$

To construct the angular momentum, we need

$$\sum_i \left(x_{i\alpha} \frac{\partial \lambda_\mu}{\partial x_{i\beta}} - x_{i\beta} \frac{\partial \lambda_\mu}{\partial x_{i\alpha}} \right) \cong i \sum_\nu \lambda_\nu (M_{\alpha\beta})_{\nu,\mu}$$

$(M_{\alpha\beta})_{\mu,\nu}$ being equal to the matrix element $(j\mu | J_{\alpha\beta} | j\nu)|_{j=2}$ of the angular momentum component $J_{\alpha\beta}$. The total angular momentum, referred to the center of mass, appears thus approximately as the sum of two terms

$$L_{\alpha\beta} = L_{C,\alpha\beta} + L'_{\alpha\beta}$$

with $L'_{\alpha\beta} = \sum_i (x'_{i\alpha} p'_{i\beta} - x'_{i\beta} p'_{i\alpha})$, the intrinsic angular momentum and $L_{C,\alpha\beta} = i \sum_{\mu\nu} (M_{\alpha\beta})_{\mu\nu} \lambda_\mu p_\nu$, the angular momentum of collective motion. We repeat here the statement made in part II.3a, that the $x'_{i\alpha}$ and $p'_{i\alpha}$ are not independent variables, and not canonically conjugate. To complete the Hamiltonian, the potential energy has to be written in terms of the ξ_σ and λ_μ:

$$V(\cdots x_i \cdots) = V(\cdots (1 + w) x_i' \cdots)$$
$$= V(\cdots x_i' \cdots) + \sum_{\alpha\beta} w_{\alpha\beta}(\lambda) \sum_i x'_{i\alpha} \frac{\partial V}{\partial x'_{i\beta}} + \cdots .$$

Now again the question comes up to see to which extent an approximate factorization of the wave function appears possible. Beginning this discussion with the consideration of even nuclei with zero angular momentum in their ground states, we are tempted to write:

$$\psi(x_i - X) \cong \phi(\xi_\sigma) v(\lambda_\mu)$$

and to construct a Hamiltonian averaged over ϕ, which is assumed to be a state with zero intrinsic angular momentum:

$$J'\phi = \left(L' + \frac{1}{2} \sum_i \mathfrak{d}_i \right) \phi \cong 0$$

For such a state, we have

$$\left\langle \frac{1}{2m}\sum_i p_{i\alpha'^2} + V \right\rangle_\phi = E_0 + \frac{1}{2}C\sum_\mu \lambda_\mu^*\lambda_\mu + \cdots$$

The constant C is partly of kinetic, partly of potential origin, but its value can, of course, not be reliably obtained by direct evaluation. It follows that

$$\langle H\rangle_\phi = \frac{1}{2B}\sum_\mu p_\mu^* p_\mu + \frac{C}{2}\sum_\mu \lambda_\mu^*\lambda_\mu \qquad 29.^9$$

with C defined above, and $B^{-1} = \langle 1/mI_0\rangle_\phi \cong 5/mNR_0^2$ for a homogenous mass distribution over a sphere of Radius R_0. $\langle H\rangle_\phi$ is a simple oscillator Hamiltonian, with *eigenvalues*

$$E_N = \hbar\omega\sum_\mu (N_\mu + \tfrac{1}{2}) \qquad \omega = \sqrt{C/B}$$

and N_μ being integers. The z-component of the collective angular momentum $L_{Cz} = i\sum_\mu \mu\lambda_\mu p_\mu$ is diagonal in the same representation and has the *eigenvalues* $\sum_\mu \mu N_\mu$. We infer that the quanta of collective vibrational excitation (the "surfons") have angular momentum 2 (corresponding to the maximum value of L_{Cz} for a one quantum excitation). This is confirmed by verifying that for any one quantum state $\langle L_C^2\rangle = 6\hbar^2$. For higher excited states, there is a degeneracy of the energy levels with respect to L_C^2, and the energy eigenstates are not automatically angular momentum eigenstates.

We now take up the question whether such an averaging as carried out above is indeed permissible: (*a*) As necessary conditions we have clearly first the zero angular momentum in the ground state. Then, in addition, we require the smallness of the collective excitation energy compared with energy of the lowest mode of intrinsic excitation. The experimental material seems to indicate that this condition is never fulfilled; even the lowest collective vibrational excitations are of the order of 1 Mev in medium weight nuclei (26, 51). (*b*) Under the assumption of the validity of the approximation, B and C may be determined from measured values of $\hbar\omega$ (which is $\propto\sqrt{B/C}$ and of the $E2$-transition rates, which are $\propto\sqrt{BC}$ (See part III.3b). The fact that the values of B obtained this way are systematically an order of magnitude larger than the value mI_0 sheds considerable doubt on the validity of this approximation.

The obvious refinement of the approximation consists in treating in a more explicit way the dynamic interplay of collective modes and the most easily excitable intrinsic modes. As in the case of rotational levels this may be achieved by dividing the system in core- and extra-nucleons, with the λ_μ describing the collective motion of the core. The core furnishes a λ-dependent potential well for the extra-particles:

$$U(x_\tau, \lambda) = \left\langle \sum_{\text{core}} V(x_\tau - x_i) \right\rangle_\phi$$

[9] Our B differs from Bohr's B_2 (50) by a factor $8\pi/15$, stemming from a difference in scale between our λ_μ and the corresponding α_μ of Bohr.

$$= \left\langle \sum_{\text{core}} V(x_r - x_i'(\xi)) \right\rangle_\phi$$

$$+ \sum_{\alpha\beta} w_{\alpha\beta}(\lambda) \left\langle \sum_i x'_{i\alpha} \frac{\partial}{\partial x'_{i\beta}} V(x_r - x_i'(\xi)) \right\rangle_\phi + \cdots$$

$$\cong U_0(x_r) + \sum_\mu \lambda_\mu \sum_{\alpha\beta} C_{\alpha\beta}{}^\mu x_{r\alpha} \frac{\partial U_0}{\partial x_{r\beta}} + \cdots$$

The ϕ-averaged Hamiltonian is now

$$\langle H \rangle_\phi = \frac{1}{2B} \sum_\mu p_\mu{}^* p_\mu + \frac{C}{2} \sum_\mu \lambda_\mu{}^* \lambda_\mu + \sum_r \left(\frac{p_r{}^2}{2m} + U(x_r; \lambda) \right) + \sum_{r,s} V_{r,s}. \qquad 3($$

This Hamiltonian has exactly the structure of the model Hamiltonian cor sidered by Bohr (50) and more will be said about it in part III.

4. The Method of Redundant Variables

a) Illustration of the method; centre of mass motion.—In part II it ha been shown that if the collective variables are isolated by means of a canon cal transformation, none of the remaining intrinsic variables may be ident fied with particle position coordinates. This is particularly unpleasant i connection with any attempt to describe the intrinsic motion in terms c an "independent particle" picture. The present method is an attempt t combine the advantages of the method of canonical transformations wit the advantages of dealing with particle coordinates to describe intrinsi motion. To achieve this, the actual dynamical system is embedded in a enlarged system, whose number of degrees of freedom is larger by th number of collective variables to be introduced. A canonical transformatio is then carried out leading to intrinsic variables which are in one-to-one cor respondence with the original particle coordinates. A subset of solutions c the enlarged system, satisfying suitable subsidiary conditions, is the equivalent to the solutions of the original system. Various approximatio techniques may then be based on the possibility of defining a new zero orde Hamiltonian, and wave functions which only approximately satisfy th subsidiary conditions.

As an illustration, we first deal with the centre of mass problem for particles of equal mass m. This problem has been discussed by Lipkin *et a* (52), Tamura (53), and Skinner (54). The method consists of replacing th usual transformation:

$$x_{i\alpha} = X_\alpha + x'_{i\alpha}(\xi_v); \qquad \sum_i x'_{i\alpha}(\xi_v) \equiv 0; \qquad 31a$$

$$p_{i\alpha} = \frac{1}{N} P_\alpha + p'_{i\alpha}(\pi_v); \qquad \sum_i p'_{i\alpha} \equiv 0, \qquad 31b$$

in which neither the $x'_{i\alpha}$ nor the $p'_{i\alpha}$ are independent variables, by a tran formation in an augmented system of variables $(x_{i\alpha}, p_{i\alpha})$ and (F_α, G_α):

$$x_{i\alpha} = X_\alpha + x'_{i\alpha} - \frac{1}{N} \sum_k x'_{k\alpha}; \qquad F_\alpha = \frac{1}{N} \sum_k x'_{k\alpha} \qquad 32a$$

and

$$p_{i\alpha} = \frac{1}{N} P_\alpha + p'_{i\alpha} + \frac{1}{N} \sum p'_{k\alpha}; \qquad G_\alpha = \sum_K p'_{k\alpha} \tag{32b.}$$

in Eq. 32 the $(x'_{i\alpha}, p'_{i\alpha})$ are now independent canonical pairs. We also see that the original transformation corresponds to the restriction $F_\alpha \equiv 0$, $G_\alpha \equiv 0$. It is interesting to observe the complete symmetry in Eq. 32 between new and old variables. In fact, the inversion of Eq. 32 is obtained simply by the substitution, in Eq. 32, of:

$$x \rightleftarrows x' \qquad F \rightleftarrows X \qquad p \rightleftarrows p' \qquad G \rightleftarrows P$$

The kinetic energy $T = \sum_i p_i^2/2m$ becomes, in terms of the new variables

$$\tilde{T} = \frac{P^2}{2mN} + \frac{\sum p_i'^2}{2m} - \frac{(\sum p_i')^2}{2mN}$$

In an X-F-space representation, the exact eigenfunctions of $H(p_i, x_i)$ are of type $U(F)\psi_n(x_i)$, with an arbitrary function $U(F)$ and ψ_n satisfying $G\psi_n = -i\hbar\partial\psi_n/\partial F = 0$. [We present here only the quantum mechanical methods. A general classical theory of redundant variables has been given by Watanabe (55).] Instead, we are going to use approximate solutions of type $U(F)\psi(x_i, F)$, opening the possibility that

$$\psi(x_i(X, x_i'), F(x_i')) = \tilde{\psi}(X, x_i')$$

has a simpler structure, say, may be of independent particle type

$$\tilde{\psi} = f(X) \operatorname{Det} \|\varphi_K(x_i')\|$$

The expectation value of the energy depends now on the structure of $U(F)$ and to minimize the error, we may choose

$$|U(F)|^2 = \delta(F),$$

(or even determine $U(F)$ from a variational principle).

The same statements can be made, *mutatis mutandis*, for any representation. In particular, the problem can also be formulated in momentum space, using $\psi(p_i, G)$. Such general aspects of the technique are discussed in a paper by Lipkin (56).

Using momentum space, and trial functions $\psi(p_i, G)$ the expectation value of the kinetic energy T, in the original variables, may be written as:

$$\langle T \rangle = \int dp_i \int dG\delta(G)\psi^*(p_i, G)T\psi(p_i, G)$$

an expression which is easily rewritten in terms of the new variables.

$$\langle T \rangle = \int dP \int dp_i'\delta(\sum p_i')\tilde{\psi}^*(P, p_i')\tilde{T}\tilde{\psi}(P, p_i')$$

The structure of $\tilde{T}$ allows to separate off the centre of mass energy in the usual way. The remaining intrinsic kinetic energy is given by

$$\int dp_i'\delta(\sum p_i')\tilde{\psi}^*(p_i')\left[\sum p_i'^2/2m\right]\tilde{\psi}(p_i') = \int d\xi \int dx_i'\tilde{\psi}^*(x_i')T_{op}\tilde{\psi}(x_i'+\xi) \quad 33.$$

The term $-(\sum p_i')^2/2mN$ does not contribute, due to the choice of $U(G$ This result shows that the method in this form of application gives the sam result for the intrinsic energy as the method of part II.2, except tha the collective energy $P^2/2mN$ is exactly separated by a coordinate tran formation. An exact calculation of the intrinsic energy would have implie the use of a wave function $\tilde{\psi}(p_i')$ satisfying $\sum_i \partial\tilde{\psi}/\partial_{p'i}=0$, a condition whic cannot be satisfied by an independent particle wave function. We see tha the present formalism, despite relaxing the subsidiary condition, gives a improved value for the intrinsic energy, apart from the exact value for th "collective" centre of mass kinetic energy. In this respect, it is seen to b superior to the method of part II.2; in fact, it somehow combines th advantages of the methods of the two previous paragraphs.

It is also seen that a different result is obtained by starting with a X-F-representation and using $|U(F)|^2=\delta(F)$: The result obtained in tha case is

$$\langle T_{\text{intr}}\rangle = \int dx_i'\delta(\sum x_i')\tilde{\psi}^*(x_i')\left[\sum_i p_i'^2/2m - (\sum p_i')^2/2mN\right]\tilde{\psi}(x_i')$$

These alternatives illustrate the flexibility of the method.

b) *Rotational energy.*—The method as illustrated above is easily adapte to the case of collective rotation. [Collective vibrations have been treate with similar methods by Süssmann (58), Marumori *et al.* (59, 60).] On important difference with the previous case, however, remains. The rota tional kinetic energy cannot be separated by a transformation of variable based on a purely kinematic argument, and all we are going to get is a resul similar to that of part II.3. But we are now in a much better position t carry out additional transformations, aiming at an optimum decoupling o the collective motion, than with the formalism of II.3; in particular, w can now approximate the interaction by a single particle potential and us independent particle wave functions, since the particle coordinates remai independent variables.

In designing the actual transformation, the method applied in part II.4 is copied as closely as possible. An essential point there was the high degre of symmetry between old and new variables, as expressed in the equation

$$x_i - X = x_i' - F, \qquad X = \sum_i x_i; \qquad F = \sum_i x_i'. \quad 34$$

X is the centre of mass in x-space, F the centre of mass in x'-space. x_i and F are the "old," x_i' and X the new variables. In analogy to this we introduc two collective rotations, given by the two sets of Euler angles ϵ_s and θ_s.

[10] To assure proper normalization, such an expansion has, of course, to be divided by $\int d\xi \int dx_i'\tilde{\psi}^*(x_i')\tilde{\psi}(x_i'+\xi)$.

ϵ_s is the "redundant" variable, and the transformation is of type

$$(x_{i\alpha}, p_{i\alpha}); \quad (\epsilon_s, G_s) \rightarrow (x'_{in}, p'_{iA}); \quad (\theta_s, \Pi_s).$$

x'_{iA} and θ_s being the new variables, x'_{iA} the new intrinsic positions, θ_s the Euler angles of the collective rotation, and conjugate to the total angular momentum. $\theta_s = \theta_s(x_{i\alpha})$ and $\epsilon_s = \epsilon_s(x'_{iA})$ are now chosen in such a way as to bring the mass tensors

$$I_{\alpha\beta} = \sum_i x_{i\alpha} x_{i\beta} \quad \text{and} \quad I_{AB} = \sum_i x'_{iA} x'_{iB}$$

to principal axes, respectively

$$R_{P\alpha}(\theta) R_{Q\beta}(\theta) I_{\alpha\beta} = R_{PA}(\epsilon) R_{QB}(\epsilon) I_{AB} = I_P \delta_{PQ} \qquad 35.$$

This equation is the analogue of Eq. 34 for the centre of mass case. We have then

$$\begin{aligned} x_{i\alpha} &= R_{P\alpha}(\theta) R_{PA}(\epsilon(x'_{iA})) x'_{iA} \\ \epsilon_s &= \epsilon_s(x'_{iA}) \text{ by Eq. 35} \end{aligned} \qquad 36.$$

and the inversion

$$\begin{aligned} x'_{iA} &= R_{PA}(\epsilon) R_{P\alpha}(\theta_s(x'_{iA})) x_{i\alpha} \\ \theta_s &= \theta_s(x_{i\alpha}) \text{ by Eq. 35.} \end{aligned} \qquad 37.$$

The expressions for momentum $p_{i\alpha}$ and angular momentum $L_{\alpha\beta}$ are now easily obtained in the new variables, and read:

$$p_{i\alpha} = R_{P\alpha}(\theta) R_{PA}(\epsilon) p'_{iA} + \frac{1}{2} \sum_{P,Q} M_{i\alpha,PQ} (L_{PQ} - L'_{PQ})$$

$$L_{\alpha\beta} = R_{P\alpha}(\theta) R_{Q\beta}(\theta) L_{PQ}$$

L_{PQ} is the angular momentum projected on the principal axes P, Q, R and a linear function of the momenta Π_s conjugate to the θ_s:

$$\Pi_s = \sum_{P<Q} \left(\frac{\partial R_{P\alpha}}{\partial \theta_s} R_{Q\alpha} \right) L_{PQ}$$

(analogous to Eq. 19). L' is the intrinsic angular momentum

$$L_{PQ}' = R_{PA}(\epsilon) R_{QB}(\epsilon) L_{AB}' \qquad L_{AB}' = \sum_i (x_{iA}' p_{iB}' - x_{iB}' p_{iA}')$$

Notice that now L' is an ordinary angular momentum, satisfying the usual commutation relations, in contrast to the property of L' in part II.3. $M_{i\alpha,PQ}$ is defined by Eq. 21a. In forming the kinetic energy, the cross terms do not vanish, as they did in part II.3:

$$\tilde{T} = \frac{1}{2m} \sum_{i,A} p_{iA}'^2 + \frac{1}{2} \sum_P Q_{PP} (L\hat{P} - L'\hat{P})^2 + \frac{1}{2m} \sum_{P,Q} \frac{\{N'_{PQ}, L_{PQ} - L'_{PQ}\}}{I_P - I_Q} \qquad 38.$$

with

$$N'_{PQ} = R_{PA} R_{QB} \sum_i (x'_{iA} p'_{iB} + x'_{iB} p'_{iA})$$

and Q_{PP} given by the "hydrodynamic" value of Eq. 23. This result was derived by Lipkin *et al.* (52), Tamura (53) and Nataf (57).

Finally, we give the operator G_s in the new variables:

$$G_s = -\sum_{A<B} \left(\frac{\partial R_{PA}(\epsilon)}{\partial \epsilon_s} R_{PB}(\epsilon) \right) L'_{AB}$$

The result (Eq. 38) is the analogue of Eq. 21, which was obtained witho introducing redundant variables. We can simplify, however, the expressic for the expectation value of T by a judicious choice of U.

$$| U(G_s) |^2 \to \prod_s \delta(G_s)$$

It is easily seen, in analogy with Eq. 33 that this corresponds to

$$\langle \tilde{T} \rangle = \int d\Omega_\mathcal{E} \int d\Omega_\theta \int \prod_i dx_i' \psi^*(x_i'; \theta_s) \tilde{T} \psi(R(\epsilon) x_i'; \theta_s) \qquad 3$$

$R(\mathcal{E})$ being a rotation of x'-space. In this way again a connection is estal lished with the Peierls-Yoccoz method. Now, the operators L'_{PQ} may b thought of operating on ϵ in Eq. 39 and therefore taken out of the $x_i' - \theta$ integral. They therefore do not contribute to the expectation value of $\tilde{T}$ which operator may therefore be replaced by

$$\tilde{T}_{\text{eff}} = \sum_i \frac{p_i'^2}{2m} + \frac{1}{2} \sum_P \frac{L\tilde{P}^2}{Q_{PP}} + \sum_{PQ} \frac{L_{PQ} N'_{PQ}}{m(I_P - I_Q)} \qquad 38$$

There do not seem to be any developments which go further than thi The programme is fairly evident, though: an additional transformation the Hamiltonian may separate the collective terms. Such transformatio can now easily be formulated since the (x_i', p_i') are good canonical pair However, the decoupling problem depends on the potential energy too, an its solution therefore requires specific assumptions concerning the inte action. Assuming a single particle potential, this programme should b fairly straightforward.

III. MODELS

1. The Spheroidal Independent Particle Model

Although this model is mainly a stepping stone towards more elabora schemes, we present it here separately, as it has a certain *raison d'être* of i own, distinguished by many characteristic aspects from the shell model.

As mentioned already, we have only rudiments of a really self-consiste calculation of nuclear levels and associated single-particle potentia [Rotenberg (17)]. Already a phenomenological analysis makes clear, hov ever, that (*a*) the potential well follows the matter distribution qui closely, being at most a fraction of 10^{-13} cm. wider than the latter (17, 6 62, 63). Neutron- and proton-distributions are almost proportional: t former may have a radius slightly larger than the latter (64, 65, 66, 67 [It has been suggested (71) that in strongly deformed nuclei a separation protons from neutrons may be responsible for the apparent difficulty of t

Bohr-Mottelson Model with the moment of inertia question. We do not support this idea.] (*b*) The most favoured shape is the "bathtub" shape, with slanted walls of a width of 2 to 3×10^{-13} cm. (68). (*c*) The potential is velocity dependent (69). It remains questionable whether this velocity dependence may adequately be described in terms of an effective mass (21, 70) m; such an attempt leads to $m^* \gtrsim \frac{1}{2}m$. The relevance of this problem in connection with the calculation of moments of inertia is obvious.

The usual approximate determinations of the single partical potential $U(x)$ are based on a parameterization, with subsequent adjustment of parameters to fit the energy:

$$\langle E\rangle = \sum_i \langle T_i\rangle + \frac{1}{2}\sum_i \langle U_i\rangle$$

(the factor $\frac{1}{2}$ being due to the assumed predominance of two body forces), and to give shell closure, etc.

In parameterizing U_i for nonspherical shapes, the condition of "volume preservation" is usually introduced. This is justified by the small compressibility of nuclear matter (62, 72, 73), but should perhaps not be too rigorously enforced for very large deviations from spherical symmetry. This volume preservation allows a minimization of $\langle E\rangle$ with respect to deformation, and therefore, in principle, a calculation of electric quadrupole moments.

We present now a sample of cases that have been investigated and give the most significant results:

a) *Harmonic oscillator without spin-orbit coupling.*—This somewhat academic case has the advantage of being completely transparent, and serves to point out some basic results (74, 75).

With the single particle Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m[\Omega_1^2(x^2+y^2) + \Omega_3^2 z^2] \qquad 40.$$

we have the energy levels

$$E_n = \hbar((n_1+n_2+1)\Omega_1 + (n_3+1/2)\Omega_3)$$

and $\langle E\rangle = \frac{3}{4}\sum E_n$. Volume preserving deformations may be defined by $\Omega_1^2\cdot\Omega_3 = \Omega_0^3 = \text{const.}$:

$$\Omega_1 \cong \Omega_0(1+\zeta/3) \qquad \Omega_3 \cong \Omega_0(1-2\zeta/3+\zeta^2/3) \qquad 41.$$

giving a deformation energy

$$\mathcal{E}(\zeta) = 3/4(E(\zeta)-E(0))/\hbar\Omega_0 = 1/4[\sum(n_1+n_2-2n_3)\zeta + \sum(n_3+1/2)\zeta^2] \qquad 42.$$

In accordance with established custom, we define a deformation parameter β and a r.m.s. radius R_0 through the equations

$$\langle\sum r^2\rangle = 3/5AR_0^2, \qquad Q = 3/\sqrt{5\pi}ZR_0^2\beta,$$

Q being the electric quadruple moment. For a nonclosed shell nucleus, for which ζ is determined by the minimum of $\mathcal{E}(\zeta)$ (Eq. 42), it follows that

[redacted] If, on the other hand, the occupation numbers satis $\sum n_1 = \sum n_2 = \sum n_3$, as for instance in the case of shape oscillations about spherical equilibrium shape, then [redacted] In particular this implies that f the harmonic oscillator case, one half of the electric quadrupole moment due to the deformed filled oscillator shells, the other half due to the ext nucleons (76).

Equation 42 does not account for the preponderance of positive qua rupole moments between closed shells. Instead, the intrinsic moments a positive in the first half of shell filling, negative in the second half. T order of magnitude of the deformations is correct, however.

b) *Harmonic oscillator with spin-orbit term.*—In this model, the potenti energy is modified by the addition of a spin-orbit coupling term, $C(\mathbf{l}\cdot\mathbf{s})$, an a further term of type Dl^2, whose function is to simulate greater "squarenes of the well [Nilsson (77)]. The corresponding Hamiltonian is diagonalize in an approximation that neglects the off-diagonal elements of the two abo mentioned terms between different values of $n = n_1 + n_2 + n_3$ (oscillator shells

The constants are adjusted separately for protons and neutrons, such to give the right single particle level structure near closed shells. As a realist model, it may be tested against experiment.

c) *Square well.*—The infinite well (deformed box) has been considere with and without $(\mathbf{l}\cdot\mathbf{s})$ coupling, by Moszkowski (78); the finite well wit out $(\mathbf{l}\cdot\mathbf{s})$ by Uretszky (79) and with $(\mathbf{l}\cdot\mathbf{s})$ by Gallone & Salvetti (80) and b Gottfried (81). The latter's work is the most detailed. His (numerica calculations are based on an expansion of the deformed potential:

$$U(\vec{r};\beta) = U_0(r) - \beta Y_{20}(\theta)\left(r\frac{dU_0}{dr}\right) + \frac{1}{2}\beta^2\left\{\frac{4}{5\pi}(1 - 2\sqrt{4\pi/5}Y_{20}(\theta))r\frac{dU_0}{dr} + Y_{20}^2(\theta)\left(r^2\frac{d^2U_0}{dr^2} - r\frac{dU_0}{dr}\right)\right\}$$

U_0 being an ordinary square well. The spin orbit term is chosen to give representative level ordering for $\beta = 0$; it is found that for large values c $\beta(\beta \gtrsim 0.3)$ the level structure is insensitive to the details of level spacin within a shell. The Hamiltonian is then diagonalized in a finite space c function $|nj\Omega\pm\rangle$. (Ω being the eigenvalue of j_3, the component along th symmetry axis of the well.)

d) *Comparison with experiment.*[11]—(i) Ground state spins: According t the coupling scheme for strongly deformed nuclei, the ground state spin I_0 i equal to J_3, the projection of the intrinsic angular momentum the nuclea symmetry axis. In the models presented here, J_3 is equal to the Ω of the las odd particle. On the basis of this, ground state spins may be read off th energy level diagram as a function of deformation β, the values of β bein taken or interpolated from measured values of Q_0. The agreement with ex periment for both Gottfried's and Nilsson's calculation is very good, in th

[11] These investigations cover the region $150 < A < 190$.

sense that at the assumed value of β the level giving the expected I_0 value lies actually lowest or is close to the lowest (the exact crossing point of levels being a detail which the model may not reliably supply) (81, 82).

(ii) Quadrupole moments: Gottfried's calculation turns out not to be reliable enough to calculate the equilibrium deformations. On the other hand, Nilsson's level scheme (77) allows to construct a curve $\beta(A)$ for $150 < A < 190$, which agrees well with the experimental data (82, 15), and, in particular, reproduces the steep increase of β above $A = 150$.

(iii) Magnetic moments: Magnetic moments may be calculated under the assumption that the state of the last odd particle determines the g-factor g_Ω, and that the g-factor of collective motion has a given value, say $g_c = 0.4$. (See Eq. 3.) Values of g_Ω have been calculated on that basis by Gottfried, and are presented, together with the experimental values, in Table II (see under part III.4, Conclusion).

This calculation of g_Ω also implies the assumption that there is no further disturbance of the states in the deformed well through the collective rotation of the latter, so that a wave function of the type of Eq. 78 may be used, with $\psi_{\Omega-K}(x)$ being the state of the last odd particle in the deformed well. It follows then that

$$g_\Omega = g_l + (g_s - g_l)\frac{(\mathbf{s}\cdot\mathbf{L})}{I^2}.$$

The investigations on the Inglis model shed some doubt on the correctness of this approximation: Investigating the moment of inertia problem (part III.2c) one finds that the Coriolis-effect on an unfilled shell amounts to a considerable perturbation, which must make itself felt in the magnetic moments too, affecting the values of both g_c and g_Ω. No systematic investigation of this problem has been made yet, however.

2. The Inglis (Cranking) Model

a) *Presentation of the model.*—This model consists in a refinement of the previous picture of independent particles in a spheroidal well. The well is assumed to rotate with a constant angular velocity about an axis perpendicular to the axis of symmetry (83). In this way, one simulates, in a semi-classical way, the effect of the precession of the well about the axis of total angular momentum (which is indeed perpendicular to the body-symmetry axis for the rotational level series $K = \Omega = 0$; see section III.3). The energy of the system of A independent particles in the well then assumes a correction term which depends quadratically on the angular velocity ω:

$$E(\omega) = E(0) + \tfrac{1}{2}\Im_{eff}\omega^2 + \cdots \qquad 43.$$

This correction term is interpreted as the classical description of the rotational energy

$$E_{rot} = \frac{(\mathrm{I})^2}{2\Im_{eff}} \qquad 44.$$

where, of course, use is made of the relation $\omega = \partial E/\partial I = I/\Im_{\text{eff}}$. Assuming rotation along the x-axis, we have a time dependent potential

$$U(x, y, z; t) = U(x, y'(t), z'(t); 0),$$

y' and z' being the coordinates in the body-fixed system and given by $y' = y \cos \omega t + z \sin \omega t$, $z' = z \cos \omega t - y \sin \omega t$. To have a stationary potential the Schroedinger equation is written in the body-fixed system, using the relations $\psi(x, y, z; t) = \varphi(x, y', z'; t)$ and

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y'} \frac{\partial y'}{\partial t} + \frac{\partial \varphi}{\partial z'} \frac{\partial y'}{\partial t} \right) = i\hbar \frac{\partial \varphi}{\partial t} + \omega l_x \varphi.$$

Hence, in the body-fixed system

$$i\hbar \frac{\partial \varphi}{\partial t} = \left(\frac{p^2}{2m} + U(x, y', z') - \omega l_x \right) \varphi = \tilde{H} \varphi \qquad 45$$

The ω-dependence of the eigenvalues E_n of $\tilde{H}$ follows from a straightforward perturbation calculation:

$$E_n(\omega) = E_n^0 - \omega^2 \sum_m \frac{|(m|l_x|n)|^2}{E_m^0 - E_n^0} + \cdots \qquad 46$$

(the states φ_n^0, being eigenstates of $l_{z'}$, give $(n|l_x|n) = 0$). E_n is, however, not the energy of the system, which is given by

$$\langle E \rangle_n = E_n(\omega) + \omega \langle l_x \rangle_n = E_n^0 + \omega^2 \sum_m \frac{|(m|l_x|n)|^2}{E_m^0 - E_n^0} \qquad 47$$

From Eq. 43 and 47 we extract an effective moment of inertia

$$\Im_{\text{eff}} = 2 \sum_{n(\text{occ.})} \sum_m \frac{|(m|l_x|n)|^2}{E_m^0 - E_n^0} \qquad 48.^{12}$$

This is the basic equation of the model, at least in the original version of strictly independent particles. It is clear from the derivation, that if spin and spin-orbit coupling is taken into account, l_x is simply replaced by j_x in Eq. 48. We emphasize here the fact that a parameter of collective motion, the moment of inertia $\Im_{\text{eff}}$, is calculated on the basis of a strict independent particle picture, in which the only collective feature is the size and shape of the well that contains the particles. Of course, a self-consistent determination of the well parameter is understood here.

b) *Results for noninteracting particles.*—We first apply Eq. 48 to particles in a deformed harmonic oscillator well, Eq. 40, neglecting spin and spin orbit coupling (76, 83, 84).

The operator l_x has matrix-elements within an oscillator shell $n = n_1 + n_2 + n_3$:

$$(n_2 - 1, n_3 + 1 | l_x | n_2 n_3) = (i\hbar/2)\sqrt{n_2(n_3 + 1)} \frac{\Omega_1 + \Omega_3}{\sqrt{\Omega_1 \Omega_3}} \qquad 49a$$

[12] It is interesting to compare this equation with an exact formula for the moment of inertia, derived by Lipkin, de Shalit & Talmi (85).

Due to the deformations of the well, non-zero matrix elements of l_x between shells n and $n \pm 2$ appear, whose value is proportional to the deformation:

$$(n_2 + 1, n_3 + 1 \,|\, l_x \,|\, n_2 n_3) = (i\hbar/2)\sqrt{(n_2 + 1)(n_3 + 1)}\,\frac{\Omega_3 - \Omega_1}{\sqrt{\Omega_1 \Omega_3}} \qquad 49b.$$

It follows that the effective moment of inertia is given by:

$$\Im_{eff} = \frac{\hbar}{2}\left(\frac{\Omega_1 + \Omega_3}{\Omega_1 \Omega_3}\right)\left\{\sum_{occ.}(n_2 + n_3 + 1)\left(\frac{\Omega_3 - \Omega_1}{\Omega_3 + \Omega_1}\right)^2 + \sum_{occ.}(n_2 - n_3)\left(\frac{\Omega_3 + \Omega_1}{\Omega_3 - \Omega_1}\right)\right\} \qquad 50.$$

To have a reference value, we compare this with the moment of the mass distribution ("rigid-body"-moment):

$$\Im_{rig} = m\sum \langle y'^2 + z'^2 \rangle = \frac{\hbar}{2}\left(\frac{\Omega_1 + \Omega_3}{\Omega_1 \Omega_3}\right)\left\{\sum_{occ.}(n_2 + n_3 + 1) + \sum_{occ.}(n_2 - n_3)\left(\frac{\Omega_3 - \Omega_1}{\Omega_3 + \Omega_1}\right)\right\} \qquad 51.$$

We see that it is convenient to split the expression for $\Im_{eff}$ into two parts, a first part giving the contribution of the filled oscillator shells, and for which therefore $\sum(n_2 - n_3) = 0$, and a second part, giving the contribution of the nucleons occupying unfilled oscillator shells (and being responsible for the deformation). We see that the contribution of the first part is

$$\Im_1 = \left(\frac{\Omega_3 - \Omega_1}{\Omega_3 + \Omega_1}\right)^2 \Im_{rig} \cong \beta^2 \Im_{rig} \qquad 52.$$

To calculate the second part, we observe that from Eq. 42, it follows that at equilibrium deformation

$$\sum (n_2 - n_3) \cong \left(\frac{\Omega_3 - \Omega_1}{\Omega_3 + \Omega_1}\right)\sum(n_2 + n_3 + 1)$$

so that the contribution $\Im_2$ of the extra nucleons is

$$\Im_2 \cong \Im_{rig} \qquad 53.$$

The physical reason of this phenomenon is clear. In an unfilled shell, the perturbation ωl_x induces transitions between almost degenerate levels (energy difference $\hbar(\Omega_1 - \Omega_3)$), so that the dimensionless expansion parameter of the perturbation series $\omega/(\Omega_1 - \Omega_3)$ is not so small.[13] Such transitions are, of course, prohibited for filled shells by the exclusion principle. $\Im_1$ is therefore due entirely to virtual transitions into the overnext, empty, oscillator shell, with a corresponding energy jump $\hbar(\Omega_1 + \Omega_3)$.

This radically different behaviour of $\Im_1$ and $\Im_2$ sheds some light on the question of how a separation should be carried out in the "unified" model between core and extra nucleons. We come back to this in part III.3.

[13] Clearly, one is not allowed to go to the limit of a spherical potential here; for that eventuality, one has to handle the problem by the methods for degenerate systems; the energy shift is then $\propto \omega^2/(\Omega_1 - \Omega_3)^2 + \omega^2$ (86), so that no rotational spectrum occurs where the expansion is not permissible.

We now discuss the dependence of these results on the simplifying assumptions made to derive the moment of inertia formula: (*a*) Neglect of spin-orbit coupling; (*b*) Neglect of distinction between protons and neutrons (*c*) Unrealistic well shape; (*d*) Effective mass effects; and (*e*) Neglect of residual interactions.

It is quite obvious that *a*, *b*, and *c*, although certainly affecting the results as to detail, do not interfere with the basic mechanism which makes $\Im_1$ proportional to the square of the deformation, and $\Im_2$ of the order of the rigid body value, and these results survive therefore in any more realistic model improved according to *a*, *b*, and *c*. The situation is quite different with *d*. It is still questionable to which extent the effective mass concept may simply be inserted into such a model [Blin-Stoyle (87)]. One is tempted to insert a value of $m^* \cong m/2$ for the effective mass into the above calculations this would bring down the value of $\Im_2$ to the order of $\frac{1}{2}\Im_{\text{rigid}}$, agreeing rather well with the maximum observed values of $\Im_{\text{eff}}$ (15, 88). Finally, with regards to point *e*, it is obvious that processes which establish the value of $\Im_{\text{eff}}$ are very sensitive to any interparticle coupling. To see this, it is sufficient to assume any kind of "pairing energy" which has to be overcome in the virtual transition $n_2 n_3 \rightarrow n_2 \pm 1,\ n_3 \pm 1$, induced by the Coriolis forces (88). Very roughly, such a coupling has the effect of adding a constant term $\gamma\hbar\Omega_0$ to the small energy denominator, so that

$$\Im_2 \alpha \sum (n_2 - n_3) \frac{1}{\beta + \gamma}$$

With $\sum(n_2 - n_3) \alpha \beta$, the trend of $\Im_2(\beta)$ is now a linear increase with β for $\beta \ll \gamma$, stabilizing towards a value $\cong \Im_{\text{rigid}}$ for $\beta \gg \gamma$. This is indeed roughly the behaviour of $\Im_{\text{eff}}$ borne out by experiments [see (15, 88).] It is clear that a careful study of this effect, along with a critical use of the effective mass concept, opens up the possibility of a better determination of the degree of "independence" of the single particle motion in nuclei.

c) *Relation of Inglis model to exact problem.*—We have presented here the Inglis model as a means of (i) extracting the moments of inertia of rotational levels; (ii) studying the conditions under which rotational levels may be supposed to occur. Clearly the model lends itself to considerable refinement and appears to be a powerful tool to investigate the connection between collective behaviour and single particle dynamics. It is therefore desirable to have some insight into the relation between the model and the actual nucleus (36, 89). For this purpose, we make use of the transformed particle Hamiltonian (Eq. 24), together with Eq. 23, that is, the "hydrodynamic" value of Q_{AA}. The separation of core and extra nucleons is assumed to be made such as in part *b* of this chapter; the single-particle dynamics should then provide the contribution $\Im_2$ to the effective moment of inertia, whereas the core should give $\Im_1 \cong \Im_{\text{rig}}\beta^2$, which is indeed just the value of Q_{11}^{-1} and Q_{22}^{-1} for the axially symmetric (spheroidal) case (Q_{33}^{-1} being zero). In addition, the particular way of dividing the system allows the assumption that

an intrinsic wave function $\phi(\xi)$ of approximately zero angular momentum ($J'\phi(\xi)\cong 0$) may be factored off. Averaging over the intrinsic motion and assuming spheroidal symmetry of the mass distribution, we have:

$$\langle H\rangle_\phi = E_{00} + C\beta^2 + \frac{1}{2\Im_0\beta^2}\Big(L_\perp - \sum_r j_{r\perp}\Big)^2 + \sum_r \Big(\frac{p_r'^2}{2m} + U(x_r';\beta)\Big) + \sum_{r,s} V_{r,s} \qquad 54.$$

Notice that $(L_3 - \sum_r j_{r3})$ is a constant of motion in case of spheroidal symmetry, and takes the value zero only.

The connection with the Inglis model rests now upon the following assumptions: (*a*) The total angular momentum is considered as a classical vector. A quantum mechanical perturbation calculation [Lüders (89)] based on Eq. 54, shows that this approximation implies neglect of the energy $\hbar^2/2\Im_{\text{eff}}$, compared to the level splitting $\Delta E(\beta)$ (e.g., $\hbar(\Omega_3 - \Omega_1)$, for an oscillator potential). The approximation is justified for large deformations β, where the rotational excitation energies are low and L reaches large (classical) values. (*b*) The coupling between particles is neglected, and $U(x_r';\beta)$ is representative for the actual potential acting upon the r'th extra nucleon. This approximation becomes poor if the number of extra particles (that is, particles occupying nonfilled shells) is large; this is a general drawback of this unsymmetric method (and also of the Bohr-Mottelson "unified model"). But for the mere problem of understanding the fundamental connection between Eq. 54 and the Inglis model, we may always assume the number of extra nucleons to be small.

With *a* and *b*, we can now use as trial function a determinant of single particle functions, $\varphi_n(x_r)$, giving

$$\langle E\rangle = E_{00} + C\beta^2 + \frac{L_\perp^2}{2\Im_0\beta^2} + \sum_n \Big\langle \frac{p^2}{2m} + U(x;\beta)\Big\rangle_n + \frac{1}{2\Im_0\beta^2}\Big\{\Big(\sum_n \langle j_\perp\rangle_n\Big)^2 - 2L_\perp\cdot\sum_n \langle j_\perp\rangle_n - \sum_{\substack{n\\(000)}}\sum_{\substack{m\\(\text{empty})}} \langle j_\perp\rangle_{nm}\langle j_\perp\rangle_{mn}\Big\} \qquad 55.$$

Neglecting the very last term, and taking variational derivatives, we obtain the single particle equations:

$$\Big(\frac{p^2}{2m} + U(x;\beta)\Big)\varphi_n(x) - \Big(\frac{L_\perp - \sum_{n'}\langle j_\perp\rangle_{n'}}{\Im_0\beta^2}\Big)\cdot j_\perp\varphi_n(x) = E_n\varphi_n(x). \qquad 56.$$

This equation clearly has the structure of Eq. 45 of the Inglis model, with

$$\omega_\perp = \Big(L_\perp - \sum_{n'}\langle j_\perp\rangle_{n'}\Big)\Big/\Im_0\beta^2$$

Now, from the previous discussion it is clear that

$$\sum_n E_n(\omega) = \sum_n E_n(0) - \frac{1}{2}\omega^2\Im_2$$

and

$$\sum_{n'}\langle j_\perp\rangle_{n'} = \omega_\perp\Im_2$$

and hence

$$\omega_\perp = L_\perp/(\Im_0\beta^2 + \Im_2)$$

Inserting these results into Eq. 55, it follows that

$$\begin{aligned}\langle E\rangle &= E_{00} + C\beta^2 + \sum_n E_n(\omega) + \left(L_\perp{}^2 + \left(\sum_{n'}\langle j_\perp\rangle_{n'}\right)^2\right)\Big/ 2\Im_0\beta^2 \\ &= E_{00} + C\beta^2 + L_\perp{}^2/2(\Im_0\beta^2 + \Im_2)\end{aligned} \qquad 57.$$

This clearly shows that a solution of the *eigenvalue* problem of the transformed Hamiltonian, Eq. 24 subject to the approximations stated above gives a rotational moment of inertia

$$\Im_{\text{eff}} = \Im_0\beta^2 + \Im_2$$

that is, essentially the result of the Inglis model. But one also realizes that the Inglis method is superior to this present calculation, since a higher degree of self-consistency between particle configuration and potential shape is possible, there, due to the symmetric handling of all nucleons, which the present method cannot achieve. The unsymmetric splitting of the system into two parts handled by different methods is always unnatural. The appeal of the method of redundant variables becomes particularly apparent here.

d) *The Inglis model for collective vibrations.*[14]—By the same techniques as outlined in part 2a, we can handle the problem of a single particle potential which classically oscillates about a spherical equilibrium shape (90, 91). This then, is an alternative approach to the problem discussed in part II,3b where such vibrations are described in terms of the variables λ_μ. Let us observe here that by proper combination of modes λ_μ we may produce actual vibrations (which go through spherical shapes) and also "tidal waves," which actually are rotations; for these latter, nothing new can be said here and their "Inglis" treatment has been given in part III,2b. We discuss here the case of a vibration, $\mu=0$: in analogy to Eq. 43 we shall obtain a collective energy

$$E(\dot\lambda) = E(0) + \tfrac{1}{2}B\dot\lambda^2 + \cdots, \qquad 58.$$

an expression which defines the intertial parameter B; for an irrotational flow pattern we should have, according to Eq. 28, $B = mI_0 \approx m\langle\sum x^2\rangle$. Specializing to the harmonic oscillator case, with frequencies $\Omega_1(t)$ in the x, $\Omega_3(t)$ in the z direction, we write

$$\begin{aligned}\Omega_1(t) &= \Omega_0(1 + \sqrt{2/3}\lambda(t)) \\ \Omega^3(t) &= \Omega_0(1 - 2\sqrt{2/3}\lambda(t) + \cdots).\end{aligned}$$

This normalization makes λ identical with λ_0 of Eq. 26a. Introducing new variables: $x'(t) = (1+\sqrt{2/3}\lambda(t))^{1/2}x$, etc. and the corresponding momenta p',

[14] A method related to the one reported on here has been used by Araujo (92).

and also $\psi(x, y, z; t) = \varphi(x', y', z'; t')$, so that:

$$\frac{\partial \psi}{\partial t} = \frac{\partial \varphi}{\partial t} + \left(\frac{\partial \varphi}{\partial x'} \frac{\partial x'}{\partial t} + \cdots \right),$$

we get the Schroedinger equation for φ:

$$i\hbar \frac{\partial \varphi}{\partial t} = H^{(0)}(t)\varphi + \sqrt{1/6}\lambda D\varphi \tag{59.}$$

$$H^{(0)}(t) = (1 + \sqrt{2/3}\lambda)[(p_x'^2 + p_y'^2)/2m + 1/2m\Omega_0^2(x'^2 + y'^2)] + (1 - 2\sqrt{2/3}\lambda + \cdots)[p_z'^2/2m + 1/2m\Omega_0^2 z'^2] \tag{60.}$$

$$D = (x'p_x' + y'p_y' - 2z'p_z') \tag{61.}$$

It follows then that

$$E_n^{(0)}(t) = \hbar\Omega_1(t)(n_1 + n_2 + 1) + \hbar\Omega_3(t)(n_3 + 1/2)$$

and, by the same argument as used for rotations,

$$B = \frac{1}{3} \sum_{n(\text{occ.})} \frac{|(m|D|n)|^2}{E_m^0 - E_n^0} \tag{62.}$$

Now the only nonvanishing matrix-element of D are from n to $n \pm 2$, so that $E_n' - E_n = 2\hbar\Omega_1$ or $2\hbar\Omega_3$. It follows that

$$B = \frac{1}{3}\left(\frac{\hbar}{\Omega_0}\right)\left\{ \sum_{\text{occ.}} (2n_3 + 1) + \frac{1}{2} \sum_{\text{occ.}} (n_1 + n_2 + 1) \right)$$

$\rightarrow (\hbar/\Omega_0) \sum_{\text{occ.}} (n_1 + \frac{1}{2})$, for a closed shell configuration. This is exactly the required value.

It is clear that the above result does not sensitively depend upon shell closure. However, Mouhasseb (91) has observed that these results are profoundly modified by spin orbit coupling. The small value of B was due to the fact that the operator D had no nonvanishing matrix-elements within an oscillator shell. This is no longer so if an $(\mathbf{l} \cdot \mathbf{s})$-coupling: $-2\kappa(\hbar\Omega_0)(\mathbf{l} \cdot \mathbf{s})$ splits the levels of a shell, with the result that nonclosed shells give a contribution of order κ^{-2}. In this way it is again possible for B to exceed the hydrodynamic value mI_0 by a large factor, since κ is $\ll 1$ (0.05 in the case of Nilsson's potential).

As a final remark we observe that the validity of the expansion (Eq. 58), depends on the smallness of $\dot{\lambda}/\Omega_0$. This condition is certainly not fulfilled for closed oscillator shells (90), but becomes more nearly satisfied as the core is "softened" by the addition of nucleons, the more as then $\dot{\lambda}$ is reduced by the increase of B.

3. The "Unified" Model of Bohr-Mottelson

a) Introduction.—This model is based on the attempt to describe the dynamical interplay between "valence nucleons" and the nuclear "core" (here understood as any essentially spherically symmetric structure with ground state spin $I = 0$) in a simple and transparent way. The core is there-

fore replaced first by a continuum, a "liquid drop," and whose lowest modes of excitation are therefore independent of the particle structure. It is now well-known that this is a too drastic assumption; we shall come back to this point again. Specifically, these modes of excitation are assumed to be "phonons," corresponding to an irrotational, incompressible flow pattern about a spherical equilibrium shape. One has then a velocity field satisfying $\mathbf{v}(x) = -\nabla\chi(x)$; $\nabla^2\chi = 0$. The regular solutions of these equations are the harmonic polynomials; as simplest case we consider

$$\chi(r, \theta, \varphi) = -\frac{1}{2} r^2 \sum_m \dot{\alpha}_m Y_{2m}(\theta, \varphi), \tag{63}$$

corresponding, for small values of $\alpha (\alpha \ll 1)$, to a nuclear boundary

$$R(\theta, \varphi) = R_0 \left(1 + \sum_m \alpha_m Y_{2m}(\theta, \varphi)\right) \tag{64}$$

Kinetic energy and angular momentum of the vibrating drop are

$$T = \frac{1}{2} \int d\tau \rho (\nabla\chi)^2 = \frac{1}{2} B_2 \sum_m \dot{\alpha}_m^* \dot{\alpha}_m \tag{65a}$$

$$\vec{L}_c = \int d\tau \rho (r \times \nabla\chi) = i B_2 \sum_{m,m'} \alpha_m^* \dot{\alpha}_{m'} \vec{M}_{m,m'} \tag{65b}$$

with

$$B_2 = \frac{1}{2} \rho_0 R_0^5 = \frac{3}{8\pi} m A R_0^2 \tag{66}$$

and $\mathbf{M}_{mm'}$, the angular momentum matrix-elements defined in part II.3b. The potential energy is of type

$$\frac{1}{2} C_2 \sum_m \alpha_m^* \alpha_m. \tag{67}$$

C_2 may be taken from the empirical surface energy (93, 94), but it is more prudent to keep it as an adjustable parameter. The dynamics of Eq. 65, 66, and 67 has already been discussed in part II.3. In general, then, the nucleus must be divided into a "core," described as above, and extra nucleons. How this division has to be effected is not so clear from the outset. Making use of our experience, we can now say that all nucleons (or holes in unfilled subshells, thus contributing to the deformation pressure, should be treated as "extra" particles. The core, by this criterion, need not be magic as closed subshells may be included, as long as they remain fully occupied at the deformation in question.

To describe the coupling of the extra nucleons to the liquid drop it is assumed that shape of the drop closely coincides with shape of the potential for the extra nucleons. This potential is therefore of type

$$U(r, \theta, \varphi) = U_0(r/R(\theta, \varphi)) = U_0(r/R_0) - \left(r \frac{dU_0}{dr}\right) \sum_m \alpha_m Y_{2m}(\theta, \varphi) + \cdots \tag{68}$$

The Hamiltonian is then the sum of the liquid drop (oscillator) Hamiltonian, the particle Hamiltonian and a coupling, given by the last term of Eq. 68. The dynamics of this coupled system is relatively simple in two limiting situations: (*a*) Weak coupling case (nucleus in neighbourhood of closed shells; one or a few extra nucleons). The interactions between drop oscillation and nucleon motion may be described as a perturbation. The amplitude of the 1 phonon component of the ground state wave function is small, and as a consequence, the expectation value of $\langle\alpha\rangle$ (the equilibrium deformation) is small compared to the zero point amplitude of the drop:

$$x = \frac{\langle\alpha\rangle}{\sqrt{\langle\alpha^2\rangle}} \sim \frac{\left\langle r \dfrac{dU}{dr} \right\rangle}{\sqrt{\hbar\omega C}} \ll 1 \qquad 69.$$

(*b*) Strong coupling case (medium and heavy nuclei well between major shell closings: $150 < A < 190$ $A \gtrsim 225$). The combined effects of extra nucleons lead to a value $x \gg 1$, so that we can speak of a nonspherical equilibrium shape. Rotational spectra.

b) *Pure collective excitation.*—For even nuclei in the weak coupling region there is the temptation to throw all nucleons in the core, thus approximating the system by a pure collective oscillator. The resulting pure collective excitations should then be characterized by:

(i) the level spectrum:

$$E_n = (n + 5/2)\cdot\hbar\sqrt{\frac{C}{B}} \qquad 70.$$

(ii) the angular momenta I_n:

$$I_0 = 0, \qquad I_1 = 2, \qquad I_2 = 0, 2, 4 \text{ (degenerate)}$$

(iii) the electromagnetic transitions, which are of pure E_2 type, with the selection rules $|\Delta I| \leq 2, \Delta n \pm 1$. This last rule prevents cross-over transitions from the second excited into the ground state. (iv) the absolute transition rates. The first excited state has a reciprocal lifetime.

$$\tau^{-1} = \frac{3}{200\pi}\frac{Z^2}{137}\omega\left(\frac{R_0}{\lambda}\right)^4 \frac{\hbar}{\sqrt{C_2 B_2}} \qquad 71.$$

As a possible application of this scheme we consider the medium weight nuclei in the range $22 < N < 90$, and the heavy group $114 \leq N \leq 134$ (centreed around the double magic $N = 126$, $Z = 82$). Way, *et al.* (26) have pointed out the regularity in the behaviour of the members of this group (except at N or Z magic): (i) The ratio of excitation energies E_2/E_1 is ≈ 2.2, and E_1 is of order 60 Mev/A (75 Mev/A) for the medium (heavy) members of the group. (ii) The sequence of angular momenta and parities is

$$0^+ - 2^+ \begin{cases} 2^+, 4^+ \text{ (medium)} \\ 4^+ \text{ (heavy)} \end{cases}$$

(iii) The transitions are predominantly E_2 (51, 95) and strongly enhanced; but the cross-over transition ($E_2 \rightarrow E_0$) not nearly as much as the stop-over ($E_2 \rightarrow E_1$), indicating a relative "forbiddeness" of the former (96) (rule of Kraushaar & Goldhaber). (iv) The transition $E_1 \rightarrow E_0$ has a favoured factor F of order 10 to 40 (95).

The evidence certainly speaks in favour of a degree of truth in the picture. Two observations may be added:

First: Points ii and iii can be taken care of without invoking the collective model at all (96); iii, in particular, by invoking the seniority selection rule. On the other hand, i and iv do not follow naturally from pure shell model considerations.

Second: The numerical values entering i and iv do not fit the simple collective picture either. [For an attempt to interpret the situation through an interpolation between weak and strong coupling description, see Wilets & Jean (97).] Calculations of C_2 and B_2 from experimental data by i and iv give values of B_2 systematically an order of magnitude too large, compared with Eq. 66. The values of C_2 fluctuate about the hydrodynamic value, showing strong correlation with shell closure.

We must conclude from this that the simple collective oscillator is too crude a picture to describe the low-lying levels of the even nuclei in the group considered here. The experimental values of B_2 indicate considerable departure from irrotational flow, indicating that the nucleons in unifilled shells should be described as individuals. It is such a hybrid description (51) which gets to the data with the least amount of complexity.

c) *Weak coupling situation.*—The cases considered here are mainly those for which the usual shell-model picture gives a reasonable approximation. The lowest excited states are definitely particle excitations, and their excitation energy is less than $\hbar\omega$. The weak coupling picture is thus essentially a refinement of the description of these states. The main effects of the coupling to the collective oscillation appear in the modifications of magnetic moments, electric quadrupole moments and lifetimes of excited states. The problem of a single odd nucleon, coupled to the collective modes, has been studied by Foldy & Milford (98), Kerman (99), Ford & Levinson (100), the weak coupling situation in the neighbourhood of Pb^{208} by True (101), and in light nuclei, for the Ca-isotopes, by Levinson & Ford (102).

The most pronounced corrections occur in the quadrupole moments and transition rates. Calling a_0 and a_1 the amplitudes of the zero- and one-phonon components of the wave function, $(a_0^2 + a_1^2 = 1)$, we have for the correction to the quadrupole moment

$$Q_{\text{coll}} = \pm \frac{6Z}{\sqrt{5\pi}} R_0^2 \sqrt{\frac{\hbar^2/2B_2}{\hbar\omega}} \sqrt{\frac{j(2j-1)}{(j+1)(2j+3)}} \, |a_1| \qquad 72.$$

(+ for hole, − for particle), the important point being linearity in a_1 and proportionality to Z.

The magnetic moment correction is proportional to $|a_1|^2$ and thus considerably less important in the weak coupling approximations, $|a_1|^2 \ll 1$:

$$\mu = \mu_{s.p.} + \frac{|a_1|^2}{2(j+1)}[(g_c - g_{l'j'})(j(j+1) - j'(j'+1) + 6) + 2(g_{l'j'} - g_{lj})j(j+1)] \quad 73.$$

(l', j' being the particle angular momenta in the one phonon component). These corrections are in general much too small to enable the weak surface coupling to account for the observed deviations from the Schmidt Lines [for presentation and discussion of the experimental material, see Blin-Stoyle (103)]. Other mechanisms, for the weak coupling region, in particular configuration mixing induced by the direct internucleon forces, is presumably the main agent for the magnetic moment deviations (104).

Finally, γ-transitions of E_2 type are enhanced considerably even for very weak coupling. Transition probabilities have been discussed by Bohr & Mottelson (16), Ford & Levinson (100), Barker (105). [For experimental data, see (25, 95).]

Odd nuclei are characterized by great fluctuations in the degree of "anomaly" they represent, as measured by magnetic moment deviation, favoured factors in γ-transitions, etc. One is tempted to take this as an indication for a rather complex situation, with the nature of the states mainly determined by the internucleon forces. The fact that collective effects dominate certain γ-transitions and quadrupole moments should not be taken as an indication that dynamically phonon excitation is the essential element of the description.

d) *Strong coupling situation.*—In this case the combined effect of a sufficiently large number of extra nucleons results in a nonspherical equilibrium shape of the drop. It is then more appropriate to express the energy of the collective flow in terms of new variables, namely the orientation of the principal axes of the deformed drop (of ellipsoidal shape) in terms of three Euler angles θ_s, and the deformations β_A, along the three principal axes:

$$R_A = R_0\left(1 + \sqrt{\frac{5}{4\pi}}\,\beta_A\right); \qquad \beta_A = \beta\cos(\gamma - 2\pi A/3)$$

θ_s, β and γ are the five new independent collective variables. The transformation to these new variables is reminiscent of the techniques of strong coupling field theory, and has been carried out in this fashion by Wentzel (38) and Coester (47). The collective energy (Eq. 65a, 67) becomes

$$H_C = \frac{1}{2}B_2(\dot\beta^2 + \beta^2\dot\gamma^2) + \frac{1}{2}C_2\beta^2 + \sum_A \frac{L_{C,A}^2}{8B\beta^2\sin^2\left(\gamma - \frac{2\pi}{3}A\right)} = H_{\text{vibr.}} + H_{\text{rot.}} \quad 74.$$

$L_{C,A}$ being the projection of the collective angular momentum L_C on the principal axis A.

To H_C has to be added the Hamiltonian of the extra nucleons. If the description of these extra particles is given in the body fixed frame of the liquid drop, we have (see part II for details):

$$H = \text{vibr.} + \sum_{rA}\left(\frac{p_r^2}{2m} + U(x_r;\beta,\gamma)\right) + \sum_A \frac{\left(L_A - \sum_r j_{r,A}\right)^2}{8B\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}A\right)} \qquad 75.$$

L_A represents now the total angular momentum; it is the same operator on the θ_s as was L_{CA} before, and given by Eq. 19.

The close resemblance between Eq. 75 and 24 is obvious.[15] It must be made clear, however, that Eq. 24 does not provide a justification for 75 and therefore of the model, with its assumption of irrotational in flow. This is true despite the fact that the same zero order moments appear both equations, since

$$m\frac{(I_B - I_C)^2}{I_B + I_C} \cong 4B_2\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}A\right)$$

These moments have nothing to do with the measured moments as long as the collective rotation is not dynamically decoupled from the intrinsic motion. Now the degree of decoupling is certainly not the same in the two equations. In fact, the assumption of irrotational flow, leading to Eq. 75 can be expressed in Eq. 24 by the condition

$$J_A'\phi(\xi_\sigma, x_{rA}) = 0 \qquad 76.$$

an equation which has no reason to hold in general, except perhaps for a magic core. Except for such a case, in which Eq. 76 holds at least approximately, the model does not appear to be justified from general principles.

The solutions of the *eigenvalue* problem of the Hamiltonian (Eq. 75) have been considered by Bohr (50), Bohr & Mottelson (16), Ford (106), Marty (37) and others.

Let us assume here that the system is γ-stable [γ-unstable situations have been considered by Jean & Wilets (97)], and stabilize at $\gamma = 0$ (prolate) or $\gamma = \pi$ (oblate), which appears to be the predominant case. We have then cylindrical symmetry, and the operators L^2, L_z, $L_3 - \sum_r j_{r,3}$ are constants of the motion, the latter having to be identically zero. In lowest approximation it is then generally assumed that L_3 and $\sum_r j_{r,3}$ are separately con-

[15] It is, of course, possible to derive Eq. 75 exactly from the Hamiltonian Eq. 54; in that case, one should keep in mind all the approximations and assumptions that have gone into the derivation of this latter equation, especially the assumption $J'\phi_{intr} \cong 0$. In contrast to this, Eq. 24 is an exact equation, and it would be easy there to isolate the collective vibrations too. However, it is not important for the questions to be discussed here.

stants, with *eigenvalues* K and $\Omega = K$ respectively. In this approximation the rotational energy is decoupled, and has the value

$$\left\langle \frac{1}{8B\beta^2} \right\rangle (I(I+1) - K^2); \qquad I \geq K; \qquad \Omega = K \neq 1/2. \tag{77.}$$

[For the special case $\Omega = K = \frac{1}{2}$, see (15, 16)].

The corresponding zero order wave functions are

$$\Psi^{(0)} = v(\beta, \gamma)(\mathfrak{D}_{MK}^{I*}(\theta_s)\psi_K(x_r) + (-1)^{I-\Sigma ir}\mathfrak{D}_{M,-K}^{I*}(\theta_s)\psi_{-K}(x_r)) \tag{78.}$$

with $I = K$ for the ground state. The symmetrization of the wave function, necessary because of the ambiguity of the definition of the body fixed axes, has been discussed in detail by Bohr (50) and by Marty (37). For even nuclei, with ground states $\Omega = K = 0$, Eq. 78 restricts the values of I to 0, 2, 4, $\cdots$.

With $\Psi^{(0)}$, electromagnetic moments and transition-probabilities follow in a straightforward way. Calling $\sum \mathbf{j}_r \equiv \mathbf{J}$, and g_c the g-factor associated with collective motion, one has

$$\vec{\mu} = g_c(\mathbf{L} - \mathbf{J}) + \mu_{\text{intr}}$$
$$\mu_{\text{intr}} = \sum (g_l \mathbf{l} + g_s \mathbf{s})$$

which, in the above approximation, except for $K \approx \frac{1}{2}$, gives a magnetic moment

$$\mu = (\mu \cdot \mathbf{L})/I + 1 = g_c \frac{I}{I+1} + g_\Omega \frac{I^2}{I+1} \tag{79a.}$$

g_Ω being defined as

$$\sum \langle g_l l_3 + g_s s_3 \rangle / I. \tag{79b.}$$

The electric quadrupole moment is essentially due to the core deformation. It is given, in the body fixed system, by

$$Q_0 = \frac{3}{\sqrt{5\pi}} Z R_0^2 \langle \beta \cos \gamma \rangle$$

The operator of the spectroscopic quadrupole moment is then

$$Q = P_2(\cos \theta) Q_0$$

and

$$\langle Q \rangle = \langle P_2(\cos \theta) \rangle_{I,M=I} Q_0 = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0 \tag{80.}$$

Finally, we mention the $E2$-transition probabilities, which, due to the predominance of the collective contribution, are directly expressible in terms of Q_0

$$B_{E2}(I_i \to I_f) = \frac{5}{16\pi} l^2 Q_0^2 (I_i 2KO \mid I_i 2 I_f K)^2 \tag{81.}$$

TABLE I

QUADRUPLE MOMENTS AND MOMENTS OF INERTIA IN THE REGION $150 < A < 190$ AND $A > 225$

Nucleus	I_0	$\lvert Q_0 \rvert$ (barns)*	$3\hbar^2/\mathfrak{I}$(kev)†	β_B‡	$\beta_E{}^2/\beta_B{}^2$§
Sm^{152}		5.7	122	0.28	5
Eu^{153}	5/2	7.7	71		
Sm^{154}		6.7	83	0.33	4.6
Gd^{154}		6.3	123	0.30	
Gd^{155}	3/2	8.0	72		
Gd^{156}		8.8	89	0.41	
Gd^{157}	3/2	7.7	66		
Gd^{158}		10	79	0.46	
Tb^{159}	3/2	6.9	70		
Gd^{160}		10	76	0.47	
Dy^{160}		7.8	86	0.35	4.8
Dy^{162}		8.2	82	0.36	
Dy^{164}		9.5	73	0.41	
Er^{164}		7.8	90	0.33	5.4
Ho^{165}	7/2	7.8	63		
Tm^{169}	1/2	8.9	74		
Yb^{170}		7.5	84	0.30	5.6
Lu^{175}	7/2	8.2	76		
Hf^{176}		7.5	89	0.29	5.5
Hf^{177}	7/2	7.5	75		
Hf^{178}		8.1	91	0.31	4.0
Hf^{179}	9/2	7	66		
Hf^{180}		7.1	93	0.27	6.0
Ta^{181}	7/2	6.8	91		
W^{182}		7.1	100	0.26	6.5
W^{183}	1/2	—	78		
W^{184}		6.5	112	0.24	
Re^{185}	5/2	5.4	108		
W^{186}		6.5	124	0.24	
Os^{186}		5.5	137	0.20	6.2
Re^{187}	5/2	5.0	116		
Os^{188}		5.1	155	0.18	6.6
Th^{232}		10	52	0.25	3.4
U^{233}	5/2	14	35		
U^{235}	7/2	9	31		
N^{237}	5/2	9	28		
U^{238}		11	44	0.28	3.4
Pu^{239}	1/2	8.3	37		

These data are copied from Ref. 15, except for the last column, which are taken from Ref. 26.

* The values of Q_0 are obtained either from Coulomb-exictation cross sections [Heydenberg & Temmer (25)], or from fast $E2$-transition probabilities [Sunyar (109)].

B_{E2} being the reduced $E2$-transition probability (93), and the last factor in Eq. 81 the square of a vector addition coefficient

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m)$$

In assessing the predictions of the strong coupling model, we must clearly distinguish between two classes of statements. There are statements which mainly derive from the mere fact of the existence of a rotational spectrum, without strongly involving other aspects of the dynamics. Such statements are: (*a*) Ratios of $E2$-transition probabilities within a rotational band (107), (*b*) Selection rules, (*c*) Ratios of transition probabilities to the states of a rotational band in α-decay (31, 108), (*d*) Relation between Q_0 and B_{E2}.

These statements only invoke the assumptions of the existence of a sufficiently strong equilibrium deformation and the adiabatic slowness of collective rotation, which guarantee the equivalence of the intrinsic structure for all levels of a rotational band. The agreement between observed and predicted values is indeed good. Opposed to these results are "sensitive" statements such as: (*a*) Moments of inertia, (*b*) Magnetic dipole moments, (*c*) Magnetic transistions. Here the basic assumptions of the model enter in a critical way, and even a state which actually diagonalizes the model Hamiltonian (Eq. 75) may not be adequate, for the reasons stated previously.

We conclude this section by giving, in form of tables, some experimental data to illustrate these statements.

4. Conclusion

Trying to assess the contribution which the collective point of view has made to our understanding of nuclear dynamics one is impressed by two facts: (*a*) The essential simplicity of the collective dynamics, which

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It is seen that the values of Q_0 for $150 < A < 190$ follow a regular pattern, increasing rapidly to a maximum at $A \simeq 160$, then decreasing slowly; there is no marked difference between the even and odd nuclei. This behaviour is explained almost quantitatively by Nielsson's model (part III.1b).

† The values of the moment of inertia $\Im$ are strongly correlated with the value of Q_0, that is β. Moments of inertia of odd nuclei are systematically larger than those of neighbouring even nuclei, a feature is qualitatively understood on the basis of the Inglis model with residual interaction of the type of a pairing energy.

‡ β_B is the deformation calculated from Q_0: $Q_0 \cong \sqrt{9/5\pi} Z R_0^2 \beta$; assuming $R_0 = 1.2 A^{1/3} \times 10^{-13}$ cm.

§ β_E^2 is defined by the $\Im_{exp} = 3B_2\beta_E^2$; $B_2 = 3/8\pi\ MAR_0^2$ (Eq. 66); β_E is the (fictitious) deformation, which, in connection with the model of irrotational flow, gives the observed moments of inertia. The values of this column illustrate the inadequacy of the hydrodynamic description of the core. In comparison, the rigid body moments are a better approximation.

TABLE II

MAGNETIC MOMENTS IN REGION $150 < A < 190$

Nucleus	Odd part	I_0	μ_{exp}	g_c*	g_Ω*	$g_\Omega{}^{th}$†	μ^{th}‡
Eu^{153}	$Z = 13$	5/2	1.5	0.5	0.6	0.23	0.69
Tb^{159}	$Z = 65$	3/2	1.5	0.1	1.6	2.3	2.34
Lu^{175}	71	7/2	2.6	0.3	0.9	1.54	4.5
Ta^{181}	73	7/2	2.1	0.25	0.70	0.42	1.46
Re^{185}	75	5/2	3.14	0.5	1.6	1.64	3.22
Re^{187}	75	5/2	3.18	0.5	1.6	1.64	3.22
Gd^{155}	$N = 91$	3/2	−0.3	0.28	−0.53	−0.34	−0.07
Gd^{157}	93	3/2	−0.37	0.27	−0.61	−1.06	−0.71
Hf^{177}	105	7/2	0.61	0.21	0.06		
Hf^{179}	107	9/2	−0.47	0.2	−0.2		

* These experimental values of g_c and g_Ω are taken from Eq. 79a for μ, and the expression for the MI-transition-probability, which gives $(g_c - g_\Omega)^2$, [see (15, 25)]. There are two sets of solutions, and we give here the one which agrees better with the theoretical values.

† This is a value calculated by Gottfried, on the basis of Eq. 79b, considering the last odd nucleon only.

‡ This is the theoretical moment on the basis of Eq. 79a, with Gottfried's values for g_Ω, and $g_c = 0.4$.

gives energy levels, moments, etc., in terms of very few parameters; and the accuracy with which this collective picture reproduces some data. (*b*) The fact is that we have today only the rudiments of a really unified description of single particle and collective motion. The main point here is to recognize the total inadequacy of the hydrodynamic approximation for the nuclear core, and the promise held out to be able, eventually, to calculate the parameters of the collective picture (deformations, moments of inertia, *g*-factors) on the basis of the independent particle dynamics.

The main result to be taken from our present insight into the situation is in fact the recognition of the crucial role, which a correct description of the "independent" particle dynamics (including inter-particle coupling) plays for the quantitative understanding of the collective motion.

Although such a synthesis of the collective and the particle aspect of nuclear dynamics is rather easily achieved in words, by simply combining results borrowed from various models, a decent mathematical formulation of the same programme is far from easy. We venture here the opinion that the method of "redundant variables" holds the greatest promise to achieve this goal, but the work is still very much in its beginnings.

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LITERATURE CITED

1. Mayer, M. G., *Phys. Rev.*, **75**, 1969 (1949)
2. Haxel, O., Jensen, J. H. D., and Suess, H. E., *Phys. Rev.*, **75**, 1766 (1949)
3. Rainwater, J., *Phys. Rev.*, **79**, 432 (1950)
4. Feshbach, H., Porter, C., and Weisskopf, V. F., *Phys. Rev.*, **90**, 166 (1953)
5. Friedman, F. L., and Weisskopf, V. F., *Niels Bohr and the Development of Physics*, 134–162 (Pauli, W., Ed., Pergamon Press, Ltd., London, England, 195 pp., 1955)
6. *Statistical Aspects of the Nucleus*, Brookhaven National Laboratory Report, BNL 331 (C-21) (1951)
7. Flowers, B. H., *Progr. in Nuclear Phys.*, **2**, 235 (1952)
8. Pryce, M. H. L., *Repts Progr. in Phys.*, **XVII**, 1 (1954)
9. Jensen, J. H. D., *Beta and Gamma Ray Spectroscopy*, pp. 414–432 (K. Siegbahn, Ed., North Holland Publishing Co., Amsterdam, Holland, 959 pp., 1955)
10. Mayer, M. G., and Jensen, J. H. D.: *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, New York, N. Y., 284 pp., 1955)
11. Feenberg, E., *Shell Theory of the Nucleus* (Oxford University Press, London, England, 222 pp., 1955)
12. Hill, D. L., and Wheeler, J. A., *Phys. Rev.*, **89**, 1102 (1953)
13. Bohr, A., *Rotational States of Atomic Nuclei*, Doctoral Thesis (Ejnar Munksgaard Forlag, Copenhagen, Denmark, 1954)
14. Bohr, A., and Mottleson, B. R., in *Beta and Gamma Ray Spectroscopy*, 468–493 (K. Siegbahn, Ed., North Holland Publishing Co., Amsterdam, Holland, 959 pp., 1955)
15. Alder, K., Bohr, A., Huus, T., Mottleson, B. R., and Winther, A., *Rev. Mod. Phys.*, **28**, 432 (1956)
16. Bohr, A., and Mottelson, B. R., *Kgl. Danske Videnskab. Selskab, Mat. fys. Medd.*, **27**(16) (1953)
17. Rotenberg, M., *Phys. Rev.*, **100**, 439 (1955)
18. Bethe, H. A., *Phys. Rev.*, **103**, 1353 (1956)
19. Brueckner, K. A., and Levinson, C. A., *Phys. Rev.*, **97**, 1344 (1955)
20. Eden, R. J., and Francis, N. C., *Phys. Rev.*, **97**, 1366 (1955)
21. Bethe, H. A., and Goldstone, J., *Proc. Roy. Soc.* (London), **A238**, 551 (1957)
22. Brueckner, K. A., Eden, R. J., and Francis, N. C., *Phys. Rev.*, **99**, 76 (1955); **100**, 891 (1955)
23. Bell, J. S., Eden, R. J., and Skyrme, T. H. R., *Nuclear Phys.*, **2**, 586 (1957)
24. Bohr, A., *Phys. Rev.*, **81**, 134 (1951)
25. Heydenburg, N. P., and Temmer, G. M., *Ann. Rev. Nuclear Sci.*, **6**, 77 (1956)
26. Way, K., Kundu, D. N., McGinnis, C. L., and van Lieshout, R., *Ann. Rev. Nuclear Sci.*, **6**, 129 (1956)
27. Davidson, J. P., and Feenberg, E., *Phys. Rev.* **89**, 856 (1953)
28. Bohr, A., and Mottelson, B. R. (Private communication)
29. Bohr, A., in *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy*, Vol. **2**, 151 (United Nations, New York, N. Y., 471 pp., 1956)
30. Wheeler, J. A., in *Niels Bohr and the Development of Physics*, 163–184. (Pauli, W., Ed. Pergamon Press, Ltd., London, England, 195 pp., 1955)
31. Goldin, L. L., Novikova, G. I., and Tretyakov, E. F., *Phys. Rev.* **103**, 1004 (1956)
32. Johansson, S. A. E., *Phys. Rev.*, **105**, 189 (1957)

33. Griffin, J., and Wheeler, J. A. (To be published)
34. Peierls, R. E., and Yoccoz, J. (To be published)
35. Yoccoz, J. (To be published)
36. Villars, F., *Nuclear Phys.*, **3,** 240 (1957)
37. Marty, C., *Nuclear Phys.* **1,** 85 (1956)
38. Wentzel, G., *Lecture on Strong Coupling Meson Theory,* University of Rochester, 1954 (Unpublished data)
39. Elliott, J. P., and Skyrme, T. H. R., *Proc. Roy. Soc. (London),* **A232,** 561 (1955)
40. Skyrme, T. H. R. (To be published)
41. Edmonds, A. R., *Angular Momentum in Quantum-Mechanics, CERN-55-26,* Geneva (1955)
42. Redlich, M. G., and Wigner, E. P., *Phys. Rev.*, **95,** 122 (1954)
43. Nataf, R., *Compt. rend.*, **240,** 2510 (1955); **241,** 31 (1955)
44. Tomonaga, S., *Progr. Theor. Phys.*, **13,** 467 (1955)
45. Coester, F., *Nuovo Cimento,* **IV**(10) 1307 (1956)
46. Miyazima, T., and Tamura, T., *Progr. Theor. Phys.*, **15,** 255 (1956)
47. Coester, F., *Phys. Rev.*, **99,** 170 (1955)
48. Nielsen, H. H., *Rev. Mod. Phys.*, **23,** 90 (1951)
49. Kronig, R. de L., *Band Spectra and Molecular Structure* (Cambridge Univ. Press, London, England, 163 pp., 1930)
50. Bohr, A., *Kgl. Danske Vibenskal. Selskab Mat.fys Medd.*, **26** (14) (1952)
51. Scharff-Goldhaber, G., and Weneser, J., *Phys. Rev.*, **98,** 212 (1955)
52. Lipkin, H. J., de Shalit, A., and Talmi, I., *Nuovo Cimento* **2**(10), 773 (1955)
53. Tamura, T., *Nuovo Cimento* **4**(10) 713 (1956)
54. Skinner, R. S., *Can. J. Phys.*, **34,** 901 (1956)
55. Watanabe, Y., *Progr. Theor. Phys.*, **16,** 1 (1956)
56. Lipkin, H. J. (To be published)
57. Nataf, R., *Nuclear Phys.*, **2,** 497 (1957)
58. Süssmann, G., *Z. Physik,* **139,** 543 (1954)
59. Marumori, T., Yukawa, J., and Tanaka, R., *Progr. Theor. Phys.*, **13,** 442 (1955)
60. Marumori, T., and Yamada, E., *Progr. Theor. Phys.*, **13,** 557 (1955)
61. Green, A. E. S., Lee, K., and Bukley, R. J., *Phys. Rev.*, **104,** 1625 (1956)
62. Berg, R. A., and Wilets, L., *Phys. Rev.*, **101,** 201 (1956)
63. Brueckner, K. A., *Phys. Rev.*, **103,** 1124 (1946)
64. Wilets, L., *Phys. Rev.*, **101,** 1805 (1956)
65. Mittelstaedt, P., *Z. Naturforsch.*, **10a,** 379 (1955)
66. Wildermuth, K., *Z. Naturforsch.*, **9a,** 1047 (1954)
67. Swiatecki, W. J., *Phys. Rev.*, **98,** 204 (1955)
68. Ford, K. W., and Hill, D. L., *Ann. Rev. Nuclear Sci.*, **5,** 25 (1955)
69. Weisskopf, V. F. (Privately Circulated Manuscript), Seattle Conference (1956)
70. Brueckner, K. A., and Wada, W., *Phys. Rev.*, **103,** 1008 (1956)
71. Miyatake, O., and Goodman, C., *Phys. Rev.*, **99,** 1040 (1955)
72. Feenberg, E., *Rev. Mod. Phys.*, **19,** 239 (1947)
73. Wilets, L., Hill, D. L., and Ford, K. W., *Phys. Rev.*, **91,** 1488 (1953)
74. Pfirsch, D., *Z. Physik,* **132,** 409 (1952)
75. Gallone, S., and Salvetti, C., *Nuovo Cimento,* **10**(9), 145 (1953)
76. Moszkowski, S. A., *Phys. Rev.*, **103,** 1328 (1956)
77. Nilsson, S. G., *Kgl. Danske Videnskab. Selskab Mat fys. Medd.*, **29**(16) (1955)
78. Moszkowski, S. A., *Phys. Rev.*, **99,** 803 (1955)

79. Uretzky, J., *Deformed Nuclei and the Finite Square Well* (Doctoral Thesis, Massachusetts Institute of Technology, Cambridge, Mass. 1956)
80. Gallone, S., and Salvetti, C., *Nuovo Cimento*, **8**(9) 970 (1951)
81. Gottfried, K., *Phys. Rev.*, **103**, 1017 (1956)
82. Mottelson, B. R., and Nilsson, S. G., *Phys. Rev.*, **99**, 1615 (1955)
83. Inglis, D. R., *Phys. Rev.*, **96**, 1059 (1954)
84. Inglis, D. R., *Phys. Rev.*, **103**, 1786 (1956)
85. Lipkin, H. J., de Shalit, A., and Talmi, I., *Phys. Rev.*, **103**, 1773, (1956)
86. Valatin, T. J., *Proc. Roy. Soc. (London)*, **A238**, 132 (1956)
87. Blin-Stoyle, R. J., *Nuclear Phys.*, **2**, 169 (1956)
88. Bohr, A., and Mottelson, B., *Kgl. Danske Videnskab. Selskab Mat fys. Medd.*, **30**(1) (1955)
89. Lüders, G., *Z. Naturforsch.*, **111**, 617 (1956)
90. Inglis, D. R., *Phys. Rev.*, **97**, 701 (1955)
91. Mouhasseb, A., *Compt. Rend.*, **243**, 1289 (1956)
92. Araujo, J. A., *Nuclear Phys.* **1**, 259 (1956)
93. Blatt, J. M., and Weisskopf, V. F., *Theoretical Nuclear Physics* (John Wiley and Sons, New York, N. Y., 864 pp., 1952)
94. Green, A. E. S., *Phys. Rev.*, **95**, 1006 (1954)
95. de Shalit, A., and Goldhaber M., *Phys. Rev.*, **92**, 1211 (1953)
96. Kraushaar, J. J., and Goldhaber, M., *Phys. Rev.* **89**, 1081 (1953)
97. Wilets, L., and Jean, M., *Compt. Rend. Acad. Sci. (Paris)*, **241**, 1108 (1955); *Phys. Rev.*, **102**, 788 (1956)
98. Foldy, L. L., and Milford, F. J., *Phys. Rev.*, **80**, 751 (1950)
99. Kerman, A., *Phys. Rev.*, **92**, 1176 (1953)
100. Ford, K. W., and Levinson, C. A., *Phys. Rev.*, **100**, 1 (1955)
101. True, W. W., *Phys. Rev.*, **101**, 1342 (1956)
102. Levinson, C., and Ford, K. W., *Phys. Rev.* **100**, 13 (1955)
103. Blin-Stoyle, R. J., *Rev. Mod. Phys.*, **28**, 75 (1956)
104. Arima, A., and Horie, H., *Progr. Theor. Phys.* **12**, 623 (1954)
105. Barker, C. F., *Phil. Mag.*, [8], **1**, 329 (1956)
106. Ford, K. W., *Phys. Rev.*, **90**, 29 (1953)
107. Alaga, G., Alder, K., Bohr. A., and Mottelson, B. R., *Kgl. Danske Videnskab. Selskab Mat. fys Medd.*, **29**(9) (1955)
108. Bohr, A., Fröman, P. O., and Mottelson, B. R., *Kgl. Danske Videnskab. Selskab Mat. fys Medd.*, **29**(10) (1955)
109. Sunyar, A. W., *Phys. Rev.*, **98**, 653 (1955)