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THE MOMENT OF INERTIA OF ROTATING NUCLEI

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Abstract: A method developed for superconductivity and the Meissner effect in refs. [8, 9]) is applied to the calculation of the rotational moment of inertia of superfluid systems. It is checked that the correct mass is obtained in an analog translational problem. The expressions for the moment of inertia are equivalent to those of A. B. Migdal [7]).

1. Introduction

In heavy nuclei far from closed shells, one observes low-lying rotational levels [1]), i.e., energies of states J satisfying to a good approximation the formula

$$E_J = \text{constant} + \frac{\hbar^2}{2\mathscr{I}} J(J+1). \tag{1.1}$$

The experimental level spacing determines the "nuclear moment of inertia" $\mathscr{I}$. This quantity may be compared with the "rigid" moment of inertia

$$\mathscr{I}_r = \tfrac{2}{5} M R^2, \tag{1.2}$$

where M is the nuclear mass and R an average nuclear radius. Generally, the experimental moments of inertia are only a fraction of $\mathscr{I}_r$, thereby suggesting that the nuclear system is superfluid.

Belyaev has applied the Bogoliubov [2]) methods of the theory of superconductivity to this nuclear problem, with considerable qualitative success. He has deduced a formula for the moment of inertia by applying the Bogoliubov transformation to the Inglis "cranking model" [3, 4]). The numerical results of Belyaev are in satisfactory agreement with experiment, considering the general qualitative nature of the calculation. Quantitative calculations by Griffin and Rich also yield satisfactory agreement [5]).

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Nevertheless, there is a difficulty implicit in Belyaev's formula, which has been emphasized recently by Prange [6]). Belyaev's method of derivation can be applied not only to a nucleus in forced rotation, so as to get the moment of inertia $\mathscr{I}$, but also to a nucleus in forced linear translation, so as to get the total *mass* M. In this "pushing model", one transforms to a coordinate system moving with linear velocity V; the Hamiltonian becomes

$$H(V) = H - VP = H - V \sum_{i=1}^{2N} p_i, \tag{1.3}$$

where p_i is the linear momentum of the i^{th} particle, along the direction of the enforced motion, and H is the Hamiltonian in the stationary coordinate system. We use N for the number of pairs, so that the total number of particles is $2N$. Let Ψ be the ground state wave function of H, Ψ' be the ground state of $H(V)$ †. Then the total linear momentum of the pushed system is

$$\langle P' \rangle = (\Psi', P\Psi') = MV, \tag{1.4}$$

since $\langle P \rangle = (\Psi, P\Psi) = 0$ if Ψ is the ground state of H. The coefficient M must equal the total mass,

$$M = 2Nm. \tag{1.5}$$

The identity (1.5) is a general consequence of Galilean invariance [4]), not dependent on any specific assumptions about H and Ψ.

However, when Belyaev's method is used to compute the mass M, the result is less than (1.5). This follows from the fact that the Belyaev technique is inconsistent with Galilean invariance, and throws doubt on Belyaev's result for the moment of inertia $\mathscr{I}$.

Migdal [7]) has derived an alternative formula for $\mathscr{I}$, by the use of Green's function techniques. Migdal's technique gives the correct mass M in the pushing model, and his formula for $\mathscr{I}$ differs from Belyaev's formula by correction terms which are numerically small, even though important in principle; thus the agreement with experiment found by Belyaev is maintained in spite of the corrections.

Migdal's derivation is not easy to understand, and he makes a rather restrictive assumption concerning the interaction between particles (δ-function interaction). Thus, although Migdal's results are quite correct, it apperead worth while to construct an alternative and clearer derivation. This is done in the present paper, by the use of methods developed in connection with the theory of superconductity [8, 9]).

† There is no need to restrict oneself to the ground state here, but we do so for the sake of simplicity. See ref. [4]) for the pushing and cranking models in statistical mechanics.

The operator for creating a Fermion pair in the quantum state with wave function $\varphi(k, k')$ (k is an arbitrary index here) is

$$b^\dagger = \frac{1}{\sqrt{2}} \sum_{k, k'} \varphi(k, k') a_{k'}^\dagger, a_k^\dagger. \tag{1.6}$$

In the Bogoliubov theory [2]), the pair wave function is restricted to "simple pairs": that is, the one particle states k are paired up, in such a way that to each state k there is assigned a unique "partner" $\tilde{k}$ (usually $\tilde{k}$ is the time-reversed state to k, but this is not necessary). One then assumes the "simple pairing condition":

$$\varphi(k, k') = 0, \quad \text{unless } k' = \tilde{k}. \tag{1.7}$$

Unfortunately, the transformation to a rotating coordinate system destroys the simple pairing; thus, in the rotating coordinate system, it is still possible to write the $2N$-particle wave function Ψ in the form (unnormalized):

$$\Psi = (b^\dagger)^N |0\rangle, \tag{1.8}$$

where N is the number of pairs, and $|0\rangle$ is the vacuum state; but it is *not* possible to insist on the restriction (1.7) on the pair wave function φ.

In ref. [8]), expressions are developed for the expectation values of single-particle and two-particle operators over the wave function (1.8), for an arbitrary pair wave function φ in (1.6). These results are used in ref. [9]) to give a completely gauge-invariant derivation of the Meissner effect, and they are used in this paper to derive an expression for the moment of inertia in the cranking model, consistent with the correct mass in the pushing model †.

The basic assumptions underlying the calculation are stated in section 2. Section 3 contains the necessary formulas for expectation values, section 4 the verification of the correct mass in the pushing model, section 5 the zero order (no rotation) results, section 6 the change in the pair wave function $\varphi(k, k')$ induced by a small perturbation. The moment of inertia is derived in section 7, and the specialization to Migdal's formula (for a δ-function interaction) is given in an Appendix. Section 8 contains some discussion of the theoretical basis for calculations of this type for nuclei.

2. Basic Assumptions

We assume that the wave function Ψ_N of the ground state of the 2N-particle system of Fermions can be approximated by the (unnormalized) variational trial function

$$\Psi_N = (b^\dagger)^N |0\rangle, \tag{2.1}$$

† The pushing model test in moment of inertia calculations is closely related to the gauge-invariance test in superconductivity.

where $b^\dagger$ is given by (1.6), and the pair wave function $\varphi(k, k')$ is antisymmetric and normalized,

$$\varphi(k, k') = -\varphi(k', k), \sum_{k, k'} |\varphi(k, k')|^2 = 1. \tag{2.2}$$

Formula (2.1) means that the system is composed of N pairs, all pairs being in the same quantum state φ, i.e., an extreme Bose-Einstein condensation. The justification for this choice of a trial function is given in earlier work [10]). We propose to determine $\varphi(k, k')$ by minimizing the expectation value

$$\langle H \rangle = \frac{(\Psi, H\Psi)}{(\Psi, \Psi)}, \tag{2.3}$$

where H is the full Hamiltonian. We do *not* assume a "reduced interaction" but retain all matrix elements throughout the calculation. *The "model assumption is an assumption on the trial wave function, not an assumption on the Hamiltonian.*

A slow rotation or translation is a small perturbation which has the form of a sum of single-particle operators. Within the frame-work of our general "model" assumption it is therefore consistent to insist that the new groundstate, in the presence of the perturbation, must still be a Bose-condensed pair state, i.e., the alteration of the Hamiltonian does not lead to breaking up of pairs, but rather leads to a first-order change in the pair wave function $\varphi(kk')$. This is the assumption on which the calculation is based. The transformation to a coordinate system rotating with angular velocity ω about the x-axis replaces the original Hamiltonian H by [4])

$$H(\omega) = H - \omega L_x. \tag{2.4}$$

We determine the new pair wave function

$$\varphi_\omega(k, k') = \varphi(k, k') + \delta\varphi(k, k') \tag{2.5}$$

by minimizing (2.3) for $H(\omega)$; this leads to an integral equation for $\delta\varphi(k, k')$; this integral equation is linear if we restrict ourselves to slow rotations, i.e., keep only terms linear in ω.

The average angular momentum around the axis of rotation is given by

$$L_x(\omega) = \frac{(\Psi(\omega), L_x\Psi(\omega))}{(\Psi(\omega), \Psi(\omega))}, \tag{2.6}$$

and the moment of inertia is defined by

$$\mathscr{I} \equiv \lim_{\omega \to 0} \left[\frac{L_x(\omega)}{\omega}\right]. \tag{2.7}$$

The corresponding formulas for the total mass M in the pushing model are obtained by the replacements in (2.4)—(2.7), of ω by the linear velocity V, and of L_x by the total linear momentum P_x. Unlike the cranking model, however, the integral equations in the pushing model can be solved explicitly and generally.

3. Expressions for Expectation Values

Let Ψ_N be the wave function (2.1), describing N Bose-condensed pairs. Let J be a single-particle operator

$$J = \sum_{k,k'} J_{kk'} a_k^\dagger a_{k'}, \tag{3.1}$$

and let K be a two-particle operator

$$K = \sum_{k,l,k'l'} K_{kl,k'l'} a_k^\dagger a_l^\dagger a_{l'} a_{k'}. \tag{3.2}$$

We define the average values

$$\Omega_N \equiv (\Psi_N, \Psi_N), \tag{3.3}$$

$$J_N \equiv (\Psi_N, J\Psi_N), \tag{3.4}$$

$$K_N \equiv (\Psi_N, K\Psi_N), \tag{3.5}$$

and their generating functions

$$\Omega(v) \equiv \sum_{N=0}^{\infty} \frac{v^N}{(N!)^2} \Omega_N, \tag{3.6}$$

$$J(v) \equiv \sum_{N=0}^{\infty} \frac{v^N}{(N!)^2} J_N, \tag{3.7}$$

$$K(v) \equiv \sum_{N=0}^{\infty} \frac{v^N}{(N!)^2} K_N. \tag{3.8}$$

These definitions differ from the ones of ref. [8]) by an extra factor $N!$ in the denominator. This simplifies the final expressions for (3.6)—(3.8) without loss of generality.

We now quote the necessary results from ref. [8]). We introduce an unnormalized one-particle density matrix ρ by

$$\rho_{kk'} = 2v \sum_{k''} \varphi(k, k'')\varphi^*(k', k''). \tag{3.9}$$

The one-particle matrix $h_{kk'}$ is defined by

$$h_{kk'} = \left(\frac{\rho}{1+\rho}\right)_{kk'} \tag{3.10}$$

and the two-particle operator p by

$$p_{kl,k'l'} = h_{kk'}h_{ll'} - h_{kl'}h_{lk'}. \tag{3.11}$$

We also need the unnormalized wave function $\psi(k, k')$ defined by

$$\psi(k, k') = (2v)^{\frac{1}{2}} \sum_{k''} \left(\frac{1}{1+\rho}\right)_{kk''} \varphi(k'', k'). \tag{3.12}$$

The quantities (3.10)—(3.12) are functions of the parameter v, see (3.9).

Using the techniques of ref. [8]) we find the following explicit and exact expressions:

$$\Omega(v) = \exp[\tfrac{1}{2} \operatorname{tr} \ln(1+\rho)], \tag{3.13}$$

$$J(v) = \Omega(v) \operatorname{tr}(hJ), \tag{3.14}$$

$$K(v) = \Omega(v)[\operatorname{tr}_2(pK) + (\psi, K\psi)], \tag{3.15}$$

where tr is a trace over the one-particle space, tr_2 is a trace over the two-particle space, and $(\psi, K\psi)$ is the expectation value of K over the two-particle wave function ψ.

We can recover the values for a given number of pairs N by complex contour integration, e.g.,

$$\Omega_N = \frac{(N!)^2}{2\pi i} \oint \frac{\Omega(v)}{v^{N+1}} \, dv, \tag{3.16}$$

where the contour encloses the origin. If N is large, such integrals can be evaluated by the saddle-point method. We then get

$$\langle J \rangle = \frac{J_N}{\Omega_N} = \operatorname{tr}(hJ) + \text{order}\left(\frac{1}{N}\right), \tag{3.17}$$

$$\langle K \rangle = \frac{K_N}{\Omega_N} = \operatorname{tr}_2(pK) + (\psi, K\psi) + \text{order}\left(\frac{1}{N}\right), \tag{3.18}$$

where the parameter v in h, p, and ψ is determined by the condition

$$\operatorname{tr}(h) = 2N. \tag{3.19}$$

We shall use the simplified expressions (3.17)—(3.19) through most of this paper; it is quite possible, however, to find correction terms if N is not very large, by doing the saddle-point evaluation of (3.16) more carefully. For very small N, of course, it is easy to expand (3.13)—(3.15) directly in powers of v.

It should be noted that our $h_{kk'}$ and $\psi(k, k')$ are essentially equal to Migdal's [7]) Green's functions G and F, respectively, when the latter are evaluated for equal times.

4. The Total Mass

We shall now check that our procedure gives the correct total mass in the pushing model. The transformation to the moving system changes the Hamiltonian according to eq. (1.3).

We choose the x-space representation for treatment of this problem, and we shall suppress spin indices as being unessential for the present argument. Corresponding to any pair wave function $\varphi(x_1 x_2)$ in the stationary system, we define the pair wave function $\varphi'(x_1 x_2)$ in the moving system by

$$\varphi'(x_1 x_2) \equiv \exp[i\boldsymbol{\varkappa} \cdot (\mathbf{x}_1+\mathbf{x}_2)]\varphi(x_1 x_2), \tag{4.1}$$

where

$$\boldsymbol{\varkappa} = m\mathbf{V}/\hbar. \tag{4.2}$$

It should be noted that (4.1) is a special case of the gauge transformation, eq. (4.2) of reference [9]), provided we select the gauge function $\Lambda(x)$ to be

$$\Lambda(x) = \frac{\hbar c}{e}(\boldsymbol{\varkappa} \cdot \mathbf{x}). \tag{4.3}$$

This shows the close relationship between the pushing model check and the gauge invariance check.

Substitution of (4.1) into the definitions (3.9)—(3.12), and identification of the formal index k with the space coordinate x, yields

$$\langle x|h'|x'\rangle = \exp[i\boldsymbol{\varkappa} \cdot (\mathbf{x}-\mathbf{x}')]\langle x|h|x'\rangle, \tag{4.4}$$

$$\langle x_1 x_2|p'|x'_1 x'_2\rangle = \exp[i\boldsymbol{\varkappa} \cdot (\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}'_1-\mathbf{x}'_2)]\langle x_1 x_2|p|x'_1 x'_2\rangle, \tag{4.5}$$

$$\psi'(x_1 x_2) = \exp[i\boldsymbol{\varkappa} \cdot (\mathbf{x}_1+\mathbf{x}_2)]\psi(x_1 x_2). \tag{4.6}$$

The model Hamiltonian in the stationary system has the form

$$\begin{aligned} H = &\int \psi^\dagger(x)\left[-\frac{\hbar^2}{2m}\nabla_x^2+\Phi(x)\right]\psi(x)\mathrm{d}^3 x \\ &+\tfrac{1}{2}\iint \psi^\dagger(x_1)\psi^\dagger(x_2)v(x_1 x_2)\psi(x_2)\psi(x_1)\mathrm{d}^3 x_1\,\mathrm{d}^3 x_2, \end{aligned} \tag{4.7}$$

where $\psi(x)$ is the second-quantized wave operator, $\Phi(x)$ is some single-particle potential †, and $v(x_1, x_2)$ is an arbitrary two-particle interaction. The essential assumptions are:

1) the potentials $\Phi(x)$ and $v(x_1 x_2)$ are *local*;
2) the actual mass m appears in the kinetic energy operator.

† We include such a term for the sake of generality, since it is often employed in shell model calculations. The exact Hamiltonian in the absence of external fields contains no such term, of course.

We now evaluate the expectation value of H over the wave function of the moving system, by the use of (3.17)—(3.19) †.
Substitution of (4.5) and (4.6) into (3.18) gives,

$$\begin{aligned}\langle K'\rangle &= \tfrac{1}{2}\iint \mathrm{d}^3x_1\,\mathrm{d}^3x_2\langle x_1x_2|p'|x_1x_2\rangle v(x_1x_2)\\ &\quad+\tfrac{1}{2}\iint \mathrm{d}^3x_1\,\mathrm{d}^3x_2|\psi'(x_1x_2)|^2 v(x_1x_2)=\langle K\rangle.\end{aligned}\tag{4.8}$$

In the single-particle term (first line of (4.7)), we get

$$\mathrm{tr}(h'\Phi)=\int \mathrm{d}^3x\langle x|h'|x\rangle\Phi(x)=\mathrm{tr}(h\Phi).\tag{4.9}$$

Thus the only term which alters is the kinetic energy. Using (4.4), this can be transformed as follows

$$\begin{aligned}&-\frac{\hbar^2}{2m}\int \mathrm{d}^3x'(\boldsymbol{\nabla}_x{}^2\langle x|h'|x'\rangle)_{x=x'}\\ &=-\frac{\hbar^2}{2m}\int \mathrm{d}^3x'[\boldsymbol{\nabla}_x{}^2\langle x|h|x'\rangle+2i\boldsymbol{\varkappa}\cdot\boldsymbol{\nabla}_x\langle x|h|x'\rangle-\varkappa^2\langle x|h|x'\rangle]_{x=x'}.\end{aligned}\tag{4.10}$$

The first term in the square bracket yields the kinetic energy in the stationary system. The second term is proportional to the expectation value of the total linear momentum in the stationary system; this is zero for the ground state. The last term in (4.10) gives an additional kinetic energy equal to

$$\frac{\hbar^2\varkappa^2}{2m}\mathrm{tr}(h)=\tfrac{1}{2}mV^2(2N),\tag{4.11}$$

where $2N$ is the number of particles in the system (N is the number of pairs).

We also need the expectation value of the linear momentum P_x over the moving wave function; this is

$$-i\hbar\int \mathrm{d}^3x'(\boldsymbol{\nabla}_x\langle x|h'|x'\rangle)_{x=x'}=-i\hbar\int \mathrm{d}^3x'[i\boldsymbol{\varkappa}\langle x|h|x'\rangle+\boldsymbol{\nabla}_x\langle x|h|x'\rangle]_{x=x'}.\tag{4.12}$$

The second term on the right again vanishes in the ground state, and the first term yields

$$\hbar\boldsymbol{\varkappa}\,\mathrm{tr}(h)=m\mathbf{V}(2N).\tag{4.13}$$

Combining the results so far, and using (1.3), we get the identity

$$\frac{(\Psi(V),H(V)\Psi(V))}{(\Psi(V),\Psi(V))}=\frac{(\Psi,H\Psi)}{(\Psi,\Psi)}-\tfrac{1}{2}mV^2(2N).\tag{4.14}$$

† The exact expressions (3.13)—(3.15) could be used equally well.

Since the additional term, $-NmV^2$, is a constant, independent of the pair wave function $\varphi(x_1x_2)$ assumed for the stationary system, this term does not affect the minimization of $\langle H\rangle$ or of $\langle H(V)\rangle$. Thus, *if a certain choice of the pair wave function* $\varphi(x_1x_2)$ *minimizes* $\langle H\rangle$, *then its Galilei transform* $\varphi'(x_1, x_2)$, *equation* (4.1), *minimizes* $\langle H(V)\rangle$; *and the linear momentum in the moving system is given by* (4.13), *i. e., the total mass is correct.*

Since all subsequent equations in this paper are derived from minimization of $\langle H\rangle$, this establishes that our procedure yields the correct linear mass in all cases.

5. Wave Function of the Stationary System

From here on, we employ the methods and results of ref. [9]) to carry out the minimization. We shall only quote the necessary equations, referring to ref. [9]) for the derivations. We introduce the un-normalized pair wave function $\chi(k, k')$ by

$$\chi(k, k') = (2v)^{\frac{1}{2}}\varphi(k, k') \tag{5.1}$$

and minimize the quantity $\langle H\rangle - \mu\,\mathrm{tr}(h)$ (μ = chemical potential) with respect to the components $\chi(k, k')$, treated as being independently variable. The resulting equation is † (M, 5.14) which we repeat here for ready reference. First we define some auxiliary quantities:

$$J_{kk'} = \langle k|-\frac{\hbar^2}{2m}\nabla^2+\Phi|k'\rangle - \mu\delta_{kk'} \tag{5.2}$$

is the single-particle part of the operator $H-\mu N$. Letting $K_{kl,\,k'l'}$ be one half of the matrix element of the interaction between particles, we define

$$L_{kk'} = \sum_{ll'} h_{l'l}(K_{k'l',\,kl} - K_{k'l',\,lk}). \tag{5.3}$$

It should be noted that $L_{kk'}$ involves matrix elements other than those of the "reduced interaction". However, it turns out that the terms involving $L_{kk'}$ are not of qualitative importance. The important terms arise from the quantity

$$\Delta_{kl} = -2\sum_{k'l'} K_{kl,\,k'l'}\psi(k'l'). \tag{5.4}$$

If the pair wave function obeys the simple pairing condition (1.7), then the matrix elements $K_{kl,\,k'l'} = K_{\tilde{k}k,\,k'\tilde{k}'}$ in (5.4) are the ones of the so-called "reduced interaction".

In terms of the quantities defined so far, the minimization equation (M, 5.14)

† We shall denote ref. [9]) by the latter M, for Meissner effect; eq. (M, 5.14) is eq. (5.14) of ref. [9]).

becomes

$$\sum_{k''}(J_{kk''}+2L_{kk''})\chi(k'', k')-\tfrac{1}{2}\Delta_{kk'}+\tfrac{1}{2}\sum_{l, l'}\Delta^*_{ll'}\chi(l, k)\chi(l', k') = 0. \quad (5.5)$$

Following Belyaev [2]) and Migdal [7]), we now *assume* that the stationary system permits simple pairing, i.e., that there exists a set of single-particle states k with a unique pairing $k \leftrightarrow \tilde{k}$ such that the variationally best pair wave function $\varphi(kk')$ has the property

$$\varphi(k, k') = \delta_{k'\tilde{k}}\varphi_k, \quad \varphi_{\tilde{k}} = -\varphi_k. \quad (5.6)$$

This assumption implies

$$\chi(k, k') = \delta_{k'\tilde{k}}\chi_k, \quad \chi_k = (2v)^{\frac{1}{2}}\varphi_k, \quad (5.7)$$

$$\rho_{kk'} = \delta_{kk'}\rho_k, \quad \rho_k = |\chi_k|^2, \quad (5.8)$$

$$h_{kk'} = \delta_{kk'}h_k, \quad h_k = \frac{\rho_k}{1+\rho_k}, \quad (5.9)$$

$$\psi(kk') = \delta_{k'\tilde{k}}\psi_k, \quad \psi_k = -\psi_{\tilde{k}} = \frac{\chi_k}{1+\rho_k}. \quad (5.10)$$

We also *assume* that the quantity

$$J_{kk'}+2L_{kk'} \equiv \left\langle k\left|-\frac{\hbar^2}{2m}\nabla^2+\Phi\right|k'\right\rangle-\mu\delta_{kk'}+2L_{kk'} = \xi_k\delta_{kk'} \quad (5.11)$$

is diagonal in the k-representation; ξ_k is defined by (5.11), and is nearly equal to $\varepsilon_k-\mu$, where ε_k is the eigenvalue of the single-particle part of the Hamiltonian. The correction term $2L_{kk'}$ is ignored by Migdal [7]).

Finally, we *assume* that Δ_{kl} also obeys a simple pairing condition

$$\Delta_{kl} = \delta_{k\tilde{l}}\Delta_k. \quad (5.12)$$

It should be noted that the simple pairing assumptions (5.6), (5.11), (5.12) can be proved rigorously for an infinite uniform system; in that case the pairing $k \leftrightarrow \tilde{k}$ connects opposite momenta and, usually, opposite spins. For finite systems, (5.5), (5.11) and (5.12) are additional assumptions, which may or may not be consistent with the general minimization eq. (5.5).

Using the simple pairing assumptions, (5.5) reduces to the equation

$$\xi_k\chi_k-\tfrac{1}{2}\Delta_k+\tfrac{1}{2}\Delta_k\chi_k^2 = 0, \quad (5.13)$$

with the solution

$$\chi_k = -\frac{\xi_k}{\Delta_k}+\sqrt{1+\left(\frac{\xi_k}{\Delta_k}\right)^2}. \quad (5.14)$$

This is just the usual Bogoliubov [3]) solution; the connection with Bogoliubov's notation is

$$\chi_k = \frac{v_k}{u_k}, \quad \psi_k = u_k v_k = \frac{\Delta_k}{2E_k}, \tag{5.15}$$

where

$$E_k = \sqrt{\xi_k{}^2 + \Delta_k{}^2}. \tag{5.16}$$

Also we have, for Bogoliubov's u_k and v_k,

$$u_k{}^2 = \tfrac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right) = \frac{1}{1+\rho_k}, \quad v_k{}^2 = \tfrac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right) = \frac{\rho_k}{1+\rho_k} = h_k. \tag{5.17}$$

We shall use the simple pairing solution for the stationary system, and we shall allow for the destruction of the simple pairing by the rotation.

6. A Weak Perturbation

We introduce a weak perturbation of the single-particle type,

$$V = \sum_{kk'} V_{kk'} a_k{}^\dagger a_{k'}. \tag{6.1}$$

Treating V to first order, we obtain from (5.5), (5.7), (5.11), and (5.12)

$$[V_{k\tilde{l}} + 2(\delta L)_{k\tilde{l}}]\chi_{\tilde{l}} + [\xi_k + \tfrac{1}{2}\Delta_k{}^*\chi_k + \tfrac{1}{2}\Delta_l{}^*\chi_l]\delta\chi(k,l) - \tfrac{1}{2}(\delta\Delta)_{kl} + \tfrac{1}{2}(\delta\Delta^*)_{\tilde{k}\tilde{l}}\chi_k\chi_l = 0. \tag{6.2}$$

Here δ denotes the first-order change of a quantity. Using the methods of section 7 of ref. [9]), we get the following expressions for the first-order changes δL and $\delta\Delta$, in terms of $\delta\chi$:

$$(\delta L)_{kk'} = \sum_{ll'} (K_{k'l',kl} - K_{k'l',lk}) \frac{\delta\chi^*(l\tilde{l}')\chi_{l'} - \chi_l{}^*\delta\chi(\tilde{l}l')}{(1+\rho_l)(1+\rho_{l'})}, \tag{6.3}$$

$$(\delta\Delta)_{kl} = -2\sum_{k'l'} K_{kl,k'l'}\delta\psi(k'l'), \tag{6.4}$$

$$\delta\psi(kl) = \frac{\delta\chi(k,l) - \chi_k\chi_l\delta\chi^*(kl)}{(1+\rho_k)(1+\rho_l)}. \tag{6.5}$$

Eq. (6.2), with the definitions (6.3)—(6.5), is a linear integral equation for the unknown first-order change $\delta\chi(kl)$ in the variationally best pair wave functions, induced by the change (6.1) in the Hamiltonian. (6.2) must be solved subject to the antisymmetry condition $\delta\chi(k,l) = -\delta\chi(l,k)$.

We can obtain a simple first approximation to $\delta\chi(kl)$ by ignoring, in (6.2), all terms in which $\delta\chi$ appears under an integral (summation) sign, i.e., we

ignore the terms involving $(\delta L)_{kl}$, $(\delta\Delta)_{kl}$, and $(\delta\Delta^*)_{kl}$, so as to get (with $\chi_l = -\chi_{\bar{l}}$)

$$[\xi_k + \tfrac{1}{2}(\Delta_k^* \chi_k + \Delta_l^* \chi_l)]\delta\chi(k, l) \cong V_{k\bar{l}}\chi_l. \tag{6.6}$$

In the rotation problem, the perturbation $V = -\omega L$ is odd under time reversal; if the pairing $k \leftrightarrow \bar{k}$ involves time-reversed states (as we shall assume), we have the identity

$$V_{k\bar{l}} = -V_{l\bar{k}}. \tag{6.7}$$

We then get a properly antisymmetric solution of (6.6) by subtracting, from it, the same equation with k and l interchanged, thereby obtaining

$$(\xi_k + \xi_l + \Delta_k^* \chi_k + \Delta_l^* \chi_l)\delta\chi(k, l) \approx V_{k\bar{l}}\chi_l - V_{l\bar{k}}\chi_k.$$

Using the identity

$$\xi_k + \Delta_k \chi_k = E_k \tag{6.8}$$

obeyed by the unperturbed solution (5.14), the approximate expression for $\delta\chi(k, l)$ is

$$\delta\chi(k, l) \approx \frac{V_{k\bar{l}}\chi_l - V_{l\bar{k}}\chi_k}{E_k + E_l} \equiv \delta\chi^{(0)}(k, l), \tag{6.9}$$

which is properly antisymmetric according to (6.7). We shall see, later on, that the approximation (6.9) is equivalent to that used by Belyaev [2]).

An improved approximation can be obtained by ignoring only the term $(\delta L)_{k\bar{l}}$ in (6.2); this is the first-order change in a term which itself is unimportant to start with. Again subtracting from (6.2) the equation with k and l interchanged, we obtain the improved expression

$$\delta\chi(k, l) \approx \frac{V_{k\bar{l}}\chi_l - V_{l\bar{k}}\chi_k + (\delta\Delta)_{kl} - (\delta\Delta^*)_{\bar{k}\bar{l}}\chi_k\chi_l}{E_k + E_l}. \tag{6.10}$$

This expression involves the, so far unknown, change $\delta\Delta$. Substituting it in (6.5) and using (see (M, 7.5), (M, 7.6))

$$\delta h_{kk'} = \frac{\delta\rho_{kk'}}{(1+\rho_k)(1+\rho_{k'})} = \frac{\delta\chi(k\bar{k}')\chi_{k'}^* + \chi_k \delta\chi^*(k'k)}{(1+\rho_k)(1+\rho_{k'})}, \tag{6.11}$$

we arrive at

$$\delta h_{mm'} = \frac{V_{mm'}(\xi_m\xi_{m'} + \Delta_m\Delta_{m'} - E_m E_{m'}) - \delta\Delta_{m\tilde{m}'}\xi_m\Delta_{m'} + \delta\Delta^*_{\tilde{m}m'}\xi_{m'}\Delta_m}{2E_m E_{m'}(E_m + E_{m'})}, \tag{6.12}$$

$$\delta\psi_{m\tilde{m}'} = \frac{-V_{mm'}(\xi_m\Delta_{m'} - \xi_{m'}\Delta_m) - \delta\Delta_{m\tilde{m}'}(E_m E_{m'} + \xi_m\xi_{m'}) - \delta\Delta^*_{\tilde{m}m'}\Delta_m\Delta_{m'}}{2E_m E_{m'}(E_m + E_{m'})}. \tag{6.13}$$

These equations are identical with Migdals eqs. (15), (16). The expression of δh is in terms of $\delta\Delta$ which should in turn be determined from the integral equation

$$\delta\Delta_{mm'} = -2\sum_{ll'} K_{mm'll'}\delta\psi_{ll'}, \tag{6.14}$$

where the expression for $\delta\psi$ should be taken from (6.13). It is possible to reintroduce the L terms so far neglected into expressions (6.12) (6.13) by the substitution

$$V_{mm'} \to V_{mm'} + 2\delta L_{mm'}. \tag{6.15}$$

If we do this, (6.14) alone is no longer enough to determine both $\delta\Delta$ and δL and we must therefore use it together with (6.3). We then get a coupled pair of integral equations for the unknown quantities $\delta\Delta$ and δL, equivalent to the original integral equation (6.2) for $\delta\chi$.

7. The Moment of Inertia

The operator L_x for the angular momentum is a sum of single-particle operators. Hence its average value is given by the general formula (3.17). In the stationary state, $h_{kk'}$ is diagonal (see (5.9)), and $h_k = h_{\tilde{k}}$. Since L_x is odd under time reversal, its diagonal elements obey

$$(L_x)_{kk} = -(L_x)_{\tilde{k}\tilde{k}}, \tag{7.1}$$

so that (the prime on the sum indicates we are summing over half the states only):

$$\langle L_x\rangle = \mathrm{tr}(hL_x) = \sum_k{}' h_k[(L_x)_{kk} + (L_x)_{\tilde{k}\tilde{k}}] = 0. \tag{7.2}$$

This is the expected result for the stationary system.

As a result of the transformation to the rotating coordinate system, $h_{kk'}$ changes into

$$h_{kk'} = h_k\delta_{kk'} + (\delta h)_{kk'}, \tag{7.3}$$

where, to first order in ω, δh is given by (6.11).

The expectation value of L_x in the rotating system is

$$L_x(\omega) = \mathrm{tr}\,(\delta h L_x) = \sum_{kk'} \frac{\delta\chi(k\tilde{k}')\chi^*_{k'} + \chi_k\delta\chi^*(k'\tilde{k})}{(1+\rho_k)(1+\rho_{k'})}\langle k'|L_x|k\rangle. \tag{7.4}$$

If we use the approximation (6.9) for $\delta\chi$, together with $V = -\omega L_x$, (7.4) reduces considerably. Straightforward substitution of (6.9) into (7.4), use of the Hermiticity and odd time-reversal character of $V = -\omega L_x$, and of the reality of the zero-order function χ_k, leads to

$$\mathscr{I} = \frac{L_x(\omega)}{\omega} \approx \sum_{kk'} \frac{|\langle k|L_x|k'\rangle|^2(\chi_k - \chi_{k'})^2}{(1+\rho_k)(1+\rho_{k'})(E_k+E_{k'})} \equiv \mathscr{I}_0. \tag{7.5}$$

By the use of (5.14) for χ_k and (5.17a) for $(1+\rho_k)^{-1}$, it is possible to show that (7.5) is identically equal to the Belyaev expression [2])

$$\mathscr{I}_0 = \sum_{kk'} \frac{|\langle k|L_x|k'\rangle|^2(E_k E_{k'} - \xi_k \xi_{k'} - \Delta_k \Delta_{k'})}{2E_k E_{k'}(E_k + E_{k'})}, \tag{7.6}$$

or,

$$\mathscr{I}_0 = \sum_{kk'} \frac{|\langle k|L_x|k'\rangle|^2(u_k v_{k'} - v_k u_{k'})^2}{E_k + E_{k'}}. \tag{7.7}$$

An improved approximation can be generated by using (6.10) for $\delta\chi$. Straightforward substitution yields

$$\mathscr{I} = \mathscr{I}_0 + \mathscr{I}_\Delta, \tag{7.8}$$

where $\mathscr{I}_0$ is given by (7.7) and

$$\mathscr{I}_\Delta = \sum_{kk'} \frac{(\delta\Delta)_{k\tilde{k}'} \xi_k \Delta_{k'} - (\delta\Delta^*)_{\tilde{k}k'} \Delta_k \xi_{k'}}{2E_k E_{k'}(E_k + E_{k'})} \langle k'|L_x|k\rangle. \tag{7.9}$$

This is Migdal's result; the correction term (7.9) can be evaluated explicitly only after the integral equation for $\delta\Delta$ has been solved.

Finally, we can solve the full integral equation (6.2) for $\delta\chi$, and substitute the result into (7.4) to obtain the best value for the moment of inertia deducible from this model. It is not expected that the improvement over Migdal's result (7.9) would be significant.

8. Discussion

Since Belyaev [2]) and Migdal [7]) have already given approximate numerical evaluations of eqs. (7.6) and (7.8—9), we need not repeat their comparison with experiment. Rather, we would like to comment briefly on the basis of the theoretical framework used in these calculations.

In an infinite system, e.g., a superconductor, there is a distinction in principle between independent-particle wave functions and the Bose-condensed pair wave functions of the Bogoliubov theory. In the limit $N \to \infty$, the wave function (1.8) is not obtainable from any *finite* superposition of independent-particle configurations. If an infinite number of configurations is superposed so as to yield (1.8), the evaluation of the expectation value of the Hamiltonian over (1.8) requires a non-trivial interchange of the order of summation of infinite series. This shows itself in the non-analytic character of $(\psi, H\psi)$ as a function of the coupling constant.

The situation is completely different in the Belyaev-Migdal theory, which deals with a finite and not very large system (the nucleus), and in which the sums over single-particle states k are usually restricted to the outermost (not

completely filled) shell of the nuclear shell model. As a result, all expressions must be analytic in the coupling constant, and can be obtained in principle by a straightforward shell model calculation.

In particular, if the sums over k are restricted even further, to the states arising from only one value of j, then the paired wave function of Belyaev is essentially the same as the shell model function of zero seniority in the Racah [11]) scheme.

The difference between the two descriptions is therefore primarily conceptual, rather than being a strict mathematical distinction. A "simple" wave function in one scheme is "complicated" in the other, and vice versa. To illustrate: if the unfilled shell contains j-values j' j'' j''', say, then a reasonable pair wave function for the ground state of an even nucleus might be

$$\begin{aligned}\varphi(1,2) = {} & C' \sum_{m'>0} u_{j'm'}(1)u_{j',-m'}(2) \\ & + C'' \sum_{m''>0} u_{j''m''}(1)u_{j'',-m''}(2) \\ & + C''' \sum_{m'''>0} u_{j'''m'''}(1)u_{j''',-m'''}(2)\end{aligned} \tag{8.1}$$

where the u_{jm} are independent-particle wave functions and C', C'', C''' are constants. A wave function (1.8) for the $2N$-particle (outside closed shells) nucleus, built from (8.1), is "simple" in the Belyaev scheme, but would be considered a very complicated instance of "configuration mixing" in the usual shell model description. Thus the Belyaev scheme is an approximate method of circumventing the need for very elaborate shell model calculations, and a way of describing conceptually, in simple terms, the type of wave function such an elaborate calculation would probably yield if carried out. But, unlike the situation in the theory of superconductivity, the Belyaev scheme is not qualitatively out of the reach of shell model calculations.

This suggests that a direct check can, and should, be carried out so as to test the accuracy of the form (1.8) for the trial wave function of the ground state. By choosing some reasonable interaction between particles, and a value of N within range of present-day electronic computers, the shell-model energy matrix can be diagonalized directly, and the ground state vector compared with the form (1.8). This is a non-trivial check, since the arguments in favour of the extreme Bose-Einstein condensation of pairs (1.8), contained in ref. [10]), apply directly only to the limit $N \to \infty$, and it is not known how big an error is made by assuming the form (1.8) for finite, and not very large N †.

On the basis of the ideal Bose gas analogy, we would expect to find the "irrotational" moment of inertia. Migdal [7]) shows that this is indeed the case

† This question is quite separate from, and much more difficult to answer than, the question of the $1/N$ corrections to the limiting forms (3.17)—(3.19) of expectation values over the function (1.8). This latter question is answered by the exact expressions (3.13)—(3.16).

(his equation (3.7)) provided the size R of the system greatly exceeds the average distance a_0 between the two particles in the pair.

On the other hand, if $R \approx a_0$, the pairs are "squeezed" by the boundaries of the system, and free irrotational flow of the centres of mass of the pairs becomes impossible. Thus the moment of inertia increases, until it reaches the "rigid" value for $R \ll a_0$ (Migdal's eq. (36)) †.

The actual nuclear size is between these extremes, so that the nuclear moments of inertia lie between the irrotational and rigid values.

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Appendix A

MIGDAL's FORMULATION

By substitution of the approximate form (6.10) into equations (6.5) and (7.4), we obtain approximations for $\delta\psi(k, l)$ and for $(\delta h)_{kk'}$, respectively. The approximate form for $\delta\psi$ is identical with Migdal's eq. (16), and the approximate form for δh is identical with his eq. (15).

Instead of writing down a general integral equation, as we have done in section 6, Migdal prefers to incorporate certain assumptions into his theory so as to make it more suitable for rough calculations. In this appendix, we shall indicate briefly how Migdal's form of the integral equation for $\delta\Delta$ may be obtained.

Working in a representation where the spin is separated from the other quantum numbers we put $m = \mu, s$. Let us define the x-space pair wave function by

$$\psi(rs, r's') \equiv \sum \psi_{\mu s\mu' s'} \varphi_\mu(r)\varphi_\mu(r'), \tag{A.1}$$

and the x-space representation of $\Delta_{mm'}$ by

$$\Delta(rs, r's') \equiv K(r, r')\psi(rs, r's'), \tag{A.2}$$

where $\varphi_\mu(r)$ is the wave function of the state μ and $K(rr')$ is the interaction. It is easily checked that

$$\begin{aligned} \Delta(rs, r's') &= \sum_{\mu\mu'} \Delta_{\mu s\mu' s'} \varphi_\mu(r)\varphi_{\mu'}(r') \\ &= \sum_{\mu} \Delta_{\mu s} \varphi_\mu(r)\varphi_\mu{}^*(r'). \end{aligned} \tag{A.3}$$

If we assume a δ function interaction

$$K(rr') = -K\delta(r-r'), \tag{A.4}$$

† Under these circumstances also the paired wave function (2.1) becomes identical with a simple shell model wave function.

eq. (A.3) reduces to

$$\Delta(rss')\delta(r-r') = \sum_{\mu} \Delta_{\mu s}\varphi_{\mu}(r)\varphi_{\mu}^{*}(r'). \tag{A.5}$$

This implies that $\Delta_{\mu s} = \Delta_s$ and is independent of μ. This further means that $\Delta(rss')$ is independent of r. We thus have

$$\Delta(ss') = \begin{cases} \Delta_s & \text{when} \quad s \neq s', \\ 0 & \text{when} \quad s = s'. \end{cases} \tag{A.6}$$

Using (A.2) we get

$$\Delta_s = K \sum_{\mu} \frac{\Delta_s}{2E_{\mu}} \varphi_{\mu}(r)\varphi_{\mu}^{*}(r), \tag{A.7}$$

or,

$$1 = K \sum_{\mu} \frac{1}{2E_{\mu}} \varphi_{\mu}(r)\varphi_{\mu}^{*}(r). \tag{A.8}$$

Similarly treating $\delta\psi$ and $\delta\Delta$ we get

$$\delta\Delta(rs, r's') = \delta\Delta(rss')\delta(r-r'). \tag{A.9}$$

Consider

$$\begin{aligned} &\sum_{\mu\mu'} \delta\Delta_{\mu s\mu' s'} \frac{1}{2E_{\mu}} \varphi_{\mu}(r)\varphi_{\mu'}^{*}(r') \\ &\qquad = \int \mathrm{d}r_1 \delta\Delta(r_1 ss') \sum_{\mu} \frac{1}{2E_{\mu}} \varphi_{\mu}^{*}(r_1)\varphi_{\mu}(r) \sum_{\mu'} \varphi_{\mu'}^{*}(r_1)\varphi_{\mu}(r) \\ &\qquad = \frac{\delta\Delta(rss')}{K} = -\sum \delta\psi_{\mu s\mu' s'}\varphi_{\mu}(r)\varphi_{\mu'}(r), \end{aligned} \tag{A.10}$$

or

$$\sum_{\mu\mu'} \left\{ \delta\Delta_{\mu\uparrow,\mu'\downarrow} \left[\frac{1}{4E_{\mu}} + \frac{1}{4E_{\mu'}} \right] + \delta\psi_{\mu\uparrow\tilde{\mu}'\downarrow} \right\} \varphi_{\mu}(r)\varphi_{\mu'}^{*}(r) = 0. \tag{A.11}$$

It further follows from (A.9) that

$$\delta\Delta_{\mu\uparrow\tilde{\mu}'\downarrow} = \delta\Delta^{*}_{\tilde{\mu}\downarrow\mu'\uparrow} \equiv if_{\mu\mu'}. \tag{A.12}$$

Substituting (A.12) into (A.11) we get:

$$\sum \frac{V_{\mu\mu'}(\xi_{\mu}-\xi_{\mu'}) - if_{\mu\mu'}\frac{1}{2}(\xi_{\mu}-\xi_{\mu'})^2}{2E_{\mu}E_{\mu'}(E_{\mu}+E_{\mu'})} \varphi_{\mu}(r)\varphi_{\mu}^{*}(r) = 0. \tag{A.13}$$

This is Migdal's equation for $\delta\Delta$. The solution of (A.13) is unique only if we require the additional condition on the form of $\delta\Delta$, namely (A.9). For the case

of translation when $V = -vP$ the simple solution

$$f_{\mu\mu'} = -2\Delta i \frac{V_{\mu\mu'}}{\xi_\mu - \xi_{\mu'}} = -2i\Delta v \frac{P_{\mu\mu'}}{\xi_\mu - \xi_{\mu'}}$$
$$= \frac{2\Delta vm}{\hbar} X_{\mu\mu'} \tag{A.14}$$

satisfies this condition. It can be checked that by its use the correct mass is obtained.

Appendix B

PERTURBATIONS EVEN UNDER TIME REVERSAL

For a perturbation which is even under time reversal, the integral equation (6.2) simplifies enormously and allows a very accurate approximate solution. We now *add* to (6.2) the equation obtained by interchanging the indices k and l. The terms in δ drop out identically. The terms in δL are small corrections of little qualitative importance. If we ignore those terms, we arrive at the explicit solution

$$\delta\chi(k, l) = \frac{V_{k\tilde{l}}\chi_l + V_{l\tilde{k}}\chi_k}{\xi_k - \xi_l}. \tag{B.1}$$

This is antisymmetric if $V_{k\tilde{l}} = V_{l\tilde{k}}$, which is true for a perturbation even under time reversal. The corresponding approximation for δh, eq. (7.4), can be reduced to

$$(\delta h)_{kk'} = \frac{h_k - h_{k'}}{\xi_k - \xi_{k'}} V_{kk'} \tag{B.2}$$

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