DESCRIPTION OF THE WOBBLING MOTION THROUGH A BOSON METHOD

Robert Poenaru

DFT, IFIN-HH
Doctoral School of Physics, UB

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Scientific Coordinator: A. A. Raduta

Outline

Nuclear Deformation

Nuclear Radius

The **shape** of the nucleus is most generally described in terms of the *nuclear radius*:

$$R(\theta, \varphi; t) = R_0 \left(1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}(t) Y_{\lambda}^{\mu}(\theta, \varphi)\right) \tag{1}$$

- The $\alpha_{\lambda\mu}$ are collective coordinates $\Longrightarrow$ *vibrations of the nucleus*.
- Y_{λ}^{μ} are the spherical harmonics.

Nuclear shapes

Most nuclei are spherical or axially symmetric in the ground state.

Figure 1: **Spherical:** $\beta_2 = 0$; **Prolate:** $\beta_2 > 0$; **Oblate:** $\beta_2 < 0$

Quadrupole deformations

- Most relevant vibrational degrees of freedom in nuclei.
- Play a crucial role in the rotational spectra of nuclei.

Quadrupole radius

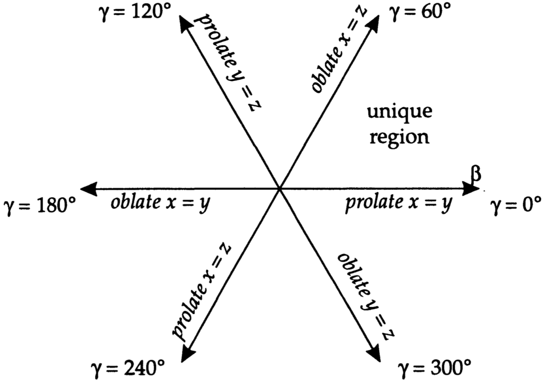
For pure quadrupole deformations:

$$R(\theta,\varphi)=R_0\left(1+\sum_{\mu}\alpha_{2\mu}Y_2^{\mu}(\theta,\varphi)\right)\ , \tag{2}$$

Using A. Bohr's description, the coordinates $\alpha_{2\mu}$ can be reduced to only two *deformation parameters*: β_2 (*eccentricity*) and γ (**triaxiality**).

Nuclear triaxiality

- Besides the axially symmetric shapes (i.e., spherical, prolate, and oblate), nuclei can be **triaxial** $\Longrightarrow$ lack of symmetry along any of the principal axes.
- The asymmetry is given by the non-zero value of γ.



Triaxial ellpsoid

Schematic example with a triaxial ellipsoid ($\gamma \neq 0$) $\beta_2 > 0$.

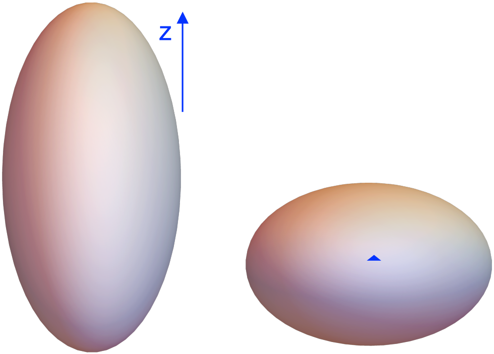


Figure 2: **Left:** side-view. **Right:** top view.

Nuclear triaxiality

- Probing triaxiality experimentally is a real challenge (e.g., large and complex detector setups).
- Only two fingerprints known so far: **chiral motion** (Frauendorf, 1997) and **wobbling motion** (Bohr and Mottelson, 1975).

Wobbling Motion (WM)

- Collective effect $\rightarrow$ *unique* to triaxial nuclei.
- Predicted almost 50 years ago, first experimental confirmation: in 2001 (Odegard et al.) for ^{163}Lu.
- In present, few wobblers are experimentally confirmed in the mass regions: $A \approx 130, 160, 180$ $\rightarrow$ A list of all known wobblers will be available in my PhD thesis (Chapter 3).

Wobbling motion

Triaxial nuclei

A triaxial nucleus can rotate about any of the three axes.

The rotational angular momentum (a.m.) is NOT aligned along any of the body-fixed axes $\Longrightarrow$ **precesses** and **wobbles** around the axes with the largest MOI.

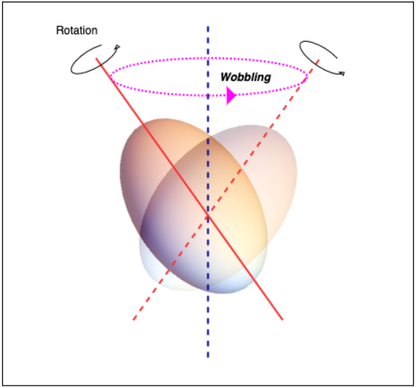


Figure 3: Schematic representation for the nuclear wobbling motion.

Wobbling Bands

Wobbling bands

Sequences of $\Delta I = 2\hbar$ rotational bands that are built on different *wobbling phonon excitations* $(n_w = 0, 1, \dots)$.
Oscillatory behavior, with a *tilting angle* for the angular momentum proportional to $n_w \longrightarrow$ **harmonic-like motion**.

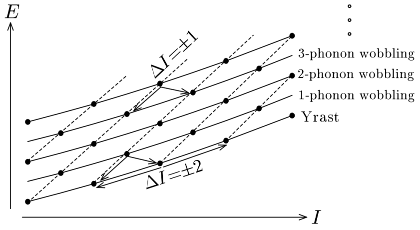


Figure 4: Rotational-band structures of the wobbling motion.

- For ^{163}Lu: $n_w = 0, 1, 2, 3$ wobbling phonon numbers, respectively.
- Nuclei have **large** quadrupole moments
- **Strong E2** character for the electro-magnetic transitions.

Even-Even vs. Even-Odd Nuclei

Theoretical frameworks for **even-mass** nuclei

1. *Harmonic Approximation(s)* (Bohr and Mottelson, 1975)
2. Triaxial-Rotor-Model (Davydov and Filippov, 1958)
3. Boson-approximations (Tanabe, 1971)

Theoretical frameworks for **odd-mass** nuclei

1. Particle Rotor Model (Hamamoto, 2002)
2. Tilted-axis wobbling (Frauendorf and Meng, 1997)
3. RPA, Mean-Field Theories, GCM+AMP...

Recent work on wobbling motion

- RPA for ^{163}Lu, Raduta et al (PRC, 2017)
- Tilted-axis wobbling for ^{135}Pr, R. Budaca (PRC, 2018)
- PRM for ^{163}Lu, R. Poenaru (IJMPE, 2021)

Theoretical formalism - Even-Odd Nuclei

Description of WM for an even-odd nucleus

- one single-particle (nucleon) *coupled* to an *even-even* triaxial core.
- the nucleon is moving in a quadrupole deformed mean-field generated by the core
- particle + rotor coupling drives the entire system to large (and stable) deformations ($\epsilon \sim 0.2 - 0.4$).

Hamiltonian:

$$\hat{H}_{\text{rot}} = \sum_{k=1}^{3} A_k \left(\hat{I}_k - \hat{j}_k \right)^2 \quad . \tag{3}$$

$A_k \rightarrow$ inertia parameters: $A_k = (2\mathcal{I}_k)^{-1}$.

Theoretical Formalism - Rotational Hamiltonian

Expanding $\hat{I}_2$ up to first order (particle is *rigidly coupled* to the core):

$$\hat{I}_2 = I\left(1 - \frac{1}{2}\frac{\hat{I}_1^2 + \hat{I}_3^2}{I^2}\right) , \tag{4}$$

can help re-write the initial Hamiltonian:

Rigid coupling Hamiltonian: $\mathbf{j} = (j\cos\theta, j\sin\theta, 0)$

$$\hat{H}_{\text{rot}} = AH' + H_{sp} + \text{s.t.} , \tag{5}$$

$$H' = \hat{I}_2^2 + u\hat{I}_3^2 + 2v_0\hat{I}_1 , \; H_{sp} = \sum_{k=1}^{2} A_k \hat{j}_k^2 , \; \text{s.t.} = A_1 I^2 - A_2 j_2 I , \tag{6}$$

$$1 > u = \frac{A_3 - A_1}{A} > -1 , \; v_0 - \frac{A_1 j_1}{A} , \; A = A_2\left(1 - \frac{j_2}{I}\right) > 0 \tag{7}$$

Rotational Hamiltonian

- H' looks like the Hamiltonian for a triaxial rigid rotator + constrained (cranked) to move along the 1-axis.
- The a.m. algebra is defined as:

$$\hat{I}_{\pm} = \hat{I}_2 \pm i\hat{I}_3 \ , \ \hat{I}_0 = \hat{I}_1 \ , \tag{8}$$

$$\left[\hat{I}_-, \hat{I}_+\right] = 2\hat{I}_0 \ , \ \left[\hat{I}_{\mp}, \hat{I}_0\right] = \mp\hat{I}_{\mp} \ . \tag{9}$$

Hamiltonian + Schrodinger Equation

$$H' = a\left(\hat{I}_+^2 + \hat{I}_-^2\right) + b\left(\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+\right) + c\hat{I}_0 \ , \tag{10}$$

$$H' \left|\Psi\right\rangle = E\left|\Psi\right\rangle \ . \tag{11}$$

Angular Momentum Representation

The a.m. ladder operators are re-defined in terms of new variables $q, d/dq$:

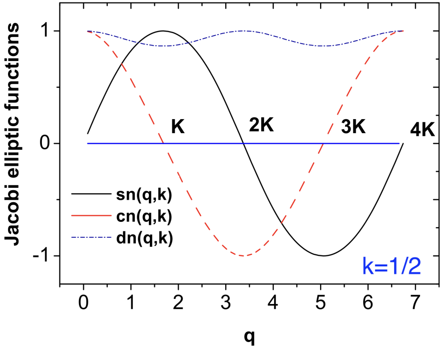
$$\hat{I}_{\mp} = i\frac{c \pm d}{s}\left(I \mp \hat{I}_0\right) \ , \ \hat{I}_0 = Icd - s\frac{d}{dq} \ . \tag{12}$$

s, c, d are the **Jacobi Elliptic Functions**:

$$s = \mathsf{sn}(q,k) \ , \ c = \mathsf{cn}(q,k) \tag{13}$$

$$d = \mathsf{dn}(q,k) \ , \tag{14}$$

The functions are periodic in q, with the periods $4K$ (s), $4K$ (c), and $2K$ (d).



Coordinate representation

The variable q is defined in terms of k ($0 < k^2 < 1$):

$$q = \int_0^{\varphi} \left(1 - k^2 \sin^2(t)\right)^{-1/2} dt = F(\varphi, k)\ ,\ \varphi = F^{-1}(q, k)\ , \tag{15}$$

$$s = \sin\varphi\ ,\ c = \cos\varphi\ ,\ d = \sqrt{1 - k^2 s^2}\ ,\ k = \sqrt{|u|}\ . \tag{16}$$

New Hamiltonian

$$H' = -\frac{d^2}{dq^2} - 2v_0 s \frac{d}{dq} + I(I+1)s^2k^2 + 2v_0 cdI\ , \tag{17}$$

with the associated *Schrodinger Equation* (fully separated Kinetic and Potential terms):

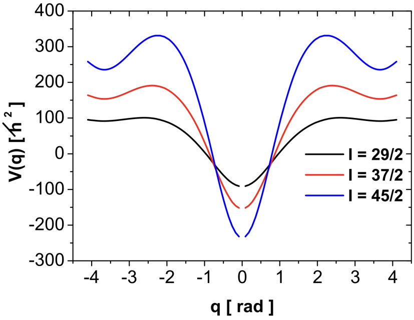
$$\left[\frac{d^2}{dq^2} + V(q)\right]\Psi = E\Psi \tag{18}$$

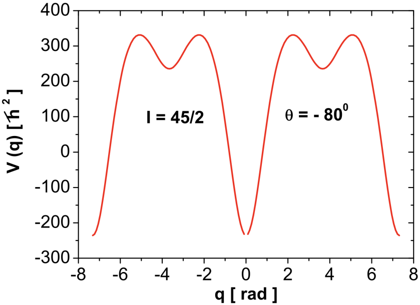
The "Elliptic" Potential

The expression of $V(q)$

With the elliptic functions s, c, d, and arbitrary k:

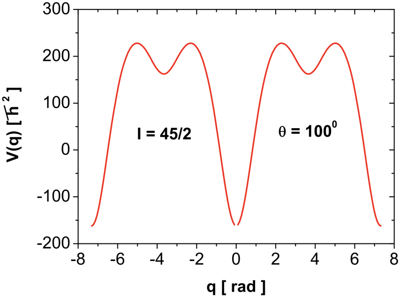
$$V(q) = \left[I(I+1)k^2 + v_0^2\right] s^2 + (2I+1)v_0 cd = V(-q) \ . \tag{19}$$





Local minima states: meta-stable. *Deepest well* states: degenerate.

The "Elliptic" Potential



- The deepest minima appear at $q = 0, \pm 4K$.
- Meta-stable (local) minima appear at $q = \pm 2K$.

Bargmann Mapping - Boson description

The variables q and d/dq can be mapped to a pair of **boson operators** $(b, b^\dagger)$ via the Bargmann representation of the angular momentum:

$$q \to b^\dagger \ , \ \frac{d}{dq} \to b \ . \tag{20}$$

New angular momentum operators

This mapping leads to *the first boson expansion of the angular momentum components in literature*.

$$\hat{I}_+ = i\frac{cb^\dagger - db^\dagger}{sb^\dagger}\left(I + Icb^\dagger db^\dagger - sb^\dagger b\right) \ , \tag{21}$$

$$\hat{I}_- = i\frac{cb^\dagger + db^\dagger}{sb^\dagger}\left(I - Icb^\dagger db^\dagger + sb^\dagger b\right) \ , \tag{22}$$

$$\hat{I}_0 = Icb^\dagger db^\dagger - sb^\dagger b \ . \tag{23}$$

Wobbling spectrum

Harmonic oscillator

Expanding $V(q)$ around the deepest minima up to second order in q, the spectrum of H_{rot} is obtained:

$$E_n = A_1 I^2 - (2I+1)A_1 j_1 - I A_2 j_2 + \sum_{k=1}^{2} A_k j_k^2 + \hbar\omega\left(n + \frac{1}{2}\right) . \quad (24)$$

- The frequency $\hbar\omega$ is the so-called *wobbling frequency*.

$$\hbar\omega_I = f(A_1, A_2, A_3; I) \quad (25)$$

- n is the wobbling phonon number $n = 0, 1, \ldots$.

Wobbling spectrum

- Recall H_{rot} contains the H' Hamiltonian.
- The $\hbar\omega$ frequency corresponds to H_{rot}.
- Similarly, $\hbar\omega'$ is the frequency for E'_n, corresponding to H'.

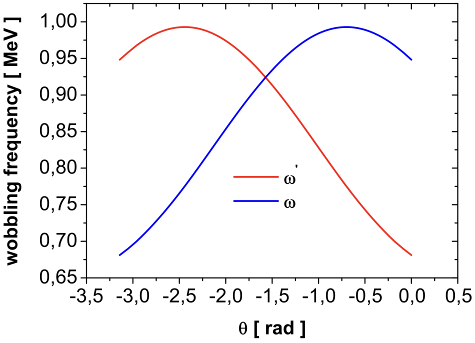


Figure 5: $j = 13/2$, $I = 55/2$.

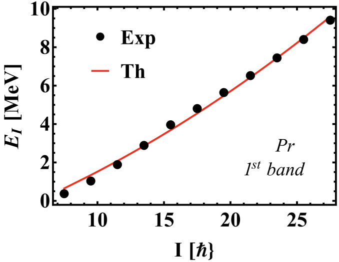
Numerical Results

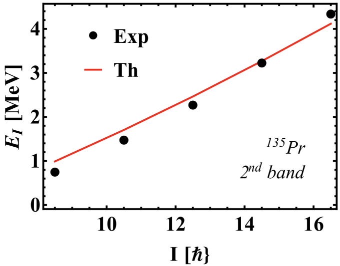
- Find the excitation energies for a nucleus using the obtained analytical results E_n.
- Applied the formalism on the wobbling spectrum of ^{135}Pr.
- Experimental confirmation for three wobbling bands (Matta et al. 2015).

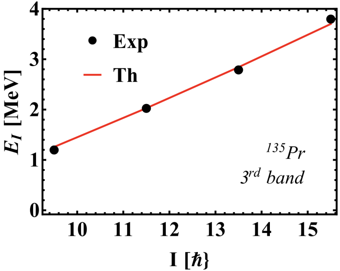
Fitting procedure

- Moments of inertia $\mathcal{I}_{1,2,3}$ are set as free parameters.
- Angle θ is also a free variable.
- ^{135}Pr ($Z = 59$, $N = 76$) has $j = 11/2$ (proton from the $h_{11/2}$ orbital).
- **Fitted MoI:** $\mathcal{I}_1 : \mathcal{I}_2 : \mathcal{I}_3 = 91 : 9 : 51$ [$\hbar$MeV^{-1}].
- **Fitted** θ: -119 degrees (-2.07694 rad).
- $E_{\text{rms}} = 0.170$ MeV.

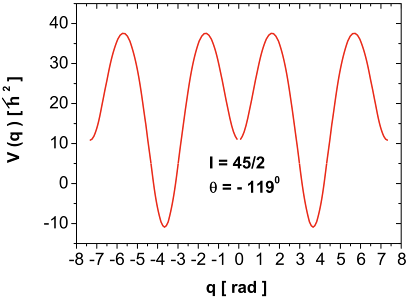
Excitation Energies

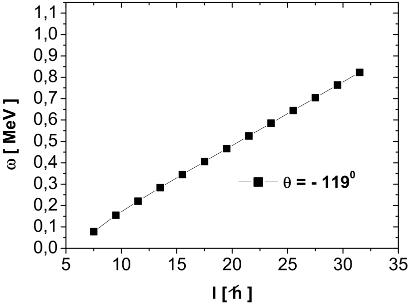






Elliptic potential





- Contribution coming from the wobbling frequency is rather small.
- Potential exhibits local minima and deep minima.
- Stability of the wobbling motion is characterized by $q \sim 3 - 4$ rad.

Conclusions

- The Hamiltonian for an odd-mass nucleus H_{rot} is described based on the coupling between *triaxial even-even core* and an odd particle.
- H_{rot} is split up in three terms: H', H_{sp}, and a spin term.
- Treating H' through the variables $q, d/dq$, one obtained a Schrodinger equation with separated kinetic and potential terms.
- The potential term $V(q)$ is expressed in terms of the Jacobi elliptic functions s, c, d.
- Bargmann mapping changes the a.m. components to boson operators.
- Expansion of $V(q)$ up to second order in q is used to obtain E_n.
- Spectrum of ^{135}Pr is described, reproducing the energies for the three wobbling bands.

Thank you for your attention!