Energy paraboloid for a triaxial rigid rotor

3D Contour Plot: odd-A case

Calculations done within Cartesian coordinate system. Energy of the odd nucleus is given by the Hamiltonian with formula:

$$H' = x_2^2 + ux_3^2 + 2v_0x_1$$

The above equation expresses a quadratic surface, namely, an ***elliptical paraboloid***. so the paraboloid generated is:

$$\mathcal{P}: \qquad vx_1 = x_2^2 + ux_3^2$$

with $v = -2v_0$.

- $\mathcal{P}$ is linear in x_1, so the paraboloid will always be oriented along the 1-axis, that is x_1.
- Since the *rotational scale factors* (u, v) can be positive and negative, depending on the values, the paraboloid will be oriented along the positive ox axis or the negative ox axis, respectively.
- Values of the scale parameters are given by the moments of inertia $\mathcal{I}_1$,$\mathcal{I}_2$,$\mathcal{I}3$, or, accordingly, by the rotational parameters A_1, A_2, A_3.

Tasks:

1. find the graphical representation of the angular momentum I of the system.
2. find the graphical representation of the energy paraboloid
3. represent the intersection of the energy paraboloid and the angular momentum sphere -> **intersection provides the classical trajectories for the system**.

Approach

A) *Angular momentum sphere.*
Using `ContourPlot3D`, one can easily plot the equation $x_1^2 + x_2^2 + x_3^2 = I^2$, which will result in a sphere of radius $r = I$. The terms x_1, x_2, x_3 are the components of the angular momentum, expressed in cartesian coordinates.
B) *Energy paraboloid.*
With the same `ContourPlot3D` function, it is possible to plot the contour surface for the equation $x_2^2 + ux_3^2 + 2v_0x_1 = E$, where E is the energy of the system: arbitrary value.

The scale parameters are fixed in such a way that there are three main cases:

- maximal moment of inertia is $\mathcal{I}_1$ - rotation around the 1-axis
  - In terms of the rotational parameters, the ordering is: $A_3 > A_2 > A_1$.
  - the numerical values for all the constants involved can be seen in the table below.

A_1	A_2	A_3	j	θ
0.0079	0.0152	0.0385	11/2	-112

- maximal moment of inertia is $\mathcal{I}_2$ - rotation around the 2-axis
  - In terms of the rotational parameters, the ordering is: $A_1 > A_3 > A_2$.
  - the numerical values for all the constants involved can be seen in the table below.

A_1	A_2	A_3	j	θ
0.0417	0.0093	0.0278	11/2	-98

- maximal moment of inertia is $\mathcal{I}_3$ - rotation around the 3-axis
  - In terms of the rotational parameters, the ordering is: $A_2 > A_1 > A_3$.
  - the numerical values for all the constants involved can be seen in the table below.

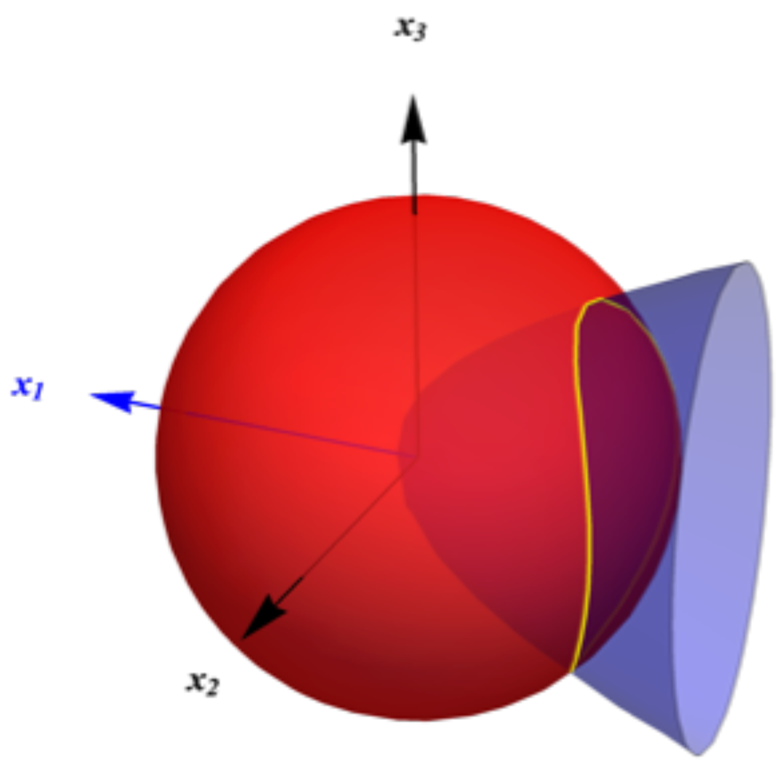
A_1	A_2	A_3	j	θ
0.0357	0.0625	0.0104	11/2	94

Once the scale parameters, odd particle angular momentum, and coupling angle are fixed, the `ContourPlot3D` is applied for arbitrary values of $E \in [0, E_{max}]$. The obtained paraboloid is represented in terms of the cartesian coordinates x_k , $k = 1, 2, 3$ where the range of the coordinates is inside $[-I, I]$.

In this numerical case, the value of the total spin was fixed to $I = 19/2$.

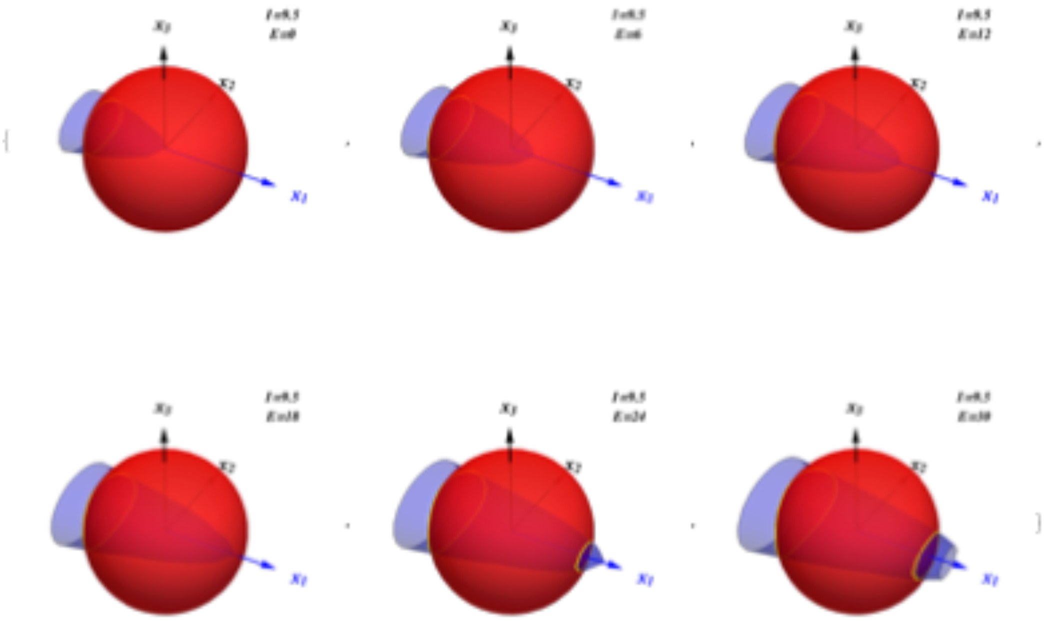
Graphical representation - results

Sketch example:



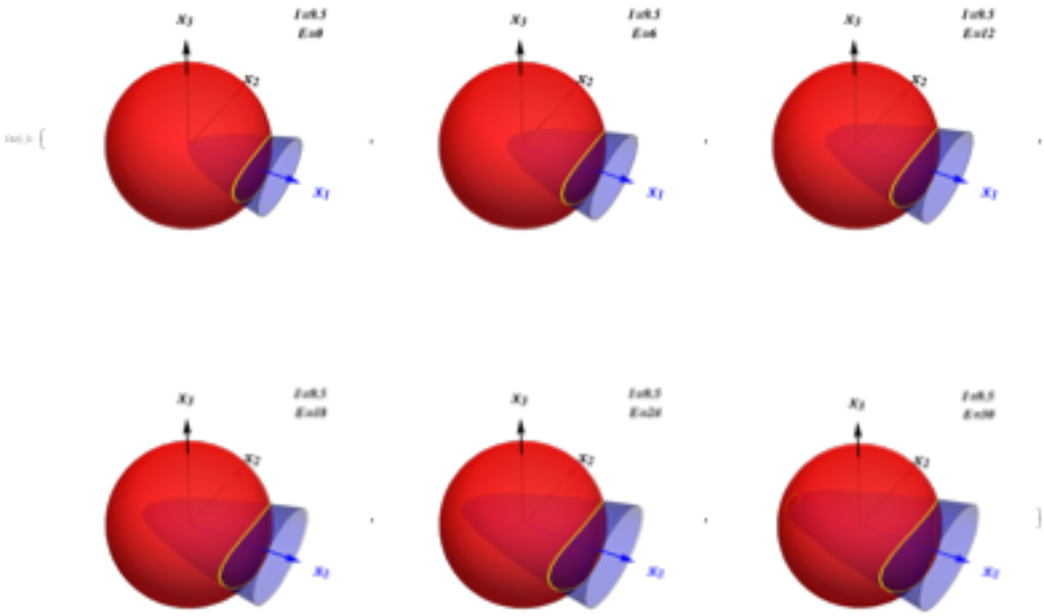
The plot represents the angular momentum sphere with radius $I = 19/2$, with the components of the angular momentum I, namely x_1, x_2, x_3 which point to the corresponding directions. The 1-axis is colored with blue to emphasize the fact that the paraboloid is oriented along it. The values of (u, v) determine if the orientation of calculated paraboloid is along the positive or the negative side. ***In this example, paraboloid points in the negative x_1 direction.*** *Intersection of the sphere with the energy paraboloid is marked by the yellow line, which is a curve that represents the classical trajectories for the odd-A system.*

MOI maximal along 1-axis



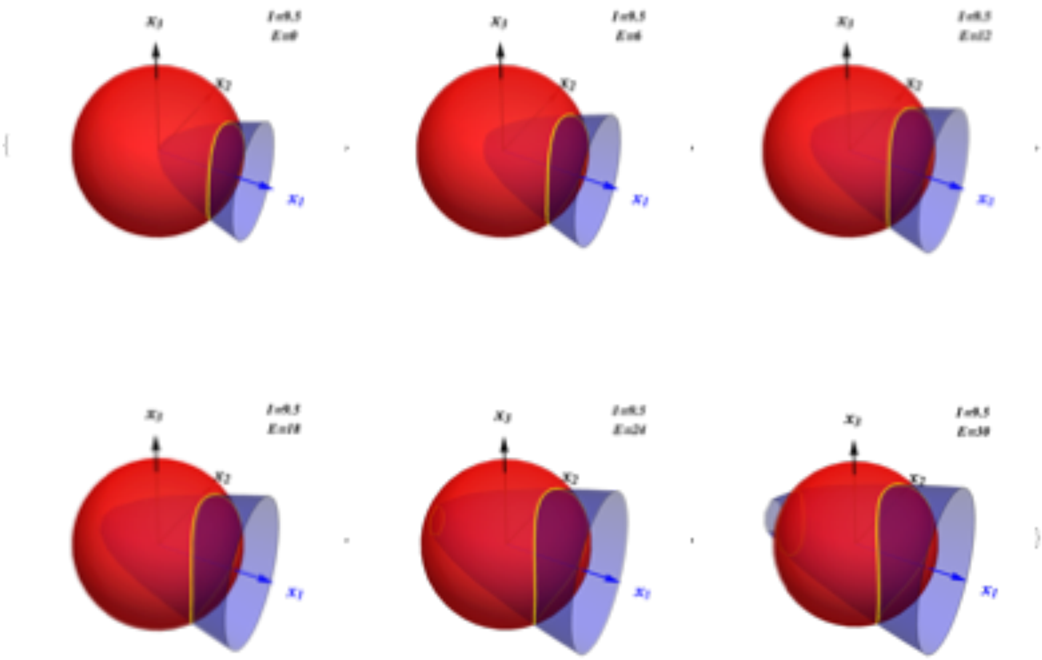
The classical trajectories (yellow curves on the surface of the a.m. sphere) for an odd-A system which has the largest moment of inertia around 1-axis.

MOI maximal along 2-axis



The classical trajectories (yellow curves on the surface of the a.m. sphere) for an odd-A system which has the largest moment of inertia around 2-axis.

MOI maximal along 3-axis



The classical trajectories (yellow curves on the surface of the a.m. sphere) for an odd-A system which has the largest moment of inertia around 3-axis.

Transition of the rotational axis

For sufficiently large energies E, the system can actually change the rotational axis.

- The trajectory of the system evolves in such a way that at some value E_s, the bottom of the paraboloid touches the angular momentum sphere, and with the increase of the energy, the paraboloid extends across any of the other two axes (e.g. 2 or 3) beyond the sphere. (example below).

![the paraboloid extended along x_2 axis beyond the edges of the angular momentum sphere of radius I. Beyond the *critical point*, system starts to rotate around x_3 axis, the yellow curves on the sphere suggesting such motion.][the paraboloid extended along x_2 axis beyond the edges of the angular momentum sphere of radius I. Beyond the critical point, system starts to rotate around x_3 axis, the yellow curves on the sphere suggesting such motion.](./images/

Energy unit

The energy of the nucleus is further expressed in terms of a value which should correspond to the type of motion:

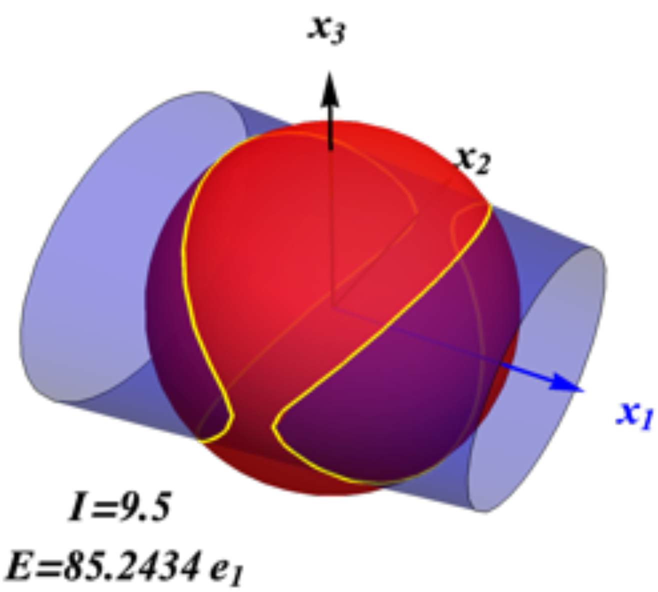
- since the nucleus rotates around the axis with the largest moment of inertia, most of the energy will be given by the rotational term: $E = A_k(I_k - j_k)^2$, where k corresponds to the axis with largest moment of inertia.
- in the present case, $j_3 = 0$, and for each axis the maximal value for I_k is equal with the total spin I.

As a result:

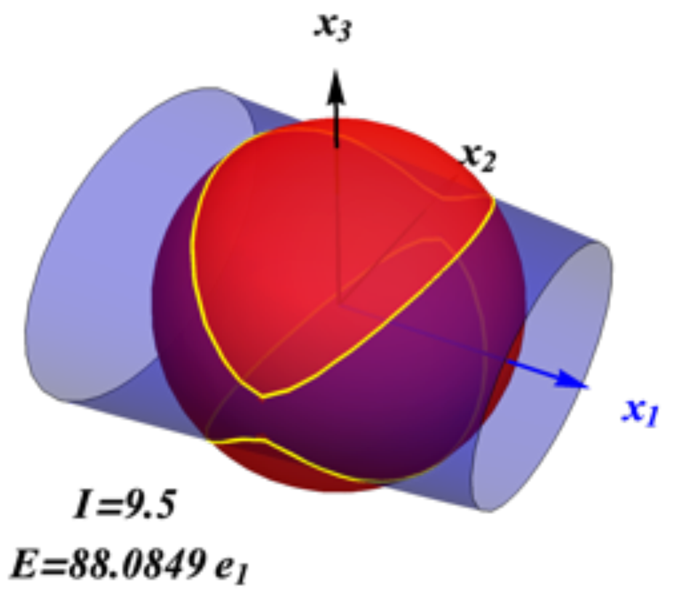
- 1-axis quantization: energy unit is given in terms of $E_{\text{unit}} = A_1(I - j_1)^2 \equiv e_1$
- 2-axis quantization: energy unit is given in terms of $E_{\text{unit}} = A_2(I - j_2)^2 \equiv e_2$
- 3-axis quantization: energy unit is given in terms of $E_{\text{unit}} = A_3 I^2 \equiv e_3$

Furthermore, the critical energies are expressed in terms of the newly declared units, just for consistency.

Critical energy for 1-axis case

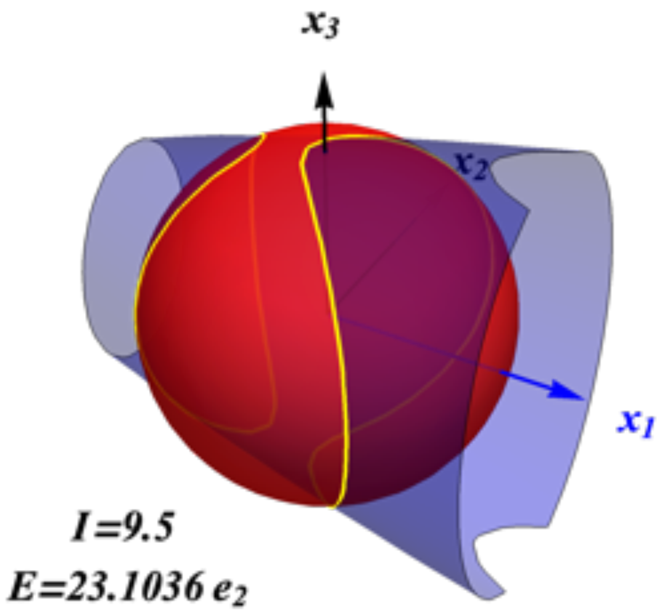


Rotation of the odd nucleus at an energy which is close to the change in the rotational axis. Energy is given in the units discussed above.

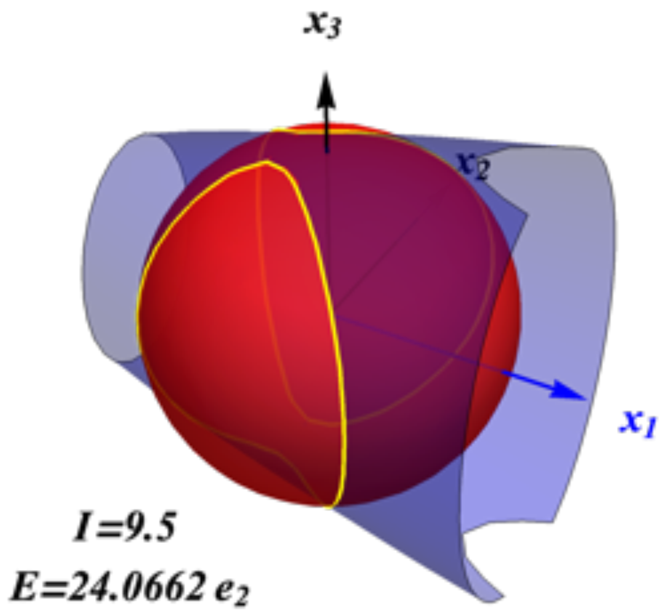


Rotation of the odd nucleus at a critical energy, where the nucleus changes its trajectory.

Critical energy for 2-axis case

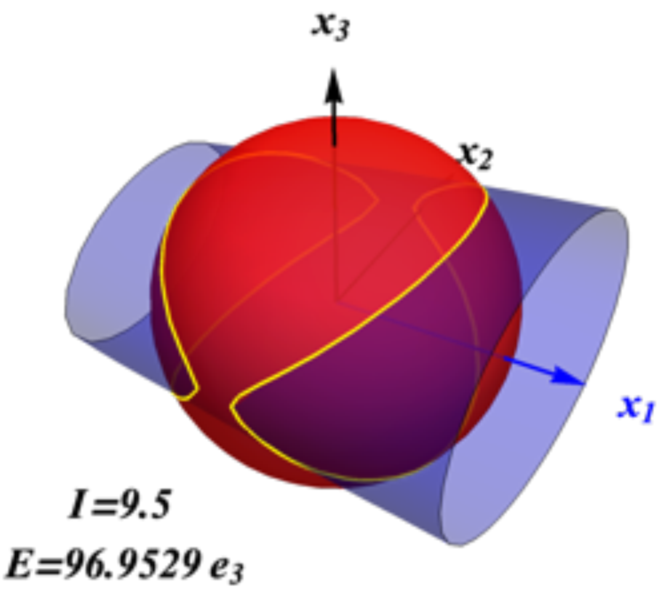


Rotation of the odd nucleus at an energy which is close to the change in the rotational axis. Energy is given in the units discussed above.

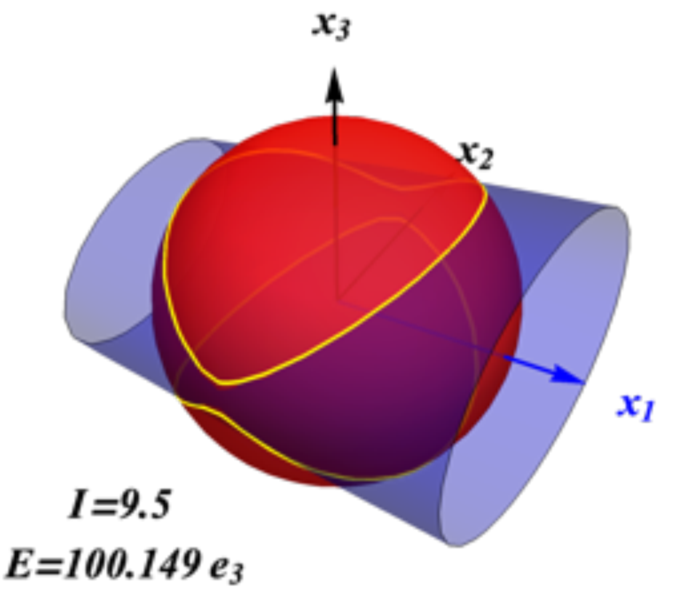


Rotation of the odd nucleus at a critical energy, where the nucleus changes its trajectory.

Critical energy for 3-axis case



Rotation of the odd nucleus at an energy which is close to the change in the rotational axis. Energy is given in the units discussed above.



Rotation of the odd nucleus at a critical energy, where the nucleus changes its trajectory.

Issues to discuss in detail

Even though the classical trajectories for an odd-A system were found, for an arbitrary spin value and for different MOI orderings, there are some unique features that arise from this investigation which are worth looking into more details.

1. The classical trajectories are given by the intersection of the energy paraboloid with the angular momentum sphere.
   1. The specific shape and orientation of the paraboloid (dictated by the values of the scale parameters (u, v)) makes the intersection to take place at lower energies for one of the two sides of 1-axis (see yellow elliptic curve on one of the sides of the sphere), then, with the increase in energy E, the paraboloid will start to extend towards the opposite direction, touching the a.m. sphere at a certain value.
   2. This mechanism is different from the one from *Petrache*: In that case, the trajectory was created by the intersection of the sphere with an ellipsoid $\rightarrow$ shape of the trajectories were the same across both directions of the

rotational axis.

   3. ***How can one explain such a difference*** while both models aim at giving the trajectory for the same nuclear system?

2. The shape of the paraboloid can not be easily related to the inertia moments (or ratios between them), because there is also an implication from j and the coupling angle θ.
3. A clear physical implication must be found for the present results, and to point the differences between *transverse* and *longitudinal* wobbling.
4. Evolution of the energy paraboloid with the change in spin must be considered.