Formal analysis of the E/A_- function

1 Function Definition

We will simply denote E/A_- by E throughout this document. The function is defined as:

$$E(r,t,I,S) = t_1(r,t,I,S) + t_2(r,t,I,S) \tag{1}$$
$$\tag{2}$$

where:

$$t_1(r,t,I,S) = -2\sqrt{rt(2I-r)(2S-t)} \tag{3}$$
$$t_2(r,t,I,S) = \frac{I-1}{2S}t + \frac{S-1}{2I}r + \frac{r}{2IS}t \tag{4}$$

For fixed parameters I and S, we can express the function as:

$$E(r,t) = -2\sqrt{rt(2I-r)(2S-t)} + \frac{I-1}{2S}t + \frac{S-1}{2I}r + \frac{r}{2IS}t \tag{5}$$

2 Critical Points

The critical points of the E function are found by setting the first-order partial derivatives equal to zero:

$$\frac{\partial E}{\partial r} = \frac{-t(2I-r)(2S-t)}{2r\sqrt{rt(2I-r)(2S-t)}} + \frac{S-1}{2I} + \frac{t}{2IS} = 0\ , \tag{6}$$
$$\frac{\partial E}{\partial t} = \frac{-r(2I-r)(2S-t)}{2t\sqrt{rt(2I-r)(2S-t)}} + \frac{I-1}{2S} + \frac{r}{2IS} = 0 \tag{7}$$

These equations determine the critical points (r_c, t_c) where the gradient of the function is zero.

3 Hessian Matrix

The Hessian matrix of the E function with respect to variables r and t (treating I and S as fixed parameters) is:

$$H_E(r,t) = \begin{pmatrix} \frac{\partial^2 E}{\partial r^2} & \frac{\partial^2 E}{\partial r \partial t} \\ \frac{\partial^2 E}{\partial t \partial r} & \frac{\partial^2 E}{\partial t^2} \end{pmatrix} = \begin{pmatrix} \frac{t(2S-t)(3r-2I)}{4r(r-2I)^2\sqrt{rt(2I-r)(2S-t)}} & \frac{1}{2IS} - \frac{(2I-r)(2S-t)+rt}{4rt\sqrt{rt(2I-r)(2S-t)}} \\ \frac{1}{2IS} - \frac{(2I-r)(2S-t)+rt}{4rt\sqrt{rt(2I-r)(2S-t)}} & \frac{r(2I-r)(3t-2S)}{4t(t-2S)^2\sqrt{rt(2I-r)(2S-t)}} \end{pmatrix} \tag{8}$$

4 Function Domain

It is important to have the domains for the variables r, t, since finding the minimum of E would require some constraints. The ranges for r and t can be determined by analyzing the square root from t_1 given by Eq.(4).

In order for the function E to be real, the term in the square root must be non-negative, thus:

$$rt(2I-r)(2S-t) \geq 0\ ,$$

which provides the valid regions for r, t as:

$$r \in [0, 2I]\ ,\ t \in [0, 2S]$$

4.1 Critical points