Formal analysis of the E/A_- function

1 Function Definition

We will simply denote E/A_- by E throughout this document. The function is defined as:

$$E(r,t,I,S) = t_1(r,t,I,S) + t_2(r,t,I,S) \tag{1}$$

where:

$$\begin{aligned} t_1(r,t,I,S) &= -2\sqrt{rt(2I-r)(2S-t)}\ , \\ t_2(r,t,I,S) &= \frac{I-1}{2S}t + \frac{S-1}{2I}r + \frac{rt}{4IS}\ . \end{aligned} \tag{2}$$

For fixed parameters I and S, we can express the function as:

$$E(r,t) = -2\sqrt{rt(2I-r)(2S-t)} + \frac{I-1}{2S}t + \frac{S-1}{2I}r + \frac{rt}{4IS} \tag{3}$$

2 Critical Points

The critical points of the E function are found by setting the first-order partial derivatives equal to zero:

$$\begin{aligned} \frac{\partial E}{\partial r} &= \frac{S-1}{2I} + \frac{t}{4IS} - \frac{(2S-t)t(2I-r) - (2S-t)tr}{\sqrt{(2S-t)t(2I-r)r}} = 0\ , \\ \frac{\partial E}{\partial t} &= \frac{I-1}{2S} + \frac{r}{4IS} - \frac{(2I-r)r(2S-t) - (2I-r)rt}{\sqrt{(2I-r)r(2S-t)t}} = 0 \end{aligned} \tag{4}$$

These equations determine the critical points (r_c, t_c) where the gradient of the function is zero.

3 Hessian Matrix

The Hessian matrix of the E function with respect to variables r and t (treating I and S as fixed parameters) is:

$$H_E(r,t) = \begin{pmatrix} \frac{\partial^2 E}{\partial r^2} & \frac{\partial^2 E}{\partial r \partial t} \\ \frac{\partial^2 E}{\partial t \partial r} & \frac{\partial^2 E}{\partial t^2} \end{pmatrix}\ , \tag{5}$$

with the derivatives:

$$\frac{\partial^2 E}{\partial r^2} = -2\left(-\frac{(2S-t)t}{\sqrt{(2S-t)t(2I-r)r}} - \frac{((2S-t)t(2I-r) - (2S-t)tr)^2}{4\left((2S-t)t(2I-r)r\right)^{3/2}} \right) \tag{6}$$

$$\frac{\partial^2 E}{\partial t^2} = -2\left(-\frac{(2I-r)r}{\sqrt{(2I-r)r(2S-t)t}} - \frac{((2I-r)r(2S-t) - (2I-r)rt)^2}{4\left((2I-r)r(2S-t)t\right)^{3/2}} \right) \tag{7}$$

$$\frac{\partial^2 E}{\partial r \partial t} = \frac{1}{4IS} - \frac{(2I-r)(2S-t) - r(2S-t) - (2I-r)t + rt}{\sqrt{(2I-r)r(2S-t)t}} \tag{8}$$

4 Function Domain

It is important to have the domains for the variables r, t, since finding the minimum of E would require some constraints. The ranges for r and t can be determined by analyzing the square root from t_1 given by Eq.(2).

In order for the function E to be real, the term in the square root must be non-negative, thus:

$$rt(2I-r)(2S-t) \geq 0\ ,$$

which provides the valid regions for r, t as:

$$r \in [0, 2I]\ ,\ t \in [0, 2S]$$

4.1 Critical points

We solve via `NSolve` the system of equations from Eq.(4) and only consider the real solutions. The points provided by the system are denoted by $\{r_c, t_c\}$ (to emphasize the critical points). Moreover, by studying second-order partial derivatives of E w.r.t to r and t, we can finally determine if the critical points represent local minimum or maximum.

5 Numerical Analysis

All the numerical values for the minimum points are provided below.

I	r_c	t_c	$\frac{\partial^2 E}{\partial r^2}$	$\frac{\partial^2 E}{\partial t^2}$
1	0.888889	0.888889	2.025	2.025
2	1.89609	0.81856	0.987397	4.20029
3	2.89829	0.79726	0.653949	6.38622
4	3.89935	0.786972	0.488987	8.57447
5	4.89998	0.780911	0.390516	10.7636
6	5.90039	0.776915	0.325067	12.9532
7	6.90068	0.774083	0.278412	15.143
8	7.9009	0.771971	0.24347	17.3329
9	8.90107	0.770335	0.216321	19.523
10	9.9012	0.769031	0.194621	21.7131

Table 1: Critical points and second-order derivatives for fixed $S = 1$.

S	r_c	t_c	$\frac{\partial^2 E}{\partial r^2}$	$\frac{\partial^2 E}{\partial t^2}$
1	0.888889	0.888889	2.025	2.025
2	0.81856	1.89609	4.20029	0.987397
3	0.79726	2.89829	6.38622	0.653949
4	0.786972	3.89935	8.57447	0.488987
5	0.780911	4.89998	10.7636	0.390516
6	0.776915	5.90039	12.9532	0.325067
7	0.774083	6.90068	15.143	0.278412
8	0.771971	7.9009	17.3329	0.24347
9	0.770335	8.90107	19.523	0.216321
10	0.769031	9.9012	21.7131	0.194621

Table 2: Critical points and second-order derivatives for fixed $I = 1$.

As we can see from Table 1 and 2 the critical points are all **minima**, since the second derivative is positive. The value for (r_c, t_c) for fixed spin values $I, S = 8, 8$ is the following tuple:

$$(r_c, t_c; I = 8, S = 8) = (7.76608\ ,\ 7.76608)$$