Formal analysis of the E/A_{-} function

1 Function Definition

We will simply denote E/A_{-} by E throughout this document. The function is defined as:

$$E(r,t,I,S) = t_1(r,t,I,S) + t_2(r,t,I,S) \tag{1}$$
$$\tag{2}$$

where:

$$t_1(r,t,I,S) = -2\sqrt{rt(2I-r)(2S-t)} \tag{3}$$
$$t_2(r,t,I,S) = \frac{I-1}{2S}t + \frac{S-1}{2I}r + \frac{r}{2IS}t \tag{4}$$

For fixed parameters I and S, we can express the function as:

$$E(r,t) = -2\sqrt{rt(2I-r)(2S-t)} + \frac{I-1}{2S}t + \frac{S-1}{2I}r + \frac{r}{2IS}t \tag{5}$$

2 Critical Points

The critical points of the E function are found by setting the first-order partial derivatives equal to zero:

$$\begin{aligned}\frac{\partial E}{\partial r} &= \frac{-t(2I-r)(2S-t)}{2r\sqrt{rt(2I-r)(2S-t)}} + \frac{S-1}{2I} + \frac{t}{2IS} = 0\ ,\\ \frac{\partial E}{\partial t} &= \frac{-r(2I-r)(2S-t)}{2t\sqrt{rt(2I-r)(2S-t)}} + \frac{I-1}{2S} + \frac{r}{2IS} = 0\end{aligned} \tag{6}$$

These equations determine the critical points (r_c, t_c) where the gradient of the function is zero.

3 Hessian Matrix

The Hessian matrix of the E function with respect to variables r and t (treating I and S as fixed parameters) is:

$$H_E(r,t) = \begin{pmatrix} \frac{\partial^2 E}{\partial r^2} & \frac{\partial^2 E}{\partial r \partial t} \\ \frac{\partial^2 E}{\partial t \partial r} & \frac{\partial^2 E}{\partial t^2} \end{pmatrix} = \begin{pmatrix} \frac{t(2S-t)(3r-2I)}{4r(r-2I)^2\sqrt{rt(2I-r)(2S-t)}} & \frac{1}{2IS} - \frac{(2I-r)(2S-t)+rt}{4rt\sqrt{rt(2I-r)(2S-t)}} \\ \frac{1}{2IS} - \frac{(2I-r)(2S-t)+rt}{4rt\sqrt{rt(2I-r)(2S-t)}} & \frac{r(2I-r)(3t-2S)}{4t(t-2S)^2\sqrt{rt(2I-r)(2S-t)}} \end{pmatrix} \tag{7}$$

4 Function Domain

It is important to have the domains for the variables r, t, since finding the minimum of E would require some constraints. The ranges for r and t can be determined by analyzing the square root from t_1 given by Eq.(4).

In order for the function E to be real, the term in the square root must be non-negative, thus:

$$rt(2I-r)(2S-t) \geq 0\ ,$$

which provides the valid regions for r, t as:

$$r \in [0, 2I]\ ,\ t \in [0, 2S]$$

4.1 Critical points

We solve via `NSolve` the system of equations from Eq.(6) and only consider the real solutions. The points provided by the system are denoted by $\{r_c, t_c\}$ (to emphasize the critical points). Moreover, by studying second-order partial derivatives of E w.r.t to r and t, we can finally determine if the critical points represent local minimum or maximum.

Numerical Analysis

As we can see from Table 1 and 2 the critical points are all **minima**, since the second derivative is positive. The value for (r_c, t_c) for fixed spin values $I, S = 8, 8$ is the following tuple:

$$(r_c, t_c; I = 8, S = 8) = (7.7509\ ,\ 7.7509)$$

I	r_c	t_c	$\frac{\partial^2 E}{\partial r^2}$	$\frac{\partial^2 E}{\partial t^2}$
1	0.8	0.8	2.08333	2.08333
2	1.80364	0.767656	0.986868	4.32622
3	2.80429	0.761959	0.651659	6.53469
4	3.80452	0.759986	0.487129	8.73419
5	4.80462	0.759076	0.389108	10.9301
6	5.80468	0.758583	0.323989	13.1241
7	6.80471	0.758286	0.277566	15.3171
8	7.80473	0.758093	0.242792	17.5095
9	8.80475	0.757961	0.215767	19.7014
10	9.80476	0.757867	0.19416	21.893

Table 1: Critical points and second-order derivatives for fixed $S = 1$

S	r_c	t_c	$\frac{\partial^2 E}{\partial r^2}$	$\frac{\partial^2 E}{\partial t^2}$
1	0.8	0.8	2.08333	2.08333
2	0.767656	1.80364	4.32622	0.986868
3	0.761959	2.80429	6.53469	0.651659
4	0.759986	3.80452	8.73419	0.487129
5	0.759076	4.80462	10.9301	0.389108
6	0.758583	5.80468	13.1241	0.323989
7	0.758286	6.80471	15.3171	0.277566
8	0.758093	7.80473	17.5095	0.242792
9	0.757961	8.80475	19.7014	0.215767
10	0.757867	9.80476	21.893	0.19416

Table 2: Critical points and second-order derivatives for fixed $I = 1$