In spite of these deficiencies, the boson model presented so far has been extensively used over the years and has proved quite successful in explaining many features of the nuclear spectra [EG 70]. The reason for this is that many low-lying nuclear excitations have a rather collective character and can be represented to a good approximation as bosons. The parameters B_λ and C_λ may not be given very well in the liquid drop model, but, as we will shown in Chapters 8 and 9, there are microscopic theories for fermions which allow more reliable calculations of B_λ and C_λ; therefore, we should consider B_λ and C_λ more as open adjustable parameters than as determined by the LDM.

The microscopic theories also will show that the harmonic approximations (1.17) have only a very rough validity in the limit of very small vibrations. For collective motions with larger amplitudes, one has to take into account anharmonic terms such as

$$\alpha^4, \alpha^2 \cdot \dot{\alpha}^2, \dot{\alpha}^4, \ldots .$$

Again the corresponding parameters can be adjusted to experiment or calculated from a microscopic many-body theory (see Chap. 9).

1.5 Rotations and Vibrations for Deformed Shapes

1.5.1 The Bohr Hamiltonian

The pure liquid drop model has a stable equilibrium only for spherical surfaces. As we shall see later (Sec. 2.8) it can happen as a consequence of quantum mechanics—i.e., shell effects—that the potential $V(\alpha)$ in the collective Hamiltonian has minima at finite non-vanishing values of $\alpha = \alpha_0$. In such cases the nucleus can have a stable ground state deformation.

In this case, the nucleus can exhibit *rotations* which can be described by time-dependent surface parameters $\alpha_{\lambda\mu}$ in the laboratory frame. We shall call these rotations collective ones: This kind of rotation will not be possible around an axis of symmetry, because we cannot distinguish the rotated system from the original one in our variables $\alpha_{\lambda\mu}$.

In a quantum mechanical description, a system with an axis of symmetry (for example, the z-axis) is given by a wave function which is an eigenfunction of the angular momentum operator J_z, and any rotation about this axis produces only a phase. The rotating system has, therefore, the same wave function as the ground state, and the same energy.

This does not mean that there are no other degrees of freedom in the system that can be excited (for instance, single-particle degrees of freedom) and carry angular momentum parallel to the symmetry axis. Such a "rotation," however, we do not call collective rotation.

Since in almost all nuclei the quadrupole degree of freedom plays a fundamental role, we will restrict the following considerations to the case $\lambda = 2$.

Assuming that the nucleus has a stable ground state deformation, it is preferable to transform to the body-fixed system, defined by the principal axes of the mass distribution (as discussed in Sec. 1.3).

After this transformation, we have five dynamical variables Ω, β, γ* instead of the variables $\alpha_{2\mu}$ ($\mu = -2, -1, 0, +1, +2$). We start again from the Hamiltonian (1.17). Only the potential in (1.17) is changed. It now has the form:

$$V(\beta,\gamma)=\tfrac{1}{2}C_{20}\big(a_{20}(\beta,\gamma)-a_{20}^0\big)^2+C_{22}\big(a_{22}(\beta,\gamma)-a_{22}^0\big)^2. \qquad (1.46)$$

This corresponds to a quadratic approximation in the vicinity of a deformed minimum β_0, γ_0. The idea is that the nucleus has this deformation in its ground state, and the excitations are rotations and small oscillations around this equilibrium deformation.

Microscopic calculations of potential landscapes show, for certain nuclei, well pronounced minima of V for finite values $\beta_0 \simeq 0.2-0.3$ and $\gamma_0 = 0$. These axially symmetric shapes are, therefore, the most important ones, and we will restrict a large part of our discussion to them.

The next step is the transformation of the kinetic energy in Eq. (1.17) to the body-fixed system. Applying Eq. (1.11), we have to differentiate the variables a_{20}, a_{22}, and Ω with respect to the time. Since the derivation is quite lengthy, and the intention in this chapter is not to be complete, we simply give the result (for the derivation, see, for example, [EG 70 Vol. I, Chap. 5]) for the so-called Bohr Hamiltonian [Bo 52]:

$$T = T_{\text{rot}} + \tfrac{1}{2}B_2(\dot\beta^2+\beta^2\dot\gamma^2) \qquad (1.47)$$

with

$$T_{\text{rot}} = \tfrac{1}{2}\sum_{\kappa=1}^{3}\vartheta_\kappa\omega_\kappa^2,$$

where ω_κ is the angular velocity around the body-fixed κ-axis and ϑ_κ are functions of β and γ given by

$$\vartheta_\kappa = 4B_2\beta^2\sin^2\Big(\gamma-\frac{2\pi}{3}\kappa\Big), \qquad \kappa = 1,2,3. \qquad (1.48)$$

In case we have fixed deformations β, γ, T_{rot} is the kinetic energy of a rotor with the moment of inertia ϑ_κ. As soon as we allow for changes of β and γ, the rotational and vibrational degrees of freedom are coupled by the deformation dependence of the moments of inertia. In this case, we no longer have a pure rotor. In fact, we see that in the case of $\beta_0 = 0$ the system can be transformed back to a harmonic vibrator (1.17) which has a harmonic spectrum.

However, even in the well deformed case with large stiffness parameters C and finite β_0 (where a constant $\beta = \beta_0$ is a rather good approximation),

* Unfortunately, the set of Euler angles $\Omega = (\alpha, \beta, \gamma)$ also contains the letters β and γ. However, we do not want to change this nomenclature [Ed 57] and we, as far as possible, use the abbreviation Ω.

$\mathcal{I}_\kappa$ are not the moments of inertia of a rigid rotor. Using (1.30), we get from Eq. (1.48) the so-called irrotational moment of inertia:

$$\mathcal{I}_\kappa^{\mathrm{irr}} = \frac{3}{2\pi} mAR_0^2\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}\kappa\right), \qquad \kappa = 1, 2, 3.$$

It differs from the moment of inertia of a rigid body with the same deformation

$$\mathcal{I}_\kappa^{\mathrm{rig}} = \frac{2}{5} mAR_0^2\left(1 - \sqrt{\frac{5}{4\pi}}\ \beta\cos\left(\gamma - \frac{2\pi}{3}\kappa\right)\right) \tag{1.49}$$

in the following ways.

(i) In the γ-dependence (Fig. 1.9), $\mathcal{I}^{\mathrm{irr}}$ vanishes about the symmetry axes.

(ii) $\mathcal{I}^{\mathrm{irr}}$ shows a strong dependence on the deformation ($\sim\beta^2$), whereas $\mathcal{I}^{\mathrm{rig}}$ changes much less with β (its main part is the moment of inertia of a rigid sphere).

(iii) The experimental moment of inertia $\mathcal{I}^{\mathrm{exp}}$ can be found from the energy of the first 2^+ state of a rotational band [see Eq. (1.64)] $\mathcal{I}^{\mathrm{exp}} = 3/E_{2^+}$ [MeV^{-1}]. Applying the empirical rule (1.40) and the formula (1.74) for the $BE2$ value, we get a connection between the deformation parameter β and the moment of inertia

$$\mathcal{I}^{\mathrm{exp}} = \frac{27}{80\pi^2} r_0^4 \frac{A^{4/3}\cdot\beta^2\cdot Z^2e^2}{E_{2^+}\cdot BE2} \approx \frac{27(1.2)^4\cdot\beta^2 A^{7/3}}{80\pi^2 25}$$

$$\approx \frac{\beta^2 A^{7/3}}{400}\ [\mathrm{MeV}^{-1}]. \tag{1.50}$$

In the case of well deformed nuclei ($\beta\sim 0.2-0.4$), $\mathcal{I}^{\mathrm{irr}}$ is usually smaller by a factor of 2–3 than the experimental values. On the other side, the values of $\mathcal{I}^{\mathrm{rig}}$ are a factor of 2 too large:

$$\mathcal{I}^{\mathrm{irr}} < \mathcal{I}^{\mathrm{exp}} < \mathcal{I}^{\mathrm{rig}}. \tag{1.51}$$

This shows that the flow structure within the nuclei is certainly not irrotational. On the other hand, it is not a rigid rotor, either.

The next step is again a quantization of the classical Hamiltonian (1.47). It is well known that there is no unique prescription for the quantization of a classical Hamiltonian in the general case [Me 61]. The ambiguity comes from the freedom in ordering noncommutable operators.* Commonly one adopts the Pauli prescription [Pa 33], which calculates the Laplace operator in curvilinear coordinates. If the classical kinetic energy has the form

$$T = \tfrac{1}{2}\sum_{ij}^{f} g_{ij}(\xi)\dot{\xi}_i\dot{\xi}_j, \tag{1.52}$$

* A discussion with a list of references on the quantization problem is found in [MD 73].

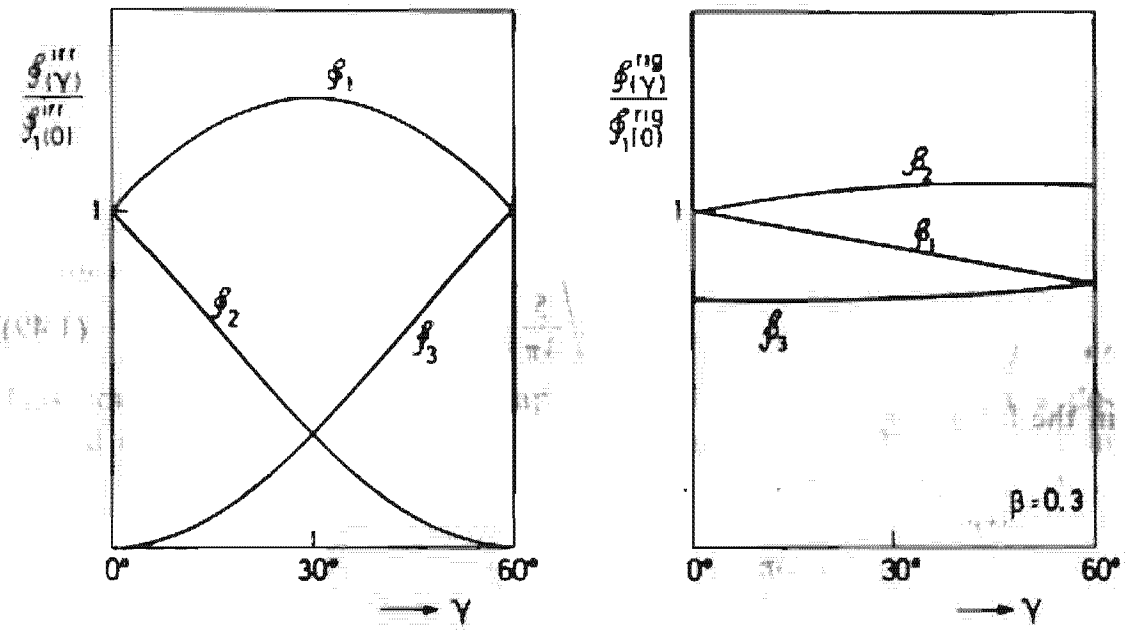

Figure 1.9. The γ-dependence of the irrotational (ϑ_k^{irr}) and the rigid (ϑ_k^{rig}) moments of inertia for fixed values of β.

then the corresponding quantized form is*

$$\hat{H}_{\text{kin}} = -\frac{\hbar^2}{2}\sum_{ij} g^{-1/2}\frac{\partial}{\partial\xi_i}\, g^{1/2}(g^{-1})_{ij}\frac{\partial}{\partial\xi_j}, \tag{1.53}$$

where g is the determinant and g^{-1} is the inverse of the matrix g_{ij}.

Applying this prescription to the Bohr Hamiltonians (1.46) and (1.47), we obtain:

$$\hat{H}_{\text{coll}} = \frac{-\hbar^2}{2B_2}\left[\beta^{-4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right) + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma}\right)\right] + \hat{T}_{\text{rot}} + V(\beta,\gamma), \tag{1.54}$$

where the rotational energy is found to be

$$\hat{T}_{\text{rot}} = \frac{\hat{I}_1^2}{2\vartheta_1} + \frac{\hat{I}_2^2}{2\vartheta_2} + \frac{\hat{I}_3^2}{2\vartheta_3}. \tag{1.55}$$

The operators $\hat{I}_\kappa$ are the projections of the total angular momentum $\hat{\mathbf{I}}$ represented in the Euler angles onto the body-fixed axes (for details, see Appendix A). Figure (1.10) shows the total angular momentum and its components $\hat{I}_z = M$ and $\hat{I}_3 = K$. The eigenfunctions of $\hat{\mathbf{I}}^2, \hat{I}_z, \hat{I}_3$ are given by

$$|IMK\rangle = \sqrt{\frac{2I+1}{8\pi^2}}\; D^{I*}_{MK}(\Omega). \tag{1.56}$$

Since H_{coll}, $\hat{\mathbf{I}}^2$ and $\hat{I}_z$ commute, the eigenfunctions of the collective Hamil-

*To see that the Hamiltonian (1.53) is Hermitian, one has to take into account the volume element $\sqrt{g}\; d\xi_1 \ldots d\xi_f$.

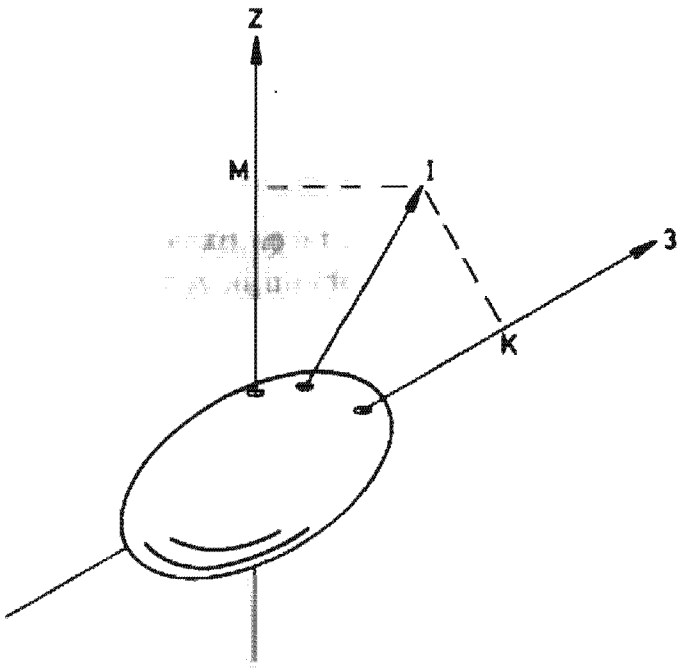

Figure 1.10. The relation between the total angular momentum **I** and its projection M onto the laboratory z axis and its projection K onto the body-fixed 3-axis.

tonian (1.54) have the general form

$$|\Psi^I_M\rangle = \sum_K g_K(\beta,\gamma)|IMK\rangle.$$

The triaxial rotor has certain discrete symmetries. $\hat{H}_{\text{coll}}$ is invariant under the point group D_2 [Bo 52]. Therefore, one classifies the eigenstates according to the irreducible representations of this group, and one can derive from this some properties of the spectra [Da 68].

One example is the rotation of 180° around the 1-axis,

$$\mathcal{R}_1 = e^{i\pi \hat{I}_1}, \tag{1.57}$$

which is equivalent to a reflection with respect to the 2, 3-plane together with a parity transformation. It commutes with $\hat{H}_{\text{coll}}$ and we require our eigenfunctions to have eigenvalue $+1$*:

$$\mathcal{R}_1|\Psi^I_M\rangle = |\Psi^I_M\rangle. \tag{1.58}$$

Using relation (A.24) of Appendix A, we find that this is only possible with

$$g_K(\beta,\gamma) = (-)^I g_{-K}(\beta,\gamma). \tag{1.59}$$

If we require, in the same way, symmetry with respect to

$$\mathcal{R}_2 = e^{i\pi \hat{I}_2}, \tag{1.60}$$

we get

$$g_K(\beta,\gamma) = (-)^{I+K} g_{-K}(\beta,\gamma). \tag{1.61}$$

* As we shall see in Eq. (11.123), the microscopic intrinsic wave function of the system is an eigenstate of this operation with eigenvalue $+1$ (see also [Bo 76 b]).

1.5.2 The Axially Symmetric Case

The Hamiltonian $\hat{H}_{\text{coll}}$ (1.54) is still very general. We shall restrict ourselves to cases of very pronounced minima in the potential surface at axially symmetric deformations $\beta=\beta_0$ and $\gamma=0$. We expect rotations and small vibrations of the nuclear surface. Expanding $T_{\text{rot}}-\hat{I}_3^2/2\vartheta_3$ (1.55) around the point $\beta=\beta_0$, $\gamma=0$, we obtain in zeroth order the Hamiltonian of an axially symmetric rotor with the moment of inertia $\vartheta_0=\vartheta_1(\beta_0,0)=\vartheta_2(\beta_0,0)$,

$$T'_{\text{rot}}=\frac{\hat{I}^2-\hat{I}_3^2}{2\vartheta_0}. \tag{1.62}$$

First-order terms are proportional to the deviations $(\beta-\beta_0)$ and γ. They mix rotational and vibrational degrees of freedom (rotational–vibrational coupling terms) and will be neglected here. The only remaining term in H_{coll} that still couples rotations and vibrations is $\hat{I}_3^2/2\vartheta_3$. It cannot be expanded, since ϑ_3 vanishes for $\gamma=0$. However (as T'_{rot}), it commutes with $\hat{I}_3$ and K is therefore a good quantum number.

We now have to distinguish

(I) $K=0$ **bands** ($I_3=0$). In this case, the rotational and vibrational motions decouple. The wave function is of the type

$$|\Psi^I_{M0}\rangle=g_0(\beta,\gamma)|IM0\rangle. \tag{1.63}$$

They are eigenfunctions of the rotational part of the Hamiltonian (1.54). $\mathfrak{R}_1$-symmetry (1.59) requires the spin sequence $I=0,2,4,\ldots$. A detailed investigation of the vibrational part of H_{coll} [EG 70, Vol. I, Chap. 6] shows that it is easier to handle in the variables a_{20} and a_{22} (1.13). In the first step one neglects terms in the potential $V(a_{20},a_{22})$, which couple these two degrees of freedom. In this case, the motion in the coordinate a_{20} (usually called β-vibration) decouples from the motion in the coordinate a_{22} (usually called γ-vibration). Axial symmetry with respect to the 3-axis is preserved by the β-vibration (quantum number n_β), but violated by the γ-vibration (quantum number n_γ). Both types of motion are shown qualitatively in Fig. 1.11.

Superimposed on each vibrational state (n_β,n_γ) is a rotational band. The spectrum is given by (see [EG 70, Vol. I, Chap. 6])

$$E_{n_\beta n_\gamma}(I)=E_{n_\beta n_\gamma}(0)+\frac{\hbar^2}{2\vartheta_0}I\cdot(I+1) \tag{1.64}$$

with the band head

$$E_{n_\beta n_\gamma}(0)=\hbar\omega_\beta(n_\beta+1/2)+\hbar\omega_\gamma(2n_\gamma+1),$$
$$n_\beta=0,1,2,\ldots,\qquad n_\gamma=0,1,2,\ldots, \tag{1.65}$$

where ω_β and ω_γ are the frequencies of β- and γ-vibrations.

$$\omega_\beta=(C_{20}/B_2)^{1/2}\qquad \omega_\gamma=(C_{22}/B_2)^{1/2}$$

In fact, such bands have been observed in many even–even nuclei, in particular the ground state band ($n_\beta = n_\gamma = 0$) and the "β-band" ($n_\beta = 1$, $n_\gamma = 0$). However, as we have already discussed, the constants $\mathcal{I}_0$ and $\omega_\beta, \omega_\gamma$ of the hydrodynamical model do not agree with the experimental data.

(II) $K \neq 0$ bands. Together with $\mathfrak{R}_1$-symmetry [Eq. (1.58)], we see that the wave function now has the form

$$|\Psi^I_{MK}\rangle = g_K(\beta,\gamma)\frac{1}{\sqrt{2}}\left\{|IMK\rangle + (-)^I|IM-K\rangle\right\}. \tag{1.66}$$

The $\mathfrak{R}_1$ and $\mathfrak{R}_2$ symmetries ((1.59) and (1.61)) give the selection rule: K must be even. Such $K \neq 0$ bands have, therefore, the spin sequence $I = |K|$, $|K|+1$, $|K|+2, \ldots$. The motion in a_{20} (β-vibration) can again be separated from the rest. However, the term $\hat{I}_3/2\mathcal{I}_3 = K^2/16B_2a_{22}^2$ couples the γ-vibration with the rotation around the 3-axis. We can easily understand this fact if we realize that a γ-vibration can be represented as a superposition of two rotations of a triaxial nucleus around the 3-axis, but with opposite K-values [BM 75, p. 656].

The following spectrum is obtained [EG 70, Vol. I, Chap. 6]

$$E_{K,n_\beta,n_\gamma}(I) = E_{K,n_\beta,n_\gamma}(0) + \frac{\hbar^2}{2\mathcal{I}_0}\left(I(I+1) - K^2\right) \tag{1.67}$$

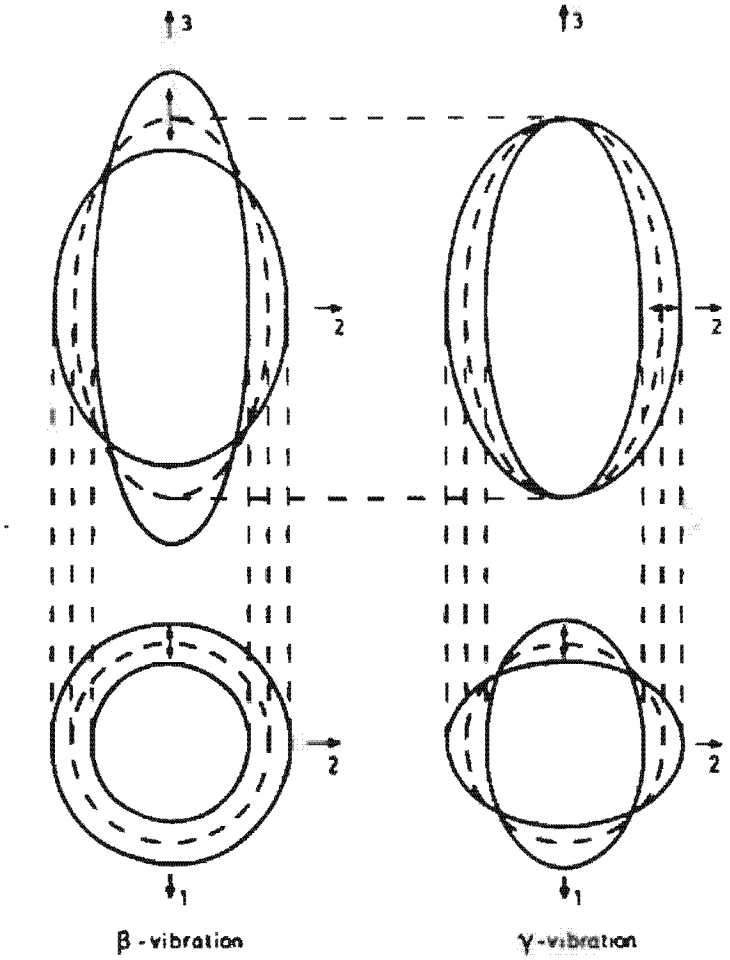

Figure 1.11. Schematical representation of β- and γ-vibrations by a cut along the (1, 3) and (1, 2) planes.

with the bandheads

$$E_{K,n_\beta,n_\gamma}(0)=\hbar\omega_\beta\left(n_\beta+\frac{1}{2}\right)+\hbar\omega_\gamma\left(2n_\gamma+1+\frac{|K|}{2}\right). \tag{1.68}$$

In fact, such bands have also been observed, especially the so-called "γ-band" in many deformed nuclei, which has the quantum numbers $K=2$, $n_\beta=0$, $n_\gamma=0$. This γ-band has the vibrational quantum number $n_\gamma=0$, however, one is not allowed to apply the classical picture of no vibration in this case. It would correspond to $\gamma=0$ and imply $\vartheta_3=0$, which would forbid a rotation with $K\neq 0$. Only the quantum mechanical zero point vibration in the γ-direction makes this possible.

Figure 1.12 shows the qualitative structure of the collective ($\lambda=2$) excitation in deformed and spherical nuclei.

The spherical nuclei have a harmonic spectrum. In the deformed nuclei we observe several rotational bands built on the ground state, on the β-vibrational state $K=0$, $n_\beta=1$, $n_\gamma=0$, and on the γ-vibrational state $K=2$, $n_\beta=n_\gamma=0$. However, these pure cases are not exactly realized in nature. In fact, we already observe in spherical nulcei a splitting of the two-boson triplet $(0^+,2^+,4^+)$ and in the deformed nuclei deviations from the $I(I+1)$ law. There is also a wide range of transitional nuclei in between these two limits. Going from isotope to isotope, one can sometimes observe a gradual transition from a vibrational to a rotational spectrum (for instance, in the Os region [SDG 76]). This is indicated by dashed lines in Fig. 1.12.

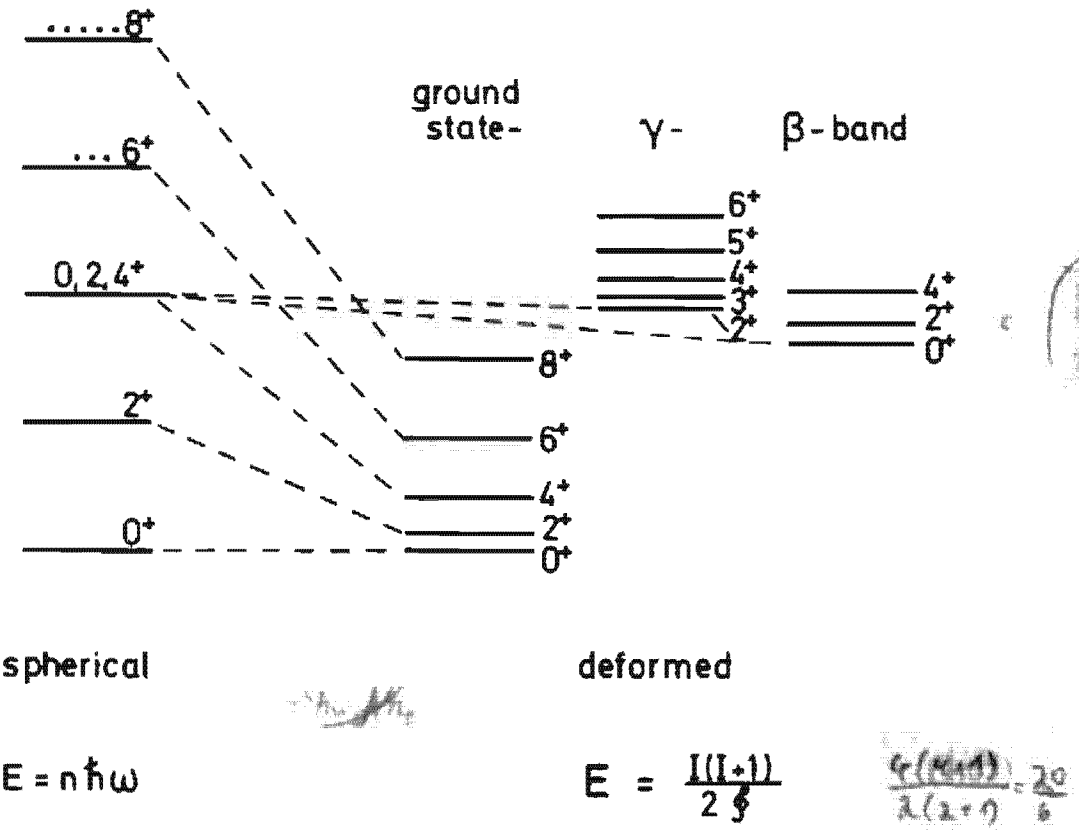

Figure 1.12. Schematic level schemes of spherical and deformed nuclei. (From [SDG 76].)

For a theoretical description of deviations from the $I\cdot(I+1)$ law, one has taken into account the rotational-vibrational coupling terms in $\hat{H}_{\text{coll}}$ which one gets by expanding ϑ_κ in powers of the deviations $(\beta-\beta_0)$ and γ[FG 62, FG 64, FGS 65, 66, ABP 68]. This rotation–vibration interaction causes changes in the moment of inertia of a band and corresponds to a change of the nuclear shape under the influence of the rotation (stretching effect). However, there also exist quite different excitations of nuclei in this energy region, for instance, two quasi-particle states and pairing vibrations (see Chap. 8), which have a much stronger influence on the rotational bands, and which are not taken into account in this simple model of a liquid drop with surface oscillations.

Before we leave this section, we want to discuss very briefly how to calculate electromagnetic moments and transition probabilities. In Eqs. (1.35) and (1.36) we defined the electric multipole and the magnetic moment operator in the coordinates $\alpha_{\lambda\mu}$. They are obviously written down in the laboratory system. In a deformed nucleus it is usually very easy to calculate the moments in the intrinsic system. To get the moments in the laboratory frame—the experimental values—one has to apply the transformation (1.11) of spherical tensors:

$$\hat{Q}^{\text{intr}}_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu} \hat{Q}^{\text{lab}}_{\lambda\mu'}. \tag{1.69}$$

Since Q^{intr} does not depend on the Euler angles, we get from [Ed 57, Eqs. 4.6.2 and 5.4.1] the reduced matrix elements with respect to $|IMK\rangle$:

$$\langle I_1 K_1 \| \hat{Q}^{\text{lab}}_{\lambda} \| I_2 K_2 \rangle = \sum_{\mu'} Q^{\text{intr}}_{\lambda\mu'} (-)^{I_1 - K_1} ((2I_1+1)(2I_2+1))^{1/2} \begin{pmatrix} I_1 & \lambda & I_2 \\ -K_1 & \mu' & K_2 \end{pmatrix}. \tag{1.70}$$

We restrict ourselves now to pure K-bands and calculate only intraband $E2$-transitions ($n_{\gamma_1} = n_{\gamma_2}, n_{\beta_1} = n_{\beta_2}$). For the reduced matrix element we find

$$\langle n_\beta n_\gamma I_1 K \| \hat{Q}^{\text{lab}}_2 \| n_\beta n_\gamma I_2 K \rangle = \sqrt{\frac{5}{16\pi}}\, Q_0(n_\beta, n_\gamma) \sqrt{(2I_1+1)(2I_2+1)}\, (-)^{I_1 - K} \begin{pmatrix} I_1 & 2 & I_2 \\ -K & 0 & K \end{pmatrix}, \tag{1.71}$$

where $Q_0(n_\beta, n_\gamma)$ is the intrinsic quadrupole moment of this band. In the ground state band with fixed β-value, and $\gamma = 0$ we have [Eqs. (1.13) and (1.35)]

$$Q_0 = \sqrt{\frac{16\pi}{5}}\, \frac{3}{4\pi} Ze \cdot R_0^2 \cdot \beta. \tag{1.72}$$

From (B.73) and (1.71) we obtain, for example, for the so-called stretched $BE2$-values in a rotational band,

$$B(E2, I+2 \to I) = Q_0^2 \frac{5}{16\pi} \left| C^{I+2\ 2\ I}_{K\ 0\ K} \right|^2, \tag{1.73}$$

which, for $K=0$ bands, gives

$$B(E2, I+2\to I) = Q_0^2 \frac{5}{16\pi}\frac{3}{2}\frac{(I+1)(I+2)}{(2I+3)(2I+5)}. \tag{1.74}$$

For the spectroscopic quadrupole moments $Q=\sqrt{16\pi/5}\,\langle IIK|\hat{Q}_{20}^{\text{lab}}|\,IIK\rangle$ [Eq. (B.32)], we get

$$Q = Q_0 \frac{3K^2 - I\cdot(I+1)}{(2I+3)(I+1)}. \tag{1.75}$$

The quotient of $Q : Q_0$ is the expectation value of $D_{00}^2 = P_2(\cos\beta)$ in the state $M=I$. This means that one cannot measure the internal quadrupole moment Q_0, but only the value averaged over the rotational motion. In fact, Q_0 is not a physical observable. Its definition depends on the introduction of a body-fixed system which moves with the nucleus and has a model character. For the band head we usually have $I=K$. That means the spectroscopic quadrupole moment Q of ground states with $I=0$ vanishes and we can get information about Q_0 only for the excited states (for instance, by the reorientation effect in Coulomb excitation [BE 68], which gives the sign and the absolute value of Q_0). Another way to determine the absolute value of Q_0 is the measurement of the $B(E2)$-values (1.73) in the transitions within a band. Recent measurements up to high spin states [WCL 76, HJE 78] have shown that in many deformed nuclei the value of Q_0 stays fairly constant within these bands, even in cases where the spectrum shows deviations from the $I(I+1)$ character. This is a hint that these deviations are not caused by the change of deformation (stretching effect).

1.5.3 The Asymmetric Rotor

The rotational–vibrational interaction model discussed so far has been on the basis of a symmetric rotor. Further attempts to explain the deviations from the $I(I+1)$ law and the low-lying second 2^+ states in many nuclei have been undertaken by Davydov et al. using the picture of a pure triaxial rotor [DF 58, DR 59, Da 59, Da 65b]. As a first step they do not consider the vibrational excitations and diagonalize only the rotational energy (1.55). With the moments of inertia (1.48), this operator is proportional to β^{-2} and one can diagonalize it for all values of γ. The constant factor can afterwards be adjusted so as to reproduce the first 2^+ state. Using the symmetries $\mathfrak{R}_1$ and $\mathfrak{R}_2$ (1.59 and 1.61), the wave functions have the form

$$|\Psi_M^I\rangle = \sum_{K=0,2,\ldots,} g_K\{|IMK\rangle + (-)^I|IM-K\rangle\}. \tag{1.76}$$

Figure (1.13) shows the corresponding energy eigenvalues. For $\gamma=0°$ and $\gamma=60°$ one gets the $I(I+1)$ spectrum. Even for strong triaxial deformations one gets only slight deviations of this form. However, additional $2^+, 3^+, 4^+$ levels come down in energy. It is a characteristic of this structure to have a low-lying second 2^+ state. Although one can, with such a model, reproduce quite reasonably the experimental

data in some regions of the periodic table (for instance, for the Os-isotopes), strong objections can be raised to it:

(i) It is impossible to describe β-vibrations. In this case, one has to include additional vibrational degrees of freedom [DC 60]
(ii) Well pronounced minima in the energy surface that could justify a stable γ-deformation have not been found. Microscopic calculations for such transitional nuclei usually show only very shallow valleys in the γ direction [KB 68, ASS 69, GPA 72, GG 78].*

In recent years, stable γ-deformations have found new interest, because:

(i) Experimental data in odd–even transitional nuclei can be very well reproduced by a model of a particle coupled to an asymmetric rotor (see Chap. 5 and [MSD 74, Me 75, TF 75]).
(ii) Theoretical calculations for very high spin states (see Sec. 1.7 and [ALL 76]) show that nuclei can become triaxial in certain spin regions.

One can get a rough estimate of the level structure of triaxial nuclei by assuming a maximal triaxiality ($\gamma = 30°$). In this case, $\vartheta_2 = \vartheta_3 = \frac{1}{4}\vartheta_1 = \frac{1}{3}\vartheta_0$ (ϑ_0 is the moment of inertia at $\gamma = 0$), and we have symmetry about the 1-axis (in the kinetic energy). The projection α of I on to this axis is a good quantum number and one finds

$$E_\alpha(I) = \frac{3\hbar^2}{8\vartheta_0}(4I\cdot(I+1) - 3\alpha^2). \tag{1.77}$$

* Wilets and Jean [WJ 56] proposed a model which is completely γ-soft, i.e., $C_{22} = 0$ in Eq. (1.46).

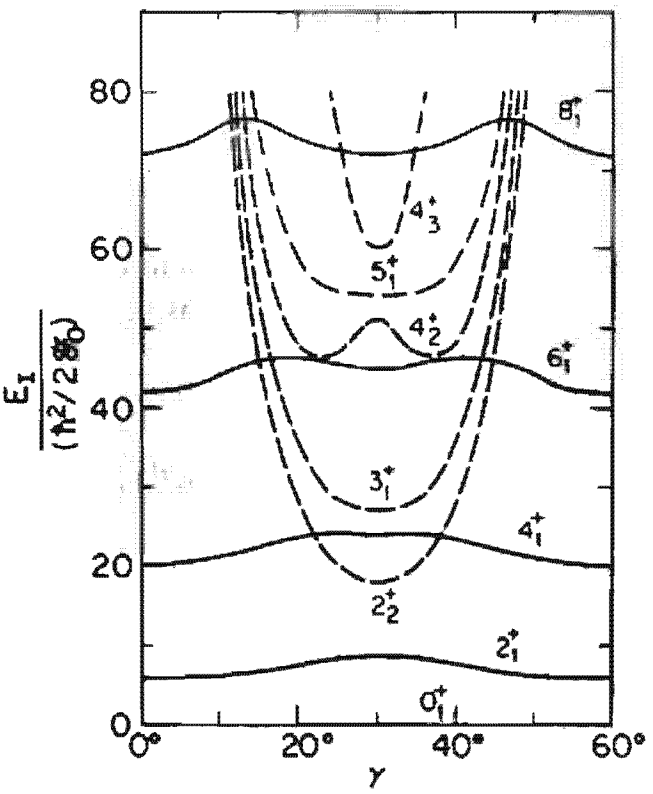

Figure 1.13. The energy eigenvalues of a deformed, asymmetric rotor with the hydrodynamic moments of inertia. (From [Me 75].)

$\mathfrak{R}_1$ and $\mathfrak{R}_2$ symmetry requires I and α to be even. For each I the level with the lowest energy (the so-called "yrast" level) has $\alpha = I$. Next in energy comes the band with $\alpha = I-2$, then $\alpha = I-4$, and so on. Therefore, we have a sequence of bands characterized by the "wobbling" quantum number $n = I-\alpha$ [BM 75]. One can calculate that the states with the same n are connected by large $E2$-transition probabilities. We get their spectrum from (1.77):

$$E_n(I) = \frac{3\hbar^2}{8\vartheta_0}(I(I+4)+3n(2I-n)). \tag{1.78}$$

However, one has to keep in mind that the structure of the spectrum (Fig. 1.13) depends drastically on the γ-dependence of the moments of inertia. There the hydrodynamical values have been used. For the rigid body values the spectrum would certainly look quite different.

1.6 Nuclear Fission

Up to now we have studied only small vibrations around the equilibrium shape in the liquid drop model. Shortly after the discovery of nuclear fission, attempts were made to understand this phenomenon using the concept of the nuclear drop [MF 39, Fr 39, BW 39].

In fact, a uniformly charged classical drop is only stable against fission (and spherical) if the Coulomb energy does not exceed a certain critical value. The Coulomb repulsion wants to deform the drop, the particles then being, on the average, further apart. The surface energy, on the contrary, being proportional to the surface of the drop, wants to keep it spherical. It is thus a subtle process of balance between these two effects (each being several hundred MeV in magnitude), which tells us whether there will be fission or not, according to a classical calculation. Of course, for such fission process, involving large deformations, one must go beyond the harmonic approximation discussed in the foregoing sections [BW 39].

The first step in describing the fission process [HW 53] is the choice of a suitable set of deformation parameters, which we call α. It has to be general enough to describe all the deformations that can occur. Cohen and Swiatecki [CS 62, 63], for instance, took as many as 18 multipolarities into account for the calculation of symmetric shapes of the form

$$R = R_0\left(1+\sum_{l=1}^{18}\alpha_l P_l(\cos\theta)\right). \tag{1.79}$$

This allows one to describe very general shapes. On the other hand, if one knows qualitatively the behavior of the droplet in the fission process, one is interested in introducing as few parameters as possible. Since one also wants to describe separated fragments, the set (1.79) is certainly not the most suitable one. For a realistic description of the fission process one needs at least three parameters [NS 65, Ni 72]:

(i) a parameter c, which describes in some way the length of the major semi-axis at the beginning of the fission process, and goes over into