and the summation runs over the occupied orbitals, $|\nu\rangle$. The total energy in the rotating system is given by

$$E^{\omega} = \sum_{\substack{\nu \\ \text{occ}}} \langle \nu | h^{\omega} | \nu \rangle = \hbar\omega_1 \Sigma_1 + \hbar\omega_\alpha \Sigma_\alpha + \hbar\omega_\beta \Sigma_\beta$$

In the cranking model, the angular momentum is identified with the sum of the expectation values of ℓ_1

$$I = \sum_{\substack{\nu \\ \text{occ}}} \langle \nu | \ell_1 | \nu \rangle = \frac{p}{(1+p^2)^{1/2}} (\Sigma_\beta - \Sigma_\alpha)$$

Thus, each configuration is associated with a maximum angular momentum, $I_{\max}$, with

$$I_{\max} = \Sigma_\beta - \Sigma_\alpha$$

The energy that will result from a measurement (in the laboratory system) is calculated as the sum of the expectation values of the static Hamiltonian h_{osc}:

$$E = \sum_{\substack{\nu \\ \text{occ}}} \langle \nu | h_{\text{osc}} | \nu \rangle = \sum_{\nu} \langle \nu | h^{\omega} + \omega \ell_1 | \nu \rangle = E^{\omega} + \hbar\omega I$$

For a fixed configuration, i.e. fixed values of Σ_1, Σ_α and Σ_β, and for a fixed spin I, we now want to find the potential shape that minimises the energy E. For this purpose, the energy is rewritten in the form

$$E = \hbar\omega_1 \Sigma_1 + \hbar\omega_2 \tilde{\Sigma}_2 + \hbar\omega_3 \tilde{\Sigma}_3$$

where

$$\tilde{\Sigma}_{2,3} = \frac{1}{2} (\Sigma_\alpha + \Sigma_\beta) \mp \frac{1}{2} \left(I_{\max}^2 - I^2 \right)^{1/2}$$

This simple formula for the energy E is obtained only if a further approximation is made, namely

$$p = \frac{\omega_2 + \omega_3}{\omega_2 - \omega_3} \frac{\omega}{(\omega_2 \omega_3)^{1/2}} \approx \frac{2\omega}{\omega_2 - \omega_3}$$

In the derivation, it is also useful to note that $p = I/(I_{\max}^2 - I^2)^{1/2}$ or $(1+p^2)^{1/2} = I_{\max}/(I_{\max}^2 - I^2)^{1/2}$.

For fixed values of $\Sigma_1, \tilde{\Sigma}_2$ and $\tilde{\Sigma}_3$, it is straightforward to find the minimum under the volume conservation constraint (cf. chapter 8):

$$\omega_1 \omega_2 \omega_3 = \overset{0}{\omega}{}_0^3$$

The following relation results:

$$\omega_1 \Sigma_1 = \omega_2 \tilde{\Sigma}_2 = \omega_3 \tilde{\Sigma}_3$$

The frequencies ω_1, ω_2 and ω_3 are obtained as

$$\omega_i = \overset{0}{\omega}_0 \left(\Sigma_1 \tilde{\Sigma}_2 \tilde{\Sigma}_3\right)^{1/3} / \tilde{\Sigma}_i$$

(with $\tilde{\Sigma}_1 \equiv \Sigma_1$). These values of the frequencies correspond to a shape adjusted to minimise the energy E for any value of the spin $I (I \leq I_{\max})$ and for any configuration specified by Σ_1, Σ_α and Σ_β (or by Σ_1, Σ_2 and Σ_3 in the limit of $I = 0$). The minimised energy E can be written in the concise form

$$E = 3\hbar \overset{0}{\omega}_0 \left(\Sigma_1 \tilde{\Sigma}_2 \tilde{\Sigma}_3\right)^{1/3} = 3\hbar \overset{0}{\omega}_0 \left[\Sigma_1 \left(\Sigma_\alpha \Sigma_\beta + \frac{1}{4} I^2\right)\right]^{1/3}$$

For $I = I_{\max}$, i.e. for maximal spin within a configuration, we note that $\tilde{\Sigma}_2 = \tilde{\Sigma}_3$. Thus, the formula for the frequencies shows that $\omega_2 = \omega_3$, which corresponds to a potential being axially symmetric around the rotation axis. For such a shape, the rotation is not collective but instead built from individual nucleons having their spin vectors quantised along the rotation axis, so-called rotation around the symmetry axis.

If the formula for the frequencies is combined with the definitions of ε and γ (chapter 8), it is possible to calculate ε and γ as functions of spin I (see problem 12.3)

$$\varepsilon = \frac{3\left(\Sigma_1^{-2} + \tilde{\Sigma}_2^{-2} + \tilde{\Sigma}_3^{-2} - \Sigma_1^{-1}\tilde{\Sigma}_2^{-1} - \tilde{\Sigma}_2^{-1}\tilde{\Sigma}_3^{-1} - \tilde{\Sigma}_3^{-1}\Sigma_1^{-1}\right)^{1/2}}{\Sigma_1^{-1} + \tilde{\Sigma}_2^{-1} + \tilde{\Sigma}_3^{-1}}$$

$$\tan\gamma = \frac{\sqrt{3}\left(\Sigma_1^{-1} - \tilde{\Sigma}_2^{-1}\right)}{\Sigma_1^{-1} + \tilde{\Sigma}_2^{-1} - 2\tilde{\Sigma}_3^{-1}}$$

We will finally consider the moment of inertia in the simple harmonic oscillator model. For collective rotation with a constant moment of inertia, $\mathcal{J}$, the energy is given by $E = (\hbar^2/2\mathcal{J})I^2$ (in the case of rotation around one axis, one should use I^2 rather than $I(I+1)$, which latter is the proper quantity for three-dimensional quantum-mechanical rotation). For a general function, $E = E(I)$, it seems natural to define moments of inertia from the derivatives (Bohr and Mottelson, 1981),

$$\frac{\hbar^2}{\mathcal{J}^{(1)}} = 2\frac{\mathrm{d}E}{\mathrm{d}I^2} = \frac{1}{I}\frac{\mathrm{d}E}{\mathrm{d}I} \approx \frac{E(I+1) - E(I-1)}{2I}$$

and

$$\frac{\hbar^2}{\mathcal{J}^{(2)}} = \frac{\mathrm{d}^2 E}{\mathrm{d}I^2} \approx \frac{E(I+2) - 2E(I) + E(I-2)}{4}$$

where we also indicate how to define these moments of inertia from measured transition energies, $E_\gamma = E(I+1) - E(I-1)$. Note that $\mathcal{J}^{(1)} = \mathcal{J}^{(2)} = \mathcal{J}$ when $E = (\hbar^2/2\mathcal{J})I^2$. The $\mathcal{J}^{(1)}$ moment of inertia is a direct measure of the transition energies while $\mathcal{J}^{(2)}$ is obtained from differences in transition energies. Except that $I(I+1)$ has been replaced by I^2 and the discrete differences are now centred around I, $\mathcal{J}^{(1)}$ is identical to the moment of inertia introduced in chapter 11 and shown in fig. 11.13.

For the harmonic oscillator in the present approximation, it is straightforward to calculate

$$\mathcal{J}^{(1)} = \frac{2\left(\Sigma_\alpha \Sigma_\beta + I^2/4\right)^{2/3}}{\Sigma^{1/3}} \frac{\hbar}{\overset{0}{\omega_0}}$$

$$\mathcal{J}^{(2)} = \mathcal{J}^{(1)} \frac{\Sigma_\alpha \Sigma_\beta + I^2/4}{\Sigma_\alpha \Sigma_\beta - I^2/12}$$

These values might be compared with the static rigid body moment of inertia

$$\mathcal{J}_{\mathrm{rig}} = M \sum_{\substack{\nu \\ \mathrm{occ}}} \langle \nu | y^2 + z^2 | \nu \rangle$$

If we follow the equilibrium shapes of the harmonic oscillator, we obtain (see problem 12.4)

$$\mathcal{J}_{\mathrm{rig}} = \frac{\Sigma_\alpha^2 + \Sigma_\beta^2 - I^2/2}{\left[\Sigma_1 \left(\Sigma_\alpha \Sigma_\beta + I^2/4\right)\right]^{1/3}} \frac{\hbar}{\overset{0}{\omega_0}}$$

It turns out that both $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ are generally below $\mathcal{J}_{\mathrm{rig}}$; at large deformations and small spins even considerably below. One could also note, however, that, for $I = I_{\max}$, $\mathcal{J}^{(1)} \equiv \mathcal{J}_{\mathrm{rig}}$.

The solution of the harmonic oscillator presented here is useful because it illustrates general features like shape changes and band terminations. On the other hand, the numerical values of moments of inertia or deformations are somewhat crude. Indeed, as mentioned above, it is possible to solve the full cranking single-particle Hamiltonian for the harmonic oscillator potential. With the resulting single-particle energies as input, it is then possible to calculate the total energy using the same procedure as above, i.e. the volume conservation condition is applied to find the deformations, ε and γ, which for each spin I minimise the total energy, E. In this case, however, no

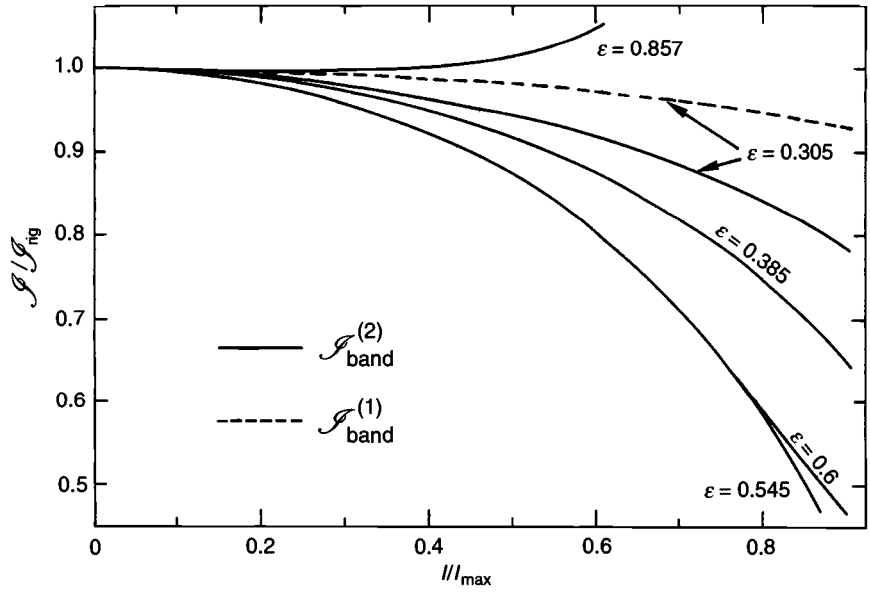

Fig. 12.1. The variation with spin of the moments of inertia $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ as obtained from the full solution of the cranked harmonic oscillator. The configurations, are labelled by their equilibrium deformations at spin zero, which are all axially symmetric. The moments of inertia and the spin are given in units of the rigid body moment of inertia at the spin zero deformation, $\mathcal{J}_{\text{rig}}$, and the maximum spin within the configuration, $I_{\max}$ (from I. Ragnarsson, 1987, *Phys. Lett.* **199B**, 317).

analytic expressions have been given for $E(I)$ but a numerical solution is straightforward. One difference (Troudet and Arvieu, 1979) compared with the approximate solution is that only those configurations that are not too deformed ($\Sigma_3/\Sigma_2 < 1.78$) for $I = 0$ really terminate, i.e. whose rotation becomes non-collective for $I = I_{\max}$. The more deformed configurations do not terminate but, instead, they become more and more elongated for very high spins. With the exception of very deformed configurations, both the moments of inertia, $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$, which start out equal to $\mathcal{J}_{\text{rig}}$ for $I = 0$, decrease with increasing spin, see fig. 12.1. In this figure, the different configurations, which are all axially symmetric for $I = 0$ ($\Sigma_x = \Sigma_y$), are labelled by their deformation ε at $I = 0$. The spin is given in units of $I_{\max}$ defined by $\Sigma_\beta - \Sigma_\alpha = \Sigma_3 - \Sigma_2$ independently of whether any real termination occurs or not, while the unit for the moments of inertia is $\mathcal{J}_{\text{rig}}(I = 0)$. With these units, the same figure can be applied to different mass regions. Note, however, that $I_{\max}$ is strongly dependent on deformation, e.g. $I_{\max}$ is much larger for an '$\varepsilon = 0.6$ configuration' than for an '$\varepsilon = 0.2$ configuration'.

In the more realistic calculations considered below, we will find that, for strongly deformed configurations in heavy nuclei (superdeformation), the spin will always be much smaller than $I_{\max}$. Therefore, we might expect that $\mathcal{J}^{(2)}$ (as well as $\mathcal{J}^{(1)}$) stays close to $\mathcal{J}_{\text{rig}}$ for all spins of physical interest. On the other hand, configurations having a smaller deformation at $I = 0$ might very well reach $I_{\max}$ at observable spins (band termination). Then, however, the special shell structure caused by the grouping in the j-shells at $\varepsilon = 0$ might strongly influence the energies and the pure oscillator can only be used to indicate the general trends.

Besides publications quoted previously in this section, one could mention the papers by Zelevinskii (1975) and by Glas, Mosel and Zint (1978), where additional aspects of the rotating harmonic oscillator are considered.

12.3 The rotating liquid-drop model

We will now for a moment ignore the quantal effects and consider the rotation of a nucleus according to the laws of classical mechanics. In such a macroscopic model, the energy is given by

$$E_{\text{macr}}(E, N, \text{def}, I) = E(Z, N, \text{def}) + \frac{\hbar^2 I^2}{2\mathcal{J}(Z, N, \text{def})}$$

The energy $E(Z, N,\text{def})$ is taken as the static liquid-drop energy, which was treated in chapter 4. The variable 'def' denotes a number of deformation parameters, e.g. $\varepsilon, \gamma, \varepsilon_4 \ldots$. For stable nuclei the liquid drop energy has a minimum for spherical shape. This minimum is caused by the surface energy, which overcomes the deforming tendencies of the Coulomb energy.

In our discussion of the harmonic oscillator, we found that the dynamical moment of inertia was essentially equal to the rigid body value. In the case of independent nucleon motion, this is what is generally expected also for potentials other than the harmonic oscillator. The fact that the experimentally observed moment of inertia is smaller than $\mathcal{J}_{\text{rig}}$ for low I can be traced back to the pairing correlations. At higher spins, however, these correlations should disappear. As the rotating liquid-drop model is relevant only at relatively high spins, we will use the rigid body moment of inertia in connection with this model.

The rotational energy becomes smaller with increasing $\mathcal{J}$. Thus, with the rigid body value, configurations with the nucleons far away from the rotation axis are favoured. This means that the rotational energy tries to deform the nucleus and this tendency will become dominating for a large enough value of the angular momentum I.