APPENDIX

4A

Particle-Rotor Model

The simple model consisting of a particle coupled to a rotor provides an approximate description of many of the properties of the low-lying bands in odd-A nuclei and, at the same time, illustrates various general features of rotating systems. It has therefore played an important role in the development of the theory of nuclear rotational spectra. The application of the model to scattering problems is considered in Appendix 5A.

4A-1 The Coupled System

The rotor is assumed to possess axial symmetry and $\mathscr{R}$ invariance (see Sec. 4-2c) and to have the quantum numbers $R_3=0$, $r=+1$, as for the ground states of even-even nuclei (see p. 27). The angular momentum of the rotor is denoted by $\mathbf{R}$, with component R_3 with respect to the symmetry axis, and the spectrum consists of the sequence $R=0,2,4,\ldots$ (see Eq. (4-15)).

The energy of the rotor is taken to be proportional to the square of the angular momentum, corresponding to the Hamiltonian

$$H_{\text{rot}}=\frac{\hbar^2}{2\mathscr{I}_0}(R_1^2+R_2^2)=\frac{\hbar^2}{2\mathscr{I}_0}\mathbf{R}^2 \qquad \text{(4A-1)}$$

as in the classical expression for a symmetric rigid body constrained to rotate about axes perpendicular to the symmetry axis (the dumbbell). The moment of inertia of the rotor is denoted by $\mathscr{I}_0$. The expression (4A-1) may be regarded as the first term in an expansion of the rotational energy in powers of the angular momentum (see p. 23); higher terms would be associated with the distortion of the rotor produced by the centrifugal and Coriolis forces.

The interaction between the rotor and the particle is described by a potential V, depending on the variables of the particle in the body-fixed system. The potential may be velocity and spin dependent and in particular may include a spin-orbit coupling, but V is assumed to be invariant under space reflection and time reversal, as well as axially symmetric and invariant under the rotation $\mathscr{R}=\mathscr{R}_2(\pi)$.

The present model ignores the possibility of terms in the potential depending on the angular rotational frequency, such as may arise from rotational perturbations of the rotor (centrifugal distortions proportional to ω_{rot}^2, as well as terms linear in ω_{rot} associated with the velocity-dependent interactions; see the discussion on p. 80 and pp. 278 ff.). The Hamiltonian for the coupled system is

$$H = H_{\mathrm{rot}} + T + V \tag{4A-2}$$

where T is the kinetic energy of the particle. The anisotropy of the potential provides the coupling between the motion of the particle and that of the rotor.

4A-2 Adiabatic Approximation

If the rotational frequency is small compared with the excitation frequencies that characterize orbits with different orientations relative to the potential V, the motion of the particle is strongly coupled to the rotor and follows the precessional motion of the axis of the rotor in an approximately adiabatic manner. The coupled system can then be described in terms of a superposition of intrinsic motion, for fixed orientation of the rotor, and a rotational motion of the system as a whole.

4A-2a Wave functions

The intrinsic motion is determined by the wave equation

$$(T+V)\Phi_K(q) = E_K \Phi_K(q) \tag{4A-3}$$

where q represents intrinsic coordinates (including the spin variable). The eigenfunctions Φ_K and eigenvalues E_K of the intrinsic motion are labeled by the quantum number K representing the components I_3 of the total angular momentum; since the rotor has $R_3 = 0$, the quantum number K is equal to the eigenvalue of the component j_3 of the particle angular momentum (which is often denoted by Ω). The intrinsic states are degenerate with respect to the sign of K, as a consequence of the $\mathscr{R}$ invariance, or $\mathscr{T}$ invariance, of the intrinsic Hamiltonian. The states with negative K are labeled by $\bar{K}$, and we employ the phase choice (see Eqs. (4-16) and (4-27))

$$\Phi_{\bar{K}}(q) = \mathscr{T}\Phi_K(q) = \mathscr{R}_{\mathrm{i}}^{-1}\Phi_K(q) = -\mathscr{R}_{\mathrm{i}}\Phi_K(q) \tag{4A-4}$$

The particle is assumed to be a fermion; hence, $\mathscr{R}_{\mathrm{i}}^2 = \mathscr{R}_{\mathrm{i}}(2\pi) = -1$, and K takes the values $1/2, 3/2, \ldots$.

The rotational motion is specified by the quantum numbers KIM, and the total wave function has the form appropriate to an $\mathscr{R}$-invariant system (see Eq. (4-19))

$$\Psi_{KIM} = \left(\frac{2I+1}{16\pi^2}\right)^{1/2} \left(\Phi_K(q)\mathscr{D}^I_{MK}(\omega) + (-1)^{I+K}\Phi_{\bar{K}}(q)\mathscr{D}^I_{M-K}(\omega)\right) \tag{4A-5}$$

where ω represents the orientation of the rotor. The wave functions (4A-5) form a complete orthonormal set, which provides a convenient basis for describing the coupled system, when the adiabatic condition is fulfilled.

The coupling between the angular momenta of the particle and the rotor that is implicit in the wave function (4A-5) can be exhibited by transforming to the representation $(jR)IM$ appropriate to a weakly coupled system. Employing the transformation (1A-34) from the intrinsic to the fixed coordinate system and the coupling relation (1A-43) for the $\mathscr{D}$ functions, one obtains

$$\langle (ljR)IM|KIM\rangle = \langle ljK|K\rangle\left(\frac{2R+1}{2I+1}\right)^{1/2}\langle jKR0|IK\rangle\frac{1}{\sqrt{2}}\left(1+(-1)^{R}\right) \quad \text{(4A-6)}$$

where the first factor represents the expansion of the intrinsic state $\Phi_K(q)$ onto a spherical basis ljK characterized by the orbital angular momentum l and the total angular momentum j of the particle. The symmetrized form of the wave function (4A-5) ensures that the amplitude (4A-6) is nonvanishing only for $R=0,2,\ldots$. The second and third factors in Eq. (4A-6) give the weight of the different states R of the rotor in the adiabatic coupling scheme.

4A-2b Energy

In order to express the Hamiltonian in the representation (4A-5), we employ the relation $\mathbf{R}=\mathbf{I}-\mathbf{j}$. The angular momentum of the rotor is thereby decomposed into a part $\mathbf{I}$, which rotates the system as a whole and thus acts only on the rotational wave function, and a part $\mathbf{j}$ acting only on the intrinsic variables.

For the energy of the rotor (4A-1), we obtain

$$\begin{aligned} H_{\mathrm{rot}} &= \frac{\hbar^2}{2\mathscr{I}_0}\left((I_1-j_1)^2+(I_2-j_2)^2\right) \\ &= \frac{\hbar^2}{2\mathscr{I}_0}\mathbf{I}^2 + \frac{\hbar^2}{2\mathscr{I}_0}(j_1^2+j_2^2-j_3^2) - \frac{\hbar^2}{2\mathscr{I}_0}(j_+I_- + j_-I_+) \end{aligned} \quad \text{(4A-7)}$$

with the notation $I_\pm = I_1 \pm iI_2$ and $j_\pm = j_1 \pm ij_2$. The first term in the Hamiltonian (4A-7) depends only on the total angular momentum and is a constant of the motion. The second term in Eq. (4A-7) represents a recoil energy of the rotor and depends only on the intrinsic variables; this term might therefore be included in equation (4A-3) determining the intrinsic wave function Φ_K. However, since the recoil energy is comparable in magnitude with rotational energies and, therefore, small compared with intrinsic energies, we may neglect the effect on Φ_K in first approximation. The third term in Eq. (4A-7) represents the Coriolis and centrifugal forces acting on the particle in the rotating coordinate system (see Eqs. (4-107) and (4-119)). This term is nondiagonal in K (has $\Delta K = \pm 1$) and provides a coupling between intrinsic and rotational motion. In the special case of $K=1/2$ bands, the Coriolis interaction connects the components with opposite K in the states (4A-5) and hence contributes to the expectation value of the energy.

The expression (4A-7) for H_{rot} leads to a total Hamiltonian, which may be written in the form (see Eq. (4A-2))

$$H = H_0 + H_c \tag{4A-8a}$$

$$H_0 = T + V + \frac{\hbar^2}{2\mathscr{I}_0}\mathbf{I}^2 \tag{4A-8b}$$

$$H_c = -\frac{\hbar^2}{2\mathscr{I}_0}(j_+I_- + j_-I_+) + \frac{\hbar^2}{2\mathscr{I}_0}(j_1^2 + j_2^2 - j_3^2) \tag{4A-8c}$$

The wave functions (4A-5) are the eigenstates of H_0, and in this basis, the expectation value of the energy is given by (see Eqs. (4-20) and (1A-93))

$$\begin{aligned} E_{KI} &= \langle KIM|H|KIM\rangle = E_K + E_{\mathrm{rot}} \\ E_{\mathrm{rot}} &= \langle KIM|H_{\mathrm{rot}}|KIM\rangle \\ &= \frac{\hbar^2}{2\mathscr{I}_0}\left(I(I+1) + a(-1)^{I+1/2}(I+1/2)\delta(K,1/2)\right) + \frac{\hbar^2}{2\mathscr{I}_0}\langle K|j_1^2 + j_2^2 - j_3^2|K\rangle \end{aligned} \tag{4A-9}$$

The parameter a in the rotational energy for $K = 1/2$ bands has the value

$$a = -\langle K = 1/2|j_+|\overline{K = 1/2}\rangle \tag{4A-10}$$

and is referred to as the decoupling parameter.

4A-2c Matrix elements

The matrix elements of the various operators, such as electric and magnetic multipole moments, can be evaluated, in the basis (4A-5), by means of the general procedure described in Sec. 4-3. For a tensor operator, the transformation to the body-fixed coordinate system is given by Eq. (4-90) and leads to matrix elements involving products of intrinsic and rotational factors.

An example is provided by the $M1$ moment, which can be expressed in terms of the g factors for the various angular momenta in the system (see Eqs. (3-36) and (3-39))

$$\begin{aligned} \mathscr{M}(M1,\mu) &= \left(\frac{3}{4\pi}\right)^{1/2}\frac{e\hbar}{2Mc}(g_R R_\mu + g_l l_\mu + g_s s_\mu) \\ &= \left(\frac{3}{4\pi}\right)^{1/2}\frac{e\hbar}{2Mc}(g_R I_\mu + (g_l - g_R)l_\mu + (g_s - g_R)s_\mu) \end{aligned} \tag{4A-11}$$

For transitions between two states in the same band, the reduced $M1$-matrix element is of the form (4-85) derived in Sec. 4-3c on the basis of symmetry arguments. The present model provides explicit expressions for the intrinsic matrix elements occurring in this general expression,

$$g_K K = \langle K | g_l l_3 + g_s s_3 | K \rangle$$
$$b(g_K - g_R) = \langle K=1/2 | (g_l - g_R) l_+ + (g_s - g_R) s_+ | \overline{K=1/2} \rangle \tag{4A-12}$$

The quantity g_K is the average g factor for the intrinsic motion. The occurrence of the parameter b, referred to as the magnetic decoupling parameter, is a special feature of $M1$ transitions within $K=1/2$ bands.

4A-3 Nonadiabatic Effects

The coupling H_c (see Eq. (4A-8)) gives rise to an interweaving of rotational and intrinsic motion that can be analyzed in terms of a mixing of different rotational bands. (The effects of the band mixing produced by the Coriolis interaction were discussed by Kerman, 1956.)

4A-3a Contributions to energy

For small rotational frquencies, the coupling H_c can be treated by a perturbation expansion (except when special degeneracies occur in the intrinsic spectrum). Acting in second order, H_c gives the energy contribution

$$\delta E_{KI}^{(2)} = -\sum_{\nu} \frac{\langle \nu IM | H_c | KIM \rangle^2}{E_\nu - E_K} \tag{4A-13}$$

where ν labels the bands that are coupled to the state KIM by the interaction H_c. The perturbation (4A-13) gives rise to a term proportional to $I(I+1)$ arising from the second-order effect of the Coriolis coupling. In addition, the expression (4A-13) contains a term independent of I and, for $K=1/2$ bands, a signature-dependent term linear in I arising from the combined effect of the Coriolis and recoil energies in the coupling H_c (see Eq. (4A-8c)). The coefficient of $I(I+1)$ is

$$\delta A = -\left(\frac{\hbar^2}{2\mathscr{I}_0}\right)^2 \sum_{\nu, K_\nu = K\pm 1} \frac{\langle \nu | j_\pm | K \rangle^2}{E_\nu - E_K} \tag{4A-14}$$

where the summation extends over all bands ν with $K_\nu = K \pm 1$. For $K=1/2$ bands, the terms in Eq. (4A-14) with $K_\nu = K-1$ are to be interpreted as excited bands with $K_\nu = 1/2$.

The contribution (4A-14) to the rotational energy can be expressed as a renormalization of the effective moment of inertia

$$\mathscr{I} = \mathscr{I}_0 + \delta\mathscr{I}$$
$$\delta\mathscr{I} = -\frac{2\mathscr{I}_0^2}{\hbar^2}\delta A = 2\hbar^2 \sum_{\nu} \frac{\langle \nu | j_1 | K \rangle^2}{E_\nu - E_K} \tag{4A-15}$$

assuming $\delta\mathscr{I}\ll\mathscr{I}_0$. The increase in the inertia of the system represents the contribution of the particle as it is dragged around by the rotor, and may also be obtained by considering a particle moving in an external field rotated with a constant frequency $\omega_{\mathrm{rot}}=(\mathscr{I}_0)^{-1}\hbar I$ (the cranking model, see pp. 75ff.).

To third order in H_c, we obtain a contribution to the energy of $K=3/2$ bands that is cubic in I,

$$\delta E_{KI}^{(3)}=(-1)^{I+3/2}(I-1/2)(I+1/2)(I+3/2)A_3$$

$$A_3=-\left(\frac{\hbar^2}{2\mathscr{I}_0}\right)^3\sum_{\substack{\nu,\nu'\\K_\nu=K_{\nu'}=1/2}}\frac{\langle K=3/2|j_+|\nu\rangle\langle\nu|j_+|\bar{\nu}'\rangle\langle\bar{\nu}'|j_+|\overline{K=3/2}\rangle}{(E_\nu-E_K)(E_{\nu'}-E_K)} \tag{4A-16}$$

In addition, the third-order effects of H_c give a contribution to the energy of $K=1/2$ bands proportional to $(-1)^{I+1/2}(I+1/2)I(I+1)$, as well as terms of lower order in I. The fourth-order perturbation contains terms proportional to $I^2(I+1)^2$, and by proceeding to higher orders, one obtains a power series expansion for the energy of the general form (4-62).

The leading-order contribution to the term in the energy expansion involving I^n arises from the Coriolis coupling acting in nth order, and the magnitude of this contribution is given by

$$\delta E^{(n)}\sim\frac{\hbar^2 j}{\mathscr{I}_0}\left(\frac{\hbar j}{\mathscr{I}_0\omega_{\mathrm{int}}}\right)^{n-1}I^n\approx\hbar\omega_{\mathrm{int}}\left(\frac{\omega_{\mathrm{rot}}j}{\omega_{\mathrm{int}}}\right)^n \tag{4A-17}$$

where ω_{int} is a measure of the intrinsic frequencies ($\hbar\omega_{\mathrm{int}}\sim E_\nu-E_K$), and where j represents the magnitude of the matrix elements of j_+. The rotational frequency ω_{rot} is the ratio between the angular momentum $\hbar I$ and the moment of inertia.

If the intrinsic spectrum is of vibrational character, the validity of the rotational coupling scheme requires the adiabatic condition $\omega_{\mathrm{rot}}\ll\omega_{\mathrm{int}}=\omega_{\mathrm{vib}}$ (see the discussion on p. 3). The estimate (4A-17) implies a more restrictive condition for the convergence of the perturbation expansion in the case of particle motion with large values of j. However, for particle orbits with large angular momentum, the elementary effects of the Coriolis interaction produce only relatively small changes in the orientation of the orbit; hence, the smallness of the perturbation of the wave function is not a necessary condition for the convergence of the expansion for the energy and matrix elements in powers of the rotational angular momentum. In fact, the estimate (4A-17) leaves open the possibility of systematic cancellations in the higher-order terms. An example is provided by the motion of a particle in a rotating harmonic oscillator potential, for which the expansion of the energy is determined by the ratio of ω_{rot} to the difference in the oscillator frequencies along different axes, irrespective of the angular momentum of the particle orbit (Valatin, 1956; Hemmer, 1962).

The recoil energy in H_c (see Eq. (4A-8c)) is independent of I and therefore contributes to the coefficient of the I^n-dependent terms only through rotational perturbations of order $n+1$, or higher. The resulting corrections to the coefficient, as estimated in nth order, are therefore of relative magnitude $(\hbar j)^2/\mathscr{I}_0(E_\nu - E_K)$, which is comparable to $\delta\mathscr{I}/\mathscr{I}_0$ (see Eq. (4A-15)). The recoil term has especially simple consequences if the intrinsic excitations involve approximately independent degrees of freedom (spin and orbit in the absence of spin-orbit coupling, two or more independent particles, etc.). In such a situation, the part of the recoil term that couples the different components gives rise to a renormalization of the moment of inertia occurring in the Coriolis coupling.

An example illustrating this effect of the recoil term is provided by the decoupling term in the absence of spin-orbit coupling. For a $K=1/2$ band, with $l_3=\Lambda=0$, the second-order energy arising from the Coriolis and recoil terms, each acting in first order, gives

$$\delta E^{(2)} = (-1)^{I+1/2}(I+1/2)\delta A_1$$

$$\delta A_1 = 2\left(\frac{\hbar^2}{2\mathscr{I}_0}\right)^2 \sum_{\nu, K_\nu=1/2} \frac{\langle K=1/2|j_1^2+j_2^2|\nu\rangle\langle\nu|j_+|\overline{K=1/2}\rangle}{E_\nu - E_K}$$

$$= -2\left(\frac{\hbar^2}{2\mathscr{I}_0}\right)^2 \sum_{\kappa, \Lambda_\kappa=1} \frac{\langle\Lambda=0|l_-|\kappa\rangle\langle\kappa|l_+|\Lambda=0\rangle}{E_\kappa - E_\Lambda} a \qquad \text{(4A-18)}$$

The intermediate states have $K_\nu = 1/2$, $\Sigma(=s_3) = -1/2$ and the orbital part, denoted by κ, has $\Lambda_\kappa = 1$. In the Coriolis matrix element, only the orbital part of j_+ contributes, and the factor a is the decoupling parameter in the unperturbed band, which equals the value of $r(=\pm 1)$ for the $\Lambda=0$ orbital state (see p. 33). The recoil matrix element receives its contribution from the part of $j_1^2+j_2^2$ involving $l_- s_+$. The sum in Eq. (4A-18) is the same as that occurring in the expression for δA for the $\Lambda=0$ band (see Eq. (4A-14) and note the factor $\sqrt{2}$ in the matrix elements (4-92) connecting a $\Lambda=0$ state with a state having $\Lambda\neq 0$). We therefore obtain

$$\delta A_1 = a\delta A \qquad \text{(4A-19)}$$

Hence, the decoupling parameter, defined as the ratio between the (renormalized) A_1 and A coefficients, remains equal to the unperturbed value r, and the spectrum, with the inclusion of the nonadiabatic effects, retains the doublet structure, as required by the absence of spin-orbit coupling.

A further illustration of the role of the recoil term is provided by the contribution to the moment of inertia, for a system consisting of a rotor with two independent particles n and p (a neutron and a proton). In this case, the third-order contribution to the coefficient of the $I(I+1)$-dependent energy that results from the recoil term acting once and the Coriolis term twice (once on n and once on p) is

$$\delta E^{(3)}=2\left(\frac{\hbar^2}{2\mathscr{I}_0}\right)^3 I(I+1)$$

$$\times\left(\sum_{\substack{\nu_n,\nu_p\\ K(\nu_n)=K_n\pm 1\\ K(\nu_p)=K_p\pm 1}}\frac{\langle K_nK_p|(j_p)_{\mp}|K_n\nu_p\rangle\langle K_n\nu_p|(j_n)_{\mp}(j_p)_{\pm}|\nu_nK_p\rangle\langle\nu_nK_p|(j_n)_{\pm}|K_nK_p\rangle}{(E(\nu_n)-E(K_n))(E(\nu_p)-E(K_p))}\right.$$

$$\left.+\sum_{\substack{\nu_n,\nu_p\\ K(\nu_n)=K_n\pm 1\\ K(\nu_p)=K_p\pm 1}}\left(\frac{\langle K_nK_p|(j_n)_{\mp}(j_p)_{\pm}|\nu_n\nu_p\rangle\langle\nu_n\nu_p|(j_p)_{\mp}|\nu_nK_p\rangle\langle\nu_nK_p|(j_n)_{\pm}|K_nK_p\rangle}{(E(\nu_n)+E(\nu_p)-E(K_n)-E(K_p))(E(\nu_n)-E(K_n))}+(\mathrm{n}\leftrightarrow\mathrm{p})\right)\right)$$

$$=\hbar^2\frac{(\delta\mathscr{I}_n)(\delta\mathscr{I}_p)}{(\mathscr{I}_0)^3}I(I+1) \tag{4A-20}$$

The same term occurs if one adds the separate contributions of n and p to the moment of inertia rather than to the energy itself.

The result (4A-20) corresponds to the fact that, in estimating the rotational energy contribution of the proton, we may consider the system in terms of the proton coupled to an effective rotor that consists of the neutron plus the core. The moment of inertia of this rotor equals $\mathscr{I}_0+\delta\mathscr{I}_n$ and, hence, the Coriolis coupling acting on the proton involves this renormalized moment.

It must be emphasized that corresponding third-order terms involving the Coriolis coupling acting twice on the neutron or on the proton will not, in general, give results equivalent to those arising from the expansion of $(\mathscr{I}_0+\delta\mathscr{I}_n+\delta\mathscr{I}_p)^{-1}$. For a single degree of freedom, the effect of the recoil term can be expressed in the above manner only if the degree of freedom can be repeated, as in the case of vibrational modes.

4A-3b Renormalization of operators

A systematic analysis of the effect of H_c on the various matrix elements for the particle-rotor system may be developed by viewing the perturbation expansion in terms of a canonical transformation

$$\begin{aligned}H'&=\exp\{iS\}H\exp\{-iS\}\\ &=H+i[S,H]-\tfrac{1}{2}[S,[S,H]]+\cdots\end{aligned} \tag{4A-21}$$

that diagonalizes the Hamiltonian in the representation (4A-5).

To first order in $I_\pm$, the transformation S is determined by the condition

$$i[S,H_0]=\frac{\hbar^2}{2\mathscr{I}_0}(j_+I_-+j_-I_+) \tag{4A-22}$$

We have ignored the recoil term in H_c, which only produces I-independent effects, to the order considered. From Eq. (4A-22), we obtain

$$S=-i(\varepsilon_+I_-+\varepsilon_-I_+) \tag{4A-23}$$

where the intrinsic operator $\varepsilon_\pm$ (denoted in Sec. 4-4 by $\varepsilon_{\pm1}$) is given by

$$\begin{aligned}[][H_0,\varepsilon_\pm]&=h_\pm\equiv-\frac{\hbar^2}{2\mathscr{I}_0}j_\pm\\ \langle K'|\varepsilon_\pm|K\rangle&=\frac{\langle K'|h_\pm|K\rangle}{E_{K'}-E_K}\end{aligned} \tag{4A-24}$$

(In Eq. (4A-22), only the nondiagonal part of the Coriolis interaction is included.)

With S given by Eq. (4A-23), the transformed Hamiltonian (4A-21) takes the form, to second order in $I_\pm$,

$$\begin{aligned}H'&=H_0+(H_c)_{\text{diag}}+\tfrac{1}{2}[(\varepsilon_+I_-+\varepsilon_-I_+),(h_+I_-+h_-I_+)]\\ &=H_0+(H_c)_{\text{diag}}+\tfrac{1}{2}(\{\varepsilon_+,h_-\}-\{\varepsilon_-,h_+\})I_3\\ &\quad+\tfrac{1}{2}([\varepsilon_+,h_-]+[\varepsilon_-,h_+])(I_1^2+I_2^2)+\tfrac{1}{2}[\varepsilon_+,h_+]I_-^2+\tfrac{1}{2}[\varepsilon_-,h_-]I_+^2\end{aligned} \tag{4A-25}$$

as follows from the relation

$$[aA,bB]=\tfrac{1}{2}\{a,b\}[A,B]+\tfrac{1}{2}[a,b]\{A,B\} \tag{4A-26}$$

for operator products in which a and b commute with A and B.

In the Hamiltonian (4A-25), the term proportional to $I_3(=j_3)$ gives a correction to the intrinsic motion. The term proportional to $I_1^2+I_2^2$ gives the renormalization (4A-15) of the moment of inertia and produces a coupling between different bands with $\Delta K=0$. Finally, the terms proportional to I_-^2 and I_+^2 provide a coupling between bands with $\Delta K=\pm2$.

The effect of the Coriolis coupling on the matrix elements of a given operator F is obtained from the transformed operator

$$F'=\exp\{iS\}F\exp\{-iS\}=F+\delta F \tag{4A-27}$$

with

$$\begin{aligned}\delta F&=i[S,F]+\cdots\\ &=[(\varepsilon_+I_-+\varepsilon_-I_+),F]+\cdots\end{aligned} \tag{4A-28}$$

The transformed operator is to be evaluated in the space of the unperturbed wave functions (4A-5), and thus the consequences of the band mixing caused by the Coriolis coupling are described by the additional term δF in the effective transition operator.

As an illustration, we consider the renormalization of a dipole moment

$$\mathscr{M}(\lambda=1,\mu)=\sum_{\nu} \mathscr{M}_{\nu}\, \mathscr{D}^{1}_{\mu\nu}(\omega) \tag{4A-29}$$

where the components $\mathscr{M}_{\nu} \equiv \mathscr{M}(\lambda=1,\nu)$ refer to the intrinsic coordinate system (see Eq. (4-90)). For the induced moment $\delta\mathscr{M}$, we obtain from Eq. (4A-28), to leading order,

$$\begin{aligned}\delta\mathscr{M}(\lambda=1,\mu) &= \Big[(\varepsilon_{+}I_{-}+\varepsilon_{-}I_{+}),\sum_{\nu}\mathscr{M}_{\nu}\mathscr{D}^{1}_{\mu\nu}\Big]\\ &= \sum_{\Delta K=-2}^{2}\delta\mathscr{M}(\lambda=1,\mu)_{\Delta K}\end{aligned} \tag{4A-30}$$

with

$$\begin{aligned}\delta\mathscr{M}(\lambda=1,\mu)_{\Delta K=0} &= \big[\varepsilon_{+}I_{-},\ \mathscr{M}_{-1}\mathscr{D}^{1}_{\mu-1}\big]+\big[\varepsilon_{-}I_{+},\ \mathscr{M}_{+1}\mathscr{D}^{1}_{\mu 1}\big]\\ &= 2^{-1/2}\big(\{\varepsilon_{+},\ \mathscr{M}_{-1}\}+\{\varepsilon_{-},\ \mathscr{M}_{+1}\}\big)\mathscr{D}^{1}_{\mu 0}\\ &\quad +2^{-1/2}\big([\varepsilon_{+},\ \mathscr{M}_{-1}]-[\varepsilon_{-},\ \mathscr{M}_{+1}]\big)\big(I_{\mu}-I_{3}\mathscr{D}^{1}_{\mu 0}\big)\\ &\quad +2^{-3/2}\big([\varepsilon_{+},\ \mathscr{M}_{-1}]+[\varepsilon_{-},\ \mathscr{M}_{+1}]\big)\big[\mathbf{I}^{2},\mathscr{D}^{1}_{\mu 0}\big]\end{aligned}$$

$$\begin{aligned}\delta\mathscr{M}(\lambda=1,\mu)_{\Delta K=\pm 1} &= \big[\varepsilon_{\pm}I_{\mp},\ \mathscr{M}_{0}\mathscr{D}^{1}_{\mu 0}\big]\\ &= 2^{-1/2}\{\varepsilon_{\pm},\ \mathscr{M}_{0}\}\,\mathscr{D}^{1}_{\mu\pm 1}\\ &\quad +2^{-1/2}[\varepsilon_{\pm},\ \mathscr{M}_{0}]\big([\mathbf{I}^{2},\mathscr{D}^{1}_{\mu\pm 1}]\mp\{I_{3},\mathscr{D}^{1}_{\mu\pm 1}\}\big)\end{aligned} \tag{4A-31}$$

$$\begin{aligned}\delta\mathscr{M}(\lambda=1,\mu)_{\Delta K=\pm 2} &= \big[\varepsilon_{\pm}I_{\mp},\ \mathscr{M}_{\pm 1}\mathscr{D}^{1}_{\mu\pm 1}\big]\\ &= [\varepsilon_{\pm},\ \mathscr{M}_{\pm 1}]I_{\mp}\mathscr{D}^{1}_{\mu\pm 1}\end{aligned}$$

We have assumed that the intrinsic moments $\mathscr{M}_{\nu}$ commute with $I_{\pm}$, as is the case for operators (such as the electric dipole moment) that do not explicitly depend on the rotational angular momentum. The $M1$ moment contains a term proportional to $\mathbf{I}$ (see Eq. (4A-11)), but since this term commutes with S, the induced moment may be obtained from the foregoing equations with

$$\mathscr{M}_{\nu}=\left(\frac{3}{4\pi}\right)^{1/2}\frac{e\hbar}{2Mc}\big((g_{l}-g_{R})l_{\nu}+(g_{s}-g_{R})s_{\nu}\big) \tag{4A-32}$$

In the evaluation of the terms in Eq. (4A-31), we have employed the relation (1A-91) for the commutator of $I_{\pm}$ and $\mathscr{D}^{\lambda}_{\mu\nu}$, and the anticommutator relation (see Eqs. (1A-88) and (1A-89)),

$$\begin{aligned}\left\{I_{-},\mathscr{D}^{1}_{\mu-1}\right\}-\left\{I_{+},\mathscr{D}^{1}_{\mu1}\right\}&=2^{1/2}\sum_{\nu=\pm1}\left\{I_{\nu},\mathscr{D}^{1}_{\mu\nu}\right\}\\&=2^{3/2}\left(I_{\mu}-I_{3}\mathscr{D}^{1}_{\mu0}\right)\end{aligned}\tag{4A-33}$$

as well as the further identity

$$\begin{aligned}\left[\mathbf{I}^{2},\mathscr{D}^{\lambda}_{\mu\nu}\right]=\nu\left\{I_{3},\mathscr{D}^{\lambda}_{\mu\nu}\right\}&+\tfrac{1}{2}\left(\lambda(\lambda+1)-\nu(\nu+1)\right)^{1/2}\left\{I_{+},\mathscr{D}^{\lambda}_{\mu,\nu+1}\right\}\\&+\tfrac{1}{2}\left(\lambda(\lambda+1)-\nu(\nu-1)\right)^{1/2}\left\{I_{-},\mathscr{D}^{\lambda}_{\mu,\nu-1}\right\}\end{aligned}\tag{4A-34}$$

The induced moment $\delta\mathscr{M}$ given by Eq. (4A-31) partly contains I-independent terms, which are equivalent to a renormalization of the unperturbed intrinsic moments $\mathscr{M}_{\nu}$. In addition, $\delta\mathscr{M}$ involves terms linear in $I_{\pm}$, which give rise to matrix elements with an I dependence different from that of the unperturbed moments. Moreover, these terms contribute to the K-forbidden transitions with $\Delta K=2$, for which the matrix element vanishes in the absence of the Coriolis coupling.

For $M1$ transitions within a band, the two first terms with $\Delta K=0$ in Eq. (4A-31) represent renormalizations of the intrinsic and rotational g factors by the amounts (see Eqs. (4A-11) and (4A-12))

$$\begin{aligned}K\delta g_{K}&=\langle K|\{\varepsilon_{+},((g_{l}-g_{R})l_{-}+(g_{s}-g_{R})s_{-})\}|K\rangle\\\delta g_{R}&=\langle K|[\varepsilon_{+},((g_{l}-g_{R})l_{-}+(g_{s}-g_{R})s_{-})]|K\rangle\end{aligned}\tag{4A-35}$$

while the first term with $\Delta K=1$ in Eq. (4A-31) gives a renormalization of the b parameter in $K=1/2$ bands (see Eq. (4A-12)),

$$\delta((g_{K}-g_{R})b)=-\langle K=1/2|\{\varepsilon_{+},((g_{l}-g_{R})l_{3}+(g_{s}-g_{R})s_{3})\}|\overline{K=1/2}\rangle\tag{4A-36}$$

The terms in Eq. (4A-31) involving $[\mathbf{I}^{2},\mathscr{D}^{1}_{\mu\nu}]$ do not contribute within a band, since the intrinsic matrix elements vanish as a consequence of the behavior of the operators under Hermitian conjugation and time reversal. It is seen that the $M1$ operators in a band, up to terms linear in the Coriolis coupling, have the general form represented by Eq. (4-89).

4A-3c Transformation of coordinates

The canonical transformation that leads to a diagonalization of the Hamiltonian may be viewed as a coordinate transformation to a new set of variables, in terms of which the Hamiltonian is diagonal in the representation (4A-5). If we denote the original variables by $x=(qp,\omega I_\kappa)$, where q and p represent the particle variables relative to the orientation of the rotor, the new variables are

$$x'=\exp\{-iS\}x\exp\{iS\} \tag{4A-37}$$

When expressed in these variables, the Hamiltonian has the form (see Eq. (4A-21))

$$\begin{aligned} H(qp,I_\kappa) &= \exp\{-iS\}H'(qp,I_\kappa)\exp\{iS\} \\ &= H'(q'p',I'_\kappa) \\ &= H'_{\text{intr}}(q'p')+H'_{\text{rot},\nu}(I'_\kappa) \end{aligned} \tag{4A-38}$$

The eigenstates of the effective intrinsic Hamiltonian H'_{intr} are labeled by a set of quantum numbers ν that includes K. The effective rotational Hamiltonian depends on ν and can be expressed as a function of the components I'_κ of the angular momentum with respect to the intrinsic coordinate system with orientation ω'. (The Hamiltonian, being a scalar, cannot depend explicitly on orientation angles.)

The stationary states written in terms of the coordinates (q,ω) are superpositions of wave functions of the form (4A-5) and describe sets of coupled rotational bands. In the new variables (q',ω'), the stationary states retain the form (4A-5), and the couplings are now expressed in terms of the structure of the operators

$$F(x)=F'(x') \tag{4A-39}$$

where F' is the transformed operator (4A-27). For example, the electromagnetic moments, which are rather simple functions of x, have a more complicated form when expressed in the x' variables, as illustrated by the renormalization of the dipole moments given by Eq. (4A-31).

For the first-order transformation (4A-23), the renormalization of the coordinates is given by

$$\begin{aligned} \delta q &= q'-q=-[\varepsilon_+,q]I_- -[\varepsilon_-,q]I_+ \\ \delta\omega &= \omega'-\omega=-\varepsilon_+[I_-,\omega]-\varepsilon_-[I_+,\omega] \end{aligned} \tag{4A-40}$$

For the Euler angles, the evaluation of the commutators yields

$$\begin{aligned} \delta\theta &= \varepsilon_+ e^{i\psi}-\varepsilon_- e^{-i\psi} \\ \delta\phi &= \frac{-i}{\sin\theta}(\varepsilon_+ e^{i\psi}+\varepsilon_- e^{-i\psi}) \\ \delta\psi &= i\cot\theta(\varepsilon_+ e^{i\psi}+\varepsilon_- e^{-i\psi}) \end{aligned} \tag{4A-41}$$

This renormalization implies that the collective orientation angles depend to some extent on the particle variables. In turn, the new intrinsic coordinates q' depend on the rotational frequency. (For a further discussion of collective coordinates for rotation, see Bohr and Mottelson, 1958.)