The beta- and gamma- bands in rotational nuclei

Axially symmetric case

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1.5.2 The Axially Symmetric Case

The Hamiltonian $\hat{H}_{\text{coll}}$ (1.54) is still very general. We shall restrict ourselves to cases of very pronounced minima in the potential surface at axially symmetric deformations $\beta=\beta_0$ and $\gamma=0$. We expect rotations and small vibrations of the nuclear surface. Expanding $T_{\text{rot}}-\hat{I}_3^2/2\vartheta_3$ (1.55) around the point $\beta=\beta_0$, $\gamma=0$, we obtain in zeroth order the Hamiltonian of an axially symmetric rotor with the moment of inertia $\vartheta_0=\vartheta_1(\beta_0,0)=\vartheta_2(\beta_0,0)$,

$$T'_{\text{rot}}=\frac{\hat{I}^2-\hat{I}_3^2}{2\vartheta_0}. \tag{1.62}$$

First-order terms are proportional to the deviations $(\beta-\beta_0)$ and γ. They mix rotational and vibrational degrees of freedom (rotational–vibrational coupling terms) and will be neglected here. The only remaining term in H_{coll} that still couples rotations and vibrations is $\hat{I}_3^2/2\vartheta_3$. It cannot be expanded, since ϑ_3 vanishes for $\gamma=0$. However (as T'_{rot}), it commutes with $\hat{I}_3$ and K is therefore a good quantum number.

We now have to distinguish

(I) $K=0$ **bands** ($I_3=0$). In this case, the rotational and vibrational motions decouple. The wave function is of the type

$$|\Psi^I_{M0}\rangle=g_0(\beta,\gamma)|IM0\rangle. \tag{1.63}$$

They are eigenfunctions of the rotational part of the Hamiltonian (1.54). $\mathfrak{R}_1$-symmetry (1.59) requires the spin sequence $I=0,2,4,\ldots$. A detailed investigation of the vibrational part of H_{coll} [EG 70, Vol. I, Chap. 6] shows that it is easier to handle in the variables a_{20} and a_{22} (1.13). In the first step one neglects terms in the potential $V(a_{20},a_{22})$, which couple these two degrees of freedom. In this case, the motion in the coordinate a_{20} (usually called β-vibration) decouples from the motion in the coordinate a_{22} (usually called γ-vibration). Axial symmetry with respect to the 3-axis is preserved by the β-vibration (quantum number n_β), but violated by the γ-vibration (quantum number n_γ). Both types of motion are shown qualitatively in Fig. 1.11.

Superimposed on each vibrational state (n_β,n_γ) is a rotational band. The spectrum is given by (see [EG 70, Vol. I, Chap. 6])

$$E_{n_\beta n_\gamma}(I)=E_{n_\beta n_\gamma}(0)+\frac{\hbar^2}{2\vartheta_0}I\cdot(I+1) \tag{1.64}$$

with the band head

$$E_{n_\beta n_\gamma}(0)=\hbar\omega_\beta(n_\beta+1/2)+\hbar\omega_\gamma(2n_\gamma+1),$$
$$n_\beta=0,1,2,\ldots,\qquad n_\gamma=0,1,2,\ldots, \tag{1.65}$$

where ω_β and ω_γ are the frequencies of β- and γ-vibrations.

$$\omega_\beta=(C_{20}/B_2)^{1/2}\qquad \omega_\gamma=(C_{22}/B_2)^{1/2}$$

In fact, such bands have been observed in many even–even nuclei, in particular the ground state band ($n_\beta = n_\gamma = 0$) and the "β-band" ($n_\beta = 1$, $n_\gamma = 0$). However, as we have already discussed, the constants $\mathcal{I}_0$ and $\omega_\beta, \omega_\gamma$ of the hydrodynamical model do not agree with the experimental data.

(II) $K \neq 0$ bands. Together with $\mathfrak{R}_1$-symmetry [Eq. (1.58)], we see that the wave function now has the form

$$|\Psi^I_{MK}\rangle = g_K(\beta,\gamma)\frac{1}{\sqrt{2}}\left\{|IMK\rangle + (-)^I|IM-K\rangle\right\}. \tag{1.66}$$

The $\mathfrak{R}_1$ and $\mathfrak{R}_2$ symmetries ((1.59) and (1.61)) give the selection rule: K must be even. Such $K \neq 0$ bands have, therefore, the spin sequence $I = |K|$, $|K|+1$, $|K|+2, \ldots$. The motion in a_{20} (β-vibration) can again be separated from the rest. However, the term $\hat{I}_3/2\mathcal{I}_3 = K^2/16B_2a_{22}^2$ couples the γ-vibration with the rotation around the 3-axis. We can easily understand this fact if we realize that a γ-vibration can be represented as a superposition of two rotations of a triaxial nucleus around the 3-axis, but with opposite K-values [BM 75, p. 656].

The following spectrum is obtained [EG 70, Vol. I, Chap. 6]

$$E_{K,n_\beta,n_\gamma}(I) = E_{K,n_\beta,n_\gamma}(0) + \frac{\hbar^2}{2\mathcal{I}_0}\left(I(I+1) - K^2\right) \tag{1.67}$$

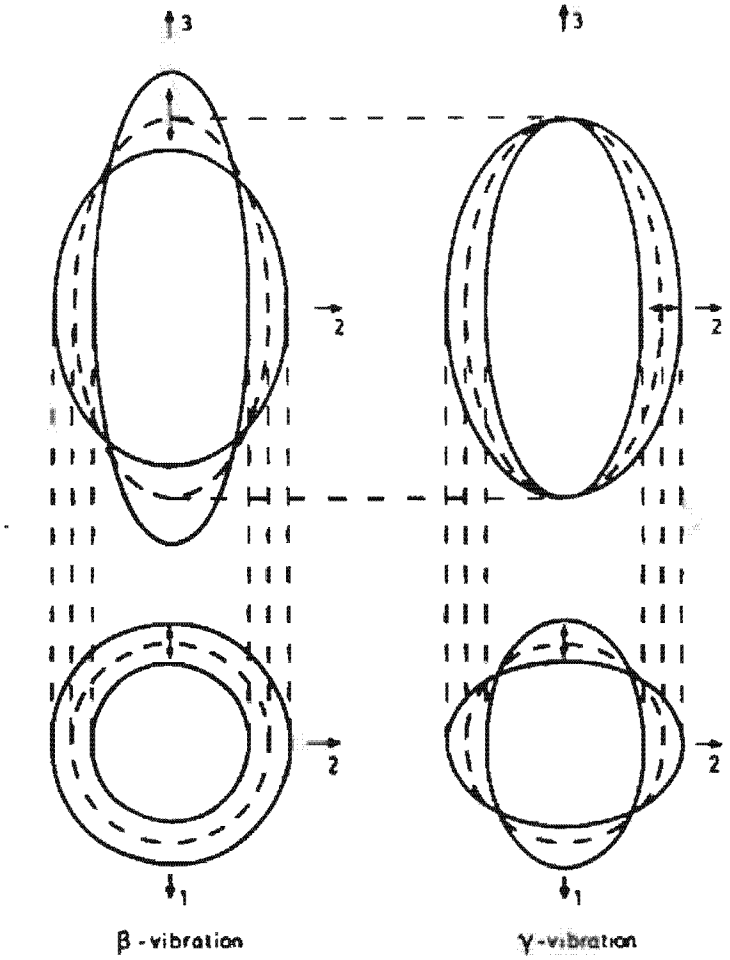

Figure 1.11. Schematical representation of β- and γ-vibrations by a cut along the (1, 3) and (1, 2) planes.

with the bandheads

$$E_{K,n_\beta,n_\gamma}(0)=\hbar\omega_\beta\left(n_\beta+\frac{1}{2}\right)+\hbar\omega_\gamma\left(2n_\gamma+1+\frac{|K|}{2}\right). \tag{1.68}$$

In fact, such bands have also been observed, especially the so-called "γ-band" in many deformed nuclei, which has the quantum numbers $K=2$, $n_\beta=0$, $n_\gamma=0$. This γ-band has the vibrational quantum number $n_\gamma=0$, however, one is not allowed to apply the classical picture of no vibration in this case. It would correspond to $\gamma=0$ and imply $\vartheta_3=0$, which would forbid a rotation with $K\neq 0$. Only the quantum mechanical zero point vibration in the γ-direction makes this possible.

Figure 1.12 shows the qualitative structure of the collective ($\lambda=2$) excitation in deformed and spherical nuclei.

The spherical nuclei have a harmonic spectrum. In the deformed nuclei we observe several rotational bands built on the ground state, on the β-vibrational state $K=0$, $n_\beta=1$, $n_\gamma=0$, and on the γ-vibrational state $K=2$, $n_\beta=n_\gamma=0$. However, these pure cases are not exactly realized in nature. In fact, we already observe in spherical nulcei a splitting of the two-boson triplet $(0^+,2^+,4^+)$ and in the deformed nuclei deviations from the $I(I+1)$ law. There is also a wide range of transitional nuclei in between these two limits. Going from isotope to isotope, one can sometimes observe a gradual transition from a vibrational to a rotational spectrum (for instance, in the Os region [SDG 76]). This is indicated by dashed lines in Fig. 1.12.

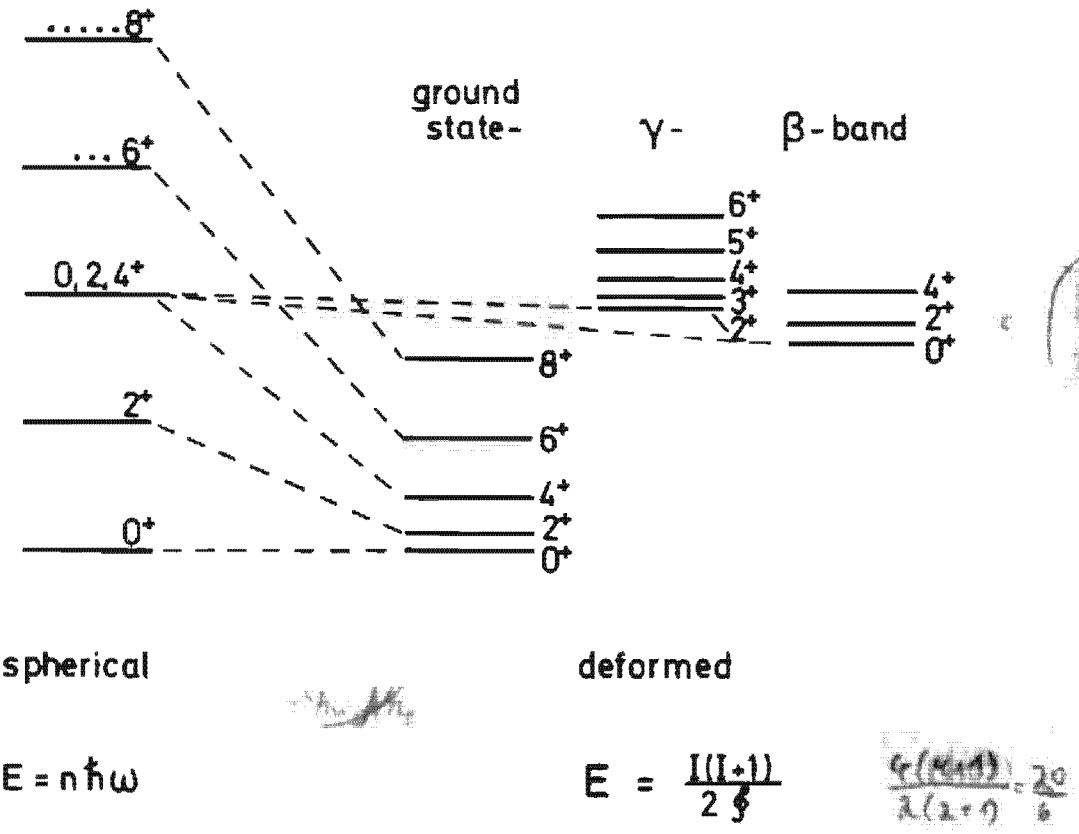

Figure 1.12. Schematic level schemes of spherical and deformed nuclei. (From [SDG 76].)

For a theoretical description of deviations from the $I\cdot(I+1)$ law, one has taken into account the rotational-vibrational coupling terms in $\hat{H}_{\text{coll}}$ which one gets by expanding ϑ_κ in powers of the deviations $(\beta-\beta_0)$ and γ[FG 62, FG 64, FGS 65, 66, ABP 68]. This rotation–vibration interaction causes changes in the moment of inertia of a band and corresponds to a change of the nuclear shape under the influence of the rotation (stretching effect). However, there also exist quite different excitations of nuclei in this energy region, for instance, two quasi-particle states and pairing vibrations (see Chap. 8), which have a much stronger influence on the rotational bands, and which are not taken into account in this simple model of a liquid drop with surface oscillations.

Before we leave this section, we want to discuss very briefly how to calculate electromagnetic moments and transition probabilities. In Eqs. (1.35) and (1.36) we defined the electric multipole and the magnetic moment operator in the coordinates $\alpha_{\lambda\mu}$. They are obviously written down in the laboratory system. In a deformed nucleus it is usually very easy to calculate the moments in the intrinsic system. To get the moments in the laboratory frame—the experimental values—one has to apply the transformation (1.11) of spherical tensors:

$$\hat{Q}^{\text{intr}}_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu} \hat{Q}^{\text{lab}}_{\lambda\mu'}. \tag{1.69}$$

Since Q^{intr} does not depend on the Euler angles, we get from [Ed 57, Eqs. 4.6.2 and 5.4.1] the reduced matrix elements with respect to $|IMK\rangle$:

$$\langle I_1 K_1 \| \hat{Q}^{\text{lab}}_{\lambda} \| I_2 K_2 \rangle = \sum_{\mu'} Q^{\text{intr}}_{\lambda\mu'} (-)^{I_1 - K_1} ((2I_1+1)(2I_2+1))^{1/2} \begin{pmatrix} I_1 & \lambda & I_2 \\ -K_1 & \mu' & K_2 \end{pmatrix}. \tag{1.70}$$

We restrict ourselves now to pure K-bands and calculate only intraband $E2$-transitions ($n_{\gamma_1} = n_{\gamma_2}, n_{\beta_1} = n_{\beta_2}$). For the reduced matrix element we find

$$\langle n_\beta n_\gamma I_1 K \| \hat{Q}^{\text{lab}}_2 \| n_\beta n_\gamma I_2 K \rangle = \sqrt{\frac{5}{16\pi}}\, Q_0(n_\beta, n_\gamma) \sqrt{(2I_1+1)(2I_2+1)}\, (-)^{I_1 - K} \begin{pmatrix} I_1 & 2 & I_2 \\ -K & 0 & K \end{pmatrix}, \tag{1.71}$$

where $Q_0(n_\beta, n_\gamma)$ is the intrinsic quadrupole moment of this band. In the ground state band with fixed β-value, and $\gamma = 0$ we have [Eqs. (1.13) and (1.35)]

$$Q_0 = \sqrt{\frac{16\pi}{5}}\, \frac{3}{4\pi} Ze \cdot R_0^2 \cdot \beta. \tag{1.72}$$

From (B.73) and (1.71) we obtain, for example, for the so-called stretched $BE2$-values in a rotational band,

$$B(E2, I+2 \to I) = Q_0^2 \frac{5}{16\pi} \left| C^{I+2\ 2\ I}_{K\ 0\ K} \right|^2, \tag{1.73}$$

which, for $K=0$ bands, gives

$$B(E2, I+2\to I) = Q_0^2 \frac{5}{16\pi}\frac{3}{2}\frac{(I+1)(I+2)}{(2I+3)(2I+5)}. \tag{1.74}$$

For the spectroscopic quadrupole moments $Q=\sqrt{16\pi/5}\,\langle IIK|\hat{Q}_{20}^{\text{lab}}|\,IIK\rangle$ [Eq. (B.32)], we get

$$Q = Q_0 \frac{3K^2 - I\cdot(I+1)}{(2I+3)(I+1)}. \tag{1.75}$$

The quotient of $Q : Q_0$ is the expectation value of $D_{00}^2 = P_2(\cos\beta)$ in the state $M=I$. This means that one cannot measure the internal quadrupole moment Q_0, but only the value averaged over the rotational motion. In fact, Q_0 is not a physical observable. Its definition depends on the introduction of a body-fixed system which moves with the nucleus and has a model character. For the band head we usually have $I=K$. That means the spectroscopic quadrupole moment Q of ground states with $I=0$ vanishes and we can get information about Q_0 only for the excited states (for instance, by the reorientation effect in Coulomb excitation [BE 68], which gives the sign and the absolute value of Q_0). Another way to determine the absolute value of Q_0 is the measurement of the $B(E2)$-values (1.73) in the transitions within a band. Recent measurements up to high spin states [WCL 76, HJE 78] have shown that in many deformed nuclei the value of Q_0 stays fairly constant within these bands, even in cases where the spectrum shows deviations from the $I(I+1)$ character. This is a hint that these deviations are not caused by the change of deformation (stretching effect).

1.5.3 The Asymmetric Rotor

The rotational–vibrational interaction model discussed so far has been on the basis of a symmetric rotor. Further attempts to explain the deviations from the $I(I+1)$ law and the low-lying second 2^+ states in many nuclei have been undertaken by Davydov et al. using the picture of a pure triaxial rotor [DF 58, DR 59, Da 59, Da 65b]. As a first step they do not consider the vibrational excitations and diagonalize only the rotational energy (1.55). With the moments of inertia (1.48), this operator is proportional to β^{-2} and one can diagonalize it for all values of γ. The constant factor can afterwards be adjusted so as to reproduce the first 2^+ state. Using the symmetries $\mathfrak{R}_1$ and $\mathfrak{R}_2$ (1.59 and 1.61), the wave functions have the form

$$|\Psi_M^I\rangle = \sum_{K=0,2,\ldots,} g_K\{|IMK\rangle + (-)^I|IM-K\rangle\}. \tag{1.76}$$

Figure (1.13) shows the corresponding energy eigenvalues. For $\gamma=0°$ and $\gamma=60°$ one gets the $I(I+1)$ spectrum. Even for strong triaxial deformations one gets only slight deviations of this form. However, additional $2^+, 3^+, 4^+$ levels come down in energy. It is a characteristic of this structure to have a low-lying second 2^+ state. Although one can, with such a model, reproduce quite reasonably the experimental