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NUCLEAR MODELS

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CONTENTS

1. INTRODUCTION

THE objects of this survey are firstly to describe the main features of different nuclear models and secondly to consider how far each model can be regarded as representing a particular aspect of an integrated picture of the nucleus. The choice of nuclear models to be discussed has been made partly on the basis of their success in classifying a range of experimental results, partly on whether they throw light on the qualitative behaviour of a nucleus under various experimental conditions, and partly on their contribution to understanding the theoretical basis for nuclear structure and for nuclear models.

In describing a nuclear model the following topics will be considered: (1) the assumptions on which the model is based and the resulting physical picture of the model, (2) the experimental evidence in favour of the model and the experimental domain in which the model is useful, (3) refinements of the model to extend its usefulness and its essential limitations which cannot be remedied by refinements, (4) the implications of the model for nuclear structure or nuclear behaviour, (5) the theoretical basis for the model and its relation to a detailed theory of the nucleus. These considerations cannot altogether be kept separate in view of the relations between different models, and some will be postponed to Section 10 of the survey which is concerned with the nuclear many-body problem.

The nuclear shell model is described in Section 2 in its simplest form of independent particle motion. This form is applicable to certain ground-state properties of most nuclei, and provides the basis of the shell characteristics of nuclei. In Section 3 the principal refinements of the shell model are described. These include (i) the consideration of residual two-body interactions which leads to a description of the properties of low-lying excited states of nuclei near closed shells, (ii) the spheroidal shell model which extends the model to describe nuclear quadrupole moments, (iii) the use of momentum-dependent single-particle potentials which permits the model to give a true representation of total nuclear energy.

The first method of investigating collective motion in the nucleus was based on the liquid-drop model. It is possible that this model gives a useful representation of nuclear fission, but it is now known that it gives an incorrect picture of the low-energy spectra of nuclei. The most important consequence of low-energy collective motion is the rotational spectrum and the strong $E2$ transitions of distorted nuclei. These are described by the rotational model of the nucleus which is considered in Section 4. This contains many of the characteristics of the liquid drop model but the picture is different. The motion can be pictured as a rotating distorted-shell model but not as oscillations of a liquid drop. The moment of inertia estimated from the spheroidal shell model with energy-dependent potential (and possibly also residual interactions) appears to be of the right magnitude to fit observed rotational spectra. Section 5 contains some concluding remarks on models for nuclear structure at low energies.

The compound-nucleus model for nuclear reactions is described in Sections 6 and 7. Section 6 is concerned with the generalizations of the theory which lead to a framework for describing nuclear reactions rather than to detailed predictions. Section 7 describes the extra assumptions which lead to the statistical theory of nuclear reactions.

The optical model in which a target nucleus is represented by a complex potential is described in Section 8. Particular attention is given to the relation of this model to the compound nucleus theory, and it is seen that the detailed mechanism by which a colliding nucleon interacts with a target nucleus to form a compound nucleus will be of importance in most energy ranges. The need for special models for very-high-energy nuclear reactions (greater than 80 MeV) is discussed in Section 9. This section also describes the deuteron model which provides a method of estimating certain types of correlation in the nucleus.

Finally in Section 10 a brief account is given of the theory of the nucleus when it is considered as a system of many nucleons interacting through two-body forces. It is shown how this theory leads to a model similar to the improved shell model with momentum-dependent potentials. The relation of this model to the actual nuclear wave function is not described in detail, but it is noted that the many-body theory provides a general basis for models for nuclear structure and nuclear reactions and provides an integrated picture of nuclear behaviour.

A complete list of references would be disproportionate to an article of this length; the references are therefore limited to a somewhat arbitrary selection of typical or important papers.

2. The Shell Model

In its simplest form the shell model assumes independent particle motion by nucleons in the nucleus subject only to the requirements of the exclusion principle which must be satisfied by both neutrons and protons. The corresponding wave function is a determinant

$$\Phi = \begin{vmatrix} \phi_1(1), & \phi_1(2), \ldots, & \phi_1(A) \\ \phi_2(1), & \phi_2(2), \ldots, & \phi_2(A) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \phi_A(1), & \phi_A(2), \ldots, & \phi_A(A) \end{vmatrix} \tag{2.1}$$

The 1, 2, . . . A denote the co-ordinates, spin, and isotopic spin of the corresponding particle.

The basic assumptions of the present form of the shell model were given by Mrs. M. G. Mayer (1948, 1949) and by Haxel, Jensen, and Suess (1948, 1950). A general account of the model is given by Mayer and Jensen (1955) who also list detailed references. The assumptions are:

(i) Independent particle motion in a potential which contains a strong spin-orbit interaction term.

(ii) A nuclear ground state corresponds to occupation by the neutrons and protons of the lowest single-particle energy levels which are compatible with the exclusion principle.

(iii) An even number of protons in the state of lowest energy couples to zero angular momentum and even parity and the same is true for an even number of neutrons.

(iv) For an odd A nucleus with an odd number of protons the nuclear angular momentum is usually equal to that of the last added proton; and similarly, if it is the neutron number which is odd (the only exceptions are certain light nuclei for which the nuclear angular momentum is one unit less than that of the last added nucleon).

The potential which determines the single-particle wave functions ϕ_i in (2.1) has the form

$$V_{s.p.} = V_A(r) - f_A(r)(\mathbf{l} \cdot \mathbf{s}) \tag{2.2}$$

where $V_A(r)$ and $f_A(r)$ depend only on the radial distance r and the size of the nucleus, and $(\mathbf{l} \cdot \mathbf{s})$ denotes the coupling of the nucleon spin $\mathbf{s}$ and the orbital angular momentum $\mathbf{l}$. There is also a difference between the potentials for protons and for neutrons which is attributed to Coulomb effects. The magnitude of the potential $V_{s.p.}$ is not a significant feature of the shell model in its present form for reasons to be discussed below. Its order of magnitude will be about

40 MeV and the ratio of $V_A(r)$ to $f_A(r)$ is about 10 to 1, but both may vary for different shells. The sign of the spin-orbit term is such that the level having angular momentum $j = l - \frac{1}{2}$ lies above the level having $j = l + \frac{1}{2}$.

The essential requirement on the potential $V_{s.p.}$ is that it predicts that single particle levels occur in an order which makes the model agree with the experimental shell characteristics of nuclei and with the values of nuclear angular momentum in their ground states. In addition to having the correct order the levels must fall into groups or shells of approximately the same energy. These are shown in Fig. 1, in which different shells are separated by horizontal lines.

Single-particle-level scheme

level (n,l,j)	*Parity*	*No. of states*	*Magic numbers*
1 *j* 15/2	odd	16	184
3 *d* 3/2	even	4	
4 *s* 1/2	even	2	
2 *g* 7/2	even	8	
1 *i* 11/2	even	12	
3 *d* 5/2	even	6	
2 *g* 9/2	even	10	
1 *i* 13/2	even	14	126
3 *p* 1/2	odd	2	
3 *p* 3/2	odd	4	
2 *f* 5/2	odd	6	
2 *f* 7/2	odd	8	
1 *h* 9/2	odd	10	
1 *h* 11/2	odd	12	82
3 *s* 1/2	even	2	
2 *d* 3/2	even	4	
2 *d* 5/2	even	6	
1 *g* 7/2	even	8	
1 *g* 9/2	even	10	50
2 *p* 1/2	odd	2	
1 *f* 5/2	odd	6	
2 *p* 3/2	odd	4	
1 *f* 7/2	odd	8	28
1 *d* 3/2	even	4	20
2 *s* 1/2	even	2	
1 *d* 5/2	even	6	
1 *p* 1/2	odd	2	8
1 *p* 3/2	odd	4	
1 *s* 1/2	even	2	2

Fig. 1.

We will consider next the experimental evidence which supports the shell model. It is to be expected that closed shells will correspond to nuclei of exceptional stability as represented by the observed magic numbers, that is to nuclei showing exceptional binding energy, for the most weakly bound nucleon. Of much greater significance is the agreement with the spin and the magnetic moments of nuclear ground states for odd A nuclei; MAYER and JENSEN (1955) list the comparative results of theory and experiment.

For Z or N odd and less than 84: out of 101 known spins 93 fit the single-particle picture, for 4 nuclei $J = j - 1$ and 4 others give agreement only by adjusting the level order within a shell in each case. If the nuclear magnetic moment is determined like the spin by the last odd nucleon only, it will lie on one or other of the two Schmidt lines which correspond to $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$. Out of 89 known magnetic moments 86 are nearer to the predicted Schmidt line than to the other, 2 are about half way, and Eu^{153} is nearer the wrong line (although Eu^{151} is all right).

There is similar qualitative agreement between the model and the observed log (ft) values for β decay. Generally, the single-particle predictions on whether a transition is allowed or first forbidden agree qualitatively with experiment, but quantitative agreement is no better than for magnetic moments.

Another important confirmation of the level scheme of the independent-particle shell model is the occurrence of nuclear isomers in certain regions of the periodic table; an isomer being a low lying excited state of a nucleus which is long lived because it can decay only to the ground state, and this involves a large spin change. An odd-A nucleus having neutron or proton number between 38 and 50 may have its last nucleon either in a $p^{\frac{1}{2}}$ or in a $g^{\frac{9}{2}}$ state. Since a transition between these levels requires a spin change of 4 the higher of the two will be a long-lived isomer. Experimentally 29 isomers have been observed in this region. Similar confirmation by experiment has been obtained in other regions of the periodic table where two single-particle levels of widely differing l have almost the same energy (GOLDHABER and HILL, 1952).

The successes of this simplest form of the shell model are related to angular momentum and parity properties of the nucleus. The magnetic moment and β-decay evidence shows that the last nucleon in an odd-A nucleus is not in a pure single-particle state, but it couples to other nucleons so that the total J is the same as the single particle j. If the pure independent-particle motion is modified by taking into account some residual particle-to-particle interaction, this will lead to impurities, particularly in the states of particles in a partly filled shell. This is the basis of one aspect of the "improved" shell model to be discussed in Section 3.

One of the most striking regions of disagreement between this form of the shell model and experiment is with experimental values of nuclear quadrupole moments. The simple shell model predicts a single-particle value for the quadrupole moment, and for nuclei between closed shells this is smaller, by a factor which may exceed 100, than the experimental values. The explanation of the discrepancy lies in the assumption of a potential (2.2) which is spherically symmetric. An independent particle wave function of lower energy can be obtained by considering a distorted or spheroidal potential, and in such a potential all particles contribute to the quadrupole moment. The resulting spheroidal shell model will be considered in Section 3.

The second region of major disagreement of the shell model with experiment concerns observations which depend on details of the intrinsic structure of the nucleus in its ground state such as correlations in position between nucleons. Independent-particle motion involves only very small correlations between the positions of nucleons (these are due to the exclusion principle and to the finite nuclear size). Thus the probability of a nucleon in the model having a high momentum depends primarily on the fourier transform of the single-particle wave functions. A number of high-energy reactions such as proton-nucleus collisions at 90 MeV with ejection of two protons (or proton and neutron), or meson absorption in the nucleus, depend sensitively on the high components in the momentum distribution of a nucleon in the nuclear ground state. It is found that the shell model gives much too small a probability for high-momentum components and predicts cross-sections for high-energy processes too small by a factor ~ 50. The evidence for this is summarized in a paper by BRUECKNER, EDEN, and FRANCIS (1955a). It is concluded that the detailed correlations in the nuclear ground state are much greater than those indicated by the shell model. This shows that the residual interactions between nucleons are still strong but their effects are masked for observations of quantities such as angular momentum which are not sensitive to small detailed correlations in that they involve averages over the nuclear volume. These questions will be examined further in Section 9.

The third major disagreement of the shell model with experiment lies in its implications about total binding energy. In the formulation of MAYER and JENSEN (1955) the model is considered purely phenomenologically and its applications avoid any question of the absolute values of energy, and therefore only the relative ordering of levels is relevant. However, if the model is to be regarded as representing a picture of the nucleus the question of energy cannot be avoided. The most natural assumption would be that the potential (2.2) is some kind of average of the nucleon-nucleon interactions. If this was so then the total nuclear energy would be

$$E = \sum_{i=1}^{A} (T_i) + \tfrac{1}{2} \sum_{i=1}^{A} (V_i) \tag{2.3}$$

where the first term denotes the single-particle kinetic energy and the second is half the expectation value of the potential energy (the factor $\frac{1}{2}$ allows for the fact that in $\sum V_i$ every two-nucleon interaction is counted twice). The observed binding of the most weakly bound nucleon in a nucleus is approximately 8 MeV. If the well depth is assumed constant and the last added nucleon is to be 8 MeV below the top of the well, it can be shown that the well depth must be approximately 33 MeV (WIGNER, 1950). In evaluating the total nuclear energy from formula (2.3) the most weakly bound nucleon will therefore contribute $+\ 8\frac{1}{2}$ MeV to the summation ($8\frac{1}{2}$ MeV $= 25$ MeV $+ \frac{1}{2}(-33$ MeV$)$). Although the most tightly bound particles will contribute negative values to the summation (2.3), it turns out that the total energy (2.3) will be near to zero or positive. In other words the model gives incorrect binding.

There are two possible explanations for this paradox. One is that the formula (2.3) for the total energy is incorrect; this would be the case only if three-body forces (or higher many-body forces) were important, but there is no evidence in their favour either from theory or experiment. The other explanation, which seems more probable, is that the potentials V_i are not the same for all particles in a given nucleus. Thus it would be possible to have the most weakly bound particle move in a potential well of depth 33 MeV as is required by experiment, but to have the most tightly bound particles moving in a well of much greater depth. Such a variation of the potential depth need not affect the basic requirement of the shell model that it predicts the single particles in a particular order as shown in Fig. 1. Such potentials are usually called "momentum dependent," they will be discussed further in Section 3 (*c*).

Apart from the three topics discussed above the independent particle form of the shell model appears to give a good first approximation to the ground states of most nuclei. A more accurate representation will be given by allowing for perturbations due to residual nucleon-nucleon interactions. The latter become particularly important if the model is to be extended to low-lying states of a particular nucleus. The model ceases to be useful in regions where the residual interactions become dominant over the average potential V_i such as in highly excited states of a nucleus.

We will consider next some of the refinements of the shell model which extend its range of application to experiment.

3. The Extended Shell Model

(*a*) *Residual interparticle forces*

The qualitative success of the independent-particle shell model led to efforts to refine it to give quantitative agreement with experiment. These refinements include the consideration of two-body forces between nucleons outside a closed shell. The effects of various two-body interactions have been studied by Flowers (1952), Edmonds (1952), Pryce (1952), Inglis (1953), Talmi (1952), and Kurath (1953). These authors assumed definite configurations of two, three, or four particles outside a closed shell. Without two body interactions between these nucleons the ground state of the model would be degenerate due to the different possible ways in which the single particle angular momenta can be combined. The perturbing two-body interactions determine the coupling which gives the lowest total energy and also provides the energy separation between states for which the coupling differs.

This work was extended by relaxing the assumption that only one configuration of the independent particles was involved. Configuration mixing has been considered by Inglis (1953), de Shalit and Goldhaber (1953), Redlich

(1954), Elliott and Flowers (1955), and Ford and Levinson (1955). These authors allow for the possibility that the two-body interactions will involve other single particle states than those which would have the lowest energy in the simple independent particle picture.

The assumption that a closed shell core has little effect on the interactions of the extra two, three, or four nucleons, (beyond providing an average field) is supported by three features of the shell model: (i) the energy difference between particles in the closed shell and the extra particles, (ii) the different parity between the highest occupied single particle states in O^{16} (and Ca^{40}) and the states of the extra particles, (iii) the "completeness" and spherical symmetry of a closed shell of nucleons will lead to nearly uniform effects on different energy levels of external nucleons. The shell-model calculations with configurations mixing for O^{16} and Ca^{40} plus two or three nucleons give quite close agreement with the experimental values of relative separation of energy levels, magnetic moments, and transition probabilities. The values of the single particle energy levels of O^{17}, etc. are taken from experiment and so also are some two-particle levels of O^{18}, etc. The work of Elliott and Flowers assumes a two-body interaction with Rosenfeld exchange mixture and they were not able to fit the data with attractive forces and Serber exchange mixture. There are no quantitative results based on calculations using a two-body potential having a repulsive core.

It is very hard to assess the implications of these shell-model calculations. The disagreement between shell model total energy and observed binding energies is avoided by taking experimental values of single particle levels. The possibility of a uniform shift in two-body levels through core interactions is not involved since only relative energies are considered. The agreement with experiment reduces therefore primarily to a test of the predominant influence of the two or three extra-shell particles on the observables involved. Until further investigations have been made on the effects of a repulsive core in the two-body force nothing can be concluded about the nature of the two-body potential which is implied by these shell model calculations.

The essential limitations on calculation with residual two-body forces are computational. It seems improbable that the method by itself will prove useful except in the neighbourhood of closed shells although, taken together with some other method such as that for collective motion, its range of application may be extended further.

(b) *Independent-particle motion in a spheroidal well*

The use of spheroidal potentials was first suggested by Rainwater (1951). The investigation of a model in which each nucleon moves independently in a spheroidal well has been carried out by Nilsson (1955) and by Gottfried (1955). The need to consider a potential which departs from spherical symmetry is emphasized by the large nuclear quadrupole moments which indicate considerable nuclear distortion in regions midway between closed shells. The assumption of an intrinsic nuclear distortion implies the existence of new dynamical variables which describe the distortion; these will be considered in a later section on a collective model. The present section will be concerned with the nuclear properties in the intrinsic reference frame defined by the spheroidal well.

The use of a spheroidal well assumes axial symmetry. The assumption is confirmed for some nuclei by observed rotational spectra but there seems to be no *a priori* reason why it should be true for all nuclei.

The spheroidal potential of Nilsson which determines the single-particle states is

$$V_i = V_0\{(1 + \tfrac{2}{3}\delta)(x_i^2 + y_i^2) + (1 - \tfrac{4}{3}\delta)z_i^2\} + C\mathbf{l}_i \cdot \mathbf{s}_i + Dl_i^2 \qquad (3.1)$$

The model neglects surface effects due to a well of finite depth, and the ratio of C and D to V_0 is chosen to make the level order agree with that of the spherical shell model when the distortion δ is zero.

A particular nucleus is represented by filling up the single particle levels in order of increasing energy. Its distortion δ is determined by minimizing the expectation value of the kinetic energy plus half the potential energy

$$\left(\sum_{i=1}^{A} T_i + \tfrac{1}{2}\sum_{i=1}^{A} V_i\right) \qquad (3.2)$$

The expectation value is taken with respect to the corresponding independent particle wave function and is minimized by varying δ. The minimum value corresponds to an "equilibrium distortion."

The equilibrium distortion obtained by this method is in reasonable agreement with the nuclear distortion deduced from observed quadrupole moments. The physical meaning of the method is not very clear, since it uses a well of constant depth and it is known (see Section 2) that this cannot lead to the correct value for total nuclear energy. However if the method of NILSSON and GOTTFRIED was modified to take account of a potential of varying depth such as that described in Section 3 (*c*), it would become a form of Hartree-Fock method. It is probable that this change would not greatly affect their results since the amount of nuclear distortion appears to be dominated by the last partly filled shell.

(*c*) *Shell model with varying well depth*

In order to make the total binding energy (3.2) correspond to the experimental value (about 8 MeV per nucleon) and at the same time make the binding energy of the last added nucleon correct (about 8 MeV also), it is necessary for particles in different shells in a given nucleus to be in potentials of different depths. The highest occupied shell in the nuclear ground state lies 8 MeV below zero, and this implies that the nucleons in this shell must move in a potential of depth about 33 MeV. The lowest shell must correspond to a deeper potential well; its actual depth cannot be determined empirically without making some assumption about the variation of the depth between successive shells in a given nucleus. If this variation is proportional to the kinetic energy of the particle the depth for particles in the lowest shell of a medium weight nucleus in a harmonic oscillator potential is about 70 MeV. The values are similar in magnitude for a Fermi gas model (WEISSKOPF, private communication).

For a nuclear medium of infinite extent the single particle states can be labelled by the momentum k_i and the potential can only be a function $V(k_i)$ of the momentum. If k_i is the momentum for one of the occupied levels it may be permissible to expand $V(k_i)$ as a power series

$$V(k_i) = V_0 + bk_i^2 + \ldots \tag{3.3}$$

The total energy of the i'th particle is then

$$E_i = T_i + V(k_i) = \frac{1}{2m} k_i^2 + V_0 + bk_i^2 + \ldots \tag{3.4}$$

$$= \frac{1}{2m^\star} k_i^2 + V_0 + \ldots \tag{3.5}$$

If terms of higher order than k_i^2 are neglected, (3.5) can be interpreted by saying that the nucleon has an effective mass $m^\star$ (BRUECKNER, 1955a). The empirical value of $m^\star$ can be determined so that the total binding energy and the binding energy of the last-added nucleon both agree with experiment. This gives $m^\star$ an empirical value equal to about half the actual nucleon mass. An important consequence of the momentum dependence of the potential is that the energies of different shells are more widely separated than they would be for a constant potential well.

4. Collective Motion: The Rotational Model

A number of models of collective motion in the nucleus have been suggested but none of them have given quantitative agreement with experiment. However they have led to a general understanding of the coupling between collective and independent particle motion and also to formulae whose structure appears to be qualitatively correct although the model may be wrong in detail. The principal work on collective motion has been carried out by BOHR and MOTTELSON (1953, 1954, 1955), and from a somewhat different viewpoint by HILL and WHEELER (1953).

The strongest evidence of collective motion in the nucleus is given by the existence of rotational spectra of strongly deformed nuclei and by the existence of many $E2$ transitions which are 10^2 or even 10^4 times as strong as single particle transitions.

The nature of the variables which describe the collective motion is not known although it seems probable that they will have a fairly close relation to the shape of the nucleus. The simplest type of collective motion which has been identified experimentally is connected with rotations of deformed nuclei. If the rotational motion is sufficiently slow it will not affect the internal structure of the system and its energy will be proportional to the square of the angular velocity ω, so that

$$E_{rot} = \tfrac{1}{2}\mathfrak{J}\omega^2 \tag{4.1}$$

We will refer to a model in which rotational and intrinsic energy are separable as the "rotational model." The parameter $\mathfrak{J}$ which denotes an effective moment of inertia of the nucleus, can be calculated only if a specific model is assumed for the internal structure. For a given nuclear distortion a classical picture of the nucleus suggests two extreme models. The first consists of treating the nucleus as a solid or rigid structure so that all parts contribute additively to the moment of inertia and give $\mathfrak{J} = \mathfrak{J}_{rigid}$. The second represents the nucleus as a liquid drop in which the distortion is carried round by a surface wave which carries the minimum possible energy for rotation of a given distortion. This requires that the liquid in the drop is in irrotational motion and gives a moment of inertia $\mathfrak{J} = \mathfrak{J}_{irrot} = \beta^2 \mathfrak{J}_{rigid}$ where β describes the amount of distortion and is less than one.

It is reasonable to assume that the same two models will set an upper and lower limit for $\mathfrak{J}$ also in the actual nucleus,

$$\mathfrak{J}_{irrot} < \mathfrak{J} < \mathfrak{J}_{rigid} \tag{4.2}$$

The rotational spectrum is obtained from (4.1) by quantising the angular momentum and for even-even nuclei this gives

$$E_{rot} = \frac{\hbar^2}{2\mathfrak{J}} I(I+1) \tag{4.3}$$

where it is assumed that the angular momentum of the intrinsic motion is zero. The nuclear deformation will be symmetric with respect to reflection in the nuclear centre and this restricts the permissible values of I to even numbers (Bohr and Mottelson, 1955),

$$I = 0,\ 2,\ 4,\ 6\ \ldots \tag{4.4}$$

Apart from the assumption that the rotation is slow enough not to affect the intrinsic structure, the formula (4.3) is independent of the detailed character of the collective motion which corresponds to a rotation. This assumption requires the spacing of the rotational levels to be small compared with those of the intrinsic motion. Taking the intrinsic motion to be characterized by the shell model, the inequality (4.2) suggests that optimum conditions for separable rotational motion will occur mid-way between closed shells where the nuclear distortion is a maximum. This prediction was first made by A. Bohr and has been strikingly confirmed by experiment. Both the ratio of the level spacing predicted by (4.3) and their even spin, even parity, character agree with the empirical data. However the actual magnitudes of the excitation energies show decisively that $\mathfrak{J}$ is larger than $\mathfrak{J}_{irrot}$ by a factor of order 3 when the nuclear distortion is estimated from the experimental value of the quadrupole moment. This is clear evidence that the collective motion does not correspond to the irrotational motion of the liquid drop model. It is much more probable that the collective motion is closely related to nuclear shell structure. This will be discussed below.

The rotational spectrum for odd-A nuclei analogous to (4.3) is also qualitatively model independent and has the form

$$E_{rot} = \frac{\hbar^2}{2\mathfrak{J}}\{I(I+1) - I_0(I_0+1)\} \tag{4.5}$$

where

$$I = I_0,\ I_0 + 1,\ I_0 + 2, \ldots \tag{4.6}$$

and I_0 is the quantum number of the ground-state spin. (When the momentum of particle motion is $\frac{1}{2}$, formula (4.6) is replaced by a more complex one, and the ground state may then have spins $\frac{1}{2}$ or $\frac{3}{2}$.) This type of spectrum has also been observed experimentally (see Bohr, 1954, and Bohr and Mottelson, 1956 for references). It is also to be expected that a spectrum of the type (4.5) will be associated with various excited states of the intrinsic motion.

The rotational model is applicable in varying degrees to low excited states of all but the nearly spherical nuclei. Rotational levels can be populated by most nuclear processes, in particular they occur systematically in the α spectra of heavy elements ($A > 220$), (Bohr, Fröman, and Mottelson, 1955). A powerful method of studying rotational spectra is given by Coulomb excitation since these spectra are most clear and have the lowest energies when the nuclear quadrupole moment and hence the cross-section for Coulomb excitation is large (Bohr and Mottelson, 1956).

The results (4.3) and (4.5) are based solely on the existence of some rotational model and do not depend on the detailed representation of the model in terms of the intrinsic nuclear structure. This is adequate so long as the rotational and intrinsic motion are clearly separable since the deficiencies of the theory can then be made good by using the experimental value of the effective moment of inertia $\mathfrak{J}$. Where the two types of motion are not clearly separable it is necessary to have a more detailed form of rotational model in order to understand the essential coupling between the rotational and intrinsic motion (i.e. coupling additional to Coriolis type of coupling through the angular momenta).

It seems almost essential that the rotational model should be closely linked to the shell model. Both describe the low-energy properties of the nucleus, and the detailed successes of the shell model as well as the theoretical considerations of Section 10 are a strong indication that the shell model closely resembles the actual nucleus at low energies for all observables which are not sensitive to detailed particle-to-particle correlations at small distances. These detailed short distance correlations do not have a major effect on angular momentum or on energy levels and it is unlikely that they will radically change the effective distortion or the moment of inertia of the nucleus.

The first requirement of a more detailed model is that it should give agreement with the experimental values of moments of inertia. There is some difficulty in defining the effective moment of inertia since the algebraic form of the variables describing the rotation is unknown. The most plausible method is in terms of a "cranking model."

The Cranking Model assumes that the intrinsic nuclear shape is determined as in the spheroidal shell model by an average potential in which the individual nucleons move, together with residual interactions between the nucleons. When the average potential is rotated with angular velocity ω which is slow compared with the single particle frequencies the additional energy which is required for the system to follow the rotation will be proportional to ω^2,

$$E_{rot} = \tfrac{1}{2}\mathfrak{J}_{crank}\omega^2 \tag{4.7}$$

The constant of proportionality is the "crankic" moment of inertia.

If residual interactions are neglected, the intrinsic structure will be similar to the independent particle model with spheroidal well investigated by NILSSON (1955). For this model the crankic moment of inertia is equal to the solid moment of inertia (BOHR and MOTTELSON, private communication) except for closed shell nuclei for which it is zero. A more realistic picture must take into account residual nucleon-nucleon interactions. It seems possible that due to their short-range character the residual interactions will tend to favour a state of spherical symmetry and will therefore reduce the value of $\mathfrak{J}_{crank}$ to a value nearer the experimental moment of inertia. A more important effect than residual interactions comes from the variation in depth of the potential well which must be introduced into the shell model to make the total energy correct. The moment of inertia of a rotating nucleus (INGLIS, 1954) is given by

$$\mathfrak{J}_x = 2\hbar \sum_{\alpha} \frac{\left|(\alpha|\sum_{i=1}^{A} j_x^{(i)}|0)\right|^2}{E_\alpha - E_0} \tag{4.8}$$

where $|0)$ denotes the ground state and $(\alpha|$ an excited state of the model whilst $j_x^{(i)}$ is the x component of angular momentum of the ith particle. The variation of well depth with energy makes the levels E_α, E_0 further apart than for a constant well and this will reduce the corresponding value of $\mathfrak{J}_x$. This fact has also been noted by BLYN-STOYLE and WEISSKOPF (private communication) who estimate the reduction to be by a factor of a half (using the effective-mass approximation (3.5)). The resulting value of $\mathfrak{J}$ would not then be substantially different from its empirical value calculated from the observed spectra.

The simplest other collective oscillations are quadrupole vibrations, and the experimental evidence for these is presented by SCHARFF-GOLDHABER and WESENER (1955). In general they would be higher in energy than the rotational levels.

5. CONCLUDING REMARKS ON MODELS FOR NUCLEAR STRUCTURE AT LOW ENERGIES

At low energies, nuclear structure is dominated by the shell structure corresponding to independent particle motion in a potential well containing a central and a spin orbit potential. Most detailed analysis of the effects of residual nucleon-nucleon interactions has been carried out for a spherically symmetric well and is limited to nuclei near closed shells. In this form quite close agreement is obtained with experimental values of angular momenta, relative spacing of energy levels, magnetic moments and decay constants. This model does not agree with total nuclear binding energy, with nuclear quadrupole moments, or with experiments which depend sensitively on two-body (or higher) correlations in the nuclear ground state.

A more realistic model allows the nucleons to move in a spheroidal well whose distortion is adjusted to give a state of minimum energy for each nucleus whilst maintaining the correct ordering of single particle levels in order to fit the ground state data on nuclei. This model gives good agreement with experimental values of nuclear quadrupole moments. The effects of nucleon-nucleon interactions on this model have not yet been compared with experiment in any detail.

The shell model can be made to give agreement with total nuclear binding energy if the single particle potential is made energy dependent so that the well is deeper for the lowest single particle levels than for the highest. The detailed consequences on other observations of this assumption would depend on the origin and interpretation of the energy dependence (see Section 10).

The rotational model assumes that a distorted nucleus can rotate without appreciably changing the intrinsic structure. This assumption alone gives qualitative agreement with the rotational spectra which are observed for nuclei having an appreciable number of nucleons outside a closed shell. For quantitative agreement the effective moment of inertia $\mathfrak{J}$ has to be calculated from a more detailed model of the intrinsic structure. It seems likely that $\mathfrak{J}$ will be given correctly by the shell model in a distorted well provided that variation of well depth between shells is taken into account and possibly also the effect of residual interactions.

Taken together, the shell model in a spheroidal well with residual interactions and the rotational model which allows the well to rotate, cover a wide range of nuclear states. There may also be more complex modes of nuclear oscillations such as vibrational excitations, or special types of collective motion of α-particle character. At the present time it seems unlikely that the latter oscillations will be sufficiently weakly coupled to the individual particle motion for them to have more than limited application.

6. The Compound Nucleus Model for Nuclear Reactions

The detailed cross-sections for nuclear reactions at low energies show very sharp resonances. It was proposed by N. Bohr that a nuclear reaction could be studied as a two-stage process, (*a*) the incident particles combine to form a compound nucleus, (*b*) the compound nucleus decays into the products of the reaction. The compound nucleus model is based on the assumption that these two stages are independent, that is to say, the mode of decay of the compound nucleus depends only on its energy, angular momentum and parity but not on the precise way in which it was formed.

The compound nucleus model provides a general scheme for describing nuclear reactions at low and medium energies where an incident particle has time to react strongly with the nucleons in a nucleus. It does not apply for example to the scattering of high-energy particles on nuclei since for these the nucleus may be almost transparent and the interaction time relatively short.

The compound nucleus theory has developed along two lines. One has been concerned with changing the assumptions on which it is based to a form suitable for a rigorous mathematical treatment; these developments are due mainly to WIGNER and EISENBUD (1947) and to earlier work by BREIT (1940). The compound nucleus assumption is replaced by the assumption that the wave function in the region where the incident particle and the nucleus are close together is a linear combination of energy independent wave functions, the combination having energy dependent coefficients. In this form the compound nucleus theory becomes so general that almost any experimental results could be fitted into it and rather than being a model for nuclear reactions it becomes simply a useful framework for describing them.

It is natural to ask what physical content is contained in such general framework, and this question has now largely been answered. WIGNER's theory expresses the reaction cross-sections in terms of a derivative matrix R_{ab} and shows that this is related to reduced widths γ_{Ca} by an equation of the form

$$R_{ab} = \sum_C \frac{\gamma_{Ca}\gamma_{Cb}}{E - E_C} \tag{6.1}$$

It has now been shown (SCHUTZER and TIOMNO, 1951) that this special form for the R matrix as a function of energy is an expression of the causal character of the reaction process. In nonrelativistic form causality requires that there can be no outgoing wave for the reaction products until the incident wave has reached the region in which there is interaction between the incident and target particles. The only physical assumption additional to causality is the requirement of conservation of probability as expressed by the unitarity of the S matrix.

The general compound nucleus theory is applicable without further assumptions to the energy regions where well defined isolated resonances occur. In these only one term of the sum (6.1) is important near a resonance corresponding to a definite energy level E_C of the compound nucleus and near resonance the cross-section is given by the Breit-Wigner one-level formula

$$\sigma_l(a,b) \approx \frac{(2l+1)}{2\pi^2 k_a^2} \cdot \frac{\Gamma_{Ca}\Gamma_{Cb}}{(E - E_{Cl})^2 + (\frac{1}{2}\Gamma_C)^2} \tag{6.2}$$

where k_a is the wave number of the incident particle and Γ_C, Γ_{Ca}, Γ_{Cb} denote total width and partial widths for decay from the compound state. The accuracy of this representation of resonances has been extensively confirmed by experimental analysis and classification of the properties of the compound states. The widest applications are for reactions in light nuclei and for the capture and scattering of slow neutrons.

In regions of overlapping energy levels the Breit-Wigner formula has to be extended to many levels and the resulting cross-section becomes dependent on unknown phase relations between the resonances. In order to make use of the theory further simplifying assumptions must be made; these form the basis of the statistical model for nuclear reactions (WEISSKOPF and EWING, 1940) which is the second principal line of development of the compound nucleus theory.

7. THE STATISTICAL MODEL

When the energy levels of the compound nucleus are distinct the additional assumptions of the statistical model take the especially simple form that, after removal of their energy dependence, the disintegration probabilities from different states of a given compound nucleus are the same for the same product nuclei. Thus it assumes that the values of reduced partial widths are essentially constant.

More generally when the resonance levels overlap the statistical model is intended to apply to experiments in which the incident beam has an energy range much larger than the distance between levels, or where many levels overlap, so that an average over levels is being measured. The model then assumes: (*a*) the compound nucleus theory applies and gives the characteristic form (6.1) for the derivative matrix, (*b*) the partial widths are constant, (*c*) each element of the scattering matrix has undetermined phase and the signs of these phases (plus or minus) are random. The additional assumptions (*b*) and (*c*) are called the statistical assumptions.

The cross-section for a given process is given by the square of the scattering matrix element and in general different processes will interfere. However, with the statistical assumption the interference terms have random sign and similar magnitude so they add up to zero. Thus one consequence of the statistical assumption is that the cross-section factorizes,

$$\sigma(a,b) = \sigma_C(a)\,\frac{\Gamma_{Cb}}{\Gamma_C} \tag{7.1}$$

where $\sigma_C(a)$ is the cross-section for formation of a compound system from incident particle a and Γ_{Cb}/Γ_C is the decay probability of the compound system with emission of particle b.

In practice the statistical model is usually formulated by assuming (7.1), (called by WEISSKOPF (1955) the independence assumption) and then assuming a particular mechanism to permit calculation of the formation and decay probabilities which are related (according to the statistical assumptions) by the principle of detailed balance. Various mechanisms have been assumed but the one most commonly adopted corresponds to a black nucleus. Then the absorption cross-section is obtained by computing the probability that the incident particle penetrates the nuclear surface (FESHBACH and WEISSKOPF, 1949). The decay probability is given by the inverse process,

$$d\Gamma_b = \frac{m_b}{2\pi^2\hbar^2\rho_C(E_0)}\,\sigma_C(\varepsilon_b)\rho_B(E)\varepsilon_b d\varepsilon_b \tag{7.2}$$

where m_b and ε_b are the mass and kinetic energy of the decay particle b. E_0 and E are the energies of the compound nucleus C and of the residual nucleus

B. The next assumption is a particular form for the level densities ρ. WEISSKOPF (1947) assumed this would be given by the Fermi gas model, which leads to a Maxwell distribution.

The statistical model gives qualitative agreement with many features of nuclear reactions, including the yield-energy curves of proton- or α-induced reactions (BLASER, 1951); the relative yields of single or double neutron emission following neutron, proton, or α bombardment; and the energy distribution of evaporated particles from nuclei bombarded with 14-MeV neutrons (GUGELOT, 1951; GRAVES and ROSEN, 1953).

However the quantitative predictions of the statistical model are much less satisfactory. There is in addition major disagreement shown with the observed total cross-section for a neutron beam having poor energy resolution (BARSCHALL, 1952), and the charge of the decay particles does not always agree with the predictions of the model (COHEN and NEWMAN, 1955), and the angular distribution of reaction products shows forward asymmetry contrary to the prediction of the model (GUGELOT, 1954). It is possible that the quantitative disagreement is in part due to the special mechanism assumed for calculation, but the work of BARSCHALL, COHEN, and GUGELOT indicates a much more serious breakdown of the model which involves the independence assumption (7.1).

The independence assumption is expected to apply to regions where resonances are sharp and distinct, since then only one state of the compound system is formed (provided the incident beam has sufficiently well-defined energy) and the properties of a state in quantum theory are always independent of its mode of formation. At intermediate energies resonance levels overlap and several states of the compound system are involved whose relative phases will depend on the mode of excitation. At higher energies many states of the compound system would be involved and they may behave as if their phases are random. At still higher energies the cross-sections between the incident particle and the nucleons in the target decrease, as does the transit time across the target nucleus and decay time of the compound system, so there may not be time for a true compound state to be formed. It is possible that the same is true also for some reactions at intermediate energies and some interaction mechanisms have been proposed which involve only part of the target nucleus (BETHE, 1938).

The above considerations suggest that the mechanism preceding the formation of chaotic conditions in the compound system may be important at medium energies ($\sim$ 20 MeV) and may play a dominant role at higher energies ($>$ 80 MeV). We shall see in the next section that the mechanism of formation of the compound state also plays an important part at low energies and determines in particular the broad structure of neutron or proton scattering on nuclei.

8. THE OPTICAL MODEL

The use of a complex potential or "cloudy crystal ball model" to represent a nucleus was first proposed by FERNBACH, SERBER, and TAYLOR (1949) to explain high energy nucleon scattering. It was applied to low-energy neutron scattering by FESHBACH, PORTER, and WEISSKOPF (1954) to explain the experiments of BARSCHALL (1952). These experiments determine the elastic scattering cross-section for a neutron beam whose energy spread is large compared with the spacing of the resonance levels of the compound nucleus. The total cross-section for any nucleus is then found to be a smooth function of neutron energy and has a characteristic "giant" resonance of width about 2 MeV. The position of the giant resonance varies systematically with the size of the target nucleus, and its character is very much that given by a particle scattering on a simple potential well except that the width is somewhat broader than the single-particle value.

The average neutron cross-sections of BARSCHALL can be explained in terms of the optical model. This describes the effect of the target nucleus on the incoming neutron by a potential well $V(r)$ given by

$$V(r) = -[V_0(r) + iV_1(r)]. \tag{8.1}$$

The real part of the potential $V_0(r)$ is analogous to the single-particle potential of the shell model; the imaginary part $iV_1(r)$ allows for the possibility of the neutron being removed from its entrance channel by absorption into a compound state and decay through some other channel. The best values of the potential are

$$\left.\begin{aligned} V_0(r) &= 40 \text{ MeV} \\ 1 \text{ MeV} < V_1(r) &< 2 \text{ MeV} \end{aligned}\right\} \text{for } r < 1{\cdot}45 \times 10^{-13} \text{ A}^{1/3} \text{ cm} \tag{8.2}$$

with both zero outside the nuclear radius. When the edges of the square well are rounded to give a more realistic potential the imaginary part of the potential iV_1 needs to be larger, possibly by as much as a factor 2.

The total cross-section σ_{tot}^{op} calculated from the model is identified with the mean total cross-section $\bar{\sigma}_{tot}$ as measured by a neutron beam having suitable energy spread. The absorption cross-section σ_{abs}^{op} calculated from the model is identified with the mean cross-section for formation of the compound nucleus $\bar{\sigma}_c$, the latter includes both the mean reaction cross-section σ_r and the mean compound elastic cross-section $\bar{\sigma}_{ce}$ (i.e. elastic scattering through the compound state),

$$\sigma_{tot}^{op} = \bar{\sigma}_{tot} \tag{8.3}$$

$$\sigma_{abs}^{op} = \bar{\sigma}_r + \bar{\sigma}_{ce} \tag{8.4}$$

We see that the model predicts the mean total cross-section directly, and at low energies this depends quite sensitively on the parameters V_0, V_1 and the nuclear radius. The experimental total cross-sections show good agreement with the prediction of the model. The elastic scattering at low energies (< 2 MeV) can only be obtained from the model by making additional assumptions to estimate the mean compound elastic cross-section $\bar{\sigma}_{ce}$, but at higher energies $\bar{\sigma}_{ce}$ tends to zero. There is reasonable agreement both with elastic cross-sections (WALT and BARSCHALL, 1954) and with angular distributions (WALT and BEYSTER, unpublished).

With increasing energy the imaginary potential increases in accordance with the greater absorption cross-section to about 10 MeV, but above 30 MeV it decreases again due to the decrease in the neutron-nucleon cross-section which makes the nucleus appear more transparent (TAYLOR, 1953).

A similar model has been found to give fair agreement with proton scattering on the nuclei, but the agreement is a less sensitive test than for low-energy neutrons.

In the light of the success of the cloudy crystal ball model the process of neutron reactions with nuclei has to be pictured in more detail than was suggested by the Bohr model. The target nucleus consists of nucleons in approximately independent particle states determined by an average potential. The incoming neutron encounters firstly this average potential which leads in first approximation to a scattered wave matched to a single particle wave inside the potential. In second approximation the wave inside the potential is modified by interaction of the neutron first with one nucleon in the target nucleus and then with others until a state of the compound system is formed. The compound system then decays as in the statistical theory. The modification of the potential due to absorption can be estimated from the second step of the compound nucleus formation in which the incident neutron reacts with one other nucleon to form a two particle excited state in the shell picture before proceeding to a more complicated state. There is negligible probability of the two-particle excitation returning to the entrance channel so the transition probability from a one- to a two-particle excitation gives a direct measure of the absorption into the compound state. This transition probability has been computed by LANE and WANDEL (1955) and gives fair agreement with the values determined by fitting the model to experiment.

It is clear that these details of the mechanism of forming a compound state lead simply to an absorption at low energies. At energies above the threshold for neutron or proton ejection from the nucleus this mechanism will include the possibility of direct ejection of a nucleon from a shell model state (analogous to a *p-d* reaction) and this process will compete with the more complicated reactions which proceed via chaotic conditions in the compound system.

The paradox of the average neutron cross-section having single-particle character whilst the fine structure shows sharp resonances at low energy has been explained by WEISSKOPF (1955). There will be no interference between that part of the neutron wave function which is scattered by the potential, and the part which goes through compound nucleus formation provided the incident neutron consists of a pulse which is of short duration compared with the decay time of the compound state. The latter is of order $2\pi\hbar/D$ where D is the mean resonance spacing. Hence the neutron wave packet must have an energy spread which is large compared with the level spacing D. These are just the conditions in which the cloudy crystal ball model is applicable. If the energy of the neutron beam is made precise there will be strong interference between the compound nucleus formation and the single-particle scattering and this will give rise to the characteristic neutron resonances.

9. Models for High-energy Nuclear Reactions

At energies greater than 80 MeV a nucleus will appear partly transparent to an incident nucleon owing to the small value of the nucleon-nucleon cross-section. The mean free path of a nucleon in the nucleus becomes comparable with the nuclear radius and the importance of compound nucleus formation decreases relative to direct nucleon-nucleon interaction. To explain the scattering of high energy neutrons on nuclei, Fernbach, Serber, and Taylor (1949) introduced the concept of the optical model with a complex potential in which the imaginary part represents the perturbation due to compound nucleus formation. This model agrees fairly well with the observed total neutron cross-section.

More detailed observations of reactions at these energies certainly involve direct single particle knock-on or pickup, multiple scattering and at least partial formation of a chaotic state of the compound system. The latter is indicated by the ejection of α-particles in greater numbers than can be explained by successive pickup but no satisfactory detailed picture of the process is known.

Direct scattering of the incident nucleon on only one particle in the nucleus forms one of a variety of reactions in which it is assumed that only one nucleon in the target nucleus is directly involved. These include deuteron pickup, proton-nucleus scattering, π-meson production and annihilation, and photo-nuclear reactions. These processes and their implications are summarized by Brueckner, Eden, and Francis (1955a). The principal feature of the theory is that the relevant cross-sections depend only on the momentum distribution of the nucleons in the target nucleus. In particular they depend on the probability $N(k)$ that the nucleon in the target nucleus has a momentum k comparable with that of the incident particle. It should be noted that k refers to a particle in the target nucleus in its ground state. If $N(k)$ is calculated on the basis of the shell model the resulting predicted cross-sections are smaller by an order of magnitude than those observed experimentally. This anomaly is resolved by recalling that the shell model is only an approximate representation of the nucleus which neglects correlations or fluctuations of the nuclear wave function over short distances.

As a first approximation to take account of these correlations Bethe and Heidmann (1950) suggested a model which allows for the two-body interaction between pairs of nucleons inside the nucleus. The model is known as the deuteron model as it estimates the nucleon momentum distribution by using a deuteron wave function within the nuclear volume to indicate the two-body correlations. It is important to note that the deuteron model is not intended as a realistic picture of the nucleus but is relevant only to the momentum distribution for high values of momentum.

It is probable that most processes at energies > 100 MeV are strongly affected not only by a collision with a single nucleon in the target but also by subsequent collisions. A general theory of multiple collisions in the nucleus has been formulated by WATSON (1953) and WATSON and FRANCIS (1953). However, relatively little progress has since been made in quantitative calculations.

10. A MANY-BODY THEORY OF THE NUCLEUS

Considerable progress has recently been made towards developing a general theory of the nucleus based on the interactions between individual nucleons. This work was initiated by WATSON (1953) and WATSON and FRANCIS (1953) in their formal theory of multiple scattering in the nucleus. It was extended to the problem of nuclear saturation by BRUECKNER, LEVINSON, and MAHMOUD (1954) and its more general applications were pointed out by BRUECKNER and LEVINSON (1955) and EDEN and FRANCIS (1955). Applications of the theory to (*a*) high-energy nuclear reactions, (*b*) low-energy nuclear structure, and (*c*) low-energy neutron scattering on nuclei were made by BRUECKNER, EDEN, and FRANCIS (1955a,b,c). Further developments have been made by BRUECKNER (1954, 1955a,b), BRUECKNER and WADA (1956), EDEN (1955, 1956). These developments have been reviewed and extended by BETHE (1956), whose work gives the most complete account of the theory to date.

In this section the principal features of the theory will be outlined as simply as possible. The method is intended to relate properties of the nucleus to the two-body forces which act between pairs of nucleons. Three-body and higher multi-body forces are neglected for simplicity and because there is no evidence that they are important, but the method could in principle be easily extended to include them. The two-body forces between nucleons are determined empirically from observation of nucleon-nucleon scattering in free space and it is assumed that they are expressible in terms of a potential. If the nucleons are labelled 1, 2, . . . A, this potential can be written

$$v_{ij} = v(\mathbf{r}_i - \mathbf{r}_j,\ \boldsymbol{\sigma}_i,\ \boldsymbol{\tau}_i,\ \boldsymbol{\sigma}_j,\ \boldsymbol{\tau}_j) \tag{10.1}$$

where $\mathbf{r}_i$, $\boldsymbol{\sigma}_i$, $\boldsymbol{\tau}_i$ are the co-ordinates spin and isotopic spin variables describing the ith nucleon.

It is convenient to consider the many-body theory in two parts. The first consists of the construction, from the two-body potential v_{ij}, of a nuclear model which is closely analogous to the improved shell model. The second part expresses the actual nuclear wave function in terms of the model wave function and shows how to evaluate corrections to the model; these corrections are unimportant for some observables such as binding energy but are very important for others which depend on detailed correlation in the nucleus.

(*a*) *Construction of the model*

The model is based on a single particle potential V_i which has to be determined from a set of self-consistent relations. In order to define these, a trial potential V_i must be assumed. This will depend on both the co-ordinates and the momentum of the ith particle (also on the charge and spin but for simplicity we will not distinguish neutrons and protons and will neglect spin dependence).

In co-ordinate representation such a potential is represented by a nondiagonal matrix

$$(r_i|V_i|r_i'). \tag{10.2}$$

The trial form (10.2) for the single-particle potential will determine a complete set of single-particle wave functions ϕ_α, $\alpha = 1, 2, \ldots$ from the Schrodinger equation

$$\left\{E_\alpha + \frac{\hbar^2}{2m}\nabla^2{}_i\right\}\phi_\alpha(r_i) = \int (r_i|V_i|r_i')\phi_\alpha(r_i')dr_i' \tag{10.3}$$

From the complete set of wave functions ϕ_α a set of A wave functions is selected, these will be called the *chosen set* and denoted by

$$\phi_1(r),\ \phi_2(r),\ \ldots,\ \phi_A(r) \tag{10.4}$$

These A wave functions define a configuration of the model which is called the *chosen configuration* and corresponds to a model wave function $\Phi(1, 2, \ldots A)$

$$\Phi(1, 2, \ldots, A) = (A\,!)^{-\frac{1}{2}} \begin{vmatrix} \phi_1(r_1), & \phi_1(r_2), \ldots, & \phi_1(r_A) \\ \phi_2(r_1), & \phi_2(r_2), \ldots, & \phi_2(r_A) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \phi_A(r_1), & \phi_A(r_2), \ldots, & \phi_A(r_A) \end{vmatrix} \tag{10.5}$$

The actual nucleon-nucleon potentials v_{ij} contain a strongly singular part corresponding to a repulsive core and for this reason they cannot be used directly to determine V_i. Instead a *pseudopotential* t_{ij} is first defined from v_{ij} and then used to determine V_i.

The pseudopotential t_{ij} is defined so that it produces the same interaction energy between particles i and j in the chosen configuration as would be produced by v_{ij}. Thus if two particles having wave functions ϕ_i, ϕ_j had an additional interaction v_{ij} switched on this would lead to an energy shift $(\Delta E)_{ij}$. The pseudopotential t_{ij} is chosen so that it would lead to the same energy shift.

The mathematical definition of t_{ij} is expressible as an integral equation,

$$t_{ij} = v_{ij} + \sum_{\phi_\alpha\phi_\beta} v_{ij}|\phi_\alpha\phi_\beta)\frac{Q}{(E_i + E_j - E_\alpha - E_\beta)}(\phi_\alpha\phi_\beta|t_{ij}. \tag{10.6}$$

The quantity Q is a projection operator defined to exclude from the summation all states for which the energy denominator is zero, and also excludes all states for which either ϕ_α or ϕ_β is the same as one of the other $(A - 2)$ wave functions in the chosen set.

The single-particle potential V_i is to be obtained from t_{ij} by taking the sum of all interactions on particle i due to all other particles in the chosen configuration,

$$V_i = \sum_{\substack{\phi_j \\ j\neq i}} (\phi_j|t_{ij}|\phi_j) \tag{10.7}$$

Unlike v_{ij}, the pseudopotential t_{ij} depends on the momentum difference $(k_i - k_j)$ as well as on the co-ordinate difference $(r_i - r_j)$. This means that V_i must be nonlocal in co-ordinate space. The object of the method is to find by trial and error a suitable trial potential (10.2) so that the potential V_i defined by (10.7) is as nearly as possible the same as the trial potential. Then the eqs. (10.3), (10.6), and (10.7) will be as nearly as possible self-consistent.

The resulting model will have the characteristics of the spheroidal shell model with the depth of the well depending on the kinetic energy of the particle. The calculations of BRUECKNER and GAMMEL (1957; private communication) indicate that for an infinite nuclear medium the model has an energy and density about the same as the empirical volume energy and density of a nucleus. At the time of writing no solutions have been obtained for a finite sized nucleus.

(*b*) *The nuclear wave function*

The actual nuclear wave function $\Psi(1, 2, \ldots, A)$ is related to the model wave function by an equation

$$\Psi(1, 2, \ldots, A) = F\Phi(1, 2, \ldots, A) \tag{10.8}$$

where F is called the *model operator*. The operator F is rather complicated and will not be written down in detail here. The most precise definition of the model operator F and the self-consistent equations for the model has been given by GOLDSTONE (1957). Various other forms of model operator F for which the Φ of (10.8) is approximately equal to the Φ of (10.5) have been given by BETHE (1956), EDEN (1956), and RIESENFELD and WATSON (1956).

These model operators have been used to show (i) that the energy of the model defined above in (*a*) is in fact very nearly the same as the actual energy of the nucleus. They also show (ii) that the wave function Ψ resulting from (10.8) is a good approximation to the solution of the many-body Schrodinger equation for the nuclear wave function which is

$$\left(E + \sum_{i=1}^{A} \frac{\hbar^2}{2m} \nabla_i^2 - \sum_{i<j}^{A} v_{ij}\right) \Psi(1, 2, \ldots, A) = 0 \tag{10.9}$$

The model operator F introduces position correlations into the nuclear wave function Ψ which are not present in the model wave function Φ. One consequence of this is that the calculated momentum distribution of nucleons in the nucleus is much greater for large momenta than is given by the model. The predictions are closely related to those of the deuteron model described in Section 9, and show that the theory is in reasonable accord with high-energy data (BRUECKNER, EDEN, and FRANCIS, 1955a).

The detailed theory shows also (BETHE, 1956) that the simplicity of the shell model is a consequence of the operation of the exclusion principle between nucleons in the nucleus. The way in which this operates can be described as follows: The model operator F in (10.8) changes the model wave function Φ by mixing into it other states of different energy from the chosen configuration.

These states must also obey the exclusion principle so that when Φ is near the ground state of the model, most of them will differ substantially in energy from Φ, and their probability will be correspondingly reduced. This mixing will only be important for observables which depend on detailed correlations in the position of nucleons, and will have relatively little effect on such observables as the angular momentum or the magnetic moment of the nucleus. However, the effects of admixing states which are degenerate with the chosen configuration are now largely contained in a secular equation for Φ (using the pseudopotential t_{ij} in the secular equation). For the ground state the degeneracy is very slight and is due only to different ways of combining angular momenta; then the resulting secular equation is simple and the model wave function Φ is simple. At higher energies there are many states with two or more particles excited out of the ground state which have the same energy as one particle excited. Then there is a very great mixing of 1, 2, 3, . . . particle excited states in the model which therefore ceases to be simple. This almost complete mixing is a characteristic of the resonance region of nuclear reactions and is also important at higher energies.

However in the region of low energy neutron reactions with nuclei the compound nucleus state has to go through a formation process. The theory corresponds essentially to the mechanism described in Section 8. This gives a method of computing the absorption potential V_1 of the cloudy crystal ball model (Brueckner, Eden, and Francis, 1955c), and it also indicates a mechanism by which more complicated nuclear interactions can be studied.

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