cannot take a once-and-for-all fixed single-particle potential and hope to find the corresponding single-particle states realized very accurately, be it even over a very limited range of neighboring nuclei.

We should keep these precautions in mind when talking about the shell model. As we said, the filling of the shells is without ambiguity, if we have closed shells. When we start filling neutrons and protons in *unfilled shells* these states will be degenerate, because the j-shells have a $(2j+1)$-fold degeneracy. The configuration of the nucleus can then be characterized by two numbers, κ and λ, which stand for the proton and neutron numbers, respectively, in the partially filled j-shell. Let the partially filled neutron shell be characterized by the quantum numbers $(n\ l\ j)$, and the partially filled proton shell by $(n'\ l'\ j')$. One then denotes the configuration by

$$(\nu nlj)^{\kappa}(\pi n'l'j')^{\lambda}.$$

Because of the $2j+1$-fold degeneracy of each j-shell, all possible shell model states corresponding to this configuration are also degenerate. The number of antisymmetric, linearly independent products is given by the product of the binomial coefficients

$$\binom{2j+1}{\kappa}\binom{2j'+1}{\lambda}. \tag{2.37}$$

The degeneracy of all these states will, of course, be removed in reality due to the action of the residual interaction V_R (2.35), which is neglected in the shell model. Taking one of the phenomenological nucleon–nucleon forces, as discussed in Chapter 4, one can diagonalize V_R in the subspace (2.37). Usually one takes not only this subspace into account, but also the one which corresponds to the nearly degenerate levels of a whole major shell. The $s_{1/2}$, $d_{5/2}$, $d_{3/2}$ levels of the $s-d$ shell is such a case, covering nuclei from ^{16}O up to ^{40}Ca. One can easily be convinced that the dimension of the matrices to be diagonalized becomes exceedingly large for more than two particles in open shells. Special procedures have been developed to diagonalize such huge matrices [Wh 72, SZ 72, WWC 77].

To reduce the size of these matrices, symmetries such as isospin or angular momentum (see Sec. 2.6) can be of great help (see, for instance, [FHM 69, WMH 71, HMW 71, GED 71, VGB 72, Wi 76]).

2.6 Symmetry Properties

2.6.1 Translational Symmetry

For any solution of the eigenvalue problem (2.19) we must require that a series of symmetry or invariance properties are fulfilled. Among these are, for example, translational and rotational invariance.* In the shell model

*Besides these exact symmetries, in some regions of the periodic table one often also has approximate symmetries, as, for instance, the isospin (see Sec. 2.6.3), which can be used for a classification of spectra (see [He 73a]).

one of these invariances is always violated: the *translational invariance*. This comes from the fact that we have to fix the potential in space in contradiction to the homogeneity of space. The most serious consequence of this violation is the appearance of spurious states in the excitation spectrum of the system. This occurs because we have introduced redundant degrees of freedom. If we fix the nucleus in space, it has only $3A-3$ spatial degrees of freedom left. The shell model, however, contains $3A$ variables. These spurious states are therefore not true excitations of the system, but correspond to motions of the nucleus as a whole. Almost all approximation schemes in nuclear physics have inherent symmetry violations. In Chapter 11 we will, therefore, show in fair detail how such violations can be removed.

2.6.2 Rotational Symmetry

The spherical shell model Hamiltonian H_0 (**ls** term included) conserves rotational symmetry. Therefore, it is possible to construct eigenstates of the total angular momentum

$$\mathbf{J} = \sum_{i=1}^{A} \mathbf{j}^{(i)} \qquad \text{and} \qquad J_z = \sum_{i=1}^{A} j_z^{(i)} \tag{2.38}$$

by a linear combination of the Slater determinants (2.23). The closed shell ground state is not degenerate; the only nondegenerate angular momentum eigenstate has $I=0$, which is therefore identified with the ground state. This consequence is experimentally confirmed with no exception. It is then clear that, having only one nucleon outside closed shells, the ground state angular momentum of such even–odd nuclei will correspond to the j-value of the odd nucleon. The same, of course, is true if there is one nucleon missing (a hole) in a filled shell. This rule is also experimentally confirmed with only very few exceptions.

If we fill (remove) more than one particle into (from) an unfilled (filled) j-level, the situation gets more complicated, because different I-values will be degenerate. Again, we can remove this degeneracy by diagonalizing the residual interaction. The matrices are now much smaller as we get one for each I value.

The construction of eigenfunctions of $\mathbf{J}^2$ will be shown explicitly for a very simple example (for more complicated situations, see [ST 63]). Suppose that in a j-shell there are only two protons, the configuration of which is then $(\pi j)^2$. Out of the degenerate two-particle states (which we want to denote by $|m_1 m_2\rangle$, m being the magnetic quantum number), we construct, according to the rules of angular momentum coupling, an eigenstate $|IM\rangle$ of $\mathbf{J}^2$ and J_z with

$$\mathbf{J}^2 = (\mathbf{j}_1 + \mathbf{j}_2)^2. \tag{2.39}$$

We obtain

$$|IM\rangle = \frac{1}{\sqrt{2}} \sum_{m_1 m_2} C^{j\ j\ I}_{m_1 m_2 M} |m_1 m_2\rangle. \tag{2.40}$$

Using the symmetry properties of the Clebsch–Gordan coefficients, (see [Ed 57, Eq. (3.5.14)]), we have with proper normalization

$$|IM\rangle=\frac{1}{\sqrt{8}}\sum_{m_1m_2}\left(C^{j\ j\ I}_{m_1\,m_2\,M}+(-)^{2j-I}C^{j\ j\ I}_{m_2\,m_1\,M}\right)|m_1m_2\rangle \tag{2.41}$$

which, upon noting the antisymmetry $|m_2m_1\rangle=-|m_1m_2\rangle$ yields

$$|IM\rangle=\frac{1}{\sqrt{8}}\sum_{m_1m_2}C^{j\ j\ I}_{m_1\,m_2\,M}\left(1+(-)^{2j-I+1}\right)|m_1m_2\rangle. \tag{2.42}$$

We see from Eq. (2.42) that $|IM\rangle$ is only different from zero for

$$2j-I+1=2n \quad \text{or} \quad I=2n; \qquad n=0,1,2\ldots, \tag{2.43}$$

that is , for even angular momenta. Taking as a definite example $j=3/2$, we see that the six independent components

$$|m_1m_2\rangle:|\tfrac{3}{2}\tfrac{1}{2}\rangle,|\tfrac{3}{2}-\tfrac{1}{2}\rangle,|\tfrac{3}{2}-\tfrac{3}{2}\rangle,|\tfrac{1}{2}-\tfrac{1}{2}\rangle,|\tfrac{1}{2}-\tfrac{3}{2}\rangle,|-\tfrac{1}{2}-\tfrac{3}{2}\rangle$$

have been transformed by a unitary transformation to the six angular momentum coupled components, which are degenerate among themselves:

$$|IM\rangle:|00\rangle,|22\rangle,|21\rangle,|20\rangle,|2-1\rangle,|2-2\rangle.$$

In the general case these considerations are a little tedious. In Table 2.1 the possible total angular momenta for a pure proton configuration are presented.

The factor $1/\sqrt{2}$ in Eq. (2.40) is a normalization in the case in which both particles are in the same shell. In general, we have for the coupling of two particles:

$$(a^+_{nlj}a^+_{n'l'j'})_{JM}=\frac{1}{\sqrt{1+\delta_{jj'}\delta_{ll'}\delta_{nn'}}}\sum_{mm'}C^{j\ j'\ J}_{m\ m'\ M}a^+_{nljm}a^+_{n'l'j'm'}. \tag{2.44}$$

Special care has to be applied in coupling hole states. The operator a^+_{nljm} transforms like the eigenfunction ϕ_{nljm}, that is cogredient, under a rotation of the coordinate system and therefore like a spherical tensor of rank j (i.e., with $D^j_{mm'}$; see Appendix A). On the other hand, the annihilation operator a_{nljm} transforms with $D^{j*}_{mm'}$, that is, contragredient. The normal coupling rules (2.44) apply only for tensors, which are both cogredient or both contragredient. We can therefore only couple a^+_{nljm} with the time reversed operator (see [Me 61]),

$$\bar{a}_{nljm}=Ta_{nljm}T^+=(-)^{l+j-m}a_{nlj-m}, \tag{2.45}$$

which is, of course, cogredient. The ph coupling rule is therefore

$$(a^+_{nlj}a_{n'l'j'})_{JM}=\sum_{mm'}(-)^{j'-m'}C^{j\ \ j'\ \ J}_{m\ -m'\ M}a^+_{nljm}a_{n'l'j'm'}, \tag{2.46}$$

where we have left out the unimportant phase $(-)^{l'+1}$.

From pure angular momentum coupling one cannot as yet decide which of the degenerate states corresponds to the ground state. For that we have, as we have said, to diagonalize V_R in a certain subspace. This confirms, in general, the experimentally observed rule that even–even groundstates have $I=0$.

Another experimentally found coupling rule which the pure shell model cannot explain without configuration mixing is the fact that even odd nuclei far from closed shells have ground state spins equal to the j-value of the odd particle. We will see in Chapter 6 how this finds a natural explanation by taking correlations of the nucleons into account.

Table 2.1 List of possible total angular momenta I for the configuration $(j)^\kappa$ ([MJ 55, p. 64])

	κ	
$j=\frac{1}{2}$		
	1	$\frac{1}{2}$
$j=\frac{3}{2}$		
	1	$\frac{3}{2}$
	2	0, 2
$j=\frac{5}{2}$		
	1	$\frac{5}{2}$
	2	0, 2, 4
	3	$\frac{3}{2}, \frac{5}{2}, \frac{9}{2}$
$j=\frac{7}{2}$		
	1	$\frac{7}{2}$
	2	0, 2, 4, 6
	3	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{15}{2}$
	4	0, 2 (twice), 4 (twice), 5, 6, 8
$j=\frac{9}{2}$		
	1	$\frac{9}{2}$
	2	0, 2, 4, 6, 8
	3	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$, (twice), $\frac{11}{2}, \frac{13}{2}, \frac{15}{2}, \frac{17}{2}, \frac{21}{2}$
	4	0 (twice), 2 (twice), 3, 4 (3 times), 5, 6 (3 times), 7, 8 (twice), 9, 10, 12
	5	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ (twice), $\frac{7}{2}$ (twice), $\frac{9}{2}$ (3 times), $\frac{11}{2}$ (twice), $\frac{13}{2}$ (twice), $\frac{15}{2}$ (twice), $\frac{17}{2}$ (twice), $\frac{19}{2}, \frac{21}{2}, \frac{25}{2}$
$j=\frac{11}{2}$		
	1	$\frac{11}{2}$
	2	0, 2, 4, 6, 8, 10
	3	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$ (twice), $\frac{11}{2}$ (twice), $\frac{13}{2}, \frac{15}{2}$ (twice), $\frac{17}{2}, \frac{19}{2}, \frac{21}{2}, \frac{23}{2}, \frac{27}{2}$
	4	0 (twice), 2 (3 times), 3, 4 (4 times), 5 (twice), 6 (4 times), 8 (4 times), 9 (twice), 10 (3 times), 11, 12 (twice), 13, 14, 16
	5	$\frac{1}{2}, \frac{3}{2}$ (twice), $\frac{5}{2}$ (3 times), $\frac{7}{2}$ (4 times), $\frac{9}{2}$ (4 times), $\frac{11}{2}$ (5 times), $\frac{13}{2}$ (4 times), $\frac{15}{2}$ (5 times), $\frac{17}{2}$ (4 times), $\frac{19}{2}$ (4 times), $\frac{21}{2}$ (3 times), $\frac{23}{2}$ (3 times), $\frac{25}{2}$ (twice), $\frac{27}{2}$ (twice), $\frac{29}{2}, \frac{31}{2}, \frac{35}{2}$
	6	0 (3 times), 2 (4 times), 3 (3 times), 4 (6 times), 5 (3 times), 6 (7 times), 7 (4 times), 8 (6 times), 9 (4 times), 10 (5 times), 11 (twice), 12 (4 times), 13 (twice), 14 (twice), 15, 16, 18

2.6.3 The Isotopic Spin

Up to now we have always considered the neutrons and protons separately. As a consequence we have, for example, in Eq. (2.23), a product of two Slater determinants—one for protons and one for neutrons. Apart from their electromagnetic interactions, protons and neutrons have practically the same physical properties. We will see, for instance, in Chapter 4, that nuclear forces are to a large extent independent of whether we consider protons or neutrons—that is, they are charge independent. As long as the influence of the Coulomb force on the nuclear properties can be neglected, we can consider the proton and the neutron as just two different manifestations of the same particle: the nucleon. Mathematically we say that the nucleon can be in two different states, the basis vectors of which may be written as $\pi = \binom{0}{1}$ and $\nu = \binom{1}{0}$, thus forming a two-dimensional space. This then is very similar to what we have in the spin case. Instead of a quantum number indicating whether the nucleon spin is up or down, we now have an additional quantum number, indicating whether the nucleon is a proton or a neutron, called the isotopic spin because of its formal analogy to ordinary spin.* With this additional quantum number we can write the nuclear shell model wave function as a single determinant. However one should realize that, contrary to the ordinary spin, the isospin has nothing to do with rotations in the coordinate space. Therefore, the treatment of isopsin is much simpler than that of ordinary spin which, unlike the isospin, has to be coupled to the orbital angular momentum.

As in the case of ordinary spin, one can set up the usual spin algebra:

$$t_3\nu = \tfrac{1}{2}\nu, \qquad t_3\pi = -\tfrac{1}{2}\pi, \tag{2.47}$$

with

$$t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.48}$$

The lowering and raising operators t_- and t_+ change the neutron into a proton and vice versa, respectively,

$$t_-\nu = \pi, \quad t_+\nu = 0, \qquad t_-\pi = 0, \quad t_+\pi = \nu, \tag{2.49}$$

with

$$t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.50}$$

Like ordinary spin, the isospin vector operator is formed out of its three cartesian components

$$\mathbf{t} = \{t_1, t_2, t_3\}. \tag{2.51}$$

We can therefore define the total isospin of a system of A nucleons

$$\mathrm{T} = \sum_{i=1}^{A} \mathbf{t}^{(i)} \tag{2.52}$$

and its 3-component

$$T_3 = \sum_{i=1}^{A} t_3^{(i)}. \tag{2.53}$$

Since we have the same algebra for isotopic spin as for ordinary spin, the same coupling rules can be applied.

The 3-component of the total isospin is a measure of the total neutron

*The concept of isospin was introduced originally by Heisenberg [He 32]. Later on it was much developed by Wigner [Wi 37].