Appendix D
Angular Correlations

Traditionally, angular correlation measurements have been used in nuclear physics as a powerful tool in order to determine the angular momenta of states participating in nuclear transitions. It also turns out that angular correlations are sensitive to the ratios of nuclear matrix elements (that is, mixing ratios; see later) that correspond to different possibilities of coupling angular momenta in a specific transition. We will not attempt here to summarize this vast field, but will focus on aspects that are of primary importance in low-energy nuclear astrophysics measurements.

Uncertainties in thermonuclear reaction rates are caused by contributions from resonances or nonresonant reaction processes that are as yet unobserved. The goal of the experimentalist is to measure such contributions. If the detection system covers the entire solid angle (4π sr), the measured intensities represent angle-integrated yields. These may then be converted to cross sections or resonance strengths (Sections 4.8 and 4.9). However, in most experimental setups the detector(s) will cover only a fraction of the full solid angle. What is measured in such cases are *differential* yields that may be influenced by angular correlation effects. It should be pointed out that the angular momenta for many levels participating in astrophysically important reactions are known or, at least, have been restricted to a certain range of values by previous nuclear structure studies. Hence, it becomes in principle possible to estimate angular correlation effects by making reasonable assumptions and, if necessary, to correct the measured differential yields appropriately.

A comprehensive theory of angular correlations is beyond the scope of the present work. The interested reader is referred to the specialized literature (see, for example, Devons and Goldfarb 1957). The focus of this section is on angular correlations in astrophysically important reactions, that is, processes such as $A(a,b)B$ or $A(a,\gamma)B$, where a and b denote particles with rest mass. We will briefly explain the origin of angular correlations in such processes and examples of the application of angular correlations to specific cases will be given. In this section, all angles θ refer to the center-of-mass system.

Nuclear Physics of Stars. Christian Iliadis

ISBN: 978-3-527-40602-9

D.1 General Aspects

For the discussions in this section, we will make the following assumptions: (i) the beam is unpolarized and the target nuclei are randomly oriented; (ii) the nuclear levels involved in the transitions at each stage have unique spin and well-defined parity; (iii) the polarization of the detected radiations is not observed. These assumptions apply to most cases of interest here. The term *radiation* denotes bombarding (incident) particles or γ-rays as well as emitted (outgoing) particles or γ-rays. An angular correlation between two radiations (for example, between the incident beam and an outgoing radiation, or between two successive outgoing radiations) is the result of the alignment of a particular nuclear level. An aligned level of spin J is prepared by some process that populates its $2J + 1$ magnetic substates *unequally* with the condition that the population of the $+m$ substate will be equal to the population of the $-m$ substate (since we assume unpolarized beam and target nuclei). Particles or γ-rays that are emitted from a specific substate m of the aligned level and that populate a substate m_f of a final level will then have a characteristic radiation pattern, or *angular correlation*, with respect to some (z-)axis of quantization, depending on the value of $\Delta m = m - m_f$. The total radiation pattern will consist of the superposition of all allowed transitions $m \rightarrow m_f$ between substates. The alignment in reactions of type $A(a, b)B$ or $A(a, \gamma)B$ is achieved by the fact that the orbital angular momentum carried by the incident radiation is perpendicular to its direction of motion. This simple circumstance, plus the additional fact that angular momentum is conserved, forms the foundation of the angular correlation theory for unpolarized radiations.

As a simple example, we will consider an excited level of spin and parity $J^\pi = 1^-$ that decays to a 0^+ ground state via emission of electric dipole (E1; $L = 1$) radiation (Fig. D.1). The spatial distribution of the emitted photons will depend on the magnetic quantum numbers m and m_f of the decaying and the final level, respectively, where each allowed value of $\Delta m = m - m_f$ gives rise to a different radiation pattern. In our example, the decaying level consists of $(2 \cdot 1 + 1) = 3$ substates and the final level has only $(2 \cdot 0 + 1) = 1$ substate. The allowed transitions are then described by $m - m_f = 0 - 0 = 0$ and $m - m_f = \pm 1 - 0 = \pm 1$. The corresponding radiation patterns are given by $W_{\Delta m=0}(\theta) \sim \sin^2\theta$ and $W_{\Delta m=\pm 1}(\theta) \sim (1 + \cos^2\theta)/2$, respectively (Jackson 1975). These are plotted as polar intensity diagrams in Fig. D.1. Suppose first that the $J^\pi = 1^-$ level is populated by the β-decay of a parent state and that the β-particles are not detected. Under such conditions, the β-decay populates the magnetic substates *equally* that is, with a probability of $p(m) = 1/(2J + 1) =$

1/3. The total photon radiation pattern is thus given by

$$W(\theta) = \sum_m p(m) W_{m \to m_f}(\theta)$$
$$\sim \frac{1}{3} \cdot \frac{1}{2}(1 + \cos^2\theta) + \frac{1}{3}\sin^2\theta + \frac{1}{3} \cdot \frac{1}{2}(1 + \cos^2\theta) = \text{const} \qquad \text{(D.1)}$$

and hence becomes isotropic. Now suppose that the $J^\pi = 1^-$ level is instead populated as a resonance in a capture reaction $A(a, \gamma)B$ involving target and projectile spins and parities of $j_A = 0^+$ and $j_a = 0^+$. The resonance can only be formed by absorption of particles a with an orbital angular momentum of $\ell_a = 1$ (Example B.1). Provided the incident particle beam is well collimated, the projection of the orbital angular momentum vector along the incident beam direction is zero (Fig. 2.4). The allowed range of magnetic substates of the resonance that can be populated in this type of capture reaction is then given by $m_{\text{res}} \leq j_A + j_a$ (see Eqs. (B.3) and (B.4)). It follows that, among the three different magnetic substates of the resonance, only the $m = 0$ substate can be populated in the reaction. In other words, we obtain $p(0) = 1$ and $p(\pm 1) = 0$, and the γ-ray decay must proceed from $m = 0$ to $m_f = 0$. Consequently, the total radiation pattern is given by the $\Delta m = m - m_f = 0$ transition only

$$W(\theta) = \sum_m p(m) W_{m \to m_f}(\theta) \sim \sin^2\theta \qquad \text{(D.2)}$$

The alignment in this example is exceptionally strong and thus the variation of the γ-ray counting rate with angle is relatively large. If the beam or target nuclei have nonzero spin, then the alignment will be weaker, but angular correlation effects are in general nevertheless observed.

In certain situations involving nuclear reactions, all magnetic substates are populated equally, independent of the mode of formation. For example, the capture of unpolarized protons by spin-zero target nuclei leading to a $J = 1/2$ resonance will always populate the $m = \pm 1/2$ magnetic substates of the resonance uniformly. As a result, the total radiation pattern will be isotropic. Similar arguments apply to a resonance of spin $J = 0$. In this case, only one magnetic substate exists and the transitions to the various substates in the final state proceed with equal probabilities. As a result, the total radiation pattern must necessarily be isotropic.

We considered so far only the angular correlation caused by the alignment of levels produced in nuclear reactions (also termed *angular distribution*). Another type of angular correlation occurs if an excited level de-excites to a final state through an intermediate level by emitting two successive radiations (for example, two photons). In this case, measurement of the direction of the first radiation will produce an aligned intermediate state. The result is again a nonuniform intensity distribution of the second radiation with respect to the

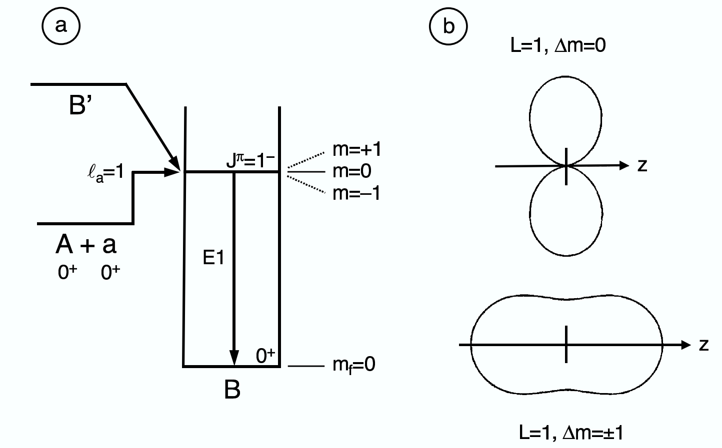

Fig. D.1 (a) Level scheme for an excited state ($J^\pi = 1^-$) that can be populated either via β-decay from nucleus B' or via the capture reaction $A + a \rightarrow B + \gamma$. Both the target and the projectile have spins and parities of 0^+. The state decays via E1 emission to the ground state ($J^\pi = 0^+$). In the first case, the radiation pattern will be isotropic, while in the second case, the pattern is anisotropic because of a strong alignment. (b) Dipole radiation pattern for $\Delta m = 0$ (top) and $\Delta m = \pm 1$ (bottom).

measured direction of the first radiation. We encountered this situation in the discussion of angular correlation effects for γ-ray detector summing corrections (Section 4.5.2). As will be seen in the following, the angular correlation formalism is quite general and describes this situation as well.

The summation over magnetic quantum numbers is performed explicitly in Eq. (D.1). In more complicated situations involving a number of unobserved or coupled orientations, such a calculation becomes very tedious. Much more convenient, but equivalent, expressions have been developed where the magnetic substates are not explicitly introduced and where the sums over substates are automatically performed. A number of different formalisms and expressions can be found in the literature. Here, we will follow the work of Biedenharn (1960).

Any correlation where only two directions of motion are measured can be expressed as a Legendre polynomial series in the angle between those directions (see also Eqs. (A.9)–(A.14)). We write

$$\begin{aligned} W(\theta) &= \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n P_n(\cos\theta) \\ &= 1 + \frac{b_2}{b_0} P_2(\cos\theta) + \frac{b_4}{b_0} P_4(\cos\theta) + \cdots + \frac{b_{n_{\max}}}{b_0} P_{n_{\max}}(\cos\theta) \end{aligned} \tag{D.3}$$

If the process under consideration is a nuclear reaction, then $W(\theta)$ is related to the differential and total cross section by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\theta} = \frac{1}{4\pi}\,\sigma\,W(\theta) \tag{D.4}$$

An isotropic differential cross section implies $W(\theta) = 1$. The sum in Eq. (D.3) is restricted to even values of n because we are making the assumption that the reaction (or the successive decay) involves at each stage nuclear states of well-defined parity. The wave function describing the exit channel must then have the same parity as the resonance (or the intermediate state). The corresponding intensity of the emitted radiation (that is, the square of the wave function) has even parity and is unchanged by the inversion $\vec{r} \to -\vec{r}$, or more specifically, by the substitution $\theta \to \pi - \theta$ (since for unpolarized beams and randomly oriented target nuclei the intensity does not depend on the azimuthal angle ϕ). The condition $W(\theta) = W(\pi - \theta)$ implies that $W(\theta)$ is symmetric about $\theta = 90°$ and, consequently, all odd Legendre polynomial terms in Eq. (D.3) must vanish.

The coefficients b_n in Eq. (D.3) depend on the angular momenta and nuclear matrix elements involved in the process. Theoretical expressions for b_n are given in the following. They can be factored into components referring separately to each transition. Each of these components, in turn, is expressed in terms of vector coupling (Clebsch–Gordan and Racah) coefficients. We will be using the coefficients F_n, defined by (Biedenharn 1960)

$$F_n(LL'jJ) \equiv (-)^{j-J-1}\sqrt{(2L+1)(2L'+1)(2J+1)}\,(L1L'-1|n0)W(JJLL';nj) \tag{D.5}$$

where j and J are angular momenta (spins) of nuclear states and L and L' are orbital angular momenta (for particles) or multipolarities (for photons) of radiations; $(L1L'-1|n0)$ and $W(JJLL';nj)$ denotes a Clebsch–Gordan and a Racah coefficient, respectively. A tabulation of the functions $F_n(LjJ) \equiv F_n(LLjJ)$ is given in Biedenharn and Rose (1953). Numerical values of the mixed correlation coefficients $F_n(LL'jJ)$ for $L \neq L'$ can be found in Appel (1968). For $n = 0$, we obtain $F_0(LL'jJ) = 0$ and $F_0(LjJ) = 1$. In order to determine how many terms have to be taken into account in the sum of Eq. (D.3), it is useful to consider the symmetry properties of the functions $F_n(LL'jJ)$ which follow directly from those of the Clebsch–Gordan and Racah coefficients. For given values of L, L', and J, we obtain $F_n(LL'jJ) \neq 0$ only for $|L - L'| \leq n \leq \min(2J, L + L')$. It follows that $F_n(LjJ) \neq 0$ only for $0 \leq n \leq \min(2J, 2L)$.

D.2
Pure Radiations in a Two-Step Process

We start by considering a two-step process, where each step proceeds via a pure transition. An intermediate state of spin J is formed from an initial state of spin j_1 via absorption or emission of some radiation of angular momentum L_1. The intermediate state decays then to the final state of spin j_2 via emission of radiation with angular momentum L_2. We write symbolically $j_1(L_1)J(L_2)j_2$. The angular correlation function between the directions of the two radiations is then given in terms of the coefficients $A_n(i)$ and the particle parameters $a_n(i)$ by

$$W(\theta) = \sum_{n=0,2,\ldots} [a_n(1)A_n(1)][a_n(2)A_n(2)]P_n(\cos\theta) \tag{D.6}$$

$$\text{for photons:} \qquad a_n(i) = 1; \qquad A_n(i) = F_n(L_i j_i J) \tag{D.7}$$

$$\text{for } s = 0 \text{ particles:} \qquad a_n(i) = \frac{2L_i(L_i+1)}{2L_i(L_i+1) - n(n+1)}; \qquad A_n(i) = F_n(L_i j_i J) \tag{D.8}$$

$$\text{for } s \neq 0 \text{ particles:} \qquad a_n(i) = \frac{2L_i(L_i+1)}{2L_i(L_i+1) - n(n+1)}; \qquad A_n(i) = F_n(L_i j_s J) \tag{D.9}$$

For photons or particles, L_i denotes the γ-ray multipolarity or the orbital angular momentum, respectively. If a particle has a nonzero spin s, then the channel spin given by $\vec{j}_s = \vec{j}_i + \vec{s}$ and $|j_i - s| \leq j_s \leq j_i + s$ replaces the initial state spin j_i. The sum in Eq. (D.6) is restricted to $0 \leq n \leq \min(2L_1, 2L_2, 2J)$.

Example D.1

The β-decay of ^{60}Co populates a 4^+ level in the ^{60}Ni daughter nucleus. This level decays to an intermediate state of spin 2^+, which in turn decays to the ground state of spin 0^+ (Fig. D.2a). Calculate the angular correlation between the two de-excitation γ-rays.

We encountered this case in Section 4.5.2 and Fig. 4.30. The β-decay electron is emitted into a random direction and is not observed. Thus, the initial 4^+ level populated in the daughter nucleus is not aligned. The first γ-ray is also emitted into a random direction. If it is detected in a counter, then a line connecting the radioactive source with the detector represents a preferred direction relative to which the second γ-ray is emitted. Both transitions in this direction–direction correlation are γ-rays and θ represents the angle between their correlated emission directions. Both the first and the second γ-ray decay can only proceed via an E2 transition (Example B.4). Thus, we have to consider the angular momentum sequence $j_1(L_1)J(L_2)j_2 \to 4(2)2(2)0$. From Eqs. (D.6) and (D.7) we obtain

$$W(\theta) = \sum_{n=0,2,...} F_n(L_1 j_1 J) F_n(L_2 j_2 J) P_n(\cos\theta) \quad \text{with} \quad 0 \leq n \leq \min(2L_1, 2L_2, 2J)$$

Hence

$$\begin{aligned} W(\theta) &= \sum_{n=0,2,4} F_n(242) F_n(202) P_n(\cos\theta) \\ &= 1 + F_2(242) F_2(202) P_2(\cos\theta) + F_4(242) F_4(202) P_4(\cos\theta) \\ &= 1 + (-0.1707)(-0.5976) P_2(\cos\theta) + (-0.0085)(-1.069) P_4(\cos\theta) \\ &= 1 + 0.1020 P_2(\cos\theta) + 0.0091 P_4(\cos\theta) \end{aligned}$$

Example D.2

A resonance with spin and parity of $J^\pi = 2^+$ is populated in the ^{32}S(α,γ)^{36}Ar reaction. The resonance decays to a final state with $J^\pi = 0^+$ (Fig. D.2b). Calculate the expected angular correlation between the incident beam (α-particles) and the emitted γ-radiation.

The ^{32}S target nuclei and the α-particles have both a spin and parity of 0^+. Therefore, the $J^\pi = 2^+$ resonance can only be formed from an α-particle orbital angular momentum of $\ell_\alpha = 2$ (Example B.1). Furthermore, the γ-ray transition can only be of E2 character (Example B.4). The angular momentum sequence is therefore given by $j_1(L_1)J(L_2)j_2 \to j_{^{32}S}(\ell_\alpha)J(L_\gamma)j_{^{36}Ar} \to 0(2)2(2)0$. We obtain from Eqs. (D.6) and (D.8)

$$W(\theta) = \sum_{n=0,2,...} \frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)} F_n(L_1 j_1 J) F_n(L_2 j_2 J) P_n(\cos\theta)$$

From $0 \leq n \leq \min(2L_1, 2L_2, 2J)$ we find n = 0, 2, and 4. Hence

$$\begin{aligned} W(\theta) &= \sum_{n=0,2,4} \frac{2\cdot 2(2+1)}{2\cdot 2(2+1) - n(n+1)} F_n(202) F_n(202) P_n(\cos\theta) \\ &= 1 + \frac{12}{12-6} F_2(202) F_2(202) P_2(\cos\theta) + \frac{12}{12-20} F_4(202) F_4(202) P_4(\cos\theta) \\ &= 1 + 2(-0.5976)(-0.5976) P_2(\cos\theta) + (-1.5)(-1.069)(-1.069) P_4(\cos\theta) \\ &= 1 + 0.7143 P_2(\cos\theta) - 1.7143 P_4(\cos\theta) \end{aligned}$$

D.3 Mixed Radiations in a Two-Step Process

Sometimes the angular momentum coupling in a sequential nuclear decay or in a nuclear reaction allows for different possibilities, each involving a unique combination of angular momenta. In general, these pure transitions will interfere, that is, their contributions to the total angular correlation add either incoherently or coherently. In either case, new parameters have to be introduced that describe quantitatively the degree of mixing. These *mixing ratios* are usually determined experimentally by fitting the data and are eventually interpreted in terms of some nuclear model.

Incoherent interference applies, for example, to the channel spin j_s. Since we assumed that the beam and target nuclei are unpolarized, the channel spin is *randomly oriented*. As a consequence, the total angular correlation is given by the sum of the individual (pure) correlations, each weighted according to the probability for a particular channel spin value to occur. We write $W(\theta) = W_{j_s}(\theta) + \delta_c^2 W_{j_s'}(\theta)$, where the *channel spin mixing ratio* $\delta_c^2 \equiv P_{j_s'}/P_{j_s}$ is defined as the ratio of probabilities for forming (or of decay from) the intermediate state via the channel spins j_s' and j_s, where $j_s' > j_s$.

Coherent interference occurs when definite phase relationships are important. This is the case if several possible values of multipolarites are allowed for a specific γ-ray transition, or if the intermediate state can be formed (or decay) by several possible values of orbital angular momenta. In practice, only the smallest two allowed values of γ-ray multipolarities (L_i and $L_i + 1$) or orbital angular momenta (ℓ_i and $\ell_i + 2$) need to be considered (Example B.4). In such cases we have to use in Eq. (D.6) the expression

$$a_n(i)A_n(i) = a_n(L_iL_i)F_n(L_ij_iJ) + 2\delta_i a_n(L_iL_i')F_n(L_iL_i'j_iJ) + \delta_i^2 a_n(L_i'L_i')F_n(L_i'j_iJ) \tag{D.10}$$

$$\text{for photons:} \qquad a_n(i) = 1 \tag{D.11}$$

$$\text{for particles:} \quad a_n(L_iL_i') = \cos(\xi_{L_i} - \xi_{L_i'})\frac{(L_i0L_i'0|n0)}{(L_i1L_i'-1|n0)}$$
$$= \cos(\xi_{L_i} - \xi_{L_i'})\frac{2\sqrt{[L_i(L_i+1)][L_i'(L_i'+1)]}}{L_i(L_i+1) + L_i'(L_i'+1) - n(n+1)} \tag{D.12}$$

where the primed quantities refer to the higher value of angular momentum (particle orbital angular momentum or γ-ray multipolarity). For particles with spin, the channel spin j_s replaces again the initial state spin j_i in Eq. (D.10).

The *γ-ray multipolarity mixing ratio δ_γ* is defined by the relation $\delta_\gamma^2 \equiv \Gamma_{\gamma L+1}/\Gamma_{\gamma L}$, with $\Gamma_{\gamma L}$ the γ-ray partial width for the transition with multipolarity L (see Eq. (1.31)). The total angular correlation not only depends on the value but also on the phase (plus or minus) of δ_γ. Hence, the sign convention (that is, the definition of δ_γ in terms of the nuclear matrix elements) becomes important when interpreting the data. We will adopt here the convention used by Biedenharn (1960). See Ferguson (1965) for a different sign convention.

For the mixing of particle orbital angular momenta, one introduces the *orbital angular momentum mixing ratio*, defined by $\delta_a^2 \equiv \Gamma_{L+2}/\Gamma_L$, with Γ_L the particle partial width for orbital angular momentum L. For charged particles, the phase shifts ξ_L are given by (Ferguson 1965)

$$\xi_L = -\arctan\left(\frac{F_L}{G_L}\right) + \sum_{n=1}^{L} \arctan\left(\frac{\eta}{n}\right) \qquad \text{(D.13)}$$

where F_L and G_L are the regular and irregular Coulomb wave functions, respectively, and η is the Sommerfeld parameter (see Section 2.4.3 and Appendix A.3). The first term in the above expression is the hardsphere phase shift and the second term is the Coulomb phase shift which is absent for neutral particles. It is obvious that the phase shift ξ_L is energy dependent.

Note that if a transition is mixed with a mixing parameter of δ_i^2, then the total angular correlation is normalized to $(1+\delta_i^2)$ instead of unity. If two or more different mixing processes are present with mixing parameters of δ_i^2, δ_{i+1}^2, δ_{i+2}^2, . . ., and so on, then $W(\theta)$ is normalized to the product $(1+\delta_i^2)(1+\delta_{i+1}^2)(1+\delta_{i+2}^2)\ldots$, and so on.

Example D.3

A resonance with spin and parity of $J^\pi = 1^-$ is formed in the ^{31}P(p,α)^{28}Si reaction. The α-particle emission populates the ground state in the final ^{28}Si nucleus (Fig. D.2c). Calculate the angular correlation between the incident proton beam and the emitted α-particles.

Both the ^{31}P target nucleus and the proton have a spin and parity of $J^\pi = 1/2^+$. Thus, the angular momentum coupling of the target and projectile can produce either one of two channel spin possibilities: $|1/2-1/2| \leq j_s \leq 1/2+1/2$, hence $j_s = 0$ or 1. The value of the orbital angular momentum is unique for the incoming and outgoing reaction channel ($\ell_\text{p} = 1$ and $\ell_\alpha = 1$). First, the angular correlations for the pure transitions will be calculated, that is, each channel spin case will be treated separately. We have to consider the angular momentum sequences $j_1(L_1)J(L_2)j_2 \rightarrow j_s(\ell_\text{p})J(\ell_\alpha)j_{^{28}Si} \rightarrow 0(1)1(1)0 \quad (j_s = 0)$ and $\rightarrow 1(1)1(1)0 \quad (j_s = 1)$. For either channel spin the sum in Eq. (D.6) is restricted to $0 \leq n \leq \min(2 \cdot 1, 2 \cdot 1)$, that is, $n = 0$ and 2. We obtain

$$\begin{aligned} W_{j_s=0}(\theta) &= \sum_{n=0,2} \frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)} F_n(L_1 j_s J) \frac{2L_2(L_2+1)}{2L_2(L_2+1) - n(n+1)} \\ &\quad \times F_n(L_2 j_2 J) P_n(\cos\theta) \\ &= 1 + \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(101) \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(101) P_2(\cos\theta) \\ &= 1 + (-2)(0.7071)(-2)(0.7071) P_2(\cos\theta) = 1 + 2P_2(\cos\theta) \end{aligned}$$

Similarly

$$\begin{aligned} W_{j_s=1}(\theta) &= 1 + \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(111) \frac{2 \cdot 1 \cdot 2}{2 \cdot 1 \cdot 2 - 2 \cdot 3} F_2(101) P_2(\cos\theta) \\ &= 1 + (-2)(-0.3536)(-2)(0.7071) P_2(\cos\theta) = 1 - P_2(\cos\theta) \end{aligned}$$

The total angular correlation is given by the sum of the correlations for the individual channel spins, each weighted according to the probability of the particular j_s value. Thus

$$\begin{aligned} W(\theta) &= W_{j_s=0}(\theta) + \delta_c^2 W_{j_s=1}(\theta) = [1 + 2P_2(\cos\theta)] + \delta_c^2 [1 - P_2(\cos\theta)] \\ &= 1 + \delta_c^2 + [2 - \delta_c^2] P_2(\cos\theta) \end{aligned}$$

with $\delta_c^2 = P_{j_s=1}/P_{j_s=0}$ the ratio of the probabilities, or the ratio of the squares of the matrix elements, of forming the resonance via $j_s = 1$ relative to $j_s = 0$.

Example D.4

A resonance of spin and parity of $J^\pi = 1^-$ is populated in the ^{29}Si(p,γ)^{30}P reaction. The resonance decays via γ-ray emission to a final state in ^{30}P with spin and parity of $J^\pi = 1^-$ (Fig. D.2b). Calculate the angular correlation of the emitted γ-rays with respect to the incident proton beam direction.

The spin and parity of both the ^{29}Si target nucleus and the proton is $1/2^+$. Thus, two values for the channel spin are allowed, j_s = 0 and 1. The only allowed value for the orbital angular momentum of the proton is ℓ_p = 1. The γ-ray decay may proceed either via a M1 or E2 transition. Hence, the angular correlation expression will contain two additional parameters, the channel spin mixing ratio δ_c and the γ-ray multipolarity mixing ratio δ_γ. We will first consider the two channel spins separately and write symbolically

$$j_1(L_1)J(L_2)j_2 \rightarrow j_s(\ell_p)J(L_\gamma)j_{^{30}P} \rightarrow 0(1)1\begin{pmatrix}1\\2\end{pmatrix}1 \quad \text{and} \quad \rightarrow 1(1)1\begin{pmatrix}1\\2\end{pmatrix}1$$

For either channel spin the sum in Eq. (D.6) is restricted to $0 \leq n \leq 2J$, that is, n = 0 and 2. We obtain

$$\begin{aligned}
W_{j_s=0}(\theta) &= \sum_{n=0,2} \left[\frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)} F_n(L_1 j_s J) \right] \\
&\quad \times [F_n(L_2 j_2 J) + 2\delta_\gamma F_n(L_2 L_2' j_2 J) + \delta_\gamma^2 F_n(L_2' j_2 J)] P_n(\cos\theta) \\
&= (1 + \delta_\gamma^2) + \left[\frac{2 \cdot 1(1+1)}{2 \cdot 1(1+1) - 2(2+1)} F_2(101) \right] \\
&\quad \times [F_2(111) + 2\delta_\gamma F_2(1211) + \delta_\gamma^2 F_2(211)] P_2(\cos\theta) \\
&= (1 + \delta_\gamma^2) + [(-2)0.7071] \\
&\quad \times [(-0.3536) + 2\delta_\gamma(-1.0607) + \delta_\gamma^2(-0.3535)] P_2(\cos\theta) \\
&= (1 + \delta_\gamma^2) + (0.5 + 3\delta_\gamma + 0.5\delta_\gamma^2) P_2(\cos\theta)
\end{aligned}$$

Similarly

$$\begin{aligned}
W_{j_s=1}(\theta) &= (1 + \delta_\gamma^2) + \left[\frac{2 \cdot 1(1+1)}{2 \cdot 1(1+1) - 2(2+1)} F_2(111) \right] \\
&\quad \times [F_2(111) + 2\delta_\gamma F_2(1211) + \delta_\gamma^2 F_2(211)] P_2(\cos\theta) \\
&= (1 + \delta_\gamma^2) + [(-2)(-0.3536)] \\
&\quad \times [(-0.3536) + 2\delta_\gamma(-1.0607) + \delta_\gamma^2(-0.3535)] P_2(\cos\theta) \\
&= (1 + \delta_\gamma^2) + (-0.25 - 1.5\delta_\gamma - 0.25\delta_\gamma^2) P_2(\cos\theta)
\end{aligned}$$

The total angular correlation is given by the incoherent sum of the expressions for the individual channel spins,

$$\begin{aligned}
W(\theta) &= W_{j_s=0}(\theta) + \delta_c^2 W_{j_s=1}(\theta) \\
&= (1 + \delta_\gamma^2) + (0.5 + 3\delta_\gamma + 0.5\delta_\gamma^2) P_2(\cos\theta) \\
&\quad + \delta_c^2 [(1 + \delta_\gamma^2) + (-0.25 - 1.5\delta_\gamma - 0.25\delta_\gamma^2) P_2(\cos\theta)] \\
&= (1 + \delta_\gamma^2) + \delta_c^2 (1 + \delta_\gamma^2) \\
&\quad + (0.5 + 3\delta_\gamma + 0.5\delta_\gamma^2 - 0.25\delta_c^2 - \delta_c^2 1.5\delta_\gamma - \delta_c^2 0.25\delta_\gamma^2) P_2(\cos\theta) \\
&= (1 + \delta_\gamma^2)(1 + \delta_c^2) + 0.5(1 + 6\delta_\gamma + \delta_\gamma^2)(1 - 0.5\delta_c^2) P_2(\cos\theta)
\end{aligned}$$

The channel spin and γ-ray multipolarity mixing ratios are given by $\delta_c^2 = P_{j_s=1}/P_{j_s=0}$ and $\delta_\gamma^2 = \Gamma_{\gamma E2}/\Gamma_{\gamma M1}$, respectively.

Example D.5

Consider the ^{19}F(p,γ)^{20}Ne reaction, populating a resonance with a spin and parity of $J^\pi = 2^-$. The resonance decays to a lower lying state in ^{20}Ne with a spin and parity of $J^\pi = 1^+$ (Fig. D.2b). Calculate the angular correlation of the emitted γ-rays with respect to the incident proton beam direction.

Both the ^{19}F target nucleus and the proton have a spin and parity of $1/2^+$. The channel spin has two allowed values, j_s = 0 and 1. However, the 2^- resonance cannot be formed from j_s = 0 since total angular momentum and parity must be conserved simultaneously. Hence, only the channel spin j_s = 1 plays a role in this process. The resonance can be formed via orbital angular momenta of ℓ_{p} = 1 and 3 and thus this transition is mixed. For the sake of simplicity, we will assume that the γ-ray decay proceeds via an E1 transition only. We write symbolically

$$j_1(L_1)J(L_2)j_2 \to j_s(\ell_{\text{p}})J(L_\gamma)j_{^{20}Ne} \to 1\begin{pmatrix}1\\3\end{pmatrix}2(1)1$$

The sum in Eq. (D.6) is restricted to $n \le 2$ since we assumed L_γ = 1. It follows

$$\begin{aligned}
W(\theta) = \sum_{n=0,2} &\left[\cos(\xi_{L_1} - \xi_{L_1})\frac{2L_1(L_1+1)}{2L_1(L_1+1) - n(n+1)}F_n(L_1 j_s J)\right.\\
&+2\delta_a \cos(\xi_{L_1} - \xi_{L_1'})\frac{2\sqrt{[L_1(L_1+1)][L_1'(L_1'+1)]}}{L_1(L_1+1) + L_1'(L_1'+1) - n(n+1)}F_n(L_1 L_1' j_s J)\\
&\left.+\delta_a^2 \cos(\xi_{L_1'} - \xi_{L_1'})\frac{2L_1'(L_1'+1)}{2L_1'(L_1'+1) - n(n+1)}F_n(L_1' j_s J)\right]\\
&\times F_n(L_2 j_2 J)P_n(\cos\theta)\\
= [1\cdot 1\cdot 1 + \delta_a^2 \cdot 1\cdot 1\cdot 1]\cdot 1 &+ \left[1\cdot\frac{2\cdot 1\cdot(1+1)}{2\cdot 1\cdot(1+1) - 2(2+1)}F_2(112)\right.\\
&+2\delta_a\cos(\xi_{\ell=1} - \xi_{\ell=3})\frac{2\sqrt{[1(1+1)][3(3+1)]}}{1(1+1)+3(3+1)-2(2+1)}F_2(1312)\\
&\left.+\delta_a^2\cdot 1\cdot\frac{2\cdot 3(3+1)}{2\cdot 3(3+1) - 2(2+1)}F_2(312)\right]F_2(112)P_2(\cos\theta)\\
= 1 + \delta_a^2 + &\Big[1\cdot(-2)(0.4183) + 2\delta_a\cos(\xi_{\ell=1} - \xi_{\ell=3})(1.2247)(0.2390)\\
&+\delta_a^2\cdot 1(1.333)(-0.7171)\Big](0.4183)P_2(\cos\theta)\\
= 1 + \delta_a^2 + &[-0.35 + 0.25\delta_a\cos(\xi_{\ell=1} - \xi_{\ell=3}) - 0.4\delta_a^2]P_2(\cos\theta)
\end{aligned}$$

The orbital angular momentum mixing ratio is given by $\delta_a^2 = \Gamma_{\ell=3}/\Gamma_{\ell=1}$.

D.4 Three-Step Process with Unobserved Intermediate Radiation

It is sometimes of interest in a particle capture reaction to determine the angular correlation of secondary γ-rays with respect to the incident beam direction. In this case, we have a three-step process, involving: (i) the formation of a resonance with spin J through the capture of an incident particle with orbital angular momentum L_1, (ii) the first (primary) γ-ray decay of multipolarity L to an intermediate level of spin $\overline{J}$, and (iii) finally the subsequent secondary γ-ray decay of multipolarity L_2 to the final state of spin j_2 (see Fig. D.2d). Only the incident beam and the secondary γ-ray transition are observed, while the primary γ-ray transition is unobserved. We write symbolically

$$j_1 \begin{pmatrix} L_1 \\ L_1' \end{pmatrix} J \begin{pmatrix} L \\ L' \end{pmatrix} \overline{J} \begin{pmatrix} L_2 \\ L_2' \end{pmatrix} j_2 \tag{D.14}$$

The angular correlation expression is then given by

$$W(\theta) = \sum_{n=0,2,\ldots} [a_n(1)A_n(1)]C_n[a_n(2)A_n(2)]P_n(\cos\theta) \tag{D.15}$$

$$C_n = \sqrt{(2J+1)(2\overline{J}+1)}W(JnL\overline{J};J\overline{J}) \tag{D.16}$$

The first link ($j_1 \rightarrow J$) and last link ($\overline{J} \rightarrow j_2$) are described by the terms $a_n(1)A_n(1)$ and $a_n(2)A_n(2)$, respectively, and are handled as before. The term C_n describes the unobserved primary radiation. Unobserved γ-rays of multipolarities L and L' mix incoherently, that is, the total correlation is given by $W(\theta) = W_L(\theta) + \delta^2_{\gamma LL'}W_{L'}(\theta)$. Furthermore, the sum over n is restricted by the condition $0 \leq n \leq \min(2L_1, 2L_2, 2J, 2\overline{J})$. In particular, the angular correlation becomes isotropic for either J or $\overline{J}$ equal to 0 or 1/2. Note that the multipolarity L of the unobserved primary radiation does not limit the sum over n.

Example D.6

Consider the ^{11}B(p,γ)^{12}C reaction leading to the formation of a resonance with spin and parity of $J^\pi = 2^+$ (Fig. D.2d). The resonance γ-ray decays to an intermediate state ($J^\pi = 2^+$) which, in turn, decays to the ^{12}C ground state ($J^\pi = 0^+$). Calculate the angular correlation of the second γ-ray transition with respect to the incident beam direction.

The spin and parity of the ^{11}B ground state is $J^\pi = 3/2^-$. The two possible channel spins are j_s = 1 and 2. Of the two allowed proton orbital angular momenta (ℓ_p = 1 and 3), we will consider only the lower ℓ_p value. Similarly, of the two γ-ray multipolarities for the unobserved primary transition (M1 and E2) we will only consider the M1 case. Only one possibility is allowed for the multipolarity of the secondary γ-ray transition (E2). Symbolically we write

$$j_s \begin{pmatrix} L_1 \\ L_1' \end{pmatrix} J \begin{pmatrix} L \\ L' \end{pmatrix} \overline{J} \begin{pmatrix} L_2 \\ L_2' \end{pmatrix} j_2 \to 1(1)2(1)2(2)0 \quad \text{and} \quad \to 2(1)2(1)2(2)0$$

For either channel spin, the summation is restricted to $n \leq 2$ (because of $\ell_\mathrm{p} = 1$). The angular correlation is given by

$$\begin{aligned} W_{j_s=1}(\theta) &= \sum_{n=0,2,\ldots} [a_n(1)A_n(1)]C_n[a_n(2)A_n(2)]P_n(\cos\theta) \\ &= \sum_{n=0,2} \frac{2L_1(L_1+1)}{2L_1(L_1+1)-n(n+1)} F_n(L_1 j_s J)\sqrt{(2J+1)(2\overline{J}+1)} \\ &\quad \times W(JnL\overline{J};J\overline{J})F_n(L_2 j_2 J)P_n(\cos\theta) \\ &= 1\cdot 1\cdot \sqrt{(2\cdot 2+1)(2\cdot 2+1)}W(2012;22)\cdot 1 \\ &\quad + \frac{2\cdot 1\cdot 2}{2\cdot 1\cdot 2-2\cdot 3}F_2(112)\sqrt{(2\cdot 2+1)(2\cdot 2+1)} \\ &\qquad \times W(2212;22)F_2(202)P_2(\cos\theta) \\ &= 1\cdot 5\cdot 0.2\cdot 1 + (-2)(0.4183)\cdot 5\cdot 0.1\cdot(-0.5976)P_2(\cos\theta) \\ &= 1 + 0.25P_2(\cos\theta) \end{aligned}$$

Similarly

$$\begin{aligned} W_{j_s=2}(\theta) &= 1\cdot 1\cdot \sqrt{(2\cdot 2+1)(2\cdot 2+1)}W(2012;22)\cdot 1 \\ &\quad + \frac{2\cdot 1\cdot 2}{2\cdot 1\cdot 2-2\cdot 3}F_2(122)\sqrt{(2\cdot 2+1)(2\cdot 2+1)} \\ &\qquad \times W(2212;22)F_2(202)P_2(\cos\theta) \\ &= 1\cdot 5\cdot 0.2\cdot 1 + (-2)(-0.4183)\cdot 5\cdot 0.1\cdot(-0.5976)P_2(\cos\theta) \\ &= 1 - 0.25P_2(\cos\theta) \end{aligned}$$

The total angular correlation is given by the incoherent sum of the expressions for the individual channel spins,

$$\begin{aligned} W(\theta) &= W_{j_s=1}(\theta) + \delta_c^2 W_{j_s=2}(\theta) \\ &= [1+0.25P_2(\cos\theta)] + \delta_c^2[1-0.25P_2(\cos\theta)] \\ &= 1 + \delta_c^2 + 0.25(1-\delta_c^2)P_2(\cos\theta) \end{aligned}$$

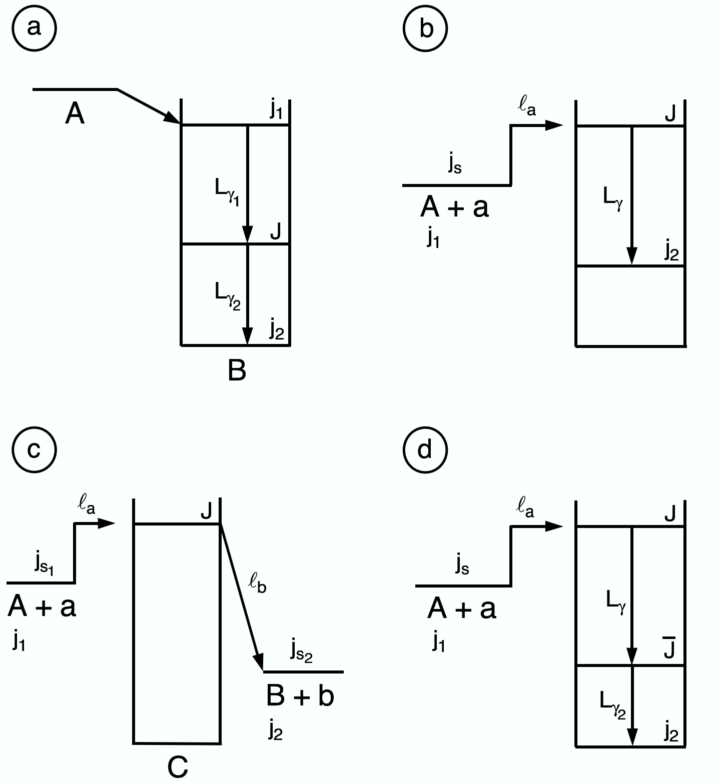

Fig. D.2 Schematic level diagrams indicating the quantum numbers involved in various angular correlation schemes. See the text.

D.5
Experimental Considerations

Experimental angular correlations and differential yields measured in the laboratory system must have both their intensities and angles converted to the center-of-mass system (Appendix C) before they can be compared to the theoretical expressions given above. Another important correction has to be performed since, strictly speaking, the theoretical angular correlation of Eq. (D.3) applies only to an ideal detector of negligible size. In an actual experiment, the measured intensities are obtained by integrating the theoretical angular correlation over the finite solid angle subtended by the detector. Hence, the effect of the finite solid angle is to reduce the anisotropy. For a detector of axial symmetry and for its symmetry axis pointing toward the source of the emitted radiation (Fig. 4.32), it can be shown that the form of the angular correlation function remains unchanged, but each term in the series of Eq. (D.3) becomes multiplied by a correction factor. For example, if radiation originating from a nuclear reaction is detected, then the experimental angular correlation measured by a specific detector is given by

$$W_{\exp}(\theta) = \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n Q_n P_n(\cos\theta) \tag{D.17}$$

Similarly, the experimental angular correlation between two emitted radiations a and b measured with two different detectors (or with the same detector, as was the case for coincidence summing in Section 4.5.2) can be written as

$$W_{\exp}(\theta) = \frac{1}{b_0} \sum_{n=0}^{n_{\max}} b_n Q_n^{(a)} Q_n^{(b)} P_n(\cos\theta) \tag{D.18}$$

The *attenuation factors* Q_n are given by (Rose 1953)

$$Q_n = \frac{\int_0^{\beta_{\max}} P_n(\cos\beta)\eta(\beta, E) \sin\beta\, d\beta}{\int_0^{\beta_{\max}} \eta(\beta, E) \sin\beta\, d\beta} \tag{D.19}$$

with β the angle between the radiation incident on the detector and the detector symmetry axis, $\beta_{\max}$ the maximum angle subtended by the detector, and $\eta(\beta, E)$ the detector efficiency for the radiation of energy E at angle β. It is apparent that the factors Q_n depend on the detector geometry, the energy of the radiation, and the kind of event that takes place in the detection process (for example, total versus partial energy deposition for γ-rays; see Section 4.5.2).

If the intrinsic detector efficiency is unity, as is generally the case for charged particle detectors, then the attenuation factor reduces to (Rose 1953)

$$Q_n = \frac{P_{n-1}(\cos\beta_{\max}) - \cos\beta_{\max} P_n(\cos\beta_{\max})}{(n+1)(1-\cos\beta_{\max})} \tag{D.20}$$

Attenuation factors calculated from this expression are displayed in Fig. D.3a for values of n = 1, 2, 3, and 4.

In the case of γ-ray detectors, where the efficiency for detecting an incident photon is smaller than unity, the attenuation factors will be larger than given by Eq. (D.20), that is, they will be closer to unity and, consequently, the difference between measured and theoretical angular correlation will be smaller. The *total efficiency* attenuation factors can be estimated with the same method used for calculating total efficiencies (Section 4.5.2). One simply substitutes $\eta^T(\beta, E) = 1 - e^{-\mu(E)x(\beta)}$ for the total efficiency in Eq. (D.19) and solves the integrals numerically. Similarly, the *peak efficiency* attenuation factors can be estimated if the peak efficiency $\eta^P(\beta, E)$ is first obtained from a Monte Carlo calculation. Peak efficiency attenuation factors estimated in this way for a HPGe detector are displayed in Fig. D.3b. The curves show values of Q_n versus γ-ray energy for a fixed source-detector distance of 1.6 cm. As expected,

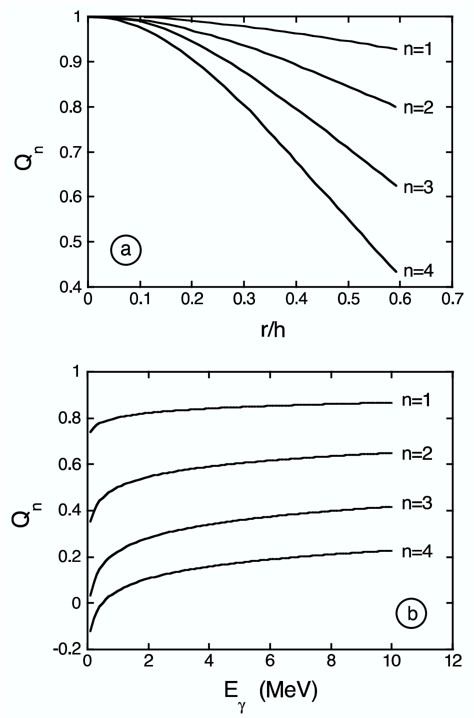

Fig. D.3 (a) Attenuation factors for an intrinsic detector efficiency of unity (for example, a silicon charged-particle counter). The horizontal axis displays the ratio r/h, with r and h the radius of the detector aperture and the source-detector distance, respectively. Note that $\tan \beta_{\max} = r/h$. The curves represent different values of n. After Gove (1959). (b) Attenuation factors for a HPGe detector versus γ-ray energy. The detector volume and source-detector distance amount to 582 cm^3 and 1.6 cm, respectively. The curves represent different values of n and are obtained by calculating peak efficiencies in Eq. (D.19) with the Monte Carlo code GEANT4. Courtesy of Richard Longland.

for decreasing photon energy the peak efficiency $\eta^P(\beta, E)$ increases and hence Q_n becomes smaller.

D.6 Concluding Remarks

We conclude this section with a few useful remarks. Since the angular momenta in low-energy nuclear reactions are rather small, the symmetry properties of the functions F_n restrict the series in Eq. (D.3) to a small number of terms. In practice, terms beyond $n = 4$ are rarely encountered. If for some reason the $n = 4$ term is zero or negligible, then we obtain $W(\theta = 55°) \approx 1$ or $W(\theta = 125°) \approx 1$, since the $P_2(\cos\theta)$ term is equal to zero at these angles (see Eq. (A.12)). Hence, the angle-integrated yield can be measured with a single detector located at a center-of-mass angle of $\theta = 55^\circ$ or $\theta = 125^\circ$. This circumstance has major practical advantages if very small yields need to be measured with a single detector in very close geometry to the target.

It is sometimes possible to simplify the theoretical angular correlation by making reasonable assumptions about the nuclear transition matrix elements. Mixtures of M1/E2 γ-ray multipolarites occur frequently, but E1/M2 mixtures are rarely important. In the latter case, it is often safe to assume that the E1 multipolarity dominates the γ-ray transition strength, hence $\delta_{\gamma M2/E1} \approx 0$. Similar arguments apply to the mixing of orbital angular momenta. Because of parity conservation (Appendix B), interfering orbital angular momenta must differ by two units, that is, ℓ_i and ℓ_{i+2}. The penetration factors decrease strongly for increasing values of orbital angular momentum, as can be seen from Fig. 2.21. Therefore, unless the reduced width (or spectroscopic factor) of the ℓ_{i+2} component is much larger than that of the ℓ_i component, the degree of orbital angular momentum mixing will be small, that is, $\delta_{a\ell_{i+2}/\ell_i} \approx 0$. Both of these simplifying assumptions should be treated with caution if the purpose of an angular correlation measurement is the determination of unknown nuclear spins. However, they are quite useful in nuclear astrophysics measurements if the level spins are known and if one is mostly interested in estimating angular correlation corrections for measured differential yields.

Sometimes a single γ-ray detector is placed at $\theta = 0^\circ$ in very close geometry to the target in order to maximize counting efficiency. Angular correlation effects may be significant for a specific primary γ-ray transition, but it may prove difficult to calculate the angular correlation if, for example, certain mixing ratios are unknown. In such cases, it could be of advantage to analyze instead the intensity of a corresponding secondary γ-ray decay for the calculation of the total yield. This is especially useful if the secondary γ-ray transition proceeds from a level with a spin of 0 or 1/2 since then its angular correlation is isotropic.

We pointed out that the series of Eq. (D.3) will contain only terms with n = even if the correlation involves an intermediate state of well-defined parity. However, if a reaction proceeds through two or more overlapping resonances of opposite parity, then the resulting angular correlation will not be symmetric about 90° anymore and terms with n = odd will appear in the series of Eq. (D.3). We will not consider here the more involved angular correlation resulting from the interference of two overlapping resonances. The interested reader is referred to Biedenharn (1960). Expressions for the angular correlation in direct radiative capture, and for the interference between resonant and direct contributions, are given in Rolfs (1973).

Example D.7

A resonance at E_r^{lab} = 519 keV is excited in the ^{17}O(p,γ)^{18}F reaction. The strongest primary transition occurs to the ^{18}F level at E_x = 1121 keV ($E_\gamma \approx$ 5 MeV, B_γ = 0.55 $\pm$ 0.03). The theoretical angular correlation is given by

$$W_\gamma(\theta) = 1 - 0.10 P_2(\cos\theta)$$

The γ-ray counter is located at $\theta = 0^\circ$ with respect to the proton beam direction in very close geometry to the target. A (peak) attenuation factor of Q_2 = 0.62 $\pm$ 0.05 is estimated from Eq. (D.19) for this geometry. The measured peak intensity is $\mathcal{N}_\gamma$ = 1530 $\pm$ 47 for a certain total number of incident protons. The peak efficiency at $E_\gamma \approx$ 5 MeV amounts to η_γ^P = 0.015 ($\pm$5%). Calculate the total number of reactions that took place. Ignore coincidence summing effects.

The measured angular correlation is given by

$$W_{\exp,\gamma}(\theta) = 1 - 0.10 Q_2 P_2(\cos\theta) = 1 - 0.10(0.62 \pm 0.05) P_2(\cos\theta)$$

At $\theta = 0^\circ$ we obtain $P_2(\cos\theta) = 1$ and hence

$$W_{\exp,\gamma}(0) = 1 - 0.10(0.62 \pm 0.05) \cdot 1 = 0.94(\pm 5\%)$$

From Eq. (4.69) we find

$$\mathcal{N}_R = \frac{\mathcal{N}_\gamma}{B_\gamma \eta_\gamma^P W_\gamma} = \frac{1530(\pm 3\%)}{[0.55(\pm 5\%)][0.015(\pm 5\%)][0.94(\pm 5\%)]} = 197292(\pm 9\%)$$