One can show [Ho 75] that $f(r)$ is peaked at the nuclear surface. By analogy with the electronic case, one quite often chooses $f(r)$ related to the spin independent part of the average potential in the following way—

$$f(r)=\lambda\frac{1}{r}\frac{dV}{dr}; \qquad \lambda\simeq -0.5\ [\mathrm{fm}^2] \tag{2.18}$$

—but also other surface-peaked radial dependences of $f(r)$ can be envisaged. It is interesting to note that the use of the Skyrme force (see Chap. 5) yields a spin orbit dependence for the average potential with $f(r)\sim(1/r)\cdot(d\rho/dr)$, where ρ is the single particle density. Since $V(r)$ roughly follows the form of ρ, this is consistent with Eq. (2.18).

2.5 The Shell Model Approach to the Many-Body Problem

The single-particle model takes into account the individual nucleons. It therefore provides a microscopic description of the nucleus. This is certainly only an approximation of the exact many-body problem. We will see, however, in the following, that the shell model can be used as a basis for more elaborate many-body theories, so before we talk about further details of the model, we want to discuss some general properites of the single particle model.

The microscopic theory of the nucleus is usually based on the following three properties.

(i) The nucleus is a quantum mechanical many-body system.
(ii) The velocities in the nucleus are small enough so that one can neglect relativistic effects $[(v/c)^2\sim 1/10]$.
(iii) The interaction between the nucleons has a two-body character.

A full microscopic theory of the nucleus would then be given by the solution of the many-body Schrödinger equation

$$H\Psi=\left\{\sum_{i=1}^{A}-\frac{\hbar^2}{2m}\Delta_i+\sum_{i<j}^{A}v(i,j)\right\}\Psi(1,\ldots,A)=E\Psi(1,\ldots,A), \tag{2.19}$$

where i represents all coordinates of the ith nucleon, for instance,

$$(i)=(\mathbf{r}_i,s_i,t_i), \tag{2.20}$$

where t_i will be $\frac{1}{2}$ for neutrons and $-\frac{1}{2}$ for protons. With the assumption of the nuclear shell model, the above equation reduces to the much simpler equation

$$H_0\Psi=\left\{\sum_{i=1}^{A}h_i\right\}\Psi=\sum_{i=1}^{A}\left\{-\frac{\hbar^2}{2m}\Delta_i+V(i)\right\}\Psi=E\Psi. \tag{2.21}$$

The solutions Ψ of Eq. (2.21) are anti-symmetrized products of single-particle functions, which are eigenfunctions to the single-particle Hamilto-

nian h_i:

$$h_i\phi_k(i)=\epsilon_k\phi_k(i). \tag{2.22}$$

The functions ϕ_k provide an orthogonal basis for an occupation number representation within the framework of second quantization (see Appendix C). To each level k corresponds a pair of creation and annihilation operators a_k^+, a_k which create or annihilate particles with wave function ϕ_k. Since nucleons are Fermions, each level can be occupied only once, and the operators a_k, a_k^+ obey Fermi commutation relations (C. 23).

The shell model Hamiltonian H_0 has the form

$$H_0=\sum\epsilon_k a_k^+ a_k.$$

Using the bare vacuum $|-\rangle$ its eigenfunctions can be represented as

$$|\Phi_{k_1\ldots k_A}\rangle=a_{k_1}^+\ldots a_{k_A}^+|-\rangle.$$

They are Slater determinants

$$\Phi_{k_1\ldots k_A}(1,\ldots,A)=\begin{vmatrix}\phi_{k_1}(1) & \cdots & \phi_{k_1}(A)\\ \vdots & & \vdots\\ \phi_{k_A}(1) & & \phi_{k_A}(A)\end{vmatrix} \tag{2.23}$$

with eigenvalues

$$E_{k_1\ldots k_A}=\epsilon_{k_1}+\cdots+\epsilon_{k_A}. \tag{2.24}$$

In the ground state the levels are filled successively according to their energy (see Fig. 2.6)

$$|\Phi_0\rangle=a_1^+\ldots a_A^+|-\rangle. \tag{2.25}$$

Thus we have for closed shells the following unique prescription for the construction of the A particle ground state as well as for the A particle excitation spectrum: Starting with the $(1s_{1/2})$ level, one has to occupy each level $|nsljm\rangle$ with just one particle until all A particles are used up. We thus obtain an A nucleon ground state where all different quantum states are occupied with just one particle up to the Fermi level (the highest occupied level); above the Fermi level all levels are unoccupied.

The independent particle picture of the nucleus is different from that in an atom in the sense that in a nucleus there are *two* different kinds of particles, the proton and the neutron, whereas in an atom there is only the

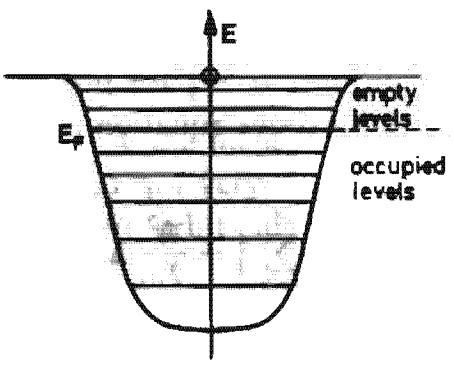

Figure 2.6. Shell model potential and Fermi level.

electron. Protons and neutrons feel different average potentials for two reasons:

(i) Protons also interact via the *Coulomb force*. One therefore usually adds the potential of a homogeneously charged sphere

$$V_C(\mathbf{r}) = \begin{cases} \dfrac{Ze^2}{R}\,\dfrac{1}{2}\left(3-\left(\dfrac{r}{R}\right)^2\right) & r \leqslant R, \\ \dfrac{Ze^2}{r} & r > R. \end{cases} \tag{2.26}$$

Sometimes (see Sec. 2.8), this feature is approximated by using different potential parameters for protons and neutrons.

(ii) The *symmetry energy* [see Eq. (1.4)] favors a configuration with an equal number of protons and neutrons. Because of the Coulomb repulsion for heavier nuclei, one has a neutron excess: If, in the nucleus, we replace a neutron by a proton, we gain symmetry energy and lose Coulomb energy. Since the Coulomb energy is already taken into account by Eq. (2.26), there must be an additional difference between the single-particle potential for protons and neutrons, which is caused by the symmetry energy. The *nuclear* part of the proton potential is therefore deeper (see Fig. 2.7, dashed line).

These two effects go in opposite directions, but they do not cancel. In the end, the Fermi surfaces for protons and neutrons must be equal, otherwise protons would turn into neutrons by β-decay or vice versa, whichever is energetically favored.

In $N \neq Z$ nuclei, energy levels with the same quantum numbers for protons and neutrons are therefore shifted with respect to one another by an amount Δ_ϵ resulting from a positive contribution Δ_C from the Coulomb force and a negative contribution $-\Delta_S$ from the symmetry energy

$$\epsilon_{nlj}^{(p)} - \epsilon_{nlj}^{(n)} = \Delta_\epsilon = \Delta_C - \Delta_S. \tag{2.27}$$

In heavy nuclei, this difference is such that the protons and neutrons at the Fermi surface belong to different major shells.

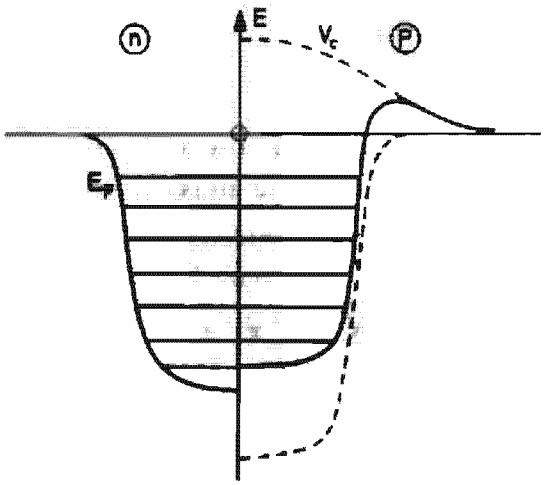

Figure 2.7. Comparison of the shell model potential for neutrons and protons in a nucleus with neutron excess.

Strong support of the independent particle idea comes from the experimental fact that magic numbers are the same for protons and neutrons (see Fig. 2.2; the magic number 126 exists only for neutrons, since for the heaviest nucleus known so far we only have $Z = 103$). If correlations played a major role, then, for example, the neutron excess in heavier nuclei would eventually influence the proton magic numbers in these nuclei (the nuclear force is almost charge independent; see Chap. 4) to be different from the corresponding neutron numbers. However, as we have said, this is not the case. The subshells of a major shell have, in some cases, a different order.

In the shell model, the *excitations of the system* are given by analogy with the free Fermi gas by a transfer of nucleons from below the Fermi level to levels above it. In the case of only a single nucleon transfer, we talk of an $1p-1h$ state with excitation energy of $\sim\hbar\omega_0$. For $^{40}_{20}Ca_{20}$ such a state is, for example, given by

$$(2s\tfrac{1}{2})^{-1}(1f\tfrac{7}{2}).$$

The Fermi level coincides in this case with the $1d_{3/2}$ level (see Fig. 2.5).

If we use the indices i, j for the levels below the Fermi surface ($\epsilon_i < \epsilon_F$), and the indices m, n for the levels above the Fermi surface ($\epsilon_n > \epsilon_F$), the lowest excitations in the shell model are then *ph* excitations of the form

$$|\Phi_{mi}\rangle := a_m^+ a_i |\Phi_0\rangle = \pm a_m^+ a_1^+ \ldots a_{i-1}^+ a_{i+1}^+ \ldots a_A^+ |-\rangle \qquad (2.28)$$

with excitation energy $\epsilon_{mi} = \epsilon_m - \epsilon_i$.

In fact one has observed such states in magic nulcei. They are, however, not the lowest states. As we have already seen in Chapter 1, there are low-lying collective states which cannot be explained in the independent particle model.

The Slater determinants (2.23) form a *complete set* of states for the A nulceon system [Lö 55]. Each state of the system is characterized by the distribution of the nucleons among the levels of the single particle potential, that is, by the "occupation numbers" of the levels. It is usual to classify all excited states by taking the ground state as a reference state. The nucleons that are missing in the ground state are denoted by holes, those above the Fermi levels by particles. A typical multiparticle–multihole configuration is shown in Fig. 2.8

Starting from a magic nucleus with the mass number A, we can add a particle and obtain a nucleus with the mass number $A+1$. If we put the

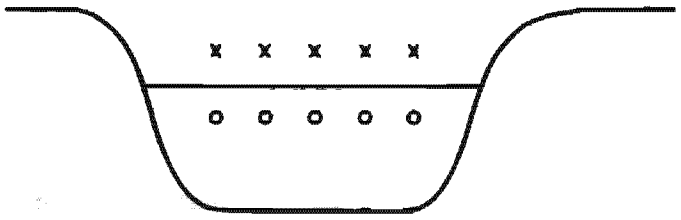

Figure 2.8. Schematic representation of a typical five-particle (crosses), five-hole (open circles) state.

particle into the level m, the wavefunction is

$$|\Phi_m\rangle = a_m^+ |\Phi_0\rangle \tag{2.29}$$

and we get the energy difference

$$\epsilon_m = E_m(A+1) - E_0(A). \tag{2.30}$$

In this way one is able to measure the single-particle energies (see Sec. 2.7). These are the simplest states in $A+1$ nuclei. More complicated states have a $2p-1h$ structure, and so on. In complete analogy, there are $1h$, $1p-2h$, etc., states in $A-1$ nuclei.

It often turns out to be very convenient to define quasiparticles by the operators

$$\begin{aligned} &\alpha_m^+ = a_m^+, && \alpha_m = a_m, && \text{for } \epsilon_m > \epsilon_F; \\ &\alpha_i^+ = a_i, && \alpha_i = a_i^+, && \text{for } \epsilon_i \leqslant \epsilon_F. \end{aligned} \tag{2.31}$$

These quasiparticles are again fermions. They are particles for states above, and holes for states below, the Fermi surface, so that we have

$$\alpha_k |\Phi_0\rangle = 0, \tag{2.32}$$

that is, the ground state of the magic nucleus is a "vacuum" with respect to these quasi-particles; *ph* states are two-quasi-particle states, and so on. The multi-quasi-particle states

$$|\Phi_{k_1 \ldots k_N}\rangle = \alpha_{k_1}^+ \ldots \alpha_{k_N}^+ |\Phi_0\rangle \tag{2.33}$$

form a complete orthogonal set in the many-body Hilbert space.

This basis is often used for further investigation of the many-body Hamiltonian H (2.19). In the shell model, one decomposes H,

$$H = T + \sum_{i<j} v(i,j) = H_0 + V_R, \tag{2.34}$$

with the residual interaction

$$V_R = \sum_{i<j} v(i,j) - \sum_i V(i) \tag{2.35}$$

in such a way that V_R is as small as possible and can be neglected. More elaborate theories investigate V_R in the basis in which H_0 is diagonal, the shell model basis (see Chap. 8).

The exact ground state wave function of a magic nucleus has the form

$$|\Psi_0\rangle = C_0 |\Phi_0\rangle + \sum_{mi} C_{mi} \alpha_m^+ \alpha_i^+ |\Phi_0\rangle + \tfrac{1}{4} \sum_{\substack{mi \\ m'i'}} C_{mim'i'} \alpha_m^+ \alpha_{m'}^+ \alpha_i^+ \alpha_{i'}^+ |\Phi_0\rangle + \cdots. \tag{2.36}$$

If the shell model is a good approximation to the nucleus, the coefficients C_{mi}, $C_{mim'i'}$, etc. should be small (see Fig. 10.3).

At this point we would like to again stress the fact that we have always been talking about a *spherical shell model* potential. Since, as we shall see, spherical nuclei exist only in the neighborhood of magic nuclei, by the same token this means that we have restricted our discussion to such nuclei. As this spherical average potential is created by the nucleons themselves, it may depend (though not abruptly) on the nucleon number A in quite a subtle way, in contrast to the atomic case. It is such that we

cannot take a once-and-for-all fixed single-particle potential and hope to find the corresponding single-particle states realized very accurately, be it even over a very limited range of neighboring nuclei.

We should keep these precautions in mind when talking about the shell model. As we said, the filling of the shells is without ambiguity, if we have closed shells. When we start filling neutrons and protons in *unfilled shells* these states will be degenerate, because the j-shells have a $(2j+1)$-fold degeneracy. The configuration of the nucleus can then be characterized by two numbers, κ and λ, which stand for the proton and neutron numbers, respectively, in the partially filled j-shell. Let the partially filled neutron shell be characterized by the quantum numbers $(n\ l\ j)$, and the partially filled proton shell by $(n'\ l'\ j')$. One then denotes the configuration by

$$(\nu nlj)^{\kappa}(\pi n'l'j')^{\lambda}.$$

Because of the $2j+1$-fold degeneracy of each j-shell, all possible shell model states corresponding to this configuration are also degenerate. The number of antisymmetric, linearly independent products is given by the product of the binomial coefficients

$$\binom{2j+1}{\kappa}\binom{2j'+1}{\lambda}. \tag{2.37}$$

The degeneracy of all these states will, of course, be removed in reality due to the action of the residual interaction V_R (2.35), which is neglected in the shell model. Taking one of the phenomenological nucleon–nucleon forces, as discussed in Chapter 4, one can diagonalize V_R in the subspace (2.37). Usually one takes not only this subspace into account, but also the one which corresponds to the nearly degenerate levels of a whole major shell. The $s_{1/2}$, $d_{5/2}$, $d_{3/2}$ levels of the $s-d$ shell is such a case, covering nuclei from ^{16}O up to ^{40}Ca. One can easily be convinced that the dimension of the matrices to be diagonalized becomes exceedingly large for more than two particles in open shells. Special procedures have been developed to diagonalize such huge matrices [Wh 72, SZ 72, WWC 77].

To reduce the size of these matrices, symmetries such as isospin or angular momentum (see Sec. 2.6) can be of great help (see, for instance, [FHM 69, WMH 71, HMW 71, GED 71, VGB 72, Wi 76]).

2.6 Symmetry Properties

2.6.1 Translational Symmetry

For any solution of the eigenvalue problem (2.19) we must require that a series of symmetry or invariance properties are fulfilled. Among these are, for example, translational and rotational invariance.* In the shell model

*Besides these exact symmetries, in some regions of the periodic table one often also has approximate symmetries, as, for instance, the isospin (see Sec. 2.6.3), which can be used for a classification of spectra (see [He 73a]).