

## COLLECTIVE NUCLEAR MOTION AND THE UNIFIED MODEL

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## § 1. Collective Oscillations of a Shell Structure

While a large body of empirical data has revealed the existence of a nuclear shell structure (cf. Ch. XV and XVI of this book), other features of the nuclear phenomena exhibit the influence of collective nuclear motion<sup>1</sup>. Indeed, it is to be expected from very general arguments that a system of particles held together by their mutual attraction can perform collective oscillations which resemble the degrees of freedom possessed by matter in bulk<sup>2</sup>.

The generalization of shell models to include collective motion can be effected in a straightforward manner. In fact, the shell model operates with an average field in which the particles move and which is generated collectively by all the particles. In the usual shell models, this field is considered a static quantity, but a more complete description must take into account the variation of the field associated with collective oscillations.

In the atomic shell model, this aspect of the structure is of relatively minor significance due to the stabilizing influence of the heavy central nucleus. However, the lack of such an external stabilization of the nuclear field implies that the collective degrees of freedom are of fundamental significance for the nuclear dynamics.

Since the collective oscillations involve variations in the average nuclear field, they are strongly coupled to the motion of the individual nucleons. The unified model thus describes the nucleus in terms of a coupled system of particle and collective degrees of freedom. In many respects, the dynamics of such a system is similar to that of molecules, where one has a coupled system of electronic motion and collective rotations and vibrations of the structure as a whole.

The most important collective degrees of freedom for the low energy nuclear properties are expected to be those associated with oscillations in shape with approximate preservation of the nuclear volume. For a nucleus of constant density, these oscillations have the character of surface waves.

<sup>1</sup> Cf., e.g., N. Bohr and J. A. Wheeler, Phys. Rev. **56**, (1939) 426; J. Rainwater, Phys. Rev. **79**, (1950) 432; A. Bohr, Phys. Rev. **81**, (1951) 134 and Dan. Mat. Fys. Medd. **26**, (1952) no. 14; D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, (1953) 1102; A. Bohr and B. R. Mottelson, Dan. Mat. Fys. Medd. **27**, (1953) no. 16. The latter reference will be denoted by BM in the following and its sections and equations designated by BM § IIb, iii and BM VII, 19, etc.

<sup>2</sup> N. Bohr and F. Kalckar, Dan. Mat. Fys. Medd. **14**, (1937) no. 10.

Oscillations of compressive character (sound waves) and those involving a motion of the neutrons with respect to the protons (dipole oscillations) are generally of appreciably higher frequency and are thus of lesser importance for the low energy nuclear properties considered in this chapter.

For small amplitudes of oscillation, the normal modes of vibration are approximately of the harmonic oscillator type, and can be classified by their multipole order  $(\lambda, \mu)$ , corresponding to an expansion of the nuclear shape in spherical harmonics. In general, the oscillations in shape of lowest order,  $\lambda=2$ , are of primary importance. A deformation of order  $\lambda=1$  simply corresponds to a displacement of the whole system.

The frequencies of oscillation depend on the structure of the collective flow and on the nuclear binding forces. While a quantitative determination in terms of the interactions in the system is difficult, an orientation in the magnitudes involved may be obtained by considering the simplified case of an incompressible uniformly charged nucleus (hereafter referred to as the incompressible model<sup>3</sup>) and relating the nuclear deformability to the surface tension as deduced from the empirical binding energies. The quanta of the shape oscillations are referred to as phonons and their energies estimated from the incompressible model are shown in Fig. 1.

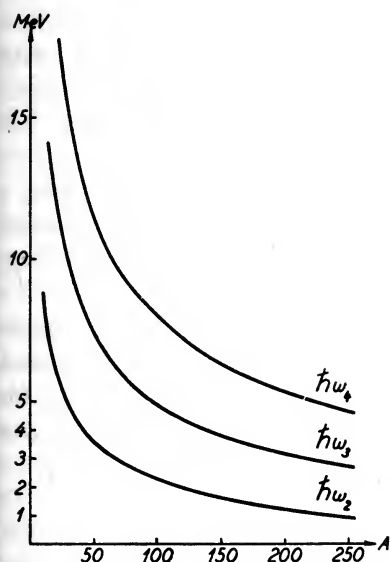


Fig. 1. *Phonon energies*

The quantum energies  $\hbar\omega_\lambda$  for nuclear shape oscillations, of multipole orders  $\lambda = 2, 3$  and  $4$ , are shown as functions of the mass number  $A$ . The above estimates refer to the incompressible model, and the parameters are the same as used in BM, § IIa, ii. The actual oscillation energies are expected to depend to some extent on the nucleonic configuration and to be somewhat larger than this average estimate in the case of closed shell configurations.

The coupling between particle and collective motion reflects the dependence of the particle energy on the nuclear deformation and in turn implies

<sup>3</sup> In BM, this simplified model was referred to as "the hydrodynamical approximation".

a tendency of the particle structure to produce a deformation in the nuclear field <sup>4</sup>. The properties of the nucleus depend in an essential way on the magnitude of this deformation.

In the next paragraph, we shall discuss the nuclear coupling scheme and the excitation spectrum arising from the coupled motion of individual particles and collective oscillations. In subsequent paragraphs, we discuss the implications of this unified model for various nuclear properties of interest in connection with investigations of nuclear  $\beta$ - and  $\gamma$ -radiation. A more comprehensive discussion of the model and of the available empirical evidence has been presented elsewhere (cf. BM, which also contains references to other papers on this subject).

## § 2. Nuclear Coupling Scheme

### A. REGION OF CLOSED SHELLS

For a nucleus in which both neutrons and protons form closed shells, the spherical shape represents a stable equilibrium. In the immediate neighbourhood of such a configuration, the deformations are therefore small, and one retains many of the features of the static shell model. The system possesses, however, the additional degrees of freedom associated with the oscillations in shape. These degrees of freedom are expected to manifest themselves in the nuclear energy spectrum as excitation quanta with spin  $\lambda$ , parity  $(-)^{\lambda}$ , and energy  $\hbar\omega_{\lambda}$ . These states would be characterized, among other things, by enhanced electric transition probabilities (cf. § 4 B, i, below).

The coupling between the particles and the nuclear shape implies a slight interweaving of these vibrational modes with the particle motion. The resulting modifications in the nuclear properties can be studied by means of a perturbation calculation.

Already for configurations involving a few particles or holes outside of closed shells, the nuclear states involve appreciable interweaving between particle and collective motion. Solutions of the coupled equations in this region are usually difficult to obtain, but some exploratory studies have been made <sup>5</sup>.

### B. STRONGLY DEFORMED NUCLEI

For configurations sufficiently far removed from closed shells, the nucleus acquires a large equilibrium deformation resulting from the

<sup>4</sup> J. Rainwater, Phys. Rev. **79**, (1950) 432.

<sup>5</sup> Cf. L. L. Foldy and F. J. Milford, Phys. Rev. **80**, (1950) 751; BM, IIb, i and iii; A. K. Kerman, Phys. Rev. **92**, (1953) 1176; D. C. Choudhury, Dan. Mat. Fys. Medd. **28**, (1954) no. 4; F. J. Milford, Phys. Rev. **93**, (1954) 1297.

deforming effects of many nucleons; the nuclear coupling scheme then acquires a relative simplicity resembling that of a linear molecule. Thus, one obtains an approximate solution by considering first the relatively fast motion of the particles with respect to the deformed nuclear field, considered as fixed in space, and subsequently the relatively slow vibration and rotation of the entire system.

The equilibrium shape of the nuclear field is determined by considering the energy of the particles and of the field as a function of the deformation, considered as fixed, and minimizing this energy.

For most configurations, the equilibrium shape possesses axial symmetry so that orbits of the individual particles may be characterized by their component of angular momentum,  $\Omega_p$ , along the symmetry axis; the total angular momentum of a particle is in general not a constant of the motion for such non-spherical fields. The quantum numbers  $\Omega_p$  take on half integer values, positive or negative. States differing only in the sign of  $\Omega_p$  are degenerate since they are identical, except for the opposite sense of rotation about the axis. The  $\Omega_p$  of the individual particles add up to a total component,  $\Omega$ , of particle angular momentum along the nuclear symmetry axis.

The ground state of a nucleus is obtained by filling the particles pairwise in states of opposite  $\Omega_p$ . Thus, in an even-even nucleus, one obtains a total  $\Omega=0$ , while in an odd- $A$  nucleus the total  $\Omega$  is equal to the  $\Omega_p$  of the last odd particle. In an odd-odd nucleus, the last odd neutron and proton each contribute an  $\Omega_p$  and the total is either the sum or difference of the two.

The next step in the solution of the equations of motion for strongly deformed nuclei involves the consideration of the oscillations around the equilibrium shape. A necessary condition for the validity of the solution is that the amplitudes of these vibrations are small compared to the total magnitude of the deformation. If we restrict ourselves to quadrupole deformations, there are two types of normal vibrations. The first preserves cylindrical symmetry and only implies variations in the eccentricity of the spheroidal shape. The second leads to deviations from cylindrical symmetry. For small oscillations about the equilibrium shape, these modes are expected to be approximately harmonic and we denote their quantum numbers by  $n_\beta$  and  $n_\gamma$ , respectively. The excitation energies are of the order of magnitude of  $\hbar\omega_2$  (cf. Fig. 1), but should depend somewhat on the configuration of the nucleons. An anomalously low energy associated with  $n_\gamma$  may result for special particle configurations which do not strongly prefer deformations possessing cylindrical symmetry<sup>6</sup>.

<sup>6</sup> Cf. e.g., BM, Ap. III, ii; B. Segall, Phys. Rev. 95, (1954) 605 (A); M. Jean, to be published.

Finally, the system as a whole may rotate with the preservation of the shape and the internal structure. This motion gives rise to a spectrum of the same type as that of rotating rigid bodies, although the collective motion that generates the nuclear rotation is essentially different from that of a rigid body and may best be pictured as a wave travelling around the nucleus (cf. Fig. 2). The moment of inertia associated with such a

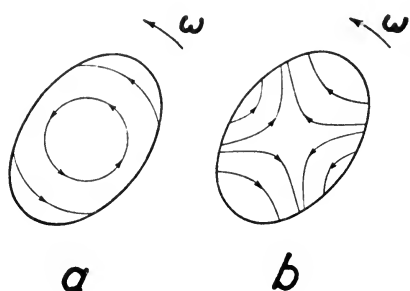


Fig. 2. *Velocity fields for collective rotations*  
 Fig. 2a illustrates the rotation of a rigid body, while in Fig. 2b the velocity field for the wave-like rotation of an irrotational fluid is shown. While the former motion refers to a system in which each particle is located near an equilibrium position, the latter type of rotation is characteristic of a shell structure, like the nucleus, in which the individual particles freely traverse the entire volume of the system.

wave-like rotation is small compared to the total moment characterizing the rigid rotation; it is proportional to the square of the amplitude of the wave, i.e. to the square of the nuclear deformation with respect to the axis of rotation.

For a nucleus with axial symmetry, the spectrum is that of a symmetric top and the energy of a state with total angular momentum  $I$  is given by

$$E_{\text{rot}} = \frac{\hbar^2}{2J} I(I+1), \quad (1)$$

where the moment of inertia is

$$J = \frac{2}{5} MA (\Delta R)^2 \quad (2)$$

for a spheroidal deformation of the incompressible model. In this expression,  $\Delta R$  represents the difference between the major and minor semi-axis of the spheroid, which is assumed to be small compared to the mean radius  $R_0$ <sup>7</sup>. The nucleon mass is denoted by  $M$ , and  $A$  is the nuclear mass number.

The rotations of the symmetric nucleus are characterized by the three quantum numbers  $I$ ,  $K$ ,  $M$ , where  $K$  and  $M$  are the components of  $I$  along the intrinsic symmetry axis and a space fixed axis, respectively.

The coupling scheme for strongly deformed nuclei is illustrated in Fig. 3. The wave function for such a system can be constructed in analogy to that for diatomic molecules and consists of a product of three parts

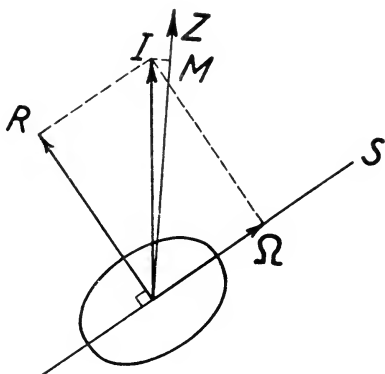
<sup>7</sup> In BM, the quantity  $\beta = \frac{4}{3} \sqrt{\pi/5} (\Delta R/R_0)$  has been employed to describe the magnitude of the deformation.

representing the particle motion, the collective vibration, and rotation, respectively.

In the nuclear ground state,  $K = \Omega$ , and since the moment of inertia about an axis of symmetry vanishes, changes in  $K$  involve the excitation of asymmetric vibrations characterized by the quantum number  $n_\nu$ <sup>8</sup>. The value of  $I$  must be at least equal to the numerical value of  $K$ .

Fig. 3. *Coupling scheme for strongly deformed nuclei*

For a strongly deformed nucleus possessing axial symmetry, the individual particles are coupled separately to the symmetry axis in states characterized by the projection  $\Omega_p$  of their angular momentum on the symmetry axis  $S$ . The sum of the individual  $\Omega_p$  gives the resultant  $\Omega$ , which for the lowest rotational band is equal to the projection  $K$  of the total angular momentum  $I$  on  $S$ . In this case, the collective rotational angular momentum  $R$  is perpendicular to  $S$ . The component of  $I$  on a fixed  $Z$ -axis is denoted by  $M$ .



The possible quantum states of the nucleus are restricted by the reflection symmetry of the deformation, which implies that states labelled by  $(K, \Omega)$  must be combined in a definite way with those labelled by  $(-K, -\Omega)$ <sup>9</sup>. Moreover, the reflection symmetry implies that the collective motion has even parity, and the parity of a nuclear state is, therefore, determined by that of the particle structure.

Thus, for an even-even nucleus, in which the ground state has  $K = \Omega = 0$ , the lowest rotational band is given by (1) with spins and parities

$$I = 0, 2, 4, 6 \dots \text{ even parity.} \quad (3)$$

The odd values of  $I$  are excluded by the above mentioned symmetry condition, which requires the wave function to be invariant to a rotation of  $180^\circ$ .

In an odd- $A$  or an odd-odd nucleus (or in an even-even nucleus with an excited particle structure with  $\Omega \neq 0$ ), the ground state has normally  $I_0 = K = \Omega$  and a rotational spectrum given by (1) with

$$I = I_0, I_0 + 1, I_0 + 2, \dots \text{ same parity as ground state.} \quad (4)$$

A special case occurs for an odd- $A$  nucleus with  $\Omega = 1/2$ ; the intrinsic spin

<sup>8</sup> Cf. BM, § VIc, iv.

<sup>9</sup> Cf. BM, II, 15.

of the particle is then partly decoupled from the rotational motion, which implies a modified rotational spectrum

$$E_{\text{rot}} = \frac{\hbar^2}{2J} [I(I+1) + a(-)^{I+1/2} (I+1/2)] \quad (5)$$

with

$$a = \sum_j (-)^{j-1/2} (j+1/2) |c_j|^2, \quad (5a)$$

where  $|c_j|^2$  represents the probability that the last odd particle has an angular momentum  $j$ . This correction term leads to an anomalous rotational spectrum and may also yield a ground state spin  $I_0 \neq \Omega$ .

Thus, the spectrum for a strongly deformed nucleus is expected to exhibit three distinct types of excitation. The first corresponds to a change in the particle motion with respect to the deformed field and is characterized by a change in the quantum numbers  $\Omega_p$ ; the energies involved in such excitations are determined by the position of particle levels in a spheroidal field<sup>10</sup>, which have an average spacing of about 100 keV in a heavy nucleus. The second mode of excitation is associated with the collective vibrations characterized by the quantum numbers  $n_\beta$  and  $n_\gamma$ , and for a heavy nucleus the vibrational energies are expected to be about 1 or 2 MeV (cf. Fig. 1). The third mode of excitation is the nuclear rotation with preservation of shape and is characterized by the quantum number  $I$ ; the rotational energies depend on the nuclear deformation and become very much smaller than the vibrational energies for the strongly deformed nuclei.

This simple separation of the spectrum represents a limiting case; for less deformed nuclei, the more rapid rotational motion will somewhat distort the shape of the nucleus and will also perturb the particle structure due to the inability of the particles to follow the rotations completely adiabatically. While the latter effect depends specifically on the particle configurations in question, the former, which is analogous to the rotation-vibration interaction in molecules, can be expressed in terms of the vibrational energies. The resulting correction term is, to leading order, given by

$$\Delta E = -2 \left[ \frac{3}{(\hbar\omega_\beta)^2} + \frac{1}{(\hbar\omega_\gamma)^2} \right] \frac{\hbar^2}{J} E_{\text{rot}}^2, \quad (6)$$

where  $\hbar\omega_\beta$  and  $\hbar\omega_\gamma$  are the vibrational quanta associated with the quantum numbers  $n_\beta$  and  $n_\gamma$ , respectively. The rotational energy,  $E_{\text{rot}}$ , in (6) is given by (1) for  $\Omega \neq 1/2$  and by (5) for  $\Omega = 1/2$ .

While the details of the expressions (6) depend on the assumptions of the incompressible model, the dependence of the vibration-rotation interaction on the spin,  $I$ , is expected to be more general.

<sup>10</sup> For a calculation of the single-particle level spectrum in a spheroidal potential, cf. S. G. Nilsson, to be published in Dan. Mat. Fys. Medd.

### § 3. Spins and Moments

#### A. GROUND STATE SPINS

The ground state spin is determined in general by the competition of the direct forces between particles outside of closed shells and the forces coupling the particles to the nuclear deformation. For small deformations, as near major closed shells, the former forces may dominate, coupling the individual particle angular momenta together to a resultant  $J$ . The particular  $J$  value favoured depends somewhat on the range of the forces and their exchange character (cf. Ch. XV of this book). With increasing deformation the interaction of the particles with the nuclear shape dominates, and the particles become coupled independently to the nuclear symmetry axis in states characterized by  $\Omega_p$ . The nuclear ground state spin then has the value  $I_0 = \Omega = \sum \Omega_p$ .

The various couplings have been studied in some detail for only certain especially simple configurations involving a few particles outside of closed shells and on the assumption that the particles can be characterized by a definite  $j$ <sup>10a</sup>.

For a single particle, one finds  $I_0 = j$ , irrespective of the nuclear deformation.

For two equivalent particles, both particle forces and the coupling to the deformation favour the ground state spin  $I_0 = 0$ .

For three equivalent particles, two-body forces of the expected range produce  $I_0 = j$ , while the three lowest  $\Omega_p$  values are  $\Omega_p = \pm j, j-1$ , which yield  $I_0 = \Omega = j-1$ <sup>11</sup>. In this case, the resulting ground state spin is expected to depend on the magnitude of the deformation. The observed spin values for these configurations exhibit both values  $I_0 = j$  and  $j-1$ , and the trends may be interpreted in terms of the influence of the deformation<sup>12</sup>.

For more complicated many-particle configurations, the effect of particle forces and their competition with the coupling to the nuclear deformation may present very difficult problems, especially since the rather strongly deformed shapes encountered for these configurations imply that the particle orbitals are essentially different from those in a spherical potential.

<sup>10a</sup> More general calculations of the coupling scheme for interacting particles moving in a spherical potential have been made for certain light nuclei; cf. e.g., D. Inglis, *Rev. Mod. Phys.* 25 (1953) 390 and B. H. Flowers, reported at Glasgow Conference, July 1954.

<sup>11</sup> The coupling to the collective motion can also be studied for small deformations by the perturbation method and yields the ground state spins:  $I_0 = 5/2$  for  $(j = 5/2)^3$ ;  $I_0 = 5/2$  for  $(j = 7/2)^2$ ;  $I_0 = 7/2$  for  $(j = 9/2)^3$ . We are indebted to Drs. I. Talmi and C. Schwartz for pointing out an error in these results as quoted in BM, p. 35.

<sup>12</sup> Cf. BM, § III, iii.



However, for sufficiently large deformations, the problem again simplifies, since the particles become coupled individually to the nuclear axis. Thus, for any even-even nucleus, the total  $\Omega$  vanishes and one obtains the ground state spin  $I_0=0$ . For an odd- $A$  nucleus, the ground state spin equals the  $\Omega_p$  of the last odd particle and could thus be obtained from the single-particle level scheme in the deformed field. (Note, however, the anomalous case of  $\Omega=1/2$ ; cf. p. 473 above).

In an odd-odd nucleus, the total  $\Omega$  of the lowest states may equal either the sum or the difference of the  $\Omega_p$ 's of the last odd proton and last odd neutron. One thus obtains two close lying states of the particle structure.

## B. QUADRUPOLE MOMENTS

The occurrence of nuclear quadrupole moments an order of magnitude larger than those associated with single-proton orbits gives some of the most direct evidence for the cooperative behaviour of the nucleons. At the same time, the nuclear shell structure manifests itself in the signs and the variations in magnitude of the quadrupole moments<sup>13</sup>. The moments are small in the region of closed shells and become larger with the addition of more particles. It is possible to account for these trends in terms of the tendency of the particle structure to produce a collective deformation of the nucleus<sup>14</sup>.

While the spectroscopically determined quadrupole moment,  $Q$ , characterizes the asymmetry in the averaged charge density in the nuclear state, the shape of the nucleus is more directly described in terms of the intrinsic quadrupole moment with respect to the nuclear symmetry axis. For the incompressible model, the intrinsic quadrupole moment is

$$Q_0 = \frac{4}{5} Z R_0^2 \frac{4R}{R_0}. \quad (7)$$

The relationship between  $Q$  and  $Q_0$  depends on the nuclear coupling scheme. For the strongly deformed nuclei, one has

$$Q = \frac{I_0}{I_0+1} \frac{2I_0-1}{2I_0+3} Q_0 \quad (8)$$

for the ground state ( $I_0=K$ ). This reduction of  $Q$  as compared with  $Q_0$  is a consequence of the quantum fluctuations in the direction of the nuclear axis.

<sup>13</sup> C. H. Townes, H. M. Foley and W. Low, Phys. Rev. **76**, (1949) 1415.

<sup>14</sup> J. Rainwater, Phys. Rev. **79**, (1950) 432; for references to more detailed discussions, cf. BM, Chapter V.

## C. MAGNETIC MOMENTS

In the unified model, the magnetic moments result in part from the collective motion and in part from the motion of the particles. The sharing of angular momentum between particle and collective motion, and the modification of the particle orbitals resulting from motion in a non-spherical field, both cause appreciable shifts of the magnetic moments from those corresponding to single-particle motion in a spherical potential.

For strongly deformed nuclei, the magnetic moment has an especially simple structure and, for a nuclear ground state with  $I_0 = \Omega \neq 1/2$ ,

$$\mu = \frac{I_0^2}{I_0 + 1} g_\Omega + \frac{I_0}{I_0 + 1} g_R, \quad (9)$$

where  $g_\Omega$  is the  $g$ -factor characterizing the state of the last odd particle in the deformed potential, and where  $g_R$  is the  $g$ -factor for the collective rotational motion, given by

$$g_R = \frac{Z}{A} \quad (10)$$

for the incompressible model.

## § 4. Nuclear Transition Probabilities

As discussed in § 2, one expects, according to the unified model, in the region of closed shells as well as for the strongly deformed nuclei, to be able to distinguish between particle excitations, which are associated with a change in the quantum state of the particle structure, and collective excitations, which involve vibrations or rotations of the nucleus as a whole, leaving the particle structure unaffected. In an intermediate region with relatively small deformations, however, the particle and collective motion may be interwoven in a more complex manner.

The measurement of transition probabilities provides an important tool for distinguishing these two modes of excitation. Such evidence comes from the  $ft$ -values of  $\beta$ -transitions and from the reduced transition probability for  $\gamma$ -decay<sup>15</sup>. More recently, the Coulomb excitation reaction has provided an alternative method for determining electromagnetic transition probabilities<sup>16</sup>.

<sup>15</sup> For the definition of the reduced transition probability  $B(L; I_i \rightarrow I_f)$ , cf., e.g., Ch. XIII.

<sup>16</sup> The Coulomb excitation cross-section can be written as a product of two factors, of which the first depends only on the bombarding particles and on the nuclear charge and excitation energy, while the second is identical with the reduced transition probability which characterizes the radiative transition of corresponding multipole order. For a recent review of the theory and applications of the Coulomb excitation reaction, cf. K. Alder, A. Bohr, T. Huus, B. R. Mottelson, A. Winther and Č. Zupanič, to be submitted to the Rev. Mod. Phys.

## A. PARTICLE TRANSITIONS

The spin and parity of the lowest particle configurations, of a given nucleus, are in many cases, especially in the region of closed shells, unaffected by the deformation, and it is thus often possible to classify particle transitions in terms of the shell model.

Striking examples are the interpretation of the M4 and E3 isomeric transitions (cf. Ch. XVI) and the classification of the  $\beta$ -decays into orders of forbiddenness (cf. Ch. XVI).

Even where such a shell model classification is possible, however, the nuclear deformations may influence the quantitative estimate of transition probabilities in a number of ways.

### i) *The Unfavoured Factor*

Since the combining states in a particle transition will in general have somewhat different shapes, one expects a reduction in the transition probability associated with this lack of overlap (cf. the Franck-Condon principle in molecules). The effect is difficult to estimate quantitatively; it depends exponentially on the difference in the nuclear shapes and may become very large when this difference exceeds the amplitude of the zero-point oscillations. Further, the modification of the particle wave functions, implied by the non-spherical potential, in general reduces the transition probability (at the same time as it breaks down selection rules on  $l$  and  $j$  (cf. § 4A, iii)).

An additional reduction in the transition probability may result when more than one particle participates in the transition<sup>17</sup>.

The isomeric particle transitions and  $\beta$ -decays are observed to be in general retarded by a factor in the range of 5 to 100 (the unfavoured factor) as compared with single-particle estimates<sup>17</sup>.

Valuable information on the origin of the observed unfavoured factors would be provided by a determination of the variation of transition probabilities with the trends of nuclear deformations. The interpretation of the unfavoured factor would be further clarified by the measurement of branching ratios to different vibrational states associated with the same particle structure.

In special cases, the combining states are expected to have almost identical shapes, in which case the transition matrix elements depend only on the particle structure. Examples of such unhindered transitions are the  $\beta$ -decays between mirror nuclei. Another case should be provided by the  $\gamma$ -transitions between the two close lying particle states char-

<sup>17</sup> Cf., e.g., S. A. Moszkowski, Phys. Rev. 89, (1953) 474; BM, VIIId, ii and VIIIc, ii and iv.

acterized by  $\Omega = |\Omega(\text{proton}) \pm \Omega(\text{neutron})|$  in strongly deformed odd-odd-nuclei.

## ii) Branching Ratios to Rotational States

For the very deformed nuclei which possess excited states that can be described in terms of a rotational motion of the whole nucleus, the transition probabilities to states of the same rotational family obey simple relations similar to the intensity rules encountered in the multiplets of atomic spectra<sup>18</sup>. Thus, for electromagnetic transitions of multipole order  $L$ , from a state characterized by  $I_i, K_i = \Omega_i$  to members of a rotational sequence characterized by  $K_f = \Omega_f$  and spins  $I_f, I'_f$  etc. (cf. Fig. 4), the reduced transition probabilities satisfy the relation

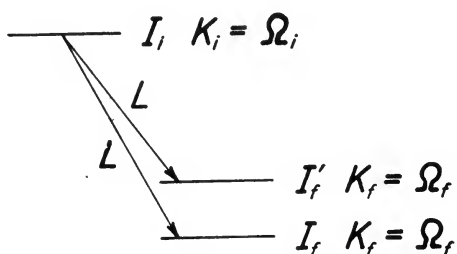
$$\frac{B(L; I_i \rightarrow I_f)}{B(L; I_i \rightarrow I'_f)} = \left( \frac{\langle I_i L K_i K_f - K_i | I_i L I_f K_f \rangle}{\langle I_i L K_i K_f - K_i | I_i L I'_f K_f \rangle} \right)^2, \quad (11)$$

where the quantities on the right are Clebsch-Gordon coefficients for the addition of angular momenta<sup>19, 20</sup>. The same relation holds for the ratio of inverse  $ft$ -values for  $\beta$ -transitions associated with a tensor operator of multipole order  $L$  (cf. Ch. X of this book).<sup>21</sup>

Fig. 4. Branching ratios to rotational states

The figure illustrates the quantum numbers involved in a branching of a  $\beta$ - or  $\gamma$ -decay to two members of a rotational family. An example of the intensity rules for such transitions is provided by the  $\beta$ -decay of  $^{170}\text{Tm}$  to the ground state ( $I_f = K_f = \Omega_f = 0$ ) and first excited state ( $I'_f = 2; K_f = \Omega_f = 0$ ) of  $^{170}\text{Yb}$ . The observed ratio of the  $ft$ -values for the transitions ( $I_f = 2$ ) and ( $I_f = 0$ ) is  $1.9 \pm 0.2$  (R. L. Graham, J. L. Wolfson and R. E. Bell, Can. J. Phys. 30, (1952) 459), which agrees with the value 2.0 calculated

from (11), assuming  $I_i = K_i = \Omega_i = 1$  and  $L = 1$ .



<sup>18</sup> Cf. e.g., E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra*. The University Press, Cambridge (1935), pp. 241 ff.

<sup>19</sup> We use a notation similar to that of Condon and Shortley, loc. cit., pp. 76 ff., in which  $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$  is the coefficient for addition of the angular momentum vectors  $j_1$  and  $j_2$  to form a resultant  $j$  with component  $m$ .

<sup>20</sup> The appearance of components with  $(-K, -\Omega)$  as well as  $(K, \Omega)$  in the nuclear wave function (cf. p. 473) may lead to additional terms in (11), but only in the rather unusual case of  $L \geq K_i + K_f$  with neither  $K_i$  nor  $K_f$  equal to zero; cf. reference 21.

<sup>21</sup> For a more detailed discussion of intensity rules for  $\beta$ - and  $\gamma$ -transitions to nuclear rotational states, cf. G. Alaga, K. Alder, A. Bohr, and B. R. Mottelson, Dan. Mat. Fys. Medd., in press. Similar intensity rules apply also to the  $\alpha$ -decay fine

Such simple rules governing branching ratios in nuclear transitions provide a useful tool in the classification of nuclear states <sup>21</sup>. Examples illustrating these intensity rules are discussed in § 5 below (cf. also caption to Fig. 4 and reference 21).

### iii) *Selection Rules*

While very general selection rules on the multipole order of  $\gamma$ -transitions and degree of forbiddenness of  $\beta$ -transitions are associated with the total angular momentum,  $I$ , and the parity of the combining states (Ch. X and XIII), additional partial selection rules may result from the existence of other angular momentum quantum numbers in the nuclear states. Since these latter quantum numbers are not exact constants of the motion, the corresponding selection rules have the effect of retarding the transitions without entirely forbidding them.

In the region of closed shells, where the nuclei are essentially spherical, the angular momenta,  $l$  and  $j$ , of the individual particles may be such approximate constants of the motion. Reductions in transition probabilities have been observed in certain transitions and ascribed to such  $l$ - and  $j$ -forbiddenness <sup>21a</sup>. With increasing nuclear deformation these selection rules are expected to become less effective.

For the very deformed nuclei, where the coupling scheme again simplifies, partial selection rules are expected to result from the rotational constant of the motion  $K$  (and  $\Omega$ ). Especially for transitions to high members of a rotational band, where  $I$ , is very different from  $K$ , such a  $K$ -forbiddenness may be very effective (for an example, cf. p. 487 below).

### iv) *Polarization of the Nucleus by Particle Transitions*

In the same way as a static quadrupole moment is induced in the nuclear shape by the mass quadrupole moment of the particle state, a particle transition of multipole order  $L$  induces a corresponding oscillating moment in the nuclear shape. Even in cases where the induced deformation is small, the corresponding electric moment may be large compared with that of the particle. Since the induced moment is proportional to the mass multipole moment of the particle transition, the transition matrix element is simply multiplied by a factor that depends on the coupling between particle and collective motion and on the nuclear deformability. Estimates of this effect <sup>22</sup> indicate that electric transitions in general proceed primarily structure; cf. A. Bohr, P. Fröman, and B. R. Mottelson, Dan. Mat. Fys. Medd., in press.

<sup>21a</sup> Cf., e.g., L. Nordheim, Rev. Mod. Phys. 23, (1951) 322; BM § VIIId, ii and § VIIc, iii.

<sup>22</sup> BM, § VIIb, iii.

by means of collective radiation. An important consequence is that the electric radiation of a neutron transition is expected to be of the same order of magnitude as for a proton transition (cf. the 870 keV E2 transition in  $O^{17\ 22a}$  and the electric transitions in  $Pb^{206\ 22b}$ ). In the region of closed shells, such polarization effects may give rise to transitions which are appreciably faster than pure particle transitions in a spherical field (cf. the E3 radiative decays of the 3-states in  $O^{16\ 23}$  and  $Pb^{208\ 24}$ ). Further away from closed shells, such enhancement of the electric radiation may be obscured by the unfavoured factor (cf. §4 A, i), but is expected to manifest itself in the occurrence of mixed multipole transitions, especially of the M1 + E2 type<sup>21</sup>.

For E1 transitions, there is no directly corresponding effect due to the lack of surface oscillations with  $\lambda = 1$ . However, the coupling of the particle motion to the collective dipole oscillations<sup>24a</sup>, which manifest themselves in the nuclear photo-effect, is expected to have an important influence on the particle transition probabilities. Since the coupling is such as to cause the neutron density to follow the motion of a proton and vice versa, it implies a reduction in the transition probabilities.

## B. COLLECTIVE $\gamma$ -TRANSITIONS

### i) *Phonon Transitions*

The radiation emitted by free oscillations of the nuclear shape is of electric multipole type of the same multipole order,  $\lambda$ , as the nuclear deformation. For the decay of a one-phonon to a non-phonon state, the reduced transition probability is given for the incompressible model by

$$B(E\lambda) = \frac{3\lambda}{8\pi} Z^2 e^2 R_0^{2\lambda} \frac{\hbar^2}{AMR_0^2} \frac{1}{\hbar\omega_\lambda}. \quad (12)$$

The cooperative nature of such a transition, as expressed in the appearance of the factor  $Z^2$  in (12), in general leads to a much faster decay than for a corresponding particle transition.

### ii) *Vibrational Transitions*

For the strongly deformed nuclei, the collective transitions are of vibrational and rotational type. While the E2 transition probabilities of

<sup>22a</sup> J. Thirion and V. L. Telegdi, Phys. Rev. 92, (1953) 1253.

<sup>22b</sup> D. E. Alburger and M. H. L. Pryce, in press.

<sup>23</sup> S. Devons, reported at the Glasgow Conference, July 1954.

<sup>24</sup> L. G. Elliott, R. L. Graham, J. Walker and J. L. Wolfson, Phys. Rev. 93, (1954) 356.

<sup>24a</sup> M. Goldhaber and E. Teller, Phys. Rev. 74, (1948) 1046; H. Steinwedel and J. H. D. Jensen, Z. f. Naturforsch. 5a, (1950) 413.

the latter depend on the static  $\lambda=2$  deformation and become much larger than (12), those of the former remain of order (12). For the transition from a member of the  $n_\beta=1$  rotational band with spin  $I_i$  to a member of the  $n_\beta=0$  band with spin  $I_f$ , one has

$$B(E2; I_i \rightarrow I_f) = B(E2)_{\text{ph}} \langle I_i 2 K 0 | I_f 2 I_f K \rangle^2, \quad (13)$$

where  $B(E2)_{\text{ph}}$  is the reduced transition probability for the  $\lambda=2$  phonon transition (12). For the corresponding transitions from  $n_\gamma=1$ ,  $K_i=K \pm 2$  to  $n_\gamma=0$ ,  $K_f=K$ , one has

$$B(E2; I_i \rightarrow I_f) = B(E2)_{\text{ph}} \begin{cases} \langle I_i 2, K \pm 2, \mp 2 | I_f 2 I_f K \rangle^2 & K \neq 0 \\ 2 \langle I_i 2, 2, -2 | I_f 2 I_f 0 \rangle^2 & K = 0. \end{cases} \quad (14)$$

For vibrational transitions, magnetic radiation is forbidden, even though  $\Delta I$  may be zero or one.

### iii) Rotational Transitions

Transition probabilities within a rotational band may be related to the static electric and magnetic moments of the nuclear states. Thus, for E2 transitions from a rotational state with  $(I_i K)$  to a state with  $(I_f K)$ , the reduced transition probability becomes

$$B(E2; I_i \rightarrow I_f) = \frac{5}{16\pi} e^2 Q_0^2 \langle I_i 2 K 0 | I_f 2 I_f K \rangle^2 \quad (15)$$

in terms of the intrinsic quadrupole moment of the nuclear shape (cf. § 3 B, above).

For the ground state rotational band in an even-even nucleus, the consecutive states have  $\Delta I=2$ , and the transitions proceed by a cascade of pure E2 radiation. In an odd-A nucleus, however, the consecutive states have  $\Delta I=1$ , and the radiation will in general be a mixture of M1 + E2. The reduced transition probability for the M1 radiation can be expressed in terms of the gyromagnetic ratios  $g_\Omega$  and  $g_R$ , appearing in the static magnetic moments (cf. (9)),

$$B(M1; I_i \rightarrow I_f) = \frac{3}{4\pi} \left( \frac{e\hbar}{2Mc} \right)^2 (g_\Omega - g_R)^2 \Omega^2 \langle I_i 1 K 0 | I_f 1 I_f K \rangle^2. \quad (16)$$

In the special case of rotational states with  $\Omega=K=1/2$ , which possess an anomalous spectrum (cf. (5)), there are correction terms to (16), which depend on the details of the particle structure. However, (15) remains valid even in this case.

For pure particle transitions of such low energy, M1 radiation would strongly predominate over the E2; however, the strong enhancement of the E2 radiation in these collective transitions may lead to an appreciable

admixture of E2 in the  $\Delta I = 1$  transitions, and to frequent  $\Delta I = 2$  cross-over transitions.

## § 5. Systematics of Nuclear Rotational Structure

In recent years, a considerable body of data has revealed the existence of a rotational structure in the nuclear excitation spectrum<sup>25, 26</sup>. The simple rotational motion is characteristic of the strongly deformed nuclei

<sup>25</sup> A. Bohr and B. R. Mottelson, *Physica* 18, (1952) 1066; *Phys. Rev.* 89, (1953) 316; *Phys. Rev.* 90, (1953) 717; BM, § VIc, ii; K. W. Ford, *Phys. Rev.* 90, (1953) 29; F. Asaro and I. Perlman, *Phys. Rev.* 91, (1953) 763; *Phys. Rev.* 93, (1954) 1423; *Phys. Rev.* 94, (1954) 381; F. Asaro, S. G. Thompson and I. Perlman, *Phys. Rev.* 92, (1953) 694; F. Asaro, F. Stephens and I. Perlman, *Phys. Rev.* 92, (1953) 1495; T. Huus and Č. Zupančič, *Dan. Mat. Fys. Medd.* 28, (1953) no. 1; J. O. Newton and B. Rose, *Phil. Mag.* 45, (1954) 58; cf. also J. O. Newton, *Nuclear Properties of Very Heavy Elements*, contribution to "Progress in Nuclear Physics", vol. 4, ed. by O. R. Frisch; J. O. Rasmussen, *Ark. f. Fysik* 7, (1954) 185; R. S. Caird and A. C. G. Mitchell, *Phys. Rev.* 94, (1954) 412; J. W. Mihelich, G. Scharff-Goldhaber and M. McKeown, *Phys. Rev.* 94, (1954) 794 (A); J. R. Arnold, *Phys. Rev.* 93, (1954) 743; T. Huus and J. Bjerregård, *Phys. Rev.* 92, (1953) 1579; G. Temmer and N. P. Heydenburg, *Phys. Rev.* 93, (1954) 351 and 906; E. L. Church and M. Goldhaber, *Phys. Rev.* 95, (1954) 626 (A); C. L. McClelland, H. Mark and C. Goodman, *Phys. Rev.* 93, (1954) 904; G. M. Temmer and N. P. Heydenburg, *Phys. Rev.* 94, (1954) 1399; J. T. Eisinger, C. F. Cook and C. M. Class, *Phys. Rev.* 94, (1954) 735 and *Phys. Rev.*, in press; P. Falk-Vairant, J. Teillac, G. Valladas and P. Benoist, *C. R.* 238, (1954) 1409 and *C. R.*, in press; J. Milsted, S. Rosenblum and M. Valadares, *C. R.* 239, (1954) 259; F. Boehm, P. Marmier and J. W. M. DuMond, *Phys. Rev.* 95, (1954) 864; T. Huus and A. Lundén, *Phil. Mag.*, 45, (1954) 966; R. F. Christy, *B.A.P.S.* 29, (1954) no. 6 (L6); N. P. Heydenburg and G. M. Temmer, *Phys. Rev.* 95, (1954) 861; W. I. Goldburg and R. M. Williamson, *Phys. Rev.* 95, (1954) 767.

For a recent discussion of the theory and a review of the experimental data, cf. A. Bohr, *Rotational States of Atomic Nuclei*, Ejnar Munksgaard, Copenhagen (1954).

<sup>26</sup> The excitation spectra for even-even nuclei have also been considered in terms of the recoupling of particles outside of closed shells:

H. Horie, M. Umezawa, Y. Yamaguchi and S. Yoshida, *Progr. Theor. Phys.* 6, (1951) 254; B. H. Flowers, *Phys. Rev.* 86, (1952) 254; P. Preiswerk and P. Stähelin, *Nuovo Cimento* 10, (1953) 1219; A. de Shalit and M. Goldhaber, *Phys. Rev.* 92, (1953) 1211.

It has been shown that, for an even group of equivalent particles, two-body forces of the expected character lead to a ground state with spin 0 and a first excited state of spin 2. This result of the particle couplings may play an important role in influencing the spins in the regions of closed shells. A similar analysis of excitation spectra further away from closed shells would require the inclusion of configuration mixings (cf. de Shalit and Goldhaber, loc. cit.). Moreover, the necessity of including excitations of the closed shell core is implied by the magnitude of the observed quadrupole moments. The existence of spectra of the rotational type corresponds to the possibility of obtaining simple solutions to the equations of motion by introducing appropriate collective coordinates.



(cf. § 2B) and therefore especially well defined in regions far removed from closed shells. The rotational structure has been studied most extensively for the heavy elements ( $A > 140$ ), where the distances between shell-closings are great and where the deformations are also known to be especially large.

## A. EVEN-EVEN NUCLEI

### i) *Energies of First Excited States*

In even-even nuclei, the lowest particle configuration has  $\Omega = 0$  and the rotational spectrum is given by (3). This sequence of states is in accordance with the well-known empirical rule that even-even nuclei have a ground state spin  $I_0 = 0$ , and also with the more recent findings that the first excited state has  $I = 2$  and even parity<sup>27</sup>. Only very few exceptions to this latter rule have been encountered, all referring to essentially spherical nuclei, for which (3) does not apply.

The rotational excitation energy (1) with the moment of inertia (2) is expected to decrease as one moves away from closed shells, corresponding to the increasing nuclear deformation produced by the increasing number of nucleons outside of closed shells.

The observed energies of the first excited states of even-even nuclei are found to exhibit a marked regularity just corresponding to the expected trends. For the heavy elements ( $A > 140$ ), the data are shown in Fig. 5. In the regions away from closed shells, the energies become very small, but inversions of the  $0+$  and  $2+$  levels are never encountered, attesting to the intimate relation between the two levels. In these regions, especially for  $155 < A < 185$  and  $A > 225$ , the slowness of the rotational motion as compared with other frequencies in the system, makes the rotational description very accurate. Thus, correction terms of the type (6) contribute less than one per cent of the rotational energy.

From the observed energies, one can estimate the moment of inertia,  $J$ , which is a measure of the deformation (cf. (2)). The intrinsic quadrupole moment,  $Q_0$ , also depending on the deformation (cf. (7)), should exhibit similar trends as  $J$ , and a correlation has been observed<sup>28</sup>. The quantitative comparison of  $J$  and  $Q_0$  shows systematic deviations from the

<sup>27</sup> M. Goldhaber and A. W. Sunyar, Phys. Rev. 83, (1951) 906; H. Horie, M. Umezawa, Y. Yamaguchi and S. Yoshida, Progr. Theor. Phys. 6 (1951) 254; P. Preiswerk and P. Stähelin, Helv. Phys. Acta 24, (1951) 623; Physica 18, (1952) 1118; Nuovo Cimento 10, (1953) 1219; G. Scharff-Goldhaber, Physica 18, (1952) 1105; Phys. Rev. 90, (1953) 587; F. Asaro and I. Perlman, Phys. Rev. 87, (1952) 393; S. Rosenblum and M. Valadares, C. R. 235, (1952) 711; A. H. Wapstra, Physica 18, (1952) 799.

<sup>28</sup> K. W. Ford, Phys. Rev. 90, (1953) 29.

relationship given by (2) and (7), which may be associated with the occurrence of higher multipole moments in the nuclear shape or the inadequacy of the incompressible model. Especially significant as a test of the former possibility would be the observation in the Coulomb excitation process of collective transitions of higher order (e.g., E4 transitions).

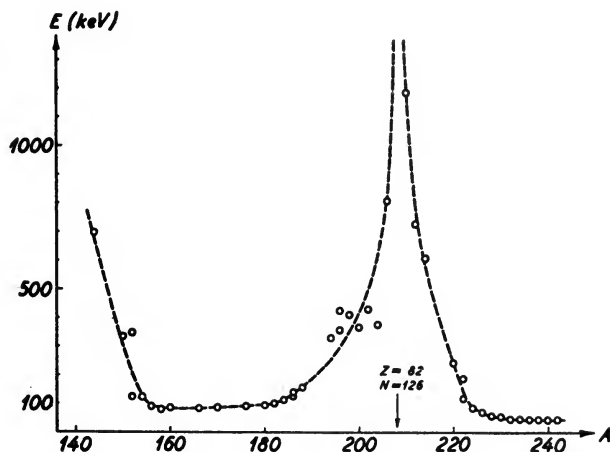


Fig. 5. Energies of first excited states in even-even nuclei with  $A > 140$

The data are taken from the compilations of G. Scharff-Goldhaber (ref. 27), P. Stähelin and P. Preiswerk (ref. 27), J. M. Hollander, I. Perlman and G. T. Seaborg, *Rev. Mod. Phys.* **25**, (1953) 469. For  $W^{182, 184}$ , cf. T. Huus and J. Bjerregård, *Phys. Rev.* **92**, (1953) 1579 and C. McClelland, H. Mark and C. Goodman, *Phys. Rev.* **93**, (1954) 904; for  $Po^{210}$ , cf. Hoff, UCRL 2325, Berkeley (1953).

Similar curves, extending also to the region of lower  $A$ , have been given by G. Scharff-Goldhaber (loc. cit.) and by P. Stähelin and P. Preiswerk (loc. cit.).

With the approach to closed shells, the rotational motion becomes more rapid and distortion terms increase in importance. In the vicinity of closed shells, the excitation spectrum changes completely (cf. § 2A). In these regions, the first excited states may represent particle excitations, collective vibrations, or combinations of both, and the energy may vary less regularly. The nature of a particular state reveals itself especially in the transition probability to the ground state. Thus, the relatively long lifetime of the first excited state of  $Pb^{204}$ <sup>29</sup> identifies it as an excitation of the particle structure<sup>30</sup>. In these regions, and especially for nuclei possessing closed shells in both neutrons and protons, particle excitations may give

<sup>29</sup> Cf., e.g., V. E. Krohn and S. Raboy, *Phys. Rev.* **95**, (1954) 1354.

<sup>30</sup> The recent measurement of the magnetic moment of this state (H. Frauenfelder, J. S. Lawson Jr., and W. Jentschke, *Phys. Rev.* **93**, (1954) 1126) is in agreement with this interpretation.

rise to first excited states having spins other than  $2+$ , as are indeed observed for  ${}^8\text{O}^{16}$  <sup>31</sup>,  ${}^{32}\text{Ge}^{72}$  <sup>31</sup>, and  ${}^{82}\text{Pb}^{208}$  <sup>32</sup>.

Similar regularities in the energies of the first excited states of even-even nuclei, as illustrated in Fig. 5, have been found in the lighter nuclei. While, in the heavy elements, the energy is approximately a function of  $A$  only, due to the simultaneous filling of neutron and proton shells at  $\text{Pb}^{208}$ , the trends are somewhat more complicated in the lighter elements and must be considered a function of the two variables  $N$  and  $Z$  <sup>33</sup>. In the regions between closed shells, the energies often become rather small compared with the expected vibrational energies (cf. Fig. 1), and a rotational structure is expected, although not with as great accuracy as in the heavy elements.

## ii) Higher Rotational States

The analysis of the first excited states in even-even nuclei is confirmed by the observation of the expected higher members of the rotational sequence. Thus, in regions where the energy of the first excited state,  $E_2$ ,

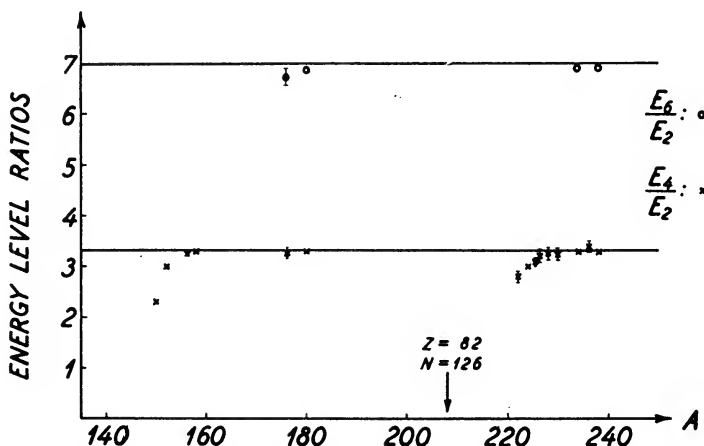


Fig. 6. Energy ratios for rotational states in even-even nuclei with  $A > 140$ . The horizontal lines give the theoretical ratios  $E_4:E_2$  and  $E_6:E_2$ , obtained from (1) for the limit of large deformations. The estimated uncertainties in the measured energy ratios are indicated in the figure; where no error is shown, the uncertainty is believed to be less than one per cent. For reference to the empirical data, cf. footnote 25. A figure similar to this was first given by F. Asaro and I. Perlman, Phys. Rev. 91, (1953) 763.

<sup>31</sup> Cf., e.g., G. Scharff-Goldhaber, Phys. Rev. 90, (1953) 587.

<sup>32</sup> L. G. Elliott, R. L. Graham, J. Walker and J. L. Wolfson, Phys. Rev. 93, (1954) 356.

<sup>33</sup> Cf. especially G. Scharff-Goldhaber, loc. cit., and P. Preiswerk and P. Stähelin, Nuovo Cimento 10, (1953) 1219.

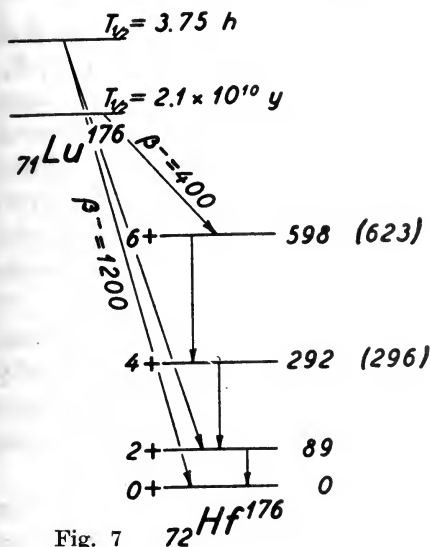


Fig. 7

 ${}^{72}\text{Hf}^{176}$ Fig. 7. Excitation of rotational states in the  $\beta$ -decay of  $\text{Lu}^{176}$ 

The data for the decay of the  $\text{Lu}^{176}$  ground state are taken from J. R. Arnold (ref. 25). The energies in parenthesis for the 4+ and 6+ rotational states in  $\text{Hf}^{176}$  are obtained from the energy of the 2+ state by means of (1). The deviations between the observed and calculated energies are of the order of the experimental uncertainty, but from the systematics of the rotational spectra in this region (cf. Fig. 8) one expects deviations of the order of -1 % and -2 % for the 4+ and 6+ states, resulting from the rotation-vibration interaction (cf. (6)). The 400 keV  $\beta$ -transition has a  $\log f_0 t$ -value of 18.5; the appreciable retardation, resulting from the  $K$ -forbiddenness (cf. § 4A,iii) of this transition may imply a second forbidden transition ( $\Delta I = 2$  or 3; no), despite the large  $f_0 t$ -value.

For the decay of the 3.75 h  $\text{Lu}^{176}$ , cf. M. Goldhaber and R. D. Hill, Rev. Mod. Phys. 24, (1952) 179. If the spin of this state is 1, one expects about 75 % of the transitions to go to the ground state and 25 % to the first excited state (cf. (11) and Fig. 4). A greater intensity of the branch to this latter state would indicate a 2- assignment for the  $\text{Lu}^{176}$  isomeric state and imply a weak  $\beta$ -transition to the 290 keV 4+ level in  $\text{Hf}^{176}$  (cf. the similar case of  $\text{Ho}^{166}$  discussed in Alaga et al. (ref. 21)).

The  $\gamma$ -decay of the 89 keV level in  $\text{Hf}^{176}$  (F. K. McGowan, Phys. Rev. 87, (1952) 542) is strongly enhanced (cf. Table I).

Fig. 8. Decay of 5.5 h isomeric state of  $\text{Hf}^{180}$ 

The  $\gamma$ -energies following the decay of 5.5 h  $\text{Hf}^{180}$  are taken from Mihelich et al. (ref. 25). The numbers in parenthesis are those obtained from (1) with the moment of inertia fitted to the first excited state. The observed minor deviations from these energies show a systematic trend and are of the sign and order of magnitude expected from the rotation-vibration interaction (6). The energies in square brackets are calculated rotational energies with the inclusion of the rotation-vibration effect. The vibrational energies in (6) have been adjusted to give the best fit to the observed levels, and are found to be about 3 MeV, in agreement with the expected order of magnitude (cf. Fig. 1).

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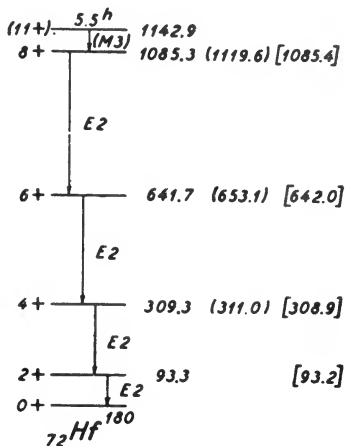


Fig. 8

 ${}^{72}\text{Hf}^{180}$

is very low, corresponding to a large deformation, one expects to find higher excited states with spins  $4+$ ,  $6+$ , etc. and energies  $E_4 = \frac{10}{3} E_2$ ,  $E_6 = 7 E_2$ , etc. (cf. (1) and (3)). States with these characteristics have been found to occur systematically in the expected regions, and the observed energy ratios are shown in Fig. 6. It is seen that, in regions far from closed shells, the rotational energy ratios are rather accurately realized. Moreover, the small deviations observed show a systematic trend increasing with the approach to closed shells and have the sign and order of magnitude expected from the rotation-vibration effect (cf. (6)).

The rotational states are populated in a variety of nuclear processes; in particular, the states up to  $6+$  have been found to occur systematically in the  $\alpha$ -spectra of the heavy elements<sup>34</sup>. High rotational states may also be excited in the  $\beta$ - or  $\gamma$ -decay of a nuclear state with high spin value (cf. Figs. 7 and 8).

The expected occurrence of rotational structures in lighter elements in regions removed from closed shells finds some support in the identification of  $4+$  states with energies approximating those expected. (Cf., e.g.,  $\text{Mg}^{24}$ ,  $\text{Fe}^{56}$ ,  $\text{Xe}^{130}$ ).

Apart from the rotational band associated with the ground state, one expects additional bands for each intrinsic excitation of the nucleus (particle or vibrational excitation). For examples of such excited rotational bands, cf., e.g., the level structure of  $\text{W}^{182\ 34a}$ .

### iii) Transition Probabilities

A number of E2 transition probabilities have been determined for the transition between the ground state and the first excited,  $2+$ , state in even-even nuclei. The data are listed in Table I and are obtained partly from lifetime determinations and partly from Coulomb excitation cross-sections.

The intensity of the transitions is indicated by the quantity,  $F$ , in column 3, which represents the ratio of the observed transition probability to that expected for a corresponding particle transition. The largeness

<sup>34</sup> F. Asaro and I. Perlman, Phys. Rev. 91, (1953) 763.

<sup>34a</sup> G. Alaga, K. Alder, A. Bohr, and B. R. Mottelson, Dan. Mat. Fys. Medd., in press.

#### *Continuation caption Fig. 8*

The long-lived 57 keV transition has been tentatively classified as M3 on the basis of its lifetime and conversion data (Mihelich et al., loc. cit.). The high degree of K-forbiddenness (cf. § 4A, iii) of this transition implies, however, a strong retardation, which may suggest an M2 transition, having an unfavoured factor of  $F \sim 10^{-2}$ ; the conversion data appear also to be consistent with such a classification (G. Scharff-Goldhaber, private communication).

of the  $F$ -factors clearly exhibits the collective nature of the transitions.

From the observed transition probability the intrinsic quadrupole moment can be obtained from (15). The values thus derived are listed in column 4 and show the expected trend, decreasing regularly with the approach to closed-shell configurations.

The nuclear quadrupole moments obtained from E2 transition probabilities may be compared with the static moments obtained from spectroscopic measurements in neighbouring odd- $A$  nuclei (cf. column 5). While the two quadrupole moment determinations give values of the same order of magnitude, the latter are somewhat larger than the former, and also fluctuate more. The difference, however, may not be significant in view of the considerable uncertainty that attaches to most of the spectroscopic determinations.

TABLE I  
ROTATIONAL TRANSITIONS IN EVEN-EVEN NUCLEI

Nucleus	$E(\text{keV})$	$F$	$Q_0 (10^{-24} \text{ cm}^2)$	
			transition	spectroscopic
$^{60}\text{Nd}^{148}$	300	10	2.6	$3(^{63}\text{Eu}^{153})$
$^{60}\text{Nd}^{150}$	131	30	4	
$^{62}\text{Sm}^{152}$	122	60	6	$7(^{63}\text{Eu}^{153})$
$^{66}\text{Dy}^{160}$	85	80	7	
$^{68}\text{Er}^{168}$	80	100	8	$\sim 20(^{68}\text{Er}^{167})$
$^{70}\text{Yb}^{170}$	84	100	8	$11(^{70}\text{Yb}^{173})$
$^{72}\text{Hf}^{176}$	89	90	8	$14(^{71}\text{Lu}^{175})$
$^{72}\text{Hf}^{180}$	93	80	7	
$^{74}\text{W}^{182}$	100	80	7	$14(^{73}\text{Ta}^{181})$
$^{74}\text{W}^{184}$	113	80	7	
$^{74}\text{W}^{186}$	124	80	7	$8(^{75}\text{Re}^{185})$
$^{76}\text{Os}^{186}$	137	50	6	$7(^{75}\text{Re}^{187})$
$^{76}\text{Os}^{188}$	155	40	5	$\sim 10(^{76}\text{Os}^{189})$
$^{90}\text{Th}^{232}$	50	170	9	$\sim 20(^{92}\text{U}^{235})$
$^{92}\text{U}^{238}$	44	110	8	

For references to the empirical data, cf. A. Bohr (ref. 24, table I) and A. W. Sunyar, Phys. Rev. 95, (1954) 627 (A); B. E. Simmons, D. M. van Patter, K. F. Famularo, and R. V. Stuart, Phys. Rev., in press; N. P. Heydenburg and G. Temmer, Phys. Rev. 93, (1954), 351 and 906, and private communication; K. L. V. Sluis and J. R. McNally, J. Opt. Soc. Am. 44, (1954), 87. The  $F$ -factor represents the ratio of the observed transition probability to that estimated for a corresponding particle transition. The  $Q_0$ -values in column four are intrinsic nuclear quadrupole moments calculated from the observed transition probabilities (cf. (15)), while those in column five are derived from spectroscopic data in neighbouring odd- $A$  nuclei by means of (8).

## B. ODD-A NUCLEI

Recently, evidence has become available, principally from the Coulomb excitation studies, for the systematic occurrence of rotational spectra in odd- $A$  isotopes in the region of strongly deformed nuclei. These states are characterized, as are rotational states in even- $A$  isotopes, by the regularities in their energies and by their enhanced E2 transition probabilities.

In Table II are listed the energy levels in odd- $A$  nuclei in the region  $150 < A < 190$ , which have been identified by means of their large Coulomb excitation cross-sections as rotational excitations. The energy of the first rotational state,  $E^{(1)}$ , is listed in column 3 and that of the second rotational member,  $E^{(2)}$ , appears in column 4. The value  $E^{(2)}$  calculated from (1) and (3) by using the moment of inertia implied by  $E^{(1)}$  is given in parenthesis. Columns 5 and 6 compare the observed moment of inertia with that of neighbouring even-even nuclei. The quantity listed,  $3\hbar^2/J$ , represents

Fig. 9. Rotational Spectrum of  $\text{Ta}^{181}$

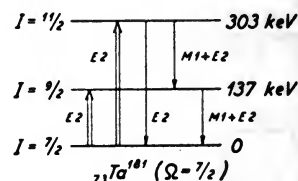
The best studied rotational spectrum of an odd- $A$  nucleus is that of  ${}_{73}\text{Ta}^{181}$ . By means of the Coulomb excitation process (double arrows in the figure), the two first rotational states have been excited. The ground state spin of  $\text{Ta}^{181}$  is known from spectroscopic data to be  $I = 7/2$  (cf., e.g., J. E. Mack, Rev. Mod.

Phys. 22, (1950) 64) and the first and second rotational states are thus expected to have  $I = 9/2$  and  $11/2$ , respectively. These spin assignments have been confirmed by angular correlation measurements (F. K. McGowan, Phys. Rev. 93, (1954) 471; J. T. Eisinger, C. Cook and C. M. Class, Phys. Rev., 94, (1954) 735.

The energies and radiative decays of the rotational states are shown in the figure. The study of these transitions by the Coulomb excitation process (T. Huus and Č. Zupančič, Dan. Mat. Fys. Medd. 28, no. 1 (1953); T. Huus and J. Bjerregård, Phys. Rev. 92, (1953) 1579), has provided a number of tests of the theory of rotational states and yielded various information on the intrinsic nuclear properties. We list the quantities on which empirical evidence is available.

- 1)  $E_{11/2} : E_{9/2} = 2.21 \pm 0.02$  (theor. values 2.22; cf. (1)).
- 2)  $E_{9/2} = 137$  keV (theor. estimate, using  $J$  of  $\text{Hf}^{180}$ , gives  $E_{9/2} \approx 140$  keV; cf. Fig. 8).
- 3)  $B(E2; 7/2 \rightarrow 9/2) : B(E2; 7/2 \rightarrow 11/2) = 3.8 \pm 0.7$  (theor. value 3.89; cf. (11)).
- 4)  $B(E2; 7/2 \rightarrow 9/2)$  yields  $|Q_0| \approx 7 \times 10^{-24}$  cm<sup>2</sup> (cf. (15)), in accordance with the deformation expected from the data in Table I; spectroscopic determinations (T. Schmidt, ZS. f. Phys. 121, (1943) 63; B. M. Brown and D. H. Tomboulia, Phys. Rev. 88, 1050 and 91, 1580, 1952) have led to an average  $Q_0$  value of about  $14 \times 10^{-24}$  cm<sup>2</sup>, but with a rather large uncertainty.
- 5)  $B(M1; 11/2 \rightarrow 9/2) : B(M1; 9/2 \rightarrow 7/2) = 2.5 \pm 1.0$  \* (theor. value 1.53; cf. (11)).
- 6)  $B(M1; 11/2 \rightarrow 9/2)$  together with  $\mu = 2.1$  (B. M. Brown and D. H. Tomboulia, loc. cit.) yield  $g_D \approx 0.75$  and  $g_R \approx 0.25$ ; cf. (9) and (16).

\* A recent reinterpretation of the angular correlation data of McGowan (loc. cit.) yields for this ratio the smaller value of about 0.7, but again with a considerable uncertainty (Alaga et al., ref. 21).



the excitation energy of the  $(2+)$  rotational state in an even-even nucleus.

The observed energy ratios,  $E^{(2)} : E^{(1)}$ , agree well with the rotational value. The comparison between the moments of inertia in odd- $A$  and even-even elements seems to indicate a tendency for the former to somewhat exceed the latter, which may imply larger deformations in odd- $A$  nuclei. The largest difference occurs for  $\text{Eu}^{153}$  with neutron number  $N=90$ , in which region the nuclear deformations are known to vary rapidly (cf. the ratio of quadrupole moments of  $\text{Eu}^{151}$  and  $\text{Eu}^{153}$ , and the excitation energies in the even-even isotopes of Nd, Sm, and Gd).

The ground states of  $\text{Tm}^{169}$  and  $\text{W}^{183}$  have  $\Omega=1/2$  and therefore the anomalous spectrum (5). The observed levels determine the moment of inertia as well as the coefficient  $a$  (given in column 5 of Table II). The expected higher members of these rotational spectra appear to agree with observed  $\gamma$ -transitions following  $K$ -capture in  $\text{Yb}^{169\ 35}$  and the  $\beta$ -decay of  $\text{Ta}^{183\ 36}$ .

The E2 transition probabilities for the levels listed in Table II, as found from the Coulomb excitation cross-sections, exceed those estimated for single-particle transitions by factors of the order of one hundred, and the derived values of  $Q_0$  (cf. (15)) are comparable with those in neighbouring even-even isotopes (cf. Table I).

The radiative decay of the rotational states in odd- $A$  nuclei also contains M1 radiation, the intensity of which can be related to the gyromagnetic ratios of the particle and collective motion (cf. (16)). The best studied case is that of  $\text{Ta}^{181}$ , the analysis of which is discussed in Fig. 9.

In the region well beyond  $\text{Pb}^{208}$ , rotational bands are again identified in the spectra of the odd- $A^{37}$  as well as of the even-even elements. The  $\alpha$ -decay process provides an especially convenient means of studying such sequences in this region of elements. The simple coupling scheme for the particle motion in these strongly deformed nuclei (cf. § 2B) also makes possible an interpretation of many features of the observed intensity relations in the  $\alpha$ -decay fine structure  $^{38}$ .

The study of elements lighter than those included in Table II, by means of the Coulomb excitation process, has also revealed the existence of

<sup>35</sup> D. S. Martin Jr., E. N. Jensen, F. J. Hughes and R. T. Nichols, *Phys. Rev.* **82**, (1951) 579.

<sup>36</sup> J. W. M. DuMond, H. C. Hoyt, P. E. Marmier and J. J. Murray, *Phys. Rev.* **92**, (1954) 202; J. J. Murray, P. Snelgrove, P. E. Marmier and J. W. M. DuMond, *B.A.P.S.* **29**, (1954) no. 6 (L5); R. F. Christy, *B.A.P.S.* **29**, (1954) no. 6 (L6).

<sup>37</sup> J. O. Rasmussen, *Ark. f. Fysik* **7**, (1954) 185; F. Asaro and I. Perlman, *Phys. Rev.* **93**, (1954) 1423; J. Milsted, S. Rosenblum and M. Valadares, *C. R.* **239**, (1954) 259.

<sup>38</sup> A. Bohr, P. O. Fröman and B. R. Mottelson, *Dan. Mat. Fys. Medd.*, in press.



rotational excitations in other regions of the periodic table. Interesting examples are afforded by the isotopes of Ag and Rh, which exhibit the pattern expected for  $\Omega=1/2$  nuclei<sup>39</sup>.

In still lighter elements, low lying collective excitations have been identified by means of their strongly enhanced E2 transition probabilities<sup>40</sup>. In some of these latter cases, the rotational motion appears to be coupled to certain intrinsic modes with a resulting more complex excitation spectrum.

<sup>39</sup> N. P. Heydenburg and G. Temmer, Phys. Rev. **95**, (1954) 861; T. Huus and A. Lundén, Phil. Mag., **45**, 966 (1954)

<sup>40</sup> N. P. Heydenburg and G. Temmer, Phys. Rev., in press.

TABLE II  
ROTATIONAL STATES IN ODD-*A* NUCLEI

Nucleus	$I_0$	$E^{(1)}$	$E^{(2)}$	$3\hbar^2/J$	$3\hbar^2/J$ (even-even)	References
<sup>63</sup> Eu <sup>153</sup>	$5/2$	84	194 (192)	72	122	(2), (3), (4)
<sup>65</sup> Tb <sup>159</sup>	$3/2$	57	138 (137)	69	79	(2), (4)
<sup>66</sup> Dy <sup>161</sup> <sup>163</sup>	$7/2$	76	166 (169)	50	85	(2), (3), (7)
<sup>67</sup> Ho <sup>165</sup>	$7/2$	93	205 (207)	63	76	(2), (3)
<sup>68</sup> Er <sup>167</sup>	$7/2$	79	174 (175)	53	80	(2), (3), (8)
<sup>69</sup> Tm <sup>169</sup>	$1/2$	10	120	76	80	(2), (4)
				$(a = -0.74)$		
<sup>70</sup> Yb <sup>173</sup>	$5/2$	81	180 (185)	69		(2), (3)
<sup>71</sup> Lu <sup>175</sup>	$7/2$	113	251 (251)	75	82	(2), (3), (4)
<sup>72</sup> Hf <sup>177</sup> <sup>179</sup>	$(7/2)$	112	240 (249)	75	89	(2), (3), (4), (9)
<sup>73</sup> Ta <sup>181</sup>	$7/2$	137	303 (304)	91	93	(1)
<sup>74</sup> W <sup>183</sup>	$1/2$	46	99	78	100	(4), (6)
				$(a = +0.19)$		
<sup>75</sup> Re <sup>185</sup>	$5/2$	123	(281)	105	113	(2), (4)
<sup>75</sup> Re <sup>187</sup>	$5/2$	134	(306)	114	123	(2), (4)
<sup>77</sup> Ir <sup>191</sup>	$3/2$	129	(310)	155		(2), (4), (10)
<sup>77</sup> Ir <sup>193</sup>	$3/2$	138	(331)	166		(2), (4), (10)
<sup>79</sup> Au <sup>197</sup>	$3/2$	279	555 (670)	335	358	(2), (5)

The Table lists the states in the region  $150 < A < 200$ , identified from their large Coulomb excitation cross-sections, as rotational excitations. The two first rotational states are denoted by  $E^{(1)}$  and  $E^{(2)}$ , and the calculated value of  $E^{(2)}$  obtained from (1) is listed in parenthesis; all energies are in keV. The spin assignments of the rotational band are based on the ground state spins (cf. (4)) listed in column two; they are taken from J. E. Mack, Rev. Mod. Phys. **22**, (1950) 64, except where additional references are listed. The two last columns compare the moments of inertia,  $J$ , with those of the neighbouring even-even nuclei, lacking the odd particle. The quantity listed,  $3\hbar^2/J$ , represents the excitation energy of the  $(2+)$  rotational state in an even-even nucleus (cf. Fig. 5). Tm<sup>169</sup> and W<sup>183</sup> with  $I_0=1/2$  have the anomalous spectrum

characteristic of a nucleus with  $\Omega=1/2$ . From the energies of the first two rotational states, one may determine the moment of inertia as well as the parameter,  $a$  (cf. (5)).

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