Appendix 14B
The moment of inertia

It was mentioned in chapter 11 (fig. 11.2) that, because of the pairing correlations, the observed nuclear moment of inertia is much smaller than the rigid body value. In this appendix, we will derive the corresponding formula. The derivations are made within the cranking model in perturbation theory with the rotational frequency, ω, as the small parameter (Inglis, 1954).

Consider the cranking Hamiltonian for rotation around the 1-axis (chapter 12):

$$H^{\omega} = H_0 - \hbar\omega I_1$$

The ground state energy is given by E_0 and the corresponding (many-particle) wave function is $|\Psi_0\rangle$:

$$H_0 |\Psi_0\rangle = E_0 |\Psi_0\rangle$$

For classical rotation, the rotational energy is given by

$$E - E_0 = \frac{1}{2}\mathcal{J}\omega^2 = \frac{1}{2\mathcal{J}}I^2$$

which formula thus defines the moment of inertia, $\mathcal{J}$.

In first order perturbation theory, the cranking wave function is given by

$$|\Psi^{\omega}\rangle = |\Psi_0\rangle - \hbar\omega \sum_{\Psi' \neq \Psi_0} |\Psi'\rangle \frac{\langle \Psi' | I_1 | \Psi_0 \rangle}{E_0 - E'}$$

where Ψ' is a part of a complete set of wave functions such that $\langle \Psi' | I_1 | \Psi_0 \rangle \neq 0$. Furthermore,

$$H_0 |\Psi'\rangle = E' |\Psi'\rangle$$

For the non-rotating ground state, it is obvious that $\langle \Psi_0 | I_1 | \Psi_0 \rangle = 0$. Thus,

to lowest order in ω

$$\langle \Psi^{\omega} | I_1 | \Psi^{\omega} \rangle = 2\hbar\omega \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

Using second order perturbation theory, the energy in the rotating system E^{ω} is obtained as

$$E^{\omega} = E_0 - (\hbar\omega)^2 \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

The energy E in the laboratory system is calculated as the expectation value of $H_0 = H^{\omega} + \hbar\omega I_1$;

$$E = E_0 + (\hbar\omega)^2 \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

leading to the following formula for the moment of inertia:

$$\mathscr{J} = 2\hbar^2 \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

The same formula is obtained if the relation

$$I = \mathscr{J}\omega/\hbar$$

is compared with the expectation value of I_1 derived above.

For a pure single-particle configuration (with no pairing correlation), the ground state wave function is given by

$$\Psi_0 = \prod_{\nu} a_{\nu}^{+} |0\rangle$$

where the product is over all occupied orbitals (ν as well as $\bar{\nu}$). In the second-quantisation formalism, the angular momentum operator is written as

$$I_1 = \sum_{\nu\nu'} \langle \nu | j_1 | \nu' \rangle \, a_{\nu'}^{+} a_{\nu}$$

The only states Ψ' that have $\langle \Psi' | I_1 | \Psi_0 \rangle \neq 0$ are the one-particle-one-hole states

$$\Psi' = a_{\mu'}^{+} a_{\mu} \Psi_0$$

This state corresponds to one particle being excited from the occupied orbital μ to the empty orbital μ' and its excitation energy is given as

$$E' - E_0 = e_{\mu'} - e_{\mu}$$

where $e_{\mu'}$ and e_μ are the single-particle energies. It is now easy to calculate

$$\langle \Psi' | I_1 | \Psi_0 \rangle = \sum_{\nu\nu'} \langle \nu | j_1 | \nu' \rangle \left\langle \Psi_0 a_\mu^+ a_{\mu'} | a_{\nu'}^+ a_\nu | \Psi_0 \right\rangle = \langle \mu | j_1 | \mu' \rangle$$

and thus for the moment of inertia

$$\mathscr{J} = 2\hbar^2 \sum_{\mu}^{(\text{occ})} \sum_{\mu'}^{(\text{empty})} \frac{|\langle \mu | j_1 | \mu' \rangle|^2}{e_{\mu'} - e_\mu}$$

where thus the first sum runs over occupied orbitals only and the second over empty orbitals. This formula is valid for an arbitrary single-particle potential and can be applied for example to the rotating harmonic oscillator discussed in chapter 12. It is then possible to distinguish between the $\Delta N = 0$ terms (couplings of j_1 within an oscillator shell) and the $\Delta N = 2$ terms. If only the former are considered, we arrive, of course, at the $\omega = 0$ limit of the formula given in chapter 12.

We will now also incorporate pairing with the BCS function as the nuclear ground state (Belyaev, 1959; Migdal, 1959; Griffin and Rich, 1960; Nilsson and Prior, 1961). The derivation becomes somewhat more involved but gives a good insight into the calculation of general matrix elements within the quasiparticle formalism.

The first step is to express the I_1 operator in quasiparticle operators (cf. problem 14.15)

$$I_1 = \sum_{\nu\nu'} \langle \mu | j_1 | \nu' \rangle \left(U_\nu \alpha_\nu^+ + V_\nu \alpha_{\bar{\nu}} \right) \left(U_{\nu'} \alpha_{\nu'} + V_{\nu'} \alpha_{\bar{\nu}'}^+ \right)$$

As the BCS ground state is the zero-quasiparticle state ($\alpha \left| \tilde{0} \right\rangle = 0$), it is evident that it is only the $\alpha^+\alpha^+$ term ,

$$I_1 = \sum_{\nu\nu'} \langle \nu | j_1 | \nu' \rangle U_\nu V_{\nu'} \left(\alpha_\nu^+ \alpha_{\bar{\nu}'}^+ + \ldots \right)$$

which contributes in the moment of inertia formula. It furthermore follows that the Ψ' which should be considered in the moment of inertia formula are the two-quasiparticle states. In the present context it is convenient to write them as

$$\Psi' = \alpha_\mu^+ \alpha_{\bar{\mu}}^+ \left| \tilde{0} \right\rangle$$

The matrix element to evaluate is

$$\begin{aligned} \left\langle \Psi' | I_1 | \tilde{0} \right\rangle &= \left\langle \tilde{0} \right| \alpha_{\bar{\mu}'} \alpha_\mu \sum_{\nu\nu'} \langle \nu | j_1 | \nu' \rangle U_\nu V_{\nu'} \alpha_\nu^+ \alpha_{\bar{\nu}'}^+ \left| \tilde{0} \right\rangle \\ &= \langle \mu | j_1 | \mu' \rangle U_\mu V_{\mu'} - \langle \bar{\mu}' | j_1 | \bar{\mu} \rangle (-1) U_{\mu'} V_\mu \end{aligned}$$

where the (-1) in the last term depends on the sign change $\left|\bar{\nu}'\right\rangle = -|\nu'\rangle$. The two single-particle matrix elements in the formula are essentially the same. With the state $|\mu\rangle$ being given as an expansion over spherical states

$$|\mu\rangle = \sum c^{\mu}_{N\ell j\Omega}\,|N\ell j\Omega\rangle$$

the conjugate state is given as

$$|\bar{\mu}\rangle = \sum (-)^{j-\Omega} c^{\mu}_{N\ell j\Omega}\,|N\ell j-\Omega\rangle$$

The following relation is now derived

$$\begin{aligned}\langle\bar{\mu}\,|j_1|\,\bar{\mu}'\rangle &= \sum\sum c^{\mu}_{N\ell j\Omega} c^{\mu'}_{N'\ell' j'\Omega'} (-1)^{j+j'-\Omega-\Omega'}\,\langle N'\ell' j'-\Omega'\,|j_1|\,N\ell j-\Omega\rangle \\ &= -\langle\mu\,|j_1|\,\mu'\rangle\end{aligned}$$

We have used the fact that the matrix element of $j_1 = (j_+ + j_-)/2$ is different from zero only if $j' = j$ and $\Omega' = \Omega \pm 1$. Furthermore, the matrix element remains the same if the sign of both Ω and Ω' is changed. Thus, as j is half-integer, the equality follows. As j_1 is a Hermitian operator with real matrix elements it also follows that

$$\langle\mu\,|j_1|\,\mu'\rangle = \langle\mu'\,|j_1|\,\mu\rangle$$

and thus for the matrix element of I_1:

$$\left\langle\Psi'\,|I_1|\,\tilde{0}\right\rangle = \langle\mu\,|j_1|\,\mu'\rangle\,(U_\mu V_{\mu'} - V_\mu U_{\mu'})$$

We finally insert in the moment of inertia formula to obtain

$$\mathscr{J} = 2\hbar^2 \sum_{\mu>0,\mu'} \frac{|\langle\mu\,|j_1|\,\mu'\rangle|^2}{E_\mu + E_{\mu'}}\,(U_\mu V_{\mu'} - V_\mu U_{\mu'})^2$$

With a full summation over both μ and μ', we had counted each state of Ψ' twice. This could have been avoided by dividing by two but, for computational reasons, we prefer instead to exclude the conjugate states in the sum over μ ($\mu > 0$).

Let us consider the case where μ is far above the Fermi surface ($V_\mu \approx 0$, $U_\mu \approx 1, E_\mu = [(e_\mu - \lambda)^2 + \Delta^2]^{1/2} \approx (e_\mu - \lambda)$) while μ' is far below ($V_{\mu'} \approx 1$, $U_{\mu'} \approx 0, E_{\mu'} \approx (\lambda - e_{\mu'})$). This leads to a denominator of $(e_\mu - e_{\mu'})$ and a UV-factor of one in agreement with the non-paired case. Similarly, one concludes that when both μ and μ' are far above or far below the Fermi surface, the UV-factor is essentially zero, and the same contribution as in the non-paired case is again obtained. It is, however, the orbitals close to the Fermi surface that give the largest contributions to the moment of inertia. For such orbitals, being partly occupied and partly empty, the

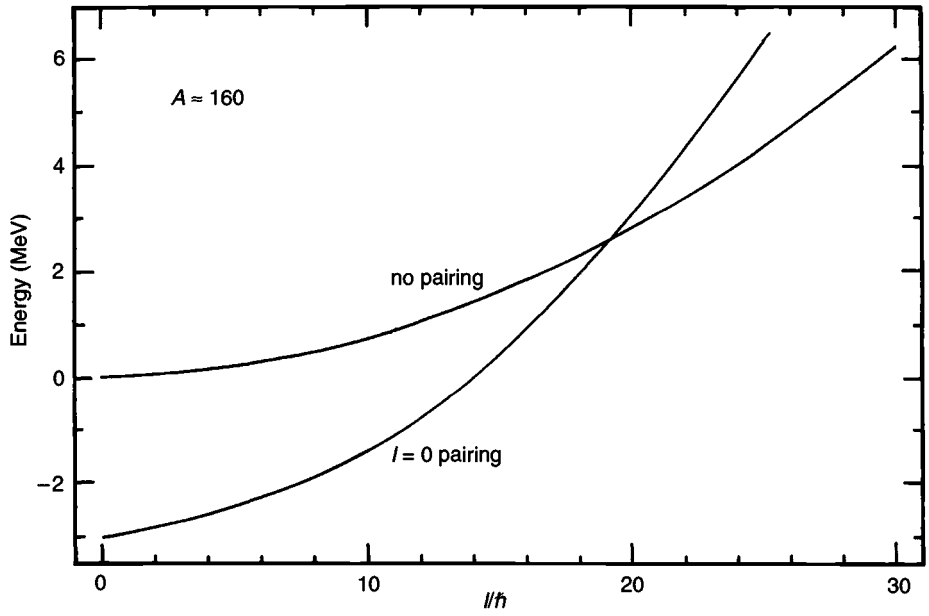

Fig. 14B.1. Schematic yrast lines for a rotational $A \approx 160$ nucleus. The 'no pairing' curve is drawn using the rigid body moment of inertia while the empirical ground state pairing energy is subtracted and the '$I = 0$ moment of inertia' is used for the '$I = 0$ pairing' curve. The figure suggests that, for high enough spin values, there will be no pairing energy in the yrast states.

UV-factor is essentially smaller than one. Furthermore, for such orbitals, the energy denominator becomes much larger than in the non-pairing case, $E_\mu + E_{\mu'} \approx 2\Delta >> e_\mu - e_{\mu'}$. This is understood from the large energy required to break up the strongly correlated BCS ground state. It is now easy to understand the general appearance of fig. 11.2 and the reduction of the moment of inertia compared with the rigid body value.

It was mentioned in chapter 12 that the pairing correlations are expected to disappear at high spins. This is referred to as Coriolis antipairing (CAP) or the Mottelson–Valatin (1960) effect and is easy to understand qualitatively. Let us take the nucleus ^{164}Er as an example. From fig. 14.5, we find that its total pairing correlation energy is around 3 MeV. Furthermore, the observed moment of inertia is $(2/\hbar^2)\mathcal{J} \approx 70$ MeV^{-1} (fig. 11.2). Assuming this moment of inertia for all spin, we get the energy versus spin as shown in fig. 14B.1. Similarly, we can draw the energy versus spin curve in the absence of pairing correlations in which case we expect to observe the rigid body moment of inertia, $(2/\hbar^2)\mathcal{J} \approx 150$ MeV^{-1}. The two curves intersect for $I \approx 20$, which is thus a very crude estimate of the spin at which the pairing

correlations should disappear. A more careful analysis will show that the disappearance is gradual and is accompanied by spin alignment giving rise to two-quasiparticle states, four-quasiparticle states, etc. (fig. 11.10). The process by which the pairing correlations become less important with increasing spin I is complicated and still not understood in its detail.