symbol invariant, while an odd permutation of rows or columns introduces a phase factor $(-1)^{\Sigma j}$, where Σj is the sum of all the nine angular momenta.

If one of the j values vanishes, the $9j$ symbol reduces to a $6j$ symbol,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_3 \\ j_6 & j_6 & 0 \end{Bmatrix} = (-1)^{j_2+j_3+j_4+j_6}((2j_3+1)(2j_6+1))^{-1/2} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} \tag{1A-26}$$

corresponding to the recoupling relation (see Eqs. (1A-22) and (1A-20))

$$\begin{aligned} &\langle (j_1 j_3)J', (j_2 j_4)J'; 0 \,|\, (j_1 j_2)J, (j_3 j_4)J; 0 \rangle \\ &= \langle (j_1 j_3)J', j_2\,; j_4 \,|\, (j_1 j_2)J, j_3\,; j_4 \rangle \\ &= (-1)^{j_2+j_3+J+J'}((2J+1)(2J'+1))^{1/2} \begin{Bmatrix} j_1 & j_2 & J \\ j_4 & j_3 & J' \end{Bmatrix} \end{aligned} \tag{1A-27}$$

For systems involving more than four angular momenta, the recoupling coefficients can be expressed in terms of higher invariants ($12j$, $15j$ symbols, and so on). These have properties analogous to those of $6j$ and $9j$ symbols, and can be expressed in terms of sums of products of coefficients of lower order. The different invariants also obey various recursion relations, which may be exploited in the numerical evaluation. (For references to tables of $6j$ and $9j$ symbols, see the directory by Way and Hurley, 1966. The simplest coefficients, in which one of the j values is small, are given in many textbooks.)

1A-4 Rotation Matrices. $\mathscr{D}$ Functions

The angular momentum operator is associated with the transformation of states under rotations of the coordinate system, in the manner indicated in Sec. 1-2a. We label the states by the total angular momentum I, the projection $M(=I_z)$, and additional quantum numbers α describing properties that are independent of the orientation of the coordinate frame (scalars).

If we introduce a new coordiante system $\mathscr{K}'$ obtained from the original system $\mathscr{K}$ by a rotation about a given axis (specified by the rotation vector χ), the transformation of states is given by (see Eq. (1-10))

$$\begin{aligned} |\alpha I M'\rangle_{\mathscr{K}'} &= \mathscr{R}(\chi)|\alpha I M'\rangle_{\mathscr{K}} \\ &= \sum_M |\alpha I M\rangle_{\mathscr{K}} \;\langle I M|\exp\{-i\chi\cdot\mathbf{I}\}|I M'\rangle \end{aligned} \tag{1A-28}$$

where $|\alpha I M'\rangle_{\mathscr{K}'}$ is the state with magnetic quantum number M', as viewed from $\mathscr{K}'$, that is, with $I_{z'} = M'$. For the states labeled $\mathscr{K}$, the magnetic quantum number refers to the eigenvalue of I_z. We have omitted the index $\mathscr{K}$ on the matrix element of $\mathscr{R}(\chi)$, since the value of the matrix element is independent of the coordinate system in which it is evaluated, as long as the M quantum numbers and the components of $\mathbf{I}$ in $\mathscr{R}(\chi)$ refer to the same set of axes. (The components of χ are in this connection to be regarded as a fixed set of numbers, equal to the components of the rotation vector in $\mathscr{K}$ or $\mathscr{K}'$ (the components are the same in these two systems).)

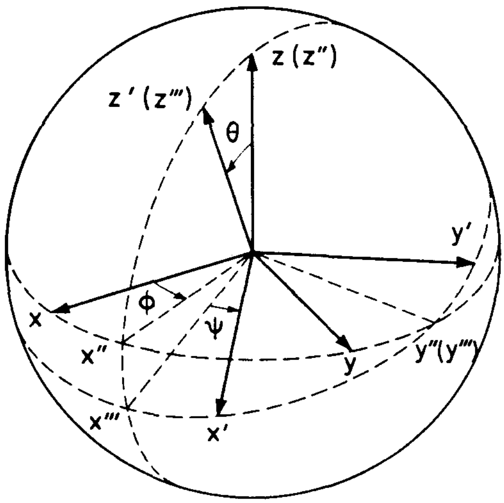

Figure 1A-1 Euler angles. The rotation from $\mathcal{K}(x, y, z)$ to $\mathcal{K}'(x', y', z')$ can be decomposed into three parts: a rotation by ϕ about the z axis to $\mathcal{K}''(x'', y'', z'')$, a rotation of θ about the new y axis (y'') to $\mathcal{K}'''(x''', y''', z''')$, and finally a rotation of ψ about the new z axis (z'''). It is seen that the Euler angles (ϕ, θ, ψ) are so defined that (θ, ϕ) are the polar angles of z' in $\mathcal{K}$, while $(\theta, \pi - \psi)$ are the polar angles of z in $\mathcal{K}'$. The Euler angles are, collectively, denoted by ω.

The transformation from $\mathcal{K}$ to $\mathcal{K}'$ can also be accomplished by a single rotation specified by the vector $\boldsymbol{\chi}$. The direction of $\boldsymbol{\chi}$ given by the polar angles $(\vartheta_\chi, \varphi_\chi)$ represents the axis of rotation, while the length χ is the angle of rotation, and we have the relationship

$$\cos\frac{\chi}{2} = \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}$$

$$\sin\frac{\chi}{2}\sin\vartheta_\chi = \sin\frac{\theta}{2}$$

$$\varphi_\chi = \frac{\phi-\psi}{2} + \frac{\pi}{2}$$

between the parameters χ, ϑ_χ, φ_χ and the Euler angles ϕ, θ, ψ.

The transformation from $\mathcal{K}$ to a system $\mathcal{K}'$ with arbitrary orientation with respect to $\mathcal{K}$ can be decomposed into three rotations of the type (1A-28), each about a coordinate axis. The angles of rotation correspond to the three Euler angles $\omega = (\phi, \theta, \psi)$ needed to specify the orientation (see Fig. 1A-1)

$$\mathcal{K} \xrightarrow[\phi(z)]{} \mathcal{K}'' \xrightarrow[\theta(y'')]{} \mathcal{K}''' \xrightarrow[\psi(z''')]{} \mathcal{K}' \qquad (1A\text{-}29)$$

Each step is characterized by the angle and (in parenthesis) the axis of rotation. We thus obtain

$$|IM'\rangle_{\mathscr{K}'} = \sum_{M'''} |IM'''\rangle_{\mathscr{K}'''}\langle IM'''|\exp\{-i\psi I_z\}|IM'\rangle$$

$$|IM'''\rangle_{\mathscr{K}'''} = \sum_{M''} |IM''\rangle_{\mathscr{K}''}\langle IM''|\exp\{-i\theta I_y\}|IM'''\rangle \tag{1A-30}$$

$$|IM''\rangle_{\mathscr{K}''} = \sum_{M} |IM\rangle_{\mathscr{K}}\langle IM|\exp\{-i\phi I_z\}|IM''\rangle$$

and, for the total transformation,

$$|IM'\rangle_{\mathscr{K}'} = \sum_{M} |IM\rangle_{\mathscr{K}}\langle IM|\mathscr{R}|IM'\rangle \tag{1A-31}$$

with

$$\mathscr{R}(\omega) = \exp\{-i\phi I_z\}\exp\{-i\theta I_y\}\exp\{-i\psi I_z\} \tag{1A-32}$$

expressing the rotation operator in terms of the Euler angles.

The matrix elements of $\mathscr{R}$ occurring in Eq. (1A-31) define the $\mathscr{D}$ functions,[6]

$$\begin{aligned}\mathscr{D}^I_{MM'}(\omega) &\equiv \langle IM|\mathscr{R}(\omega)|IM'\rangle^* \\ &= \langle IM'|\exp\{i\psi I_z\}\exp\{i\theta I_y\}\exp\{i\phi I_z\}|IM\rangle\end{aligned} \tag{1A-33}$$

Since $\mathscr{R}$ is a unitary operator, the relation (1A-31) can also be written

$$|\alpha IM\rangle_{\mathscr{K}} = \sum_{M'} \mathscr{D}^I_{MM'}(\omega)|\alpha IM'\rangle_{\mathscr{K}'} \tag{1A-34}$$

In the following we note some of the important properties of the $\mathscr{D}$ functions.

The matrix element in Eq. (1A-33) defining the $\mathscr{D}$ function is evaluated in the representation of the angular momentum matrices given in Sec. 1A-1. The dependence on the angles ϕ and ψ is especially simple

$$\mathscr{D}^I_{MM'}(\omega) = e^{iM\phi} d^I_{MM'}(\theta) e^{iM'\psi} \tag{1A-35}$$

and the θ dependence is given by

$$d^I_{MM'}(\theta) \equiv \langle IM'|\exp\{i\theta I_y\}|IM\rangle \tag{1A-36}$$

The d function is real, since I_y is a purely imaginary matrix, and has the symmetries

$$\begin{aligned} d^I_{MM'}(\theta) &= (-1)^{M-M'} d^I_{M'M}(\theta) \\ &= (-1)^{M-M'} d^I_{-M,-M'}(\theta) \\ &= d^I_{M'M}(-\theta) \end{aligned} \tag{1A-37}$$

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[6] We follow here a definition of the $\mathscr{D}$ functions that is extensively employed in nuclear physics problems, in particular in the description of rotational wave functions. The $\mathscr{D}$ function employed by Rose (1957) is the complex conjugate of that defined by Eq. (1A-33). The convention of Rose has been used, for example, by Jacob and Wick (1959) in their discussion of scattering theory based on the helicity representation, and has become customary in elementary particle physics. For the functions $d^I_{MM'}(\theta)$, which are real, the convention of Rose is the same as that employed here. The $d^I_{MM'}$ matrix employed by Edmonds (1957) is the transpose of that used here, and the $\mathscr{D}^I_{MM'}$ functions of Edmonds differ from Eq. (1A-33) by the phase factor $(-1)^{M-M'}$.

Thus, complex conjugation of a $\mathscr{D}$ function gives

$$\begin{aligned}\mathscr{D}^{I*}_{MM'}(\phi, \theta, \psi) &= (-1)^{M-M'}\mathscr{D}^{I}_{-M,-M'}(\phi, \theta, \psi)\\ &= (-1)^{M-M'}\mathscr{D}^{I}_{MM'}(-\phi, -\theta, -\psi)\end{aligned} \tag{1A-38}$$

Closed expressions for the $\mathscr{D}$ functions and their explicit form for small values of I can be found in many textbooks discussing rotational invariance.

For fixed I and ω, the $\mathscr{D}$ functions form a unitary matrix

$$\begin{aligned}\sum_{M} \mathscr{D}^{I*}_{MM_1}(\omega)\mathscr{D}^{I}_{MM_2}(\omega) &= \delta(M_1, M_2)\\ \sum_{M'} \mathscr{D}^{I*}_{M_1M'}(\omega)\mathscr{D}^{I}_{M_2M'}(\omega) &= \delta(M_1, M_2)\end{aligned} \tag{1A-39}$$

and for the inverse rotation ω^{-1}, characterizing the orientation of $\mathscr{K}$ with respect to $\mathscr{K}'$, we have

$$\begin{aligned}&\mathscr{D}^{I}_{MM'}(\omega^{-1}) = (\mathscr{D}^{I}_{M'M}(\omega))^*\\ &(\phi, \theta, \psi)^{-1} = (-\psi, -\theta, -\phi) = (\pi - \psi, \theta, -\pi - \phi)\end{aligned} \tag{1A-40}$$

The $\mathscr{D}$ functions form a complete orthogonal set of basis functions in ϕ, θ, ψ space, with the normalization

$$\begin{aligned}&\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \mathscr{D}^{I*}_{MM'}(\omega)\mathscr{D}^{I_1}_{M_1M'_1}(\omega)\\ &\qquad = \frac{8\pi^2}{2I+1}\delta(I, I_1)\,\delta(M, M_1)\,\delta(M', M'_1)\end{aligned} \tag{1A-41}$$

(The $\mathscr{D}$ functions with half-integer I change sign under rotations through 2π, and to obtain one-valued functions, one must therefore double the angular domain, for example by letting ϕ vary from 0 to 4π. Such an extension of the domain of integration in Eq. (1A-41) is necessary to obtain orthogonality if one of the angular momenta (I or I_1) is half-integer and the other integer. See also the comment on the double-valuedness of the representations of the rotation group in Sec. 1C-3b.)

The $\mathscr{D}$ functions represent generalizations of the spherical harmonics. (The spherical harmonics constitute a complete orthonormal set of functions on a sphere, that is, in θ, ϕ space or in θ, ψ space.) Thus, for $M' = 0$ (or $M = 0$), the $\mathscr{D}$ functions reduce to spherical harmonics

$$\begin{aligned}\mathscr{D}^{I}_{M0}(\omega) &= \left(\frac{4\pi}{2I+1}\right)^{1/2} Y_{IM}(\theta, \phi)\\ \mathscr{D}^{I}_{0M}(\omega) &= (-1)^M\left(\frac{4\pi}{2I+1}\right)^{1/2} Y_{IM}(\theta, \psi)\\ \mathscr{D}^{I}_{00}(\omega) &= P_I(\cos\theta)\end{aligned} \tag{1A-42}$$

By considering the relation (1A-34) for a coupled system $(I_1I_2)IM$, we obtain the coupling rule for $\mathscr{D}$ functions, which is extensively employed in the text,

$$\begin{aligned}&\sum_{M_1M_2} \langle I_1M_1I_2M_2 \mid IM\rangle \mathscr{D}^{I_1}_{M_1M'_1}(\omega)\mathscr{D}^{I_2}_{M_2M'_2}(\omega)\\ &\qquad = \langle I_1M'_1I_2M'_2 \mid IM'\rangle \mathscr{D}^{I}_{MM'}(\omega)\end{aligned} \tag{1A-43}$$

A similar relation can be obtained by coupling the $\mathscr{D}$ functions through their second index $(M'_1 M'_2)$. By an application of the orthogonality relation (1A-7) for the vector addition coefficients, the relation (1A-43) can also be written in the form

$$\mathscr{D}^{I_1}_{M_1 M'_1}(\omega)\mathscr{D}^{I_2}_{M_2 M'_2}(\omega) = \sum_{I=|I_1-I_2|}^{I=I_1+I_2} \langle I_1 M_1 I_2 M_2 | I M_1 + M_2\rangle \langle I_1 M'_1 I_2 M'_2 | I M'_1 + M'_2\rangle \mathscr{D}^I_{M_1+M_2,\, M'_1+M'_2}(\omega) \tag{1A-44}$$

If we introduce a coordinate system $\mathscr{K}_1$ with orientation ω_1 with respect to $\mathscr{K}$ and ω'_1 with respect to $\mathscr{K}'$, we obtain for the composition of rotations (by steps similar to those leading to Eq. (1A-31))

$$\begin{aligned} \mathscr{R}(\omega_1) &= \mathscr{R}(\omega)\mathscr{R}(\omega'_1) \\ \mathscr{D}^I_{MM'}(\omega_1) &= \sum_{M_1} \mathscr{D}^I_{MM_1}(\omega)\mathscr{D}^I_{M_1 M'}(\omega'_1) \end{aligned} \tag{1A-45}$$

For $M' = 0$, the relation (1A-45) gives the transformation of spherical harmonics under rotations of the coordinate system, and for $M = M' = 0$, we obtain the addition theorem for spherical harmonics,

$$\begin{aligned} Y_{IM}(\theta, \phi) &= \sum_{M'} \mathscr{D}^I_{MM'}(\omega) Y_{IM'}(\theta', \phi') \\ P_I(\cos\theta_{12}) &= \frac{4\pi}{2I+1} \sum_M Y^*_{IM}(\theta_1, \phi_1) Y_{IM}(\theta_2, \phi_2) \end{aligned} \tag{1A-46}$$

In the second expression, θ_{12} denotes the angle between the directions $\theta_2\phi_2$ and $\theta_1\phi_1$ corresponding, respectively, to ω'_1 and ω^{-1} in Eq. (1A-45); see also Eq. (1A-40).

The relation (1A-45) expresses the group property of the rotation operators and the $\mathscr{D}$ functions. The $\mathscr{D}$ functions are the irreducible representations of the rotation group (irreducible because it is not possible from the set of $(2I + 1)$ states IM with fixed I to construct subsets that transform separately under all rotations.)

In defining phase conventions for basis states in the angular momentum representation (see p. 19), it is convenient to employ the operator $\mathscr{R}_y(\pi)$ for rotations through an angle π about the y axis,

$$\begin{aligned} \mathscr{R}_y(\pi) &= \exp\{-i\pi I_y\} \\ \langle IM' | \mathscr{R}_y(\pi) | IM\rangle &= d^I_{M'M}(\pi) \\ &= (-1)^{I-M}\,\delta(M, -M') \end{aligned} \tag{1A-47}$$

Since the operation $\mathscr{R}_y(\pi)$ inverts the direction of the z axis, the state IM goes into $I\ -M$, and the phase factor may be obtained by first considering the state $M = I$. This state can be represented by the parallel coupling of $2I$ spin 1/2 systems, for each of which we have

$$\begin{aligned} \mathscr{R}_y(\pi) &= \exp\{-i\pi s_y\} = \exp\left\{-i\frac{\pi}{2}\sigma_y\right\} \\ &= \cos\frac{\pi}{2} - i\sigma_y \sin\frac{\pi}{2} = -i\sigma_y \end{aligned} \tag{1A-48}$$

where σ_y is the Pauli matrix. Thus, the phase factor for each spin, and for the total system, is $+1$. Since, moreover,

$$\mathscr{R}_y(\pi)(I_x + iI_y) = -(I_x - iI_y)\mathscr{R}_y(\pi) \tag{1A-49}$$

the phase factor in Eq. (1A-47) changes sign for each lowering of M by one unit.

1A-5 Spherical Tensors and Reduced Matrix Elements

1A-5a Definition of spherical tensors

One may characterize operators by the amount of angular momentum they transfer to the state on which they act. A spherical tensor of rank λ is a set of operators $T_{\lambda\mu}$ ($\mu = \lambda, \lambda - 1, \ldots, -\lambda$) transferring an angular momentum λ with the different components μ. For example, if the tensor operates on a state of angular momentum zero (specified by further quantum numbers α), we obtain

$$T_{\lambda\mu}\,|\alpha, I=0\rangle = \mathscr{N}\,|\gamma, I=\lambda, M=\mu\rangle \tag{1A-50}$$

where $\mathscr{N}$ is a normalization constant depending on the properties of the tensor and of the state α. The different μ components of the tensor are to have the same intrinsic properties and, thus, the states on the right-hand side of Eq. (1A-50) differ only in M, while they have the same specification γ and normalization $\mathscr{N}$.

If we generalize to the action of $T_{\lambda\mu}$ on states of arbitrary angular momentum I_1, we obtain

$$\begin{aligned} T_\lambda\,|\alpha, I_1\rangle_{(I_1\lambda)I_2M_2} &\equiv \sum_{\mu M_1} \langle I_1 M_1 \lambda\mu | I_2 M_2\rangle\, T_{\lambda\mu}\,|\alpha, I_1 M_1\rangle \\ &= \mathscr{N}\,|\gamma, I_2 M_2\rangle \end{aligned} \tag{1A-51}$$

with $\mathscr{N}$ independent of M_2. We are here employing a notation, by which the operators and states are written without magnetic quantum numbers, while the coupling scheme is specified in a subscript. This notation has the flexibility of allowing a coupling of the angular momenta in arbitrary order (such as $(\lambda I_1)I_2 M_2$ instead of $(I_1\lambda)I_2 M_2$ as in Eq. (1A-51); see also Eq. (1A-15)).

The tensor property of an operator may also be expressed in terms of its transformation under a rotation of the coordinate system. By applying the relation (1A-34) to both sides of Eq. (1A-51), one obtains, by means of Eq. (1A-43),

$$T_{\lambda\mu} = \sum_{\mu'} \mathscr{D}^{\lambda}_{\mu\mu'}(\omega) T'_{\lambda\mu'} \tag{1A-52}$$

where T' is the tensor in the rotated coordinate system. Hence, $T'_{\lambda\mu} = T_{\lambda\mu}(x \rightarrow x')$ where x and $x' = x'(x, \omega)$ represent the dynamical variables, such as position and spin of a particle, referred to the coordinate systems $\mathscr{K}$ and $\mathscr{K}'$. Thus, from Eq. (1-4) together with Eq. (1A-52) follows

$$\mathscr{R}^{-1}(\omega) T_{\lambda\mu} \mathscr{R}(\omega) = \sum_{\mu'} \mathscr{D}^{\lambda}_{\mu\mu'}(\omega) T_{\lambda\mu'} \tag{1A-53}$$

(One expresses the property (1A-53) by saying that the operators $T_{\lambda\mu}$ transform as an irreducible representation of the rotation group.)