

- ▼ The values of $B(M 1)$ in Table 4-21, if interpreted in terms of the first-order effect (4-254), thus yield $(\partial g_R / \partial \beta) \approx 0.5$.

Similarly, the matrix element of the effective coupling operator h_0 can be expressed in terms of the dependence of the moment of inertia on the eccentricity (see Eq. (6-296)),

$$\langle 0_2 | h_0 | 0_1 \rangle = \frac{\partial}{\partial \beta} \left(\frac{\hbar^2}{2\mathcal{I}(\beta)} \right)_{\beta=\beta_0} \langle 0_2 | \beta - \beta_0 | 0_1 \rangle \quad (4-266)$$

and the empirical value of h_0 (see Eq. (4-257)) yields

$$\left(\frac{\beta}{\mathcal{I}} \frac{\partial \mathcal{I}}{\partial \beta} \right)_{\beta=\beta_0} \approx 1.5 \quad (4-267)$$

This value is much greater than the eccentricity dependence of the rigid moment of inertia (see Eq. (4-104)), but the effect of the pairing on the moment of inertia is rather strongly dependent on the vibrational amplitude. Such a dependence arises partly from the fact that the excitation energies of the single-particle transitions induced by the Coriolis interaction depend on the nuclear eccentricity. In the simple estimate (4-128) for the moment of inertia, the single-particle transition energy is proportional to β , and the dependence of \mathcal{I} on this energy implies

$$\begin{aligned} \frac{\beta}{\mathcal{I}} \left(\frac{\partial \mathcal{I}}{\partial \beta} \right)_{\Delta \text{const}} &= - \frac{x}{(1 - g(x))} \frac{dg}{dx} \\ &\approx 0.9 \end{aligned} \quad (4-268)$$

$$\left(x = \frac{\hbar \omega_0 \delta}{2\Delta} \approx 1.3 \right)$$

assuming the values of $\hbar \omega_0 \delta$ and Δ given on p. 83. An additional contribu-

- ▲ tion to $\partial \mathcal{I} / \partial \beta$ may arise if the pairing parameter Δ depends on the vibrational amplitude β .

4-5 ROTATIONAL SPECTRA FOR SYSTEMS WITHOUT AXIAL SYMMETRY

The considerations in the preceding sections apply to systems with axial symmetry, for which the collective rotational motion is restricted to take place about axes perpendicular to the symmetry axis. In any quantal system, there will be fluctuations in the shape, which involve momentary excursions away from axial symmetry. These fluctuations are treated above as intrinsic excitations, and their effect on the rotational motion is included

in the general expressions derived in Sec. 4-3 (see also the $\Delta K=2$ coupling discussed in Sec. 4-4).

If the system possesses a stable equilibrium shape that deviates from axial symmetry by an amount exceeding the zero-point shape fluctuations, it becomes possible to go a step further in the separation between rotational and intrinsic motion and to consider collective rotations about all axes of the intrinsic structure. In such a situation, the rotational families contain an added dimension, and the rotational relationships are correspondingly more extensive.

The study of rotational motion in nuclei with asymmetric shapes is potentially a field of broad scope. Although, at present, there are no well-established examples of nuclear spectra corresponding to asymmetric equilibrium shapes, it appears likely that such spectra will be encountered in the exploration of nuclei under new conditions (large deformations, angular momentum, isospin, etc.). The discussion in the present section is directed partly toward these potential applications to nuclear spectra, but is also motivated by the fact that the analysis of the more general rotational motion of a nonaxial system gives a broader view of many of the problems treated in the previous sections. (The quantal theory of asymmetric rotors has been extensively developed in connection with the analysis of the spectra of polyatomic molecules; see, for example, the texts by Herzberg, 1945, and by Townes and Schawlow, 1955. The possibility of exploiting the asymmetric rotor in the interpretation of nuclear spectra has been especially emphasized by Davydov and Filippov, 1958; see also the review by Davidson, 1965.)

4-5a Symmetry Classification for Even A

In the analysis of rotational motion for a system with an even number of fermions, we shall assume a nondegenerate intrinsic state, as may be expected if the intrinsic Hamiltonian possesses no higher symmetry than rotations of π about one or more axes, in addition to space and time reversal invariance. In fact, in such a situation, as will be discussed below, the intrinsic invariance group has only one-dimensional irreducible representations. (In contrast, for an odd- A system, time-reversal invariance in itself implies a twofold degeneracy. For the even- A system, multidimensional irreducible representations may occur if the intrinsic invariance group includes an axis of order $n \geq 3$ (an axis of order n signifies invariance with respect to rotations of $2\pi/n$). In fact, an axis of order $n \geq 3$ together with time reversal (which has nondiagonal matrix elements between the

states with complex conjugate eigenvalues for the rotations) or together with an additional rotational axis implies that the intrinsic invariance group contains noncommuting elements.)

Symmetry of Hamiltonian

If we assume the deformation to preserve time-reversal invariance, the nondegenerate intrinsic states will be eigenstates of \mathcal{T} and will have vanishing expectation values for intrinsic operators that are odd under \mathcal{T} . Hence, the rotational Hamiltonian, expressed as a function of the components I_κ of the total angular momentum with respect to intrinsic axes, is invariant with respect to the inversion $I_\kappa \rightarrow -I_\kappa$. To leading order, H_{rot} is thus a bilinear expression in the I_κ , which can be brought to a diagonal form by a transformation to principal axes,

$$H_{\text{rot}} = \sum_{\kappa=1}^3 A_\kappa I_\kappa^2 \quad (4-269)$$

The coefficients A_κ , which represent expectation values for the intrinsic state, may be expressed in terms of the moments of inertia,

$$A_\kappa = \frac{\hbar^2}{2J_\kappa} \quad (4-270)$$

The Hamiltonian (4-269) is invariant with respect to rotations through the angle π about each of the three principal axes,

$$\mathcal{R}_\kappa(\pi) = \exp\{-i\pi I_\kappa\} \quad (4-271)$$

The commutation relations for the intrinsic components I_κ (see Eq. (1A-91)) imply the identities

$$\mathcal{R}_1(\pi) = \mathcal{R}_3(\pi) \mathcal{R}_2(\pi)$$

$$\mathcal{R}_2(\pi) \mathcal{R}_3(\pi) = \mathcal{R}_3(\pi) \mathcal{R}_2(\pi) \mathcal{R}(2\pi) \quad (4-272)$$

and cyclic permutations

where

$$\mathcal{R}(2\pi) = (\mathcal{R}_1(\pi))^2 = (\mathcal{R}_2(\pi))^2 = (\mathcal{R}_3(\pi))^2 = (-1)^A \quad (4-273)$$

is a rotation of 2π . For even values of A , such a rotation equals the identity, and the operations $\mathcal{R}_\kappa(\pi)$ commute; the more general relations (4-272) also apply to odd- A systems to be considered below.

For even A , the symmetry group of the Hamiltonian consisting of the commuting elements 1, $\mathcal{R}_1(\pi)$, $\mathcal{R}_2(\pi)$, and $\mathcal{R}_3(\pi)$ is the point group conventionally denoted by D_2 . The eigenvalues of $\mathcal{R}_\kappa(\pi)$ equal $r_\kappa = \pm 1$, and since $r_1 = r_2 r_3$, the symmetry group has the four (one-dimensional) representations

$$(r_1 r_2 r_3) = \begin{cases} (+++) \\ (+--) \\ (-+-) \\ (--+) \end{cases} \quad (4-274)$$

The symmetry quantum numbers r_κ thus label the eigenstates of the asymmetric rotor Hamiltonian (4-269).

Eigenstates

For a system with three different moments of inertia, none of the components I_κ are constants of the motion, but eigenstates with symmetry quantum number r_κ involve only components with even values of I_κ (for $r_\kappa = +1$) or odd I_κ (for $r_\kappa = -1$). In the representation in which I_3 is diagonal (with eigenvalues K), the expansion of the rotational wave function with specified symmetry $r_2 r_3$ takes the form

$$\varphi_{r_2 r_3; \tau I M}(\omega) = \sum_{\substack{K=0, 2, \dots, (r_3 = +1) \\ K=1, 3, \dots, (r_3 = -1)}} c(r_2 r_3; \tau I K) \chi_{r_2 r_3; I K M}(\omega) \quad (4-275)$$

where τ labels the different states in the band with the same value of I and where

$$\chi_{r_2 r_3; I K M}(\omega) = \left(\frac{2I+1}{16\pi^2} \right)^{1/2} \left(\mathcal{D}_{MK}^I(\omega) + r_2 (-1)^{I+K} \mathcal{D}_{M-K}^I(\omega) \right) (1 + \delta(K, 0))^{-1/2} \quad (4-276)$$

The relative phase of the components with opposite K is given by the quantum number r_2 (see Eq. (4-18), where the operation $\mathcal{R}_2(\pi)$ is denoted by \mathcal{R}_e). Components with $K=0$ occur only for $r_3 = +1$ and $(-1)^I = r_2$; from this selection rule, it follows that the eigenstates of the rotational Hamiltonian with specified point symmetry $(r_1 r_2 r_3)$ comprise the sets of I values

$$I = \begin{cases} 0, 2^2, 3, 4^3, 5^2, \dots \\ 1, 2, 3^2, 4^2, 5^3, \dots \end{cases} \quad (r_1 r_2 r_3) = \begin{cases} (+++) \\ (+--) \\ (-+-) \\ (--+) \end{cases} \quad (4-277)$$

The four symmetry classes together give $(2I+1)$ states for each value of I .

Constraints associated with intrinsic D_2 symmetry

The form (4-269) for the leading-order rotational Hamiltonian, which leads to the classification of the states in terms of D_2 symmetry, was based only on the assumed time-reversal invariance of the deformation. If, in addition, the deformation, and thus the intrinsic Hamiltonian, possesses D_2 invariance, as for shapes with ellipsoidal symmetry, the rotational degrees of freedom are restricted by constraints similar to those discussed in Sec. 4-2 for axially symmetric systems.

For a deformation with D_2 symmetry, orientations that differ by a rotation $\mathcal{R}_\kappa(\pi)$ are indistinguishable, and these rotations become part of the intrinsic degrees of freedom. The corresponding reduction in the rotational degrees of freedom can be expressed by the constraint $(\mathcal{R}_\kappa(\pi))_e = (\mathcal{R}_\kappa(\pi))_i$, which is similar to the relation (4-11). Thus, for an intrinsic state with quantum numbers r_κ , the rotational spectrum contains only the states belonging to the symmetry class with the same values of r_κ . In contrast, for a deformation that violates D_2 symmetry, the rotations $\mathcal{R}_\kappa(\pi)$ are no longer part of the intrinsic degrees of freedom, and the rotational spectrum contains all the states (4-277) belonging to the four different symmetry classes.

Additional intrinsic symmetries

If the deformation is invariant with respect to finite rotations that are not contained in the D_2 group, the rotational Hamiltonian possesses the invariance of the extended symmetry group that includes the D_2 group, as well as the additional elements of invariance of the deformation. In such a situation, the tensor of inertia will have axial symmetry; in fact, the extended group must contain at least one axis of order $n > 2$, and such an axis is seen to be a symmetry axis for the inertial tensor. The existence of a symmetry axis for the inertial tensor implies that the moments of inertia in directions perpendicular to the symmetry axis are equal, and that the component of angular momentum along this axis is a constant of the motion. A system with these properties is referred to as a symmetric top. If the intrinsic structure possesses two or more axes of order $n > 2$ (as for tetrahedral or cubic symmetry), the tensor of inertia acquires spherical symmetry ($A_1 = A_2 = A_3$), and the system is referred to as a spherical top. (The occurrence of a symmetry axis of order $n \geq 3$ may lead to degeneracy in the intrinsic motion (see p. 176) and thus to the occurrence of terms in the rotational Hamiltonian that are linear in I_κ .)

The eigenstates of the rotational Hamiltonian can be classified in terms of the representations of the extended symmetry group. For a given intrinsic state, the rotational spectrum is restricted to the representations that have the same symmetry quantum numbers as the intrinsic state with respect to the rotations that leave the intrinsic structure invariant. For example, if the intrinsic state has the eigenvalue $\exp\{-2\pi i v/n\}$ for a rotation of $2\pi/n$ about an axis of order n , the rotational spectrum contains only states with $K = v, v \pm n, v \pm 2n, \dots$, with respect to this axis. If the intrinsic state belongs to a multidimensional irreducible representa-

tion of the intrinsic invariance group (as for axial symmetry combined with $\mathcal{R}_2(\pi)$ invariance), the wave function involves a linear combination of the different intrinsic states belonging to the given representation, obtained by a symmetrization procedure analogous to that employed for the axially symmetric nuclei (see Eq. (4-19)).

One may also consider the possibility of deformations that are invariant with respect to a combination of a finite rotation and a reflection in space or time, without being invariant under these operations separately. In such a situation, the rotational degrees of freedom are coupled to those associated with the inversion in the manner discussed in Sec. 4-2f. For example, if a deformation of symmetry $Y_{32} + Y_{3-2}$ is superposed on an ellipsoidal deformation, the intrinsic structure is invariant under $\mathcal{R}_3(\pi)$ and the combination $\mathcal{S}_2 = \mathcal{R}_2(\pi)\mathcal{P}$, and the intrinsic states can be labeled by the eigenvalues r_3 and s_2 of these two operations. The rotational spectrum contains the representations $(r_3, r_2 = \pm 1)$ that have the same value of r_3 as the intrinsic state, and the two representations have opposite parity ($\pi = r_2 s_2$).

Molecules with identical nuclei

In molecular spectra, restrictions on the rotational degrees of freedom arise when a permutation of identical nuclei can be accomplished by a rotation of the molecule. (For a discussion of the symmetry classification of the states of polyatomic molecules, see for example, Landau and Lifschitz, 1958, pp. 383ff.) The rotations that either leave the equilibrium positions of the nuclei invariant, or result in permutations of identical nuclei, constitute a subgroup G of the rotations in three dimensions. The group G is contained in the "molecular symmetry group" that characterizes the invariance of the Hamiltonian describing the electronic and vibrational degrees of freedom with respect to the intrinsic coordinate system; in turn, the molecular symmetry group is contained in the invariance group of the rotational Hamiltonian. The elements of G can be expressed either in terms of overall (external) rotations or as internal operations consisting of rotations of electronic and vibrational variables and permutations of identical nuclei. In analogy to the constraints discussed above for nuclear systems, the molecular spectrum is restricted by the condition

$$D(G; \text{rotation}) = D(G; \text{electronic}) \otimes D(G; \text{vibration}) \otimes D(G; \text{permutation}) \quad (4-278)$$

where the representations $D(G)$ characterize the symmetry with respect to the group G of the different components in the molecular wave function. The factor $D(G; \text{permutation})$ refers to the spatial permutation symmetry of the nuclei, which in turn is determined by the statistics and the wave function for the nuclear spins.

As an example, we consider the molecule C_2H_4 (ethylene, $\text{>} C=C \text{<}$), in which the four protons lie at the corners of a plane rectangle with the two carbon nuclei (assumed to be ^{12}C) placed symmetrically on an axis bisecting two opposite sides of the rectangle. For this molecule, the symmetry group G consists of the identity and the three rotations $\mathcal{R}_\kappa(\pi)$ about axes through the center of the molecule; each of these rotations involves a pair of transpositions of the four protons, $(P_{12}P_{34}), (P_{13}P_{24}), (P_{14}P_{23})$. Two of the rotations also involve an interchange of the ^{12}C nuclei, but the Bose statistics and the spin 0 for these nuclei

imply that this interchange always gives a factor +1. The invariance group is seen to be $G = D_2$, whose representations are one-dimensional, and the condition (4-278) therefore reduces to the corresponding relation for the eigenvalues, $r_\kappa(\text{rotation}) = r_\kappa(\text{electronic}) \times r_\kappa(\text{vibration}) \times r_\kappa(\text{permutation})$. Permutations of the H atoms constitute a subgroup of the symmetric group S_4 and, for the protons with spin 1/2 and Fermi statistics, the possible spatial permutation symmetries are $[f]=[1111]$, $[211]$, and $[22]$, corresponding to the conjugate spin permutation symmetries $[f]=[4], [31]$, and $[22]$ associated with the total spin quantum number $S=2, 1$, and 0, respectively (see Appendix 1C, Vol. I, pp. 115 and 120). The determination of the quantum numbers $r_\kappa(\text{permutation})$ involves the decomposition of the representations of S_4 into representations of the subgroup that consists of the identity and the three pairs of transpositions (and which is isomorphic with D_2). In the present case, this decomposition can be immediately obtained by noting that the three representations of D_2 involving negative values of r_κ (see Eq. (4-274)) must occur together and that these representations do occur in $[f]=[211]$, for which one of the basis functions may be taken to be symmetric in particles 1 and 2 and antisymmetric in 3 and 4 (see Table 1C-1, Vol. I, p. 128) and therefore an eigenfunction of $P_{12}P_{34}$ with eigenvalue -1. Since the dimensions of $[f]$ are 1, 3, and 2 for $[1111]$, $[211]$, and $[22]$, respectively, as can be seen from the rules given in Vol. I, p. 111, the following decomposition results: A proton spin state with $S=2$, corresponding to $[f]=[1111]$, implies $r_\kappa(\text{permutation})=(+++)$, while $S=1$ ($[f]=[211]$) yields the representations $(+-+)$, $(-+-)$, $(--+)$, and $S=0$ ($[f]=[22]$) the symmetric representation $(+++)$ occurring twice. Thus, from a knowledge of the quantum numbers r_κ for the electronic and vibrational motion, one can determine the possible symmetries r_κ (rotation), for given value of S . The set of states contained in the rotational spectrum then follows from Eq. (4-277).

4-5b Energy Spectra

The determination of the eigenvalues of the rotational Hamiltonian (4-269) with three different moments of inertia requires diagonalization of matrices with dimensions increasing with I . The energy spectrum can be expressed in the form

$$E_{\text{rot}} = \frac{1}{2}(A_1 + A_3)I(I+1) + \frac{1}{2}(A_1 - A_3)E_{\tau I}(\kappa) \quad (4-279)$$

where $E_{\tau I}(\kappa)$ are the eigenvalues of the matrix

$$H(\kappa) = I_1^2 + \kappa I_2^2 - I_3^2 \quad (4-280)$$

depending on a single asymmetry parameter, conventionally denoted by κ ,

$$\kappa = \frac{2A_2 - A_1 - A_3}{A_1 - A_3} \quad (4-281)$$

Since we can label the principal axes such that

$$\begin{aligned} A_1 &\leq A_2 \leq A_3 \\ (\mathcal{I}_1 &\geq \mathcal{I}_2 \geq \mathcal{I}_3) \end{aligned} \quad (4-282)$$

it is sufficient to determine the eigenvalues of $H(\kappa)$ in the interval

$$-1 \leq \kappa \leq 1 \quad (4-283)$$

For $\mathcal{I}_2 = \mathcal{I}_1$, corresponding to $\kappa = 1$, the ellipsoid of inertia is an oblate spheroid, while for $\mathcal{I}_2 = \mathcal{I}_3$ and $\kappa = -1$, the ellipsoid of inertia is prolate. (If the moments of inertia have the classical rigid-body values, a prolate (oblate) shape implies an oblate (prolate) ellipsoid of inertia, but this relationship is not in general valid for quantal systems; for example, if the system possesses axial symmetry, the moment of inertia with respect to this axis vanishes, and the ellipsoid of inertia is therefore prolate, irrespective of whether the shape is prolate or oblate.)

The spectrum of eigenvalues of $H(\kappa)$ as a function of κ is shown in Fig. 4-33 for a few values of I . The states are labeled by the symmetry quantum numbers $(r_1 r_2 r_3)$. The set of eigenvalues $E_{\tau I}$ for negative κ is the same, apart from opposite sign, as for positive κ ; the inversion involves an interchange of the 1 and 3 axes. (Tables of eigenvalues $E_{\tau I}$ have been given by King *et al.*, 1943 and 1949 ($I \leq 12$) and by Erlandsson, 1956 ($13 \leq I \leq 40$); the properties of the spectrum for large I will be discussed further below.)

If two of the moments of inertia are equal (axially symmetric rotor, $\kappa = \pm 1$), the spectrum consists of a succession of bands with specified K representing the component of the angular momentum along the symmetry axis of the inertial tensor, and the energy within each band is linear in $I(I+1)$. For $\kappa = 1$, the symmetry axis is the 3 axis and $K = I_3$, while for $\kappa = -1$, the 1 axis becomes the symmetry axis and $K = I_1$, as indicated in Fig. 4-33a. A small asymmetry can be taken into account by means of a coupling between bands with different values of K , and the spectrum can be expressed as a power series in $I(I+1)$ (see Sec. 4-5c). For intermediate values of κ , there is no longer a conserved quantum number K , and the expansion in powers of $I(I+1)$ will not in general be a useful tool. Nevertheless, for any value of κ , one can define analytic trajectories that link states with successive values of I into rotational sequences (Regge trajectories). However, the spectrum of the asymmetric rotor with its multiplicity of states for given I reveals ambiguities in the definition of the Regge trajectories. This feature is illustrated by the example considered on pp. 194 ff.

In a nucleus with a shape that is invariant with respect to $\mathcal{R}_\kappa(\pi)$, the one-particle orbits are twofold degenerate and the ground-state configuration of an even-even nucleus is obtained by a pairwise filling of conjugate orbits. Such an intrinsic state has the quantum numbers $r_1 = r_2 = r_3 = \pi = +1$ (see the similar arguments for axially symmetric shapes on p. 29). For an asymmetric rotor with this intrinsic symmetry, the low-energy spectrum contains the sequence $I\pi = 0+, 2+, 4+, \dots$ (as for a $K=0$ band), while the next group of excitations has $I\pi = 2+, 3+, 4+, \dots$ (as for a $K\pi = 2+$ band). Such a pattern is widely observed (see, for example, the spectrum of ^{166}Er discussed on p. 159 and the spectra of the Os isotopes in Fig. 6-32, p. 536); with the available evidence, however, it has not been possible to establish quantitative relationships in these spectra that are specific to the asymmetric rotor model. The excitation energy of the second $2+$ state is never observed to be less than 500 keV, and is therefore considerably larger than the rotational excitation energies in the strongly deformed nuclei. On this basis, the excitations have usually been described in terms of vibrational degrees of freedom, but it must be emphasized that many features of the spectra are not well understood at present (see the discussion of the quadrupole vibrations in Chapter 6, pp. 549 ff. and pp. 554 ff.). The possibility of an interpretation of some of these spectra in terms of shapes with a small asymmetry parameter is further discussed on pp. 164 ff. and in Sec. 4-5c. The crucial evidence for deciding between a rotational and a vibrational interpretation would be provided by the identification of the levels corresponding to a repeated excitation of the mode responsible for the second $I\pi = 2+$ state. In the asymmetric rotor model, these states correspond to a $K\pi = 4+$ band beginning at an energy of between $10/3$ and 4 times that of the second $I\pi = 2+$ state. (In the oblate limit ($\kappa = -1$), the second $2+$ and third $4+$ states have $I_1 = 0$ and energies $A_3 I(I+1)$; in the prolate limit ($\kappa = 1$), the states have $|I_3| = I$ and energies $A_1 I + A_3 I^2$.) For harmonic vibrations about an axially symmetric equilibrium shape, the double excitation gives rise to two bands with $K\pi = 0+$ and $4+$, with excitation energies approximately twice that of the second $I\pi = 2+$ state.

Additional tests of the coupling scheme of the asymmetric rotor could be provided by evidence on the intensity rules for transitions involving different states with the same intrinsic configuration. For wave functions of the form (4-275), the matrix elements of tensor operators can be evaluated by a transformation to the intrinsic coordinate frame, as in the derivation of the intensity rules for the axially symmetric rotor (see Sec. 4-3d). The resulting intensity relations depend on the coefficients $c(K)$, which in turn depend on the asymmetry parameter κ , and in general involve a number of

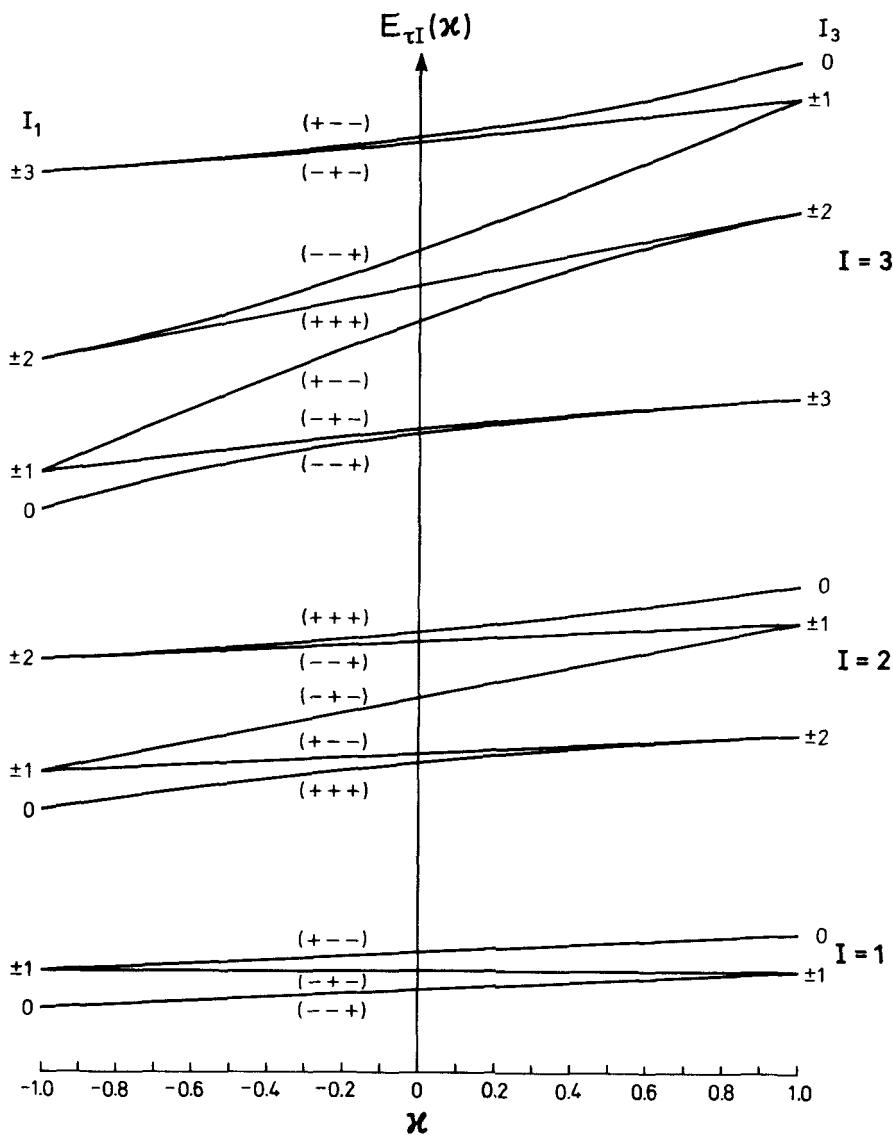


Fig. 4-33a

Figure 4-33 Eigenvalues of the characteristic matrix $H(\kappa)$ for the asymmetric rotor. The eigenvalues are taken from the tables by G. W. King, R. M. Hainer, and P. C. Cross, *J. Chem. Phys.* **11**, 27 (1943) and **17**, 826 (1949) and by G. Erlandsson, *Arkiv Fysik* **10**, 65 (1956) and are shown, for a few values of I , as a function of the asymmetry parameter κ ; the states are labeled by the quantum numbers $(r_1 r_2 r_3)$ of the D_2 group.

intrinsic matrix elements, corresponding to different components ν of the tensor operator. These intensity relations are more comprehensive than

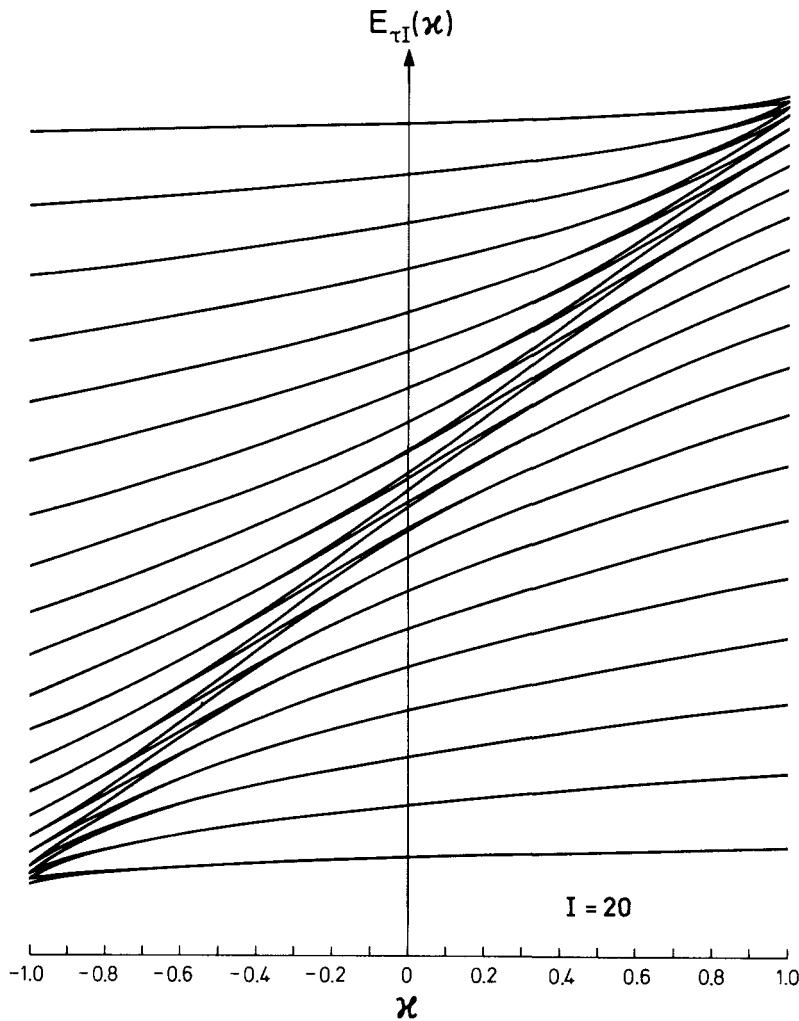


Fig. 4-33b

those for axially symmetric nuclei, on account of the greater number of rotational levels associated with a given intrinsic state of an asymmetric system.

4-5c Systems with Small Asymmetry

If two of the moments of inertia are approximately equal ($\mathcal{I}_1 \approx \mathcal{I}_2$), the spectrum can be separated into bands characterized by the quantum number K ($= I_3$). In the treatment of such a nearly symmetric rotor, it is convenient to write the Hamiltonian (4-269) in the form

$$\begin{aligned} H_{\text{rot}} &= \frac{1}{2}(A_1 + A_2)I^2 + \frac{1}{2}(2A_3 - A_1 - A_2)I_3^2 + H_c \\ H_c &= \frac{1}{4}(A_1 - A_2)(I_-^2 + I_+^2) \end{aligned} \quad (4-284)$$

The corrections to the axially symmetric band structure are described by the term H_c in Eq. (4-284), which is of the form (4-206) for effective $\Delta K=2$ couplings with

$$h_{+2} = \frac{1}{4}(A_1 - A_2) \quad (4-285)$$

From the expressions given in Sec. 4-4, one can therefore obtain the leading-order effects of the coupling, including first-order corrections to the intensity relation for $E2$ transitions between bands with $\Delta K=2$ and second-order corrections to the rotational energies and to the $E2$ transitions within the bands.

As an example of higher-order corrections, we give the coefficients in the rotational energy expansion (4-62) for the $K=0$ band (in a system with $r_2 = r_3 = +1$),

$$\begin{aligned} A &= \frac{1}{2}(A_1 + A_2) + \frac{1}{8} \frac{a^2}{b} + \dots \\ B &= -\frac{1}{16} \frac{a^2}{b} - \frac{19}{1024} \frac{a^4}{b^3} + \dots \\ C &= -\frac{3}{1024} \frac{a^4}{b^3} + \dots \\ D &= \frac{7}{4096} \frac{a^4}{b^3} + \dots \end{aligned} \quad (4-286)$$

$$a \equiv A_1 - A_2 \quad b \equiv 2A_3 - A_1 - A_2$$

and for the $K=2$ band,

$$\begin{aligned} A &= \frac{1}{2}(A_1 + A_2) + \frac{1}{8} \frac{a^2}{b} + \dots \\ A_4 &= \frac{1}{32} \frac{a^2}{b} + \dots \\ B &= \frac{1}{48} \frac{a^2}{b} + \dots \end{aligned} \quad (4-287)$$

It must be emphasized that the above expressions for the rotational energy coefficients are based on the leading-order rotational Hamiltonian (4-269), which is quadratic in the I_κ . Additional contributions may arise from higher-order terms in the rotational Hamiltonian involving fourth and higher powers of the angular momentum.

In view of the systematic occurrence of excited $K\pi=2+$ bands in the spectra of strongly deformed even-even nuclei, one may consider the

possibility of describing these spectra in terms of a rotor deviating slightly from axial symmetry. The energy of the second $2+$ state is typically an order of magnitude larger than the energy of the first excited $2+$ state, which implies $A_3 \sim 10A_1$, while the observed magnitude of the $\Delta K=2$ coupling implies $|A_1 - A_2| \sim 10^{-1}A_1$ (see the example discussed on p. 164). The assumption of a slightly asymmetric rotor does not lead to any new relation between the $E2$ -matrix elements connecting the $K=0$ and $K=2$ bands, beyond those obtained from the general analysis of $\Delta K=2$ couplings (as treated in Sec. 4-4), but the observed rotational energies may be compared with the relations (4-286) and (4-287). Examples are provided by the spectra of ^{168}Er (Table 4-1, p. 65), ^{166}Er (Fig. 4-29, p. 159), and ^{174}Hf (Table 4-20, p. 169). It is found that the observed higher-order terms in the energies exhibit quite a different pattern from that implied by the relations (4-286) and (4-287). Thus, the B terms in the $K=0$ and $K=2$ bands are observed to be rather similar, in contrast to Eqs. (4-286) and (4-287), which imply $B(K=2) \approx -\frac{1}{3}B(K=0)$. For the C and D terms in the observed $K=0$ bands, one finds $C > 0$, $D < 0$, and $|C| \gtrsim 10^2|D|$, while Eq. (4-286) implies $C < 0$, $D > 0$, and $|C| \approx \frac{12}{7}|D|$. The observed A_4 terms in the $K=2$ bands are either very small (compared with $B(K=0)$) or negative, in contrast to the result (4-287), which implies $A_4 \approx \frac{1}{2}|B(K=0)|$. Thus, the observed deviations from the leading-order rotational energies in the low-lying bands of even-even nuclei cannot be ascribed to deviations from axial symmetry in the nuclear shape.

4-5d Symmetry Classification for Odd A

The lack of definite evidence for the occurrence of asymmetric rotor patterns in the spectra of even-even nuclei makes it difficult to test the corresponding relationships in the more complex spectra of the odd- A nuclei (see, for example, the comments on the spectrum of $A=25$ on pp. 287 ff.). In the present section, we consider briefly the general features of the odd- A spectra that can be obtained from symmetry considerations. (The analysis of the spectrum of a particle coupled to a nonaxial rotor was considered by Hecht and Satchler, 1962.)

For a system with an odd number of fermions, the intrinsic states are doubly degenerate, if the deformation is invariant under time reversal (Kramers' degeneracy, see Vol. I, p. 19). The rotational Hamiltonian is therefore a 2×2 matrix in the space of the conjugate intrinsic states, which will be denoted by $|\alpha, \rho=1\rangle$ and $|\alpha, \rho=-1\rangle = -\mathcal{T}|\alpha, \rho=1\rangle$ (see Eq. (1B-20)). Such a matrix can be written as a linear combination of the unit

matrix ρ_0 and the three Pauli matrices $\rho (= \rho_1, \rho_2, \rho_3)$. Since ρ_0 is even under time reversal, while the components of ρ are odd (see Eq. (1B-23)), the rotational Hamiltonian is of the form

$$H_{\text{rot}} = \rho_0 H_0(I_\kappa) + \rho \cdot \mathbf{H}(I_\kappa) \quad (4-288)$$

where H_0 is even and \mathbf{H} odd with respect to the inversion $I_\kappa \rightarrow -I_\kappa$. If we include only the leading-order terms, linear and bilinear in I_κ , the rotational Hamiltonian can be written

$$H_{\text{rot}} = \sum_{\kappa=1}^3 A_\kappa \left(I_\kappa^2 - \sum_{k=1}^3 a_{kk} \rho_k I_\kappa \right) \quad (4-289)$$

where the intrinsic axes have been chosen to coincide with the principal axes of the inertial tensor in H_0 . The coefficients a_{kk} in Eq. (4-289) are real (since H_{rot} is Hermitian) and are analogous to the decoupling parameter in $K=1/2$ bands of axially symmetric nuclei.

If the intrinsic Hamiltonian possesses D_2 symmetry, one can choose basis states $\alpha\rho$ that are eigenstates of $\mathcal{R}_3(\pi)$; since $(\mathcal{R}_3(\pi))^2 = -1$ for an odd- A system (see Eq. (4-273)), the eigenvalues of $\mathcal{R}_3(\pi)$ are $\pm i$. The states $\rho = \pm 1$ have complex conjugate eigenvalues r_3 and, selecting the state $\rho = 1$ as the one with eigenvalue $-i$, we have $\mathcal{R}_3(\pi) = -i\rho_3$. The phases of the states may be determined in accord with the standard convention $\mathcal{R}_2(\pi)\mathcal{T} = +1$. Since $\mathcal{T} = i\rho_2$ (see Eq. (1B-22)), this condition implies $\mathcal{R}_2(\pi) = -i\rho_2$, and we therefore have

$$\mathcal{R}_\kappa(\pi) = -i\rho_\kappa \quad (4-290)$$

as a consequence of the relation $\mathcal{R}_1(\pi) = \mathcal{R}_2(\pi)\mathcal{R}_3(\pi)$ and cyclic permutations. (These relations for the operators $(\mathcal{R}_\kappa(\pi))_i$ acting on the intrinsic variables are the inverse of the relations (4-272) for the corresponding operators $(\mathcal{R}_\kappa(\pi))_e$ acting on the orientation angles.)

The invariance with respect to the operations $\mathcal{R}_\kappa(\pi)$ implies that, in the basis (4-290), the rotational Hamiltonian (4-289) takes the form

$$H_{\text{rot}} = \sum_{\kappa=1}^3 A_\kappa (I_\kappa^2 - a_{kk} \rho_\kappa I_\kappa) \quad (4-291)$$

corresponding to a diagonal decoupling matrix, $a_{kk} = a_\kappa \delta_{kk}$.

For an odd- A system, the total wave function may involve rotational components with all possible K values ($-I \leq K \leq I$) each occurring together with both the intrinsic states $\rho = \pm 1$. The $\mathcal{R}_\kappa(\pi)$ symmetry of the intrinsic

motion imposes the selection rules

$$\mathcal{R}_3(\pi) = \exp\{-i\pi K\} = -i\rho_3 = \begin{cases} -i & \rho=1 \\ i & \rho=-1 \end{cases} \quad (4-292)$$

and the further relation $\mathcal{R}_2(\pi)\mathcal{T}=1$ leads to wave functions of the form

$$\Psi_{\alpha\tau IM} = \sum_{K=\dots, -3/2, 1/2, 5/2, \dots} c(\alpha\tau IK) \psi_{\alpha I KM}$$

$$\psi_{\alpha I KM} = \left(\frac{2I+1}{16\pi^2} \right)^{1/2} \left(\Phi_{\alpha, \rho=1} \mathcal{D}_{MK}^I(\omega) - (-1)^{I+K} \Phi_{\alpha, \rho=-1} \mathcal{D}_{M-K}^I(\omega) \right) \quad (4-293)$$

(In the limit of axial symmetry, the intrinsic state denoted by Φ_K equals $\Phi_{\alpha, \rho=1}$ for $K=1/2, 5/2, \dots$ and $\Phi_{\alpha, \rho=-1}$ for $K=3/2, 7/2, \dots$. Since $\Phi_{\alpha, \rho=-1} = -\mathcal{T}\Phi_{\alpha, \rho=1}$, the minus sign between the two components of $\psi_{\alpha I KM}$ in Eq. (4-293) corresponds to the plus sign in Eq. (4-19).) For a given value of I , the states (4-293) involve $(I+1/2)$ terms, and the rotational spectrum has the components

$$I = 1/2, (3/2)^2, (5/2)^3, \dots \quad (4-294)$$

The multiplicity of states equals $(2I+1)$ multiplied by two (as a result of the twofold degeneracy of the intrinsic states) and divided by four (as a result of the invariance with respect to the D_2 group with four elements).

For a system consisting of a particle coupled to an asymmetric rotor, the rotational energy (4-289) is obtained from expression (4-269) for the rotor by replacing I_κ by $R_\kappa = I_\kappa - j_\kappa$. The operator j_κ , considered as a 2×2 matrix in the space $|\alpha, \rho = \pm 1\rangle$, is linear in ρ_κ , and hence the coefficient $a_{\kappa\kappa}$ in Eq. (4-289) equals the trace of the product $j_\kappa \rho_\kappa$. If the potential produced by the rotor possesses D_2 symmetry, the rotational Hamiltonian is given by Eq. (4-291) with

$$a_\kappa = \text{Tr}(j_\kappa \rho_\kappa) \quad (4-295)$$

The decoupling parameters a_κ depend on the symmetry r_κ of the rotor as well as on the single-particle wave functions. Thus, for example, for a particle state with $j=1/2$, we have $\mathcal{R}_\kappa(\pi) = r_\kappa \exp\{-i\pi j_\kappa\} = -2ij_\kappa r_\kappa$ (see Eq. (1A-48)), and the relation (4-290) therefore implies $a_\kappa = r_\kappa$. For these values of a_κ , the rotational Hamiltonian (4-291) yields a spectrum of doublets corresponding to vanishing coupling between the $j=1/2$ particle and the rotor with symmetry r_κ .

4-5e States with Large I

Simple and illuminating solutions for the asymmetric rotor can be obtained for the high angular momentum states in the yrast region. In the classical theory of the asymmetric rotor, the motion reduces to a simple rotation without precession of the axes, if the angular momentum is along the axis corresponding to the largest or smallest moment of inertia. Correspondingly, in the quantal theory, the states of smallest (or largest) energy for given I acquire a simple structure in the limit of large I (Golden and Bragg, 1949).

With the axes labeled such that $\mathcal{J}_1 > \mathcal{J}_2 > \mathcal{J}_3$ ($A_1 < A_2 < A_3$), the states of lowest energy for given I (the yrast states) have $|I_1| \approx I$. The coupling between the components of the rotational wave function with positive and negative values of I_1 becomes negligible for the states with $|I_1| \approx I \gg 1$, and I_1 may thus be treated as a positive quantity. The angular momentum components perpendicular to the I_1 axes obey the commutation relation

$$[I_-, I_+] = 2I_1 \approx 2I \quad (I_{\pm} \equiv I_2 \pm iI_3) \quad (4-296)$$

and can therefore be treated approximately in terms of boson creation and annihilation operators

$$\begin{aligned} c^{\dagger} &= \frac{1}{\sqrt{2I}} I_+ & c &= \frac{1}{\sqrt{2I}} I_- \\ [c, c^{\dagger}] &\approx 1 \end{aligned} \quad (4-297)$$

In terms of these variables, the rotational Hamiltonian (4-269) can be written

$$\begin{aligned} H &= A_1 \mathbf{I}^2 + \frac{1}{2}(A_2 + A_3 - 2A_1)(I_2^2 + I_3^2) + \frac{1}{2}(A_2 - A_3)(I_2^2 - I_3^2) \\ &= A_1 \mathbf{I}^2 + H' \end{aligned} \quad (4-298)$$

with

$$\begin{aligned} H' &= \frac{1}{2}\alpha(c^{\dagger}c + cc^{\dagger}) + \frac{1}{2}\beta(c^{\dagger}c^{\dagger} + cc) \\ &= \alpha(n + \frac{1}{2}) + \frac{1}{2}\beta(c^{\dagger}c^{\dagger} + cc) \\ n &= c^{\dagger}c \approx I - I_1 \end{aligned} \quad (4-299)$$

$$\alpha \equiv (A_2 + A_3 - 2A_1)I$$

$$\beta \equiv (A_2 - A_3)I$$

The number of boson excitations is denoted by n , and the boson vacuum ($n=0$) is the state with $I_1=I$; each quantum carries an angular momentum of -1 unit with respect to the I_1 axis. (For an odd- A system, the leading-order Hamiltonian contains additional terms linear in I_k (see Eq. (4-289)), which may give rise to terms linear in c^\dagger and c ; for a system with D_2 symmetry, however, the terms $a_2 I_2 \rho_2$ and $a_3 I_3 \rho_3$ can be neglected in the approximation considered, while $a_1 I_1 \rho_1$ is approximately a constant ($\approx a_1 I$)).

The Hamiltonian (4-298) can be diagonalized by a canonical transformation to the new boson variables \hat{c}^\dagger and \hat{c} ,

$$\begin{aligned} c^\dagger &= x\hat{c}^\dagger + y\hat{c} & (x^2 - y^2 = 1) \\ \hat{c}^\dagger &= xc^\dagger - yc \end{aligned} \quad (4-300)$$

and one obtains

$$\begin{aligned} H' &= \hbar\omega(\hat{n} + \frac{1}{2}) \\ \hat{n} &= \hat{c}^\dagger \hat{c} \end{aligned} \quad (4-301)$$

with the excitation quanta

$$\hbar\omega = (\alpha^2 - \beta^2)^{1/2} = 2I((A_2 - A_1)(A_3 - A_1))^{1/2} \quad (4-302)$$

and the transformation coefficients

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \left(\frac{1}{2} \left(\frac{\alpha}{(\alpha^2 - \beta^2)^{1/2}} \pm 1 \right) \right)^{1/2} \quad (4-303)$$

The stationary states can thus be characterized by the quantum number \hat{n} together with I (and M), and the energy values are

$$E(\hat{n}, I) = A_1 I(I+1) + (\hat{n} + \frac{1}{2})\hbar\omega \quad (4-304)$$

The quantum number \hat{n} describes the precessional motion of the axes with respect to the direction of I ; for small amplitudes, this motion has the character of a harmonic vibration with frequency ω . If the intrinsic state possesses D_2 symmetry, the spectrum of an even- A rotor is restricted to the states with

$$(-1)^{\hat{n}} = r_1(-1)^I \quad (4-305)$$

while for odd A , the spectrum contains states with all values of \hat{n} . The asymptotic spectrum (4-304) for an even- A system can be recognized in the example illustrated in Fig. 4-34, p. 197.

The validity of the approximate solution (4-304) requires (see Eqs. (4-296) and (4-297))

$$I^2 \gg \langle I_2^2 + I_3^2 \rangle = I \langle c^\dagger c + cc^\dagger \rangle \quad (4-306)$$

By using the explicit solution given above, the condition (4-306) can be expressed in the form

$$I \gg (2\hat{n} + 1) \frac{A_2 + A_3 - 2A_1}{2(A_3 - A_1)^{1/2}(A_2 - A_1)^{1/2}} \quad (4-307)$$

If the three moments of inertia are of comparable magnitude (and \hat{n} is small), the condition (4-307) corresponds to $I \gg 1$. If one of the inertial parameters is much larger than the other two ($A_3 \gg A_2, A_1$), as for a nucleus that deviates only slightly from axial symmetry, the transition to the coupling scheme described above, with $K = I_1 \approx I$, occurs for angular momenta of order $(A_3/(A_2 - A_1))^{1/2}$, while for lower values of I , the yrast states have $K = I_3 \approx 0$.

The $E2$ moment of the nucleus can be expressed in terms of two intrinsic quadrupole moments Q_0 and Q_2 (see Eq. (4-245)). For the states in the yrast region with $|I_1| \approx I$, we shall define an intrinsic nuclear coordinate system \mathcal{H}' with axes $x' = x_2$, $y' = x_3$, $z' = x_1$, so that ϑ, ϕ is the orientation of the 1 axis. Thus, Q_0 is the quadrupole moment with respect to the 1 axis, while Q_2 is a measure of the asymmetry in the shape with respect to this axis. In the representation in which $K = I$ is diagonal, the reduced $E2$ -matrix elements are given by

$$\begin{aligned} \langle I'K' | \mathcal{M}(E2) | IK \rangle &= (2I+1)^{1/2} \left(\frac{5}{16\pi} \right)^{1/2} e(Q_0 \langle IK20 | I'K' \rangle \\ &\quad + Q_2 (\langle IK22 | I'K' \rangle + \langle IK2-2 | I'K' \rangle)) \end{aligned} \quad (4-308)$$

For $K \approx I$, the vector addition coefficients in Eq. (4-308) have the approximate values

$$\langle IK20 | IK \rangle \approx \langle IK2 \pm 2 | I \pm 2 K \pm 2 \rangle \approx 1$$

$$\langle IK20 | I+1K \rangle \approx -\langle I+1K20 | IK \rangle \approx \left(\frac{3}{I} \right)^{1/2} (I-K+1)^{1/2} \quad (4-309)$$

$$\langle IK22 | I+1K+2 \rangle \approx -\langle I+1K+2 | I+1K+2-2 \rangle \approx -\left(\frac{2}{I} \right)^{1/2} (I-K)^{1/2}$$

ignoring terms of order I^{-1} or smaller. In this approximation, the matrix element (4-308) is conveniently expressed in terms of the variables I and

$n = I - K$. Introducing the operator $m(I_1, I_2)$ defined by

$$\langle I', K' = I' - n' \parallel \mathcal{M}(E2) \parallel I, K = I - n \rangle$$

$$= (2I+1)^{1/2} \left(\frac{5}{16\pi} \right)^{1/2} e \langle n' | m(I, I') | n \rangle \quad (4-310)$$

we obtain

$$\begin{aligned} m(I, I') &\approx Q_0 \delta(I, I') + Q_2 \delta(I \pm 2, I') \\ &+ \left(\left(\frac{3}{I} \right)^{1/2} Q_0 c^\dagger - \left(\frac{2}{I} \right)^{1/2} Q_2 c \right) \delta(I+1, I') \\ &+ \left(- \left(\frac{3}{I} \right)^{1/2} Q_0 c + \left(\frac{2}{I} \right)^{1/2} Q_2 c^\dagger \right) \delta(I-1, I') \end{aligned} \quad (4-311)$$

where the operators c^\dagger and c are the creation and annihilation operators (4-297). By means of the transformation (4-300), the operator m can be expressed in terms of the variable \hat{n} , in which the Hamiltonian is diagonal to the approximation considered.

The leading-order terms in the operator (4-311) involve no change of \hat{n} and give the static moment

$$Q \approx Q_0 \quad (4-312)$$

as well as the transitions

$$B(E2; \hat{n}I \rightarrow \hat{n}, I \pm 2) \approx \frac{5}{16\pi} e^2 Q_2^2 \quad (4-313)$$

induced by the rotation of the intrinsic Q_2 moment about the 1 axis.

The terms in Eq. (4-311) with $I' = I \pm 1$ are proportional to the vibrational amplitude and involve transitions with $\Delta\hat{n} = \pm 1$,

$$B(E2; \hat{n}I \rightarrow \hat{n}-1, I-1) = \frac{5}{16\pi} e^2 \frac{\hat{n}}{I} (\sqrt{3} Q_0 x - \sqrt{2} Q_2 y)^2 \quad (4-314)$$

$$B(E2; \hat{n}I \rightarrow \hat{n}+1, I-1) = \frac{5}{16\pi} e^2 \frac{\hat{n}+1}{I} (\sqrt{3} Q_0 y - \sqrt{2} Q_2 x)^2$$

The strength of these transitions is smaller than for the transitions (4-313) without change in the vibrational quantum number, by a factor of order \hat{n}/I , which represents the square of the amplitude of the precessional motion.

The qualitative features of the spectrum of the asymmetric rotor in the yrast region are seen to correspond to sequences of one-dimensional rota-

tional trajectories with strong $E2$ transitions along the trajectories and much weaker transitions connecting the different trajectories. In contrast to the bands of symmetric rotors, which have band heads with $I = K$ (that may give rise to K isomerism), the different trajectories of the asymmetric rotor merge into the single common band with $I = 0$ (or $I = 1/2$), for any intrinsic state. The very tentative evidence on nuclear spectra in the yrast region, for $A \sim 160$ and $I \sim 30$, may suggest the occurrence of the characteristic asymmetric rotor patterns (see pp. 72ff.).



ILLUSTRATIVE EXAMPLES TO SECTION 4-5

Regge trajectories for an asymmetric rotor (Fig. 4-34)

The concept of a Regge trajectory (see Vol. I, p. 13) is intimately connected with that of rotational band structure. One-dimensional trajectories that are analytic functions of the total angular momentum I can be trivially defined for the rigid, axially symmetric rotor, for which the energy is a linear function of $I(I+1)$. Such trajectories can also be quite generally defined in the two-body problem, since the separation between the angular and radial motion leads to a radial wave equation in which the angular momentum only appears as a parameter and can therefore be varied continuously. The characteristic feature of these systems is the existence of an internal axis of rotational symmetry with respect to which one may define the quantum number K ; for given values of the three quantum numbers KIM , the rotational motion is completely specified. For the axially symmetric rotor, K is a constant of the motion and labels the trajectories. For a system of two spinless particles, K is constrained to have the value 0, while for two particles with spin, the trajectories are obtained by a diagonalization of a matrix with a number of dimensions corresponding to the possible values of the helicities.

In more general many-body systems, the quantum number K is not a constant of the motion; indeed, there may be no unique way of introducing an intrinsic coordinate system. In such a situation, the definition of the Regge trajectories involves new features, some of which may be illustrated by a study of the asymmetric rotor.

The energy of an asymmetric rotor can be defined as an analytic function of the total angular momentum I by considering the eigenvalue problem as a matrix equation in the representation in which I_3 is diagonal.

- ▲ The invariance of the rotational Hamiltonian with respect to the D_2 group

- ▼ (see Sec. 4-5a) implies that the matrix (for an even- A system) separates into four submatrices specified by the symmetry quantum numbers $r_3 = (-1)^{I_3}$ and c_3 , of which the latter represents the relative phase of the components with $I_3 = \pm K$, where $K = |I_3|$. For the physical states, we have (see Eqs. (4-273) and (4-276))

$$c_3 = r_2(-1)^{I+K} = r_2 r_3(-1)^I = r_1(-1)^I \quad (4-315)$$

In the representation IKc_3 , the matrix elements of the Hamiltonian (4-269) are given by

$$\begin{aligned} \langle IKc_3 | H_{\text{rot}} | IKc_3 \rangle &= AI(I+1) + (A_3 - A)K^2 + \frac{1}{4}c_3(A_1 - A_2)I(I+1)\delta(K, 1) \\ \langle I, K+2, c_3 | H_{\text{rot}} | IKc_3 \rangle &= \langle IKc_3 | H_{\text{rot}} | I, K+2, c_3 \rangle \\ &= \frac{1}{4}(A_1 - A_2)((I-K-1)(I-K)(I+K+1)(I+K+2))^{1/2} \begin{cases} \sqrt{2} & K=0 \\ 1 & K \neq 0 \end{cases} \\ A_\kappa &= \frac{\hbar^2}{2\mathcal{J}_\kappa} \quad A = \frac{1}{2}(A_1 + A_2) \end{aligned} \quad (4-316)$$

In these matrices, I may be treated as a parameter that can be continuously varied, while the spectrum of K values (K even for $r_3 = +1$ and K odd for $r_3 = -1$) as well as the phase $c_3 (= \pm 1)$ are held fixed. (The state $K=0$ occurs only for the trajectory with $r_3 = +1$ and $c_3 = +1$.) For noninteger values of I , the matrix elements ($K \leftrightarrow K+2$) no longer vanish for certain K values ($K=I$ or $I-1$), and we therefore obtain infinite matrices extending to arbitrarily large positive K values. For $I-1 < K < I$, the nondiagonal matrix elements become imaginary; the analytic continuation is obtained from the symmetric (non-Hermitian) matrix, as indicated in Eq. (4-316).

The condition for the existence of discrete eigenvalues for the infinite matrix (4-316) can be determined by considering the asymptotic behavior, for large K , of the coefficients $c(K)$ in the eigenvectors (4-275). Thus, to leading order in K ,

$$(A_3 - A)c(K) \underset{K \gg I}{\approx} -\frac{1}{4}(A_1 - A_2)(c(K-2) + c(K+2)) \quad (4-317)$$

Hence, the ratio

$$\alpha = \frac{c(K+2)}{c(K)} \quad (4-318)$$

approaches a constant value given by

$$\alpha + \frac{1}{\alpha} = -2 \frac{2A_3 - A_1 - A_2}{A_1 - A_2} \quad (4-319)$$



▼ If \mathcal{I}_3 is either the largest or the smallest of the three moments of inertia, the right-hand side of Eq. (4-319) is positive and larger than 2 (assuming \mathcal{I}_2 to be the intermediate moment). The two roots α_1, α_2 are then real and positive with $\alpha_1 < 1$ and $\alpha_2 > 1$, and $c(K)$ is a superposition of an exponentially increasing and an exponentially decreasing function of K . Thus, in the usual manner, the condition that no exponentially increasing component may be present defines a set of discrete eigenvalues. For the representation based on the intermediate axis, the roots of Eq. (4-319) are complex with $|\alpha_1| = |\alpha_2| = 1$. The inclusion of terms of order K^{-1} in Eq. (4-317) shows that, in this case, $|c(K)|$ is always proportional to K^{-1} ; the infinite matrices, therefore, do not have discrete eigenvalues, but a continuous spectrum. (The discussion of the convergence of the solutions is due to R. Cutkosky, private communication. For a more detailed treatment of the analytic structure of the asymmetric rotor spectrum as a function of I , see Talman, 1971.)

For given moments of inertia labeled such that $\mathcal{I}_1 > \mathcal{I}_2 > \mathcal{I}_3$, one can thus define Regge trajectories in two different ways, employing either the I_3 or the I_1 representation. These two sets of trajectories are illustrated in Fig. 4-34 for the moment ratios $\mathcal{I}_1 : \mathcal{I}_2 : \mathcal{I}_3 = 4 : 2 : 1$. The trajectories with positive r_3 (based on even I_3 values) and those with positive r_1 (even I_1) are drawn with solid lines, while those with negative r_3 and negative r_1 are dashed. Within each set, the trajectories are labeled by a serial number K , which represents the ordering of the solutions for given I ; with increasing values of K , the energy of the I_3 trajectories increases, while the I_1 trajectories decrease in energy. (If the moments of inertia are continuously varied, so that the rotor becomes symmetric with respect to the 3 (or 1) axis, the label K on the I_3 (or I_1) trajectories becomes equal to the eigenvalue of $|I_3|$ (or $|I_1|$).) For $K \neq 0$, there are two trajectories for each value of K , corresponding to the phase c_3 (or c_1) between the components $\pm K$. As can be seen from the c_3 -dependent terms in Eq. (4-316), the lowest I_3 trajectory of given K has $r_3 c_3 = -1$; for the I_1 trajectories, the lowest member has $r_1 c_1 = +1$. The splitting of the c doublets increases with increasing values of I , but decreases with increasing K , since the c -dependent terms only affect the matrix elements in Eq. (4-316) with the lowest K values.

The physical states correspond to the intersections of two trajectories with the same set of symmetry quantum numbers $(r_1 r_2 r_3)$; see Eq. (4-315). In the figure, the physical states with symmetry $(r_1 r_2 r_3) = (+++)$ are indicated by solid circles.

The spectrum of the asymmetric rotor acquires a simple structure in the regions corresponding to the highest and lowest states of given I (and $I \gg 1$); in these asymptotic regions, the motion can be approximately described in terms of rotation about a single axis and a vibration-like precessional motion (see Sec. 4-5e). Thus, in the yrast region (lowest energies for given I), the physical states have the approximately conserved quantum number $K = I_1 \approx I$,



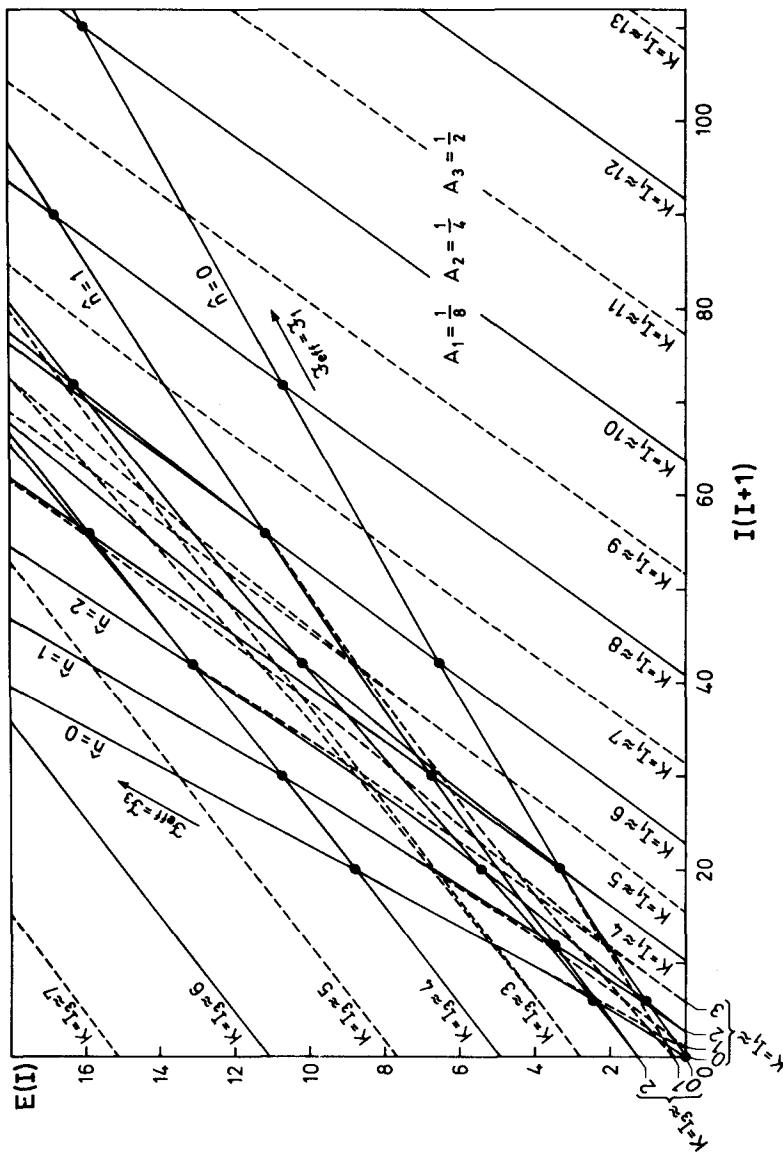


Figure 4-34 Regge trajectories for an asymmetric rotor. We wish to thank H. Lütken and J. D. Talman for help in the preparation of the figure.

- ▼ corresponding to the serial number of the I_1 trajectories. The states with different I , but with the same intrinsic vibrational motion, lie on the I_3 trajectories, which can be labeled by the vibrational quantum number \hat{n} and have a slope determined by the moment of inertia \mathcal{I}_1 (see Fig. 4-34). Hence, in this region of the spectrum, the I_3 trajectories have the characteristic properties of rotational bands, while the I_1 trajectories represent vibrational sequences.

The results obtained for the asymmetric rotor provide an example of the multiplicity of coupling schemes that may be employed in the definition of Regge trajectories in systems with many degrees of freedom. The two sets of trajectories in Fig. 4-34 correspond to the coupling schemes that characterize the simple structure of the spectrum for the lowest and highest states of given I , respectively. Each set of trajectories can be continued through the entire spectrum, and in this sense is complete and exclusive. However, the full characterization of the rotational relationships requires the use of both sets of

- ▲ trajectories.