1A-6 Transformation to Intrinsic Coordinate System

In the description of many-body systems having a shape deviating from spherical symmetry (nonspherical nuclei, molecules, etc.), it is convenient to employ an intrinsic (or "body-fixed") coordinate frame. The transformation of operators from the fixed frame (the laboratory system) to the intrinsic frame involves special features as a result of the fact that the orientation angles ($\omega = \phi, \theta, \psi$) of the intrinsic frame are to be regarded as dynamical variables. The states of orientation can be specified by the angular variables or by the associated angular momenta.

The transformation to an intrinsic coordinate system is also employed in the description of the spin polarization of a particle in terms of the helicity.

1A-6a Components of angular momentum with respect to intrinsic axes

We label the intrinsic axes by $\kappa = 1, 2, 3$, and the spherical angular momentum components I'_ν with respect to the intrinsic frame are thus

$$\begin{aligned} I'_{\nu=\pm 1} &= \mp \frac{1}{\sqrt{2}}(I_1 \pm iI_2) \\ I'_{\nu=0} &= I_3 \end{aligned} \qquad \text{(1A-88)}$$

The relation between I'_ν and the components referring to the fixed axes can be expressed in the form (see Eqs. (1A-52) and (1A-40))

$$\begin{aligned} I_{\mu} &= \sum_{\nu} \mathscr{D}^1_{\mu\nu}(\omega) I'_\nu = \sum_{\nu} I'_\nu \mathscr{D}^1_{\mu\nu}(\omega) \\ I'_\nu &= \sum_{\mu} \mathscr{D}^{1\dagger}_{\mu\nu}(\omega) I_\mu = \sum_{\mu} I_\mu \mathscr{D}^{1\dagger}_{\mu\nu}(\omega) \end{aligned} \qquad \text{(1A-89)}$$

The operators I_μ and I'_ν do not commute with the orientation angles, but the sums in Eq. (1A-89) are independent of the ordering of the $\mathscr{D}$ functions and I components, as can be seen from Eq. (1A-55) with $T_{\lambda\mu} = \mathscr{D}^\lambda_{\mu\nu}$.

Commutation relations involving the I'_ν can be found by applying Eq. (1A-64) to the spherical tensors $\mathscr{D}^\lambda_{\mu\nu}$ and I_μ,

$$\begin{aligned} [I'_\nu, I_\mu] &= 0 \\ [I'_\nu, \mathscr{D}^\lambda_{\mu\nu'}(\omega)] &= (-1)^\nu (\lambda(\lambda+1))^{1/2} \langle \lambda\nu' 1 -\nu | \lambda, \nu' - \nu \rangle \mathscr{D}^\lambda_{\mu, \nu'-\nu}(\omega) \\ [I'_\nu, I'_{\nu'}] &= \sqrt{2} \langle 1\nu 1\nu' | 1, \nu+\nu' \rangle I'_{\nu+\nu'} \end{aligned} \qquad \text{(1A-90)}$$

In terms of the Cartesian components, I_κ, the two last relations in Eq. (1A-90) can be written

$$\begin{aligned} [I_1, I_2] &= -iI_3 \quad \text{and cyclic permutations} \\ [I_1 \pm iI_2, \mathscr{D}^\lambda_{\mu\nu}(\omega)] &= ((\lambda \pm \nu)(\lambda \mp \nu + 1))^{1/2} \mathscr{D}^\lambda_{\mu, \nu\mp 1}(\omega) \\ [I_3, \mathscr{D}^\lambda_{\mu\nu}(\omega)] &= \nu \mathscr{D}^\lambda_{\mu\nu}(\omega) \end{aligned} \qquad \text{(1A-91)}$$

The commutation of I'_ν with I_μ is a simple consequence of the fact that the I'_ν components are independent of the orientation of the external system, and thus scalars with respect to the rotations generated by the I_μ. The commutation relations of the

I'_ν among themselves (and with $\mathscr{D}^{\lambda}_{\mu\nu}$) can also be obtained from geometrical considerations by interpreting the I'_ν as the generators of infinitesimal rotations about the intrinsic axes. The operators I'_ν describe the change in the state vector when the coordinate system (the fixed system) is rotated about an axis of the intrinsic frame. From this point of view there is a dissymmetry between the I'_ν and the I,, since the latter give the effect of a rotation of the fixed system about one of its own axes. However, we can also view the rotation operators in a different way, which formally restores the symmetry between the two sets of angular momentum components. Thus, the effect on the state vector produced by a rotation of the fixed system is equivalent to the effect of the opposite rotation of the intrinsic system, that is, of the body itself. We can therefore regard the I'_ν as the generators of inverse rotations by which the intrinsic system is rotated about one of its own axes.

The commutation relations of the I'_ν can now be obtained as for the I, (see Eq. (1-11)). For inverse rotations, we have

$$[\mathscr{R}_1^{-1}, \mathscr{R}_2^{-1}] = -((\mathscr{R}_1 \mathscr{R}_2)^{-1} - (\mathscr{R}_2 \mathscr{R}_1)^{-1}) \tag{1A-92}$$

which implies that the commutators for the I'_ν involve a change of sign with respect to those for I,, in accordance with Eqs. (1A-91).

Similarly, the commutation relations between I'_ν and the rotation matrix $\mathscr{D}^{\lambda}_{\mu\nu}(\omega)$ for finite rotations can be interpreted in terms of the relation (1A-64) applied to inverse rotations with respect to the intrinsic frame.

Since the intrinsic components of the angular momentum vector commute with the space-fixed components, we can choose a representation that is diagonal in I, as well as in I, and $(\mathbf{I})^2$. The eigenvalues of I_3 are denoted by K and the states may thus be labeled by the quantum numbers IKM (and an additional set α representing operators commuting with $I_{1,2,3}$ as well as with I,,,,,).

The change of sign in the commutation relations for I,,,,, as compared with those for $I_{x,y,z}$ can be taken into account by representing I,,,,, by a set of matrices that are the complex conjugates of those associated with I,,,,, . In the standard representation (see Sec. 1A-1), the complex conjugation simply implies a reversal of sign for I, as compared with I,. The eigenvalues of I, are therefore (as for I,) $K = -I$, $-I + 1, \ldots, I$, while raising and lowering operators are interchanged. The nonvanishing matrix elements of $I_{,2,3}$ are

$$\begin{gathered} \langle \alpha IKM | I_3 | \alpha IKM \rangle = K \\ \langle \alpha IK \mp 1 M | I_1 \pm i I_2 | \alpha IKM \rangle = ((I \pm K)(I \mp K + 1))^{1/2} \end{gathered} \tag{1A-93}$$

1A-6b Wave functions describing orientation of intrinsic system

The state of orientation of the body-fixed system is completely specified by the three angular momentum quantum numbers IKM representing the conjugates of the three orientation angles $\omega = (\phi, \theta, \psi)$. The transformation from the basis set $|\omega\rangle$ with sharply defined orientation to the basis set $|IKM\rangle$ may be obtained by employing the transformation (1A-34), with a $= K,$ to a coordinate system $\mathscr{K}'$ with orientation ω with respect to $\mathscr{K}$

$$|IKM\rangle_{\mathscr{K}} = \sum_{M'} \mathscr{D}^I_{MM'}(\omega)\, |IKM'\rangle_{\mathscr{K}'} \tag{1A-94}$$

The state $|\omega\rangle$ with orientation $\boldsymbol{o}$ with respect to $\mathscr{K}$ has the orientation $\text{o} = 0$ with respect to $\mathscr{K}'$,

$$|\omega\rangle_{\mathscr{K}} = |\omega = 0\rangle_{\mathscr{K}'} \tag{1A-95}$$

and the scalar product of the state vectors in Eqs. (1A-94) and (1A-95) gives the wave function

$$\begin{aligned}\Phi_{IKM}(\omega) &\equiv \langle \omega \,|\, IKM\rangle \\ &= \sum_{M'} \mathscr{D}^I_{MM'}(\omega)\langle \omega = 0\,|IKM'\rangle \\ &= \mathscr{D}^I_{MK}(\omega)\langle \omega = 0 \,|\, IKM = K\rangle \end{aligned} \tag{1A-96}$$

In fact, $(\boldsymbol{o} = 0 \,|\, IKM')$ vanishes except for M' = K, since I_3 equals I_z when acting on the state $|\,\omega = 0\rangle$ (see Eq. (1A-89)). From Eq. (1A-41), it follows that the normalized wave function (1A-96) can be written, with a suitable choice of phase,

$$\Phi_{IKM}(\omega) = \left(\frac{2I+1}{8\pi^2}\right)^{1/2} \mathscr{D}^I_{MK}(\omega) \tag{1A-97}$$

(It may be noted that the states $|IKM\rangle$ are not eigenstates of $\mathscr{R}\mathscr{T}$, since I_3 as well as I_z change sign under time reversal. This point is further discussed for the wave functions of deformed nuclei (Sec. 4-2) and for the helicity states (Sec. 3A-1).)

The $\mathscr{D}$ functions can thus also be viewed as the wave functions describing the orientation of a dynamical system with specified angular momentum quantum numbers $I,$ M, and K. In the special case of a single spin-zero particle in a potential (or the relative motion of two particles without spin), the intrinsic angular momentum K is constrained to have the value zero, and the orientation wave functions reduce to the more familiar spherical harmonics (see Eq. (1A-42)).

While $(\mathbf{I})^2$ and $I,$ are constants of the motion for any system with rotational invariance, the commutator of the Hamiltonian with I_3 depends on the intrinsic dynamics of the system, and the stationary states do not, in general, possess a definite K value. (The conditions under which I_3 is an approximate constant of the motion are discussed in Chapter 4.)

1A-6c Intrinsic components of tensor operators

For an arbitrary tensor operator $T_{\lambda\mu}$ we can define intrinsic components $T'_{\lambda\nu}$ in terms of the relations (see Eq. (1A-52))

$$\begin{aligned} T_{\lambda\mu} &= \sum_{\nu} \mathscr{D}^{\lambda}_{\mu\nu}(\omega) T'_{\lambda\nu} \\ T'_{\lambda\nu} &= \sum_{\mu} \mathscr{D}^{\lambda\dagger}_{\mu\nu}(\omega) T_{\lambda\mu} \end{aligned} \tag{1A-98}$$

If the T components and the $\mathscr{D}$ functions do not commute, the ordering in Eq. (1A-98) is significant. Instead of the products in Eq. (1A-98), we could have used symmetrized expressions in the definition of the intrinsic components.

The intrinsic components $T'_{\lambda\nu}$ are scalars with respect to rotations of the external system and thus commute with $\boldsymbol{I}$,. (The sum over μ in Eq. (1A-98) is seen to be the scalar product of the two tensors $\mathscr{D}^{\lambda}_{\mu-\nu}$ and $T_{\lambda\mu}$; see Eqs. (1A-38) and (1A-71).)

The commutation relations of the tensor components with I'_ν depend on the tensor properties of $\boldsymbol{T}$,, with respect to intrinsic rotations, which are not in general related to those characterizing the behavior under external rotations. Two examples will illustrate this point.

For $T_{\lambda\mu} = \mathscr{D}^{\lambda}_{\mu\nu_0}(\omega)$, the intrinsic components are c numbers $\left(T'_{\lambda\nu} = \delta(\nu, \text{v,})\right)$ and thus scalars with respect to internal as well as external rotations, while $\boldsymbol{T}$,, is a tensor component of rank λ with respect to internal and external rotations.

If $\boldsymbol{T}$,, is a scalar with respect to internal rotations as for the angular momentum components $\boldsymbol{I}$,, the intrinsic components $T'_{\lambda\nu}$ form a tensor of rank λ with respect to internal rotations.

1A-7 Transformation of **Fields**

A field $F(\mathbf{r})$ is an operator associated with the space point r. For given $\mathbf{r}$, the field depends on the dynamical variables x of the system (such as positions, momenta, and spins of constituent particles), and the field operator may, therefore, be written as $F(\mathbf{r}, \text{x})$. (Note that the components of $\mathbf{r}$ are c numbers while the variables x are q numbers.)

If we rotate the coordinate system from $\mathscr{K}$ to $\mathscr{K}'$, we have (in analogy to Eq. (1-4))

$$\mathscr{R}F(\mathbf{r})\mathscr{R}^{-1} = F'(\mathbf{r}) \tag{1A-99}$$

where $F'(\mathbf{r})$ is the same function of the dynamical variables x' (referred to X') as $F(\mathbf{r})$ is of x,

$$F'(\mathbf{r}) = F(\mathbf{r}, x') \tag{1A-100}$$

A scalar field $\rho(\mathbf{r})$ is characterized by its invariance with respect to rotations of the coordinate frame in the sense that its value, at a definite point in space, is the same whether described by an observer in $\mathscr{K}$ or in $\mathscr{K}'$,

$$\rho(\mathbf{r}, x) = \rho(\mathbf{r}', x') \tag{1A-101}$$

or

$$\rho(\mathbf{r}) = \rho'(\mathbf{r}') \tag{1A-102}$$

where $\mathbf{r}'$ and $\mathbf{r}$ are the coordinates, referred to $\mathscr{K}'$ and $\mathscr{K}$, of the same point in space.

Combining Eq. (1A-99) with Eq. (1A-102), we obtain

$$\mathscr{R}^{-1}\rho(\mathbf{r})\mathscr{R} = \rho(\mathbf{r}') \tag{1A-103}$$

as the formal expression for the scalar character of the field. (Note that Eq. (1A-103) involves the inverse transformation as compared with Eq. (1A-99), corresponding to