Appendix 14B
The moment of inertia

It was mentioned in chapter 11 (fig. 11.2) that, because of the pairing correlations, the observed nuclear moment of inertia is much smaller than the rigid body value. In this appendix, we will derive the corresponding formula. The derivations are made within the cranking model in perturbation theory with the rotational frequency, ω, as the small parameter (Inglis, 1954).

Consider the cranking Hamiltonian for rotation around the 1-axis (chapter 12):

$$H^{\omega} = H_0 - \hbar\omega I_1$$

The ground state energy is given by E_0 and the corresponding (many-particle) wave function is $|\Psi_0\rangle$:

$$H_0 |\Psi_0\rangle = E_0 |\Psi_0\rangle$$

For classical rotation, the rotational energy is given by

$$E - E_0 = \frac{1}{2}\mathscr{J}\omega^2 = \frac{1}{2\mathscr{J}} I^2$$

which formula thus defines the moment of inertia, $\mathscr{J}$.

In first order perturbation theory, the cranking wave function is given by

$$|\Psi^{\omega}\rangle = |\Psi_0\rangle - \hbar\omega \sum_{\Psi' \neq \Psi_0} |\Psi'\rangle \frac{\langle\Psi'|I_1|\Psi_0\rangle}{E_0 - E'}$$

where Ψ' is a part of a complete set of wave functions such that $\langle\Psi'|I_1|\Psi_0\rangle \neq 0$. Furthermore,

$$H_0 |\Psi'\rangle = E' |\Psi'\rangle$$

For the non-rotating ground state, it is obvious that $\langle\Psi_0|I_1|\Psi_0\rangle = 0$. Thus, to lowest order in ω

$$\langle \Psi^{\omega} | I_1 | \Psi^{\omega} \rangle = 2\hbar\omega \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

Using second order perturbation theory, the energy in the rotating system E^{ω} is obtained as

$$E^{\omega} = E_0 - (\hbar\omega)^2 \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

The energy E in the laboratory system is calculated as the expectation value of $H_0 = H^{\omega} + \hbar\omega I_1$;

$$E = E_0 + (\hbar\omega)^2 \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

leading to the following formula for the moment of inertia:

$$\mathscr{J} = 2\hbar^2 \sum_{\Psi' \neq \Psi_0} \frac{|\langle \Psi' | I_1 | \Psi_0 \rangle|^2}{E' - E_0}$$

The same formula is obtained if the relation

$$I = \mathscr{J}\omega/\hbar$$

is compared with the expectation value of I_1 derived above.

For a pure single-particle configuration (with no pairing correlation), the ground state wave function is given by

$$\Psi_0 = \prod_{\nu} a_{\nu}^{+} |0\rangle$$

where the product is over all occupied orbitals (ν as well as $\bar{\nu}$). In the second-quantisation formalism, the angular momentum operator is written as

$$I_1 = \sum_{\nu\nu'} \langle \nu | j_1 | \nu' \rangle \, a_{\nu'}^{+} a_{\nu}$$

The only states Ψ' that have $\langle \Psi' | I_1 | \Psi_0 \rangle \neq 0$ are the one-particle-one-hole states

$$\Psi' = a_{\mu'}^{+} a_{\mu} \Psi_0$$

This state corresponds to one particle being excited from the occupied orbital μ to the empty orbital μ' and its excitation energy is given as

$$E' - E_0 = e_{\mu'} - e_{\mu}$$