

CHAPTER 5

ROTATION AND VIBRATIONS OF NUCLEI

5.1. Vibrations of Spherical Nuclei

Surface deformations of nuclei. We shall examine the states associated with the excitation of collective degrees of freedom of nuclei. The shell model of the nucleus is based on the simplifying assumption that the individual nucleons move independently, the interaction between them being described by a self-consistent field. The liquid-drop model of the nucleus is based on the assumption of the existence of strong coupling between the nucleons, as a result of which the mean free path of a nucleon in nuclear matter is small compared with the dimensions of the nucleus. Whereas in the shell model of the nucleus one considers one-particle excitations associated with changes of the states of individual nucleons, in the liquid-drop model of the nucleus collective excitations associated with a simultaneous change of the states of many nucleus are taken into account. Lying at the basis of the so-called *generalized model of the nucleus*, which is a synthesis of the shell and liquid-drop models, is the assumption that the individual nucleons move independently in a slowly varying self-consistent field. In this model, as in the shell model, degrees of freedom associated with the motion of one nucleon or several weakly coupled nucleons in the self-consistent field are taken into account. In the generalized model, as in the liquid-drop model, collective degrees of freedom associated with change of the shape of the nucleus and of its orientation in space are taken into account. In the generalized model, it is usually assumed that the internal and collective motions are separable; they are therefore treated independently.¹

Since nuclear matter is characterized by low compressibility, the collective motion of the nucleons in nuclei reduces principally to deformations of the shapes of the nuclei, with no change in their volume. In the case of spherical nuclei, the collective excitations correspond to vibrations of the surface of the nucleus about the equilibrium shape. In non-spherical nuclei, the collective excitations can be associated with *vibrations of the surface of the nucleus* and with *rotation of the nucleus in space*.

¹The generalized model of the nucleus was proposed in papers by Bohr (1952) and Bohr and Mottelson (1953). The detailed theory of the collective excited states of nuclei is described in the monograph by Davydov (1967).

We shall consider an arbitrary deformation, described in the spherical system of coordinates by the function $R(\vartheta, \varphi)$, of the surface of a nucleus. The quantity $R(\vartheta, \varphi)$ is the distance from the centre of the nucleus to its surface in the direction of the polar angles ϑ and φ , measured in the laboratory coordinate frame. We shall expand this function as a sum over the spherical harmonics $Y_{\lambda\mu}(\vartheta, \varphi)$:

$$R(\vartheta, \varphi) = \left\{ 1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\vartheta, \varphi) \right\} R, \quad (5.1)$$

where R is the radius of the sphere of the same volume and $\alpha_{\lambda\mu}$ are the deformation parameters characterizing the shape of the deformed nucleus. Since $R(\vartheta, \varphi)$ is real, the *deformation parameters* $\alpha_{\lambda\mu}$ must satisfy the condition

$$\alpha_{\lambda\mu} = (-1)^\mu \alpha_{\lambda, -\mu}^*. \quad (5.2)$$

We shall consider small deviations from a spherical nuclear shape. In the case of small deviations from spherical shape, the deformation parameters $\alpha_{\lambda\mu}$ are small quantities. If nuclear matter is incompressible, it follows from the conservation of the volume that

$$\alpha_{00} = -(1/4\pi) \sum_{\lambda\mu} |\alpha_{\lambda\mu}|^2, \quad (5.3)$$

from which it follows that the parameter α_{00} is a second-order small quantity. The parameters $\alpha_{1\mu}$ can be assumed equal to zero, since the deformation (5.1) for $\lambda = 1$ corresponds to a displacement of the nucleus as a whole in space. We note that there is an upper bound on the possible values of λ ($\lambda < A^{1/3}$), inasmuch as the linear dimensions of an element of the nuclear surface should be greater than the mean distance between the nucleons. Thus, surface deformations of the nucleus are characterized by a finite number of deformation parameters $\alpha_{\lambda\mu}$. Ellipsoidal deformations of the nuclear surface, corresponding to the value $\lambda = 2$, are of the greatest interest.

The kinetic energy T and potential energy V of the surface vibrations of the nucleus are given, in the case of small deformations, by the following general formulae:

$$T = \frac{1}{2} \sum_{\lambda\mu} B_\lambda |\dot{\alpha}_{\lambda\mu}|^2; \quad (5.4)$$

$$V = \frac{1}{2} \sum_{\lambda\mu} C_\lambda |\alpha_{\lambda\mu}|^2, \quad (5.5)$$

where the coefficients B_λ and C_λ depend on the specific assumptions about the properties of the nuclear substance.

By regarding the nucleus as a drop of an ideal incompressible charged liquid, we can show that, in the case of irrotational flow of this liquid, the mass coefficient B_λ is equal to

$$B_\lambda = (\varrho/\lambda)R^5, \quad (5.6)$$

where ϱ is the density of the nuclear substance. The deformability coefficient C_λ in the case of a uniform distribution of the nuclear charge over the volume is given by the expression

$$C_\lambda = (\lambda - 1)(\lambda + 2)\sigma R^2 - (3/2\pi)[(\lambda - 1)/(2\lambda + 1)] \cdot (Z^2 e^2 / R), \quad (5.7)$$

where σ is the surface-tension coefficient, connected with the coefficient ε_2 in the Weizsäcker formula (3.17) by the relation

$$4\pi\sigma R_0^2 = \varepsilon_2, \quad R = R_0 A^{1/3}.$$

The potential energy (5.5) of the nucleus is the sum of the surface and Coulomb energies. The surface energy is a minimum for a sphere and increases with deviation from spherical symmetry. Because of the Coulomb repulsive forces, the nucleus, regarded as a drop of charged liquid, tends to take an asymmetric shape. According to (5.5), a nucleus in equilibrium has a spherical shape, stable against deformations, if the deformability coefficients $C_\lambda > 0$. Using (5.7), we obtain the condition for stability of a nucleus against small deformations:

$$(\lambda + 2)\sigma R^2 > 3Z^2 e^2 / 2\pi(2\lambda + 1)R. \quad (5.8)$$

Putting $\lambda = 2$ and using the values of the parameters of (3.17), we have $Z^2/A < 48$. This condition is fulfilled for all known nuclei. (For example, for elements at the end of the periodic table, the ratio Z^2/A is equal to $35 \cdot 6$ for $^{238}_{92}U$, $36 \cdot 5$ for $^{242}_{94}Pu$, $37 \cdot 8$ for $^{244}_{96}Cm$, $39 \cdot 0$ for $^{246}_{98}Cf$, $40 \cdot 3$ for $^{248}_{100}Fm$ and $41 \cdot 0$ for $^{249}_{101}Mv$).

Surface vibrations of spherical nuclei. We shall consider small vibrations of the surface of a spherical nucleus. The dynamical variables in this case are the deformation parameters $\alpha_{\lambda\mu}$, which can be regarded as generalized coordinates. The energy of the surface vibrations

$$E = \frac{1}{2} \sum_{\lambda\mu} \left\{ B_\lambda |\dot{\alpha}_{\lambda\mu}|^2 + C_\lambda |\alpha_{\lambda\mu}|^2 \right\} \quad (5.9)$$

depends on the generalized coordinates $\alpha_{\lambda\mu}$ and generalized velocities $\dot{\alpha}_{\lambda\mu}$. We can go over from the classical energy E to the quantum Hamiltonian operator \mathbf{H} by introducing the generalized momenta

$$\pi_{\lambda\mu} \equiv \partial E / \partial \dot{\alpha}_{\lambda\mu} = B_\lambda \dot{\alpha}_{\lambda\mu}^* \quad (5.10)$$

canonically conjugate to the coordinates $\alpha_{\lambda\mu}$ and postulating the following commutation relations between the quantities $\alpha_{\lambda\mu}$ and $\pi_{\lambda\mu}$, which must be regarded as operators:

$$\begin{aligned} [\pi_{\lambda\mu}, \alpha_{\lambda'\mu'}] &= (\hbar/i) \delta_{\lambda\lambda'} \delta_{\mu\mu'}; \\ [\alpha_{\lambda\mu}, \alpha_{\lambda'\mu'}] &= [\pi_{\lambda\mu}, \pi_{\lambda'\mu'}] = 0. \end{aligned} \quad \left. \right\} \quad (5.11)$$

The Hamiltonian obtained in this way for the surface vibrations of the nucleus,

$$\mathbf{H} = \frac{1}{2} \sum_{\lambda\mu} \{(1/B_\lambda) |\pi_{\lambda\mu}|^2 + C_\lambda |\alpha_{\lambda\mu}|^2\}, \quad (5.12)$$

has the form of a sum of Hamiltonians for uncoupled harmonic oscillators with eigenfrequencies

$$\omega_\lambda = \sqrt{(C_\lambda/B_\lambda)}. \quad (5.13)$$

The energy eigenvalues of the surface vibrations of the nucleus, according to (5.12), can be represented in the form of a sum of energies of individual oscillators:

$$E = \sum_{\lambda\mu} (n_{\lambda\mu} + \frac{1}{2}) \hbar \omega_\lambda, \quad (5.14)$$

where $n_{\lambda\mu}$ are positive integers. When the nucleus undergoes a transition from one state to another, the energy changes by an amount equal to a sum of multiples of the $\hbar \omega_\lambda$. Clearly, the quantity $n_{\lambda\mu}$ determines the number of elementary excitations of a given type $\lambda\mu$, characterized by the energy $\hbar \omega_\lambda$. For a given value of λ , the quantity μ takes $2\lambda + 1$ different values ($-\lambda \leq \mu \leq \lambda$); therefore, the excited states with a fixed value of λ are characterized by $(2\lambda + 1)$ -fold degeneracy.

The elementary excitations of the surface vibrations of a nucleus can be regarded as quasi-particles – *phonons*. By specifying the numbers of phonons of each type, we are thereby defining the nuclear excited state associated with surface vibrations. In treating nuclear excited states associated with surface vibrations, it is convenient to use the occupation-number representation. For this, in place of the operators $\alpha_{\lambda\mu}$ and $\pi_{\lambda\mu}$, we introduce the new operators $b_{\lambda\mu}$:

$$\left. \begin{aligned} \alpha_{\lambda\mu} &= \left(\frac{\hbar}{2B_\lambda\omega_\lambda} \right)^{1/2} [b_{\lambda\mu} + (-1)^\mu b_{\lambda,-\mu}^+]; \\ \pi_{\lambda\mu} &= i \left(\frac{\hbar B_\lambda \omega_\lambda}{2} \right)^{1/2} [b_{\lambda\mu}^+ - (-1)^\mu b_{\lambda,-\mu}]. \end{aligned} \right\} \quad (5.15)$$

Using (5.11), it is not difficult to obtain the commutation relations

$$\left. \begin{aligned} [b_{\lambda\mu}, b_{\lambda'\mu'}^+] &= \delta_{\lambda\lambda'}\delta_{\mu\mu'}; \\ [b_{\lambda\mu}, b_{\lambda'\mu'}] &= [b_{\lambda\mu}^+, b_{\lambda'\mu'}^+] = 0, \end{aligned} \right\} \quad (5.16)$$

from which it follows directly that the operators $b_{\lambda\mu}$ and $b_{\lambda\mu}^+$ can be represented in the form of matrices:

$$\begin{aligned} b_{\lambda\mu} &= \begin{bmatrix} 0 & \sqrt{1} & 0 \dots 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} \dots 0 & 0 & \dots \\ 0 & 0 & 0 \dots 0 & 0 & \dots \\ \dots & & & & \\ 0 & 0 & 0 \dots 0 & \sqrt{n_{\lambda\mu}} & \dots \\ 0 & 0 & 0 \dots 0 & 0 & \dots \\ \dots & & & & \end{bmatrix}, \\ b_{\lambda\mu}^+ &= \begin{bmatrix} 0 & 0 & 0 \dots & 0 & 0 \dots \\ \sqrt{1} & 0 & 0 \dots & 0 & 0 \dots \\ 0 & \sqrt{2} & 0 \dots & 0 & 0 \dots \\ \dots & & & & \\ 0 & 0 & 0 \dots & 0 & 0 \dots \\ 0 & 0 & 0 \dots & \sqrt{n_{\lambda\mu}} & 0 \dots \\ \dots & & & & \end{bmatrix}. \end{aligned} \quad (5.17)$$

Substituting the expressions, (5.15) into (5.12), we obtain the Hamiltonian of the system in the form

$$\mathbf{H} = \sum_{\lambda\mu} (b_{\lambda\mu}^+ b_{\lambda\mu} + \frac{1}{2}) \hbar\omega_\lambda. \quad (5.18)$$

The *wave function in the occupation-number representation* (in the case of phonons) depends on the number $N_{\lambda\mu}$ of phonons in each state $\lambda\mu$, and can be written in the form

$$\begin{aligned} &\Phi_{n_{\lambda_1\mu_1} n_{\lambda_2\mu_2} \dots n_{\lambda\mu}} (N_{\lambda_1\mu_1} N_{\lambda_2\mu_2} \dots N_{\lambda\mu} \dots) \\ &= \langle N_{\lambda_1\mu_1} N_{\lambda_2\mu_2} \dots N_{\lambda\mu} \dots | n_{\lambda_1\mu_1} n_{\lambda_2\mu_2} \dots n_{\lambda\mu} \dots \rangle, \end{aligned} \quad (5.19)$$

where $n_{\lambda\mu}$ are the quantum state indices and $N_{\lambda\mu}$ are the representation indices, that is, the arguments of the wave function. The operators $b_{\lambda\mu}$ and $b_{\lambda\mu}^+$ act on the variables $N_{\lambda\mu}$; $b_{\lambda\mu}$ describes the disappearance of one phonon of type $\lambda\mu$, and $b_{\lambda\mu}^+$ describes the creation of one phonon of type $\lambda\mu$:

$$b_{\lambda\mu}|n_{\lambda\mu}\rangle = \sqrt{n_{\lambda\mu}}|n_{\lambda\mu}-1\rangle, b_{\lambda\mu}^+|n_{\lambda\mu}\rangle = \sqrt{n_{\lambda\mu}+1}|n_{\lambda\mu}+1\rangle. \quad (5.20)$$

Here we have omitted the arguments of the wave functions, and also the quantum numbers of all states apart from the state $\lambda\mu$. According to (5.20), the wave function of the one-phonon excited state of type $\lambda\mu$ can be represented in the form

$$|1_{\lambda\mu}\rangle = b_{\lambda\mu}^+|0\rangle, \quad (5.21)$$

and the wave function of the state with $n_{\lambda\mu}$ phonons can be represented in the form

$$|n_{\lambda\mu}\rangle = (n_{\lambda\mu}!)^{-1/2}(b_{\lambda\mu}^+)^{n_{\lambda\mu}}|0\rangle, \quad (5.22)$$

that is, can be expressed in terms of the wave function $|0\rangle$ of the ground state, in which there are no phonons.

We can extend the analogy between the elementary excitations and particles if we take into account that the operator $b_{\lambda\mu}$ (like the operators $a_{\lambda\mu}$ and $\pi_{\lambda\mu}$) referring to the excitation of the type $\lambda\mu$ is a spherical tensor operator of rank λ . Therefore, each elementary excitation (phonon) of type $\lambda\mu$ has a well-defined angular momentum (spin) λ , with component μ along a specified direction, and a well-defined parity $(-1)^\lambda$, in the same way that the spherical function $Y_{lm}(\vartheta, \varphi)$ (a tensor operator of rank l) describes a particle with angular momentum l , angular-momentum component m and parity $(-1)^l$. Since in each state $\lambda\mu$ there can be any numbers of phonons, with integer spins, phonons are bosons. Therefore, the wave function (5.19) should be symmetric with respect to interchanges of phonons of any type.

It follows directly from (5.17) that the operator $b_{\lambda\mu}^+b_{\lambda\mu}$ has diagonal form in the occupation-number representation, with the possible numbers of phonons of type $\lambda\mu$ along the diagonal:

$$b_{\lambda\mu}^+b_{\lambda\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \dots \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \dots \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \dots \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \dots \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & n_{\lambda\mu} & 0 \dots \\ \dots & & & & & & \end{bmatrix}. \quad (5.23)$$

Therefore, the operator

$$\hat{n}_{\lambda\mu} \equiv b_{\lambda\mu}^+ b_{\lambda\mu} \quad (5.24)$$

is the operator of the number of phonons of a given type $\lambda\mu$; this operator is naturally diagonal in its own representation (the occupation-number representation): $\hat{n}_{\lambda\mu}|n_{\lambda\mu}\rangle = n_{\lambda\mu}|n_{\lambda\mu}\rangle$. It follows from (5.16), and also from (5.17), that

$$b_{\lambda\mu} b_{\lambda\mu}^+ = \hat{n}_{\lambda\mu} + 1. \quad (5.25)$$

By means of the phonon-number operator (5.24), we can rewrite the Hamiltonian (5.18) of the system in the form

$$\mathbf{H} = \sum_{\lambda\mu} \left(\hat{n}_{\lambda\mu} + \frac{1}{2} \right) \hbar\omega_{\lambda}, \quad (5.26)$$

whence it can be seen directly that excitations of the type $\lambda\mu$ make a contribution to the energy of the collective motions of the nucleus equal to

$$E_{n_{\lambda\mu}} = \left(n_{\lambda\mu} + \frac{1}{2} \right) \hbar\omega_{\lambda}, \quad (5.27)$$

and the total energy E is equal to a sum of expressions of the type (5.27). The operator of the projection of the total angular momentum of the system along the quantization axis z is connected with the phonon-number operator (5.24) by the relation

$$L_z = \hbar \sum_{\lambda\mu} \mu \hat{n}_{\lambda\mu}. \quad (5.28)$$

The eigenvalues of (5.28) in the states $|n_{\lambda\mu}\rangle$ are equal to

$$\langle n_{\lambda\mu} | L_z | n_{\lambda\mu} \rangle = \hbar \mu n_{\lambda\mu}. \quad (5.29)$$

We shall discuss briefly the classification of the *states of a system of several phonons*. The two-phonon wave functions $b_{\lambda\mu}^+ b_{\lambda'\mu'}^+ |0\rangle$ describe states with well-defined energy $\hbar(\omega_{\lambda} + \omega_{\lambda'})$, well-defined parity $(-1)^{\lambda+\lambda'}$ and well-defined projection $M = \mu + \mu'$ of the total angular momentum. (In this case, the square of the total angular momentum does not have a well-defined value.) The wave function of a state with well-defined angular momentum Λ lying within the limits $|\lambda - \lambda'| \leq \Lambda \leq \lambda' + \lambda'$ can be constructed by means of a linear combination of two-particle functions:

$$|\Lambda M\rangle = \sum_{\mu\mu'} (\lambda\mu\lambda'\mu' | \Lambda M) b_{\lambda\mu}^+ b_{\lambda'\mu'}^+ |0\rangle. \quad (5.30)$$

In the case of phonons of the same type ($\lambda = \lambda'$), it follows from the symmetry of the wave function $|\Lambda M\rangle$ under interchange of phonons and from the symmetry property of the Clebsch-Gordan coefficient $(\lambda\mu\lambda\mu')|\Lambda M\rangle = (-1)^{2\lambda-\Lambda}(\lambda\mu'\lambda\mu)|\Lambda M\rangle$ that the only possible values of Λ are the even numbers: $\Lambda = 0, 2, \dots, 2\lambda$. For example, for $\lambda = 2$ there are three possible values of the total angular momentum of a system of two phonons: $\Lambda = 0, 2$ and 4 . If we have a system of three phonons with $\lambda = 2$, the symmetry requirement on the wave function leads to the following values of the total angular momentum: $\Lambda = 0, 2, 3, 4$ and 6 . With increasing numbers of phonons of the same type the states are conveniently classified using group-theoretical methods, in a similar way to that used in the classification of the states of systems of equivalent nucleons. The necessity of invoking group-theoretical methods is connected with the fact that, for a complete description of a system of several collective excitations of the same type, it is no longer sufficient to specify the energy, the angular momentum and its projections, and the parity; additional quantum numbers must be introduced, as, for example, in the study of quadrupolar excitations ($\lambda = 2$) when there are more than three phonons.

Vibrational spectra of nuclei. Even-even nuclei always have spin $J = 0$ in the ground state. If the spin of the system is equal to zero, a spherically symmetric density distribution corresponds to the equilibrium state with lowest energy. However, although a spherically symmetric distribution corresponds to a condition of equilibrium, it is not always the case that the latter is stable.

In fact, if we plot the energy of the ground state of the nucleus as a function of a deformation parameter ε (for example, as a function of the eccentricity of the nucleus in the case when the nucleus has the shape of an ellipsoid of revolution), then the derivative of the energy with respect to the deformation parameter is equal to zero at zero value of the parameter. In the case of stable equilibrium for a spherical shape, the energy curve has a minimum at $\varepsilon = 0$. The doubly magic nuclei and the nearby nuclei have a spherical shape in the ground state (the dependence of the energy on the deformation parameter for these nuclei is depicted schematically by the curves 1 and 2 in Fig. 5.1). However, as the number of nucleons in the unfilled outer shell increases, the polarizing effect of these nucleons leads to violation of the spherical symmetry. In this case, the curve of the energy as a function of the deformation parameter has a minimum at $\varepsilon \neq 0$ (see Fig. 5.1, curve 3), which increases and shifts towards larger deformations ε as the number of nucleons in the unfilled outer shell increases.

In the case of spherical nuclei, the collective excitations correspond to vibrations of the surface of the nucleus about the equilibrium shape. According to (5.13), the square of the frequency ω_λ of the surface vibrations

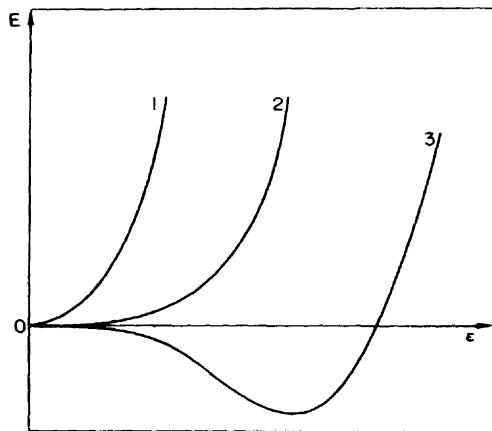


Figure 5.1. Dependence of the energy on the nuclear deformation parameter.

is given by the ratio of the coefficients C_λ and B_λ . The energy $\hbar\omega_\lambda$ of the surface vibrations increases monotonically with increasing λ , in accordance with (5.6) and (5.7). It may be expected, therefore, that the first collective excitations in the low-energy spectra of spherical nuclei will correspond to quadrupolar vibrations with $\lambda = 2$. For the set of quantum numbers defining the excited states, it is convenient to choose the number n of phonons and the angular momentum Λ . For $n = 1$ the spin and parity in the case of quadrupolar vibrations are equal to 2^+ ; for $n = 2$ there is a degenerate triplet of even states with spin values $0^+, 2^+$ and 4^+ ; for $n = 3$ there are five degenerate even states: $0^+, 2^+, 3^+, 4^+$ and 6^+ (Problem 5.16); and so on (Fig. 5.2).

In fact, for the majority of even-even nuclei, the first excited state has spin and parity 2^+ . For certain even-even nuclei ($^{110}_{46}Pd$, $^{114}_{48}Cd$, etc.), triplets of the almost degenerate states $0^+, 2^+$ and 4^+ predicted by the vibrational model have also been discovered. As an illustration, Fig. 5.2 shows the scheme of the observed levels of the nucleus $^{114}_{48}Cd$. For most nuclei, however, the expected triplets of excited states are not observed, although the second excited state for most nuclei is indeed characterized by values of the spin and parity equal to $0^+, 2^+$ or 4^+ .

Attempts to perform quantitative calculations of the spectra of the levels of even-even nuclei on the basis of the vibrational model have proved unsuccessful. In fact, using (5.13) with (5.6) and (5.7) and neglecting the Coulomb energy, we have for the quadrupolar vibrations

$$\hbar\omega_2 = a/A^{1/2}, \quad a = \hbar\sqrt{(8\sigma/\rho R_0^3)}.$$

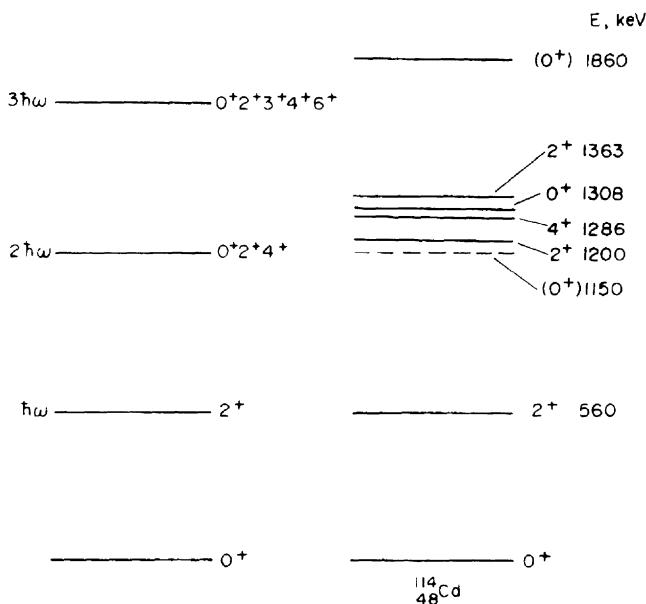


Figure 5.2. Energy spectrum of the quadrupolar vibrations of spherical even-even-nuclei.

Allowance for the Coulomb energy leads to some reduction of this quantity. Consequently, according to the vibrational model of the nucleus, the energy of the first excited level should be a monotonic function of the mass number A . But according to the empirical data, the energy of the first excited state of even-even nuclei depends strongly on the structure of the nuclei and is characterized by very large values for the magic nuclei. This indicates the limited applicability of this model of the nucleus.

Figure 5.3 shows the dependence on the number of neutrons of the ratio E_2/E_1 of the energies of the second and first excited states for even-even nuclei. According to the surface-vibration model, this ratio should be equal to two; in reality, however, there are regions in which this ratio is considerably greater. (Thus, in the ranges of mass numbers $150 < A < 190$ and $A > 220$, this ratio is somewhat greater than three.) It is found that nuclei for which the ratio E_2/E_1 deviates from the characteristic value for vibrational states possess large quadrupole moments; this is connected with the non-spherical equilibrium shape of these nuclei. Therefore, the low-energy levels of these nuclei are associated with the rotation of the nuclei rather than with the surface vibrations.

The quantities $\alpha_{\lambda\mu}$ introduced in (5.1) describe the change of shape of a nucleus relative to a coordinate frame fixed in space. In the case of a non-spherical equilibrium shape, the orientation of the nucleus in space can be

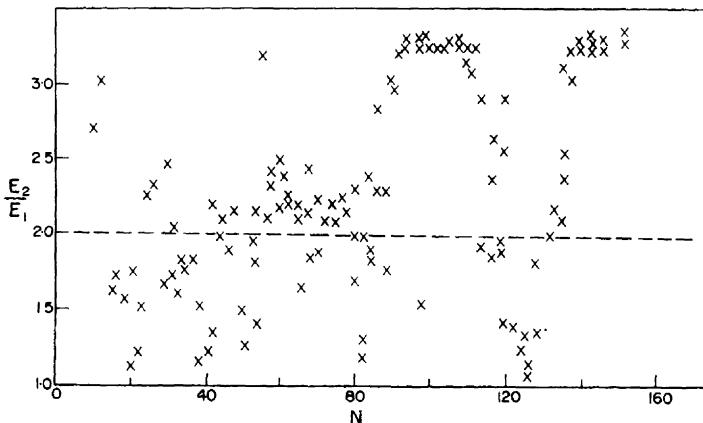


Figure 5.3. Dependence of the ratio of the energies of the second and first excited states on the number of neutrons for even-even nuclei.

described conveniently by the Euler angles, which define the positions of the principal axes of the nucleus relative to a stationary coordinate frame. On rotation of the nucleus, the parameters $\alpha_{\lambda\mu}$ defining the shape of the nucleus in the stationary coordinate frame change with time, although the actual shape of the nucleus remains unchanged. Therefore, to describe surface deformations it is necessary to introduce new parameters defining the shape of the nucleus in the proper coordinate frame associated with the nucleus.

5.2. Rotation of the Coordinate Frame

Rotations of the coordinate frame. We shall consider a certain fixed point P in three-dimensional space. We denote the coordinates of this point in the stationary coordinate frame K by $\mathbf{r} = (x, y, z)$. Correspondingly, we denote the coordinates of this point in the moving coordinate frame K' by $\mathbf{r}' = (x', y', z')$. We shall describe the orientation of the moving coordinate frame K' with respect to the stationary frame K (the origins of the two frames coincide) by the three Euler angles Φ , Θ and ψ , of which Φ and Θ are the polar angles of the z' -axis in the coordinate frame K , and ψ is the angle between the nodal line (the line of intersection of the xy and $x'y'$ planes) and the y' -axis.²

The transformation from the coordinate frame K to the coordinate frame K' can be realized by means of three consecutive rotations (Fig.

²In Wigner's book (1959), another notation for the Euler angles is used: $(\Phi, \Theta, \psi) \equiv (\gamma, \beta, \alpha)$.

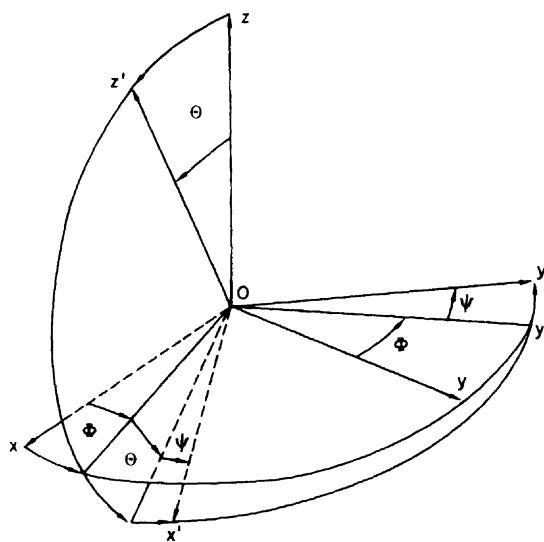


Figure 5.4. Transformation from the coordinate frame K to the coordinate frame K' .

5.4):

1. A rotation through an angle $\Phi(0 \leq \Phi < 2\pi)$ about the axis Oz in the direction from the axis Ox to the axis Oy , as a result of which the axis Oy coincides with the nodal line Oy_1 . The coordinate frame K thus goes over into the coordinate frame K_1 , in which the point P will have the coordinates $\mathbf{r}_1 = (x_1, y_1, z_1)$, which are connected with the coordinates of the same point in the frame K by the relations

$$\mathbf{r}_1 = a^z(\Phi)\mathbf{r}, \quad a^z(\Phi) = \begin{pmatrix} \cos \Phi & \sin \Phi & 0 \\ -\sin \Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.31)$$

2. A rotation through an angle $\Theta(0 \leq \Theta < \pi)$ about the nodal line Oy_1 , as a result of which the axis Oz coincides with the axis Oz' . The coordinate frame K_1 thus goes over into the coordinate frame K_2 , in which the point P will have the coordinates $\mathbf{r}_2 = (x_2, y_2, z_2)$, connected with the coordinates in the frame K_1 by the relations

$$\mathbf{r}_2 = a^{y_1}(\Theta)\mathbf{r}_1, \quad a^{y_1}(\Theta) = \begin{pmatrix} \cos \Theta & 0 & -\sin \Theta \\ 0 & 1 & 0 \\ \sin \Theta & 0 & \cos \Theta \end{pmatrix}. \quad (5.32)$$

3. A rotation through an angle $\psi(0 \leq \psi < 2\pi)$ about the axis Oz' , as a result of which the coordinate frame K_2 goes over into the coordinate

frame K' , in which the point P will have the coordinates $\mathbf{r} = (x', y', z')$, connected with the coordinates in the frame K_2 by the relations

$$\mathbf{r}' = a^{z'}(\psi) \mathbf{r}_2, \quad a^{z'}(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.33)$$

Using (5.31), (5.32) and (5.33), the coordinates of the point P in the coordinate frame K' can be expressed directly in terms of the coordinates of the point P in the coordinate frame K :

$$\mathbf{r}' = a(\Phi, \Theta, \psi) \mathbf{r}, \quad (5.34)$$

where the coefficients $a(\Phi, \Theta, \psi)$ of the linear transformation are equal to

$$a(\Phi, \Theta, \psi) \equiv a^{z'}(\psi) a^{y_1}(\Theta) a^z(\Phi). \quad (5.35)$$

By inverting the equality (5.34), we can express the coordinates of the point P in the coordinate frame K in terms of the coordinates of the same point in the coordinate frame K' :

$$\mathbf{r} = a^{-1}(\Phi, \Theta, \psi) \mathbf{r}', \quad (5.36)$$

where $a^{-1}(\Phi, \Theta, \psi)$ is the inverse of the linear transformation (5.35):

$$a^{-1}(\Phi, \Theta, \psi) = a^{z^{-1}}(\Phi) a^{y_1^{-1}}(\Theta) a^{z'^{-1}}(\psi). \quad (5.37)$$

The finite-rotation operator. We shall consider an arbitrary scalar field, characterized at each point \mathbf{r} in the coordinate frame K by some function $f(\mathbf{r})$. In the coordinate frame K' , the same field is characterized, generally speaking, by another function $f'(\mathbf{r}')$ depending on the coordinates of the point in the new coordinate frame. It is obvious that the values of the field calculated at the same spatial point P in the different coordinate frames should coincide. Therefore, the equality

$$f'(\mathbf{r}') = f(\mathbf{r}), \quad (5.38)$$

should be fulfilled if \mathbf{r}' and \mathbf{r} are connected by the relation (5.34).

We introduce the *finite-rotation operator* $D(\Phi, \Theta, \psi)$ defining the scalar-field function $f'(\mathbf{r})$ in the rotated coordinate frame K' when this function $f(\mathbf{r})$ is given in the stationary coordinate frame K :

$$f'(\mathbf{r}) = D(\Phi, \Theta, \psi) f(\mathbf{r}). \quad (5.39)$$

Using the equality (5.38), it is not difficult to obtain a relation enabling us to find an explicit expression for the finite-rotation operator:

$$D(\Phi, \Theta, \psi) f(\mathbf{r}) \equiv f(a^{-1}(\Phi, \Theta, \psi)\mathbf{r}), \quad (5.40)$$

where $a^{-1}(\Phi, \Theta, \psi)$ is the linear transformation defined by the formula (5.37). Inverting the equality (5.40), we obtain

$$D^{-1}(\Phi, \Theta, \psi) f(\mathbf{r}) \equiv f(a(\Phi, \Theta, \psi)\mathbf{r}). \quad (5.41)$$

This relation is convenient for finding the explicit form of the inverse operator $R(\Phi, \Theta, \psi) = D^{-1}(\Phi, \Theta, \psi)$.

We shall find an explicit expression for the operator of a finite rotation of the coordinate frame through an angle ψ about an axis whose direction is specified by the unit vector \mathbf{n} . For this, we first consider an infinitesimal rotation of the coordinate frame through an angle $\delta\psi$ about the axis \mathbf{n} . In an infinitesimal rotation, $\mathbf{r} \rightarrow \mathbf{r}' = a\mathbf{r} = \mathbf{r} + \delta\mathbf{r}$, where $-\delta\mathbf{r} = \delta\psi(\mathbf{n} \times \mathbf{r})$. As a result of this rotation, the scalar-field function $f(\mathbf{r})$ goes over into the function

$$f'(\mathbf{r}) = f(a^{-1}\mathbf{r}) = f(\mathbf{r} - \delta\mathbf{r}) = [1 + \delta\psi\mathbf{n} \cdot (\mathbf{r} \times \nabla)]f(\mathbf{r}).$$

By making use of the definition of the operator of the orbital angular momentum of the system, $\mathbf{l} = -i(\mathbf{r} \times \nabla)$, we can rewrite the equality obtained in the form

$$f'(\mathbf{r}) = (1 + i\delta\psi\mathbf{n} \cdot \mathbf{l})f(\mathbf{r}). \quad (5.42)$$

The expression in the brackets in the right-hand side of (5.42) can be regarded as the operator of an infinitesimal rotation of the coordinate frame through an angle $\delta\psi$. A rotation through a finite angle ψ about an axis whose direction is specified by the vector \mathbf{n} can be represented as an infinite sequence of rotations through infinitesimal angles $\delta\psi \rightarrow \psi/k$, with $k \rightarrow \infty$:

$$D^{\mathbf{n}}(\psi) f(\mathbf{r}) = \lim_{k \rightarrow \infty} \left(1 + i\frac{\psi}{k} \mathbf{n} \cdot \mathbf{l} \right)^k f(\mathbf{r}) = e^{i\psi\mathbf{n} \cdot \mathbf{l}} f(\mathbf{r}). \quad (5.43)$$

Thus, the operator of a finite rotation of the coordinate frame through an angle ψ about an axis whose direction is specified by the unit vector \mathbf{n} has, in the case of a scalar field, the form

$$D^{\mathbf{n}}(\psi) = e^{i\psi\mathbf{n} \cdot \mathbf{l}}. \quad (5.44)$$

In the general case, when the physical system possesses intrinsic angular momentum (spin) and the field functions are characterized by well-defined transformation properties, the operator $D^{\mathbf{n}}(\psi)$ of a finite rotation of the

coordinate frame is expressed in terms of the operator \mathbf{j} of the total angular momentum of the system:

$$D^{\mathbf{n}}(\psi) = e^{i\psi \mathbf{n} \cdot \mathbf{j}}, \quad (5.45)$$

where \mathbf{j} is the sum of the orbital and spin angular momenta of the system:

$$\mathbf{j} = \mathbf{l} + \mathbf{s}. \quad (5.46)$$

We now find the explicit form of the finite-rotation operator $D(\Phi, \Theta, \psi)$ that effects the transformation from the stationary frame K to the rotated frame K' . Substituting (5.37) into (5.40) and using the definition of the operator of a rotation about a given axis, we obtain

$$\begin{aligned} D(\Phi, \Theta, \psi) f(\mathbf{r}) &= f\left(a^{z^{-1}}(\Phi) a^{y_1^{-1}}(\Theta) a^{z'^{-1}}(\psi) \mathbf{r}\right) \\ &= D^z(\Phi) f\left(a^{y_1^{-1}}(\Theta) a^{z'^{-1}}(\psi) \mathbf{r}\right) = D^z(\Phi) D^{y_1}(\Theta) f\left(a^{z'^{-1}}(\psi) \mathbf{r}\right) \\ &= D^z(\Phi) D^{y_1}(\Theta) D^{z'}(\psi) f(\mathbf{r}). \end{aligned} \quad (5.47)$$

The right-hand side of the equality (5.47) contains operators of rotations about single axes in the different coordinate frames.

According to the identity (5.41), the value of a function of the coordinates of a point in the rotated frame can be expressed in terms of the value of the function of the coordinates of the same point in the stationary frame by means of the operator $D^{-1}(\Phi, \Theta, \psi)$:

$$f(\mathbf{r}') = D^{-1} f(\mathbf{r}), \quad (5.48)$$

where $\mathbf{r}' = a\mathbf{r}$. Clearly, on transforming from $f(\mathbf{r})$ to $f(\mathbf{r}')$, an arbitrary operator O transforms according to the law $O \rightarrow O' = D^{-1}OD$. Therefore, in the transformation $K_1 \rightarrow K_2$ characterized by the rotation operator $D^{y_1}(\Theta)$, we have

$$D^{z'}(\psi) = D^{y_1^{-1}}(\Theta) D^z(\psi) D^{y_1}(\Theta).$$

Analogously, in the transformation $K \rightarrow K_1$ characterized by the rotation operator $D^z(\Phi)$, we have

$$D^{y_1}(\Theta) = D^{z^{-1}}(\Phi) D^y(\Theta) D^z(\Phi).$$

Moreover, noting that $D^z(\Phi) D^z(\psi) = D^z(\psi) D^z(\Phi)$, we obtain the operator $D(\Phi, \Theta, \psi)$ effecting the transformation from the coordinate frame K to the frame K' in the following form:

$$D(\Phi, \Theta, \psi) = D^z(\psi) D^y(\Theta) D^z(\Phi). \quad (5.49)$$

Using the explicit representation (5.45) for the operators of rotations about the individual axes, we finally obtain the following expression for the finite-rotation operator:

$$D(\Phi, \Theta, \psi) = e^{i\psi j_z} e^{i\Theta j_y} e^{i\Phi j_z}. \quad (5.50)$$

It follows directly from the hermiticity of the operator of the angular momentum of the system that the finite-rotation operator (5.50) is unitary:

$$D^{-1}(\Phi, \Theta, \psi) = D^+(\Phi, \Theta, \psi). \quad (5.51)$$

Using the explicit form (5.50) of the finite-rotation operator, it is not difficult to obtain the following expressions for the derivatives of the operator $D(\Phi, \Theta, \psi)$ with respect to the Euler angles:

$$\begin{aligned} \frac{\partial}{\partial \Phi} D &= iDj_z = i(-j_x \sin \Theta \cos \psi + j_y \sin \Theta \sin \psi + j_z \cos \Theta)D; \\ \frac{\partial}{\partial \Theta} D &= iD(-j_x \sin \Phi + j_y \cos \Phi) = i(j_x \sin \psi + j_y \cos \psi)D, \quad (5.52) \\ \frac{\partial}{\partial \psi} D &= iD(j_x \cos \Phi \sin \Theta + j_y \sin \Phi \sin \Theta + j_z \cos \Theta) = ij_z D. \end{aligned}$$

Since $j_{z'} = j_x \cos \Phi \sin \Theta + j_y \sin \Phi \sin \Theta + j_z \cos \Theta$, it follows from the last equality that

$$j_z D = Dj_{z'}. \quad (5.53)$$

We introduce the angular-velocity vector

$$\boldsymbol{\omega} = \dot{\Phi} \hat{\boldsymbol{\Phi}} + \dot{\Theta} \hat{\boldsymbol{\Theta}} + \dot{\psi} \hat{\boldsymbol{\psi}}, \quad (5.54)$$

where the vector $\dot{\Phi}$ is directed along the axis Oz in the coordinate frame K , the vector $\dot{\Theta}$ is directed along the nodal line Oy_1 , and the vector $\dot{\psi}$ is directed along the axis Oz' in the coordinate frame K' . Projecting the vector $\boldsymbol{\omega}$ on to the moving and stationary axes, we find

$$\left. \begin{aligned} \omega_x &= -\dot{\Theta} \sin \Phi + \dot{\psi} \cos \Phi \sin \Theta; \\ \omega_y &= \dot{\Theta} \cos \Phi + \dot{\psi} \sin \Phi \sin \Theta; \\ \omega_z &= \dot{\Phi} + \dot{\psi} \cos \Theta \end{aligned} \right\} \quad (5.55)$$

and

$$\left. \begin{aligned} \omega_{x'} &= -\dot{\Phi} \sin \Theta \cos \psi + \dot{\Theta} \sin \psi; \\ \omega_{y'} &= \dot{\Phi} \sin \Theta \sin \psi + \dot{\Theta} \cos \psi; \\ \omega_{z'} &= \dot{\Phi} \cos \Theta + \dot{\psi}. \end{aligned} \right\} \quad (5.56)$$

Using the relations (5.52) and the definition (5.54) of the angular-velocity vector, it is not difficult to find the following expression for the time derivative of the finite-rotation operator:

$$\frac{d}{dt} D = iD(j_x \omega_x + j_y \omega_y + j_z \omega_z) = i(\omega_{x'} j_x + \omega_{y'} j_y + \omega_{z'} j_z) D. \quad (5.57)$$

Generalized spherical harmonics. We shall consider the set of eigenfunctions $\psi_{jm}(\vartheta, \varphi)$ of the operators of the square and projection of the total angular momentum of the system. Since the finite-rotation operator $D(\Phi, \Theta, \psi)$ contains only the operators j_y and j_z , the function $\psi_{jm}(\vartheta', \varphi')$ resulting from the action of the operator $D(\Phi, \Theta, \psi)$ on the function $\psi_{jm}(\vartheta, \varphi)$ can be expanded in the complete set of functions $\psi_{jm'}(\vartheta, \varphi)$. Taking into account the unitarity of the operator $D(\Phi, \Theta, \psi)$ we can write this expansion in the form

$$\begin{aligned} \psi_{jm}(\vartheta' \varphi') &= D^{-1}(\Phi, \Theta, \psi) \psi_{jm}(\vartheta, \varphi) = D^+(\Phi, \Theta, \psi) \psi_{jm}(\vartheta, \varphi) \\ &= \sum_{m'} D_{m'm}^j(\Phi, \Theta, \psi) \psi_{jm'}(\vartheta, \varphi), \end{aligned} \quad (5.58)$$

where $D_{m'm}^j(\Phi, \Theta, \psi)$ are the matrix elements of the finite-rotation operator, forming the unitary finite-rotation matrix:

$$D_{m'm}^j(\Phi, \Theta, \psi) \equiv \langle jm' | D(\Phi, \Theta, \psi) | jm \rangle = \langle jm' | e^{i\psi j_z} e^{i\Theta j_y} e^{i\Phi j_z} | jm \rangle. \quad (5.59)$$

Using the unitarity of the matrix $D_{m'm}^j(\Phi, \Theta, \psi)$ it is not difficult to obtain the inverse relation to (5.58):

$$\psi_{jm}(\vartheta, \varphi) = \sum_{m'} D_{m'm}^j(\Phi, \Theta, \psi) \psi_{jm'}(\vartheta', \varphi'). \quad (5.60)$$

According to (5.60), the value at a given point in the stationary coordinate frame K of an eigenfunction of the operators of the square and projection of the angular momentum can be expressed in terms of a linear combination of the values of the eigenfunctions of the operators of square and projection of the angular momentum taken at the same point in the rotated coordinate frame K' .

The finite-rotation matrix elements $D_{m'm}^j(\Phi, \Theta, \psi)$ are completely independent of the choice of the coordinate frame in which the functions $\psi_{jm}(\vartheta, \varphi)$ are defined, and for a fixed value of j are well-defined functions of the angles Φ, Θ and ψ . If j is an integer ($j \equiv l$), and either of the indices m and m' equals zero, the functions $D_{m'm}^l(\Phi, \Theta, \psi)$ reduce to ordinary spherical harmonics:

$$\left. \begin{aligned} D_{0m}^l(\Phi, \Theta, \psi) &= \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\Theta, \Phi); \\ D_{m'0}^l(\Phi, \Theta, \psi) &= (-1)^{m'} \sqrt{\frac{4\pi}{2l+1}} Y_{lm'}(\Theta, \psi). \end{aligned} \right\} \quad (5.61)$$

In the particular case $m = m' = 0$, we have

$$D_{00}^l(\Phi, \Theta, \psi) = P_l(\cos \Theta). \quad (5.62)$$

Therefore, the function $D_{m'm}^l(\Phi, \Theta, \psi)$ are usually called *generalized spherical harmonics*, or *Wigner D-functions*.³

The generalized spherical harmonics satisfy the following orthonormalization condition:

$$\begin{aligned} &\int_0^{2\pi} d\Phi \int_0^\pi d\Theta \sin \Theta \int_0^{2\pi} d\psi D_{m'_1 m_1}^{j_1*}(\Phi, \Theta, \psi) D_{m'_2 m_2}^{j_2}(\Phi, \Theta, \psi) \\ &= \frac{8\pi^2}{2j_1 + 1} \delta_{j_1 j_2} \delta_{m_1 m_2} \delta_{m'_1 m'_2}. \end{aligned} \quad (5.63)$$

Integrals of products of three generalized spherical harmonics are easily calculated by means of the vector-coupling formula valid for the generalized spherical harmonics:

$$D_{m'_1 m_1}^{j_1} D_{m'_2 m_2}^{j_2} = \sum_{jm'm} (j_1 m'_1 j_2 m'_2 | jm') (j_1 m_1 j_2 m_2 | jm) D_{m'm}^j. \quad (5.64)$$

We note that the generalized spherical harmonics $D_{m'm}^j(\Phi, \Theta, \psi)$ form an irreducible unitary representation of the three-dimensional rotation group.

Using the definition (5.59), we can separate out the dependence on the angles Φ and ψ explicitly in the functions $D_{m'm}^j(\Phi, \Theta, \psi)$. Since $|jm\rangle$ and $|jm'\rangle$ are eigenfunctions of the operator of the component j_z of the angular momentum, we have

$$D_{m'm}^j(\Phi, \Theta, \psi) = e^{im'\psi} d_{m'm}^j(\Theta) e^{im\Phi}, \quad (5.65)$$

³The properties of the generalized spherical harmonics are considered in detail in the books by Wigner (1959) and Edmonds (1960).

where $d_{m'm}^j(\Theta)$ is a real function defined by the equality

$$d_{m'm}^j(\Theta) \equiv \langle jm' | e^{i\Theta j_y} | jm \rangle. \quad (5.66)$$

It is easy to convince oneself that the function $d_{m'm}^j(\Theta)$ is real by using the expansion

$$e^{i\Theta j_y} = \sum_{n=0}^{\infty} (i\Theta j_y)^n / n!$$

and noting that the matrix elements of the operator j_y are purely imaginary:

$$\begin{aligned} \langle jm' | j_y | jm \rangle &= \frac{i}{2} \{ \sqrt{(j+m)(j-m+1)} \delta_{m',m-1} \\ &\quad - \sqrt{(j-m)(j+m+1)} \delta_{m',m+1} \}. \end{aligned}$$

It follows from the fact that the matrices $d_{m'm}^j(\Theta)$ are real and unitary that they are orthogonal:

$$d_{m'm}^{j-1}(\Theta) = d_{mm'}^j(\Theta), \quad (5.67)$$

whence we obtain the following symmetry property:

$$d_{m'm}^j(\Theta) = d_{mm'}^j(-\Theta). \quad (5.68)$$

By making use of the relation

$$\langle jm' | j_y | jm \rangle = -\langle j-m' | j_y | j-m \rangle,$$

we can also easily verify the following symmetry property:

$$d_{m'm}^j(\Theta) = d_{-m,-m'}^j(\Theta). \quad (5.69)$$

Finally, using the operator identity

$$e^{i\Theta j_y} \equiv e^{i\pi j_z} e^{-i\Theta j_y} e^{-i\pi j_z},$$

it is not difficult to obtain the relation

$$d_{m'm}^j(\Theta) = (-1)^{m'-m} d_{mm'}^j(\Theta). \quad (5.70)$$

The functions $d_{m'm}^j(\Theta)$ for any value of the angle Θ can be expressed in terms of the values of the functions $d_{m'm}^j(\Theta)$ at $\Theta = \pi/2$:

$$d_{m'm}^j(\Theta) = e^{i(\pi/2)(m'-m)} \sum_{m''} d_{m'm''}^j(\pi/2) e^{im''\Theta} d_{m''m}^j(-\pi/2). \quad (5.71)$$

In the derivation of this relation, the operator identity

$$e^{i\Theta j_y} \equiv e^{i(\pi/2)j_z} e^{i(\pi/2)j_y} e^{i\Theta j_z} e^{-i(\pi/2)j_y} e^{-i(\pi/2)j_z}$$

must be used. We also note the following useful relations for the functions $d_{m'm}^j(\Theta)$:

$$\left. \begin{aligned} d_{m'm}^j(\pi) &= (-1)^{j-m'} \delta_{m', -m}; \\ d_{m'm}^j(\pi - \Theta) &= (-1)^{j-m'} d_{m', -m}^j(\Theta); \\ d_{m'm}^j(\pi + \Theta) &= (-1)^{j-m'} d_{-m', m}^j(\Theta). \end{aligned} \right\} \quad (5.72)$$

To conclude, we also give explicit expressions for the time derivative of the finite-rotation matrix $D_{m'm}^j(\Phi, \Theta, \psi)$:

$$\begin{aligned} \frac{d}{dt} D_{m'm}^j &= i \sum_{k=x,y,z} \omega_k \sum_{m''} D_{m'm''}^j \langle jm''|j_k|jm\rangle \\ &= i \sum_{k'=x',y',z'} \omega_{k'} \sum_{m''} \langle jm'|j_{k'}|jm''\rangle D_{m''m}^j. \end{aligned} \quad (5.73)$$

These relations follow directly from the operator relations (5.57).

The rotation operator \mathbf{R} . We introduce the operator \mathbf{R} an infinitesimal rotation of the moving coordinate frame K' . For this, we choose as the dynamical variables the Euler angles Φ, Θ and ψ and consider an arbitrary scalar function $f(\Phi, \Theta, \psi)$ of these angles. Clearly, in rotations of a moving coordinate frame K' whose orientation is specified by these Euler angles, this function will change in the same way as a scalar-field function changes on rotation of the physical system. (A rotation of the physical system through an infinitesimal angle is equivalent to a rotation of the coordinate frame through the same angle in the opposite direction.) Therefore, in an infinitesimal rotation of the moving coordinate frame K' about an axis whose direction is specified by the unit vector \mathbf{n} (as a result to which the angles Φ, Θ and ψ go over into $\Phi' = \Phi + \delta\Phi, \Theta' = \Theta + \delta\Theta$ and $\psi' = \psi + \delta\psi$), the change of the function is equal to

$$\begin{aligned} \delta f(\Phi, \Theta, \psi) &\equiv f(\Phi + \delta\Phi, \Theta + \delta\Theta, \psi + \delta\psi) - f(\Phi, \Theta, \psi) \\ &= i \delta\varphi \mathbf{n} \cdot \mathbf{R} f(\Phi, \Theta, \psi), \end{aligned} \quad (5.74)$$

where $\delta\varphi$ is the infinitesimal angle of rotation and \mathbf{R} is the hermitian rotation operator:

$$\mathbf{n} \cdot \mathbf{R} = -i \frac{\partial}{\partial \varphi}. \quad (5.75)$$

We denote the unit vectors directed along the coordinate axes Oz , Oy_1 and Oz' by \mathbf{n}_Φ , \mathbf{n}_Θ and \mathbf{n}_ψ respectively. Then the projections of the vector \mathbf{R} along the corresponding directions can be expressed directly in terms of derivatives with respect to the Euler angles:

$$\mathbf{n}_\Phi \cdot \mathbf{R} = -i \frac{\partial}{\partial \Phi}, \quad \mathbf{n}_\Theta \cdot \mathbf{R} = -i \frac{\partial}{\partial \Theta}, \quad \mathbf{n}_\psi \cdot \mathbf{R} = -i \frac{\partial}{\partial \psi}. \quad (5.76)$$

Expressing the scalar products in the left-hand sides of these equalities in terms of the projections of the vectors in the stationary or moving coordinate frame, from (5.76) we can easily find the explicit form of the *rotation operator* \mathbf{R} in the stationary and moving coordinate frames.

The *projections of the rotation operator* \mathbf{R} along the axes of the stationary coordinate frame are given by the expressions

$$\left. \begin{aligned} \mathbf{R}_x &= -i \left\{ -\cos \Phi \cot \Theta \frac{\partial}{\partial \Phi} - \sin \Phi \frac{\partial}{\partial \Theta} + \frac{\cos \Phi}{\sin \Theta} \frac{\partial}{\partial \psi} \right\}; \\ \mathbf{R}_y &= -i \left\{ -\sin \Phi \cot \Theta \frac{\partial}{\partial \Phi} + \cos \Phi \frac{\partial}{\partial \Theta} + \frac{\sin \Phi}{\cos \Theta} \frac{\partial}{\partial \psi} \right\}; \\ \mathbf{R}_z &= -i \frac{\partial}{\partial \Phi}. \end{aligned} \right\} \quad (5.77)$$

The projections of the rotation operator \mathbf{R} along the axes of the moving coordinate frame are equal to

$$\left. \begin{aligned} \mathbf{R}_{x'} &= -i \left\{ -\frac{\cos \psi}{\sin \Theta} \frac{\partial}{\partial \Phi} + \sin \psi \frac{\partial}{\partial \Theta} + \cot \Theta \cos \psi \frac{\partial}{\partial \psi} \right\}; \\ \mathbf{R}_{y'} &= -i \left\{ \frac{\sin \psi}{\sin \Theta} \frac{\partial}{\partial \Phi} + \cos \psi \frac{\partial}{\partial \Theta} - \cot \Theta \sin \psi \frac{\partial}{\partial \psi} \right\}; \\ \mathbf{R}_{z'} &= -i \frac{\partial}{\partial \psi}. \end{aligned} \right\} \quad (5.78)$$

Since the square of a vector is an invariant quantity, we find for the operator \mathbf{R}^2 in both coordinate frames

$$\mathbf{R}^2 = - \left\{ \frac{1}{\sin^2 \Theta} \left[\frac{\partial^2}{\partial \Phi^2} - 2 \cos \Theta \frac{\partial^2}{\partial \Phi \partial \psi} + \frac{\partial^2}{\partial \psi^2} \right] \right\}$$

$$+ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left(\sin \Theta \frac{\partial}{\partial \Theta} \right) \Bigg\}. \quad (5.79)$$

Using the explicit form (5.77) of the projections of the rotation operator in the stationary coordinate frame, it is not difficult to show that they satisfy the usual commutation relations for angular-momentum components:

$$\mathbf{R}_x \mathbf{R}_y - \mathbf{R}_y \mathbf{R}_x = i \mathbf{R}_z. \quad (5.80)$$

Thus, \mathbf{R} can be regarded as the operator of the angular momentum associated with the rotation of a physical system which is rigidly attached to the coordinate frame K' . In the moving coordinate frame K' the projections of the rotation operator satisfy commutation relations differing from (5.80) in the sign in the right-hand side:

$$\mathbf{R}_{x'} \mathbf{R}_{y'} - \mathbf{R}_{y'} \mathbf{R}_{x'} = -i \mathbf{R}_{z'}. \quad (5.81)$$

The projections of the rotation operator in both the moving and stationary coordinate frames commute with the operator \mathbf{R}^2 :

$$[\mathbf{R}_i, \mathbf{R}^2] = [\mathbf{R}_{i'}, \mathbf{R}^2] = 0. \quad (5.82)$$

We now consider a quantum-mechanical system characterized by the total angular-momentum operator J . We denote the wave function of the system in K by $\psi(\vartheta, \varphi)$ and the wave function of the system in K' by $\psi(\vartheta', \varphi')$. According to (5.60), the functions $\psi(\vartheta, \varphi)$ and $\psi(\vartheta', \varphi')$ are connected by the relation

$$\psi_{JM}(\vartheta, \varphi) = \sum_{M'} D_{M'M}^J(\Phi, \Theta, \psi) \psi_{JM'}(\vartheta', \varphi'). \quad (5.83)$$

It is obvious that the operator \mathbf{R} will act only on the function $\psi(\vartheta', \varphi')$, since a rotation of the coordinate frame K' cannot affect the value the function $\psi(\vartheta, \varphi)$ in the stationary coordinate frame K ; that is,

$$\mathbf{R}\psi(\vartheta, \varphi) = 0. \quad (5.84)$$

Since the functions $\psi(\vartheta', \varphi')$ depend on the Euler angles Φ, Θ and ψ , the change of this function on an infinitesimal rotation of the coordinate frame K' through an angle $\delta\varphi$ about an arbitrary direction \mathbf{n} , according to (5.74), is equal to

$$\delta\psi(\vartheta', \varphi') = i \delta\varphi \mathbf{n} \cdot \mathbf{R} \psi(\vartheta', \varphi'). \quad (5.85)$$

Moreover, this change can be expressed in terms of the operator \mathbf{J} of the total angular momentum of the system. According to (5.45) and (5.48), this change is equal to

$$\delta\psi(\vartheta', \varphi') = -i \delta\varphi \mathbf{n} \cdot \mathbf{J} \psi(\vartheta', \varphi'). \quad (5.86)$$

Equating the right-hand side of the equalities (5.85) and (5.86), we obtain

$$(\mathbf{R} + \mathbf{J})\psi(\vartheta', \varphi') = 0. \quad (5.87)$$

Using (5.83), we can represent the equality (5.84) in the form

$$\begin{aligned} & \sum_{M'} \mathbf{R} D_{M'M}^J(\Phi, \Theta, \psi) \psi_{JM'}(\vartheta', \varphi') \\ & \equiv \sum_{M'} \left\{ (\mathbf{R} D_{M'M}^J(\Phi, \Theta, \psi)) + D_{M'M}^J(\Phi, \Theta, \psi) \mathbf{R} \right\} \\ & \times \psi_{JM'}(\vartheta', \varphi') = 0, \end{aligned} \quad (5.88)$$

where $(\mathbf{R} D_{M'M}^J(\Phi, \Theta, \psi))$ denotes the result of the action of the operator \mathbf{R} on $D_{M'M}^J(\Phi, \Theta, \psi)$ and is a certain function of the angles. Taking (5.87) into account, we can rewrite (5.88) in the form

$$\sum_{M'} \left\{ (\mathbf{R} D_{M'M}^J(\Phi, \Theta, \psi)) - D_{M'M}^J(\Phi, \Theta, \psi) \mathbf{J} \right\} \psi_{JM'}(\vartheta', \varphi') = 0. \quad (5.89)$$

By projecting the equality (5.89) on to the axis Oz' and noting that $\mathbf{J}_{z'} \psi_{JM'}(\vartheta', \varphi') = M' \psi_{JM'}(\vartheta', \varphi')$, we have

$$\sum_{M'} \left\{ (\mathbf{R}_{z'} D_{M'M}^J(\Phi, \Theta, \psi)) - M' D_{M'M}^J(\Phi, \Theta, \psi) \right\} \psi_{JM'}(\vartheta', \varphi') = 0.$$

Since the functions $\psi_{JM'}(\vartheta', \varphi')$ for different values of M' are linearly independent, if this equality is to be fulfilled it is necessary that all the coefficients vanish identically; that is,

$$\mathbf{R}_{z'} D_{M'M}^J(\Phi, \Theta, \psi) = M' D_{M'M}^J(\Phi, \Theta, \psi). \quad (5.90)$$

Analogously, we can obtain

$$\begin{aligned} & (\mathbf{R}_{x'} \pm i \mathbf{R}_{y'}) D_{M'M}^J(\Phi, \Theta, \psi) \\ & = \sqrt{(J \pm M')(J \mp M' + 1)} D_{M'\mp 1, M}^J(\Phi, \Theta, \psi). \end{aligned} \quad (5.91)$$

Using (5.90) and (5.91), it is not difficult to show that

$$\mathbf{R}^2 D_{M'M}^J(\Phi, \Theta, \psi) = J(J+1) D_{M'M}^J(\Phi, \Theta, \psi). \quad (5.92)$$

To determine the eigenvalue of a projection of \mathbf{R} in the stationary coordinate frame K , we perform the summation over M' in the second term of (5.89)

$$\sum_{M'} \left(\mathbf{R} D_{M'M}^J(\Phi, \Theta, \psi) \right) \psi_{JM'}(\vartheta', \varphi') - \mathbf{J} \psi_{JM}(\vartheta, \varphi) = 0. \quad (5.93)$$

By projecting this equality on to the axis Oz and making use of the relation $\mathbf{J}_z \psi_{JM}(\vartheta, \varphi) = M \psi_{JM}(\vartheta, \varphi)$, we find

$$\sum_{M'} \left\{ \left(\mathbf{R}_z D_{M'M}^J(\Phi, \Theta, \psi) \right) - M D_{M'M}^J(\Phi, \Theta, \psi) \right\} \psi_{JM'}(\vartheta', \varphi') = 0,$$

whence

$$\mathbf{R}_z D_{M'M}^J(\Phi, \Theta, \psi) = M D_{M'M}^J(\Phi, \Theta, \psi). \quad (5.94)$$

Analogously, it can be shown that

$$\left(\mathbf{R}_x \pm i \mathbf{R}_y \right) D_{M'M}^J(\Phi, \Theta, \psi) = \sqrt{(J \mp M)(J \pm M + 1)} D_{M', M \pm 1}^J(\Phi, \Theta, \psi). \quad (5.95)$$

It is not difficult to verify that the equality (5.92) follows also from (5.94) and (5.95).

Thus, we have shown that the generalized spherical harmonics $D_{M'M}^J(\Phi, \Theta, \psi)$ are simultaneously eigenfunctions of the operators \mathbf{R}^2 , \mathbf{R}_z and $\mathbf{R}_{z'}$, the corresponding eigenvalues being equal to $J(J+1)$, M and M' respectively.

5.3. Rotation and Surface Vibrations of Deformed Nuclei

Ellipsoidal deformations of nuclei. For small deformations of nuclei, we can confine ourselves to taking into account the second-order deformations only. In this case, the surface of the deformed nucleus is an ellipsoid, arbitrarily oriented in space:

$$R(\vartheta, \varphi) = \left\{ 1 + \sum_{\mu} \alpha_{\mu} Y_{2\mu}(\vartheta, \varphi) \right\} R \quad (5.96)$$

(for simplicity we shall omit the index $\lambda = 2$). In place of the five parameters α_μ describing the deformations, we can introduce three angles defining the orientation of the ellipsoid and two internal parameters defining its shape. (Because the volume of the nucleus is conserved, the product of the three principal axes of the ellipsoid remains constant under deformations).

We introduce the proper coordinate frame K' whose axes coincide with the principal axes of the ellipsoid. The orientation of the frame K' relative to the stationary frame K is defined by the Euler angles Φ, Θ and ψ . We write the equation of the surface in the proper coordinate frame:

$$R(\vartheta', \varphi') = \left\{ 1 + \sum_{\nu} \alpha'_{\nu} Y_{2\nu}(\vartheta', \varphi') \right\} R, \quad (5.97)$$

where α'_{ν} are the deformation parameters in the proper coordinate frame K' . Using the formula (5.60) for the transformation of spherical harmonics on going from K to K' , it is not difficult to obtain the following relation between the parameters α_μ and α'_{μ} :

$$\alpha_\mu = \sum_{\nu} D_{\nu\mu}^*(\Phi, \Theta, \psi) \alpha'_{\nu}. \quad (5.98)$$

Because of the symmetry of the ellipsoid about the principal axes in the proper coordinate frame, we have

$$\alpha'_1 = \alpha'_{-1} = 0, \quad \alpha'_2 = \alpha'_{-2}. \quad (5.99)$$

In fact, on reflection through the plane $z' = 0$ ($\vartheta', \varphi' \rightarrow \pi - \vartheta', \varphi'$), the spherical harmonics Y_{20} and $Y_{2\pm 2}$ remain unchanged while the functions $Y_{2\pm 1}$ change sign. It follows from this that $\alpha'_1 = \alpha'_{-1} = 0$. On reflection through the plane $x' = 0$ ($\vartheta', \varphi' \rightarrow \vartheta', \pi - \varphi'$), the spherical harmonics $Y_{2\mp 2}$ go over into $Y_{2\mp 2}$ whence it follows that $\alpha'_{-2} = \alpha'_2$. Thus, in the proper frame, deformations of the surface are characterized by only two real parameters α'_0 and α'_2 . In place of α'_0 and α'_2 it is convenient to introduce two new parameters β and γ by means of the relations

$$\alpha'_0 = \beta \cos \gamma, \quad \alpha'_2 = (\beta/\sqrt{2}) \sin \gamma. \quad (5.100)$$

By virtue of the unitarity of $D_{\mu\nu}$, we have

$$\sum_{\mu} |\alpha_{\mu}|^2 = \alpha'_0'^2 + 2\alpha'_2'^2 = \beta^2. \quad (5.101)$$

Consequently, the parameter β defines the total deformation of the nucleus. The parameter γ characterizes the deviation of the shape of the nucleus from an axially symmetric shape.

As can be easily verified, the principal semi-axes of the nucleus can be expressed in terms of β and γ in the following way:

$$R_{k'} = \left\{ 1 + \sqrt{\frac{5}{4\pi}} \beta \cos\left(\gamma - \frac{2\pi}{3}k'\right) \right\}, \quad k' = 1, 2, 3 \equiv x', y', z'. \quad (5.102)$$

If $\gamma = 0$ and $\beta > 0$, the nucleus is a prolate ellipsoid of revolution:

$$R_{x'} = R_{y'} = \left(1 - \frac{1}{2} \sqrt{\frac{5}{4\pi}} \beta \right) R, \quad R_{z'} = \left(1 + \sqrt{\frac{5}{4\pi}} \beta \right) R.$$

If $\gamma = \pi$ and $\beta > 0$, we have an oblate ellipsoid of revolution:

$$\mathbf{R}_{x'} = \mathbf{R}_{y'} = \left(1 + \frac{1}{2} \sqrt{\frac{5}{4\pi}} \beta \right) \mathbf{R}, \quad \mathbf{R}_{z'} = \left(1 - \sqrt{\frac{5}{4\pi}} \beta \right) \mathbf{R}.$$

If $\gamma = \pi/3$ or $2\pi/3$, the symmetry axis will be the axis Oy' or the Ox' .

Thus, in place of the five *deformation parameters* α_μ , we have introduce five new coordinates: the three *Euler angles* Φ, Θ and ψ defining the orientation of the principal axes of the ellipsoid in space, and the two *internal parameters* β and γ defining the shape of the ellipsoid. Physically, the idea of introducing the new coordinates is to make it possible to study independently the rotation of the nucleus as a whole and the vibrations associated with change of shape of the nucleus.

Vibrational energy and energy of rotation. In the case of *ellipsoidal deformations of the nucleus*, the kinetic energy is given by the general expression

$$T = (B/2) \sum_\mu |\dot{\alpha}_\mu|^2. \quad (5.103)$$

The transformation to the new coordinates ($\alpha_\mu \rightarrow \Phi, \Theta, \beta$ and γ) makes it possible to divide the total kinetic energy (5.103) of the surface deformations into rotational and vibrational energies. To find the form of the kinetic energy in the proper coordinate frame, we differentiate the equality (5.98) with respect to the time:

$$\dot{\alpha}_\mu = \sum_\nu (D_{\nu\mu}^* \dot{\alpha}'_\nu + \dot{D}_{\nu\mu}^* \alpha'_\nu).$$

The time derivative of $D_{\mu\nu}$ can be expressed directly, according to (5.73), in terms of the functions $D_{\mu\nu}$ and the time derivatives of the Euler angles. Substituting the expression for $\dot{\alpha}_\mu$ into (5.103), we can represent the kinetic energy as a sum of three terms:

$$T = (B/2) \sum_{\mu\nu\nu'} \left(D_{\nu\mu}^* D_{\nu'\mu} \dot{\alpha}'_\nu \dot{\alpha}'_{\nu'} + 2 \operatorname{Re} [D_{\nu\mu}^* \dot{D}_{\nu'\mu} \dot{\alpha}'_\nu \alpha'_{\nu'}] + \dot{D}_{\nu\mu}^* \dot{D}_{\nu'\mu} \alpha'_\nu \alpha'_{\nu'} \right). \quad (5.104)$$

The first term in (5.104), which is quadratic in $\dot{\alpha}_\nu$, determines the kinetic energy of the vibrations that change the shape of the ellipsoid but conserve its orientation in space:

$$T_v = (B/2) \sum_{\mu\nu\nu'} D_{\nu\mu}^* D_{\nu'\mu} \dot{\alpha}'_\nu \dot{\alpha}'_{\nu'}.$$

Using the unitarity of the functions $D_{\mu\nu}$, it is not difficult to bring the vibrational energy to the form

$$T_v = (B/2)(\dot{\beta}^2 + \beta^2 \dot{\gamma}^2). \quad (5.105)$$

The second term in (5.104), containing the cross-terms, vanishes, as can easily be seen by making use of the explicit form (5.73) of $\dot{D}_{\mu\nu}$:

$$\operatorname{Re} \sum_{\mu\nu\nu'} D_{\nu\mu}^* \dot{D}_{\nu'\mu} \dot{\alpha}'_\nu \alpha'_{\nu'} = \operatorname{Re} i \sum_{\nu\nu'} \sum_{k'} \omega_{k'} \langle \nu' | j_{k'} | \nu \rangle \dot{\alpha}'_\nu \alpha'_{\nu'}.$$

Since ν and ν' take only the values -2, 0 and 2, only the term $k' = z'$ can give a contribution, and so

$$\operatorname{Re} \sum_{\mu\nu\nu'} D_{\nu\mu}^* \dot{D}_{\nu'\mu} \dot{\alpha}'_\nu \alpha'_{\nu'} = \operatorname{Re} i \omega_{z'} \sum_{\nu} \nu \dot{\alpha}'_\nu \alpha'_{\nu} = 0.$$

The third term in (5.104), which is quadratic in $\dot{D}_{\mu\nu}$, gives the kinetic energy of rotation of the ellipsoid with no change of shape:

$$T_r = (B/2) \sum_{\mu\nu\nu'} \dot{D}_{\nu\mu}^* \dot{D}_{\nu'\mu} \alpha'_\nu \alpha'_{\nu'}.$$

Using the explicit form (5.73) of the derivatives $\dot{D}_{\mu\nu}$ and the unitary of the functions $D_{\mu\nu}$, we find

$$T_r = (B/2) \sum_{\nu\nu'} \sum_{k' \bar{k}'} \omega_{k'} \omega_{\bar{k}'} \langle \nu' | \mathbf{j}_{\bar{k}'} \mathbf{j}_{k'} | \nu \rangle \alpha'_\nu \alpha'_{\nu'}. \quad (5.106)$$

Since ν and ν' take only even values, only the matrix elements $\langle \nu' | \mathbf{j}_{\bar{k}'} \mathbf{j}_{k'} | \nu \rangle$ with $k' = \bar{k}'$ make a contribution to (5.106). In fact, if either of k' and \bar{k}' is equal to z' and the other is equal to x' or y' , the matrix element $\langle \nu' | \mathbf{j}_{\bar{k}'} \mathbf{j}_{k'} | \nu \rangle$ vanishes, since $\mathbf{j}_{z'}$ is diagonal, while for $\mathbf{j}_{x'}$ or $\mathbf{j}_{y'}$ the only non-zero matrix elements are for states with quantum numbers ν and ν'

differing by unity. But if either of k' and \bar{k}' is equal to x' while the other is equal to y' , the corresponding matrix elements will be purely imaginary and their sum in (5.106) will be equal to zero. Therefore, we can write the kinetic energy of rotation (5.106) in the form

$$T_r = \frac{1}{2} \sum_{k'} \omega_{k'}^2 J_{k'}, \quad (5.107)$$

where $J_{k'}$ is the moment of inertia of the nucleus about the principal axis k' :

$$J_{k'} = B \sum_{\nu\nu'} \langle \nu' | \mathbf{j}_{k'}^2 | \nu \rangle \alpha'_{\nu} \alpha'_{\nu'}.$$

Using the explicit expressions for the matrix elements of the squares of the angular-momentum components, we finally obtain the following formula for the three *moments of inertia of the nucleus*:

$$J_{k'} = 4B\beta^2 \sin^2 \left(\gamma - \frac{2\pi}{3} k' \right), \quad k' = 1, 2, 3. \quad (5.108)$$

The moments of inertia are proportional to the square of the deformation parameter β of the nucleus. In the case of an axially symmetric nucleus, $\gamma = 0$ or $\gamma = \pi$ and the moments of inertia are

$$J_{x'} = J_{y'} = 3B\beta^2, \quad J_{z'} = 0. \quad (5.109)$$

The angular momentum of the nucleus about the principal axis k' is connected with the angular velocity and moment of inertia of the nucleus by the relation

$$\hbar \mathbf{R}_{k'} = \omega_{k'} J_{k'}. \quad (5.110)$$

Therefore, the energy of rotation of the nucleus can be represented in the form

$$T_r = \sum_{k'} (\hbar^2 / 2J_{k'}) \mathbf{R}_{k'}^2. \quad (5.111)$$

According to (5.5) and (5.101), the potential energy in the case of small deviations of the shape of the nucleus from spherical shape is expressed in terms of the deformation parameter β only:

$$V = \frac{1}{2} C \beta^2. \quad (5.112)$$

Therefore, the collective Hamiltonian H of the nucleus in the case of small deformations can be represented as the sum of two parts H_v and H_r , associated with the vibrations of the surface of the nucleus and with the rotation of the nucleus as a whole:

$$H = H_v + H_r, \quad H_v = \frac{1}{2}B(\dot{\beta}^2 + \beta^2\dot{\gamma}^2) + \frac{1}{2}C\beta^2, \quad H_r = \sum_{k'} \frac{\hbar^2}{2J_{k'}} \mathbf{R}_{k'}^2; \quad (5.113)$$

that is, in the case of small deformations the surface vibrations of the nucleus and the rotation of the nucleus can be completely separated.

Potential energy of vibrations of non-spherical nuclei. We shall consider the surface vibrations of a rotating nucleus whose equilibrium shape is non-spherical. We denote the equilibrium values of the deformation parameters by $\alpha_{\lambda\mu}^0$ and introduce the deviations of the deformation parameters from their equilibrium values:

$$a_{\lambda\mu} = \alpha_{\lambda\mu} - \alpha_{\lambda\mu}^0. \quad (5.114)$$

By $\alpha_{\lambda\mu}$ (which may be large) we shall mean, as before, the coefficients in the expansion in spherical harmonics of the distance from the centre of the nucleus to a point on its surface:

$$R(\vartheta, \varphi) = \left\{ 1 + \sum_{\lambda\mu} (\alpha_{\lambda\mu}^0 + a_{\lambda\mu}) Y_{\lambda\mu}(\varphi) \right\} R. \quad (5.115)$$

The equilibrium parameters $\alpha_{\lambda\mu}^0$, defined in the stationary coordinate frame K , also describe the rotation of the nucleus and therefore depend on time. In the rotating coordinate frame K' , the parameters $\alpha'_{\lambda\mu}^0$ define only the shape of the nucleus, and are therefore constant.

The kinetic energy associated with the collective degrees of freedom of the nucleus is a quadratic function of the generalized velocities $\dot{\alpha}_{\lambda\mu}$; however, it can also depend on the generalized coordinates $\alpha_{\lambda\mu}$. The kinetic energy is associated with both the rotation of the nucleus and the vibrations of its surface. The potential energy of a non-spherical nucleus will be expressed purely in terms of the deviations of the parameters $\alpha_{\lambda\mu}$ from the equilibrium values $\alpha_{\lambda\mu}^0$, and does not depend on the rotation of the nucleus.

The quantities $a_{\lambda\mu}$, like $\alpha_{\lambda\mu}$ and $\alpha_{\lambda\mu}^0$, transform under rotations of the coordinate frame according to irreducible representations of the rotation group,

$$a_{\lambda\mu} = \sum_{\nu} D_{\nu\mu}^*(\Phi, \Theta, \psi) a'_{\lambda\nu}, \quad (5.116)$$

and are spherical tensor operators of rank λ , satisfying the hermiticity relations $a_{\lambda\mu}^* = (-1)^\mu \times a_{\lambda,-\mu}$ which follow from the reality of (5.115). Since the potential energy of the nucleus is a scalar quantity, it must be expressed in terms of invariant combinations, independent of the rotation of the coordinate frame, of the tensor components $a_{\lambda\mu}$; in other words, the potential energy must be a tensor of rank zero, constructed from the components $a_{\lambda\mu}$. The simplest such invariant is the quadratic invariant (the quadratic invariant expression constructed from the components $a_{\lambda\mu}$):

$$S_2(\lambda) = \sum_{\mu\mu'} (\lambda\mu\lambda\mu' | 00) a_{\lambda\mu} a_{\lambda\mu'} = \frac{(-1)^\lambda}{\sqrt{2\lambda+1}} \sum_\mu |a_{\lambda\mu}|^2. \quad (5.117)$$

The invariant containing the components $a_{\lambda\mu}$ to third order has the form

$$S_3(\lambda) = \sum_{\mu_1\mu_2\mu_3} (\lambda\mu_1\lambda\mu_2|\lambda\mu) (\lambda\mu\lambda\mu_3|00) a_{\lambda\mu_1} a_{\lambda\mu_2} a_{\lambda\mu_3}. \quad (5.118)$$

For a given λ it is impossible to construct any other second- and third-order invariants from the components $a_{\lambda\mu}$.

The invariants are most easily calculated in the proper coordinate frame K' moving with the nucleus. For $\lambda = 2$, choosing the axes of the coordinate frame to be along the principal axes of the ellipsoid, we introduce, as in the case of a spherical equilibrium shape, the new deformation parameters ($a'_\mu = \alpha'_\mu - \alpha'^0_\mu$):

$$\left. \begin{aligned} \alpha'_0 &= \beta \cos \gamma; & \alpha'_2 &= \alpha'_{-2} = (\beta/\sqrt{2}) \sin \gamma; & \alpha'_{\pm 1} &= 0; \\ \alpha'^0_0 &= \beta_0 \cos \gamma_0; & \alpha'^0_2 &= \alpha'^0_{-2} = (\beta_0/\sqrt{2}) \sin \gamma_0; & \alpha'^0_{\pm 1} &= 0. \end{aligned} \right\} \quad (5.119)$$

Using these definitions, we obtain the following expressions for the invariants $S_2(2)$ and $S_3(2)$:

$$S_2(2) = \frac{1}{\sqrt{5}} \left\{ \left(\beta - \beta_0 \right)^2 + 4\beta\beta_0 \sin^2 \frac{\gamma - \gamma_0}{2} \right\}; \quad (5.120)$$

$$\begin{aligned} S_3(2) = \sqrt{\frac{2}{35}} &\left(\beta \cos \gamma - \beta_0 \cos \gamma_0 \right) \left\{ 3 \left(\beta \sin \gamma - \beta_0 \sin \gamma_0 \right)^2 \right. \\ &\left. - \left(\beta \cos \gamma - \beta_0 \cos \gamma_0 \right)^2 \right\}. \end{aligned} \quad (5.121)$$

We shall consider *small vibrations of the surface of a rotating nucleus*. In this case, the mass coefficients B_λ appearing in the expression for the kinetic energy are constants:

$$T = \frac{1}{2} \sum_{\lambda\mu} B_\lambda |\dot{\alpha}_{\lambda\mu}|^2. \quad (5.122)$$

In the same approximation, the potential energy can be expressed purely in terms of the invariants $S_2(\lambda)$:

$$V = \frac{1}{2} \sum_{\lambda\mu} C_\lambda |a_{\lambda\mu}|^2. \quad (5.123)$$

In the case of *small quadrupolar* ($\lambda = 2$) vibrations, in the kinetic energy (5.122), as in the case of a spherical equilibrium shape ($\beta_0 = 0$), we can separate the energy of rotation of the nucleus from the energy of the vibrations of the nuclear surface. As a result, we obtain for the kinetic energy (for $\lambda = 2$) a sum of the vibrational and rotational energies, $T = T_v + T_r$, given by the expressions (5.105), (5.107) and (5.108), in which, however, in the general case, $\beta_0 \neq 0$ and $\gamma_0 \neq 0$, since only the differences $\beta - \beta_0$ and $\gamma - \gamma_0$ are small while β and γ (like β_0 and γ_0) may be large. For $\beta_0 = 0$ we have a spherical nucleus, and in this case the dependence on γ_0 drops out automatically.

According to (5.120), the potential energy of small quadrupolar vibrations has the form

$$\begin{aligned} V(\beta, \gamma) &= (C/2) \left\{ (\beta - \beta_0)^2 + \beta \beta_0 (\gamma - \gamma_0)^2 \right\} \\ &\approx (C/2) \left\{ (\beta - \beta_0)^2 + \beta_0^2 (\gamma - \gamma_0)^2 \right\}. \end{aligned} \quad (5.124)$$

Thus, in the case of spheroidal nuclei having non-spherical equilibrium shapes, surface vibrations of two types, usually called β - and γ -vibrations, are possible.

In the case of spherical nuclei ($\beta_0 = 0$), the potential energy does not depend on the parameter γ in the quadratic approximation. But if we take into account the anharmonicity of the vibrations, then, according to (5.121), the potential energy will depend on the transverse-deformation parameter γ even in the case of a spherical nucleus:

$$V(\beta, \gamma) = \frac{1}{2} C \beta^2 + C' \beta^3 (3 - 4 \cos^2 \gamma) \cos \gamma, \quad (5.125)$$

where C' is some constant.

5.4. Rotation of Axially Symmetric Nuclei

Rotational model. The simplest version of the generalized model of the nucleus is the so-called *rotational model*, in which the shape of the nucleus is assumed fixed. In the rotational model, in addition to the rotation of the nucleus as a whole, excitations associated with the internal degrees of freedom of the nucleus are also taken into account.⁴ The rotational model gives a good description of the level spectrum and other characteristics of deformed nuclei in the mass-number ranges $150 < A < 190$ and $A > 220$. In the case of deformed even-even nuclei, almost all the weakly excited levels are rotational levels. In the case of deformed nuclei with odd A , low-lying levels associated with the excitation of individual nucleons are also encountered. A separate rotational band is associated with each such level.

We denote the coordinate frame rotating with the deformed nucleus by K' . The Euler angles Φ, Θ and ψ defining the orientation of K' with respect to the stationary coordinate frame K can be regarded as the dynamical variables describing the rotation of the nucleus as a whole. We denote the coordinates characterizing the internal degrees of freedom of the nucleus by \mathbf{r}' .

We write the full Hamiltonian of the nucleus in the form

$$H = H_0(\mathbf{r}') + H_r + H', \quad (5.126)$$

where $H_0(\mathbf{r}')$ describes the motion of the internal degrees of freedom, H_r characterizes the rotation of the nucleus and H' takes account of the coupling between the rotation and the internal motion.

Our subsequent treatment will be based on the assumption that the motion is adiabatic, that is, that the wave function of the nucleus can be written in the form of the product

$$\Psi \sim D(\Theta)\chi(\mathbf{r}'), \quad (5.127)$$

where $D(\Theta)$ is the rotational wave function describing the collective motion of the nucleus (by Θ , we shall mean the collection of Euler angles Φ, Θ and ψ); $\chi(\mathbf{r}')$ is the wave function describing the motion of the internal degrees of freedom.

The assumption that the motion is adiabatic means that we can neglect the quantity H' compared with $H_0(\mathbf{r}')$ and H_r . However, the neglect of H' still does not imply complete separation of the collective and internal motions, since these are also coupled in H_r . Therefore, the rotational function $D(\Theta)$ is not, generally speaking, an eigenfunction of the operator H_r .

⁴The theory of the rotation of axially symmetric nuclei was developed in papers by Bohr (1952) and Bohr and Mottelson (1953).

Let \mathbf{J} be the nuclear angular momentum associated with the internal degrees of freedom and let \mathbf{R} be the angular momentum of rotation of the nucleus as a whole. The total angular momentum \mathbf{I} of the nucleus is equal to the sum of the angular momenta \mathbf{R} and \mathbf{J} :

$$\mathbf{I} = \mathbf{R} + \mathbf{J}. \quad (5.128)$$

It is clear that the total wave function (5.127) of the nucleus must be an eigenfunction of the square \mathbf{I}^2 of the total angular momentum.

Angular-momentum coupling scheme and wave function of the nucleus. We shall determine the wave function (5.127) of the nucleus, taking into account that it is an eigenfunction of the operators of the square \mathbf{I}^2 and projection I_z of the total angular momentum. Since the potential of a deformed nucleus does not possess spherical symmetry, the *internal angular momentum \mathbf{J} of the nucleus* is not a constant of the motion. Experiment shows, however, that most deformed nuclei are characterized by axial symmetry. For an *axially symmetric nucleus*, the projection of the internal angular momentum \mathbf{J} along the symmetry axis of the nucleus will be conserved. Therefore, the internal wave function will correspond to a definite value of $J_{z'}$, which we denote by Ω :

$$\langle J_{z'} \rangle = \Omega. \quad (5.129)$$

Denoting by τ all the other quantum numbers characterizing the internal state of the nucleus, we can write the internal wave function in the form

$$\chi_\Omega^\tau(\mathbf{r}'). \quad (5.130)$$

(The wave function $\chi_\Omega^\tau(\mathbf{r}')$ is assumed to be normalized to unity.)

Since the *total angular-momentum operator* $\mathbf{I} = \mathbf{R} + \mathbf{J}$ is the operator of the simultaneous rotation of the proper coordinate frame K' and of the physical system described by the function $\chi_\Omega^\tau(\mathbf{r}')$, we have

$$\mathbf{I}\chi_\Omega^\tau(\mathbf{r}') = 0. \quad (5.131)$$

Consequently, the total angular-momentum operator \mathbf{I} acts only on the rotational function $D(\Theta)$, whereas the operator \mathbf{J} of the internal angular momentum acts on $\chi_\Omega^\tau(\mathbf{r}')$.

For completeness, we give the commutation relations for the angular momenta \mathbf{J} and \mathbf{I} . The operator \mathbf{J} of the internal angular momentum acts only on the internal coordinates in the proper frame K' , and so the components of \mathbf{J} along the axes of the proper coordinate frame K' satisfy the usual commutation relations

$$\mathbf{J}_{x'}\mathbf{J}_{y'} - \mathbf{J}_{y'}\mathbf{J}_{x'} = i\mathbf{J}_{z'}. \quad (5.132)$$

The operator \mathbf{I} of the total angular momentum acts only on the angles Φ, Θ and ψ defining the orientation of the nucleus in the stationary coordinate frame K . Therefore, the components of \mathbf{I} along the axes of the stationary frame satisfy the same relations:

$$\mathbf{I}_x\mathbf{I}_y - \mathbf{I}_y\mathbf{I}_x = i\mathbf{I}_z. \quad (5.133)$$

The components of the total angular momentum \mathbf{I} along the axes of the proper coordinate frame K' satisfy the commutation relations

$$\mathbf{I}_{x'}\mathbf{I}_{y'} - \mathbf{I}_{y'}\mathbf{I}_{x'} = -i\mathbf{I}_{z'}. \quad (5.134)$$

The minus sign in the right-hand side of (5.134) is associated with the fact that, in an arbitrary rotation of the proper coordinate frame K' , the function $D(\Theta)$ on which \mathbf{I} acts transforms in the same way as in a rotation of the laboratory coordinate frame K about the same axis and through the same angle, but in the opposite direction. We note that the operators \mathbf{J} and \mathbf{I} commute.

We choose the rotational function $D(\Theta)$ in such a way that it is simultaneously an eigenfunction of the operators $\mathbf{I}^2, \mathbf{I}_z$ and $\mathbf{I}_{z'}$. Inasmuch as the operator \mathbf{J} does not act on $D(\Theta)$, this means that the function $D(\Theta)$ must be simultaneously an eigenfunction of the operators \mathbf{R}^2, R_z and $R_{z'}$. According to (5.90), (5.92) and (5.94), these properties are possessed by the function $D_{KM}^I(\Phi, \Theta, \psi)$, and so

$$\left. \begin{aligned} \mathbf{I}^2 D_{KM}^I(\Phi, \Theta, \psi) &= I(I+1)D_{KM}^I(\Phi, \Theta, \psi); \\ I_z D_{KM}^I(\Phi, \Theta, \psi) &= M D_{KM}^I(\Phi, \Theta, \psi); \\ I_{z'} D_{KM}^I(\Phi, \Theta, \psi) &= K D_{KM}^I(\Phi, \Theta, \psi). \end{aligned} \right\} \quad (5.135)$$

M and K are the projections of the total angular momentum of the nucleus along the axis Oz in the stationary coordinate frame K and along the axis Oz' in the coordinate frame K' moving with the nucleus.

Using (5.63) and assuming that the motion is adiabatic, we can write the normalized wave function of the nucleus in the form

$$\Psi = \sqrt{\frac{2I+1}{8\pi^2}} D_{KM}^I(\Phi, \Theta, \psi) \chi_\Omega^\tau(\mathbf{r}'). \quad (5.136)$$

Figure 5.5. shows the *angular-momentum coupling scheme in the rotational model* of the nucleus.

Symmetry of the wave function. The above wave function (5.136), however, does not yet satisfy the symmetry requirements stemming from

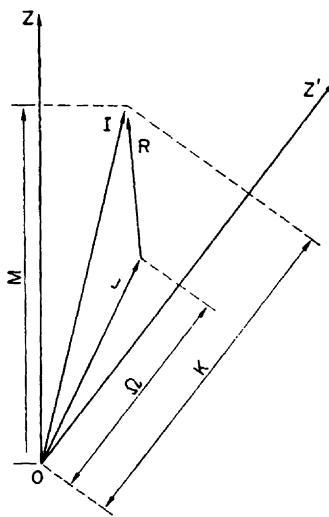


Figure 5.5. Angular-momentum coupling scheme in the rotational model of the nucleus.

the symmetry of the shape of the nucleus. The wave function (5.136) must be modified somewhat in order to satisfy these requirements.

(a) *The axial symmetry of the nucleus*. Because of the axial symmetry of the nucleus, the wave function Ψ must be unchanged on rotation of the coordinate frame associated with the nucleus about the symmetry axis of the nucleus:

$$R_{z'}\Psi = 0. \quad (5.137)$$

Noting that $\mathbf{R} = \mathbf{I} - \mathbf{J}$, we can rewrite this condition

$$\left(I_{z'} D_{KM}^I(\Phi, \Theta, \psi) \right) \chi_{\Omega}^{\tau}(\mathbf{r}') = D_{KM}^I(\Phi, \Theta, \psi) \left(J_{z'} \chi_{\Omega}^{\tau}(\mathbf{r}') \right).$$

Consequently, the requirement that the wave function be invariant under rotations about the symmetry axis of the nucleus is equivalent to the condition

$$K = \Omega. \quad (5.138)$$

The condition means that the angular momentum of rotation \mathbf{R} is perpendicular to the symmetry axis for an axially symmetric nucleus.

Thus, for an axially symmetric nucleus the wave function must have the form

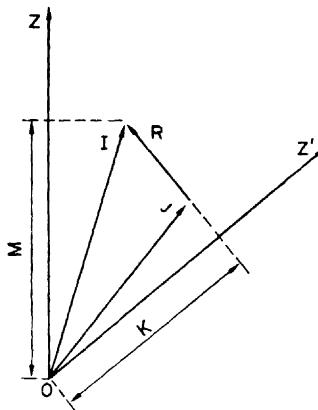


Figure 5.6. Angular-momentum coupling scheme for an axially symmetric nucleus.

$$\Psi_{IKM} = \sqrt{\frac{2I+1}{8\pi^2}} D_{KM}^I(\Phi, \Theta, \psi) \chi_K^\tau(\mathbf{r}'). \quad (5.139)$$

Figure 5.6 shows the angular-momentum coupling scheme for an axially symmetric nucleus.

(b) *The symmetry of the nucleus with respect to the plane $z' = 0$* . If the nucleus is an ellipsoid of revolution, it is transformed into itself on reflection through the principal plane perpendicular to the symmetry axis. The reflection operation p through the plane $z' = 0$ is equivalent for an axially symmetric nucleus to a rotation $R_{y'}(\pi)$ through an angle of 180° about the y' -axis. Under a rotation $R_{y'}(\pi)$, the rotational function $D_{KM}^I(\Phi, \Theta, \psi)$ transforms according to (5.58):

$$R_{y'}(\pi) D_{KM}^I = \sum_{K'} D_{K'K}^{*I}(0, \pi, 0) D_{K'M}^I. \quad (5.140)$$

Noting that

$$D_{K'K}^{*I}(0, \pi, 0) = (-1)^{I-K} \delta_{K', -K},$$

we have

$$R_{y'}(\pi) D_{KM}^I = (-1)^{I-K} D_{-KM}^I. \quad (5.141)$$

Analogously, for the internal wave function χ_K^τ we obtain

$$R_{y'}(\pi) \chi_K^\tau = (-1)^{J-K} \chi_{-K}^\tau, \quad (5.142)$$

where $(-1)^J$ must be regarded as an operator. If

$$\chi_K^\tau = \sum_J c_J \chi_{JK}^\tau,$$

then

$$(-1)^J \chi_K^\tau = \sum_J c_J (-1)^J \chi_{JK}^\tau.$$

Noting that

$$(-1)^{I+J-2K} = (-1)^{I-J+2(J-K)} = (-1)^{I-J},$$

we have

$$R_{y'}(\pi) \Psi_{IKM} = (-1)^{I-J} \Psi_{I-KM}. \quad (5.143)$$

If $K \neq 0$, the requirement of invariance under the rotation $R_{y'}(\pi)$ implies that the wave function should be a linear combination of (5.139) and (5.143). The normalized wave function has the form

$$\Psi(IKM) = \sqrt{\frac{1}{2}} \left\{ \Psi_{IKM} + (-1)^{I-J} \Psi_{I-KM} \right\}. \quad (5.144)$$

If $K = 0$, the function (5.139) already satisfies the required invariance property; here it is necessary, however, that I and J have the same parity (otherwise the function vanishes):

$$(-1)^{I-J} = 1. \quad (5.145)$$

(c) *The parity of the wave function*. The total-inversion operator \mathbf{P} includes the inversion p' of the internal coordinates with respect to the axes x', y', z' and the inversion p of the collective coordinates (the axes x', y', z'): $\mathbf{P} = pp'$. We denote the parity of the internal wave function χ_K^τ by π :

$$p' \chi_K^\tau = \pi \chi_K^\tau. \quad (5.146)$$

Since $p = \mathbf{R}_{y'}(\pi)$ for an axially symmetric nucleus, for states Ψ that are invariant under reflection in the plane $z' = 0$ we have

$$\mathbf{P}(\Psi) = \pi(\chi). \quad (5.147)$$

Thus, the parity of the wave function Ψ is determined by the parity of the internal wave function χ .

If $K = 0$, the parity of the internal wave function can be reduced directly to the value \mathbf{I} of the total angular momentum. In fact, for $K = 0$ the

function $\chi_{K=0}^\tau$ is invariant under rotations about the z' -axis, and so $p' = \mathbf{R}_{y'}(\pi)$. Using (5.142) and (5.145), we find

$$\pi(\chi) = (-1)^J = (-1)^I. \quad (5.148)$$

(d) *The wave functions.* We shall give explicit expressions for the *wave functions of a rotating axially symmetric nucleus* in states with well-defined angular momentum and parity.

If $K \neq 0$, the wave function has the form

$$\begin{aligned} \Psi(IKM) = & \sqrt{\frac{2I+1}{16\pi^2}} \left\{ D_{KM}^I(\Phi, \Theta, \psi) \chi_K^\tau(\mathbf{r}') \right. \\ & \left. + (-1)^{I-J} D_{-KM}^I(\Phi, \Theta, \psi) \chi_{-K}^\tau(\mathbf{r}') \right\}. \end{aligned} \quad (5.149)$$

The parity of the function Ψ is the same as the parity of the internal function χ . The set of states (5.149) corresponding to a given internal function χ_K^τ but to different values I of the total angular momentum forms a rotational band:

$$I = K, \quad K+1, \quad K+2, \dots \quad (5.150)$$

If $K = 0$, the wave function has the form

$$\Psi(I0M) = \sqrt{\frac{2I+1}{8\pi^2}} D_{0M}^I(\Phi, \Theta, \psi) \chi_0^\tau(\mathbf{r}'), \quad (5.151)$$

or, using (5.61),

$$\Psi(I0M) = \left(1/\sqrt{2\pi} \right) Y_{IM}(\Theta, \Phi) \chi_0^\tau(\mathbf{r}'). \quad (5.152)$$

In this case, two rotational bands are possible, for the even and odd states respectively:

$$\begin{aligned} I = & 0^+, 2^+, 4^+, \dots, \pi = +1, \\ I = & 1^-, 3^-, 5^-, \dots, \pi = -1. \end{aligned} \quad \left. \right\} \quad (5.153)$$

Rotational spectra of nuclei. Making the assumption of adiabatic motion, we write the Hamiltonian of the rotating nucleus in the form

$$H = H_0(\mathbf{r}') + H_r. \quad (5.154)$$

According to (5.111), the rotational part of the Hamiltonian is given by the expression

$$\mathbf{H}_r = \frac{\hbar^2}{2J_{x'}} \mathbf{R}_{x'}^2 + \frac{\hbar^2}{2J_{y'}} \mathbf{R}_{y'}^2 + \frac{\hbar^2}{2J_{z'}} \mathbf{R}_{z'}^2, \quad (5.155)$$

where $J_{x'}$, $J_{y'}$ and $J_{z'}$ are the moments of inertia about the principal axes and can be regarded as empirical parameters.

Assuming the nucleus to be axially symmetric and choosing the z' -axis in the direction of the symmetry axis of the nucleus, we put $J_{x'} = J_{y'} = J$. Noting that $\mathbf{R} = \mathbf{I} - \mathbf{J}$, we rewrite (5.155) in the form

$$H_r = \frac{\hbar^2}{2J} \left\{ \mathbf{I}^2 + \mathbf{J}^2 - 2\mathbf{I} \cdot \mathbf{J} \right\} + \left(\frac{\hbar^2}{2J_{z'}} - \frac{\hbar^2}{2J} \right) (I_{z'} - J_{z'})^2. \quad (5.156)$$

In the case of an axially symmetric nucleus, the projections $I_{z'}$ and $J_{z'}$ along the symmetry axis of the nucleus are equal (cf. (5.138)), and so the Hamiltonian (5.154) of the nucleus can be represented finally in the form

$$H = H_0(\mathbf{r}') + \frac{\hbar^2}{2J} \mathbf{J}^2 + \frac{\hbar^2}{2J} \mathbf{I}^2 - \frac{\hbar^2}{J} \mathbf{I} \cdot \mathbf{J}. \quad (5.157)$$

The first two terms (5.157) characterize the motion of the internal degrees of freedom of the nucleus. The internal function χ_K^τ can be assumed to be an eigenfunction of this part of the Hamiltonian. The rotational function $D_{KM}^I(\Phi, \Theta, \psi)$ is an eigenfunction of the third term in (5.157). If we neglect the fourth term in (5.157), we obtain the following simple formula for the *energy levels of a rotational band* :

$$E_K(I) = E_K^0 + \frac{\hbar^2}{2J} I(I+1). \quad (5.158)$$

where E_K^0 is the internal energy, which does not depend on I .

The term $-\hbar^2 \mathbf{I} \cdot \mathbf{J}/J$ in (5.157) takes into account the coupling between the rotational and internal motions in the nucleus. For an estimate of the contribution of this term to the energy, we represent the scalar product of the angular momenta \mathbf{I} and \mathbf{J} in the form

$$\mathbf{I} \cdot \mathbf{J} = I_{z'} J_{z'} + \frac{1}{2} (I_+ J_- + I_- J_+), \quad (5.159)$$

where

$$I_\pm = I_{x'} \pm i I_{y'}, \quad J_\pm = J_{x'} \pm i J_{y'}. \quad (5.160)$$

Since the function $\Psi(IKM)$ (5.149) is an eigenfunction of the operator $I_{z'} J_{z'}$, corresponding to the eigenvalue K^2 , allowance for this term in (5.159)

leads only to a displacement of the levels (5.158) of the rotational band, with no change in their relative positions.

The other two terms $\frac{1}{2}(I_+J_- + I_-J_+)$ in (5.159) are not diagonal in the representation (5.149), provided that $K \neq \frac{1}{2}$. Using the commutation relations (5.132) and (5.134), it is not difficult to verify that the only non-zero matrix elements of the operators I_{\pm} and J_{\pm} are

$$(K \mp 1|I_{\pm}|K) = \sqrt{(I \pm K)(I \mp K + 1)};$$

$$(K \mp 1|J_{\mp}|K) = \sqrt{(J \pm K)(J \mp K + 1)}.$$

Therefore,

$$\begin{aligned} & I_+J_- D_{KM}^I(\Phi, \Theta, \psi) \chi_{JK}^{\tau}(\mathbf{r}') \\ &= \sqrt{(I + K)(I - K + 1)(J + K)(J - K + 1)} \\ & \times D_{K-1,M}^I(\Phi, \Theta, \psi) \chi_{J-1}^{\tau}(\mathbf{r}'). \end{aligned} \quad (5.161)$$

Thus, the inclusion of I_+J_- and I_-J_+ should cause mixing of states with different values of K . If, however, the difference between the internal energies for the different rotational bands is large compared with the effective value of the coupling energy $-\hbar^2 \mathbf{I} \cdot \mathbf{J}/J$, the mixing of states with different values of K can be neglected. In this case, the energy levels in a particular rotational band are given by formula (5.158).

If $K = \frac{1}{2}$, the term $-\hbar^2 \mathbf{I} \cdot \mathbf{J}/J$ has a diagonal part, depending on I , which must be taken into account in (5.158) and which in certain cases can change the order of the levels in the rotational band. We shall denote the additional energy in the case $K = \frac{1}{2}$ by ΔE . Taking into account the symmetry of the function (5.149), we have

$$\begin{aligned} \Delta E &= -\frac{\hbar^2}{2J} \left(I \frac{1}{2} M |I_+J_- + I_-J_+| I \frac{1}{2} M \right) \\ &= -\frac{\hbar^2}{J} \left(I \frac{1}{2} M |I_+J_-| I \frac{1}{2} M \right). \end{aligned} \quad (5.162)$$

To calculate ΔE , we note that

$$I_+J_- \Psi(I \frac{1}{2} M) = \sqrt{\frac{2I+1}{16\pi^2}} \left\{ I_+ D_{(1/2)M}^I \sum_J c_J J_- \chi_{J1/2}^{\tau} \right\}$$

$$\begin{aligned}
& + I_+ D_{-(1/2)M}^I \sum_J c_J (-1)^{I-J} J_- \chi_{J-1/2}^\tau \Big\} \\
& = \sqrt{\frac{2I+1}{16\pi^2}} \left\{ \left(I + \frac{1}{2} \right) D_{-(1/2)M}^I \sum_J c_J \left(J + \frac{1}{2} \right) \chi_{J,-1/2}^\tau \right. \\
& \left. + \left(\text{term with } K = -\frac{3}{2} \right) \right\}.
\end{aligned}$$

Using the orthonormalization of the functions D_{KM}^I and χ_K^τ , we thus obtain

$$\left(I \frac{1}{2} M | I_+ J_- | I \frac{1}{2} M \right) = \frac{1}{2} \left(I + \frac{1}{2} \right) \sum_J |c_J|^2 (-1)^{I-J} \left(J + \frac{1}{2} \right). \quad (5.163)$$

Taking into account that the number $2J+1$ is even for $K = \frac{1}{2}$, we represent the expression for the energy (5.162) in the form

$$\Delta E = \frac{\hbar^2}{2J} a (-1)^{I+1/2} \left(I + \frac{1}{2} \right), \quad (5.164)$$

where a is the so-called coupling parameter. The magnitude of the *coupling parameter* a depends on the internal state of the odd nucleon and is given by the expression

$$a = \sum_J |c_J|^2 (-1)^{J-1/2} \left(J + \frac{1}{2} \right). \quad (5.165)$$

Combining (5.158) and (5.165), we finally obtain the following formula for the energy levels of a rotational band:

$$E_K(I) = E_K^0 + \frac{\hbar^2}{2J} \left[I(I+1) + a(-1)^{I+1/2} \left(I + \frac{1}{2} \right) \delta_{K,1/2} \right], \quad (5.166)$$

which is valid for both the case $K \neq \frac{1}{2}$ and the case $K = \frac{1}{2}$

Comparison with experiment. The rotational model gives a good description of the rotational spectra of strongly deformed nuclei for the rare earths $150 < A < 190$ and the heavy α -active nuclei $A > 220$, (Fig.5.7) and also for the light nuclei $20 < A < 30$.

Even-even nuclei. In the case of even-even nuclei, the lowest rotational band is the band $K = 0^+$, corresponding to the spin sequence $I=0,2,4,\dots$ We denote the excitation energy, reckoned from the ground state of the nucleus, by E_I :

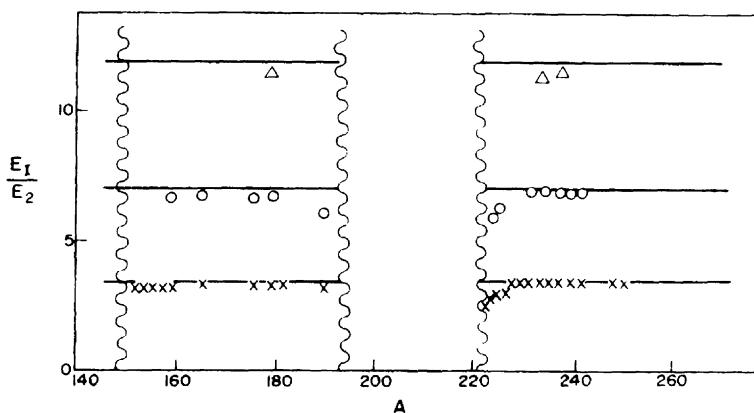


Figure 5.7. Ratios of the rotational energies of strongly deformed nuclei in the regions of the rare-earth nuclei and the heavy α -active nuclei (\times denotes E_4/E_2 , \circ denotes E_6/E_2 , and Δ denotes E_8/E_2).

$$E_I = \frac{\hbar^2}{2J} I(I+1). \quad (5.167)$$

According to (5.167), the following interval rule should be fulfilled for the levels of a rotational band:

$$E_2 : E_4 : E_6 : E_8 \dots = 1 : 10/3 : 7 : 12 \dots \quad (5.168)$$

This rule is well satisfied in the energy spectra of strongly deformed nuclei in the three ranges of A indicated.

As an illustration, the rotational spectrum of the rare-earth element $^{180}_{72}Hf$ is shown in Fig. 5.8. The excitation energy is indicated alongside each level. Using formula (5.167), we can determine the moment of inertia from the position of the first rotational level. For the nucleus $^{180}_{72}Hf$, the moment of inertia found in this way is $J = 2 \cdot 23 \times 10^{-47}$ g.cm 2 . The energy values calculated from formula (5.167) are given in brackets in Fig. 5.8. It can be seen that the discrepancy between the values amounts to about 1%.

Figure 5.9 shows the rotational spectrum of the heavy α -active nucleus $^{238}_{94}Pu$. The moment of inertia of the nucleus $^{238}_{94}Pu$ is $J = 4 \cdot 70 \times 10^{-47}$ g.cm 2 . This value is roughly twice the moment of inertia of the lighter nucleus $^{180}_{72}Hf$.

Figure 5.10 also shows the rotational spectrum of the light nucleus $^{24}_{12}Mg$. In this case, the moment of inertia is $J = 0 \cdot 152 \times 10^{-47}$ g.cm 2 .

Odd nuclei. Rotational spectra are also observed for deformed nuclei with odd A . For the individual levels of a rotational band, the spin values

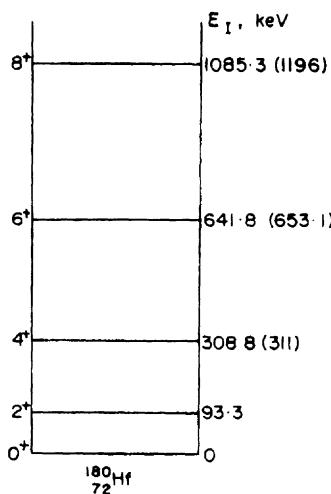


Figure 5.8. Level scheme of the $^{180}_{\text{72}}\text{Hf}$ nucleus.

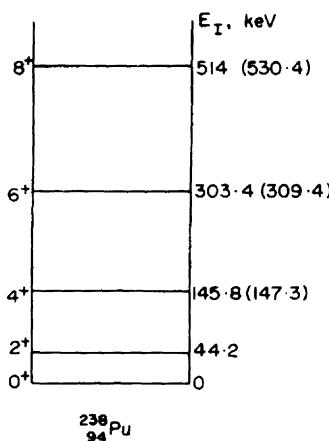


Figure 5.9. Level scheme of the $^{238}_{\text{94}}\text{Pu}$ nucleus.

are equal to $I_0 (= K)$, $I_0 + 1$, $I_0 + 2, \dots$. If $K \neq \frac{1}{2}$, we have for the ratio of the excitation energies

$$\frac{E_{I_0+2}}{E_{I_0+1}} = 2 + \frac{1}{I_0 + 1}. \quad (5.169)$$

An exact measurement of this ratio makes it possible to determine the spin I_0 of the ground state. Figure 5.11 shows the rotational spectrum of $^{181}_{\text{73}}\text{Ta}$. The observed ratio $E_{I_0+2}/E_{I_0+1} = 2 \cdot 23$; consequently, $I_0 = \frac{7}{2}$.

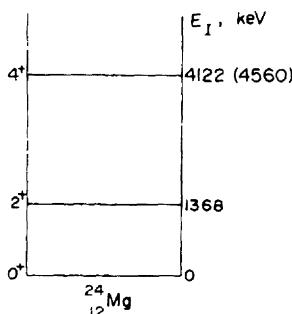


Figure 5.10. Level scheme of the $^{24}_{12}\text{Mg}$ nucleus.

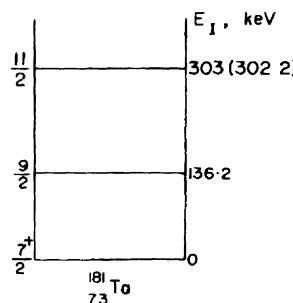


Figure 5.11. Level scheme of the $^{181}_{73}\text{Ta}$ nucleus.

From the position of the first rotational level, the moment of inertia $J = 2 \cdot 23 \times 10^{-47} \text{ g.cm}^2$, that is, has the same order of magnitude as for $^{180}_{72}\text{Hf}$.

The rotational spectrum for $K = \frac{1}{2}$. For $K = \frac{1}{2}$, the spectrum of the levels can differ substantially from a simple rotational spectrum. This is due to the coupling between the internal motion and the rotation, which is taken into account by the term with a in the formula (5.166). The energy levels in this case depend on the two parameters a and J . Figure 5.12 shows the energy-level scheme for $K = \frac{1}{2}$ as a function of the coupling parameter a .

Since $-1 < a < 1$ for the rotational spectra of heavy nuclei with $K = \frac{1}{2}$, allowance for the coupling does not lead to a change in the order of the levels for heavy nuclei. Figure 5.13, for example, shows the rotational spectrum of $^{183}_{74}\text{W}$. The empirical values of the parameters a and J , determined from the positions of the first two excited levels, are $a = 0 \cdot 19$ and $J = 2 \cdot 67 \times 10^{-47} \text{ g.cm}^2$. Using these values, we can calculate the excitation energies of the higher levels from formula (5.166). For light nuclei, however, the coupling parameter a can take large values (for example, $a = -3 \cdot 2$ for $^{25}_{13}\text{Al}$), and this can lead to a change in the order of the levels.

Magnetic moments of deformed nuclei. The operator of the magnetic dipole moment of a rotating nucleus, expressed in nuclear magnetons, is given by the expression

$$\boldsymbol{\mu} = g_R \mathbf{R} + \boldsymbol{\mu}', \quad (5.170)$$

where g_R is the gyromagnetic ratio associated with the rotation of the nucleus as a whole and $\boldsymbol{\mu}'$ is the internal magnetic moment, equal to the sum of the magnetic moments of the individual nucleons:

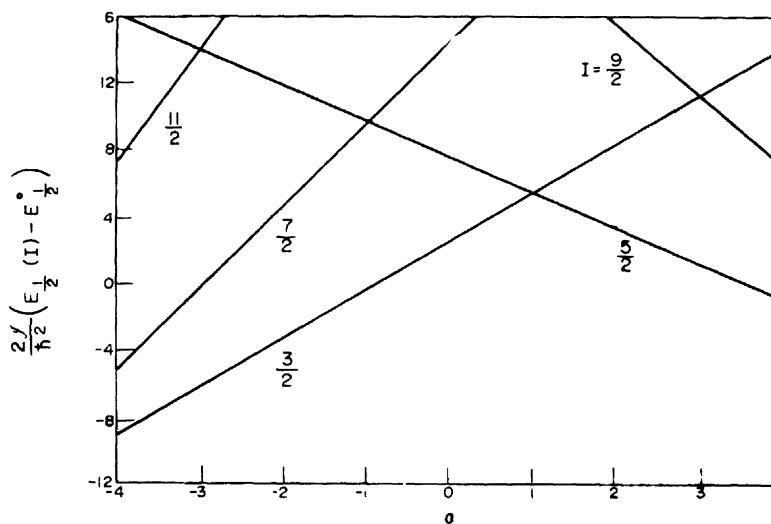


Figure 5.12. Dependence on the coupling parameter a of the rotational energy levels $E_K(I)$ for $K = \frac{1}{2}$.

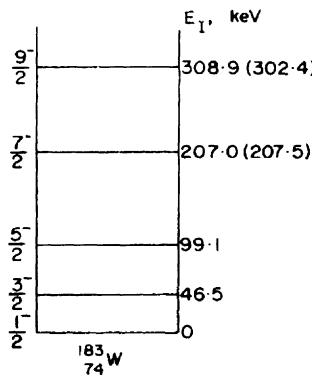


Figure 5.13. Level scheme for the $^{183}_{74}W$ nucleus.

$$\mu' = \sum(g_l l' + g_s s') \quad (5.171)$$

(note that $\sum(l' + s') = \mathbf{J}$). Expressing the angular momentum of rotation in terms of the total angular momentum of the nucleus, $\mathbf{R} = \mathbf{I} - \mathbf{J}$, we rewrite the relation (5.170) in the form]

$$\mu = g_R \mathbf{I} + (\mu' - g_R \mathbf{J}). \quad (5.172)$$

The magnetic moment μ of the nucleus is defined as the mean value of the projection of the operator μ along the stationary axis Oz in the state $M = I$. In calculating the mean value, however, the vector $(\mu' - g_R \mathbf{J})$ characterizing the internal motion is more conveniently expressed in the proper coordinate frame K' moving with the rotating nucleus.

Introducing the spherical components of any vector \mathbf{L} by means of the relations

$$L_0 = L_z, \quad L_{\pm 1} = \mp \sqrt{\frac{1}{2}} L_{\pm}, \quad (5.173)$$

we can write the relation between the components of the vector in the stationary and moving coordinate frames K and K' in the form

$$L_q = \sum_{q'} D_{q'q}^1(\Phi, \Theta, \psi) L_{q'}. \quad (5.174)$$

Thus, for the operator μ_z of the projection of the magnetic moment we obtain the expression

$$\mu_z = g_R I_z + \sum_{q'=0,\pm 1} D_{q'0}^1(\Phi, \Theta, \psi) (\mu' - g_R \mathbf{J})_{q'}. \quad (5.175)$$

Using the formula (5.149) for the function $\Psi(IKM)$, we shall determine the mean value of μ_z in the state $M = I$. The expression for the mean value of the second term in (5.175) will contain integrals of products of three D -functions. These integrals can be calculated easily by means of the relation (5.64):

$$\begin{aligned} & \int_0^{2\pi} d\Phi \int_0^\pi d\Phi \sin \Phi \int_0^{2\pi} d\psi D_{\pm KM}^{*I} D_{q'0}^1 D_{\pm KM}^I \\ &= \frac{8\pi^2}{2I+1} (IM10|IM) (I \pm K 1 q' | I \pm K). \end{aligned} \quad (5.176)$$

Opposite signs of K correspond to the cross-terms arising from the multiplication of the different terms of the function (5.149).

The coefficient $(I \pm K 1 q' | I \pm K)$ is non-zero only when the condition

$$\pm K = q' \pm K \quad (5.177)$$

is fulfilled. Therefore, if $K \neq \frac{1}{2}$, only the term $q' = 0$ makes a non-zero contribution to (5.175). Thus, for $K \neq \frac{1}{2}$, we have

$$\mu = g_R I + (IM10|IM)(IK10|IK) \left(\chi_K | \mu'_{z'} - g_R J_{z'} | \chi_K \right). \quad (5.178)$$

(The states χ_K and χ_{-K} give the same contribution to (5.178).) Introducing the internal gyromagnetic ratio g_K

$$(\chi_K | \mu'_{z'} | \chi_K) = g_K K \quad (5.179)$$

and noting that

$$(IK10|IK) = \frac{K}{\sqrt{I(I+1)}},$$

we obtain the following formula:

$$\mu = g_R I + (g_K - g_R) \frac{K^2}{I+1}. \quad (5.180)$$

If $K = \frac{1}{2}$, the condition (5.177) is also fulfilled when $q' = \pm 1$. In this case, the "cross" matrix elements between the states χ_K and χ_{-K} will be non-zero. Therefore, in (5.178) it is also necessary to take into account the term

$$\frac{1}{2} \left(II10|II \right) \left(I - \frac{1}{2} 11 | I \frac{1}{2} \right) \left(\chi_{1/2} | (\mu'_1 - g_R J_1) (-1)^{I-J} | \chi_{-1/2} \right)$$

and the complex conjugate term. Introducing the notation

$$\text{Re} (\chi_{1/2} | J_1 (-1)^{I-J} | \chi_{-1/2}) = \frac{1}{\sqrt{2}} (-1)^{-I+1/2} b,$$

we obtain the correction to (5.178) in the case $K = \frac{1}{2}$ in the form

$$\frac{1}{4} (-1)^{I-1/2} \frac{2I+1}{I+1} (g_K - g_R) b. \quad (5.181)$$

Here we have used that the fact that

$$\left(I - \frac{1}{2} 11 | I \frac{1}{2} \right) = \frac{1}{\sqrt{8}} \frac{2I+1}{\sqrt{I(I+1)}}.$$

Combining the formulae (5.180) and (5.181), we finally obtain the following expression for the *magnetic moment of a rotating nucleus*:

$$\mu = g_R I + \frac{g_K - g_R}{I+1} \left\{ K^2 + \frac{1}{4} (-1)^{I-1/2} (2I+1) b \delta_{K1/2} \right\}. \quad (5.182)$$

For the ground state of the nucleus, we must take $I = K$ in (5.182).

In the general case, the expression (5.182) for the magnetic moment contains three parameters g_K , g_R and b ; to calculate these, a more detailed theory is necessary. The experimental determination of the magnetic moment gives one independent relation between these parameters. Simultaneous measurement of the probabilities of transitions between the rotational states makes it possible to find empirical values of the parameters g_K , g_R and b for certain cases.

As an illustration, we shall determine the *magnetic moment of a deformed nucleus with odd A* by means of (5.182). By assuming that the charge distribution is uniform, we put $g_R \approx Z/A$. Assuming also that the internal moment μ' is due to the last odd nucleon, which is in a state with well-defined J , we can put $g_K \approx g_J$, where g_J is the gyromagnetic ratio for the odd nucleon, obtained from the shell model. Thus, for $I > \frac{1}{2}$, we have

$$\mu = g_R I + (g_J - g_R) \frac{I^2}{I+1}, \quad K = I. \quad (5.183)$$

As an illustration, for a series of nuclei with spin $I = \frac{5}{2}$, we give the experimental values of the magnetic moments and the values calculated from formula (5.183) (Table 5.1).

According to the shell model, the magnetic moment should be equal to $4 \cdot 79$ or $-1 \cdot 91$, depending on whether the odd particle is a proton or a neutron. Consequently, allowance for the additional magnetic moment associated with the deformation of the nuclear core appreciably improves the agreement with experiment.

Quadrupole moments of deformed nuclei. The operator of the electric quadrupole moment of a nucleus is defined by the expression

$$Q_q = \sqrt{\frac{16\pi}{5}} \sum r^2 Y_{2q}(\vartheta, \varphi), \quad (5.184)$$

TABLE 5.1.

Nucleus	μ_{exp}	μ	Nucleus	μ_{exp}	μ
$^{27}_{13}Al$	3 · 64	3 · 77	$^{25}_{12}Mg$	-0 · 86	-0 · 67
$^{121}_{51}Sb$	3 · 36	3 · 73	$^{95}_{42}Mo$	-0 · 91	-0 · 81
$^{131}_{55}Cs$	3 · 48	3 · 73	$^{105}_{46}Pd$	-0 · 6	-0 · 81
$^{187}_{75}Re$	3 · 20	3 · 71	$^{111}_{48}Cd$	-0 · 7	-0 · 81

where the summation is extended over all the protons. Going over to the coordinate frame moving with the nucleus, we have, according to (5.60),

$$Q_q = \sum_{q'} D_{qq'}^2(\Phi, \Theta, \psi) Q_{q'}. \quad (5.185)$$

The electric quadrupole moment Q of the nucleus is defined as the mean value of the operator $Q_{q=0}$ in the state $M = I$:

$$Q \equiv \langle Q_{q=0} \rangle_{M=I}. \quad (5.186)$$

Using the expression (5.149) for the wave function $\Psi(IKM)$, we thus find

$$Q = (II20|II)(IK20|IK)Q_0, \quad (5.187)$$

where Q_0 is the internal quadrupole moment of the nucleus, defined by the expression

$$Q_0 = (\chi_K|Q_{q'=0}|\chi_K). \quad (5.188)$$

(Note that, for $K = \frac{1}{2}$ or 1, in (5.185) it is necessary to include also the terms $q' \neq 0$, whose matrix elements, however, vanish for the states χ_K and χ_{-K} .) Noting that

$$(II20|II)(IK20|IK) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)},$$

we obtain the following expression for the quadrupole moment:

$$Q = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0. \quad (5.189)$$

If I is sufficiently large ($I(I+1) > 3K^2$), the quadrupole moment Q is opposite in sign to the *internal quadrupole moment*.

For the ground state $I = K$ of the rotational band, we have

$$Q = \frac{I(2I-1)}{(I+1)(2I+3)} Q_0. \quad (5.190)$$

The coefficient of Q_0 is usually called the *projecting factor*. The presence of this factor leads to a sharp reduction of the *measured quadrupole moment* Q compared with Q_0 . If $I = 0$ or $\frac{1}{2}$, the quadrupole moment Q of the nucleus vanishes, even though the internal quadrupole moment Q_0 may be non-zero. For $I = 1$, the projecting factor is equal to $\frac{1}{10}$, for $I = \frac{3}{2}$ it is equal to $\frac{1}{5}$, and on further increase of I the factor slowly tends to unity.

Assuming the charge distribution to be uniform, we can calculate the internal quadrupole moment of the nucleus:

$$Q_0 = (2Z/V) \int_V d^3r (z^2 - x^2) = (2Z/5)(R_z^2 - R_x^2). \quad (5.191)$$

Using the expression (5.102) for the principal semi-axes of the ellipsoid, we thus obtain

$$Q_0 = \frac{3}{\sqrt{5\pi}} Z R_0^2 \beta \left(1 + \frac{1}{4} \sqrt{\frac{5}{4\pi}} \beta \right), \quad (5.192)$$

where β is the nuclear deformation parameter.

Q_0 can be determined experimentally by measuring the probabilities of γ -transitions of the nucleus from one rotational state to another.

Table 5.2 shows the values of Q_0 and the corresponding values of the deformation parameter β , found from the probabilities of $E2$ -transitions, for a series of even-even nuclei.

For nuclei with odd A the quadrupole moment can also be determined from spectroscopic measurements of the hyperfine structure of the spectral lines ($I > \frac{1}{2}$). Table 5.3 shows the values of the internal quadrupole moments Q_0 for a series of odd nuclei, determined from spectroscopic measurements and measurements of $E2$ -transition probabilities.

Figure 5.14 shows the values of the quadrupole moments for strongly deformed nuclei in the mass-number range $150 < A < 190$.

One-particle states in deformed nuclei. In studying the shell model, we assumed that the self-consistent nuclear potential in which the nucleons move possesses spherical symmetry. The nuclear shape characterized by this potential was also spherically symmetric. In fact, in the case of filled shells, the equilibrium shape of real nuclei is spherically symmetric. However, nuclei in which the last shell is approximately half-filled are characterized by considerable deformations. To explain the properties of deformed nuclei, it is necessary not only to study the collective motions in them but also to

TABLE 5.2.

Nucleus	$Q_0, 10^{-24} \text{cm}^2$	β
$^{150}_{60} Nd$	4 · 8	0 · 25
$^{152}_{62} Sm$	5 · 7	0 · 28
$^{154}_{62} Sm$	6 · 7	0 · 33
$^{160}_{64} Gd$	10	0 · 46
$^{178}_{72} Hf$	8 · 1	0 · 31
$^{184}_{74} W$	6 · 5	0 · 24
$^{188}_{76} Os$	5 · 1	0 · 18

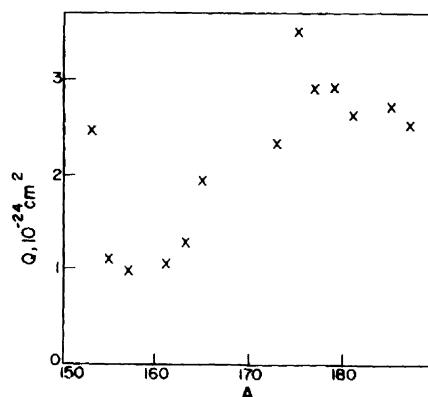


Figure 5.14. Values of the quadrupole moments of strongly deformed nuclei in the region of the rare-earths.

consider the *one-particle states of the nucleons moving in the non-spherical potential*. In the adiabatic approximation, the one-particle and collective motions in the nucleus can be separated, so that the total wave function of the nucleus can be represented in the form of a product of two functions, of which one describes the motion of the nucleus as a whole, that is, the rotation of the nucleus and the vibrations of its surface, and the other describes the internal motion in the non-spherical nucleus.

In treating the internal motion in the nucleus, we shall assume that the individual nucleons move independently in a certain fixed non-spherical field of the nucleus. The Hamiltonian of the internal motion can then be represented, as in the ordinary shell model (without allowance for the residual interaction), in the form of a sum of one-particle Hamiltonians. Thus, the problem reduces to solving the Schrödinger equation for one particle with a one-particle potential which must be chosen in such a way that it reflects the principal features of the motion of nucleons in non-spherical

TABLE 5.3.

Nucleus	I	Q_0 (spectroscopic measurements)	Q_0 (E_2 -transition)
$^{153}_{63} Eu$	$\frac{5}{2}$	7.0	7.7
$^{155}_{64} Gd$		5.5	8.0
$^{157}_{62} Gd$		5.0	7.7
$^{175}_{71} Lu$	$\frac{7}{2}$	12	8.2
$^{185}_{75} Re$	$\frac{5}{2}$	7.8	5.4

nuclei while having at the same time the simplest possible form. This problem has been solved for several potentials by various authors. The so-called *Nilsson potential* (Nilsson, 1955), which applies well to the case of medium and heavy nuclei, has the simplest form.

The Nilsson potential is chosen in the form of a sum

$$V = (M/2)(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) + C\mathbf{l} \cdot \mathbf{s} + D\mathbf{l}^2, \quad (5.193)$$

where the first term is an anisotropic-oscillator potential, which takes into account that the potential well is non-spherical and diffuse at the boundary, the second term describes the spin-orbit interaction, and the third term, proportional to \mathbf{l}^2 , is introduced to lower the energy of a one-particle state at large values l of the orbital angular momentum. Since the majority of non-spherical nuclei have a shape close to an ellipsoid of revolution, we shall assume that the field of the nucleus is characterized by axial symmetry and $\omega_1 = \omega_2 \neq \omega_3$. As we know, in this case, if the volume of the nucleus remains constant we can describe its shape by introducing only one deformation parameter. In the Nilsson potential this parameter, denoted by δ , is defined as follows:

$$\omega_1^2 = \omega_2^2 = \omega^2 \left(1 + \frac{2}{3}\delta\right), \quad \omega_3^2 = \omega^2 \left(1 - \frac{4}{3}\delta\right). \quad (5.194)$$

From the constancy of the nuclear volume, which is inversely proportional to the product of the frequencies ω_1 , ω_2 and ω_3 , it follows that the quantity ω is not constant and depends on the deformation parameter δ in the following way:

$$\omega \equiv \omega(\delta) = \omega(0) \left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{-1/6}. \quad (5.195)$$

In the case of small deformations, the quantity δ is connected with the deformation parameter β introduced earlier, by the relation

$$\delta = \frac{3}{2} \sqrt{\frac{5}{4\pi}} \beta. \quad (5.196)$$

We find the total Hamiltonian H of the nucleon by adding the kinetic-energy operator $-\hbar^2 \nabla^2 / 2M$ to (5.193). Separating out in the Hamiltonian the spherically symmetric part

$$H_0 = -\frac{\hbar^2}{2M} \nabla^2 + \frac{M\omega^2(\delta)}{2} r^2, \quad (5.197)$$

we represent the one-particle Hamiltonian in the form

$$H = H_0 + H_\delta + C\mathbf{l} \cdot \mathbf{s} + D\mathbf{l}^2, \quad (5.198)$$

where

$$H_\delta = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}M\omega^2\delta r^2Y_{20}(\mathbf{n}). \quad (5.199)$$

We note that the operators \mathbf{j}^2 and \mathbf{l}^2 of the squares of the total angular momentum and orbital angular momentum of the nucleon and the operators l_z and s_z of the components of the orbital and spin angular momenta along the z -axis (the symmetry axis of the field) do not commute with the operator H . We note that the spherically symmetric part H_0 also does not commute with the total Hamiltonian H . However, the operator $j_z = l_z + s_z$ of the component of the total angular momentum along the symmetry axis does commute with H , and so the states of a nucleon in the field (5.193) are characterized by well-defined values Ω of this component of the total angular momentum.

The next step is to diagonalize the one-particle Hamiltonian H . For this, we must choose a convenient representation and write the matrix of the Hamiltonian H in this representation. One usually chooses a representation in which the symmetric Hamiltonian H_0 is diagonal. In other words, for the basis functions in terms of which we shall expand the functions of a nucleon moving in a Nilsson potential and by means of which the matrix of the Hamiltonian H will be defined, we shall use the isotropic oscillator functions $\chi_{l\Lambda\Sigma}^N$ defined by the equation

$$H_0\chi_{l\Lambda\Sigma}^N = \left(N + \frac{3}{2}\right)\hbar\omega\chi_{l\Lambda\Sigma}^N, \quad (5.200)$$

where Λ and Σ are the projections of the orbital and spin angular momenta of the nucleon along the symmetry axis of the nucleus. Although the total one-particle Hamiltonian matrix $\langle N'l'\Lambda'\Sigma'|H|Nl\Lambda\Sigma\rangle$ calculated by means of these functions is not diagonal in any of the quantum numbers N, l, Λ and Σ , it is fairly easily calculated in this representation. In fact, two terms in the total Hamiltonian, H_0 and $D\mathbf{l}^2$, are diagonal in the representation chosen. The non-zero matrix elements of the operator $\mathbf{l} \cdot \mathbf{s}$ can be written as follows:

$$\begin{aligned} \langle N'l'\Lambda'\Sigma'|\mathbf{l} \cdot \mathbf{s}|Nl\Lambda\Sigma\rangle &= \frac{1}{2}\delta_{N'N}\delta_{l'l} \\ &\times \left\{ \sqrt{(l \mp \Lambda)(l \pm \Lambda + 1)} \delta_{\Lambda', \Lambda \pm 1} \delta_{\Sigma', -\Sigma} \pm \Lambda \delta_{\Lambda' \Lambda} \delta_{\Sigma' \Sigma} \right\}. \end{aligned} \quad (5.201)$$

The calculation of the matrix elements of H_δ is more cumbersome. The matrix element $\langle l'\Lambda'|Y_{20}|l\Lambda\rangle$ of the angular part of the operator is calculated using (4.70). To calculate the radial matrix elements $\langle N'l'|r^n|Nl\rangle$, we can make use of the explicit form (4.27) of the radial wave function of a particle in the field of an isotropic-oscillator potential. We give the result of the calculation of a more general matrix element using the functions (4.27):

$$\begin{aligned} \langle N'l'|r^n|Nl\rangle = & \left[\frac{\Gamma\left(\frac{N-l+2}{2}\right)\Gamma\left(\frac{N'-l'+2}{2}\right)}{\Gamma\left(\frac{N+l+3}{2}\right)\Gamma\left(\frac{N'+l'+3}{2}\right)} \right]^{1/2} \\ & \times \left(\frac{l-l'+n}{2} \right)! \left(\frac{l'-l+n}{2} \right)! \\ & \times \sum_{\sigma} \frac{\Gamma\left(\frac{l+l'+n+3}{2} + \sigma\right)}{\sigma! \left(\frac{N-l}{2} - \sigma \right)! \left(\frac{N'-l'}{2} - \sigma \right)! \left(\frac{l+n-N}{2} + \sigma \right)! \left(\frac{l+n-N'}{2} + \sigma \right)!}. \end{aligned} \quad (5.202)$$

After matrix $\langle N'l'\Lambda'\Sigma'|H|Nl\Lambda\Sigma\rangle$ has been calculated, the problem reduces to diagonalizing this matrix. The diagonalization has been performed, by means of computers, for several values of the deformation parameter δ and a series of fixed values of the quantum numbers N and Ω . The numerical values of C and D were chosen in such a way as to reproduce the sequence of levels of the shell model for $\delta = 0$. (In the calculations for $\delta = 0$, the quantity ω was chosen in accordance with (4.31), and $C = -0 \cdot 1\hbar\omega$.) The ratio $\mu = 2D/C$ was assumed equal to 0 for $N = 0, 1$ and 2, $0 \cdot 35$ for $N = 3$, $0 \cdot 45$ for $N = 4, 5$ and 6, and $0 \cdot 40$ for $N = 7$. We denote the eigenvalues of the part of the Hamiltonian $H - H_0$ by $-(C/2)r_\alpha^{N\Omega}$, where the index α labels these eigenvalues. Then, taking (5.200) into account, we represent the eigenvalues of the one-particle Hamiltonian (5.198) in the form

$$E_\alpha^{N\Omega}(\delta) = \left(N + \frac{3}{2} \right) \hbar\omega(\delta) - (C/2)r_\alpha^{N\Omega}(\delta). \quad (5.203)$$

We denote the eigenfunction of the Hamiltonian H corresponding to the quantum numbers N, Ω and α by $\chi_\Omega^{N\alpha}$. This function is a superposition of the functions $\chi_{l\Lambda\Sigma}^N$:

$$\chi_\Omega^{N\alpha} = \sum_{l\Lambda} a_{l\Lambda}^{N\alpha}(\delta) \chi_{l\Lambda\Sigma}^N, \quad (5.204)$$

where $\Sigma = \Omega - \Lambda$. The superposition coefficients $a_{l\Lambda}^{N\alpha}(\delta)$, which are normalized so that

$$\sum_{l\Lambda} |a_{l\Lambda}^{N\alpha}(\delta)|^2 = 1,$$

are determined simultaneously with the energy eigenvalues in the numerical diagonalization of the Hamiltonian matrix. Tables of the expansion coefficients $a_{l\Lambda}^{N\alpha}(\delta)$ have been compiled, and curves of the single-particle energies $E_{\alpha}^{N\Omega}$ as functions of the deformation parameter δ have been obtained (Nilsson, 1955).

It can be seen from the relation (5.202) that the Hamiltonian matrix is not diagonal in the quantum number N ; N' differs from N by an even number. However, in most cases, the off-diagonal matrix elements in the number N are considerably smaller in magnitude than the nearest matrix elements diagonal in N ; therefore, the coupling between states with different N can be neglected, except in certain spacial cases. Thus, the one-particle states in the Nilsson potential can be characterized not only by the projection Ω of the total angular momentum but also by the quantum number N . In the case of sufficiently small deformations, when the term H_{δ} in the Hamiltonian H can be neglected, the operators \mathbf{l}^2 and \mathbf{j}^2 commute with H , and so the one-particle states can be characterized for $\delta \rightarrow 0$ by the set of quantum numbers N, l, j and Ω . For small δ the term H_{δ} can be taken into account by perturbation theory. With increasing deformation, the quantum numbers l and j lose their meaning. For sufficiently large deformations, when we can neglect the terms with $\mathbf{l} \cdot \mathbf{s}$ and \mathbf{l}^2 compared with H_{δ} in the Hamiltonian, we obtain in the limit the Hamiltonian of an anisotropic oscillator with axial symmetry. In this case, the one-particle states can be described by sets of quantum numbers N, n_3, Λ and Σ (in this case, Σ also has a well-defined value: $\Sigma = \Omega - \Lambda$). For large deformations δ , the terms containing $\mathbf{l} \cdot \mathbf{s}$ and \mathbf{l}^2 can be taken into account by perturbation theory. We note that the superposition (5.204) contains terms with values of l differing from each other and from the number N by an even number; therefore, the one-particle states are characterized by well-defined parity π .

As an illustration, we show in Fig. 5.15 the scheme of the levels as a function of the deformation parameter δ for the range $50 < N < 82$, in which many non-spherical nuclei are found. For $\delta > 0$ the nucleus is prolate along the symmetry axis, and for $\delta < 0$ the nucleus is oblate. Quantum numbers $l_j|\Omega|^{\pi}(Nn_3|\Lambda|)$ are associated with each curve in Fig. 5.15; of these, the projection Ω of the total angular momentum (there is degeneracy in the sign of Ω) and the parity π are good quantum numbers, irrespective of the magnitude of the deformation; the oscillator energy quantum number N is, in the sense indicated above, almost a good quantum number; the quantum numbers l and j are good quantum numbers only when $\delta \rightarrow 0$, while n_3 and Λ , on the other hand, are good quantum numbers for

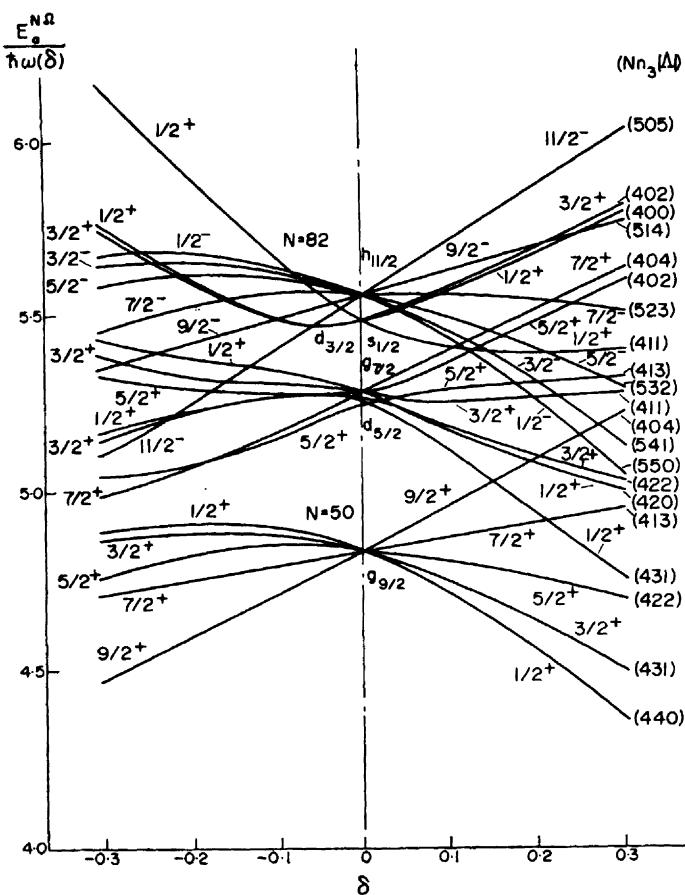


Figure 5.15. Scheme of the energy levels as a function of the deformation parameter δ for nuclei in the range $50 < N < 82$, according to the Nilsson model.

large deformations (asymptotic quantum numbers). For example, one of the curves in Fig. 5.15 is labelled in the following way: $g_{7/2}^{5+}(402)$, which means that $l = 4$ and $j = \frac{7}{2}$ when $\delta = 0$, $|\Omega| = \frac{5}{2}$ and $\pi = +1$, and $N = 4$, $n_3 = 0$ and $|\Lambda| = 2$. (The last two numbers have these values for large deformations δ .) We note that, since the parameters occurring in the one-particle potential H differ somewhat for a proton and a neutron, the level schemes and the sets of coefficients in the superposition (5.204) will also be different.

It is possible to calculate the total energy and the equilibrium deformation of nuclei on the basis of the above model. If we assume that only two-particle forces act between the nucleons, the total Hamiltonian of the

nucleus,

$$H = \sum_i T_i + \frac{1}{2} \sum_{i \neq j} V_{ij},$$

can be expressed directly in terms of the Hamiltonian of a single nucleon,

$$H_i = T_i + \sum_{j(\neq i)} V_{ij},$$

in the following way:

$$H = \frac{1}{2} \sum_i (H_i + T_i). \quad (5.205)$$

The total wave function of the nucleus is an anti-symmetrized product of the wave functions (5.204) of the individual nucleons. Therefore, to find the total energy $E(\delta)$ of the nucleus as a function of the deformation parameter δ , we must calculate the mean value of (5.205) using the calculated one-particle functions. Then the equilibrium value δ_0 of the deformation parameter can be determined from the condition that the energy $E(\delta)$ be a minimum:

$$\partial E(\delta) / \partial \delta = 0. \quad (5.206)$$

Figure 5.16 shows the values of the *deformation parameter* δ calculated in the *Nilsson model*, and the experimental values in the region of the rare-earth elements.

The Nilsson model also makes it possible to justify the rule used to determine the spins of nuclei in the ground state. According to (5.203), each one-particle energy level is doubly degenerate ($\pm \Omega$). Neutrons and protons occupy the levels independently in pairs. If the number of identical particles is even, then $\sum \Omega_i = 0$; but if this number is odd, the total Ω is equal to the Ω_i of the last, unpaired, particle. If the number of neutrons (protons) is even and the number of protons (neutrons) is odd, then for the ground state the spin is equal to $I_0 = \Omega_p(\Omega_n)$. Thus, the spins of nuclei in the ground state are determined by the properties of the last odd nucleon, and are equal to $I_0 = \Omega = K$, except in the case $\Omega = \frac{1}{2}$. If $\Omega = \frac{1}{2}$, the spin I_0 of the ground state is determined by the magnitude of the coupling parameter a (Table 5.4).

The coupling parameter a , which is defined by the relation (5.165), can be expressed directly in terms of the coefficients $a_{l\Lambda}$ appearing in the formula (5.204):

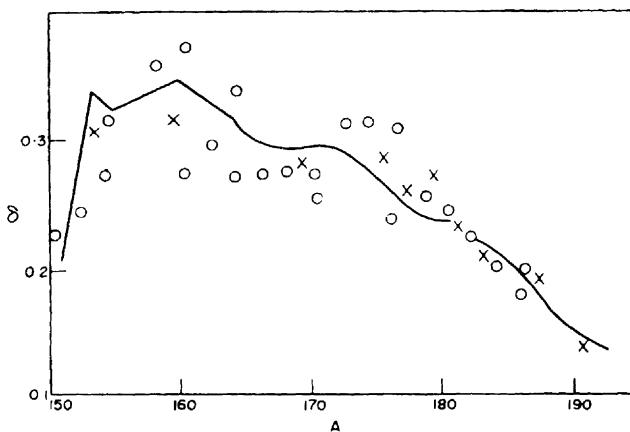


Figure 5.16. Experimental values (the points) of the deformation parameter δ and values calculated from Nilsson model (the solid curve) in the region of the rare-earths (\times denotes odd nuclei and \circ denotes even-even nuclei).

TABLE 5.4.

Range of variation of the coupling parameter a	Ground-state spin I_0	Range of variation of the coupling parameter a	Ground-state spin I_0
$-14 < a < -10$	$\frac{11}{2}$	$4 < a < 8$	$\frac{5}{2}$
$-10 < a < -6$	$\frac{7}{2}$	$8 < a < 12$	$\frac{9}{2}$
$-6 < a < -1$	$\frac{3}{2}$	$12 < a < 16$	$\frac{13}{2}$
$-1 < a < 4$	$\frac{1}{2}$		

$$a = \sum_l (-1)^l (a_{l0}^2 + 2\sqrt{l(l+1)}a_{l0}a_{l1}). \quad (5.207)$$

In an analogous way, we can find the gyromagnetic ratio g_K and the coefficient b occurring in the expression (5.182) for the magnetic moment of the nucleus, in the case $I_0 = K = \frac{1}{2}$:

$$\left. \begin{aligned} g_K &= g_l + (1/2K)(g_s - g_l) \sum_l (a_{l,K-1/2}^2 - a_{l,K+1/2}^2); \\ (g_K - g_R)b &= (g_s - g_l) \sum_l (-1)^l a_{l0}^2 + (g_l - g_R)a. \end{aligned} \right\} \quad (5.208)$$

The calculated values of the magnetic moments for deformed odd nuclei agree with the experimental data.

Moments of inertia of nuclei. The most important property distinguishing non-spherical from spherical nuclei is the presence in non-spherical nuclei of rotational energy levels. Whereas the general features of the rotational spectra (for example, the ratios of the energies, the values of the spins and parities, the symmetry of the states, etc.) are entirely determined by the rotational nature of the levels, the absolute values of the rotational energies (that is, the magnitudes of the moments of inertia) depend essentially on the internal structure of the nuclei.⁵

We shall assume that an axially symmetric nucleus is rotating slowly with a definite frequency ω about an axis perpendicular to the symmetry axis of the nucleus. The angular momentum associated with the rotation of the nucleus is equal to

$$\hbar R = \omega J,$$

where J is the moment of inertia. To determine the *moment of inertia* J , it is necessary to make use of specific ideas on the internal structure of the nucleus. As limiting cases of the rotational model of the nucleus, we can consider the rotation of a solid and the potential motion of an ideal liquid in a rotating shell. The difference between these limiting cases is very clearly manifested in the example of the rotation of a spherical system. In the case of the solid, on going over to the spherically symmetric case the moment of inertia tends to a finite value, namely, the moment of inertia of a solid sphere. In the case of the hydrodynamic model, in the rotation of the spherical shell the velocity of each point on the surface is directed along a tangent, and the normal component of the velocity is equal to zero. Therefore, only a liquid at rest will satisfy the equations of motion of ideal hydrodynamics in such a vessel (since $v = 0$ satisfies both the equations of motion and the boundary conditions), and, consequently, the moment of inertia of the system will be equal to zero. If the shell has a non-spherical shape, the normal component of the velocity at the surface will be non-zero and the liquid will be dragged along as the shell rotates. The energy of rotation for a given angular velocity will be greater the more the shape of the shell differs from that of a sphere.

We shall represent the nucleus in the form of an ellipsoid of revolution, with semi-axes of length

⁵The relation between the internal structure of the nucleus and the nature of the collective motions is discussed in detail in the review by Moszkowski (1957).

$$R_1 = \left(1 - \frac{1}{2}\sqrt{\frac{5}{4\pi}}\beta\right)R, \quad R_2 = \left(1 + \frac{1}{2}\sqrt{\frac{5}{4\pi}}\beta\right)R. \quad (5.209)$$

Here β is the deformation parameter. If the nucleus is regarded as a solid, the moment of inertia will be equal to

$$J_0 = \frac{m}{5} \left(R_1^2 + R_2^2 \right) = \frac{2}{5} m R^2 \left(1 + \frac{1}{2}\sqrt{\frac{5}{4\pi}}\beta + \frac{25}{32\pi}\beta^2 \right), \quad (5.210)$$

where m is the mass of the nucleus.

We now determine the moment of inertia for the potential motion of a liquid in an ellipsoidal vessel having the shape of the nucleus. The potential motion of the liquid is described by the potential φ , satisfying the Laplace equation

$$\nabla^2 \varphi = 0.$$

The velocity \mathbf{v} of the liquid is given by the gradient of the potential:

$$\mathbf{v} = \text{grad } \varphi.$$

In the case of an ideal liquid, the boundary conditions reduce to the requirement that the normal component of the velocity of the liquid at the surface coincide with the normal component of the velocity of the vessel wall.

The velocity of a point with position vector \mathbf{r} at the surface of the rotating vessel is equal to $\omega \times \mathbf{r}$. Noting that the normal to the surface of the ellipsoid is characterized by the components

$$\mathbf{N} = (x/R_1^2, y/R_1^2, z/R_1^2)$$

(if the symmetry axis is directed along the axis Oz), we write the condition that the normal components of the velocities \mathbf{v} and $\omega \times \mathbf{r}$ be equal at the surface in the form

$$\frac{x}{R_1^2} \left(\frac{\partial \varphi}{\partial x} - \omega z \right) + \frac{y}{R_1^2} \frac{\partial \varphi}{\partial y} + \frac{z}{R_1^2} \left(\frac{\partial \varphi}{\partial z} + \omega x \right) = 0. \quad (5.211)$$

This condition will be satisfied if for the solution of the Laplace equation we choose

$$\varphi = Axz, \quad (5.212)$$

where A is some constant. Substituting (5.212) into (5.211) we find the value of this constant:

$$A = \frac{R_2^2 - R_1^2}{R_2^2 + R_1^2} \omega. \quad (5.213)$$

We shall calculate the kinetic energy of the liquid in the rotating shell. This energy is equal to

$$T_r = (\varrho/2) \int d^3r v^2 = (\varrho/2) A^2 \int d^3r (x^2 + z^2),$$

where ϱ is the density of the liquid and the integration is performed over the volume of the ellipsoid. Substituting (5.213) for A and performing the integration, we find

$$T_r = \frac{m}{5} \frac{(R_2^2 - R_1^2)^2}{R_2^2 + R_1^2} \frac{\omega^2}{2}, \quad (5.214)$$

where m is the mass of the liquid. The coefficient at $\omega^2/2$ in (5.214) must be regarded as the *moment of inertia in the hydrodynamic model of the nucleus*:

$$J = \frac{m}{5} \frac{(R_2^2 - R_1^2)^2}{R_2^2 + R_1^2}. \quad (5.215)$$

Using the expressions (5.209) for R_1 and R_2 , we can rewrite the moment of inertia in the hydrodynamic model in the form

$$J = \frac{9mR^2}{4\pi} \frac{\beta^2 \left(1 + \frac{1}{4}\sqrt{\frac{5}{4\pi}}\beta\right)^2}{2 + \sqrt{\frac{5}{4\pi}\beta + \frac{25}{16\pi}\beta^2}}. \quad (5.216)$$

In the case of small deformations, the ratio of the moments of inertia (5.216) and (4.210) is equal to

$$\frac{J}{J_0} = \frac{45}{16\pi} \beta^2. \quad (5.217)$$

Figure 5.17 (Moszkowski, 1957) depicts the velocity field in the rotation of a solid and in the potential motion of an ideal liquid in a rigid rotating shell. Whereas in the case of a solid the entire system rotates as a whole, in the rotation of a rigid shell filled with an ideal liquid the latter is dragged by the walls only near the surface of the shell.

Figure 5.18 shows the dependence of the moments of inertia on the magnitude of the deformation parameter for some even-even rare-earth nuclei.

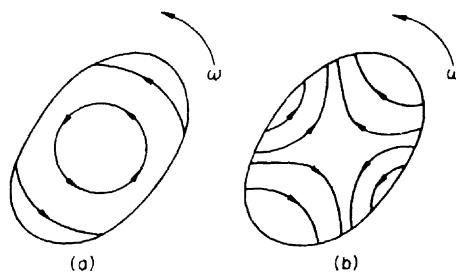


Figure 5.17. Velocity field in the rotation of a solid (a) and in the potential motion of an ideal liquid in a rigid rotating shell (b).

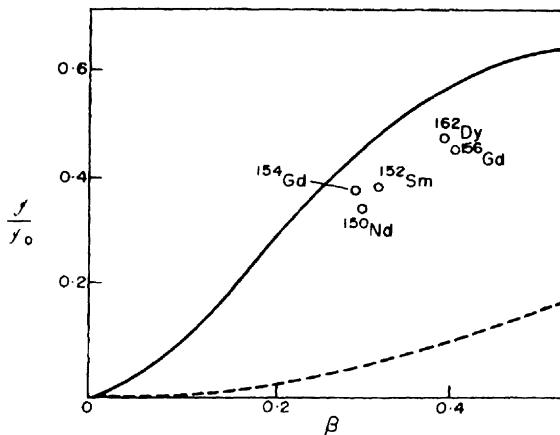


Figure 5.18. Dependence of the moments of inertia J/J_0 on the magnitude of the deformation parameter for even-even rare-earth nuclei, obtained on the basis of the superfluid model (solid curve) and the hydrodynamic model (dashed curve).

As can be seen from Fig. 5.18, for small deformations β the experimentally observed moments of inertia are considerably smaller than the values predicted by the solid model. They are considerably greater, however, than the values predicted by the hydrodynamic model.

The moment of inertia of the nucleus can be calculated more systematically on the basis of the so-called *cranking model* proposed by Inglis (1954, 1956). For this, we consider the motion of a system of particles in a non-spherical potential well that is slowly rotating about an axis that is not coincident with the symmetry axis. The Schrödinger equation describing the motion of the system has the form

$$-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = H(t)\psi, \quad (5.218)$$

where

$$H(t) = T + V(\mathbf{r}, t)$$

(T is the kinetic energy and $V(\mathbf{r}, t)$ is the potential energy, which varies slowly with time in the stationary coordinate frame). If the rotation proceeds slowly compared with the internal motion, we may speak of the energy of rotation of a system in an unchanged internal state.

We introduce the rotating coordinate frame rigidly attached to the nucleus. We choose the axis Oz along the symmetry axis of the nucleus, and let the rotation occur about the axis Ox . Then, in going over to the rotating coordinate frame, the function $\psi = \psi(\mathbf{r})$ is transformed into the function $\psi' \equiv \psi(\mathbf{r}')$, which, according to (5.48), is connected with ψ by the relation

$$\psi' = e^{-i\Theta j_x} \psi, \quad (5.219)$$

where $\Theta = \omega t$. The Hamiltonian H of the system is transformed correspondingly into H' :

$$H' = e^{-i\Theta j_x} H e^{i\Theta j_x}.$$

Since H' is the Hamiltonian of the internal motion in the proper coordinate frame, it does not depend on time. We denote the eigenvalues and eigenfunctions of H' by E_i and χ_i , so that

$$H' \chi_i = E_i \chi_i. \quad (5.220)$$

We find the equation for the function ψ' defined in the rotating coordinate frame. For this, we express the function ψ in terms of ψ' and substitute the result in (5.218). As a result, we obtain

$$-(\hbar/i)(\partial\psi'/\partial t) = \tilde{H}\psi', \quad (5.221)$$

where

$$\tilde{H} = H' + \hbar\omega j_x. \quad (5.222)$$

We introduce the notation

$$\psi' = \tilde{\chi} e^{-(i/\hbar)\tilde{E}t}. \quad (5.223)$$

Assuming the frequency ω to be sufficiently small, we can calculate the quantity \tilde{E} and the function $\tilde{\chi}$ by means of perturbation theory. Since the diagonal matrix element of j_x is equal to zero, we have

$$\left. \begin{aligned} \tilde{E}_i &= E_i - \hbar^2 \omega^2 \sum_{j \neq i} \frac{|\langle j | j_x | i \rangle|^2}{E_j - E_i}; \\ \tilde{\chi}_i &= \chi_i - \sum_{j \neq i} \frac{\langle j | j_x | i \rangle}{E_j - E_i} \chi_j. \end{aligned} \right\} \quad (5.224)$$

Knowing the function (5.223), it is not difficult to calculate the energy of the system in the stationary coordinate frame:

$$\begin{aligned} (\psi_i, H \psi_i) &= (\psi'_i, H' \psi'_i) = \left(\tilde{\chi}_i, (\tilde{H} - \hbar \omega j_x) \tilde{\chi}_i \right) \\ &= E_i + \hbar^2 \omega^2 \sum_{j \neq i} \frac{|\langle j | j_x | i \rangle|^2}{E_j - E_i}. \end{aligned} \quad (5.225)$$

It is clear that the correction to the internal energy E_i must be regarded as the energy of rotation, the moment of inertia being defined as twice the coefficient of ω^2 . Thus, we obtain the following formula:

$$J = 2\hbar^2 \sum_{j \neq i} \frac{|\langle j | j_x | i \rangle|^2}{E_j - E_i}. \quad (5.226)$$

If we use as the internal wave functions the shell functions of independent particles moving in a non-spherical well (for example, in an anisotropic oscillator well), we obtain the same moment of inertia as in the rotation of a solid. This is connected with the fact that the one-particle states of independent (uncorrelated) particles do not change during slow adiabatic rotation of the potential well, and, consequently, a collective rotation of the whole systems occurs, as in the case of a solid. The deviations of the observed values of the moments of inertia of nuclei from the values for the solid model are due to the presence of correlation between the nucleons. The residual interaction between the nucleons (which is neglected in the independent-particle model) leads to correlation between the nucleons, which tends to slow down the collective rotation and reduce the moment of inertia of the system. If the interaction between the nucleons were so strong that the mean free path was small compared with the dimensions of the nucleus, the hydrodynamic model would be applicable and the collective motion would become potential. Correlation between the nucleons due to pairing forces is taken into account in the superfluid model of the nucleus. A systematic method for calculating the moments of inertia of nuclei on the basis of the superfluid model has been developed by Migdal (1959).

5.5. Rotation of Non-axial Nuclei

The non-axial rotator model. A detailed comparison of the observed excitation spectra of deformed nuclei with the predictions of the rotational model in which the nucleus is regarded as an axially symmetric rotator has shown that, although the rotational model does reproduce well the features of the rotational spectra for many nuclei, there is, nevertheless, disagreement between the observed picture and the predictions of the simple rotational model for certain nuclei. Attempts to explain the discrepancy by including corrections for the non-adiabaticity of the rotation have been unsuccessful. Davydov and Filippov (1958) have shown that the difficulties can be removed if the assumption of axial symmetry of the deformed nuclei is discarded. They developed a theory of *non-axial nuclei*, in which the fundamental assumption is that the deformed nucleus is a triaxial ellipsoid whose rotation determines the spectrum of the collective excitations of the nucleus. This theory has made it possible to describe the principal features of the spectra of the excited states of most even-even nuclei.

In the adiabatic approximation, when the rotational motion can be separated from the vibrations of the nuclear surface and from the one-particle motion, the shape of the nucleus can be assumed fixed; then the Hamiltonian describing the rotation of an even-even nucleus can be written in the form

$$H_r = \sum_{k'} \frac{\hbar^2}{2J_{k'}} I_{k'}^2, \quad (5.227)$$

where $I_{k'}$ are the projections of the total angular-momentum operator along the principal axes of the ellipsoid and $J_{k'}$ are the moments of inertia about the principal axes:

$$J_{k'} = 4B\beta^2 \sin^2 \left(\gamma - \frac{2\pi}{3} k' \right)$$

(the three moments of inertia of the nucleus are assumed to be all different). The rotational-energy operator (5.227) acts only on the Euler angles and is the Hamiltonian operator of a rigid non-axial rotator (asymmetric top). Before finding the solutions of the Schrödinger equation for the asymmetric top and the corresponding well-defined values I and M of the total angular momentum and its projection along the axis Oz respectively,

$$(H_r - E_{I\tau}) \Phi_{IM}^\tau(\Phi, \Theta, \psi) = 0, \quad E_{I\tau} \equiv \frac{\hbar^2}{4B\beta^2} \epsilon_{I\tau} \quad (5.228)$$

TABLE 5.5.

D_2	e	$R_1(\pi)$	$R_2(\pi)$	$R_3(\pi)$	D_2	e	$R_1(\pi)$	$R_2(\pi)$	$R_3(\pi)$
A	1	1	1	1	B_2	1	-1	1	-1
B_1	1	-1	-1	1	B_3	1	1	-1	-1

(τ are the additional quantum numbers), we shall ascertain which conditions, stemming from the symmetry properties of the Hamiltonian H_r , must be imposed on the wave functions $\Phi_{IM}^\tau(\Phi, \Theta, \psi)$ of the asymmetric top.

The operator H_r and the commutation relations (5.134) for the projections $I_{k'}$ of the total angular momentum along the principal axes are invariant under the transformations of the group D_2 consisting of the three rotations through an angle of 180° about the axes Ox' , Oy' and Oz' ($R_{j'}(\pi) = e^{i\pi I_{j'}}$) and the identity element e . The invariance is easily verified, if we make use of the relations

$$R_{j'}(\pi)I_{k'}R_{j'}^{-1}(\pi) = (2\delta_{j'k'} - 1)I_{k'} \quad (5.229)$$

(in each rotation $R_{j'}(\pi)$, two of the three operators $I_{j'} (j' = 1, 2, 3)$ change their sign), from which it follows immediately that the Hamiltonian commutes with all the operations of the group D_2 . Since each element of the group D_2 forms a class of the group, and the number of classes of a group is equal to the number of irreducible representations, there are in all four one-dimensional irreducible representations of the group D_2 , which we denote by the symbols A , B_1 , B_2 , B_3 . In Table 5.5 we give their characters (which coincide in this case with the one-dimensional matrices of the irreducible representations); these can be used to determine the eigenfunctions of the operator H_r .

It is well known that there is a close connection between the eigenfunctions of the stationary states and the irreducible representations of a group of transformations which commute with the Hamiltonian. The energy states of the system can be classified according to the irreducible representations corresponding to these states. It is clear that the degeneracy of an energy level coincides with the dimensionality of the representation, that is, with the number of basis functions of the representation. All these functions belong to the same energy value. Thus, knowing the irreducible representations (of the symmetry group) according to which the stationary-state wave functions transform, we can find certain properties of these functions without solving the Schrödinger equation.

In the case under consideration, all the irreducible representations of the group D_2 are one-dimensional, and so the energy states of the corresponding physical system are non-degenerate. Since there are four irreducible representations in all, the stationary-state wave functions can be classified by their symmetry properties into four types. According to the character Table 5.5, those states which belong to the irreducible representation A are unchanged under the action of any of the operations of the group D_2 . These states are called *fully symmetric states*, and the representation A corresponding to them is called the *fully symmetric representation*. Usually, the ground state of the system belongs to the fully symmetric representation.

The wave functions of the collective motions in even-even nuclei depend only on the spatial coordinates of the nucleons. Since the spins of two identical nucleons in the same one-particle (orbital) state have opposite orientations, the coordinate wave function will be symmetric. Inasmuch as the spatial coordinates of the nucleons are interchanged by the operations $R_{j'}(\pi)$, the wave function describing the rotation of the nucleus must belong to the fully symmetric representation A of the group D_2 , since only in this case will the wave function of the collective motions be transformed into itself, with the same sign, under any operation of the group D_2 .

It is easily verified that the functions (5.149) for a rigid symmetric top (with even K) are invariant under the rotations $R_{j'}(\pi)$, if we take into account that the operators $R_{j'}(\pi)$ act on the generalized spherical harmonics $D_{MK}^I(\Phi, \Theta, \psi)$ in the following way:

$$\begin{aligned} R_1(\pi)D_{MK}^I &= (-1)^K D_{MK}^I; & R_2(\pi)D_{MK}^I &= (-1)^{I-K} D_{M,-K}^I; \\ R_3(\pi)D_{MK}^I &= (-1)^I D_{M,-K}^I. \end{aligned}$$

(We note that, for any K , the transformed function $\Psi(IKM)$ of the symmetric top will differ from (5.149) only in the replacement of the factor $(-1)^I$ in the second term by the factor $(-1)^{I+K}$, and so the functions $\Psi(IKM)$ with odd K do not possess the above-mentioned invariance.) Taking this fact into account, we can represent the solutions of eqn. (5.228) which belong to the fully symmetric representation A of the group D_2 in the form

$$\Phi_{IM}^r(\Phi, \Theta, \psi) = \sum_{\text{even } K \geq 0} g_{IK}^r \Psi(IKM). \quad (5.230)$$

Substituting (5.230) into eqn. (5.228), we obtain a system of linear homogeneous equations for the coefficients g_{IK}^r :

$$\sum_{K' \geq 0} \left\{ \langle IKM | H_r | IK'M \rangle - E_I \delta_{KK'} \right\} g_{IK'}^r = 0. \quad (5.231)$$

The number of these equations coincides with the number of possible different (but even!) values of K and is equal to $\frac{1}{2}I + 1$ for even I and $I - \frac{1}{2}$ for odd I . By equating the determinant of the system of equations to zero, we find the energy eigenvalues $E_{I\tau}$.

As an example, we shall consider the state with angular momentum (spin) $I = 2$. The possible values of K are 0 and 2, and so the system (5.231) will consist of two equations. Using the relations (5.90) and (5.91), it is not difficult to determine the matrix elements $\langle IKM|H_r|IK'M\rangle$ of the Hamiltonian operator; these can be non-zero only when $K' = K$ and when $K' = K \pm 2$. Thus, in the case $I = 2$, we obtain the system

$$\left. \begin{aligned} & \left[\frac{3}{2}(a_1 + a_2) - E_2 \right] g_{20} + \frac{\sqrt{3}}{2}(a_1 - a_2)g_{22} = 0; \\ & \frac{\sqrt{3}}{2}(a_1 - a_2)g_{20} + \left[\frac{1}{2}(a_1 + a_2 + 4a_3) - E_2 \right] g_{22} = 0, \end{aligned} \right\} \quad (5.232)$$

where

$$a_{k'} = \frac{\hbar^2}{J_{k'}}.$$

From the condition that the determinant vanish, we find the two energy values:

$$\left. \begin{aligned} E_{2\tau} &= (a_1 + a_2 + a_3) \\ &+ (-1)^\tau \sqrt{(a_1 + a_2 + a_3)^2 - 3(a_1 a_2 + a_2 a_3 + a_1 a_3)}, \\ E_{2\tau} &= \frac{\hbar^2}{4B\beta^2} \frac{3[3 + (-1)^\tau \sqrt{9 - 8 \sin^2 3\gamma}]}{\sin^2 3\gamma}. \end{aligned} \right\} \quad (\tau = 1, 2) \quad (5.233)$$

According to (5.232) and (5.233), the coefficients g_{IK}^τ , which are orthonormalized by the conditions

$$\sum_{K \geq 0} g_{IK}^\tau g_{IK'}^{\tau'} = \delta_{\tau\tau'}, \quad \sum_\tau g_{IK}^\tau g_{IK'}^{\tau} = \delta_{KK'}, \quad (5.234)$$

will be equal to (for $I = 2$)

$$g_{2,K}^{\tau=1} = \sqrt{\frac{1}{2} + (-1)^{K/2} \frac{\sin \gamma \sin 3\gamma + 3 \cos \gamma \cos 3\gamma}{2\sqrt{9 - 8 \sin^2 3\gamma}}},$$

$$g_{2,K}^{\tau=2} = (-1)^{(2-K)/2} g_{2,2-K}^{\tau=1}, \quad K = 0, 2. \quad (5.235)$$

For $I = 3$, only the value $K = 2$ is possible; therefore $g_{3,2}^{\tau=1} = 1$ and $\Phi_{3M}^{\tau=1}(\Theta_i) = \Psi(3, 2, M)$. The value of the energy is

$$E_{3,1} = \langle 3, 2, M | H_r | 3, 2, M \rangle = 2(a_1 + a_2 + a_3) = \frac{\hbar^2}{4B\beta^2} \frac{18}{\sin^2 3\gamma}. \quad (5.236)$$

(The state with $I = 3$ and $K = 0$ is absent, since $\Psi(3, 0, M) = 0$.)

Principal features of the rotational spectra of non-axial nuclei.

The rotational-energy spectrum of axially symmetric nuclei ($\gamma = 0$ or $\pi/3$) is associated with the rotation of the nucleus about an axis perpendicular to the symmetry axis of the nucleus. In non-axial nuclei, however, the rotations are more diverse, and the corresponding energy spectrum is therefore more complicated. For $\gamma \rightarrow 0$ or $\gamma \rightarrow \pi/3$, some of the levels of non-axial nucleus remain finite and go over into levels of the axially symmetric nucleus. Such levels of a non-axial nucleus form the principal rotational band in the rotational-energy spectrum. The other rotational levels of a non-axial nucleus tend to infinity as $\gamma \rightarrow 0$ or $\gamma \rightarrow \pi/3$. These levels form anomalous rotational bands.

The ratio of any two *rotational energies of a non-axial nucleus* depends only on the one parameter γ , which can be determined for a given nucleus from the ratio of the observed energies of the two lowest excited states. Knowing γ , we can predict the positions of the other energy levels in the rotational spectrum of non-axial nuclei.

The parameter γ can be found by making use of the ratio of the experimental energy values for the first two spin-2 levels, belonging to the principal and anomalous rotational bands. According to (5.233), this ratio is equal to

$$\frac{E_{2,2}}{E_{2,1}} = \frac{3 + \sqrt{9 - 8 \sin^2 3\gamma}}{3 - \sqrt{9 - 8 \sin^2 3\gamma}}. \quad (5.237)$$

As the parameter γ changes from 0 to 30° , this ratio decreases monotonically from ∞ to 2. By measuring the ratio (5.237) experimentally, we can calculate γ ; knowing γ , it is not difficult to determine the spins and positions of the levels in the principal and anomalous rotational bands.

We shall indicate the following characteristic features of the *spectrum of the excited states of a non-axial rotator*:

1. Deviation of the shape of the nucleus from an axially symmetric shape leads to violation of the interval rule (5.168) in the principal rotational

band. For example, if $\gamma = 30^\circ$, the levels of the principal rotational band will satisfy the following interval rule:

$$E_2 : E_4 : E_6 : E_8 : \dots = 1 : 2 \cdot 67 : 5 : 8 : \dots , \quad (5.238)$$

which is substantially different from (5.168).

2. The presence of non-axiality in nuclei leads not only to a change of the interval rule for the levels of the principal rotational bands but also to the appearance of new (anomalous) rotational bands. The first anomalous band consists of levels with spins 2,3,4,5,..., the second consists of levels with spins 4,5,6,7,..., and so on. If $\gamma > 23^\circ$, the first anomalous-band level with spin 2 lies below the level with spin 4 in the principal rotational band. For the rigid non-axial rotator, the anomalous bands are associated with rotation, although in reality (in a non-adiabatic treatment) they correspond to excitations of a rotational-vibrational character.

As an illustration, we show in Fig. 5.19 the ratio of the excited-state energies to the energy of the first excited state as, a function of the non-axiality parameter γ for a non-axial rotator.

The solid curves correspond to the principal rotational band, and the dashed curves to the first anomalous band. The circles indicate the experimental energy ratios for the nuclei ^{166}Er , ^{186}Os , ^{190}Os and ^{192}Os . The experimental spin values are indicated alongside the circles. If, for example, $\gamma = 22^\circ$ for the nucleus ^{190}Os , the calculated energy ratios in the principal rotational band

$$1 : 3 \cdot 01 : 5 \cdot 76 : 9 \cdot 24$$

will be very close to the experimentally observed ratios

$$1 : 2 \cdot 99 : 5 \cdot 61 : 8 \cdot 90.$$

The value $\gamma = 22^\circ$ also explains the positions of the second spin-2 level and the spin-3 level, both of which belong to the anomalous band.

The non-axial rotator model enables us to determine not only the energies of the excited states but also the wave functions. In the general case, the wave functions (5.230) for the non-axial rotator are represented in the form of superpositions of the states (5.149), which correspond to well-defined values of the projection of the total angular momentum along the axis Oz . The coefficients g_{IK}^{γ} determining the contributions of states with different K depend on the non-axiality parameter γ and are determined by the system of algebraic equations (5.231). If $\gamma < 15^\circ$, the main contribution in the sum (5.230) is made by only one term, and, consequently, K is approximately a constant of the motion. In this approximation, the value $K = 0$ corresponds to the principal rotational band; therefore, the rotation

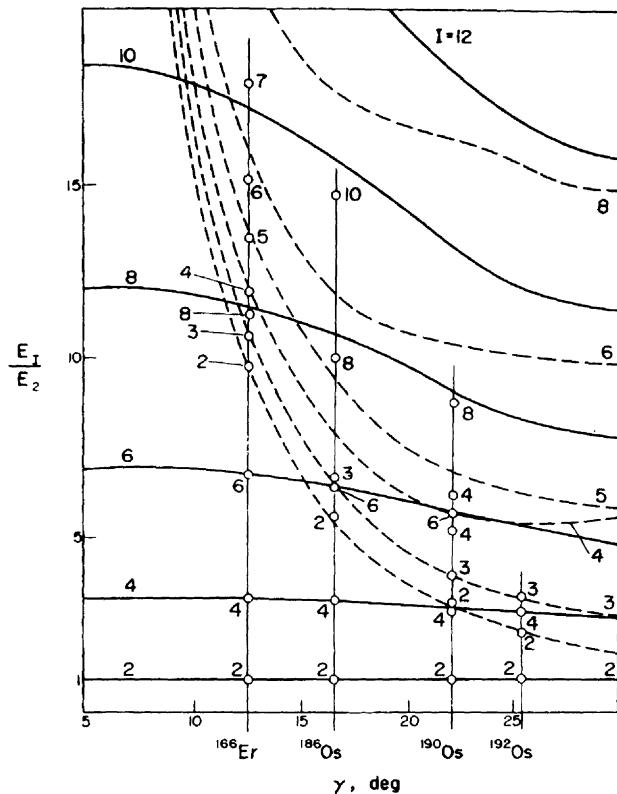


Figure 5.19. Ratios of the energies of the excited states to the energy of the first excited state as a function of the non-axially parameter γ for a non-axial rotator.

occurs about an axis perpendicular to Oz' . For $\gamma > 15^\circ$, one value $K = 2$ corresponds to the first anomalous band. The lowest level $I = 2$ of this band corresponds to rotation of the nucleus about the axis Oz' . For large spin values, a rotation about an axis perpendicular to Oz' is superimposed on the rotation about the axis Oz' . If $\gamma > 15^\circ$, the number K no longer has the meaning of a constant of the motion. The wave function in this case is a superposition of states with different values of K .

5.6. Rotational-vibrational Excitations of Nuclei

General form of the wave equation in the case of quadrupolar deformations. In the adiabatic theory of collective excitations, considered earlier, the coupling between the rotation of the nucleus and the vibrations

of its surface was neglected. This neglect is not always justified, since in many cases the energies of the rotational and vibrational excitations are of the same order. We shall consider the *non-adiabatic theory of rotational-vibrational excitations in nuclei*.⁶

When only second-order deformations ($\lambda = 2$) are taken into account, the collective motions in the nucleus are characterized by five degrees of freedom. As the generalized coordinates, it is convenient to choose the three Euler angles Φ, Θ and ψ , specifying the orientation of the deformed nucleus, and two internal parameters β and γ , defining its shape. In order to write the wave function in these variables, it is necessary to transform from the classical energy of the collective motions to the quantum Hamiltonian operator acting in the space of the five variables $\Phi, \Theta, \psi, \beta$ and γ .

The potential-energy operator $V(\beta, \gamma)$, which depends only on the coordinates β and γ , retains its original form in the quantum case. It is elementary to quantize the part of the Hamiltonian associated with the rotation, since the projections of the angular momentum on to the coordinate axes moving with the nucleus and the corresponding angles of rotation are ordinary canonically conjugate operators. Therefore, the quantum Hamiltonian of rotation H_r is obtained from the classical rotational energy (5.111) by simple replacement of the angular-momentum components $R_{k'}$ by the operators of the angular-momentum components along the principal axes of the ellipsoid:

$$H_r = \sum_{k'} \frac{\hbar^2}{2J_{k'}} R_{k'}^2, \quad J_{k'} = 4B\beta^2 \sin^2 \left(\gamma - \frac{2\pi}{3} k' \right). \quad (5.239)$$

In the case of rotation of even-even nuclei, for which the internal angular momentum $J = 0$, the rotational angular momentum \mathbf{R} coincides with the total angular momentum \mathbf{I} of the system:

$$\mathbf{R} \rightarrow \mathbf{I}.$$

The transition to the quantum description of the kinetic energy (5.105) of the surface vibrations of nuclei is considerably more complicated.

The total kinetic energy of the system, expressed in terms of the generalized coordinates q_i (the index i labels all the degrees of freedom of the system), is usually written in the form

$$T = (B/2) \sum_{ij} g_{ij} \dot{q}_i \dot{q}_j, \quad (5.240)$$

⁶The non-adiabatic theory of collective excitations in nuclei was developed by Davydov (1959, 1960, 1967).

where B is the mass of the system and g_{ij} are symmetric coefficients which can be arbitrary functions of the coordinates. We introduce the square of the line differential in the space of the generalized coordinates:

$$ds^2 = \sum_{ij} g_{ij} dq_i dq_j; \quad (5.241)$$

then the kinetic energy can be represented in the form

$$T = (B/2)(ds/dt)^2. \quad (5.242)$$

If the square of the line differential is known, the Laplacian operator ∇^2 acting in the space of the generalized coordinates is defined in the following way (Pauli, 1933):

$$\nabla^2 = \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial q_i} \left(\sqrt{g} g_{ij}^{-1} \frac{\partial}{\partial q_j} \right), \quad (5.243)$$

where g is the determinant of the matrix g_{ij} , and g_{ij}^{-1} is the inverse matrix of the matrix g_{ij} . Thus, the kinetic energy operator can be expressed in terms of the Laplacian operator as follows:

$$T = -(\hbar^2/2B)\nabla^2. \quad (5.244)$$

According to (5.105) and (5.107), we can write the total nuclear kinetic energy associated with quadrupolar deformations in the form

$$T \equiv T_r + T_v = (B/2)(ds/dt)^2; \\ ds^2 = \sum_{k'} (J_{k'}/B) d\varphi_{k'}^2 + d\beta^2 + \beta^2 d\gamma^2; \quad (5.245)$$

$d\varphi_{k'} = \omega_{k'} dt$ are the angles of rotation about the principal axes Ox' , Oy' , and Oz' of the ellipsoid in the time interval dt . In the general case, $\beta_0 \neq 0$ and $\gamma_0 \neq 0$. If $\omega_{k'}$ is expressed in terms of derivatives of the Euler angles, in accordance with (5.56), then ds^2 will be the square of the line differential in the five-dimensional space of the variables $\Phi, \Theta, \psi, \beta$ and γ . It is simpler, however, to obtain the explicit form of the Laplacian operator directly in the space of the variables $\varphi_{x'}, \varphi_{y'}, \varphi_{z'}, \beta$ and γ . In fact, in this case, the matrix g_{ij} determining the square (5.245) of the line differential has diagonal form. We denote $\varphi_{x'}, \varphi_{y'}, \varphi_{z'}, \beta$ and g , respectively, by q_i , where $i=1,2,3,4$ and 5. According to (5.245), we have

$$g_{ij} = g^{(i)} \delta_{ij}, \quad (5.246)$$

where

$$g^{(1)} = J_{x'}/B, \quad g^{(2)} = J_{y'}/B; \quad g^{(3)} = J_{z'}/B, \quad g^{(4)} = 1, \quad g^{(5)} = \beta^2.$$

Therefore, the inverse matrix g_{ij}^{-1} is also diagonal:

$$g_{ij}^{-1} = g^{(i)-1} \delta_{ij}. \quad (5.247)$$

It is not difficult to verify that the determinant of the matrix (5.246) is equal to

$$g = 4\beta^8 \sin^2 3\gamma. \quad (5.248)$$

Thus, in accordance with (5.243), the Laplacian operator in the five-dimensional space of the coordinates $\varphi_{x'}, \varphi_{y'}, \varphi_{z'}, \beta$ and γ takes the form

$$\nabla^2 = \sum_{k'} \frac{B}{J_{k'}} \frac{\partial^2}{\partial \varphi_{k'}^2} + \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma}. \quad (5.249)$$

Since the derivatives with respect to the angles $\varphi_{k'}$ determine directly the components of the angular-momentum operator \mathbf{R} in the proper coordinate frame, we can rewrite (5.249) in the form

$$\nabla^2 = - \sum_{k'} \frac{B}{J_{k'}} \mathbf{R}_{k'}^2 + \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma}. \quad (5.250)$$

The first three terms in the right-hand side of (5.250) characterize the rotational part of the Laplacian operator, and the last two characterize the vibrational part.

Using (5.244), we obtain the operator of the total kinetic energy in the form

$$T = \sum_{k'} \frac{\hbar^2}{2J_{k'}} \mathbf{R}_{k'}^2 - \frac{\hbar^2}{2B} \left[\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right]. \quad (5.251)$$

Thus, the quantum Hamiltonian operator for the collective motions of the nucleus with second-order deformation taken into account has the form

$$H = \sum_{k'} \frac{\hbar^2}{2J_{k'}} \mathbf{I}_{k'}^2 - \frac{\hbar^2}{2B} \left[\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right] + V(\beta, \gamma). \quad (5.252)$$

Here, the components of the operator \mathbf{R} have been replaced by the components of the operator \mathbf{I} . We write the corresponding wave equation for the stationary states:

$$H\Psi(\Phi, \Theta, \psi, \beta, \gamma) = E\Psi(\Phi, \Theta, \psi, \beta, \gamma). \quad (5.253)$$

The eigenfunctions $\Psi(\Phi, \Theta, \psi, \beta, \gamma)$ of the operator H are studied in the space of the five variables

$\Phi, \Theta, \psi, \beta$ and γ , which vary in the ranges $0 \leq \Phi \leq 2\pi, 0 \leq \Theta \leq \pi, 0 \leq \psi \leq 2\pi, 0 \leq \beta < \infty$, and $0 \leq \gamma \leq 2\pi$. (Sometimes a narrower range is chosen; for example, for a spherical nucleus and small surface vibrations, $0 \leq \gamma \leq \pi/3$.) The functions $\Psi(\Phi, \Theta, \psi, \beta, \gamma)$ are normalized by the condition

$$\int d\tau \Psi^* \Psi = 1, \quad (5.254)$$

where the volume element $d\tau$, which is proportional to \sqrt{g} , is equal to

$$d\tau = \beta^4 |\sin 3\gamma| \sin \Theta d\Phi d\Theta d\psi d\beta d\gamma. \quad (5.255)$$

The expression (5.252) is valid in the case of small vibrations of the nuclear surface about the equilibrium values of the parameters, $\beta = \beta_0$ and $\gamma = \gamma_0$ (the so-called β - and γ -vibrations). The Schrödinger equation (5.253) with the Hamiltonian operator (5.252) has not been solved in the general case. Below, we shall examine the solutions of this equation separately in particular cases, with certain simplifying assumptions.

We have written down the Hamiltonian operator and the corresponding wave equation for the collective motions only, neglecting the one-particle motions of the individual nucleons. However, these motions are coupled and it is not always possible, generally speaking, to separate the collective motions in the nucleus from the one-particle motions. The separation of the collective and one-particle excitations is well fulfilled, for example, for even-even nuclei, in which the minimum one-particle excitation energy is considerably greater than the energy of the collective excitations. In the following, we shall assume that the collective and one-particle motions can be separated, that is, we shall use the adiabatic approximation. In the adiabatic approximation, the total Hamiltonian of the nucleus is represented in the form of a sum of two terms, referring to the collective and one-nucleon motions respectively. The wave function in the adiabatic approximation is written in the form of a product of a wave function depending on the collective variables and a wave function depending on the one-particle variables. Using this function, we can also take into account the coupling between the collective and one-particle motions in the nucleus by perturbation theory, if we can assume this coupling to be a small correction to the Hamiltonian.

Solution of the wave equation for a spherical nucleus. If the equilibrium shape of the nucleus is spherical, then $\beta_0 = 0$ and for small vibrations of the nuclear surface the potential energy is equal to $V = \frac{1}{2}C\beta^2$. In this case, the solution of the Schrödinger equation (5.253) with the operator (5.252) is conveniently chosen in the form of a product of two functions by separating out the dependence of the function on the variable β :

$$\Psi(\Phi, \Theta, \psi, \beta, \gamma) = f(\beta)\Phi(\theta_i, \gamma), \quad (5.256)$$

where θ_i are the three Euler angles. Substituting (5.256) into (5.253), we obtain two equations:

$$\left\{ -\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{BC}{\hbar^2} \beta^2 + \frac{N}{\beta^2} - \frac{2BE}{\hbar^2} \right\} f(\beta) = 0; \quad (5.257)$$

$$\left\{ \sum_{j=1}^3 \frac{I_j^2}{4 \sin^2(\gamma - \frac{2\pi}{3}j)} - \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - N \right\} \Phi(\theta_i, \gamma) = 0, \quad (5.258)$$

where N is the separation parameter, which depends on I .

We first consider the equation for the function $f(\beta)$. We introduce the new function

$$\varphi(\beta) = \beta^2 f(\beta) \quad (5.259)$$

and the new variable

$$x = (\beta/\beta_C)^2, \quad \beta_C = \sqrt[4]{\hbar^2/BC}.$$

The function $\varphi(x)$ will then satisfy the equation

$$\left\{ x \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} - \frac{x}{4} - \frac{N+2}{4x} + \frac{E}{2\hbar\omega} \right\} \varphi(x) = 0 \quad (5.260)$$

and the boundary conditions

$$\varphi(0) = 0, \quad \lim_{x \rightarrow \infty} x^{-2} \varphi(x) = 0.$$

For small x , this equation has the solution

$$\varphi(x) = x^s, \quad x \rightarrow 0; \quad s = \frac{1 + \sqrt{9 + 4N}}{4}, \quad (5.261)$$

while for large x ,

$$\varphi(x) = e^{-x/2}, \quad x \rightarrow \infty. \quad (5.262)$$

Therefore, in the whole range of the variable x , the solution of eqn. (5.260) can be represented conveniently in the form

$$\varphi(x) = x^s e^{-x/2} P(x). \quad (5.263)$$

The function $P(x)$ satisfies the equation

$$\left\{ x \frac{d^2}{dx^2} + \left(2s + \frac{1}{2} - x \right) \frac{d}{dx} + n \right\} P(x) = 0, \quad (5.264)$$

where

$$n = (E/2\hbar\omega) - s - \frac{1}{4} = (E/2\hbar\omega) - \frac{1}{2} - \frac{1}{4}\sqrt{9+4N}. \quad (5.265)$$

According to (5.261) and (5.262), the function $P(x)$ should tend to a finite non-zero constant as $x \rightarrow 0$, and should behave as a polynomials as $x \rightarrow \infty$. Such solutions of eqn. (5.264) are possible for positive integer values of n and can be expressed in terms of the confluent hypergeometric function

$$P(x) = F(-n, 2s + \frac{1}{2}; x). \quad (5.266)$$

Finally, for the function $\varphi(\beta)$, normalized by the condition $\int_0^\infty d\beta \varphi^2(\beta) = 1$, we obtain the expression

$$\begin{aligned} \varphi_{ns}(\beta) &= \left[\frac{2\Gamma(n+2s+\frac{1}{2})}{\beta_C n! \Gamma^2(2s+\frac{1}{2})} \right]^{1/2} \left(\frac{\beta}{\beta_C} \right)^{2s} e^{-(1/2)(\beta/\beta_C)^2} \\ &\times F\left(-n, 2s + \frac{1}{2}; \frac{\beta^2}{\beta_C^2}\right). \end{aligned} \quad (5.267)$$

According to (5.265), the corresponding energy values will be

$$E_{ns} = 2\hbar\omega \left(n + s + \frac{1}{4} \right) \quad (5.268)$$

and will depend through s (or N) on the total spin I of the nucleus.

We now consider the solutions of eqn (5.258). Since the total angular momentum I and its z -component M in the laboratory coordinate frame K are conserved, we represent the function $\Phi_{IM}^\tau(\theta_i, \gamma)$ in the form

$$\Phi_{IM}^\tau(\theta_i, \gamma) = \sqrt{\frac{2I+1}{8\pi^2}} \sum_{K=-1}^I g_{IK}^\tau(\gamma) D_{KM}^I(\theta_i), \quad (5.269)$$

where $D_{KM}^I(\theta_i)$ are the generalized spherical harmonics and the symbol τ denotes the quantum numbers necessary to characterize the state completely. Before proceeding to the determination of the functions $g_{IK}^\tau(\gamma)$, we shall elucidate which symmetry conditions they satisfy.

In transforming from the variables $\alpha_{2\mu}$ to the variables θ_i, β and γ , only the deformation parameter β is uniquely determined by the variables $\alpha_{2\mu}$, whereas the four variables Φ, Θ, ψ and γ depend on the labelling of the axes x' , y' , and z' and on the choice of the reference directions relative to each of these. The function Φ_{IM}^τ , which depends on these four variables, must be invariant under the above transformations of the coordinate axes. This leads to certain conditions to be imposed on the functions g_{IK}^τ . In particular, using the properties of the functions $D_{KM}^I(\theta_i)$ under transformations of the axes, we can show that the requirement of invariance of the functions Φ_{IM}^τ leads to the following conditions which must be satisfied by the functions $g_{IK}^\tau(\gamma)$:

$$g_{IK}^\tau(\gamma) = e^{i\pi(I+K)} g_{I,-K}^\tau(\gamma); \quad (5.270)$$

$$g_{IK}^\tau(\gamma) = e^{i\pi K/2} g_{IK}^\tau(-\gamma); \quad (5.271)$$

$$g_{IK}^\tau(\gamma) = e^{i\pi K} g_{IK}^\tau(\gamma); \quad (5.272)$$

$$g_{IK}^\tau(\gamma) = \sum_{K'} D_{K'K}^I(0, \pi/2, \pi/2) g_{IK'}^\tau(\gamma - 2\pi/3). \quad (5.273)$$

It follows from (5.272) that

$$g_{IK}^\tau(\gamma) = 0 \quad \text{for odd } K, \quad (5.274)$$

while from (5.270) we obtain

$$g_{IK}^\tau(\gamma) = (-1)^I g_{I,-K}^\tau(\gamma) \quad \text{for even } K. \quad (5.275)$$

Using these conditions, we represent the function $\Phi_{IM}^\tau(\theta_i, \gamma)$ in the form

$$\Phi_{IM}^\tau(\theta_i, \gamma) = \sum_{\text{even } K \geq 0} g_{IK}^\tau(\gamma) \Phi(IKM), \quad (5.276)$$

where

$$\Phi(IKM) = \sqrt{\frac{2I+1}{16\pi^2(1+\delta_{0K})}} \left\{ D_{KM}^I(\theta_i) + (-1)^I D_{-KM}^I(\theta_i) \right\} \quad (5.277)$$

are the normalized function of a rigid symmetric top and are invariant under reflection in the plane $z' = 0$ (if K is even).

We consider the states with zero spin ($I = M = K = 0$). In this case, the function $g_{00}^{\tau}(\gamma) \equiv g(\gamma)$ satisfies the equation (there is no rotation of the nucleus)

$$\left(-\frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - N_0 \right) g(\gamma) = 0, \quad (5.278)$$

whose solutions, when the conditions (5.271) and (5.273) are taken into account, will be the Legendre polynomials

$$g_m(\gamma) = \sqrt{\frac{3(2m+1)}{2}} P_m(\cos 3\gamma), \quad (5.279)$$

if

$$N_0 = 9m(m+1), \quad m = 0, 1, 2, \dots \quad (5.280)$$

From (5.261), we obtain the connection between s and m :

$$s = 1 + \frac{3}{2}m, \quad (5.281)$$

so that, for the energies of the excited states of zero spin, we have

$$E_{nm} = \hbar\omega \left(2n + 3m + \frac{5}{2} \right), \quad n, m = 0, 1, 2, \dots; \quad (5.282)$$

that is, the energy spectrum will be equidistant.

We note that states with spin $I = 1$ are absent, since in this case the only possible value of K is $K = 0$, and from the condition (5.275) we obtain $g_{10}^{\tau}(\gamma) = -g_{10}^{\tau}(\gamma) = 0$.

A study of nuclei with $I > 1$ leads, as in the zero-spin case, to an equidistant energy spectrum for the collective excitations. As we should expect, these results are in complete agreement with the results of the theory of surface vibrations of spherical nuclei (in which internal deformation parameters and Euler angles were not specified; cf. Section 5.1).

The *doubly magic nuclei*, and nuclei close to them, have a spherical equilibrium shape. The experimentally observed energy spectrum of these nuclei differs from the spectrum predicted by the theory of quadrupolar ($\lambda = 2$) vibrations. It is possible that this difference is connected with the neglect of the anharmonicity of the vibrations and also of the interaction between the vibrations of the nuclear surface and the motion of the individual nucleons.

Rotational-vibrational excitations of non-axial nuclei. The rigid non-axial rotator model, in which the interaction between the deformations

of the nuclear shape and the rotation of the nucleus is neglected, is a fairly crude approximation to the real picture of the collective excitations in nuclei. In fact, nuclei may experience stretching on rotation, and so, in the general case, it is necessary to take into account the *coupling of the rotation of the nucleus with the deformations of its surface*.

Confining ourselves to second-order deformations ($\lambda = 2$), we shall consider the particular case of the wave equation describing the collective motions in the nucleus when the variable γ of the transverse vibrations is held fixed (we replace it by some effective value Γ) but the variable β of the longitudinal vibrations remains free. This non-adiabatic approach to the study of the *rotational-vibrational excitations in nuclei* was proposed by Davydov and Chaban (1960). In this case, the collective motions in the nucleus are characterized by four degrees of freedom, corresponding to the three Euler angles $\theta_i(\Phi, \Theta, \psi)$ and the variable β . The classical energy of the collective motions in this case has the following form:

$$E = \sum_{k'=1}^3 2B\beta^2 \sin^2 \left(\Gamma - \frac{2\pi}{3}k' \right) \omega_{k'}^2 + \frac{1}{2}B\dot{\beta} + V(\beta).$$

The associated quantum Hamiltonian operator for the collective motions differs from general expression (5.252). The matrix g_{ij} , which determines the squared differential length in the space of generalized coordinates (5.241), is the fourth order now. Knowing the matrix g_{ij} , we can easily write down the expression of Laplace operator according (5.243) for the case of four variables and the Hamilton operator, as well:

$$H_\Gamma = \sum_{k'=1}^3 \frac{\hbar^2 I_{k'}^2}{8B\beta^2 \sin^2 \left(\Gamma - \frac{2\pi}{3}k' \right)} - \frac{\hbar^2}{2B} \frac{1}{\beta^3} \frac{\partial}{\partial \beta} \beta^3 \frac{\partial}{\partial \beta} + V(\beta). \quad (5.283)$$

The wave function $\Psi(\theta_i, \beta)$, satisfying the equation

$$(H_\Gamma - E)\Psi(\theta_i, \beta) = 0, \quad (5.284)$$

normalized by the condition

$$\int d\tau \Psi^*(\theta_i, \beta)\Psi(\theta_i, \beta) = 1, \quad d\tau = d\Phi \sin \theta d\theta d\psi \beta^3 d\beta.$$

In this approximation we can separate β -variable from Euler angles θ_i . The wave function $\Psi(\theta_i, \beta)$ can be chosen as a product

$$\Psi(\theta_i, \beta) = \beta^{-3/2} \varphi(\beta) \Phi(\theta_i), \quad (5.285)$$

Substituting in equation (5.284) with Hamiltonian (5.283), we obtain two equations:

$$\left\{ \sum_{k'=1}^3 \frac{I_{k'}^2}{2 \sin^2(\Gamma - \frac{2\pi}{3} k')} - \varepsilon_{I\tau} \right\} \Phi_{I\tau}(\theta_i) = 0; \quad (5.286)$$

$$\left\{ -\frac{\hbar^2}{2B} \frac{d}{d\beta^2} + W_{I\tau}(\beta) - E_{\nu I\tau} \right\} \varphi_{\nu I\tau}(\beta) = 0, \quad (5.287)$$

$$W_{I\tau}(\beta) = V(\beta) + \frac{\hbar^2(\varepsilon_{I\tau} + \frac{3}{2})}{4B\beta^2}.$$

Equation (5.286) for the function $\Phi_{I\tau}(\theta_i)$ coincides with the Schrödinger equation for a rigid non-axial rotator considered earlier (see sec. 5.5); $\varepsilon_{I\tau} \equiv \varepsilon_{I\tau}(\Gamma)$ is the rotator dimensionless energy, I is a spin of a nucleus, τ is an additional quantum number determining the rotation band ($\varepsilon_{01} = 0$), ν is the energy quantum number associated with the β -vibration of nuclei. It will be specified below. The function $\varphi_{\nu I\tau}(\beta)$ satisfies the condition

$$\varphi_{\nu I\tau}(0) = \varphi_{\nu I\tau}(\infty) = 0$$

and is normalized according

$$\int_0^\infty d\beta \varphi_{\nu I\tau}^2(\beta) = 1.$$

The quantity $W_{I\tau}(\beta)$ in (5.287) plays a part of the effective potential energy, which depends on the quadrupole nucleus deformations, as well as on its rotations. Note, first of all, that for an arbitrary potential energy of β -vibrations $V(\beta)$, a solution $\varphi_{\nu I\tau}(\beta)$ of equation (5.287) and related energy $E_{\nu I\tau}$ can be written explicitly, near $\beta = \beta_{I\tau}$, which corresponds to a minimum of the effective potential energy $W_{I\tau}(\beta)$. At the same time, the function $\varphi_{\nu I\tau}(\beta)$ takes maximum values ($|\beta - \beta_{I\tau}| \ll 1$):

$$\varphi_{\nu I\tau}(\beta) = N_{\nu I\tau} D_\nu \left(g_{I\tau}(\beta - \beta_{I\tau}) \right), \quad g_{I\tau} = \left[\frac{4B}{\hbar^2} \lim_{\beta \rightarrow \beta_{I\tau}} \frac{d^2 W_{I\tau}(\beta)}{d\beta^2} \right]^{1/4}, \quad (5.288)$$

$$E_{\nu I\tau} \approx \left(\nu + \frac{1}{2} \right) \frac{\hbar^2 g_{I\tau}^2}{2B} + W_{I\tau}(\beta_{I\tau}),$$

where $N_{\nu I\tau}$ is the normalization factor, D_ν is the function (we express it in terms of the product of the Gauss function and Hermite function of the first kind) of a parabolic cylinder; ν can be obtained from the boundary condition $D_\nu(-g_{I\tau}\beta_{I\tau}) = 0$.

Expanding the potential energy (potential) of longitudinal quadrupole deformations $V(\beta)$ in the series with respect to small declinations of the variable β from equilibrium value β_0 and keeping the quadrupole terms only, we obtain

$$V(\beta) = \frac{C}{2} (\beta - \beta_0)^2, \quad C > 0, \quad V(\beta_0) = 0, \quad (5.289)$$

where $C \equiv C_\beta$ is the nucleus elasticity coefficient with respect to β -vibrations. Suppose now that $V(\beta)$ takes form (5.289) not only for small vibrations $\frac{\beta - \beta_0}{\beta_0} \ll 1$. In this case the model of two-parametric potential energy $V(\beta)$ gives the following expression for equation (5.287):

$$\left\{ -\frac{\hbar^2}{2B} \frac{d^2}{d\beta^2} + \frac{C}{2} (\beta - \beta_0)^2 + \frac{\hbar^2(\varepsilon_{I\tau} + \frac{3}{2})}{4B\beta^2} - E_{\nu I\tau} \right\} \varphi_{\nu I\tau}(\beta) = 0. \quad (5.290)$$

This equation describes longitudinal vibrations in the general case of a nonspherical ($\beta_0 \neq 0$) non-axial rotating (at $I \neq 0$) even-even nucleus.

On increasing the nucleus spin I , the rotation part of the effective potential energy in (5.290) $W_{I\tau}(\beta) = \frac{C}{2}(\beta - \beta_0)^2 + \frac{\hbar^2(\varepsilon_{I\tau} + \frac{3}{2})}{4B\beta^2}$ also increases. It consists of $\varepsilon_{I\tau}$, which leads to a visible displacement of the equilibrium position with respect to β , as well as the minimum of the function $W_{I\tau}(\beta)$ and change in the elasticity coefficient. Expanding $W_{I\tau}(\beta)$ in the series with respect to small departures of β from a new equilibrium value $\beta_{I\tau}$ and keeping only quadratic terms, we get

$$W_{I\tau}(\beta) = W_{I\tau}(\beta_{I\tau}) + \frac{C_{I\tau}}{2} (\beta - \beta_{I\tau})^2,$$

$$\beta_{I\tau} = \beta_0 + \frac{\hbar^2(\varepsilon_{I\tau} + \frac{3}{2})}{2BC\beta_{I\tau}^3},$$

where $C_{I\tau} = C \left[1 + \frac{3\hbar^2(\varepsilon_{I\tau} + \frac{3}{2})}{2BC\beta_{I\tau}^2} \right]$ is a new (effective) coefficient of the nucleus elasticity, and $W_{I\tau}(\beta_{I\tau}) = \frac{C}{2}(\beta_{I\tau} - \beta_0)^2 + \frac{\hbar^2(\varepsilon_{I\tau} + \frac{3}{2})}{4BC\beta_{I\tau}^2}$.

Now we introduce a new denotations:

$$p_{I\tau} = \frac{\beta_{I\tau}}{\beta_0} \geq 1, \quad \mu = \frac{1}{\beta_0} \left(\frac{\hbar^2}{BC} \right)^{1/4}, \quad \mu_{I\tau} = \frac{1}{\beta_0} \left(\frac{\hbar^2}{BC_{I\tau}} \right)^{1/4},$$

$$\omega_0 = \sqrt{\frac{C}{B}}, \quad \omega_{I\tau} = \sqrt{\frac{C_{I\tau}}{B}}.$$

It is easy to obtain the following interrelations between them

$$2(p_{I\tau} - 1)p_{I\tau}^3 = \mu^4 \left(\varepsilon_{I\tau} + \frac{3}{2} \right), C_{I\tau} = C \left[1 + \frac{3\mu^4 \left(\varepsilon_{I\tau} + \frac{3}{2} \right)}{2p_{I\tau}^4} \right].$$

Equation (5.290) can be approximately presented in the form

$$\left\{ -\frac{\hbar^2}{2B} \frac{d^2}{d\beta^2} + \frac{C_{I\tau}}{2} \left(\beta - \beta_{I\tau} \right)^2 - [E_{\nu I\tau} - W_{I\tau}(\beta_{I\tau})] \right\} \varphi_{\nu I\tau}(\beta) = 0. \quad (5.291)$$

This is equation of the parabolic cylinder function (see (5.288)). It is convenient to separate the Gauss type factor, which decreases rapidly with growth in the difference $\beta - \beta_{I\tau}$, in the solution of equation (5.291). Introducing a new variable

$$z = \frac{p_{I\tau}(\beta - \beta_{I\tau})}{\mu_{I\tau}\beta_{I\tau}} = \frac{\beta - \beta_{I\tau}}{\beta_0\mu_{I\tau}}, -\frac{p_{I\tau}}{\mu_{I\tau}} \leq z < \infty$$

and a new function $\psi(z)$:

$$\varphi_{\nu I\tau}(\beta) = \psi(z) \exp(-\frac{1}{2}z^2),$$

for the function $\psi(z)$ from (5.291), we obtain the equation

$$\left(\frac{d^2}{dz^2} - 2z \frac{d}{dz} + 2\nu \right) \psi(z) = 0, \nu = \frac{E_{I\tau} - W_{I\tau}(\beta_{I\tau})}{\hbar\omega_{I\tau}} - \frac{1}{2}, \psi\left(-\frac{p_{I\tau}}{\mu_{I\tau}}\right) = 0.$$

A solution of the last second order differential equation that satisfies the above cited boundary conditions for $\varphi_{\nu I\tau}(\beta)$ and $\psi(z)$ is the first kind Hermite function

$$H_\nu(z) = \frac{1}{2\Gamma(-\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(\frac{k-\nu}{2}\right) (2z)^k.$$

The energy quantum number ν associated above all with longitudinal β -vibrations of the nucleus is a root of the equation $H_\nu(-p_{I\tau}/\mu_{I\tau}) = 0$.

Thus, the eigenfunctions $\varphi_{\nu I\tau}(\beta)$ and eigenvalues $E_{\nu I\tau}$ for equation (5.291) takes the final form

$$\varphi_{\nu I\tau}(\beta) = \bar{N}_{\nu I\tau} H_\nu \left(\frac{\beta - \beta_{I\tau}}{\beta_0\mu_{I\tau}} \right) \exp \left[-\frac{1}{2} \left(\frac{\beta - \beta_{I\tau}}{\beta_0\mu_{I\tau}} \right)^2 \right], \quad (5.292)$$

$$E_{\nu I\tau}(\beta) = \hbar\omega_0 \left\{ \left(\nu + \frac{1}{2} \right) \sqrt{1 + \frac{3}{2} \left(\frac{\mu}{p_{I\tau}} \right)^4 \left(\varepsilon_{I\tau} + \frac{3}{2} \right)} + \frac{1}{4} \left(\frac{\mu}{p_{I\tau}} \right)^2 \times \left(\varepsilon_{I\tau} + \frac{3}{2} \right) + \frac{1}{2} \left(\frac{p_{I\tau} - 1}{\mu} \right)^2 \right\}, \quad (5.293)$$

where $\bar{N}_{\nu I\tau}$ is a normalization factor. The quantum numbers ν , that determined from the boundary conditions, in a general case are not integers. If ν is a nonnegative number n , then the Hermite function $H_\nu(z)$ transforms in the n -th order Hermite polynomial. The quantities $\varepsilon_{I\tau}$ and $p_{I\tau}$ depend on the nucleus parameter of non-axiality Γ . Hence, the energy $E_{\nu I\tau}$ specified by (5.293) depends on the parameter Γ , as well.

The above-specified parameter μ is inversely proportional to the forth power root from the nucleus elasticity coefficient C . For enough rigid (weakly deformable) nuclei when C is large, μ is a small quantity ($\mu \ll 1$). For nuclei that easy change their shape, at small C , the quantity can be order of one. It is naturally to call the parameter μ the nucleus softness parameter. At $\mu \rightarrow 0$, when $C \rightarrow \infty$, we deal with the limiting case of absolutely rigid nuclei. It corresponds to complete separation of the rotational motion from other nucleus motions, as it is assumed in the adiabatic approximation. That is why μ is called also the parameter of the nucleus unadiabaticity. According to the above-reported, we usually call the nuclei with $\mu < \frac{1}{3}$ rigid nuclei, and nuclei with $\mu > \frac{1}{3}$ soft nuclei. We can approximately divide the collective excited states in rigid nuclei into rotational and vibrational. For soft nuclei such division is quite conditionally. In soft nuclei the collective excite states may be treated as a complex mixture of rotational and vibrational motions. The energy spectrum of soft nuclei, even in the ground rotation band, can considerably differ from the adiabatic theory spectrum. In particular, it happens to be that the energy spectrum of soft nuclei is practically equidistant at some values of parameters in spite of the equilibrium shape can essentially differ from the spherical one.

Knowing the experimental ratio of energies of three levels, we can determine the nucleus parameters of softness and nonaxiality μ and Γ . By using them, we can find ratio between energies of the different nuclear states. Comparison of the theory with experiment shows that nuclei $^{56}_{26}Fe$, $^{72}_{32}Ge$, $^{108}_{46}Pd$, $^{198}_{80}Hg$ and others can be related to the soft nuclei. Their softness parameter is $\mu \gtrsim 0.5$. Nuclei $^{156}_{64}Gd$, $^{166}_{68}Er$, $^{186}_{76}Os$, $^{188}_{76}Os$, $^{230}_{90}Th$, $^{238}_{92}U$, $^{238}_{94}Pu$ with $\mu < 0.3$ relate to rigid ones. The rotational states of such nuclei are good defined. The available experimental data for some nuclei confirm that their rotational motion and vibrational motion of the surface are closely

connected. The corresponding numerical calculations are in a good agreement with these experimental data (Davydov, 1967).

Now we briefly consider solutions of equation (5.287) for other, different from (5.289), model dependencies of the potential energy (potential) of the longitudinal nucleus deformations $V(\beta)$ used in applications. For example, for quite general three-parameter potential energy $V'(\beta)$ with a minimum at $\beta = \beta_0$, we can write:

$$V'(\beta) = \frac{C'}{2}(\beta - \beta')^2 + \frac{\hbar^2 \varepsilon}{4B\beta^2} + D, \quad C' > 0, \quad \varepsilon > 0. \quad (5.294)$$

The wave function $\varphi_{\nu I\tau}(\beta)$ in the region of $W_{I\tau}(\beta)$ minimum takes form (5.288) or (5.292) with parameters determined by relations

$$g_{I\tau} = \frac{\sqrt{2}}{\beta' \mu_{I\tau}}, \quad \beta' = \beta_0 - \frac{\hbar^2 \varepsilon}{2BC'\beta_0^3}, \quad \beta_{I\tau} = \beta' + \frac{\hbar^2(\varepsilon + \varepsilon_{I\tau} + \frac{3}{2})}{2BC'\beta_{I\tau}^3},$$

$$\mu_{I\tau} = \frac{1}{\beta'} \left(\frac{\hbar^2}{BC'_{I\tau}} \right)^{1/4}, \quad C'_{I\tau} = C' + \frac{3\hbar^2}{2B\beta_{I\tau}^4} \left(\varepsilon + \varepsilon_{I\tau} + \frac{3}{2} \right), \quad p'_{I\tau} = \frac{\beta_{I\tau}}{\beta'},$$

$$2(p'_{I\tau} - 1)p'^3_{I\tau} = \mu'^4 \left(\varepsilon + \varepsilon_{I\tau} + \frac{3}{2} \right), \quad \mu' = \frac{1}{\beta'} \left(\frac{\hbar^2}{BC'} \right)^{1/4}.$$

In this case, for the energy eigenvalues of the collective motions in a nucleus we obtain the following formula

$$E'_{\nu I\tau} = D + \hbar\omega' \left\{ \left(\nu + \frac{1}{2} \right) \left[1 + \frac{3}{2} \left(\mu'/p'_{I\tau} \right)^4 \left(\varepsilon + \varepsilon_{I\tau} + \frac{3}{2} \right) \right]^{1/2} \right. \\ \left. + \frac{1}{4} \left(\mu'/p'_{I\tau} \right)^2 \left(\varepsilon + \varepsilon_{I\tau} + \frac{3}{2} \right) + \frac{1}{2} \left[\left(p'_{I\tau} - 1 \right)/\mu' \right]^2 \right\}, \quad \omega' = (C'/B)^{1/2},$$

where D is determined from the condition $V(\beta_0) = 0$. Apparently, at $\varepsilon = 0$ ($D = 0$) and $C' = C$, the potential energy $V'(\beta)$ transforms in (5.289). At the same time, $E'_{\nu I\tau}$ transforms in (5.293).

For the two-parameter dependencies of the potential energy $V(\beta)$

$$V^{(1)}(\beta) = \frac{C}{8} \beta_0^2 \left(\frac{\beta}{\beta_0} - \frac{\beta_0}{\beta} \right)^2, \quad V^{(2)}(\beta) = \frac{C}{2} \left(\frac{\beta_0}{\beta} \right)^2 \left(\beta - \beta_0 \right)^2 \quad (5.295)$$

the solutions of equations (5.287) can be written in the explicit form in the whole domain of a change of the parameter $\beta \geq 0$ in terms of the Laguerre polynomials L_n^α :

$$\begin{aligned}\varphi_{nI\tau}^{(1)}(\beta) &\equiv \varphi_{nI\tau}^{(1)}\left(\sqrt{2}\beta_{00}x\right) \\ &= \left[\sqrt{2}n!/\beta_{00}\Gamma(n+\sigma)\right]^{1/2} x^{\sigma-\frac{1}{2}} L_n^{\sigma-1}(x^2)e^{-\frac{x^2}{2}},\end{aligned}$$

$$\begin{aligned}\beta_{00} &= \left(\frac{\hbar^2}{BC}\right)^{1/4}, \quad x = \frac{\beta}{\beta_{00}\sqrt{2}} = \frac{\beta}{\beta_0}\xi, \\ \sigma &\equiv \sigma_{I\tau} = 1 + \left(1 + \xi^4 + \frac{1}{2}\varepsilon_{I\tau}\right)^{1/2};\end{aligned}$$

$$\begin{aligned}\varphi_{nI\tau}^{(2)}(\beta) &\equiv \varphi_{nI\tau}^{(2)}\left(\beta_0y\right) \\ &= \left[\frac{\eta^4 n!}{\beta_0(n+s)^2\Gamma(n+2s)}\right]^{1/2} (2\lambda y)^s L_n^{2s-1}(2\lambda y)e^{-\lambda y}, \\ y &= \frac{\beta}{\beta_0}, \quad \eta = \beta_0\left(\frac{BC}{\hbar^2}\right)^{1/4}, \quad \lambda = \frac{\eta^4}{n+s}, \\ s &\equiv s_{I\tau} = \frac{1}{2} + \left(1 + \eta^4 + \frac{1}{2}\varepsilon_{I\tau}\right)^{1/2}.\end{aligned}$$

The corresponding energy spectra are determined by expressions

$$\begin{aligned}E_{nI\tau}^{(1)} &= \hbar\omega\left(n + \frac{\sigma_{I\tau} - \xi^2}{2}\right), \quad E_{nI\tau}^{(2)} = \hbar\omega\frac{\eta^2}{2} \left[1 - \frac{\eta^4}{(n+s_{I\tau})^2}\right], \\ \omega &= (C/B)^{1/2}, \quad n = 0, 1, 2, \dots,\end{aligned}$$

To study the general wave equation of the collective motion (5.253) with the quantum Hamilton operator (5.252) for five collective dynamical variables $\Phi, \Theta, \psi, \beta, \gamma$, we recall that the potential energy (potential) of the quadrupole deformations $V(\beta, \gamma)$ in (5.252) must be expressed in terms of two invariants (5.120) and (5.121). For small β -vibrations around the equilibrium value $\beta = \beta_0$ we can take into account the only one invariant (5.120). That is why in this case, we get

$$V(\beta, \gamma) = \frac{C}{2} \left\{ \left(\beta - \beta_0 \right)^2 + 4\beta\beta_0 \sin^2 \frac{\gamma - \gamma_0}{2} \right\},$$

where C is a constant. Further, we deal with an arbitrary $\beta \geq 0$, and rewrite $V(\beta, \gamma)$ in the more general form

$$V(\beta, \gamma) = V(\beta) + \beta\beta_0 V(\gamma).$$

Here, we still have a wide range of explicit forms of potentials $V(\beta)$ and $V(\gamma)$. The last expression for $V(\beta, \gamma)$ can also be presented as follows

$$\begin{aligned} V(\beta, \gamma) &= V(\beta) + \frac{\beta_0^4}{\beta^2} V(\gamma) + \delta(\beta, \gamma), \\ \delta(\beta, \gamma) &= \left(\frac{\beta}{\beta_0} - \frac{\beta_0^2}{\beta^2} \right) \beta_0^2 V(\gamma). \end{aligned}$$

Since $\delta(\beta, \gamma) \rightarrow 0$ at $\beta \rightarrow \beta_0$, we can neglect $\delta(\beta, \gamma)$ and substitute

$$V(\beta, \gamma) \approx V(\beta) + \frac{\beta_0^4}{\beta^2} V(\gamma)$$

in equation (5.252). The contribution of order of $\delta(\beta, \gamma)$ corresponds to the bind between β - and γ -vibrations of a nucleus and can be taken into account by the perturbation theory methods. But usually, this bind is weak. Equation (5.253) with five dynamic variables allows one to split out the equation with one variable β .

In fact, let us seek a solution of general wave equation (5.253) in the factorized form

$$\Psi(\Phi, \Theta, \psi, \beta, \gamma) = \beta^{-2} \varphi(\beta) \Phi(\Theta_i, \gamma).$$

Substituting this expression in (5.253) and neglecting by the weak bind between β - and γ -vibrations ($\delta(\beta, \gamma) \approx 0$), we can obtain two independent equations for functions $\Phi(\Theta_i, \gamma)$ and $\varphi(\beta)$:

$$\begin{aligned} &\left\{ \frac{1}{4} \sum_{k'=1}^3 \frac{I_{k'}^2}{\sin^2 \left(\gamma - \frac{2\pi}{3} k' \right)} - \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \left(\sin 3\gamma \frac{\partial}{\partial \gamma} \right) + W(\gamma) - \kappa_{mI\tau} \right\} \\ &\times \Phi_{mI\tau}(\Theta_i, \gamma) = 0, \end{aligned} \tag{5.296}$$

$$W(\gamma) = \frac{2B\beta_0^4}{\hbar^2} V(\gamma);$$

$$\left\{ -\frac{\hbar^2}{2B} \frac{d^2}{d\beta^2} + V(\beta) + \frac{\hbar^2(\kappa_{mI\tau} + 2)}{2B\beta^2} - E_{\nu m I \tau} \right\} \varphi_{\nu m I \tau}(\beta) = 0, \quad (5.297)$$

where $\kappa_{mI\tau}$ is a separation parameter (eigenvalue of equation (5.296)), m – a quantum number associated with the *nucleus transverse γ -vibrations*, ν is a quantum number of the *nucleus longitudinal β -vibrations*. First of all, note that we can write down straight forward the solution of equation (5.297) $\varphi_{\nu m I \tau}(\beta)$ for the above introduced model potentials $V(\beta)$ of the nucleus longitudinal deformations (5.289), (5.294), (5.295). The point is that equation (5.297) for the function $\varphi_{\nu m I \tau}(\beta)$ is obtained from equation (5.287) after formal substitution the quantity $\varepsilon_{I\tau} + \frac{3}{2}$ by $2(\kappa_{mI\tau} + 2)$ in (5.287). Equation (5.296) for the function $\Phi_{mI\tau}(\Theta_i, \gamma)$, in a general case, is quite complex. For the nucleus spin $I \neq 0$ it can be solved numerically. Further, we consider the transverse quadrupole vibrations in a general case of a nonaxial ($\beta_0 \neq 0, \gamma_0 \neq 0$) even-even nucleus in the states with zero spin.

At $I = 0 (\tau = 1)$ the operator of rotation energy in (5.296) consisting squares of the nucleus spin projections I_k^2 give zero contribution. This allows one to simplify equation (5.296), which at $I = 0$ takes the form

$$\left\{ -\frac{1}{\sin 3\gamma} \frac{d}{d\gamma} \left(\sin 3\gamma \frac{d}{d\gamma} \right) + W(\gamma) - \kappa_m \right\} \Phi_m(\gamma) = 0,$$

$$W(\gamma) = \frac{2B}{\hbar^2} \beta_0^4 V(\gamma),$$

where $\kappa_m = \kappa_{m01}$ is an eigenvalue, and the desired wave function $\Phi_m(\gamma) \equiv \Phi_{m01}(\gamma)$ does not depend on the Euler angles. The wave function of the transverse nucleus vibration $\Phi_m(\gamma)$ is normalized by the condition

$$\int_0^{\pi/3} d\gamma \sin^3 \gamma \Phi_m^2(\gamma) = 1.$$

The equation of the function $\Phi_m(\gamma)$ is generalization of equation (5.278) for a nonspherical ($\beta_0 \neq 0$) nonaxial ($\gamma_0 \neq 0$) nucleus (Tartakovsky, 1989).

If we choose the potential energy (potential) of the transverse vibration $V(\gamma)$ as follows

$$W(\gamma) = \frac{9}{4} \left(a \operatorname{ctg} \frac{3\gamma}{2} - b \operatorname{tg} \frac{3\gamma}{2} \right)^2, \quad ab \geq 0, \quad (5.298)$$

then the solution of the equation for the transverse nucleus vibrations can be presented in the form ($0 \leq \gamma \leq \frac{\pi}{3}$):

$$\begin{aligned}\Phi_m(\gamma) &= \left[\frac{3m!(2m+a+b+1)\Gamma(m+a+b+1)}{2^{a+b+1}\Gamma(m+a+1)\Gamma(m+b+1)} \right]^{1/2} \\ &\times (1 - \cos 3\gamma)^{\frac{a}{2}} (1 + \cos 3\gamma)^{\frac{b}{2}} P_m^{(a,b)}(\cos 3\gamma),\end{aligned}\quad (5.299)$$

$$\kappa_m = 9[m(m+a+b+1) + \frac{1}{2}(a+b)], \quad m = 0, 1, 2, \dots, \quad a, b \geq -1,$$

where $P_m^{(a,b)}(\cos 3\gamma)$ are the Jacobi polynomials. Note that the potential $V(\gamma)$ has a minimum at $\gamma = \gamma_0$, where $\cos 3\gamma_0 = \frac{b-a}{b+a}$ and $V(\gamma_0) = 0$. The nucleus elasticity coefficient with respect to γ -vibrations is

$$C_\gamma \equiv \lim_{\gamma \rightarrow \gamma_0} \frac{d^2V(\gamma)}{d\gamma^2} = \frac{81\hbar^2}{4B\beta_0^4}(a+b)^2 > 0.$$

If parameters of (5.298) are equal: $a = b = \xi - \frac{1}{2}$ then we have the one-parameter potential

$$V(\gamma) = \frac{9\hbar^2}{2B\beta_0^4} \left(\xi - \frac{1}{2} \right)^2 \operatorname{ctg}^2 3\gamma$$

with $\gamma_0 = \frac{\pi}{6}$. Its wave functions $\Phi_m(\gamma)$ can be expressed in terms of the ultraspherical polynomials or the Gegenbauer polynomials

$$C_m^\xi(\cos \gamma) = \frac{\Gamma\left(\xi + \frac{1}{2}\right)\Gamma(2\xi+m)}{\Gamma\left(\xi + m + \frac{1}{2}\right)\Gamma(2\xi)} P_m^{(\xi-\frac{1}{2}, \xi-\frac{1}{2})}(\cos 3\gamma).$$

In this case, according to (5.299), the wave functions of the nucleus transverse vibrations take the form

$$\Phi_m(\gamma) = 2^{\xi-1} \left[\frac{6m!(m+\xi)}{\pi\Gamma(m+2\xi)} \right]^{1/2} (\sin 3\gamma)^{\xi-\frac{1}{2}} \Gamma(\xi) C_m^\xi(\cos 3\gamma),$$

and the corresponding eigenvalues are

$$\kappa_m = 9[m(m+2\xi) + \xi - \frac{1}{2}].$$

At $a = b = -\frac{1}{2}$ ($\xi = 0$) the functions $\Phi_m(\gamma)$ are expressed in terms of the first kind Tchebyshev polynomials

$$\begin{aligned} T_m(\cos 3\gamma) &= \cos[m a r c \cos(\cos 3\gamma)] \\ &= \frac{2^{2m}(m!)^2}{(2m)!} P_m^{(-\frac{1}{2}, -\frac{1}{2})}(\cos 3\gamma) = \cos 3m\gamma. \end{aligned}$$

Thus, in this case ($\xi = 0$) the wave functions are equal to:

$$\begin{aligned} \Phi_m(\gamma) &= \left(\frac{6}{\pi \sin 3\gamma} \right)^{1/2} T_m(\cos 3\gamma) = \left(\frac{6}{\pi \sin 3\gamma} \right)^{1/2} \cos m\gamma, \\ \kappa_m &= 9 \left(m^2 - \frac{1}{2} \right). \end{aligned}$$

(Note that $\lim_{\xi \rightarrow 0} \Gamma(\xi) C_m^\xi(\cos 3\gamma) = \frac{2}{m} \cos 3\gamma$.)

At $a = b = 0$ ($\xi = \frac{1}{2}$) the Jacobi and Gogenbauer polynomials reduce to the Legendre polynomials

$$P_m^{(0,0)}(\cos 3\gamma) = C_m^{\frac{1}{2}}(\cos 3\gamma) = P_m(\cos 3\gamma).$$

According to (5.298), this is the case of a spherical nucleus ($\beta_0 = 0$), when $W(\gamma) = 0$, and we arrive to equation (5.278) with solutions (5.279) and eigenvalues (5.280).

Wave functions (5.299) can be used, in particular, when studying the monopoly electric transitions in nuclei, i.e., EO-transitions, between the nucleus states with zero spin. These O-O-transitions are rather sensitive to the nuclear structure. Thus, we can hope to determine the parameters of the transverse γ -vibrations a and b in (5.298) and (5.299) for particular nuclei by comparison of the calculated nuclear form-factors and other characteristics with the experimental data.

5.7. Dynamical Chaos and Collective Motion in Nuclei

Regular and chaotic motion. Common usage of the term "*regular motion in physical systems*" appeals to those determinate processes where the past and future are uniquely determined by the present state and corresponding equations of motion. One usually considers regular motion in studying the dynamics of various physical systems, including atomic nuclei; however, it has turned out that, under the influence of nonlinear effects, one has to allow for other types of motion, namely, for *chaotic* (*or stochastic*)

motion discovered by Henri Poincaré as early as the end of the previous century (Poincaré, 1924). It should be noted that chaotic motion, resulting in complete randomness of behavior of the system under study and the unpredictability of its future development in terms of a deterministic description, arises in rigidly determined systems and follows immediately from the dynamics itself rather than from the complexity of the systems or the limited accuracy in assigning the initial conditions.

Therefore, different nuclear models may be governed by both regular (one-particle nuclear models) and *chaotic patterns of nuclear motion*, with the latter exhibited by some collective models such as that of a liquid droplet. Advances in the statistical theory of multinucleonic nuclei ought to be related not only to the complexity of the nuclear systems which this theory has enabled us to describe, but also to the existence of chaotic types of nuclear motion resulting from properties of nuclear interaction. At the same time, the shell structure arising in nuclei should be related not only to the Pauli principle but also to the regular type of nucleon motion. Under appropriate conditions, regular and chaotic motions in nuclei can co-exist in the same way as in other systems, with the respective contributions depending on the total energy of the system and specific type of the potential energy.

It should be noted that in a quantum system the reasons underlying the observed chaotic motion are closely related to the behaviour of the corresponding classical system with the same number of degrees of freedom and the same potential energy as those in the quantum system. Therefore, it seems advisable to begin our considerations with an examination of a classical physical system, simply to obtain an insight into the chaotic motion therein; at the same time, we will discuss the conditions under which the systems may exhibit regular or *chaotic motion*. It is convenient to proceed from the general canonical Hamilton equations which describe the dynamics of a classical system and are given by a set of ordinary first-order equations with respect to time:

$$\dot{q}_i = \{H, q_i\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \dots, s, \quad (5.300)$$

where $H \equiv H(q, p)$ is the Hamiltonian of a system with s degrees of freedom. $H(q, p)$ is a function of the independent canonically-conjugate values (variables) $q = (q_1, q_2, \dots, q_s)$ and $p = (p_1, p_2, \dots, p_s)$ which are specified by s -dimensional vectors in $2s$ -dimensional phase space (q, p) . Recall that, in a particular case, the set of q may be spatial coordinates of material particles of the system in question and the set of p may be their momenta. The brackets in (5.300) denote the classical Poisson ones for a pair of the variables f and g , i.e.

$$\{f, g\} = \sum_{i=1}^s \left(\frac{\partial f}{\partial p_i} \cdot \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \cdot \frac{\partial g}{\partial p_i} \right). \quad (5.301)$$

The dots above $q_i \equiv q_i(t)$ and $p_i \equiv p_i(t)$ stand, as usual, for derivatives with respect to time t . We confine ourselves to conservative systems where the Hamiltonian $H(q, p)$ does not depend explicitly on time. The motion of such a system is described by a *phase trajectory* in $2s$ -dimensional phase space, i.e. by a vector $z(t) = (q(t), p(t))$ with $2s$ components in this space. The value f , which is a function of the *dynamical variables* q and p , and does not depend explicitly on time, will remain constant provided that the *Poisson brackets* $\{H, f\}$ for it are zero because, consistent with (5.300),

$$\dot{f} \equiv \sum_{i=1}^s \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = \{H, f\}.$$

Such an f is referred to as an integral of motion. In particular, for $f = H$ we get $\{H, H\} = 0$, and this simply means that the total energy E of the system is one of the integrals of its motion, i.e. $H(q, p) = E = \text{const}$.

It is well known that the Hamilton equations (5.300) are integrable if there exist s independent integrals of motion $I_i = I_i(q, p)$ satisfying the following conditions

$$\{I_i, I_k\} = 0, \quad \{H, I_i\} = 0, \quad i, k = 1, 2, \dots, s. \quad (5.302)$$

Needless, to say, one can hardly determine from the general appearance of a system whether it is integrable or not; as first shown by Poincaré, the vast majority of such systems are not integrable. However, if for some physical system there exists a *canonical transformation* $(q, p) \rightarrow (\xi, \eta)$ such that the Hamiltonian has the form of a sum of s Hamiltonians of independent one-dimensional oscillators

$$H(q, p) \rightarrow H(\xi, \eta) = \sum_{k=1}^s \frac{\omega_k}{2} \left(\xi_k^2 + \eta_k^2 \right) \quad (5.303)$$

(here the masses are assumed to be unit), then the *integrability of the system* (5.300) becomes evident because one can readily write down s independent integrals of motion I_k

$$I_k = \frac{1}{2} \left(\xi_k^2 + \eta_k^2 \right), \quad k = 1, 2, \dots, s, \quad (5.304)$$

meeting the requirements (5.302). The variables ξ_k and η_k are conveniently substituted by specific canonically-conjugate variables I_k (action) and φ_k (angle):

$$\xi_k = \sqrt{2I_k} \sin \varphi_k, \quad \eta_k = \sqrt{2I_k} \cos \varphi_k, \quad k = 1, 2, \dots, s. \quad (5.305)$$

Hence, it is seen that, in particular, the integrals taken along a closed contour $\oint \eta_k d\xi_k$ are equal to $2\pi I_k$, i.e. the variables I_k are the *adiabatic invariants*, suitable for quantization of the classical system. This being the case, the Hamiltonian of the system will depend only on the *actions* I_k

$$H = \sum_{k=1}^s \omega_k I_k, \quad (5.306)$$

and the equations of motion become separable and assume the very simple form

$$\dot{\varphi}_k = \frac{\partial H}{\partial I} = \omega_k, \quad \dot{I}_k = -\frac{\partial H}{\partial \varphi} = 0. \quad (5.307)$$

Their solutions are given by

$$I_k = C_k, \quad \varphi_k = \omega_k t + \varphi_{k0}, \quad (5.308)$$

where C_k and φ_{k0} are constants.

With a single degree of freedom ($s = 1$), these solutions are exhibited by a circumference of radius $\sqrt{2I_1}$ in the phase space (or a plane); i.e. $\xi_1^2 + \eta_1^2 = 2I_1 = 2C_1$. For $s = 2$, the phase trajectory $z(t)$ in a three-dimensional space may be conceived as a curve densely wound around the surface of a two-dimensional torus with the radius $\sqrt{2C_1} = \sqrt{2I_1}$ or $\sqrt{2C_2} = \sqrt{2I_2}$. By changing the constants C_1 and C_2 whose interdependence for a given value of energy E is determined, according to (5.306), by the relation $\omega_1 C_1 + \omega_2 C_2 = E$, we can obtain an infinite set of the tori embedded into each other and referred to as the invariant ones. The specific character of motion depends on the ratio of the frequencies ω_1 and ω_2 . If these are commensurable, i.e. $\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}$ where n_1 and n_2 are integers, then the phase trajectory shall close on itself after n_1 rotations by the angle φ_1 and n_2 rotations by the angle φ_2 ; such tori are termed the rational or resonant ones. If the frequencies are incommensurable, then the phase trajectory will never close on itself in winding a torus (the irrational or nonresonant tori).

In what follows, we restrict our attention to conservative systems with two degrees of freedom ($s = 2$) and show that in this case there can arise chaotic motions for the Hamiltonians of a more complicated form than that given by expression (5.303). This takes place because the system of equations (5.300) is not generally integrable for $s > 2$, whereas at $s = 1$ a conservative system is always integrable. In studying *nonintegrable systems*

one usually presents the Hamiltonian as a sum of an integrable part, similar to that given by (5.303), and a nonintegrable remainder which is said to be a perturbation. By and large, the finite motion of an integrable system (without the perturbation) with s degrees of freedom, such as that governed by the Hamiltonian (5.303), is conditionally periodic with the frequencies ω_k , and the phase trajectories lie on the s -dimensional invariant tori; in this case, the system's motion shall be regular. However, with the perturbation switched on, a transition to chaotic motion can take place.

The criteria determining the regular and chaotic motion can be stated in terms of the phase trajectories. In fact, the perturbations mentioned above can give rise to *local instabilities* due to which, in the course of time, initially close trajectories move apart from one another in the phase space. With the regular motion that may take place even given the perturbation, the trajectories drift apart at a low rate, approximately equal to time ($\sim t$). For the *irregular motion*, however, the scattering of initially close trajectories occurs rapidly according to the exponential law ($\sim e^{\lambda t}$) (Henon, Heiles, 1964).

A rigid *Kolmogorov-Arnold-Moser theorem* (Kolmogorov, 1954; Arnold, 1963; Moser, 1968) has been proven which states that, given a small nonintegrable perturbation, the nonresonant invariant tori with phase trajectories wound around do not generally disintegrate but change their shape, still remaining invariant and embedded into each other; the resonant tori do disintegrate, yet the regular motion still governs the general evolution of the system. With sufficiently strong (nonlinear) perturbations, the nonresonant tori disintegrate as well and this leads to the so-called *global chaos* (a transition from regular to random behaviour) which was first observed in numerical and analytical simulations. Clearly, as the system proceeds from regular motion to chaos, its entropy increases (the so-called *Kolmogorov entropy* (Kolmogorov, 1959)).

Procedures for the determination of randomness of motion.

There are several general procedures applicable to the tests of arbitrary systems for the chaotic motion; they amount to calculations of the *Lyapunov exponents*, *K-entropy* and so-called *autocorrelation functions*. Yet, for conservative systems with two degrees of freedom ($s = 2$), two simple and illustrative methods, which fall slightly short of the rigorous ideal, are convenient to use for the determination of the domains where motion in the phase space is regular or chaotic, and for the evaluation of the *critical energy of the order-to-chaos transition*. One of these procedures employed for theoretical predictions of the order-to-chaos transition starts from an investigation of the geometric properties of the two-dimensional potential energy surface of the system $V = V(q_1, q_2)$, with the emphasis being usually on the examination of the *Gaussian curvature* and *singular points of*

this *surface*. The other procedure, often called that of the *Poincaré cross-sections*, enables one to find out the *domains of order and chaos* in an adequate and straight-forward way. The Poincaré cross-section are known to be given by the set of points of the intersection of a phase trajectory with one of phase planes.

Below we will consider, in more detail, the procedure based on the examination of geometric properties of the potential energy surface. To begin with, we study the stability of an arbitrary phase trajectory $z_0(t) = (q_0(t), p_0(t))$ satisfying eqns. (5.300) with $s = 2$ (Ljapunov, 1954, 1956). Consider the difference $\Delta z(t) = z(t) - z(t + \Delta t)$ where $z(t)$ is an arbitrary phase point, close to $z_0(t)$. For a locally stable trajectory $z_0(t)$ the deviation $\Delta z(t)$ will decrease exponentially with time, and for a locally unstable trajectory it will grow exponentially provided the perturbation is sufficiently great.

We linearize Eqs. (5.300) with respect to $\Delta z(t)$ by having rewritten them in the form

$$\dot{z}_n = \sum_l M_{nl} \frac{\partial H}{\partial z_l}, \quad M_{nl} = \frac{(-1)^l - (-1)^n}{2}, \\ n = 1, 2 (l = 1, 2), \quad n = 3, 4 (l = 3, 4), \quad (5.309)$$

where the following designations are used

$$z_1 = p_1, \quad z_2 = p_2, \quad z_3 = q_1, \quad z_4 = q_2.$$

Expanding the Hamiltonian H in the Taylor series with respect to Δz , we get

$$H(z) = H(z_0) + \sum_{k=1}^4 \frac{\partial H}{\partial z_k} \Delta z_k + \dots \quad (5.310)$$

and having retained the terms of the first power in Δz_k at most, we arrive at the set of four linear equations for the deviations $\Delta z(t)$

$$\dot{\Delta z}_n = \sum_k A_{nk}(t) \Delta z_k, \quad A_{nk}(t) = \sum_l M_{nl} \frac{\partial H(z_0(t))}{\partial z_l \partial z_k}. \quad (5.311)$$

If the eigenvalues of the matrix $A_{nk}(t)$ are imaginary, $\Delta z(t)$ will not grow in magnitude, therefore, the phase trajectory $z_0(t)$ will be stable. In the opposite case, when the eigenvalues of the matrix A_{nk} are real, the deviation $\Delta z(t)$ exponentially grows in magnitude and the phase trajectory shall be unstable. From here on, we assume that within a small time interval Δt

the value of $z_0(t)$ remains almost constant and, hence, the matrix A_{nk} in (5.311) may be supposed to be independent of time.

For the Hamiltonian with two degrees of freedom, assuming again unit masses and the kinetic energy independent of time

$$H(q, p) = \frac{1}{2} \left(p_1^2 + p_2^2 \right) + V(q_1, q_2), \quad (5.312)$$

we get the following equation for the eigenvalues of the matrix A_{nk}

$$\text{Det} \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ V_{11} & V_{12} & \lambda & 0 \\ V_{21} & V_{22} & 0 & \lambda \end{vmatrix} = 0, \quad V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}. \quad (5.313)$$

These eigenvalues are equal to

$$\lambda = \pm \frac{1}{\sqrt{2}} \left[-(V_{11}+V_{22}) \pm \sqrt{(V_{11}+V_{22})^2 - 4(V_{11}V_{22}-V_{12}^2)} \right]^{1/2}. \quad (5.314)$$

The roots λ are seen to be real (of both signs possible) with the proviso that

$$\frac{\partial^2 V}{\partial q_1^2} \cdot \frac{\partial^2 V}{\partial q_2^2} - \left(\frac{\partial^2 V}{\partial q_1 \partial q_2} \right)^2 < 0, \quad (5.315)$$

resulting in the exponential instability of the solutions. On the other hand, it follows from geometric definitions that the *Gaussian curvature* $K(q_1, q_2)$ of the potential energy surface $V = V(q_1, q_2)$ is determined by the relation

$$K(q_1, q_2) = \left[\frac{\partial^2 V}{\partial q_1^2} \cdot \frac{\partial^2 V}{\partial q_2^2} - \left(\frac{\partial^2 V}{\partial q_1 \partial q_2} \right)^2 \right] \cdot \left[1 + \left(\frac{\partial V}{\partial q_1} \right)^2 + \left(\frac{\partial V}{\partial q_2} \right)^2 \right]^{-1}. \quad (5.316)$$

Comparing the last expression with the instability condition (5.315), we find that the negative sign of the Gaussian curvature of the surface $V = V(q_1, q_2)$ is responsible for the local instability. Therefore, one may relate the conditions, underlying the global chaos in a two-dimensional system, to the domains of the negative Gaussian curvature appearing on the potential energy surface $V = V(q_1, q_2)$. With the energy E of the system growing, generally, the probability of a transition to the chaotic motion increases.

It would be reasonable to determine the critical energy E_C of an order-to-chaos transition as that equal to the minimum value of the potential energy $V(q_1, q_2)$ on the line of zeros of the Gaussian curvature. In most cases, the critical energy thus determined agrees well enough with the value of the critical energy E_C calculated by the Poincaré cross-section technique.

It should be noted, however, that if both the domain of the negative Gaussian curvature on the potential energy surface and the magnitude of this curvature are small, chaos does not necessarily arise. Yet, if chaos has already arisen for the energy $E \gtrsim E_C$, then calculations show that with the energy increasing further, in a number of cases (when $V(q_1, q_2)$ exhibits a localized domain of the negative Gaussian curvature) an opposite transition from chaos to the regular motion may take place; yet, it occurs at sufficiently high energies $E \gtrsim E'_C$, here E'_C stands for the second critical energy of the chaos-to-order transition; as a rule, $E'_C \gg E_C$ and the former is determined by the upper bound of the negative curvature $K(q_1, q_2)$ domain.

Chaotic oscillations of nuclear surface. According to the Weizsaecker formula (3.17), in a nucleus a considerable fraction of the total binding energy (up to one-third) may be attributed to nuclear surface oscillations and, for this reason, *chaotization of the surface oscillations* becomes of interest, too. Usually one figures the parameters of the collective Hamiltonian out of experimental data but currently some attempts have been made to evaluate these parameters from nucleon-nucleon interactions. At the same time it has turned out that nuclear surface oscillations are described by a collective Hamiltonian resulting in the nonlinear equations of motion which, in their turn, may give rise to the *chaotic motion of nuclear surface* (Bolotin, et al., 1989).

We restrict our consideration to a spherical nucleus of the radius $R = R_0 A^{1/3}$ and investigate the quadrupole oscillations of its surface. Then, in line with (5.97), the oscillating surface should be described by the following equation

$$R(\vartheta', \varphi', t) = R \left\{ 1 + \alpha'_0(t) Y_{20}(\vartheta', \varphi') + \alpha'_2(t) \times [Y_{22}(\vartheta', \varphi') + Y_{2-2}(\vartheta', \varphi')] \right\}, \quad (5.317)$$

where the time-dependent parameters of the quadrupole deformations $\alpha'_0 = \alpha'_0(t)$ and $\alpha'_2 = \alpha'_2(t)$ act as two collective variables. For the sake of convenience we redesignate these variables as

$$x = \sqrt{2}\alpha'_2, \quad y = \alpha'_0, \quad (5.318)$$

introduce the corresponding momenta p_x and p_y , respectively, and arrive at the classical Hamiltonian

$$H \equiv H(x, y, p_x, p_y) = \frac{1}{2} (p_x^2 + p_y^2) + V(x, y) \quad (5.319)$$

with two degrees of freedom which is formally similar to (5.312); yet, here we have to employ an explicit form of the potential energy of the quadrupole oscillations $V(x, y)$ and fit its parameters for a specific nucleus to experimental observations. Recall that, according to (5.318) and (5.100), the variables x and y are related to the deformation parameters β and γ by the equalities $x = \beta \sin \gamma$, $y = \beta \cos \gamma$.

As shown in sec. 5.3, even for a nonspherical nucleus the potential energy of nuclear surface oscillations must be expressed in terms of the invariants $S_2(\lambda)$ and $S_3(\lambda)$ derived in the general form, despite the fact that this potential energy is an invariant itself. In the case of the quadrupole surface oscillations ($\lambda = 2$) of a spherical nucleus we dwell upon here, these invariants can be related to the variables α'_0 and α'_2 or x and y as follows (see (5.117) – (5.125))

$$\begin{aligned} S_2(2) &= \frac{\alpha'_0{}^2 + 2\alpha'_2{}^2}{\sqrt{5}} = \frac{x^2 + y^2}{\sqrt{5}}, \\ S_3(2) &= \sqrt{\frac{18}{35}} \left(2\alpha'_0\alpha'_2 - \frac{1}{3}\alpha'_0{}^3 \right) = \sqrt{\frac{18}{35}} \left(x^2y - \frac{1}{3}y^3 \right). \end{aligned} \quad (5.320)$$

As seen from (5.320), a general polynomial presentation of the potential energy of the quadrupole surface oscillations in a spherical nucleus is given by the expression

$$V(x, y) = \sum_{m+n \geq 1} C_{mn} (x^2 + y^2)^n (x^2y - \frac{1}{3}y^3)^m, \quad (5.321)$$

where m and n are non-negative integers and the constants C_{mn} for real nuclei can be figured out of experimental data.

Now we briefly outline a general approach, irrespective of the given nucleus, to the search of domains of the chaotic and regular patterns of the oscillatory motion of nuclear surface, which stems from the technique suggested above for the studies of the potential energy (5.320) with $n \leq 3$ and $m \leq 2$:

$$\begin{aligned} V(x, y) &= \frac{a}{2}(x^2 + y^2)^2 + b(x^2y - \frac{1}{3}y^3) + c(x^2 + y^2) \\ &+ d(x^2y - \frac{1}{3}y^3)(x^2 + y^2) + e(x^2y - \frac{1}{3}y^3)^2 + f(x^2 + y^2)^3 \end{aligned} \quad (5.322)$$

and describe some of the obtained results. In so doing, we may consider the sum of all terms, except the first term of the oscillatory type $\frac{a}{2}(x^2 + y^2)$, as the perturbation (nonintegrable part of the Hamiltonian) which gives rise to nonlinearities in the equations of motion and, thus, to emerging chaos. In fact, chaos may take place even if we allow for the first three terms ($n \leq 2$, $m \leq 1$) in (5.322). In Eq. (5.322) the parameters b and d may be assumed positive because the potential energy does not change under the substitution $b \rightarrow -b$, $d \rightarrow -d$ provided that y is substituted by $-y$ at the same time. In the three-dimensional space $\{x, y, V = V(x, y)\}$, the expression (5.322) is presented by a complicated surface with minima, maxima and saddle points, which is highly sensitive to five parameters (for simplicity, the expression for $V(x, y)$ is conveniently scaled by the parameter a). Hence, using the symmetry of the Hamiltonian, one usually examines the cross-sections of the potential energy (in particular, that at $x = 0$) in the vicinity of characteristic points and investigates the behavior of $V = V(x, y)$ in the most interesting areas in the space of these parameters. If one also takes into account the requirement specifying the lines of zero Gaussian curvature (see (5.315) and (5.316))

$$\frac{\partial^2 V}{\partial x^2} \cdot \frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y} \right)^2 = 0, \quad (5.323)$$

one can find the critical energy E_C and single out the domains of positive and negative values of the Gaussian curvature $K(x, y)$.

General numerical and analytical analysis of the potential energy (5.322) with three or six terms has shown that, in the domain of positive Gaussian curvature where the motion should be regular, the Poincaré cross-sections for different energies are presented by an ordered dense set of the points of intersection of the phase trajectory with the phase plane $x = 0$. These points are clustered into isolated, usually dense curves, separated on a phase plane from one another by big voids containing no points at all. Such Poincaré cross-sections are typical for the regular motion of the system. Similar Poincaré cross-sections with ordered pointwise curves are also observed in the domain of negative Gaussian curvature, yet, at much lower energies of the system than the first critical value E_C , i.e. here the motion still remains regular. However, with E approaching E_C , an ordered arrangement of the pointwise curves in the Poincaré cross-sections is gradually destroyed, and at $E \gtrsim E_C$ points in the Poincaré cross-sections will no longer belong to some definite curves but will be chaotically or almost chaotically scattered over the phase plane, thus, being indicative of the arising chaos. In the chaotic case the whole or almost whole phase space is occupied by chaotic trajectories. With the energy E increasing further

up to the second critical energy, when $E \gtrsim E'_C$, the regular motion may be restored again, as we have already noted.

The quantitative degree of the chaotic build-up, regarded as a function of system's energy E , can be determined by that fraction $S(E)$ of the phase space which is occupied by chaotic trajectories, with the Poincaré cross-section technique being pretty illustrative in this respect. The function $S(E)$ is very small at $E < E_C$ and $E > E'_C$; it becomes approximately equal to 1/2 at $E \cong E_C$ and $E \cong E'_C$, whereas in the wide energy range $E_C < E < E'_C$ (remind that $E'_C \gg E_C$!) it is equal to unity almost everywhere. Thus, we may conclude that for a considerable number of the sets of the potential energy parameters (5.322) there exist a pretty narrow transitional energy region, where the regular motion turns into chaos, and the second transitional region, where the regular motion may be restored.

As regards real nuclei, such studies with the potential energy (5.322) were carried out for the isotopes of Kr^{74-80} , with the realistic values of parameters matched to the data reported from measurements. Some experimental results suggested that $Kr^{74,76}$ nuclei should be spherical, whereas other were indicative of a considerable deformation of these nuclei. On the whole, these data may be regarded as an experimental evidence in favor of the theoretically predicted coexistence of nuclear shape isomers (Solovyev, 1971). To put down the potential energy for nonspherical nonaxial nuclei, one ought to make use of the generalized invariants $S_2(2)$ and $S_3(2)$ which can be derived from (5.320) by the substitutions $x \rightarrow x - x_0$ and $y \rightarrow y - y_0$ where $x_0 = \beta_0 \sin \gamma_0$, $y_0 = \beta_0 \cos \gamma_0$. Here β_0 and γ_0 are the equilibrium nuclear deformation parameters (see (5.119) – (5.121)).

Strongly deformed nuclei exhibit such a geometry of the potential energy surface that nonlinear effects should show up even at low excitation energies; hence, in these nuclei chaotic motion can take place at pretty low energy, i.e. here E_C is not great. It may be also supposed that chaos may also arise at low energies when one type of the motion becomes mixed with another, for example, under the mixing of oscillations and rotations in non-spherical soft nuclei or when there is a mixing of one-particle oscillations with the collective ones.

Yet, there exist some particular special cases. For instance, in Kr^{74} the chaotic regime of surface oscillations was found to occur at $E \cong 2E_C$ rather than at $E \cong E_C$, with the energy $E \cong 2E_C$ corresponding to a saddle-point value of the potential energy. This conclusion is supported by characteristic patterns on the corresponding Poincaré cross-sections. In such cases, of prime importance become the independent evaluations of the critical energy and moment of the onset of chaos, achieved by different techniques.

The chaotic motion of surface oscillations was also reported for

$Kr^{76,78,80}$. In such nuclei this observation was attributed to the existence of a fairly large domain of the negative Gaussian curvature and great magnitudes of the curvature $K(x, y)$ therein. In $Kr^{74,76}$ nuclei, the effect of restoration of the regular motion was observed at the energies greatly exceeding the saddle-point value and later this conclusion was substantiated in studies of the Poincaré cross-sections. Thus, order-chaos-order transitions in the vibrational motion of nuclear surface were noticed to occur in a number of cases. It seems likely that the effect of restoration of the regular motion after a chaotic stage is not uncommon for the dynamical systems with a localized domain of unstable motion. In the isotopes of $Kr^{78,80}$ however, at high energies the oscillatory motion of the nuclear surface does not become regular again, i.e. here it still remains chaotic; this fact may be attributed to a peculiar geometry of the potential energy surface.

Quantum evidence of chaos in nuclei. Before preceding to the particular case of *quantum chaos emerging on a nuclear surface*, we briefly touch upon the distribution function $P(s)$ of the spacings between neighboring energy levels in a nucleus; this function specifies the probability of the neighboring levels with the same spin J and parity Π to be separated by an energy interval s (Dyson, 1962). If a dynamical system is integrable, the distribution function has the Poisson form

$$P_p(s) = \frac{1}{D} e^{-\frac{s}{D}}, \quad \int_0^\infty ds P_p(s) = 1, \quad (5.324)$$

where D is an average distance between the levels. The *Poisson distribution* (5.324) can be derived, for example, from the nuclear Fermi-gas model without the configurational mixing when the eigenvalues of energy are distributed stochastically.

Another explicit form of the distribution function $P(s)$ follows from the so-called *model of random matrices*, applicable to nonintegrable systems. In this model, one usually diagonalizes the energy matrix of a great order, supposing that matrix elements of the Hamiltonian are distributed at random. The random matrix model is consistent with the limiting case of a complete mixing of all degrees of freedom, available inside the corresponding energy interval. The energy levels, calculated as the eigenvalues of a random matrix, are governed by the distribution

$$P_w(s) = \frac{\pi s}{2D^2} \exp\left(-\frac{\pi s^2}{4D^2}\right), \quad \int_0^\infty ds P_w(s) = 1, \quad (5.325)$$

known as the Wigner function and noted for the ultimately strong mixing of configurations.

As $s \rightarrow 0$, the eigenvalues of a random matrix behave in a peculiar way: two neighboring levels push apart from each other, i.e. $P_w(0) = 0$ in contrast to $P_p(0) = 1$. In other words, the *Wigner distribution* yields zero probability for two neighboring levels to coincide, whereas for the Poisson distribution the same probability is the greatest. In the simplest case of the Hamilton matrix of the second order H_{ij} ($i, j=1,2$) whose eigenvalues $E_{1,2}$ are related to the matrix elements H_{ij} by the well-known formula

$$E_{1,2} = \frac{1}{2} \left(H_{11} + H_{22} \right) \pm \frac{1}{2} \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2}, \quad (5.326)$$

these two levels may coincide ($E_1 = E_2$) if two requirements are simultaneously fulfilled: $H_{11} = H_{22}$ and $H_{12} = 0$, this may happen only by chance as it is suggested by the random matrix model.

Naturally, only an experiment can reliably answer the question which of the two distributions (5.324) or (5.325) applies to reality. Yet, some progress in this respect has been already achieved in theoretical studies. Numerical simulations showed that the energy levels of the system where chaotic motion was supposed to be present drifted away from each other in the classical limit. Some calculations are also indicative of the fact that the following general statement is likely to be true: the distribution function of inter-level energy spacings should be of the Poisson form for systems with the regular classical motion, and it should be given by the Wigner distribution provided the system moves chaotically. Thus, it can be seen that any examination of the quantum properties related, for instance, to the regular or chaotic patterns of motion must start from investigations of the same system within the classical limit, just as it has been attempted in this Section.

In studying *statistical properties of the energy spectrum* of a certain system, it is extremely helpful to account for the symmetry of the Hamiltonian which determines, in particular, the degeneracy of energy levels; note also that a nonintegrable perturbation generally removes the degeneracy, inherent in an integrable part of the Hamiltonian.

To exemplify possible quantum evidence of the classical chaos governed mainly, as shown above, by the geometry of the potential energy surface, consider *quantum surface oscillations in a spherical nucleus*. We restrict the discussion to quadrupole oscillations and truncate the expansion of the potential energy at the terms of the fourth power of the variables x and y ; then, the quantum Hamiltonian of surface oscillations of the nucleus can be written as

$$\hat{H} = \frac{1}{2} \left(\hat{P}_x^2 + \hat{P}_y^2 \right) + V(x, y),$$

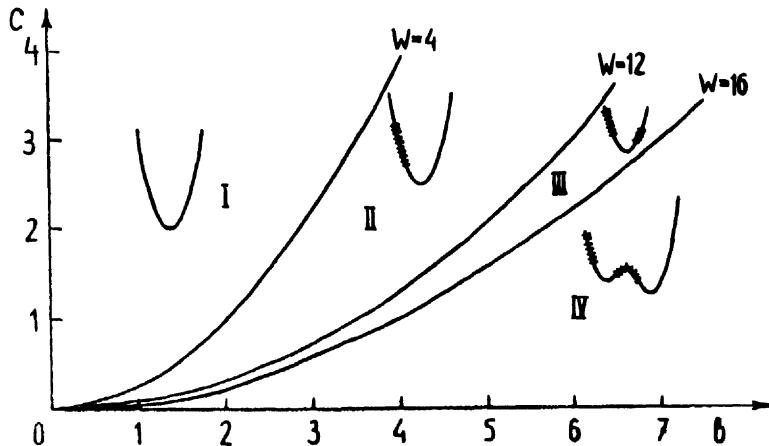


Figure 5.20. Potential energy $V(x = 0, 9)$ profiles for four domains of the space b and c separated by curves $w \equiv \frac{b^2}{c^2} = \text{const}$ with shaded domains of the Gauss negative curvature

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + b\left(x^2 y - \frac{1}{3}y^3\right) + c(x^2 + y^2)^2, \quad (5.327)$$

where $\hat{P}_x = -i\frac{\partial}{\partial x}$, $\hat{P}_y = -i\frac{\partial}{\partial y}$. As above, the parameter a in Eq. (5.322) may be put equal to unity without loss in generality. The parameters b and c , that underlie the occurrence of chaos, specify the degree of nonlinearity of the related equations of motion.

Fig. 5.20 presents, on the b - and c -parameter plane, the shapes of the potential energy $V(x = 0, y)$ of Hamiltonian (5.327) for four regions of the parameter space, separated by the curves $w = \frac{b^2}{c^2} = \text{const}$, with the shaded sections of negative Gaussian curvature where chaos may occur. In three regions I, II and III where $0 < w < 16$, the potential energy exhibits a single extremum, i.e. a minimum at the origin; at the same time in region IV there are additional minima arranged at other points. The Gaussian curvature is positive everywhere inside region I and, hence, the corresponding motion should be regular for all energies. The Poincaré cross-sections, typical for region III where $12 < w < 16$, are exemplified in Fig. 5.21 for three values of energy $E = 0.2E_C$, $E = 0.8E_C$ and $E = 1.2E_C$; they illustrate the order-to-chaos transition for Hamiltonian (5.237). For $E < E_C$, the Poincaré cross-sections are seen to constitute a totality of ordered pointwise curves, peculiar to the regular motion, and at $E = 1.2E_C$ the Poincaré cross-section is presented by the chaotic set of points on the plane $x = 0$, which is innate to chaotic motion (oscillations) of the nuclear shape.

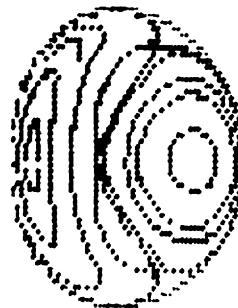
A numerical solution of the Schrödinger equation



3



2



1

Figure 5.21. The Poincaré cross-sections for three values of the system energy E : 1) $0.2E_c$, 2) $0.8E_c$, 3) $1.2E_c$

$$\hat{H}\Psi(x, y) = E\Psi(x, y) \quad (5.328)$$

with Hamiltonian (5.237) was calculated through its diagonalization on the basis of the wave functions of the two-dimensional harmonic oscillator $\varphi_\alpha(x, y)$, corresponding to Hamiltonian (5.237) with $b = c = 0$; the wave functions of nuclear shape oscillations were expanded, respectively, in terms of the basis ones

$$\Psi(x, y) = \sum_{\alpha} C_{\alpha} \varphi_{\alpha}(x, y). \quad (5.329)$$

In that way the eigenvalues of energy E and expansion coefficients C_{α} were found, and the whole set of the latter gave the wave function in the representation of the eigenfunctions of the 2D harmonic oscillator for various values of the parameters b and c of Hamiltonian (5.237) in different regions of both the regular and chaotic classical motion. Then, hundreds of the states of each pattern of motion were analyzed with the aid of computer to an accuracy of $\cong 1\%$.

These preliminary calculations were employed to construct *histograms of the distribution of inter-level spacings* for the Hamiltonian of quadrupole oscillations of the nuclear surface; afterwards, in line with the general assumption, the histograms were fitted with the Poisson distribution for $E < E_C$ (when the motion was regular) and with Wigner distribution for $E > E_C$ (when the motion was chaotic).

Fig. 5.22 presents, in the form of a histogram, the obtained distribution of the inter-level spacings for the chaotic motion region, which was calculated for the potential energy of the nuclear surface oscillations at $w = 13$ (region III in Fig. 5.20) and $E_C < E < E'_C$; for comparison, the Poisson $P_p(s)$ and Wigner $P_w(s)$ distributions are shown in the same figure. It is clearly seen that in the chaotic motion region $P(s)$ is in good agreement with the Wigner distribution. For $E < E_C$ when the motion becomes regular, the calculations conform with the Poisson distribution. The corresponding Poincaré cross-sections were also found and employed to verify the results obtained: the Poisson distributions were consistent with ordered patterns in the Poincaré cross-sections ($E < E_C$) and the Wigner ones correlated with chaotic collections of points in the cross-sections ($E_C < E < E'_C$). With E growing further ($E \geq E'_C$), initially chaotic states of these points were showing some indications of their ordering. Thus, the computer simulations have rendered good grounds to assert that, when the energy of the quantum system increases and the regular motion transforms into the chaotic behaviour, the Poisson distribution changes into its Wigner counterpart and, with the energy increasing further up to the conversion of chaos into a regularity, the Wigner distribution is

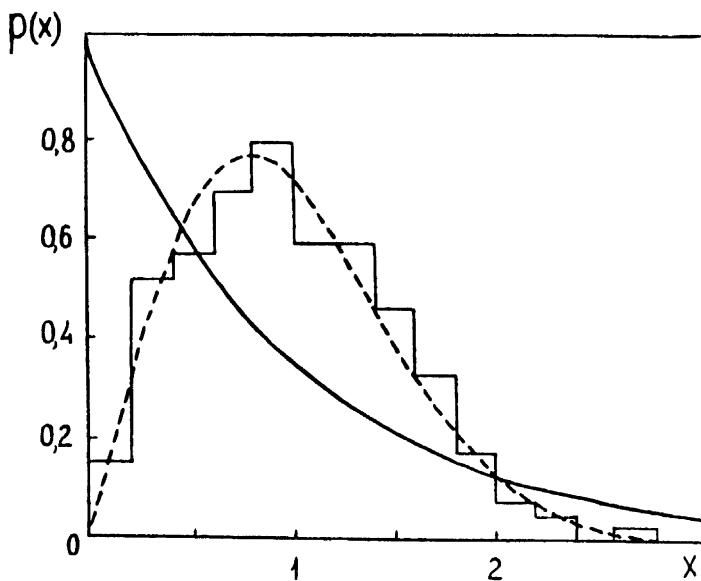


Figure 5.22. A distribution histogram of the distance between neighboring system levels for the chaotic motion (quadrupole nucleus vibrations) and the Poisson (solid line) and Wigner (dotted line) distributions

reconstructed again into the Poisson one. This finding is likely to be general no matter which quantum system namely we study.

At the same time, a peculiar dependence of the wave functions $\Psi(x, y)$ or C_α on the type of the corresponding classical motion was established. It turned out that with the advent of the classical chaos the number of the expansion coefficients C_α in (5.329), which were almost of the same, yet small, value, grew markedly. This fact can be regarded as an indicative criterion of the onset of quantum chaos. If we introduce the function $Q(C_\alpha)$ describing the distribution of the coefficients C_α over their magnitudes, then, this function must show a sharp maximum nearby $C_\alpha = 0$ in the case of the classical regular motion. It is observed for both the energies $E < E_C$ and $E > E'_C$. In the classical chaotic region ($E_C < E < E'_C$) almost all coefficients C_α shall be given by random, almost Gaussian-distributed numbers. In this case, as it was confirmed by the calculations, the distribution $Q(C_\alpha)$ exhibits much broader maximum than that found for the classical regularity. Such a gradually changing distribution $Q(C_\alpha)$ is an inherent feature of quantum chaos. This correlation between statistical properties of the quantum characteristics, such as the energy spectra or wave functions, of stationary states in a quantum system, on the one hand, and nature of the corresponding classical motion was found not only for the Hamiltonian of

nuclear surface quadrupole oscillations but also for other Hamiltonians, different from (5.237). As regards Hamiltonian (5.237) itself, it can be readily applied to many other quantum systems.

Investigations of Hamiltonian (5.237) have revealed the effect of multiple intersections of the energy levels under changes of the parameter b when system's energy becomes equal to the classical critical value E_C of the order-chaos transition. On the other hand, for $E < E_C$, when the motion is regular, such intersections are absent even under a considerable nonlinearity. Apparently, this observation is another universal quantum evidence of chaos in classically nonintegrable Hamilton systems.

One more general effect, inherent in quantum systems, has been found, namely that of the *destruction of the shell structure* (or quantum numbers) *in the course of an order-to-chaos transition*. It manifests itself in the variations of quantum numbers (analogues of the classical integrals of motion), which accompany the switch-on of the nonintegrable perturbation resulting itself in destruction of the shell structure of the energy spectrum. Observed under the perturbation switch-on, correlation between the changes of the phase space, governed by the above-mentioned Kolmogorov-Arnold-Moser theorem, and corresponding destruction of the shell structure accompanying the order-to-chaos transition can be regarded as a quantum generalization (analogy) of this theorem. The destruction of the shell structure, occurring when $E \rightarrow E_C$, is found to be attended by violations of a quasi-periodic (at $E < E_C$) dependence of system's entropy on its energy E and by a consequent monotonic entropy growth into a plateau of the limiting values, where the energy of levels already becomes much greater than the critical one.

Inasmuch as the main statements above are of a general character, starting from these, we can suggest several assumptions about motion in a multinucleonic atomic nucleus which is quite a complicated dynamical system. For example, one may try to relate the shell structure, arising in the ground and weakly excited states of nuclei, to the assertion that the nuclear system becomes similar to the integrable one due to arising self-consistent field. Then, a residual nucleon-nucleon interaction could be treated as a nonintegrable addition (perturbation) to the self-consistent field and the reason of the shell destruction in a number of nuclei. The existence of shells in many real nuclei at sufficiently strong residual interaction could be interpreted either as the presence of undestroyed nonresonant tori in the phase space, which help the regular motion to survive, or as some other unknown mechanism (apart from the Pauli principle), which restores the regular motion of nucleons in nuclei in the region of well-developed chaos. This new stand point may be helpful in explaining the applicability of drastically different models to the description of nuclei, such as the one-particle models, more

closely related to the regular motion (they apply at the excitation energies $E < E_C$), or collective models, mainly characteristic of the chaotic motion (they apply at $E > E_C$ when the internal structure of nuclei changes considerably).

In conclusion it should be noted that the *order-chaos-order transitions* can take place in pretty simple quantum systems, as, for example, the atom of hydrogen. The procedures we have discussed above apply to such systems as well. In the weak-magnetic-field limit, when the Coulomb forces prevail, the problem amounts to the integrable Kepler case and the motion should be regular. For very strong magnetic fields, when one may neglect the Coulomb forces, the problem is also integrable and one deals again with the regular motion. Yet, in the intermediate case, when the intensities of the Coulomb and magnetic forces are of the same order of magnitude, electron begins to move chaotically. Thus, as it follows from the whole foregoing discussion, the chaotic type of motion and order-chaos-order transitions can be due to the dynamics of the process itself rather than to the complexity of the given system, and we are inclined to believe that under appropriate conditions such connection is usual thing rather than an exception.

Problems

5.1. Find the transformation law for a spinor field on rotations of the coordinate frame.

Let $\chi_{s\mu}$ be an eigenfunction of the spin angular momentum and of its projection along a specified direction characterized by the unit vector \mathbf{N}

$$\left. \begin{aligned} s^2 \chi_{s\mu} &= s(s+1) \chi_{s\mu}; \\ (\mathbf{N} \cdot \mathbf{s}) \chi_{s\mu} &= \mu \chi_{s\mu}. \end{aligned} \right\} \quad (5.330)$$

The function $\chi_{s\mu}$ depends on the spatial coordinates \mathbf{r} of the particle and on the spin variable:

$$\chi_{s\mu}(\mathbf{r}, \sigma).$$

We shall consider a rotation of the coordinate frame though a certain angle φ about an axis whose direction is specified by the unit vector \mathbf{n} . As a result of the rotation, the components of any vector in the rotated coordinate frame K' and in the original coordinate frame K are connected by the linear relation

$$\mathbf{r}' = a\mathbf{r}. \quad (5.331)$$

If a particle is situated at the coordinate origin, its coordinate remain unchanged on rotation of the coordinate frame. The change of the function $\chi_{s\mu}$ in this case will be determined entirely by the dependence on the spin variable σ .

To determine the transformation law for the function $\chi_{s\mu}$ on rotations of the coordinate frame, we shall start from the requirement that the equations of the form (5.330) be invariant in different coordinate frames. In the coordinate frame K' , in place of the second equation (5.330) we have

$$(\mathbf{N}' \cdot \mathbf{s}) \chi'_{s\mu} = \mu \chi'_{s\mu}. \quad (5.332)$$

(The form of the spin matrices \mathbf{s} is the same in all coordinate frames.) The functions $\chi'_{s\mu}$ and $\chi_{s\mu}$ must be related by a certain unitary transformation

$$\chi'_{s\mu} = D_{(s)} \chi_{s\mu}, \quad (5.333)$$

whose form is easily found from the requirement that (5.332) and the second equation (5.330) be equivalent. Substituting (5.333) into (5.332), we find

$$a_{ij} N_j D_{(s)}^{-1} s_i D_{(s)} \chi_{s\mu} = \mu \chi_{s\mu}.$$

This equation is equivalent to the second equation (5.330) if

$$a_{ij} D_{(s)}^{-1} s_i D_{(s)} = s_j$$

or

$$D_{(s)}^{-1} s_i D_{(s)} = a_{ij} s_j, \quad (5.334)$$

since the matrix a_{ij} is orthogonal.

In the case of a rotation through an infinitesimal angle $\delta\varphi$ about an axis specified by the unit vector \mathbf{n} , the matrix a_{ij} has the form

$$a_{ij} = \delta_{ij} + \delta\varphi \varepsilon_{ijk} n_k, \quad (5.335)$$

where ε_{ijk} is the completely anti-symmetric unit tensor. Since $D_{(s)}(0) = 1$, the unitary operator $D_{(s)}(\delta\varphi)$ can be sought in the form

$$D_{(s)} = 1 + \delta\varphi \mathbf{n} \cdot \mathbf{A},$$

where \mathbf{A} is an operator to be determined.

We shall consider the case of spin $s = \frac{1}{2}$. The spin operator \mathbf{s} in this case is expressed in terms of the Pauli matrices

$$\mathbf{s} = \frac{1}{2} \boldsymbol{\sigma},$$

and the function $\chi_{s\mu}$ is a two-component spinor. On an infinitesimal rotation, eqn. (5.334) takes the form

$$(1 - \delta\varphi \mathbf{n} \cdot \mathbf{A}) \sigma_i (1 + \delta\varphi \mathbf{n} \cdot \mathbf{A}) = (\delta_{ij} + \delta\varphi \varepsilon_{ijk} n_k) \sigma_j,$$

or

$$\sigma_i A_j - A_j \sigma_i = -\varepsilon_{ijk} \sigma_k. \quad (5.336)$$

Using the commutation relations for the Pauli matrices, it is easily seen that the solution of eqn. (5.336) is $\mathbf{A} = (i/2)\boldsymbol{\sigma}$. Thus, the operator of a rotation through an infinitesimal angle has the form

$$D_{(s)}^{\mathbf{n}}(\delta\varphi) = 1 + \frac{i}{2} \delta\varphi \mathbf{n} \cdot \boldsymbol{\sigma}. \quad (5.337)$$

The operator of a rotation through a finite angle φ about an axis whose direction is specified by the vector \mathbf{n} can be represented as an infinite sequence of rotations through infinitesimal angles:

$$D_{(s)}^{\mathbf{n}}(\varphi) = \lim_{k \rightarrow \infty} \left(1 + \frac{i}{2} \frac{\varphi}{k} \mathbf{n} \cdot \boldsymbol{\sigma}\right)^k = e^{(i/2)\varphi \mathbf{n} \cdot \boldsymbol{\sigma}}. \quad (5.338)$$

If we now take into account the dependence of the spinor components on the spatial coordinates, then

$$\chi'_{s\mu}(\mathbf{r}', \sigma) = D_{(s)} \chi_{s\mu}(\mathbf{r}, \sigma),$$

or

$$\chi'_{s\mu}(\mathbf{r}', \sigma) = D_{(s)} \chi_{s\mu}(a^{-1}\mathbf{r}', \sigma). \quad (5.339)$$

Then, defining the unitary operator D by means of the equality

$$\chi'_{s\mu}(\mathbf{r}, \sigma) = D \chi_{s\mu}(\mathbf{r}, \sigma)$$

and using (5.339), we have

$$D \chi_{s\mu}(\mathbf{r}, \sigma) = D_{(s)} \chi_{s\mu}(a^{-1}\mathbf{r}, \sigma).$$

Hence, we finally obtain the following expression:

$$D^{\mathbf{n}}(\varphi) = e^{i\varphi \mathbf{n} \cdot [l + (1/2)\boldsymbol{\sigma}]}. \quad (5.340)$$

The operator $D^{\mathbf{n}}(\varphi)$ defines the transformation law for a two-component spinor on rotations of the coordinate frame.

5.2. Find the transformation law for a vector field on rotations of the coordinate frame.

In a rotation of the coordinate frame, defined by the linear transformation

$$\mathbf{r}' = a\mathbf{r}, \quad (5.341)$$

the vector field $\mathbf{A}(\mathbf{r})$ transforms according to the law

$$\mathbf{A}'(\mathbf{r}') = a\mathbf{A}(\mathbf{r}),$$

or

$$\mathbf{A}'_i(r'_k) = a_{ij} \mathbf{A}_j(a_{kl}^{-1} r_l'). \quad (5.342)$$

We define the finite-rotation operator D by means of the equality

$$\mathbf{A}'(\mathbf{r}) = D \mathbf{A}(\mathbf{r}). \quad (5.343)$$

Using (5.342), we obtain the following equation for D :

$$D \mathbf{A}_i(r_k) = a_{ij} \mathbf{A}_j(a_{kl}^{-1} r_l). \quad (5.344)$$

In the case of a rotation through an infinitesimal angle $\delta\varphi$ about the axis \mathbf{n} , the matrix a_{ij} is defined by the expression (5.335). Then, from (5.344) we find

$$D^{\mathbf{n}}(\delta\varphi) \mathbf{A}_i(r_k) = \left(\delta_{ij} + \delta\varphi \varepsilon_{ijm} n_m \right) \left(1 + \delta\varphi \varepsilon_{knl} n_n r_l \frac{\partial}{\partial r_k} \right) \mathbf{A}_j(r_k). \quad (5.345)$$

We now define three matrices s_i ($i = 1, 2, 3$) by means of the relation

$$(s_i)_{jk} = -i\varepsilon_{ijk} \quad (5.346)$$

and introduce the orbit angular momentum operator

$$l_i = -i\varepsilon_{ijk} r_j \frac{\partial}{\partial r_k}. \quad (5.347)$$

Keeping the terms linear in $\delta\varphi$ in (5.345), we find

$$D^{\mathbf{n}}(\delta\varphi)\mathbf{A}(\mathbf{r}) = [1 + i\mathbf{n} \cdot (\mathbf{l} + \mathbf{s})]\mathbf{A}(\mathbf{r}). \quad (5.348)$$

In the case of a rotation through a finite angle, we have

$$D^{\mathbf{n}}(\varphi) = e^{i\varphi \mathbf{n} \cdot (\mathbf{l} + \mathbf{s})}. \quad (5.349)$$

It is easily verified that the vector quantity (5.346) satisfies the commutation relations

$$[s_i, s_j] = i\varepsilon_{ijk} s_k, \quad (5.350)$$

and the eigenvalue of the square of the quantity s is equal to

$$s^2 = s(s+1) = 2.$$

Thus, we may associate a spin equal to unity ($s = 1$) with the vector field $\mathbf{A}(\mathbf{r})$. We note that the operator \mathbf{l} occurring in (5.349) characterizes the change of the vector-field function due to the coordinate dependence of the latter, while the operator \mathbf{s} characterizes the change of the function due to the vector character of the field.

If we consider a vector field whose value at each point is proportional to the position vector of the point,

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}\mathbf{r}$$

(\mathbf{A} is a constant scalar quantity), then the change of the field function on rotations of the coordinate frame will be equal to zero. This can be seen easily by verifying the relation

$$[\mathbf{n} \cdot (\mathbf{l} + \mathbf{s})]\mathbf{r} = 0, \quad (5.351)$$

where \mathbf{l} and \mathbf{s} are defined by the equalities (5.346) and (5.347).

5.3. Show that the projections of the operator \mathbf{R} along the axis Oz in the stationary coordinate frame and along the axis Oz' in the moving coordinate frame commute:

$$[R_{z'}, R_z] = 0. \quad (5.352)$$

According (5.77) and (5.78), the projections R_z and $R_{z'}$ can be expressed directly in terms of derivatives with respect to the Euler angles Φ and ψ :

$$R_z = -i \frac{\partial}{\partial \Phi}; \quad R_{z'} = -i \frac{\partial}{\partial \psi}.$$

Using (5.77), we can express the operator $\mathbf{R}_{z'}$ in terms of the projections of the operator \mathbf{R} along the axes of the stationary coordinate frame:

$$R_{z'} = (\cos \Phi \sin \Theta) R_x + (\sin \Phi \sin \Theta) R_y + (\cos \Theta) R_z.$$

Noting that

$$\begin{aligned} [(\cos \Phi \sin \Theta) R_x, R_z] &= -i\{(\sin \Phi \sin \Theta) R_x + (\cos \Phi \sin \Theta) R_y\}; \\ [(\sin \Phi \sin \Theta) R_y, R_z] &= i\{(\sin \Phi \sin \Theta) R_x + (\cos \Phi \sin \Theta) R_y\}; \\ [(\cos \Theta) R_z, R_z] &= 0, \end{aligned}$$

we convince ourselves of the validity of the relation (5.352).

5.4. Prove the operator relation

$$\begin{aligned} e^{-\tau \hat{b}} \hat{a} e^{\tau \hat{b}} &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \overbrace{[\hat{b}, [\hat{b}, \dots [\hat{b}, \hat{a}] \dots]]}^n \\ &\equiv \hat{a} - \tau [\hat{b}, \hat{a}] + \frac{\tau^2}{2!} [\hat{b}, [\hat{b}, \hat{a}]] - \frac{\tau^3}{3!} [\hat{b}, [\hat{b}, [\hat{b}, \hat{a}]]] + \dots, \end{aligned} \quad (5.353)$$

where \hat{a} and \hat{b} are arbitrary operators.

We represent the operator expression in the left-hand side of (5.353) in the form of a series in powers of the parameter τ :

$$e^{-\tau \hat{b}} \hat{a} e^{\tau \hat{b}} = \sum_{n=0}^{\infty} \tau^n f_n(\hat{b}, \hat{a}), \quad (5.354)$$

where $f_n(\hat{b}, \hat{a})$ is some function of the operators \hat{b} and \hat{a} . Differentiating the equality (5.354) with respect to τ , we find

$$-e^{-\tau \hat{b}} [\hat{b}, \hat{a}] e^{\tau \hat{b}} = \sum_{n=1}^{\infty} n \tau^{n-1} f_n(\hat{b}, \hat{a}).$$

Successively differentiating the equality (5.354) k times with respect to τ , we have

$$(-1)^k e^{-\tau \hat{b}} \overbrace{[\hat{b}, [\hat{b}, \dots [\hat{b}, \hat{a}] \dots]]}^k e^{\tau \hat{b}} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \tau^{n-k} f_n(\hat{b}, \hat{a}).$$

Putting $\tau = 0$ in this equality, we find

$$(-1)^k \overbrace{[\hat{b}, [\hat{b}, \dots [\hat{b}, \hat{a}] \dots]]}^k = k! f_k(\hat{b}, \hat{a}).$$

Substituting $f_n(\hat{b}, \hat{a})$ thus found into (5.354), we obtain the relation (5.353).

5.5. Prove the operator relation

$$e^{-i\Phi j_i} j_k e^{i\Phi j_i} = \cos \Phi j_k + \sin \Phi \epsilon_{ikl} j_l \quad (i \neq k), \quad (5.355)$$

where j_i are the components of the angular-momentum operator, satisfying the commutation relations $[j_i, j_j] = i\epsilon_{ijk} j_k$.

Using the relation (5.353) and separating the even and odd powers of the parameter Φ in the right-hand side, we find

$$\begin{aligned} e^{-i\Phi j_i} j_k e^{i\Phi j_i} &= j_k + \sum_{n=1}^{\infty} \frac{(-i\Phi)^{2n}}{(2n)!} \overbrace{[j_i, [j_i, \dots [j_i, j_k] \dots]]}^{2n} \\ &\quad + \sum_{n=0}^{\infty} \frac{(-i\Phi)^{2n+1}}{(2n+1)!} \overbrace{[j_i, [j_i, \dots [j_i, j_k] \dots]]}^{2n+1}. \end{aligned} \quad (5.356)$$

Since $[j_i, [j_i, [j_i, j_k]]] = [j_i, j_k]$, we have

$$\begin{aligned} \overbrace{[j_i, [j_i, \dots [j_i, j_k] \dots]]}^{2n} &= j_k, \quad i \neq k, \\ \overbrace{[j_i, [j_i, \dots [j_i, j_k] \dots]]}^{2n+1} &= [j_i, j_k]. \end{aligned}$$

Substituting these equalities into (5.356), we obtain the relation (5.355).

5.6. Prove the operator relation

$$e^{-i\Phi j_i} e^{i\Theta j_k} e^{i\Phi j_i} = e^{i\Theta \{(\cos \Phi) j_k + (\sin \Phi) \epsilon_{ikl} j_l\}}, \quad (5.357)$$

where j_i are the components of the angular-momentum operator.

Using the relation (5.355), we have

$$e^{-i\Phi j_i} (j_k)^n e^{i\Phi j_i} = \{(\cos \Phi) j_k + (\sin \Phi) \epsilon_{ikl} j_l\}^n.$$

Multiplying this equality by $(i\Theta)^n / n!$ and summing the left-hand and right-hand sides over n from 0 to ∞ , we obtain the relation (5.357).

5.7. Show that

$$D_{0m}^l(\Phi, \Theta, \psi) = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\Theta, \Phi). \quad (5.358)$$

We shall make use of the relation (5.60), which gives the transformation law for spherical harmonics on rotations of the coordinate frame:

$$Y_{lm}(\vartheta, \varphi) = \sum_{m'} D_{m'm}^l(\Phi, \Theta, \psi) Y_{lm'}(\vartheta', \varphi'). \quad (5.359)$$

We choose $\vartheta = 0$, that is, we consider the direction along the axis Oz' in the rotated coordinate frame. Then,

$$Y_{lm'}(0, \varphi') = \sqrt{\frac{2l+1}{4\pi}} \delta_{m'0}.$$

In the stationary coordinate frame, the direction of the axis Oz' is characterized by the angles $\vartheta = 0$ and $\varphi = \Phi$; therefore,

$$Y_l(\Theta, \Phi) = \sqrt{\frac{2l+1}{4\pi}} D_{0m}^l(\Phi, \Theta, \psi),$$

whence the relation (5.358) follows.

5.8. Obtain the addition formula for spherical harmonics by making use of the transformations law for spherical harmonics on rotations of the coordinate frame.

We shall choose an arbitrary point P on a unit sphere, with the polar coordinates ϑ and φ in the original coordinate frame and the coordinates ϑ' and φ' in the rotated coordinate frame. The polar coordinates of the axis Oz' in the original coordinate frame are Θ and Φ . By putting m equal to zero in the inverse formula to (5.359),

$$Y_{lm}(\vartheta', \varphi') = \sum_{m'} D_{mm'}^{l*}(\Phi, \Theta, \psi) Y_{lm'}(\vartheta, \varphi),$$

and substituting $D_{0m'}^{l*}(\Phi, \Theta, \psi)$ in the form (5.358), we obtain the addition formula for spherical harmonics:

$$\sqrt{\frac{2l+1}{4\pi}} Y_{l0}(\vartheta', \varphi') \equiv \frac{2l+1}{4\pi} P_l(\cos \vartheta') = \sum_m Y_{lm}^*(\Theta, \Phi) Y_{lm}(\vartheta, \varphi). \quad (5.360)$$

5.9. Find the coupling rules for D -functions on addition of angular momenta.

Let $\psi_{j_1 m_1}(\mathbf{r}_1)$ and $\psi_{j_2 m_2}(\mathbf{r}_2)$ be the wave function for two systems with well-defined values of the square and z -component of the angular momentum. On rotations of the coordinate frame, these functions transform according to the law

$$\left. \begin{aligned} \psi_{j_1 m_1}(\mathbf{r}_1) &= \sum_{m'_1} D_{m'_1 m_1}^{j_1}(\Phi, \Theta, \psi) \psi_{j_1 m'_1}(\mathbf{r}'_1); \\ \psi_{j_2 m_2}(\mathbf{r}_2) &= \sum_{m'_2} D_{m'_2 m_2}^{j_2}(\Phi, \Theta, \psi) \psi_{j_2 m'_2}(\mathbf{r}'_2). \end{aligned} \right\} \quad (5.361)$$

According to the rules for addition of angular momenta, we have

$$\psi_{j_1 m_1}(\mathbf{r}_1) \psi_{j_2 m_2}(\mathbf{r}_2) = \sum_{jm} (j_1 m_1 j_2 m_2 | jm) \psi_{jm}(\mathbf{r}_1, \mathbf{r}_2), \quad (5.362)$$

where $\psi_{jm}(\mathbf{r}_1, \mathbf{r}_2)$ is the wave function of the whole system in a state characterized by well defined values of the square and z -component of the angular momentum. On rotations of the coordinate frame, the functions $\psi_{jm}(\mathbf{r}_1, \mathbf{r}_2)$ transform analogously to (5.361):

$$\psi_{jm}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{m'} D_{m'm}^j(\Phi, \Theta, \psi) \psi_{jm}(\mathbf{r}'_1, \mathbf{r}'_2). \quad (5.363)$$

The addition rule (5.362) for angular momenta is also valid in the rotated coordinate frame; therefore,

$$\psi_{jm'}(\mathbf{r}'_1, \mathbf{r}'_2) = \sum_{m'_1 m'_2} (j_1 m'_1 j_2 m'_2 | jm') \psi_{j_1 m'_1}(\mathbf{r}'_1) \psi_{j_2 m'_2}(\mathbf{r}'_2), \quad (5.364)$$

and, consequently,

$$\begin{aligned} & \psi_{j_1 m_1}(\mathbf{r}_1) \psi_{j_2 m_2}(\mathbf{r}_2) \\ = & \sum_{\substack{j m m' \\ m'_1 m'_2}} (j_1 m_1 j_2 m_2 | jm) (j_1 m'_1 j_2 m'_2 | jm') D_{m'm}^j(\Phi, \Theta, \psi) \psi_{j_1 m'_1}(\mathbf{r}'_1) \psi_{j_2 m'_2}(\mathbf{r}'_2). \end{aligned}$$

Substituting (5.361) into the left-hand side of the equality obtained and taking into account that $\psi_{j_1 m'_1}(\mathbf{r}'_1)$ and $\psi_{j_2 m'_2}(\mathbf{r}'_2)$ are linearly independent, we find

$$\begin{aligned} & D_{m'_1 m_1}^{j_1}(\Phi, \Theta, \psi) D_{m'_2 m_2}^{j_2}(\Phi, \Theta, \psi) \\ = & \sum_{\substack{j m m' \\ m'_1 m'_2}} (j_1 m_1 j_2 m_2 | jm) (j_1 m'_1 j_2 m'_2 | jm') D_{m'm}^j(\Phi, \Theta, \psi). \end{aligned} \quad (5.365)$$

Making use of the orthogonality of the Glebsch-Gordan coefficients, we finally obtain the following coupling formula for the D -functions on addition of angular momenta:

$$\begin{aligned} D_{m'm}^j(\Phi, \Theta, \psi) = & \sum_{\substack{m_1 m'_1 \\ m_2 m'_2}} (j_1 m_1 j_2 m_2 | jm) (j_1 m'_1 j_2 m'_2 | jm') \\ & \times D_{m'_1 m_1}^{j_1}(\Phi, \Theta, \psi) D_{m'_2 m_2}^{j_2}(\Phi, \Theta, \psi). \end{aligned} \quad (5.366)$$

5.10. Calculate the matrix element

$$d_{m'm}^j(\Theta) = \langle jm' | e^{i\Theta j_y} | jm \rangle, \quad (5.367)$$

by making use of the relation between the angular-momentum operator and the creation and destruction operators for two uncoupled harmonic oscillators.

We shall consider two independent harmonic oscillators, which we characterize by the indices 1 and 2. In the occupation-number representation, with each of the oscillators we can associate operators a_i and a_i^\dagger ($i=1$ and 2) destroying and creating excitation quanta and satisfying the following commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \quad (5.368)$$

From these operators we can construct four independent quadratic operators:

$$a_1^\dagger a_1, \quad a_2^\dagger a_2, \quad a_1^\dagger a_2 \text{ and } a_2^\dagger a_1. \quad (5.369)$$

In the occupation-number representation the first two operators are diagonal and determine the numbers of excitation quanta (bosons) of the first or second type. The other two operators, $a_1^\dagger a_2$ and $a_2^\dagger a_1$, have non-zero matrix elements for transitions in which the numbers of excitation quanta change by unity. Therefore, we can associate operators of angular-momentum components with the operators (5.369), if the numbers n_1 and n_2 of excitation quanta are associated with the quantum numbers of the square and projections of the angular momentum.

It can be checked directly that the operators j_x, j_y and j_z defined by the equalities

$$\left. \begin{aligned} j_x + ij_y &= a_1^\dagger a_2; \\ j_x - ij_y &= a_2^\dagger a_1; \\ j_z &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \end{aligned} \right\} \quad (5.370)$$

satisfy the usual commutation relations for angular-momentum components. The relations (5.370) can be written compactly if we define formally the two-component spinor operators

$$a^+ = (a_1^\dagger, a_2^\dagger), \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

and introduce the two-row Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the operator \mathbf{j} of the angular-momentum vector is written in the form

$$\mathbf{j} = \frac{1}{2} a^+ \boldsymbol{\sigma} a. \quad (5.371)$$

If we substitute the unit matrix in place of $\boldsymbol{\sigma}$ in (5.371), we obtain the scalar operator

$$j = \frac{1}{2} a^+ a. \quad (5.372)$$

It is easily verified that

$$\mathbf{j}^2 = j(j+1). \quad (5.373)$$

The quantities \mathbf{j} and j are operators in the occupation-number representation. The eigenvalues of the operators j and j_z , which we denote by j and m , can be expressed in terms of the numbers n_1 and n_2 of excitation quanta:

$$j = \frac{1}{2}(n_1 + n_2), \quad m = \frac{1}{2}(n_1 - n_2). \quad (5.374)$$

Consequently, the quantum numbers n_1 and n_2 completely determine the values of the square and projection (along a specified direction) of the angular momentum.

We denote the wave function of the ground state of a system of two oscillators ($n_1 = n_2 = 0$ – the vacuum state) by $|0\rangle$. Obviously, the values

$$j = m = 0$$

correspond to this state.

By means of the creation operators a_1^+ and a_2^+ , it is not difficult to construct the wave function for a state with well-defined values of the quantum numbers j and m of the square and projection of the angular momentum. This function, normalized to unity, has the form

$$|jm\rangle = \frac{(a_1^+)^{j+m}(a_2^+)^{j-m}}{\sqrt{(j+m)!(j-m)!}} |0\rangle. \quad (5.375)$$

By means of (5.353), one can prove the following relations for the operators a_k^+ and a_k :

$$\left. \begin{aligned} e^{-i\Theta j_y} a_k^+ e^{i\Theta j_y} &= (\cos \frac{\Theta}{2}) a_k^+ - i(\sin \frac{\Theta}{2}) a_j^+ (\sigma_i)_{jk}; \\ e^{-i\Theta j_y} a_k e^{i\Theta j_y} &= (\cos \frac{\Theta}{2}) a_k + i(\sin \frac{\Theta}{2}) (\sigma_i)_{kj} a_j. \end{aligned} \right\} \quad (5.376)$$

We need two particular cases ($i = y, k=1,2$):

$$\left. \begin{aligned} e^{i\Theta j_y} a_1^+ e^{-i\Theta j_y} &= (\cos \frac{\Theta}{2}) a_1^+ - (\sin \frac{\Theta}{2}) a_2^+; \\ e^{i\Theta j_y} a_2^+ e^{-i\Theta j_y} &= (\cos \frac{\Theta}{2}) a_2^+ + (\sin \frac{\Theta}{2}) a_1^+. \end{aligned} \right\} \quad (5.377)$$

We calculate the result of the action of the operator $e^{i\Theta j_y}$ on the function (5.375), using the relation (5.377). Since

$$e^{i\Theta j_y} = |0\rangle,$$

we have

$$\begin{aligned} e^{i\Theta j_y} |jm\rangle &= \frac{1}{\sqrt{(j+m)!(j-m)!}} \\ &\times \left[\left(\cos \frac{\Theta}{2} \right) a_1^+ - \left(\sin \frac{\Theta}{2} \right) a_2^+ \right]^{j+m} \left[\left(\cos \frac{\Theta}{2} \right) a_2^+ + \left(\sin \frac{\Theta}{2} \right) a_1^+ \right]^{j-m} |0\rangle \\ &= \sqrt{(j+m)!(j-m)!} \sum_{n=0}^{j+m} \sum_{k=0}^{j-m} (-1)^{j+m-n} \left(\sin^{2j-k-n} \frac{\Theta}{2} \right) \\ &\times \left(\cos^{k+n} \frac{\Theta}{2} \right) \frac{(a_1^+)^{j-m-k+n}}{n!(j+m-n)!} \times \frac{(a_2^+)^{j+m+k-n}}{k!(j-m-k)!} |0\rangle. \end{aligned}$$

If for the summation indices we make the replacement

$$k = j - m' - \kappa, \quad n = j + m - \kappa, \quad (-j \leq m' \leq j),$$

then

$$\begin{aligned}
 e^{i\Theta j_y} |jm\rangle &= \sum_{m'=-j}^j \sum_{\kappa} (-1)^{\kappa} \\
 &\times \frac{[(j+m)!(j-m)!(j+m')!(j-m')]^{1/2}}{\kappa!(j+m-\kappa)!(j-m'-\kappa)!(\kappa+m'-m)!} \\
 &\times (\sin \frac{\Theta}{2})^{2\kappa-m+m'} (\cos \frac{\Theta}{2})^{2j+m-m'-2\kappa} |jm'\rangle. \quad (5.378)
 \end{aligned}$$

Finally, we obtain the following representation for the function $d_{m'm}^j(\Theta)$ (Wigner, 1959):

$$\begin{aligned}
 d_{m'm}^j(\Theta) &= \sum_{\kappa} (-1)^{\kappa} \frac{[(j+m)!(j-m)!(j+m')!(j-m')]^{1/2}}{\kappa!(j+m-\kappa)!(j-m'+\kappa)!(\kappa+m'-m)!} \\
 &\times (\sin \frac{\Theta}{2})^{2\kappa-m+m'} (\cos \frac{\Theta}{2})^{2j+m-m'-2\kappa}, \quad (5.379)
 \end{aligned}$$

where the sum is taken over all $\kappa \geq 0$.

5.11. Find the explicit form of the functions $d_{m'm}^j(\Theta)$ in the case $j = \frac{1}{2}$.

We make use of the definition

$$d_{m'm}^j(\Theta) = \langle jm' | e^{i\Theta j_y} | jm \rangle \quad (5.380)$$

and note that, in the case $j = \frac{1}{2}$, we have

$$j_y = \frac{1}{2}\sigma_y, \quad (5.381)$$

where σ_y is a Pauli matrix. We expand the exponential in (5.380) in a series in powers of Θ and separate the even and odd terms:

$$d_{m'm}^{1/2}(\Theta) = \left\langle \frac{1}{2}m' \mid \sum_{n=0}^{\infty} \left(\frac{i\Theta}{2} \sigma_y \right)^{2n} \frac{1}{(2n)!} + \sum_{n=0}^{\infty} \left(\frac{i\Theta}{2} \sigma_y \right)^{2n+1} \frac{1}{(2n+1)!} \mid \frac{1}{2}m \right\rangle.$$

Since $\sigma_y^2 = 1$, we have

$$\begin{aligned}
 &d_{m'm}^{1/2}(\Theta) \\
 &= \left\langle \frac{1}{2}m' \mid \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Theta}{2} \right)^{2n} \frac{1}{(2n)!} + i\sigma_y \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Theta}{2} \right)^{2n+1} \frac{1}{(2n+1)!} \mid \frac{1}{2}m \right\rangle,
 \end{aligned}$$

or

$$d_{m'm}^{1/2}(\Theta) = (\cos \frac{\Theta}{2}) \delta_{mm'} + i(\sin \frac{\Theta}{2}) (\sigma_y)_{m'm}. \quad (5.382)$$

In the representation in which the matrix σ_z is diagonal, the Pauli matrix $(\sigma_y)_{m'm}$ has the form

$$(\sigma_y)_{m'm} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, for $d_{m'm}^{1/2}(\Theta)$ we finally obtain

$$d_{m'm}^{\frac{1}{2}}(\theta) = \begin{array}{c|cc} m & \frac{1}{2} & -\frac{1}{2} \\ \hline \frac{1}{2} & \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\frac{1}{2} & -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{array} \quad (5.383)$$

The functions $d_{m'm}^j(\Theta)$ for any j can be obtained from $d_{m'm}^{1/2}(\Theta)$ by successive application of the coupling rules (5.366) for D -functions. Putting $\Phi = \psi = 0$ in (5.366), we obtain

$$d_{m'm}^j(\Theta) = \sum_{\substack{m_1 m'_1 \\ m_2 m'_2}} (j_1 m_1 j_2 m_2 | jm) (j_1 m'_1 j_2 m'_2 | jm') d_{m'_1 m_1}^{j_1}(\Theta) d_{m'_2 m_2}^{j_2}(\Theta). \quad (5.384)$$

This relation makes it possible to find $d_{m'm}^j(\Theta)$ for any j from (5.383).

5.12. Find the explicit form of the functions $d_{m'm}^j(\Theta)$ in the case $j = 1$.

We proceed in the same way as in the preceding case:

$$d_{m'm}^1(\Theta) = < 1m' | 1 + \sum_{n=0}^{\infty} \frac{(i\Theta j_y)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(i\Theta j_y)^{2n+2}}{(2n+2)!} | 1m >.$$

Since, for angular momentum equal to unity,

$$j_y(j_y^2 - 1) = 0, \quad (5.385)$$

we have

$$j_y^{2n+1} = j_y, \quad j_y^{2n+2} = j_y^2.$$

Thus, we obtain

$$d_{m'm}^1(\Theta) = < 1m' | 1 + i(\sin \Theta) j_y + j_y^2 \sum_{n=1}^{\infty} (-1)^n \frac{\Theta^{2n}}{(2n)!} | 1m >,$$

or

$$d_{m'm}^1(\Theta) = < 1m' | 1 + i(\sin \Theta) j_y + (\cos \Theta - 1) j_y^2 | 1m >. \quad (5.386)$$

In the representation in which the matrix j_z is diagonal, the matrix j_y for $j = 1$ has the form

$$j_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad j_y^2 = -\frac{1}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \quad (5.387)$$

Thus, for $d_{m'm}^1(\Theta)$ we obtain

$$d_{m'm}^1(\Theta) = \begin{array}{c|ccc} & m & l & o & -l \\ \hline m' & \begin{matrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{matrix} \\ \hline l & & & & \end{array} \quad (5.388)$$

5.13. Verify that, in the case $j = \frac{3}{2}$, the functions $d_{m'm}^{3/2}(\Theta)$ have the form

$$d_{m'm}^{3/2}(\Theta) = \begin{array}{c|ccccc} & m & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline m' & \begin{matrix} \frac{1}{2}(1+\cos\theta)\cos\frac{\theta}{2} & \frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2} & \frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2} & \frac{1}{2}(1-\cos\theta)\sin\frac{\theta}{2} \\ -\frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2} & \frac{1}{2}(1-3\cos\theta)\cos\frac{\theta}{2} & \frac{1}{2}(1+3\cos\theta)\sin\frac{\theta}{2} & \frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2} \\ \frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2} & -\frac{1}{2}(1+3\cos\theta)\sin\frac{\theta}{2} & -\frac{1}{2}(1-3\cos\theta)\cos\frac{\theta}{2} & \frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2} \\ -\frac{1}{2}(1-\cos\theta)\sin\frac{\theta}{2} & \frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2} & -\frac{\sqrt{3}}{2}(1-\cos\theta)\sin\frac{\theta}{2} & \frac{1}{2}(1+\cos\theta)\cos\frac{\theta}{2} \end{matrix} \\ \hline m & & & & & \end{array} \quad (5.389)$$

5.14. Verify that, in the case $j = 2$, the functions $d_{m'm}^2(\Theta)$ have the form

m'	2	1	0	-1	-2
2	$\frac{1}{4}(1+\cos\theta)^2$	$\frac{1}{2}\sin\theta(1+\cos\theta)$	$\sqrt{\frac{3}{8}}\sin^2\theta$	$\frac{1}{2}\sin\theta(1-\cos\theta)$	$\frac{1}{4}(1-\cos\theta)^2$
1	$-\frac{1}{2}\sin\theta(1+\cos\theta)$	$\frac{1}{2}(1+\cos\theta)(2\cos\theta-1)$	$\sqrt{\frac{3}{2}}\sin\theta\cos\theta$	$\frac{1}{2}(1-\cos\theta)(2\cos\theta+1)$	$\frac{1}{2}\sin\theta(1-\cos\theta)$
0	$\sqrt{\frac{3}{8}}\sin^2\theta$	$-\sqrt{\frac{3}{2}}\sin\theta\cos\theta$	$\frac{1}{2}(3\cos^2\theta-1)$	$\sqrt{\frac{3}{2}}\sin\theta\cos\theta$	$\sqrt{\frac{3}{8}}\sin^2\theta$
-1	$-\frac{1}{2}\sin\theta(1-\cos\theta)$	$\frac{1}{2}(1-\cos\theta)(2\cos\theta+1)$	$-\sqrt{\frac{3}{2}}\sin\theta\cos\theta$	$\frac{1}{2}(1+\cos\theta)(2\cos\theta-1)$	$\frac{1}{2}\sin\theta(1+\cos\theta)$
-2	$\frac{1}{4}(1-\cos\theta)^2$	$-\frac{1}{2}\sin\theta(1-\cos\theta)$	$\sqrt{\frac{3}{8}}\sin^2\theta$	$-\frac{1}{2}\sin\theta(1+\cos\theta)$	$\frac{1}{4}(1+\cos\theta)^2$

(5.390)

5.15. The axes of the stationary coordinate frame can be aligned along the principal axes of inertia of a physical system in twenty-four different ways (if only right-handed coordinate frames are used). In fact, we can choose the orientation of the axis Oz' in six different ways along three mutually perpendicular directions in space, and for each given direction of the axis Oz' we can choose the orientation of the axis Ox' in four different ways (the position of the axis Oy' will then be completely determined in a right-handed coordinate frame).

The only solutions of the Schrödinger equation which have physical meaning are those which are invariant under transformations corresponding to a transition from one orientation of the coordinate frame to another. Transitions from one coordinate frame to another can be described by means of three operators D_1 , D_2 , D_3 and powers of these operators.

The operator D_1 changes the directions of the axes Oz' and Oy' and corresponds to a rotation of the coordinate frame through an angle π about the axis Ox' ($D_1^2 = 1$). The operator D_2 corresponds to a rotation of the coordinate frame through $\pi/2$ about the axis Oz' ($D_2^4 = 1$). The operator D_3 corresponds to a cyclic permutation of the axes ($D_3^3 = 1$).

The operator D_1 can be represented as the sequence of rotations through the Euler angles $\Phi = 0$, $\Theta = \pi$ and $\psi = \pi$. The operator D_2 can be represented as a rotation through the Euler angle $\psi = \pi/2$. The operator D_3 can be represented as the sequence of rotations through the Euler angles $\Phi = 0$, $\Theta = \pi/2$ and $\psi = \pi/2$.

In any transformation D_k , the function $\psi(\mathbf{r})$ goes over into the transformed function

$$\psi(\mathbf{r}') = R_k \psi(\mathbf{r}),$$

where $R_k = D_k^{-1}$. According to the relations (5.58), on rotations of the coordinate frame the transformed function can be expressed in terms of the functions in the stationary coordinate frame:

$$\psi_{jm}(\mathbf{r}') \equiv R_k \psi_{jm}(\mathbf{r}) = \sum_{m'} D_{mm'}^{j*}(\Phi^{(k)}, \Theta^{(k)}, \psi^{(k)}) \psi_{jm'}(\mathbf{r}). \quad (5.391)$$

Now calculate the result of the action of the operators R_1 , R_2 and R_3 on the D -functions:

$$R_1 D_{m'm}^j(\Phi, \Theta, \psi) = \sum_{m''} D_{m'm''}^{j*}(0, \pi, \pi) D_{m''m}^j(\Phi, \Theta, \psi).$$

Since

$$D_{m'm''}^{j*}(0, \pi, \pi) = e^{-im'\pi} d_{m'm''}^j(\pi) = (-1)^{-j} \delta_{-m', m''},$$

we have

$$R_1 D_{m'm}^j(\Phi, \Theta, \psi) = (-1)^{-j} D_{-m', m}^j(\Phi, \Theta, \psi). \quad (5.392)$$

By analogy with (5.391), we have

$$R_2 D_{m'm}^j(\Phi, \Theta, \psi) = e^{-im'(\pi/2)} D_{m'm}^j(\Phi, \Theta, \psi); \quad (5.393)$$

$$R_3 D_{m'm}^j(\Phi, \Theta, \psi) = \sum_{m''} D_{m'm''}^j\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) D_{m''m}^{j*}(\Phi, \Theta, \psi). \quad (5.394)$$

Calculate the result of the action of the operators R_1 , R_2 and R_3 on the variables β and γ . Using the transformation formulae (5.58), we find

$$R_1 \alpha'_{2\mu} = \sum_{\mu'} D_{\mu\mu'}^{2*}(0, \pi, \pi) \alpha'_{2\mu'} = \alpha'_{2,-\mu}.$$

Since

$$\alpha'_{20} = \beta \cos \gamma, \quad \text{and} \quad \alpha'_{2,2} = \alpha'_{2,-2} = \frac{\beta}{\sqrt{2}} \sin \gamma,$$

we have

$$R_1 \{\gamma\} = \{\gamma\}. \quad (5.395)$$

In addition,

$$R_2 \alpha'_{2\mu} = e^{-i\mu(\pi/2)} \alpha'_{2\mu},$$

or

$$R_2 \alpha'_{20} = \alpha'_{20}, \quad R_2 \alpha'_{22} = -\alpha'_{22}.$$

Thus,

$$R_2 \{\gamma\} = \{-\gamma\}. \quad (5.396)$$

Finally,

$$R_3 \alpha'_{2\mu} = e^{-i\mu(\pi/2)} \left\{ d_{\mu 0}^2(\pi/2) \alpha'_{20} + \left(d_{\mu 2}^2(\pi/2) + d_{\mu-2}^2(\pi/2) \right) \alpha'_{22} \right\}.$$

Using the explicit form of the functions $d_{\mu\mu'}^2(\pi/2)$, we find

$$\begin{aligned} R_3 \cos \gamma &= (\sqrt{3}/2) \sin \gamma - \frac{1}{2} \cos \gamma \equiv \cos(\gamma - 2\pi/3); \\ R_3 \sin \gamma &= -\frac{1}{2} \sin \gamma - (\sqrt{3}/2) \cos \gamma \equiv \sin(\gamma - 2\pi/3), \end{aligned}$$

or

$$R_3\{\gamma\} = \{\gamma - 2\pi/3\}. \quad (5.397)$$

5.16. Show that three-phonon quadrupolar excitations are characterized by the values $\Lambda=0,2,3,4$ and 6 of the total angular momentum and have positive parity.

The total angular momentum Λ of a three-phonon excitation is equal to the sum of the angular momenta of the individual excitations:

$$\Lambda = \lambda_1 + \lambda_2 + \lambda_3.$$

Correspondingly, the projection Λ_z of the total angular momentum along the axis Oz in the laboratory coordinate frame is equal to

$$\Lambda_z = \lambda_{1z} + \lambda_{2z} + \lambda_{3z}.$$

A one-phonon quadrupolar excitation is characterized by angular momentum equal to 2 with projection μ ; the corresponding function $b_{2\mu}^+|0\rangle$, where $b_{2\mu}^+$ is a phonon creation operator, satisfies the equations

$$\Lambda^2 b_{2\mu}^+ |0\rangle = 2(2+1)b_{2\mu}^+ |0\rangle, \quad \lambda_z b_{2\mu}^+ |0\rangle = \mu b_{2\mu}^+ |0\rangle.$$

We can choose a complete set of three-phonon functions in the form

$$b_{2\mu_1}^{+(1)} b_{2\mu_2}^{+(2)} b_{2\mu_3}^{+(3)} |0\rangle, \quad (5.398)$$

where μ_1, μ_2 and μ_3 run over all possible values. Clearly, there are 125 different independent functions (5.398). From these we need to construct completely symmetric linear combinations, which must be eigenfunctions of the operators Λ^2 and Λ_z :

$$\begin{aligned} \Lambda^2 \psi_{\Lambda M} &= \Lambda(\Lambda+1)\psi_{\Lambda M}; \\ \Lambda_z \psi_{\Lambda M} &= M\psi_{\Lambda M}. \end{aligned} \quad (5.399)$$

The rules for vector addition of angular momenta permit, generally speaking, the following angular-momentum values Λ : 0, 1, 2, 3, 4, 5 and 6. In a system of three phonons, however, not all these values of the total angular momentum are allowed, since the wave function must be completely symmetric with respect to permutations of the phonons. We shall ascertain how many linearly independent functions satisfying the symmetry requirements can be constructed for each value of M .

If $M=6$ or 5, in each case we can construct only one completely symmetric function:

$$\begin{aligned} M = 6; \quad & b_{22}^{+(1)} b_{22}^{+(2)} b_{22}^{+(3)} |0\rangle, \\ M = 5; \quad & b_{22}^{+(1)} b_{22}^{+(2)} b_{21}^{+(3)} |0\rangle. \end{aligned}$$

If $M = 4$, we can construct two functions:

$$M = 4; \quad b_{22}^{+(1)} b_{22}^{+(2)} b_{20}^{+(3)} |0\rangle, \quad b_{22}^{+(1)} b_{21}^{+(2)} b_{21}^{+(3)} |0\rangle$$

If $M = 3$, we can construct three functions:

$$\begin{aligned} M = 3; \quad & b_{22}^{+(1)} b_{22}^{+(2)} b_{2-1}^{+(3)} |0\rangle, \quad b_{22}^{+(1)} b_{21}^{+(2)} b_{20}^{+(3)} |0\rangle, \\ & b_{21}^{+(1)} b_{21}^{+(2)} b_{21}^{+(3)} |0\rangle. \end{aligned}$$

If $M = 2$, or 1, in each case four functions can be constructed:

$$\begin{aligned} M = 2; \quad & b_{22}^{+(1)} b_{22}^{+(2)} b_{2-2}^{+(3)} |0\rangle, \quad b_{22}^{+(1)} b_{21}^{+(2)} b_{2-1}^{+(3)} |0\rangle; \\ & b_{22}^{+(1)} b_{20}^{+(2)} b_{20}^{+(3)} |0\rangle, \quad b_{21}^{+(1)} b_{21}^{+(2)} b_{20}^{+(3)} |0\rangle; \\ M = 1; \quad & b_{22}^{+(1)} b_{21}^{+(2)} b_{2-2}^{+(3)} |0\rangle, \quad b_{22}^{+(1)} b_{20}^{+(2)} b_{2-1}^{+(3)} |0\rangle, \\ & b_{21}^{+(1)} b_{21}^{+(2)} b_{2-1}^{+(3)} |0\rangle, \quad b_{21}^{+(1)} b_{20}^{+(2)} b_{20}^{+(3)} |0\rangle. \end{aligned}$$

Finally, if $M = 0$, we can construct five functions:

$$\begin{aligned} M = 0; \quad & b_{22}^{+(1)} b_{2-2}^{+(2)} b_{20}^{+(3)} |0\rangle, \quad b_{22}^{+(1)} b_{2-1}^{+(2)} b_{2-1}^{+(3)} |0\rangle, \\ & b_{21}^{+(1)} b_{21}^{+(2)} b_{2-2}^{+(3)} |0\rangle, \quad b_{21}^{+(1)} b_{2-1}^{+(2)} b_{20}^{+(3)} |0\rangle, \\ & b_{20}^{+(1)} b_{20}^{+(2)} b_{20}^{+(3)} |0\rangle. \end{aligned}$$

After symmetrization, each of the products gives a linearly independent function with a well-defined value of M . The fact that no additional linearly independent combination appears on going from $M = 6$ to $M = 5$ and from $M = 2$ to $M = 1$ tells us that the values $\Lambda = 5$ and 1 are forbidden.

The positive parity of each of the states follows from the fact that the functions of these states are constructed from quadrupolar one-phonon functions.

An exactly similar elementary treatment shows that two-phonon quadrupolar excitations can have only three values of the total angular momentum: $\Lambda = 0, 2$ and 4.

5.17. Show that four-phonon quadrupolar excitations have positive parity and are characterized by the following values Λ of the total angular momentum: 0, 2, 4, 5, 6 and 8; the values $\Lambda = 2$ and 4 are encountered twice and to classify these an additional quantum number – the *seniority* – is introduced.

The solution of this problem is analogous to that of the preceding problem.

5.18. Show that three-phonon octupolar ($\lambda = 3$) excitations have negative parity and the following total angular-momentum values $\Lambda: 0, 3, 4, 5, 6, 7$ and 9, the

value $\Lambda = 3$ being encountered twice (for three or more phonons, it becomes necessary to introduce an additional quantum number – the seniority – for a complete classification of the states).

The solution of this problem is analogous to the solutions of the preceding problems.

5.19. Quantize the kinetic energy associated with the rotation and surface deformations of a nucleus, selecting the Euler angles Φ, Θ and ψ and the deformation parameters β and γ as the independent variables.

The classical expression for the kinetic energy associated with the rotation and surface deformations of a nucleus has the form

$$T = \frac{1}{2} \sum_{k'} \omega_{k'}^2 J_{k'} + (B/2)(\dot{\beta}^2 + \beta^2 \dot{\gamma}^2), \quad (5.400)$$

where the components of the angular velocity along the axes of the moving coordinate frame are expressed in terms of derivatives of the Euler angles:

$$\begin{aligned} \omega_{x'} &= -\dot{\Phi} \sin \Theta \cos \psi + \dot{\Theta} \sin \psi; \\ \omega_{y'} &= \dot{\Phi} \sin \Theta \sin \psi + \dot{\Theta} \cos \psi; \\ \omega_{z'} &= \dot{\Phi} \cos \Theta + \dot{\psi}. \end{aligned} \quad (5.401)$$

As the generalized coordinates q_i , we choose the quantities

$$q_1 = \Phi, \quad q_2 = \Theta, \quad q_3 = \psi, \quad q_4 = \beta, \quad \text{and} \quad q_5 = \gamma.$$

Then the kinetic energy (5.400) can be represented in the form

$$T = \frac{B}{2} \left(\frac{ds}{dt} \right)^2, \quad (5.402)$$

where ds^2 is the square of the line differential,

$$ds^2 = g_{ij} dq_i dq_j. \quad (5.403)$$

The symmetric matrix g_{ij} in this case has the form

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix}, \quad (5.404)$$

where the non-zero matrix elements are equal to:

$$\left. \begin{aligned} g_{11} &= \frac{J_1}{B} \sin^2 \Theta \cos^2 \psi + \frac{J_2}{B} \sin^2 \Theta \sin^2 \psi + \frac{J_3}{B} \cos^2 \Theta; \\ g_{12} &= \frac{1}{B} (J_2 - J_1) \sin \Theta \sin \psi \cos \psi; \\ g_{13} &= \frac{J_3}{B} \cos \Theta; \\ g_{22} &= \frac{J_1}{B} \sin^2 \psi + \frac{J_2}{B} \cos^2 \psi; \\ g_{33} &= \frac{J_3}{B}; \\ g_{44} &= 1; \\ g_{55} &= \beta^2. \end{aligned} \right\} \quad (5.405)$$

Using (5.405), it is not difficult to find the determinant of the matrix (5.404):

$$g \equiv |g_{ij}| = \frac{J_1 J_2 J_3}{B^3} \beta^2 \sin^2 \Theta. \quad (5.406)$$

By the rules of linear algebra, we then find the matrix g_{ij}^{-1} inverse to (5.404):

$$g_{ik}^{-1} g_{kj} = \delta_{ij}.$$

This matrix has the form

$$g_{ij}^{-1} = \begin{pmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} & 0 & 0 \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} & 0 & 0 \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} & 0 & 0 \\ 0 & 0 & 0 & g_{44}^{-1} & 0 \\ 0 & 0 & 0 & 0 & g_{55}^{-1} \end{pmatrix}, \quad (5.407)$$

where

$$\left. \begin{aligned} g_{11}^{-1} &= B \left(\frac{\cos^2 \psi}{J_1} + \frac{\sin^2 \psi}{J_2} \right); \\ g_{12}^{-1} &= -B \left(\frac{1}{J_1} - \frac{1}{J_2} \right) \frac{\sin \psi \cos \psi}{\sin \Theta}; \\ g_{13}^{-1} &= -B \left(\frac{\cos^2 \psi}{J_1} + \frac{\sin^2 \psi}{J_2} \right) \cot \Theta; \\ g_{22}^{-1} &= B \left(\frac{\sin^2 \psi}{J_1} + \frac{\cos^2 \psi}{J_2} \right); \\ g_{23}^{-1} &= B \left(\frac{1}{J_1} - \frac{1}{J_2} \right) \cot \Theta \sin \psi \cos \psi; \\ g_{33}^{-1} &= B \left(\frac{\cos^2 \psi}{J_1} + \frac{\sin^2 \psi}{J_2} \right) \cot^2 \Theta + \frac{B}{J_3}; \\ g_{44}^{-1} &= 1; \quad g_{55}^{-1} = \frac{1}{\beta^2}. \end{aligned} \right\} \quad (5.408)$$

The Laplacian in the five-dimensional space is defined by the general expression

$$\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left(\sqrt{g} g_{ij}^{-1} \frac{\partial}{\partial q_j} \right).$$

Substituting the values of the determinant (5.406) and of the matrix elements of the inverse matrix (5.407) into this formula, we obtain

$$\nabla^2 = - \sum_{k'} \frac{B}{J_{k'}} R_{k'}^2 + \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma}, \quad (5.409)$$

where $R_{k'}$ are the components of the angular momentum of rotation along the moving axes:

$$\begin{aligned} R_{x'} &= -i \left(-\frac{\cos \psi}{\sin \Theta} \frac{\partial}{\partial \Phi} + \sin \psi \frac{\partial}{\partial \Theta} + \cot \Theta \cos \psi \frac{\partial}{\partial \psi} \right); \\ R_{y'} &= -i \left(\frac{\sin \psi}{\sin \Theta} \frac{\partial}{\partial \Phi} + \cos \psi \frac{\partial}{\partial \Theta} - \cot \Theta \sin \psi \frac{\partial}{\partial \psi} \right); \\ R_{z'} &= -i \frac{\partial}{\partial \psi}. \end{aligned} \quad (5.410)$$

The operator of the kinetic energy associated with the rotation and surface deformations of the nucleus has the form

$$T \equiv -\frac{\hbar^2}{2B} \nabla^2 = \sum_{k'} \frac{\hbar^2}{2J_{k'}} R_{k'}^2 - \frac{\hbar^2}{2B} \left\{ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right\}. \quad (5.411)$$

As we should expect, this formula coincides with the formula (5.251) obtained earlier.

5.20. In a non-adiabatic approximation, consider the nuclear collective motions associated with axially symmetric deformations.

If we confine ourselves to axially symmetric deformations only (that is, put $\gamma = 0$), then

$$J_{x'} = J_{y'} = 3B\beta^2, \quad J_{z'} = 0, \quad \omega_{x'}^2 + \omega_{y'}^2 = \dot{\Phi}^2 \sin^2 \Theta + \dot{\Theta}^2,$$

and the kinetic energy of the collective motions takes the form

$$T = \frac{B}{2} \{ 3\beta^2 (\dot{\Phi} \sin^2 \Theta + \dot{\Theta}^2) + \dot{\beta}^2 \}. \quad (5.412)$$

Since in the case of axial symmetry the nucleus can rotate only about directions perpendicular to the symmetry axis, the motion is characterized by only three degrees of freedom: $q_1 = \Phi$, $q_2 = \Theta$ and $q_3 = \beta$. Introducing the square of the line differential, $ds^2 = g_{ij} dq_i dq_j$, and noting that the matrix g_{ij} is diagonal

$$g_{ij} = \begin{pmatrix} 3\beta^2 \sin^2 \Theta & 0 & 0 \\ 0 & 3\beta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain, using the general rules, the three-dimensional Laplacian

$$\nabla^2 = \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} + \frac{1}{3\beta^2} \left(\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \sin \Theta \frac{\partial}{\partial \Theta} + \frac{1}{\sin^2 \Theta} \frac{\partial^2}{\partial \Phi^2} \right). \quad (5.413)$$

Taking into account that the potential energy in the case of small deformations is proportional to the square of the deformation parameter β , we write the Hamiltonian of the system in the form

$$H = -\frac{\hbar^2}{2B} \left\{ \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} + \frac{1}{3\beta^2} \left[\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \sin \Theta \frac{\partial}{\partial \Theta} + \frac{1}{\sin^2 \Theta} \frac{\partial^2}{\partial \Phi^2} \right] \right\} + \frac{1}{2} C \beta^2. \quad (5.414)$$

We note that the angular dependence of the Hamiltonian (5.414) coincides with that of the ordinary Laplacian operator, and so the Schrödinger equation

$$H\Psi = E\Psi \quad (5.415)$$

can be solved by the method of separation of variables.

We choose the function $\Psi(\Phi, \Theta, \beta)$ in the form

$$\Psi(\Phi, \Theta, \beta) = f(\beta) Y_{lm}(\Theta, \Phi). \quad (5.416)$$

Then the function $f(\beta)$ will be determined by the equation

$$-\frac{\hbar^2}{2B} \left\{ \frac{1}{\beta^2} \cdot \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} - \frac{l(l+1)}{3\beta^2} \right\} f(\beta) + \frac{1}{2} C \beta^2 f(\beta) = Ef(\beta). \quad (5.417)$$

We introduce the new variables:

$$\left. \begin{aligned} \beta &= \beta_0 \xi; & \beta_0 &= \left(\frac{\hbar^2}{BC} \right)^{1/4}; \\ E &= \frac{\lambda}{2} \hbar \omega; & \omega &= \sqrt{\frac{C}{B}}. \end{aligned} \right\} \quad (5.418)$$

With these variables,

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \xi^2 \frac{\partial}{\partial \xi} f - \frac{1}{3\xi^2} l(l+1)f - \xi^2 f + \lambda f = 0. \quad (5.419)$$

We investigate the behavior of the solution as $\xi \rightarrow 0$. Putting $f(\xi) \sim \xi^s$, we obtain for s the characteristic equation

$$s^2 + s - \frac{1}{3} l(l+1) = 0,$$

whence

$$s = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{3} l(l+1)}.$$

Since $f(\xi)$ must be finite at $\xi = 0$, we choose the first solution. Of the two asymptotic solutions as $\xi \rightarrow \infty$, we choose $f(\xi) \sim e^{-(1/2)\xi^2}$ and seek the solution of eqn. (5.419) in the form

$$f(\xi) = \xi^s u e^{-\xi^2/2}, \quad (5.420)$$

where u is a polynomial satisfying the equation

$$u'' + 2\left(\frac{s+1}{\xi} - \xi\right)u' - (2s+3-\lambda)u = 0. \quad (5.421)$$

The solution of this equation is the confluent hypergeometric function

$$u = F\left(\frac{1}{2}(s + \frac{3}{2} - \lambda/2), s + \frac{3}{2}; \xi^2\right),$$

which becomes a polynomial if

$$\frac{1}{2}(s + \frac{3}{2} - \lambda/2) = -n, \quad (5.422)$$

where n is a non-negative integer ($n=0,1,2,\dots$). In this case, the confluent hypergeometric function reduces to a Laguerre polynomial

$$F(-n, s + \frac{3}{2}; \xi^2) = \frac{n!}{(s + \frac{3}{2})} L_n^{s+(1/2)}(\xi^2),$$

and the condition (5.422) defines the possible energy values. Finally, the energy levels of the system and the corresponding wave functions can be represented in the form

$$E_{nl} = \left(\frac{3}{2} + 2n + s\right)\hbar\omega; \quad (5.423)$$

$$\Psi_{nlm}(\Phi, \Theta, \beta) = N \xi^s L_n^{s+(1/2)}(\xi^2) e^{-\xi^2/2} Y_{lm}(\Theta, \Phi), \quad (5.424)$$

where

$$s = \frac{1}{2}(-1 + \sqrt{1 + \frac{4}{3}l(l+1)}),$$

$\xi = \beta/\beta_0$ and N is a normalization constant:

$$N = \left(\frac{\Gamma(n+1)}{\beta_0^3 \Gamma(n+s+1)}\right)^{1/2}$$

From the requirement that the wave function be symmetric with respect to the plane perpendicular to the symmetry axis of the nucleus and passing through the centre of the nucleus, it follows that only even values of l are possible.

Figure 5.23 depicts the scheme of the energy levels calculated from formula (5.423). The energy levels for the nuclei $^{92}_{40}Zr$, $^{94}_{40}Zr$ and $^{150}_{62}Sm$ are plotted on the right in Fig. 5.23. The magnitude of $\hbar\omega$ for each of the nuclei was determined from the position of the first observable excited level. The model considered enables us to explain the presence of the closely spaced levels 0^+ and 4^+ . The ratio, calculated

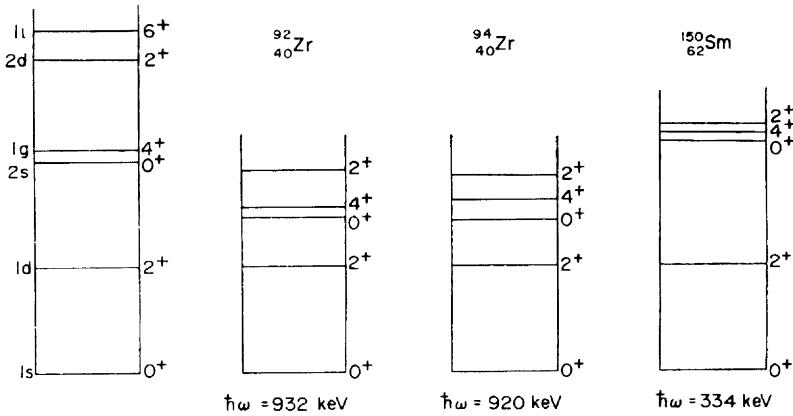


Figure 5.23. Scheme of the rotational-vibrational levels for axially symmetric deformations. Also given are the observed level schemes for the nuclei $^{92}_{40}\text{Zr}$, $^{94}_{40}\text{Zr}$ and $^{150}_{62}\text{Sm}$, for which the parameter $\hbar\omega$ is equal to 932, 920 and 334 keV respectively.

from formula (5.423), of the energies of the closely spaced levels 4^+ and 0^+ is equal to $1 \cdot 06$, whereas the ratios of the experimental values of the levels for the above three nuclei are $1 \cdot 08$ ($^{92}_{40}\text{Zr}$), $1 \cdot 12$ ($^{94}_{40}\text{Zr}$) and $1 \cdot 05$ ($^{150}_{62}\text{Sm}$).

5.21. Assuming that the self-consistent potential of the nucleus differs only slightly from a spherically symmetric potential, use perturbation theory to determine the correction, associated with the deformation of the nucleus, to the one-particle energy.

Unlike the case of large deformations, in the case of small deformations the one-particle energy levels and wave functions of nucleons owing in a weakly deformed nuclear field can be found in explicit form by using perturbation theory. As the perturbation V' , we shall take the difference of the non-spherical potential, which depends on the deformation parameters, and the spherical (shell) potential. To determine the first correction ΔE to the energy, we must calculate the diagonal matrix element of the perturbation V' using the wave functions of the shell model

$$\chi_{j\Omega}^{r=nl} = R_{nl}(r')Y_{l(1/2)j\Omega}, \quad (5.425)$$

where $Y_{l(1/2)j\Omega}$ is the spin-angle function. Choosing the potential in the form of a square well of depth V_0 for simplicity, we represent the correction ΔE to the energy in the form

$$\begin{aligned} \Delta E \equiv \langle \chi_{j\Omega}^{nl} | V' | \chi_{j\Omega}^{nl} \rangle &= \sum_{\text{spins}} \int d\sigma_{r'} \left\{ -V_0 \int_0^{R(\vartheta, \varphi)} dr' r'^2 |\chi_{j\Omega}^{nl}|^2 + V_0 \int_0^R dr' r'^2 |\chi_{j\Omega}^{nl}|^2 \right\} \\ &= -V_0 \sum_{\text{spins}} \int d\sigma_{r'} Y_{l(1/2)j\Omega}^* \times Y_{l(1/2)j\Omega} \int_R^\infty dr' r'^2 |R_{nl}(r')|^2. \end{aligned} \quad (5.426)$$

Considering only ellipsoidal deformations and taking into account that the deviations of the quantity

$$R(\vartheta, \varphi) = (1 + \sum_{\nu} \alpha'_{\nu} Y_{2\nu}(\vartheta, \varphi)) R$$

from the radius R of a sphere of the same volume are small, after summation over the spin variables we obtain the correction ΔE in the form

$$\Delta E = -V_0 R^3 |R_{nl}(R)|^2 \sum_{\nu} \alpha'_{\nu} \langle j\Omega | Y_{2\nu} | j\Omega \rangle. \quad (5.427)$$

If we had not used a square potential V , then in the perturbation V' , which for the square potential had the form

$$V' = -V_0 R \delta(r' - R) \sum_{\nu} \alpha'_{\nu} Y_{2\nu}(\vartheta, \varphi),$$

it would be necessary to replace $V_0 R \delta(r' - R)$ by the quantity $r' dV/dr'$. Noting that

$$\langle j\Omega | Y_{2\nu} | j\Omega \rangle = -\frac{1}{4} \sqrt{\frac{5}{4\pi}} \frac{3\Omega^2 - j(j+1)}{j(j+1)} \delta_{\nu 0},$$

we finally obtain for the correction to the one-particle energy the expression

$$\Delta E = c_j [3\Omega^2 - j(j+1)] \alpha'_0, \quad c_j = \frac{1}{4} \sqrt{\frac{5}{4\pi}} \frac{V_0 R^3 |R_{nl}(R)|^2}{j(j+1)}, \quad (5.428)$$

or, since $\alpha'_0 = \beta \cos \gamma$, we have ($\gamma < 15^\circ$)

$$\Delta E \equiv \Delta E_{nlj\Omega} = c_j [3\Omega^2 - j(j+1)] \beta \cos \gamma. \quad (5.429)$$

For small deformations ($\beta \ll 1$), the state of a nucleon in the nucleus can be characterized approximately by the quantum number j of the total angular momentum. It follows from (5.429) that for $j = \frac{1}{2}$ the first-order perturbation-theory correction ΔE is equal to zero. There is a degeneracy in the sign of the projection Ω of the total angular momentum.

If we have several nucleons in the shell (nlj) , the corresponding correction to the energy will be equal the sum of the separate corrections (5.429) for each of the nucleons. If the shell (nlj) is completely filled, that is, if there are $2j+1$ nucleons with different angular-momentum projections Ω in the shell, the total correction to the energy is equal to zero, since

$$\sum_{\Omega=-j}^j [3\Omega^2 - j(j+1)] = 0.$$

5.22. Determine the equilibrium deformation and the order of occupation of the one-particle states in the outer shell of a weakly deformed nucleus.

A nucleus with a completely filled outer shell has spherical shape. If we add one nucleon to such a nucleus, the new nucleus will then differ somewhat from

a sphere, the equilibrium deformation being determined by the interaction of the outer nucleon with the nuclear core formed by the filled shells. Because of this interaction, the core is polarized, and deformed, as a result, the field in which the outer nucleon moves will not possess spherical symmetry. The core resists, as it were, deformations; on the other hand, the outer nucleon tends to increase the deformation. The equilibrium deformation will thus depend on the elastic properties of the core and on the state of the outer nucleon in the deformed field of the core.

The dependence of the potential energy of the core on the deformation is determined for small deformations by the quadratic law

$$E'(\beta) = E'(0) + \frac{1}{2}C\beta^2. \quad (5.430)$$

The deformation dependence of the potential energy of the whole nucleus with the one outer nucleon is determined by the expression ($\gamma < 15^\circ$)

$$E(\beta) = E(0) + c_j[3\Omega^2 - j(j+1)]\beta \cos \gamma + \frac{1}{2}C\beta^2. \quad (5.431)$$

We shall consider deformed nuclei with axial symmetry, that is, we put $\gamma = 0$. Then we obtain the equilibrium value β_0 of the parameter β from the condition

$$\left. \frac{\partial E(\beta)}{\partial \beta} \right|_{\beta=\beta_0} = 0. \quad (5.432)$$

Substituting for $E(\beta)$ from (5.431), we obtain for the equilibrium deformation the value

$$\beta_0(\Omega) = -\frac{c_j}{C}[3\Omega^2 - j(j+1)]. \quad (5.433)$$

We obtain the energy of the nucleus in the equilibrium state by substituting (5.433) into (5.431) with $\gamma = 0$:

$$E(\beta_0(\Omega)) = E(0) - \frac{c_j^2}{2C}[3\Omega^2 - j(j+1)]^2. \quad (5.434)$$

It can be seen from this that, for one outer nucleon, the energy in equilibrium will be minimized if we take the largest possible value of $|\Omega|$, that is, if we put $\Omega = \pm j$. Then $\beta_0 < 0$, that is, the nucleus will be oblate along the symmetry axis and will have a negative quadrupole moment.

If there are several nucleons in the shell (nlj) , then, since we are assuming that they do not interact with each other, the results can be generalized easily by replacing the quantity $3\Omega^2 - j(j+1)$ in the formulae by a sum over the nucleons:

$$\sum_i [3\Omega_i^2 - j(j+1)].$$

The Pauli principle permits only two identical nucleons to have the same value of Ω_i^2 . If the shell is completely filled, the equilibrium deformation is equal to zero, as it should be:

$$\beta_0 = -\frac{c_j}{C} \sum_{\Omega_i=-j}^j [3\Omega_i^2 - j(j+1)] = 0.$$

If there are two nucleons in the shell, the energy will be minimized if the projections of their angular momenta are equal in magnitude but opposite in sign: $\Omega_1 = +j$ and $\Omega_2 = -j$, so that the total projection is equal to zero. If to the two nucleons we add a third, this will have $\Omega_3 = j-1$ or $\Omega_3 = -j+1$, and so on. The total projection $\sum_i \Omega_i$ of the total angular momentum will be equal to zero if there is an even number of nucleons in the unfilled shell; otherwise, it is equal to the projection of the angular momentum of the last odd nucleon.

When the shell is more than half full, it is energetically favorable for nucleons to occupy the states with the minimum projections of the total angular momentum: $\Omega_{1,2} = \pm \frac{1}{2}$, $\Omega_{3,5,4} = \pm \frac{3}{2}$, This can be seen clearly from the general expression for the energy of the nucleus in the equilibrium state:

$$E\left(\sum_i \beta_0(\Omega_i)\right) = E(0) - \frac{c_j^2}{2C} \left\{ \sum_i [3\Omega_i^2 - j(j+1)] \right\}^2. \quad (5.435)$$

Suppose, for example, that we have a complete shell less one nucleon. It can be seen from (5.435) that the energy will be minimized if a state with $|\Omega_i| = j$ is not occupied. In fact, in this case we have for the square of the sum in the right-hand side of the equality (5.435):

$$\begin{aligned} & \left\{ \sum_i [3\Omega_i^2 - j(j+1)] \right\}^2 \\ &= \left\{ \sum_{\Omega_i=-j}^j [3\Omega_i^2 - j(j+1)] - [3j^2 - j(j+1)] \right\}^2 = [3j^2 - j(j+1)]^2, \end{aligned}$$

which, for $j \geq \frac{3}{2}$, is the largest of all the quantities $[3\Omega_i^2 - j(j+1)]^2$. The equilibrium deformation in this case is equal to

$$\sum_i \beta_0(\Omega_i) = -\frac{c_j}{C} \sum_i [3\Omega_i^2 - j(j+1)] = \frac{c_j}{C} [3j^2 - j(j+1)],$$

that is, is a possible quantity, corresponding to a nucleus that is prolate along the symmetry axis, with a positive quadrupole moment.

Thus, we can draw the following general conclusion. If the number of nucleons in the shell does not exceed half the maximum possible number of nucleons in the given shell, the occupation of all states with $|\Omega_i| = j, j-1, j-2$, etc., down to a certain $|\Omega_i| \geq \frac{1}{2}(j+\frac{1}{2})$ is energetically favoured. In this case, the equilibrium deformation is negative, corresponding to an oblate nucleus having a negative quadrupole moment. But if in the shell there are more than half the maximum possible number of nucleons, then states with $|\Omega_i| = \frac{1}{2}, \frac{3}{2}, \dots$, etc., up to a certain value of $|\Omega_i|$ which is again greater than or equal to $\frac{1}{2}(j+\frac{1}{2})$, are occupied. In this case, the nucleus is prolate (the equilibrium deformation is positive) and its

quadrupole moment is positive. It is clear that the maximum equilibrium deformation will be reached when the shell is exactly half-filled.

The simple model described helps us to understand qualitatively the mechanism of the appearance of deformation of nuclei. Quantitatively, however, it does not explain the magnitudes of the quadrupole moments and spins of nuclei. Only for certain nuclei, close to the magic nuclei, does this model correctly predict the spins of the ground states of nuclei. But far from the magic nuclei, where the deformations can no longer be assumed to be small, we cannot assume that the quantum number j of the total angular momentum of a nucleon is, even approximately, a good quantum number. In addition, it is necessary to take into account higher powers in the expansion of the energy of the nucleus in powers of the deformation parameter.

If we plot the energy (5.435) of the nucleus as a function of the number of identical particles in the shell, we obtain a symmetric dependence about the middle of the shell, where the energy has a minimum. As we have seen using the example of the Nilsson potential, the one-particle levels plotted as functions of the deformation parameter intersect more and more frequently as the excitation energy and magnitude of the deformation increase. If we fix the number of nucleons and plot their energy as a function of the magnitude of the deformation, we shall observe discontinuities in this dependence, associated with the fact that individual nucleons will undergo transitions from certain states to others that are energetically more favoured. The change of state will occur exactly on the intersection of the one-particle energy curves. The discontinuous curve describing the energy of several nucleons as a function of the deformation usually has several minima, and it may happen that the magnitude of the energy is almost the same at some of the minima. In this case, a small change of the potential can lead to a sharp change in the order of the levels.

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