ON BOSON REPRESENTATION OF ANGULAR MOMENTUM. II

V. V. Mikhailov

The states of a system of N harmonic oscillators with fixed total number of quanta are decomposed with respect to bases of irreducible representations of SU(2). The previously introduced basis [1] is a basis with the highest dimensionality in this decomposition. For the case of three harmonic oscillators, the operators and a discrete basis of a representation of the noncompact group SU(1,1) are constructed. Bargmann's representation is considered for these states.

In [1] operators and states of angular momentum expressed in terms of operators and states of a system of N harmonic oscillators were introduced. In the present paper, the states of such a system containing a fixed total number of excitation quanta are completely decomposed with respect to bases of irreducible representations of SU(2). In some special cases, the explicit form of such bases is given and a rule for their construction in the general case is described. In a treatment along these lines for N = 3 the operators of the Lie algebra of SU(1,1) and a discrete basis of a representation of SU(1,1) appear naturally. The problem is analyzed by means of simple algebraic and combinatorial methods. Bargmann's representation for these states is found to be a source of new formulas between spherical functions.

It should be pointed out that in the present note, and also in [1], we are concerned solely with the representation of angular momentum by means of N boson operators. To each boson operator there corresponds a certain harmonic oscillator. It is only in this sense that one must understand the words "system of harmonic oscillators," which are frequently employed in the present paper.

1. Decomposition of a Representation of SU(2) by Means of a System of Harmonic Oscillators with Respect to Irreducible Representations

1. Quadratic combinations of N = 2s + 1 pairs of boson operators a_μ^+, a_μ ($\mu = -s, -s+1, \ldots, s$) of the form

$$J_3 = \sum \mu a_\mu{}^+ a_\mu, \quad J_- = \sum \sqrt{(s-\mu)(s+\mu+1)}\, a_\mu{}^+ a_{\mu+1} \tag{I}$$

and $J_+ = J_-^+$ give expressions for angular momentum operators. Then the states $|jm, s\rangle$ for $j = ls$ (l is a positive integer) can be represented as the result of the action of homogeneous polynomials of degree l in the operators a_μ^+ on the ground state $|0\rangle$:

$$|j, m, s\rangle = \sqrt{l! \Big/ \binom{2j}{j-m}} \sum_\xi \prod_\mu \frac{1}{\xi_\mu!} \left[\sqrt{\binom{2s}{s-\mu}}\, a_\mu{}^+ \right]^{\xi_\mu} |0\rangle, \tag{II}$$

where

$$\sum \xi_\mu = l, \quad \sum \mu \xi_\mu = m, \quad j = ls.$$

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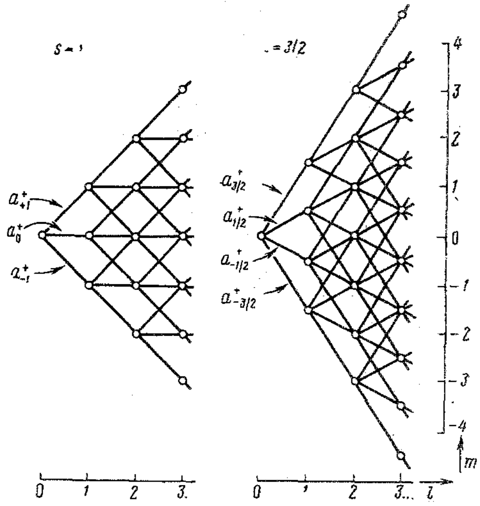

Fig. 1

2. Here we are interested in the combinatorial aspect of the construction of the states (II). For this purpose we construct graphs of "tree" type, which emanate from a single vertex and grow unrestrictedly on the right. The trees for s = 1 and s = 3/2 are shown in Fig. 1. Each vertex (intersection of lines indicated by a circle) corresponds to a definite state $|j, m; s\rangle$ of angular momentum and is denoted by (j, m, s). The lines joining neighboring vertices are called edges. To each edge of a definite slope there corresponds a definite operator a_μ^+ as one moves from the left to the right or an operator a_μ as one moves from the right to the left. Each vertex is the source of 2s + 1 edges that emanate from it and go to the right. Trees for arbitrary s can be "grown" trivially. Further, we shall give the name itinerary to a set of edges for which the beginning of each successive edge and the end of the foregoing edge are at one vertex. We consider only itineraries consisting of edges of the "left to right" type. To each such itinerary from (0, 0, s) to (j, m, s) there corresponds a certain term in the state $|j, m; s\rangle$. For example, in the state $|3, 0; 1\rangle = (1/\sqrt{15})(3a_1^+a_0^+a_{-1}^+ + a_0^{+3})|0\rangle$ the itinerary (0,0)–(1,1)–(2,1)–(3,0) corresponds to the term $a_1^+a_0^+a_{-1}^+|0\rangle$, while the itinerary (0,0)–(1,0)–(2,0)–(3,0) corresponds to the term $a_0^{+3}|0\rangle$. All the remaining itineraries are equivalent to these two, since each of them corresponds to one of these two terms. The number of all inequivalent itineraries from (0, 0, s) to (j, m, s), i.e., the number of terms in the expression (II) for $|j, m; s\rangle$, will be denoted by $P_{l,s}^m$. In other words, one can say that this number is equal to the dimensionality of the space $\mathscr{P}_{l,s}^m$, which consists of different states of a system of 2s + 1 harmonic oscillators $|\xi_s\rangle|\xi_{s-1}\rangle\ldots|\xi_{-s}\rangle$ for $\Sigma\xi_\mu = l$, $\Sigma\mu\xi_\mu = m$ and fixed l and their linear combinations. Clearly, any state in $\mathscr{P}_{l,s}^m$ is an eigenstate of the operator J_3 in (I) with eigenvalue m. Obviously, $P_{l,s}^m = 0$ for $m > ls$ and $P_{l,s}^{ls} = 1$.

The set of states of 2s + 1 harmonic oscillators containing l quanta and their linear combinations will be denoted by $\mathscr{M}_{l,s}$, and the dimensionality of this state by $M_{l,s}$. Obviously,

$$M_{l,s} = \frac{(l+N-1)!}{l!(N-1)!} = \binom{l+2s}{l}, \tag{1}$$

$$\sum_{m=-ls}^{ls} \mathscr{P}_{l,s}^m = \mathscr{M}_{l,s}, \quad \sum_{m=-ls}^{ls} P_{l,s}^m = M_{l,s}, \tag{2}$$

$$P_{l,s}^m = P_{l,s}^{-m}. \tag{3}$$

If the infinitesimal operators of SU(2) are the operators $J_1 = (J_+ + J_-)/2$, $J_2 = i(J_- - J_+)/2$, and J_3 from (I), one can say that in the space $\mathscr{M}_{l,s}$ there acts a certain representation $Q_{l,s}$ of SU(2). This representation is reducible if s > 1/2. Our task is to find the decomposition

$$Q_{l,s} = \sum_j q_{l,s}^j D_{l,s}^j. \tag{4}$$

The positive (or vanishing) integer $q_{l,s}^j$ is the multiplicity with which the corresponding irreducible representation $D_{l,s}^j$, which acts on its orthonormalized basis $|j, m; l, s\rangle$, occurs in (4). Here and everywhere below $j = ls, ls-1, \ldots, 1/2$, or 0, respectively, for half-integral or integral s. Previously, we wrote simply $j = ls$, and therefore in the designation of an angular momentum state we have now introduced the additional subscript l.

3. Following the scheme for decomposing a representation into irreducible representations [2], we attempt to determine a method of decomposing a basis of the space $\mathcal{M}_{l,s}$ with respect to the bases $|j, m; l, s\rangle$. From (II) we have the basis $|ls, m; l, s\rangle$ of the irreducible representation $D^{ls}_{l,s}$ of the maximal possible dimensionality (for given l and s) $2ls+1$. This basis determines the space $\mathcal{R}^{ls}_{l,s}$. In the orthogonal complement $\mathcal{M}_{l,s}-\mathcal{R}^{ls}_{l,s}=\mathcal{M}'_{l,s}$, whose dimensionality is $M_{l,s}-(2ls+1)$, we shall seek the basis $|j, m; l, s\rangle$ with largest j. It is not difficult to see, and we shall show this later, that $P^{ls}_{l,s} = P^{ls-1}_{l,s} = 1$, $P^{ls-2}_{l,s} = 2$ for $s \geq 1$ and $l \geq 2$, i.e., the spaces $\mathcal{P}^{ls}_{l,s}$ and $\mathcal{P}^{ls-1}_{l,s}$ are one-dimensional and both of their basis vectors belong in $\mathcal{R}^{ls}_{l,s}$. Hence, the state with largest j and m in $\mathcal{M}'_{l,s}$ is the state $|ls-2, ls-2; l, s\rangle$. From the two basis states in $\mathcal{P}^{ls-2}_{l,s}$ we construct a linear combination and multiply from the left by J_+ and equate to zero:

$$J_+(r_1 a_s^{+l-1} a^+_{s-2}+r_2 a_s^{+l-2} a^+_{s-1})|0\rangle=(\sqrt{(2s-1)}r_1+\sqrt{s}\,r_2)a_s^{+l-1}a^+_{s-1}|0\rangle=0. \tag{5}$$

Together with the normalization condition this completely determines (to within an arbitrary phase factor) the desired state:

$$|ls-2, ls-2; l, s\rangle=[(l-2)!2(2ls-1)]^{-1/2}a_s^{+l-2}(\sqrt{4s}\,a_s^+a^+_{s-2}-\sqrt{(2s-1)}\,a^{+2}_{s-1})|0\rangle. \tag{6}$$

Acting repeatedly on this state with the operator J_- [2], we construct the entire basis $|ls-2, m; l, s\rangle$, which consists of $2(ls-2)+1$ states. We subtract the space $\mathcal{R}^{ls-2}_{l,s}$ defined by this basis from $\mathcal{M}'_{l,s}$ and in the new orthogonal complement we again construct the basis with largest j.

Suppose that we have continued the operation of extracting the bases $|j, m; l, s\rangle$ from the space $\mathcal{M}_{l,s}$ to some j_0 inclusive. There remains the undecomposed space $M''_{l,s}$. On the basis of the first two stages of the decomposition we assume that the number of bases with $j \geq j_0$ is equal to $P^{j_0}_{l,s} = t$. States within the bases are orthogonal, like states from different bases. In the construction of the state with largest $j = j'$ and $m = j'$ in $\mathcal{M}''_{l,s}$, we can encounter three cases.

1. $P^{j_0-1}_{l,s} - t = 0$. The construction of each of the foregoing t bases required one basis state in each case from the space $\mathcal{P}^{j_0-1}_{l,s}$, in which, therefore, the desired $|j', j'; l, s\rangle$ cannot lie. The desired element must be sought in the space $\mathcal{P}^{j_0-2}_{l,s}$. If we nevertheless try to construct the desired state in $\mathcal{P}^{j_0-1}_{l,s}$ in accordance with (5) we obtain only vanishing coefficients in the linear combination.
2. $P^{j_0-1}_{l,s} - t = 1$. One can construct one further linear combination of states in $\mathcal{P}^{j_0-1}_{l,s}$ that is orthogonal to all the foregoing and is the desired $|j', j'; l, s\rangle$.
3. $P^{j_0-1}_{l,s} - t = g > 1$. One can construct g linear combinations of states in $\mathcal{P}^{j_0-1}_{l,s}$ that are orthogonal to one another and to t basis states in the intersection of the spaces $\mathcal{M}_{l,s}-\mathcal{M}''_{l,s}$ and $\mathcal{P}^{j_0-1}_{l,s}$ and which give g "upper" states $|j', j'; l, s\rangle_1, \ldots, |j', j'; l, s\rangle_g$, and hence g bases $|j', m; l, s\rangle_1, \ldots, |j', m; l, s\rangle_g$. This means that in the decomposition (4) the irreducible representation $D^{j_0-1}_{l,s}$ enters with multiplicity g, i.e., $q^{j_0-1}_{l,s} = g$. We can now see that the number of bases with $j \geq j_0-1$ is equal to $P^{j_0-1}_{l,s}$, and by induction this is true for any j_0.

Taking into account the condition (3), we obtain

$$q^j_{l,s}=P^j_{l,s}-P^{j+1}_{l,s}, \tag{7}$$

which holds, as is readily seen, for all three cases. We should point out that in case 3 the bases are not determined uniquely: the g orthogonal "upper" states can be chosen in many ways.

4. Thus, the problem (4) of decomposing $Q_{l,s}$ has been reduced by means of (7) to the determination of the numbers of itineraries $P^m_{l,s}$ in s-trees. Below, we shall obtain recursion relations and a generating function for these numbers. We note first that

$$P^m_{l,1/2}=1, \quad P^m_{l,1}=\left[\frac{l-|m|}{2}\right]+1. \tag{8}$$

TABLE 1

m \ l	0	1	2	3	4	5	6...	0	1	2	3	4	5	6...
	$s=1$							$s={}^3/_2$						
9														1
8														1
7													1	2
6							1					1	1	3
5						1	1					1	2	4
4					1	1	2				1	2	3	5
3				1	1	2	2			1	1	3	4	7
2			1	1	2	2	3			1	2	4	5	7
1		1	1	2	2	3	3		1	2	3	4	6	8
0	1	1	2	2	3	3	4	1	1	2	3	5	6	8
...	$s=2$							$s={}^5/_2$						
15														1
14														1
13														2
12							1						1	3
11							1						1	5
10						1	2					1	2	7
9						1	3					1	3	10
8					1	2	5					2	5	12
7					1	3	6				1	3	7	16
6				1	2	5	9				1	5	9	19
5				1	3	6	10			1	2	6	11	23
4			1	2	5	8	13			1	3	8	14	25
3			1	3	5	9	14			2	4	9	16	29
2		1	2	4	7	11	16		1	2	5	11	18	30
1		1	2	4	7	11	16		1	3	6	11	19	32
0	1	1	3	5	8	12	18	1	1	3	6	12	20	32

Note. The values of $P_{l,s}^m$ for $l=1,3,5$ and $s=3/2$ and $5/2$ correspond to $m=1/2,\ 3/2,\ 5/2$, etc.

The first equation is obvious and the second can be readily obtained from the explicit form of the states (II) for $s = 1$.

We now turn to the trees. If in an s-tree we remove all edges of one slope, for example, those corresponding to the operator $a^{\pm}_{s}$, we obtain an $(s-1/2)$-tree. On the basis of this point of view, we obtain by simple arguments the recursion formula

$$P_{l,s}^{m}=\sum_{k}P_{l-k,\,s-1/2}^{m+k(s+1/2)-l/2}. \tag{9}$$

Here $0 \le k \le [(ls-m)/2s]$, where the square brackets gives the integral part of the enclosed number. Using (9) and on the basis of (8), we can readily obtain the values of $P_{l,s}^m$ successively for $s = 1, 3/2, 2, \ldots$ (see Table 1). The relation (9) gives a simple proof of the following interesting property of the numbers $P_{l,s}^m$. For $l \ge 2s$, $s \ge s_0$ in all columns of numbers like those in Table 1 the upper $2s_0 + 1$ numbers are equal. Going to large s, one can use this to construct a certain series of numbers. Here is the start: 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, ...

The generating function for the numbers $P_{l,s}^m$ can be found by noting the analogy between this problem and the simple combinatorial problem of changing [3]. From the product of $2s + 1$ infinite series

$$[1+x^{2s}+(x^{2s})^2+\ldots][1+x^{2s-1}y+(x^{2s-1}y)^2+\ldots]\ldots[1+y^{2s}+(y^{2s})^2+\ldots]=(1-x^{2s})^{-1}(1-x^{2s-1}y)^{-1}\ldots(1-y^{2s})^{-1}. \tag{10}$$

Taking one term from each square brackets and multiplying them, we obtain

$$(x^{2s})^{\xi_s}(x^{2s-1}y)^{\xi_{s-1}}\ldots(y^{2s})^{\xi_{-s}}=x^{ls+m}y^{ls-m}. \tag{11}$$

Equality here holds if the numbers ξ_μ satisfy the conditions, which we have in mind in (II), $ls \pm m = \Sigma(s \pm \mu)\xi_\mu$. The expression (11) occurs $P_{l,s}^m$ times in the infinite series (10). Therefore, we finally have

$$\prod_{\mu}(1-x^{s+\mu}y^{s-\mu})^{-1}=\sum_{l=0}^{\infty}\sum_{m=-ls}^{ls}P_{l,s}^{m}x^{ls+m}y^{ls-m}. \tag{12}$$

This is the generating function for the numbers $P_{l,s}^m$. Trees with different s give the graphical interpretation of (12).

Further, if in (10) we make the substitution $u = y^{1/(2s-1)}x$, $v = y^{2s/(2s-1)}$, then (12) can be rewritten as

$$(1-x^{2s})^{-1}\prod_{\mu=-s}^{s-1}(1-x^{s+\mu}y^{s-\mu})^{-1}=(1-x^{2s})^{-1}\prod_{\mu=-s'}^{s'}(1-u^{s'+\mu}v^{s'-\mu})^{-1}=(1-x^{2s})^{-1}\sum P_{l,s'}^{m}u^{ls'+m}v^{ls'-m},\quad s'=s-{}^{1}/_{2},\tag{13}$$

or

$$(1-x^{2s})\sum P_{l,s}^{m}x^{ls+m}y^{ls-m}=\sum P_{l,s'}^{m}u^{ls'+m}v^{ls'-m}.\tag{14}$$

Equating coefficients of equal powers of x and y on the left- and right-hand sides of (14), we obtain a recursion relation for the numbers $P_{l,s}^{m}$. A second similar relation is obtained if in the product (10) we separate $(1-y^{2s})^{-1}$, and not $(1-x^{2s})^{-1}$ as in (13). The two recursion formulas can be expressed in the single relation

$$P_{l,s}^{m}-P_{l-1,s}^{m\pm s}=P_{l,s-1/2}^{m\mp l/2}.\tag{15}$$

Formulas (15) are simpler than (9), but using them one also cannot construct numbers for an s-tree without knowing the numbers for the (s−1/2)-tree. For this to be possible, we obtain one formula from the two formulas (15):

$$P_{l,s}^{m}-P_{l,s}^{m+l}=P_{l-1,s}^{m-s}-P_{l-1,s}^{m+l+s}.\tag{16}$$

We obtain a further recursion relation if we split the product (10) into two groups with N_1 and N_2 factors ($N_1 + N_2 = N$ or $s_1 + s_2 = s-1/2$) and make the substitution $u = x^{s/s_1}$, $v = x^{(s-s_1)/s_1}y$; $w = y^{s/s_2}$, $z = y^{(s-s_2)/s_2}x$ in the first N_1 factors and the remaining N_2 factors, respectively. After this we obtain as in the case of (14)

$$P_{l,s}^{m}=\sum_{l_1=0}^{l}\sum_{m_1,m_2}{}' P_{l_1,s_1}^{m_1}P_{l-l_1,s_2}^{m_2},\quad m={}^{1}/_{2}(l_1N-lN_2)+m_1+m_2.\tag{17}$$

Thus, the generating function (12) has enabled us to obtain various recursion relations (15)-(17), from which we can extract useful general properties of the numbers $P_{l,s}^{m}$. First, on the basis of (16), we can show that for fixed s for $l \geq l_0$ the upper l_0 numbers in any column (in Table 1) are equal; for suppose $m + l > ls$; then $P_{l,s}^{m+l} = P_{l-1,s}^{m+l+s} = 0$ and from (16) we have $P_{l,s}^{m} = P_{l-1,s}^{m-s}$, as we wish to show. Secondly, we can show that

$$P_{l,s}^{m}=P_{2s,l/2}^{m}.\tag{18}$$

For this we replace s by $l/2$ and l by 2s in the second (lower signs) formula (15) and write down the first formula (15) unchanged:

$$P_{l,s}^{m}=P_{l-1,s}^{m+s}+P_{l,s-1/2}^{m-l/2};\quad P_{2s,l/2}^{m}=P_{2s-1,l/2}^{m-l/2}+P_{2s,(l-1)/2}^{m+s}.\tag{19}$$

We now set s = 1. For $l = 3$ we see, using Table 1, that the right-hand sides in (19) are equal, so that $P_{3,1}^{m} = P_{2,3/2}^{m}$. Using this result, we set $l = 4$ and from (19) we obtain $P_{4,1}^{m} = P_{2,2}^{m}$, etc. Continuing in this manner we prove (18) for any l for s = 1. This, in its turn, enables us, beginning with $l = 4$, to argue similarly without recourse to Table 1 for s = 3/2. We then go to s = 2, s = 5/2, etc., and prove (18) for all s and l. With allowance for (4) and (7) the property (18) means that the basis of the system of 2s + 1 harmonic oscillators with l quanta can be decomposed with respect to bases of irreducible representations of SU(2) in the same way as the basis of a system of $l + 1$ harmonic oscillators with 2s quanta.

2. Some Bases of Irreducible Representations and Their Properties

1. We note first that for the states $|ls, m; l, s\rangle$ in (II) the operators a_λ are lowering in the numbers l and m:

$$a_\lambda|j,m;l,s\rangle=\sqrt{l\binom{j+m}{s+\lambda}\binom{j-m}{s-\lambda}\Big/\binom{2j}{2s}}\,|j-s,m-\lambda;l-1,s\rangle.\tag{20}$$

Here $j = ls$. In an s-tree we choose the vertex (ls, m, s). Motion from it to the right along the λ-edge denotes multiplication of the state $|ls, m\rangle$ by the creation operator with number λ: $a_\lambda^+|ls, m\rangle$. This last is not a pure state of angular momentum $|(l+1)s, m+\lambda\rangle$, since we do not take into account other possible itineraries, for example, $a_{\lambda+1}^{+}|ls, m-1\rangle \to |(l+1)s, m+\lambda\rangle$. Conversely, motion along the λ-edge to the left (this is the operation (20)) takes us from one pure state of angular momentum to another, since such motion is along the only possible itinerary. Generalizing (20), we obtain

$$\prod_{\lambda=-s}^{s}(a_\lambda)^{\xi_\lambda}|j, m; l, s\rangle = \left[\frac{l!}{(l-l')!}\prod_{\lambda}\binom{2s}{s+\lambda}^{\xi_\lambda}\binom{2j-2j'}{j-j'+m-m'}\Big/\binom{2j}{j+m}\right]^{1/2}|j-j', m-m'; l-l', s\rangle, \tag{21}$$

where $\sum_\lambda \xi_\lambda = l'$, $l's = j'$, $ls = j$, $\sum_\lambda \lambda\xi_\lambda = m'$, $l' < l$.

2. Further, suppose s = 1 [4]. From (4), (7), and (8) we obtain a method for decomposing the representation Q_l with respect to irreducible representations:

$$Q_l = D^l + D^{l-2} + \ldots + D^1 \quad \text{or} \quad D^0 \tag{22}$$

respectively, for odd or even l. The multiplicity of an arbitrary irreducible representation D^j is here 0 or 1. To construct bases of the representations D^j we introduce the operators

$$K_+ = a_1^+ a_{-1}^+ - {}^1/_2 a_0^{+2}, \quad K_- = a_1 a_{-1} - {}^1/_2 a_0^2, \tag{23}$$

$$K_3 = {}^1/_2(a_1^+a_1 + a_0^+a_0 + a_{-1}^+a_{-1} + {}^3/_2), \quad K^2 = K_3(K_3-1) - K_+K_-,$$

$$[K_+, K_-] = -2K_3, \quad [K_3, K_\pm] = \pm K_\pm, \tag{24}$$

$$[J_\rho, K_\rho] = 0, \tag{25}$$

where $\rho = +, 0, -$. The operators K_ρ are similar to the hyperbolic operators constructed in [5] by means of two boson operators. The operators K_ρ form the Lie algebra of SU(1,1), and K^2 is a Casimir operator [6].

Consider the state $K_+^n|j, j; j, 1\rangle = |\psi\rangle$. First, this is a homogeneous polynomial of degree j + 2n in the operators a_1^+, a_0^+, and $a_{\pm 1}^+$ acting on the state $|0\rangle$. Secondly, the state $|\psi\rangle$ by virtue of (25) satisfies the equations $J_+|\psi\rangle = 0$ and $J_3|\psi\rangle = j|\psi\rangle$. Thus, the state $|\psi\rangle$ has all the properties of the state $|j, j; l, 1\rangle$ for $l = j + 2n$, and since we know from the decomposition (22) that the latter is unique, $|\psi\rangle$ and $|j, j; l, 1\rangle$ can differ only by a constant factor:

$$|j, j; l, 1\rangle = C_{jn}K_+^n|j, j; j, 1\rangle, \quad l = j + 2n. \tag{26}$$

To determine the normalization constant C_{jn}, we note that the relations

$$[a_{\pm1}, K_+^n] = na_{\mp1}^+K_+^{n-1}, \quad [a_0, K_+^n] = -na_0^+K_+^{n-1}, \quad [a_\rho^+, K_+] = 0$$

and the equations $a_{-1}|j, j; j, 1\rangle = a_0|j, j; j, 1\rangle = 0$ yield

$$K_-K_+^n|j, j; j, 1\rangle = n(n+j+{}^1/_2)K_+^{n-1}|j, j; j, 1\rangle. \tag{27}$$

Using this we determine successively for n = 1, 2, . . . the normalization constant C_{jn} in (26). Note also that, since J_- and K_+ commute, (26) holds for arbitrary m. As a result,

$$|j, m; l, 1\rangle = \sqrt{\frac{2^n(2j+1)!!}{n!(2j+2n+1)!!}}K_+^n|j, m; j, 1\rangle. \tag{28}$$

Using (24) and (27), we obtain the following properties of the states (28):

$$K_+|j, m; l, 1\rangle = \sqrt{(n+1)(j+n+{}^3/_2)}|j, m; l+2, 1\rangle, \tag{29}$$

$$K_-|j, m; l, 1\rangle = \sqrt{n(j+n+{}^1/_2)}|j, m; l-2, 1\rangle, \tag{30}$$

$$K_3|j, m; l, 1\rangle = {}^1/_2(l+{}^3/_2)|j, m; l, 1\rangle, \tag{31}$$

$$K^2|j, m; l, 1\rangle = {}^1/_2(j-{}^1/_2)[{}^1/_2(j-{}^1/_2)+1]|j, m; l, 1\rangle. \tag{32}$$

The operators K_+ and K_- couple states of the system of three harmonic oscillators that have the same angular momenta and different numbers l of quanta. The states (28) form a discrete basis of a representation of SU(1,1) [6].

From (28) we can readily obtain the generating function for these states:

$$\sum_{n=0}^{\infty} \sqrt{\frac{(2n+2j+1)!!}{2^n n!(2j+1)!!}} \sigma^n |j, m; j+2n, 1\rangle = \exp(\sigma K_+) |j, m; j, 1\rangle.$$

3. In one further case the decomposition of the representation $Q_{l,s}$ in (4) into irreducible representations will have a form as simple as in (22). Namely, it follows from the property (18) that if $l = 2$ for arbitrary s then

$$O_{2,s} = D^{2s} + D^{2s-2} + \ldots + D^1 \quad \text{or} \quad D^0 \tag{33}$$

depending on the parity of the number 2s. The number of states in the bases of the representations on the left and the right in (33) is the same because

$$M_{2,s} = (2s+1)(2s+2)/2 = \sum_k [2(2s-2k)+1],$$

where $0 \leq k \leq [s]$. The "upper" (m = j) state of any basis (for $l = 2$ the index l will be omitted for the time being from the designation of the state) is, as is readily shown, of the form

$$|2s-2k, 2s-2k; s\rangle = C \sum_{\nu=-k}^{k} C_\nu a^+_{s-k-\nu} a^+_{s-k+\nu} |0\rangle. \tag{34}$$

The condition $J_+ | 2s-2k, 2s-2k; s\rangle = 0$ gives a system of linear equations for the coefficients C_ν, whose solution is

$$C_\nu = (-1)^{k+\nu} \sqrt{\binom{2k}{k-\nu}\binom{2p}{p-k} \Big/ \binom{2p}{p-\nu}},$$

where $p = 2s-k$. Now, using the relation for the binomial coefficients,

$$\sum_{i=0}^{n} \binom{\nu+i}{i}\binom{n+\nu-i}{\nu} = \binom{n+2\nu+1}{n},$$

we can determine the normalization constant C. As a result, (34) takes the form

$$|2s-2k, 2s-2k; s\rangle = \frac{1}{(p-k)!} \sqrt{\frac{(2p-2k+1)!}{2(2p+1)}}$$

$$\times \sum_\nu (-1)^{k+\nu} \sqrt{\binom{2k}{k-\nu} \Big/ \binom{2p}{p-\nu}} a^+_{s-k-\nu} a^+_{s-k+\nu} |0\rangle. \tag{35}$$

States with arbitrary m can be obtained by applying the operator J_- to (35) a total of $2s-2k-m$ times.

The examples of this section have demonstrated the value of the approach to the boson representation of angular momentum from the point of view of trees and the numbers $P^m_{l,s}$ that determine these trees.

4. In conclusion, let us consider the question of the relationship between the angular momentum states introduced in [1] and the present paper with ordinary spherical functions. In [4] this question was considered for the special case s = 1.

One says that a Bargmann representation [7] for the state $f(\{a^+_\mu\})|0\rangle$, where a^+_μ are boson operators, is defined if one introduces the correspondence $a^+_\mu = z_\mu$, $a_\mu = \partial/\partial z_\mu$, $|0\rangle = \text{const}$. Our aim is to find a parametrization $z_\mu = z_\mu(\theta, \varphi)$ such that the states (II) are consistently associated with the spherical functions $Y_{jm}(\theta, \varphi)$. This is not difficult; for if

$$a^+_{s\mu} = \mathcal{Y}_{s\mu}(\theta, \varphi) = s! \sqrt{\frac{4\pi}{(2s+1)!}} Y_{s\mu}, \quad |0\rangle = \frac{1}{\sqrt{4\pi}}, \tag{36}$$

which means $\langle \theta, \varphi | s, \mu; s\rangle = \mathcal{Y}_{s\mu}(\theta, \varphi)(1/\sqrt{4\pi})$, then

$$\langle \theta, \varphi | j, m; s\rangle = \frac{1}{\sqrt{l! 4\pi}} \mathcal{Y}_{jm}(\theta, \varphi), \tag{37}$$

where

$$\mathcal{Y}_{jm}(\theta,\varphi)=j!\sqrt{\frac{4\pi}{(2j+1)!l!}}\,Y_{jm}(\theta,\varphi).$$

The coefficients in front of the spherical functions in (36) and (37) can be readily obtained if one considers the case m = j, when $|j, m; s\rangle$ has the simple form $(a_s^+)^l|0\rangle/\sqrt{l!}$. Thus, (37) with allowance for (II) and (36) is a formula that relates spherical functions of multiple orders:

$$\sqrt{\binom{2j}{j-m}}\,\mathcal{Y}_{jm}(\theta,\varphi)=l!\sum_{\xi}\prod_{\mu}\frac{1}{\xi_\mu!}\left[\sqrt{\binom{2s}{s-\mu}}\,\mathcal{Y}_{s\mu}(\theta,\varphi)\right]^{\xi_\mu}. \tag{38}$$

Using (37), one can readily introduce Bargmann's representation for the coherent states of angular momentum in (II) as well:

$$\langle\theta,\varphi|\alpha,s\rangle=\frac{1}{\sqrt{4\pi}}\exp\left(-\frac{n^{2s}}{2}\right)\sum_{j,m}\frac{1}{l!}\alpha_{jm}\mathcal{Y}_{jm}(\theta,\varphi)$$

$$=\frac{1}{\sqrt{4\pi}}\exp\left(-\frac{n^{2s}}{2}\right)\prod_{\mu}\exp[\alpha_{s\mu}\mathcal{Y}_{s\mu}(\theta,\varphi)]=\frac{1}{\sqrt{4\pi}}\prod_{\mu}\langle\theta,\varphi|\alpha_{s\mu}\rangle. \tag{39}$$

Here, $\langle\theta,\varphi|\alpha_{s\mu}\rangle$ is a coherent state of the μ-th oscillator in Bargmann's representation. In (36)-(39), s is an integer and $j = ls$. Other formulas relating spherical functions of different and not necessarily multiple orders can be obtained similarly from the formulas for the abstract states $|j, m; l, s\rangle$ (20), (21), (28)-(32), (35), and others.

Note. In [1] in formula (2) the factor in front of the summation sign should be $\sqrt{l!\Big/\binom{2j}{j-m}}$ and not $\sqrt{l\Big/\binom{2j}{j-m}}$.

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