A Systematic Description of the Wobbling Motion in Odd-Mass Nuclei Within a Semi-Classical Formalism

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A presentation for the degree of Doctor of Philosophy

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Outline

Aim

Research Objectives

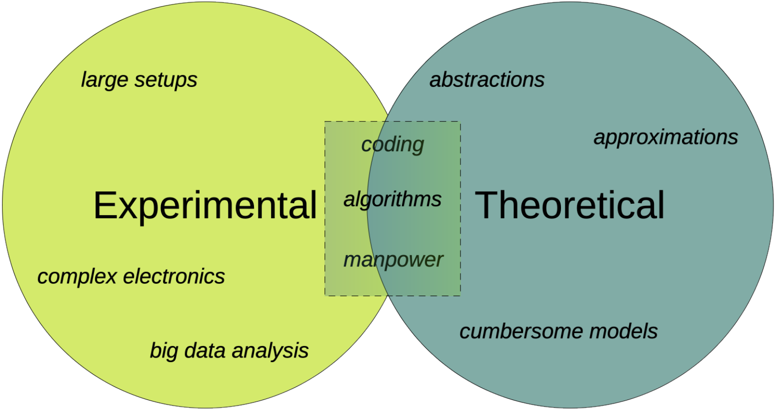
- Extend the current interpretation of the **Nuclear Triaxiality** in the context of its unique fingerprint: **Wobbling Motion**.
- Adopt a framework that is as close as possible to a **classical picture**.

Objectives exclusive to the thesis

- Sufficient context for a better understanding of the underlying concepts, methods, and results.
- create a completely *open-source* project.

Motivation

- **Nuclear Triaxiality** recently became a *hot topic* within the scientific community.
- Identifying nuclei with triaxial deformations represents a real **experimental** and **theoretical** challenge.



Outline

Nuclear Deformation

Nuclear shapes

Most generally described in terms of the **nuclear radius**:

$$R(\theta,\varphi)=R_0\left(1+\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\alpha_{\lambda\mu}Y_{\lambda}^{\mu}(\theta,\varphi)\right)$$

Quadrupole deformations $\lambda=2$

- Most relevant modes are the **quadrupole vibrations** $\lambda=2 \Longrightarrow$ *Play a crucial role in the rotational spectra of nuclei:*
- $\alpha_{2\mu}$ reduced to only two *deformation parameters*: β_2 (**eccentricity**) and γ (**triaxiality**) (*Bohr and Mottelson, 1969*).

Axial shapes

- Most of the nuclei are either **spherical** or **axially symmetric** in their ground-state.
- Nuclear moments of inertia $\mathcal{I}_{1,2,3}$: only two are equal.

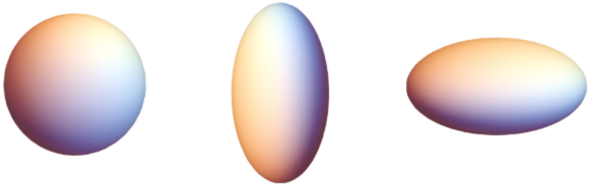
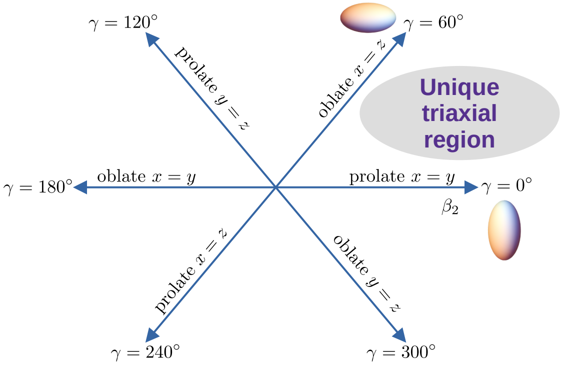
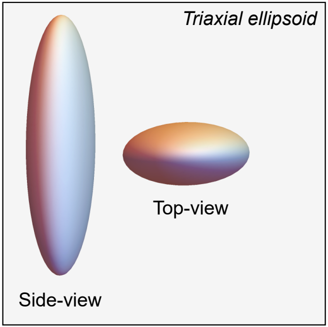


Figure: **spherical:** $\beta_2 = 0$ **prolate:** $\beta_2 > 0$ **oblate:** $\beta_2 < 0$. $(\gamma = 0^\circ)$.

Non-axial shapes

- The triaxiality parameter $\gamma \neq 0°$: departure from axial symmetry.
- Moments of inertia: $\mathcal{I}_1 \neq \mathcal{I}_2 \neq \mathcal{I}_3$.





Outline

Fingerprints of Triaxiality

Evidence

- Currently, there are **only two** well-established phenomena uniquely attributed to triaxial deformation.
  1. **Wobbling Motion** - WM (*Bohr and Mottelson, 1970s*)
  2. Chiral Motion - χM (*Frauendorf, 1997*)
- These two can be measured/detected experimentally.

Goal

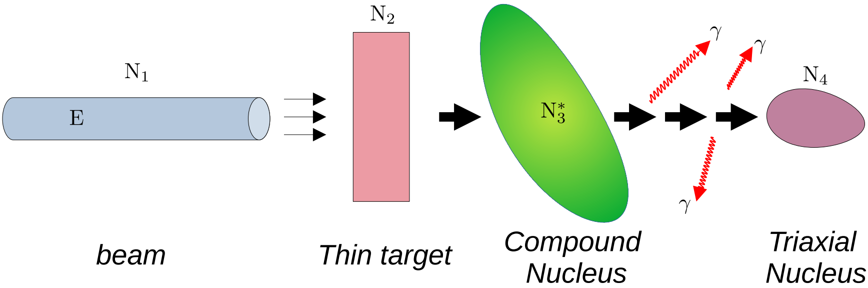
Describe the elusive character of Wobbling Motion in the context of nuclear triaxiality.

Probing triaxiality in nuclei

Triaxial nuclei can be observed/obtained in several experiments:

- Nuclear fission: $A \to B + C$
- Nuclear fusion: $X + Y \to Z$
- **Fusion-evaporation reactions**: Long-lived + enhanced deformation

$$Beam(N_1, E) + Target(N_2) \longrightarrow N_3^* \to \cdots \to triaxial(N_4)$$



Nuclear facilities

Figure: Gammasphere detector, ANL-ATLAS USA. *Source: aps.org*

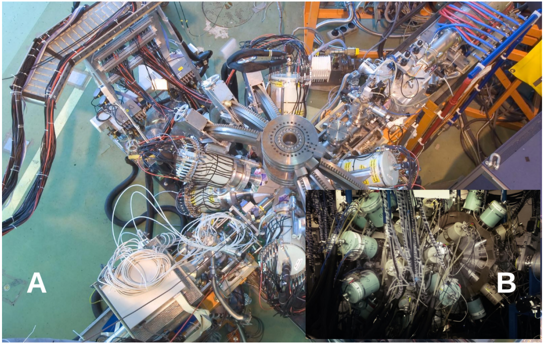
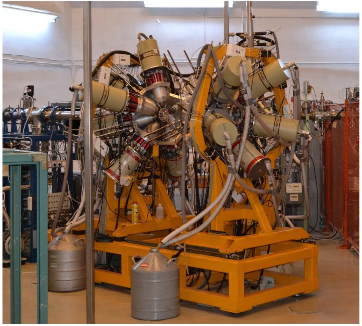


Figure: a) IDS detector, CERN. *Source: isolde.web.cern.ch* b) JUROGAM II, Finland. *Source: twitter.com*

High-Spin Physics @ IFIN-HH



Contents lists available at ScienceDirect

Nuclear Instruments and Methods in Physics Research A

journal homepage: www.elsevier.com/locate/nima

The ROSPHERE γ-ray spectroscopy array

D. Bucurescu [a], I. Căta-Danil [a], G. Ciocan [a], C. Costache [a], D. Deleanu [a], R. Dima [a], D. Filipescu [a,c], N. Florea [a], D.G. Ghiță [a], T. Glodariu [a], M. Ivaşcu [a], R. Lică [a], N. Mărginean [a], R. Mărginean [a], C. Mihai [a,*], A. Negret [a], C.R. Niță [a], A. Olăcel [a], S. Pascu [a], T. Sava [a], L. Stroe [a], A. Şerban [a,i], R. Şuvăilă [a], S. Toma [a], N.V. Zamfir [a,c], G. Căta-Danil [b], I. Gheorghe [c], I.O. Mitu [c], G. Suliman [c], C.A. Ur [c], T. Braunroth [d], A. Dewald [d], C. Fransen [d], A.M. Bruce [e], Zs. Podolyák [f], P.H. Regan [f,g], O.J. Roberts [h]

Figure: ROSPHERE, IFIN-HH. *Source: tandem.nipne.ro*

Wobbling Motion

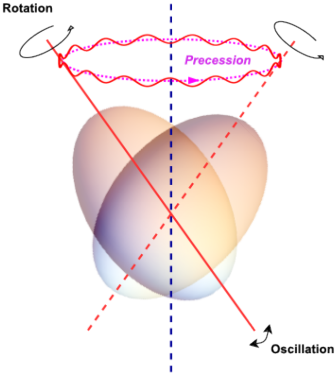
$\gamma \neq 0^\circ$

↓

MOI anisotropy

↓

main rotation around $\mathcal{J}_{\text{max}}$ is disturbed by the other two axes



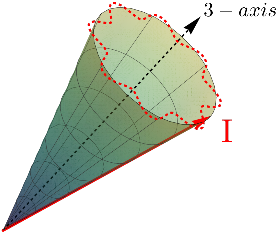
Wobbling Effect

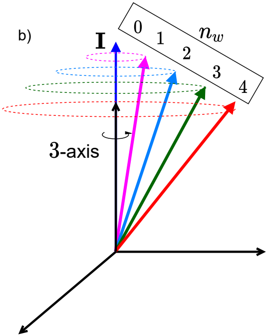
- The **total angular momentum** of the nucleus **precesses** and **oscillates** around $\mathcal{J}_{\text{max}}$.

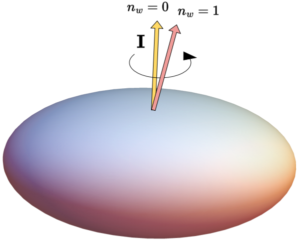
Wobbling Motion

Harmonic oscillation

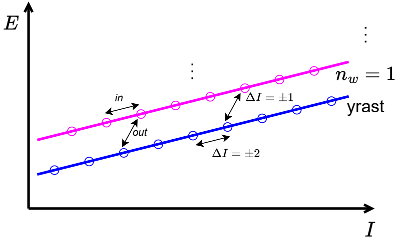
- Precession of **I** is affected by **rotational frequency** and/or **tilting**
- Tilting only by "specific" amount → **harmonic character** → **wobbling phonon**: $n_w = 0, 1, 2, \ldots$.

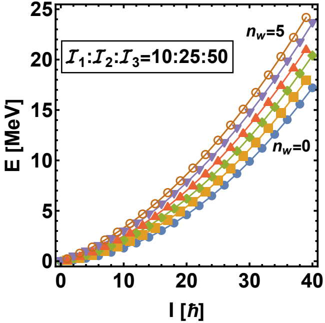


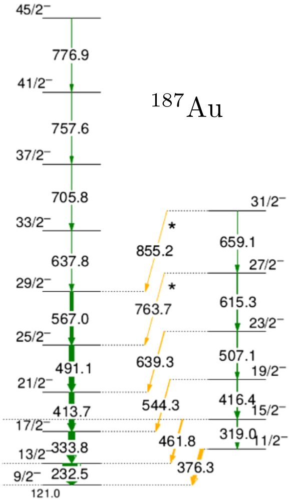




Wobbling Motion II



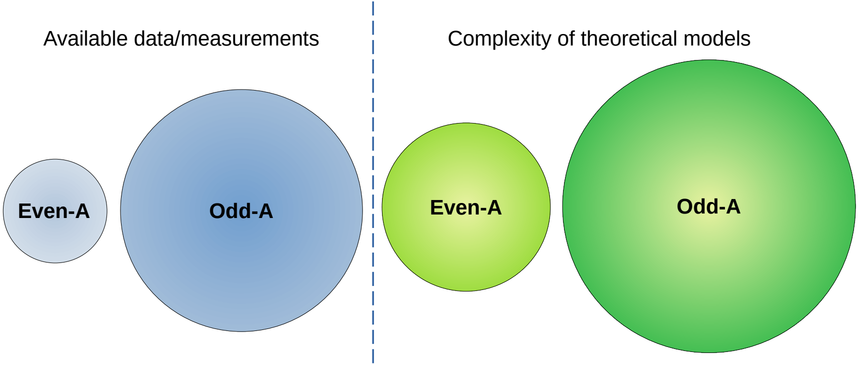




Sensharma, 2020.

Even-A vs. Odd-A Picture

- Predicted for even-A nuclei more than 50 years ago.
- First experimental evidence: ^{163}Lu (*Ødegård, 2001*).
- Current mass-regions for wobblers: $A \approx [130, 160, 180]$.



Wobbling Motion in ^{130}Ba

Experimental measurements show **two** wobbling bands (*Petrache et. al. 2019*).

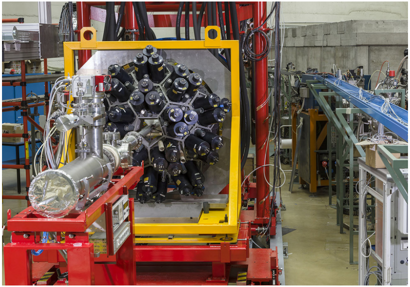
Harmonic formalism

Harmonic Approximation (*Bohr & Mottelson, 1969*):

$$E_{I,n_w} = A_3 I(I+1) + \hbar\omega_w \left(n_w + \frac{1}{2}\right),$$

$$A_3 = (2\mathcal{I}_3)^{-1}.$$

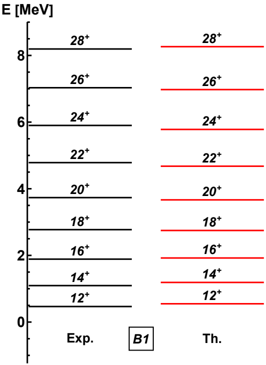
(rotational term + wobbling frequency)

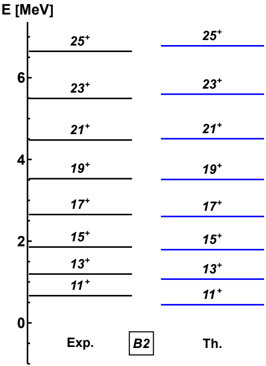


GALILEO, LNL, *Source: lnl.infn.it*

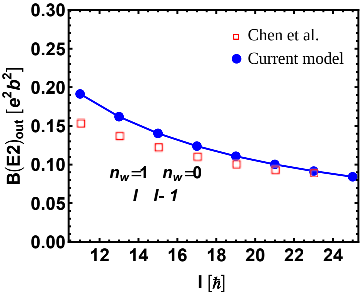
Fusion evaporation: ^{13}C beam of $E = 65$ MeV and ^{122}Sn target.

Results for ^{130}Ba





$\mathcal{P}_{\text{fit}}$			
$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	Unit
27	22	**43**	$\hbar^2\text{MeV}^{-1}$



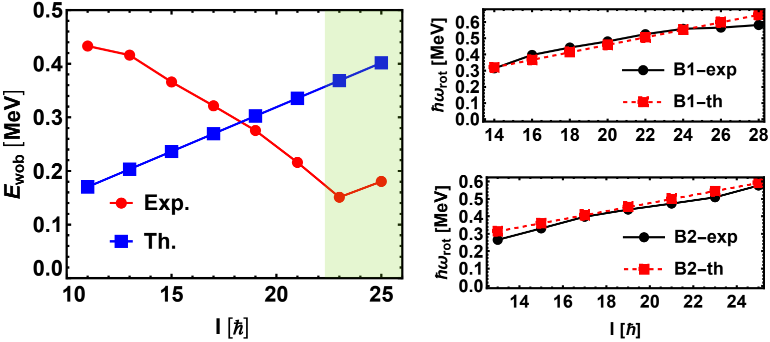
Results presented at the international conference NSP-2022, Turkey.

Results for ^{130}Ba II

Excitation energies vs. **Wobbling Energies**:

$$E_{\text{wob}}(I_{\text{even}}) = E_{I,n} - E_{I,0} \ ,$$

$$E_{\text{wob}}(I_{\text{odd}}) = E_{I,n} - \frac{1}{2}\left(E_{I-1,0} + E_{I+1,0}\right)$$



Results presented at the international conference NSP-2022, Turkey.

Outline

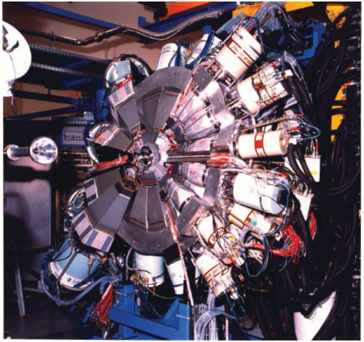
Starting Point

Framework

- **First semi-classical description** for ^{163}Lu, using the **Particle Rotor Model**.
- A. A. Raduta, **R. Poenaru**, L. Gr. Ixaru, PRC, 2017
- A. A. Raduta, **R. Poenaru**, Al. H. Raduta, JPG, 2018

Particle Rotor Model (PRM)

An odd-nucleon moving in a quadrupole deformed mean field generated by an even-even triaxial core *Hamamoto, 2002*.



Euroball IV, Strasbourg, *Source: technologysi.stfc.ac.uk.*

$^{139}La\left(^{29}Si,5n\right)^{163}Lu.$

$(E_{beam}=152\ MeV).$

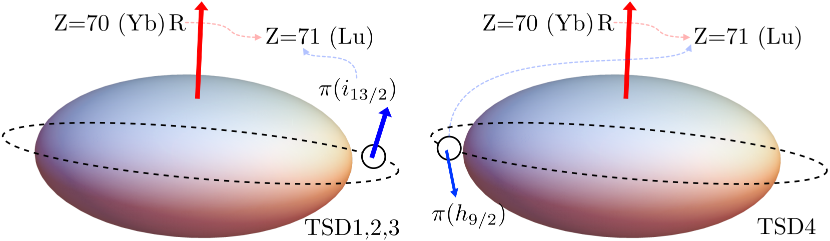
Fresh-Up 1: $\mathbf{W}_1$

Particle-Rotor Model Hamiltonian for an odd-A nucleus:

$$\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{sp}} \ , \ \hat{H}_{\text{rot}} = \sum_{k=1}^{3} A_k(\hat{I}_k - \hat{j}_k)^2,$$

$$\hat{H}_{\text{sp}} = \epsilon_j + \frac{V}{j(j+1)}\left[\cos\gamma\left(3\hat{j}_3^2 - \mathbf{j}^2\right) - \sqrt{3}\sin\gamma\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right].$$

V - single-particle potential strength $\propto \beta_2$ (*Tanabe, 2017*)



A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Variational Principle + Eqs. of Motion

Time-Dependent Variational Equation

$$\delta \int_0^t \langle \Psi_{IM;j} | \hat{H} - i\frac{\partial}{\partial t'} | \Psi_{IM;j} \rangle \, dt' = 0$$

$$\Psi_{\text{trial}} \equiv |\Psi_{IM;j}\rangle = \mathcal{N} e^{z\hat{I}_-} \cdot e^{s\hat{j}_-} |IMI\rangle \otimes |jj\rangle$$

- $(z, r, \varphi, \hat{I}, |IMI\rangle)$ - core (rotor $\mathbf{R}$)
- $(s, f, \psi, \hat{j}, |jj\rangle)$ - single-particle ($\mathbf{j}$)
- $\{z, s\}$ $\rightarrow$ **phase space coordinates**

Constant of motion:

$$\mathcal{H} \equiv \langle \Psi_{IM;j} | \hat{H} | \Psi_{IM;j} \rangle$$

Canonical equations of motion:

$$\mathcal{S}_1 : \frac{\partial \mathcal{H}}{\partial r} = \dot{\varphi} \, , \quad \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r}$$

$$\mathcal{S}_2 : \frac{\partial \mathcal{H}}{\partial f} = \dot{\psi} \, , \quad \frac{\partial \mathcal{H}}{\partial \psi} = -\dot{f}$$

The two sets of Hamilton equations are **the semi-classical description** of the initial quantal $\hat{H}$.

Wobbling frequency

Solving $\mathcal{S}_1$ and $\mathcal{S}_2$ leads to the algebraic equation:

$$\Omega^4 + B\Omega^2 + C = 0$$

four solutions $\longrightarrow$ **only two are real**:

$$\Omega_{1,2} = \left[\frac{1}{2}\left(-B \mp \sqrt{B^2 - 4C}\right)\right]^{1/2}$$

- Ω_1: wobbling frequency of the even-A core **R**
- Ω_2: wobbling frequency of the odd-nucleon **j**
- **Two wobbling phonon numbers:** n_{w_1} **and** n_{w_2}

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Energy spectrum

Spectra of odd-A nuclei within $\mathbf{W}_1$

$$E_{I,n_1,n_2} = \epsilon_j + \mathcal{H}^I_{\text{min}} + \mathcal{F}^I_{n_{w_1} n_{w_2}}$$

- Phonon factor:

$$\mathcal{F}^I_{n_{w_1} n_{w_2}} = \hbar\Omega^I_1 \left(n_{w_1} + \frac{1}{2}\right) + \hbar\Omega^I_2 \left(n_{w_2} + \frac{1}{2}\right)$$

- $\mathcal{H}^I_{\text{min}}$ - Classical Energy Function taken in its minimum point: $p_0 = (0, I; 0, j)$.
- ϵ_j - single-particle energy

A new interpretation for TSD1 and TSD2

Previous models

$TSD1 =$ zero-phonon wobbling band
$TSD2 =$ one-phonon wobbling band...

Redefinition

$TSD1$ and $TSD2$ are **Signature Partner Bands** (in favor of Ultimate Cranker calculations, *Jensen, 2004*).

$$\left(\alpha_{fav} = +\frac{1}{2}\right) : \{TSD1\} \equiv \{[0^+, 2^+, 4^+, \ldots] \otimes [j^\pi = 13/2^+]\}$$
$$\left(\alpha_{unfav} = -\frac{1}{2}\right) : \{TSD2\} \equiv \{[1^+, 3^+, 5^+, \ldots] \otimes [j^\pi = 13/2^+]\}$$

$TSD4$ is a **ground-state wobbling band**, $\pi(h_{9/2})$ configuration.

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

A new band structure for ^{163}Lu

Band	Spins	π	α	$\pi(I_j)$	$\mathbf{W_0}$: $\mathscr{C}+\mathscr{Q}_p$	$\mathbf{W_1}$: $\mathscr{C}+\mathscr{Q}_p$
TSD1	$13/2, 17/2\dots 97/2$	$+$	$+1/2$	$\pi(i_{13/2})$	$0^+, 2^+, 4^+, \dots$	$0^+, 2^+, 4^+, \dots$
TSD2	$27/2, 31/2\dots 91/2$	$+$	$-1/2$	$\pi(i_{13/2})$	$\mathrm{TSD1}+1\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$
TSD3	$33/2, 37/2\dots 85/2$	$+$	$+1/2$	$\pi(i_{13/2})$	$\mathrm{TSD1}+2\Gamma^\dagger$	$\mathrm{TSD2}+\Gamma^\dagger$
TSD4	$47/2, 51/2\dots 83/2$	$-$	$-1/2$	$\pi(h_{9/2})$	$\mathrm{TSD1}+3\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$

Bands	n_{w_1}	n_{w_2}	$\mathcal{F}^I_{n_{w_1} n_{w_2}}$	I_0	I_t
TSD1	0	0	$\mathcal{F}^I_{00}=\frac{1}{2}\left(\Omega^I_1+\Omega^I_2\right)$	$13/2^+$	$97/2^+$
TSD2	**0**	**0**	$\mathcal{F}^I_{00}=\frac{1}{2}\left(\Omega^I_1+\Omega^I_2\right)$	$27/2^+$	$91/2^+$
TSD3	1	0	$\mathcal{F}^{I-1}_{10}=\frac{3}{2}\Omega^{I-1}_1+\frac{1}{2}\Omega^{I-1}_2$	$33/2^+$	$85/2^+$
TSD4	**0**	**0**	$\mathcal{F}^I_{00}=\frac{1}{2}\left(\Omega^I_1+\Omega^I_2\right)$	$47/2^-$	$83/2^-$

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Extension to $A \approx 160$ mass region

The model was further applied to the other Lu isotopes.

^{161}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$21/2^+, 25/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$4^+, 6^+, 8^+ \ldots$	$(0,0)$	$21/2$
TSD2	$31/2^+, 35/2^+, \ldots, 79/2^+$	$j^\pi = 13/2^+$	$9^+, 11^+, 13^+ \ldots$	$(0,0)$	

^{165}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \ldots$	$(0,0)$	$25/2$
TSD2	$35/2^+, 39/2^+, \ldots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \ldots$	$(0,0)$	
TSD3	$41/2^+, 45/2^+, \ldots, 81/2^+$	$j^\pi = 13/2^+$	$\mathrm{TSD2} + \Gamma^\dagger$	$(1,0)$	

^{167}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \ldots$	$(0,0)$	$25/2$
TSD2	$35/2^+, 39/2^+, \ldots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \ldots$	$(0,0)$	

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$\mathbf{W}_1$ — Numerical Results

Free parameters in the model $\rightarrow \mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

Fitting procedure

$$\chi^2 = \frac{1}{N_T} \sum_i \frac{\left(E_{\text{exp}}^{(i)} - E_{\text{th}}^{(i)}\right)^2}{E_{\text{exp}}^{(i)}}$$

^{163}Lu-TSD4: separate fitting procedure (different nucleon configuration)

Isotope	Bands	$\mathcal{I}_1$ $[\hbar^2/\text{MeV}]$	$\mathcal{I}_2$ $[\hbar^2/\text{MeV}]$	$\mathcal{I}_3$ $[\hbar^2/\text{MeV}]$	V [MeV]	γ [°]	n.o.s	E_{rms} [MeV]
^{161}Lu	TSD1-2	87.555	2.773	22.744	2.933	20	29	0.168
^{163}Lu	TSD1-3	63.2	20	10	3.1	17	52	0.264
	TSD4	67	34.5	50	0.7	17	10	0.057
^{165}Lu	TSD1-3	77.295	16.184	4.399	1.673	20	42	0.125
^{167}Lu	TSD1-2	87.032	10.895	3.758	8.167	19.48	30	0.165

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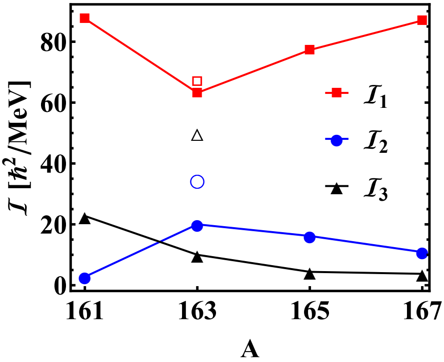
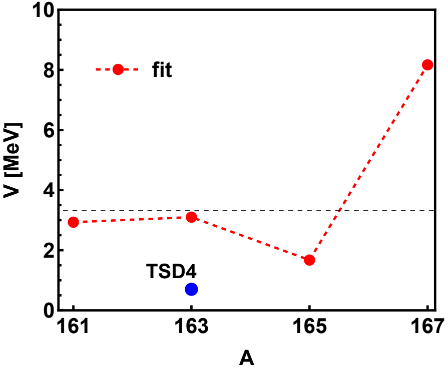
Graphical representation for MOIs and V





$V \approx 3$ MeV $\rightarrow$ agreement with other calculations (*Tanabe, 2017*)

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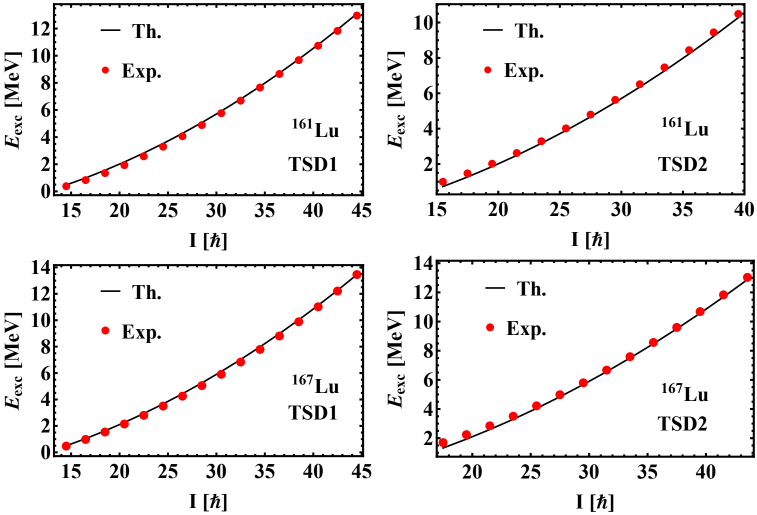
Excitation Energies — ^{161,167}Lu



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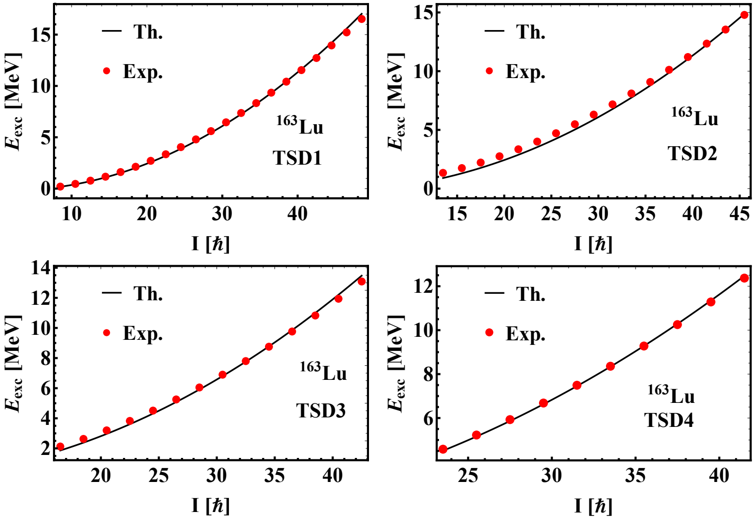
Excitation Energies — ^{163}Lu



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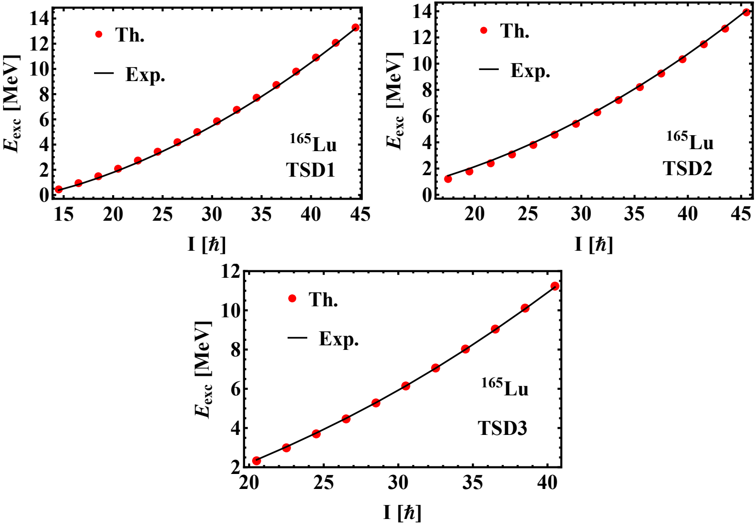
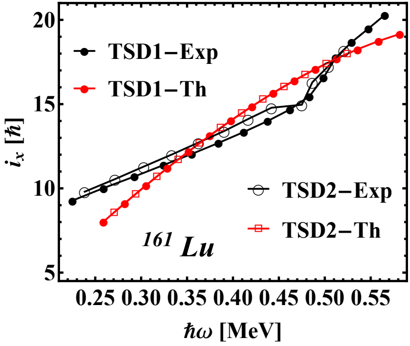
Excitation Energies — ^{165}Lu

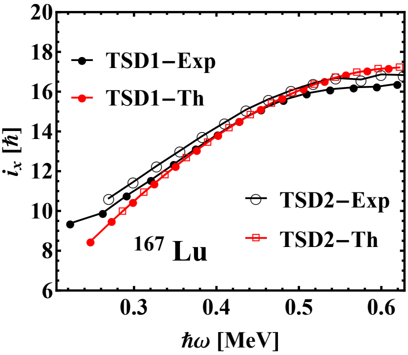


A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Alignment — ^{161,167}Lu

$$i_x = I - I_{\text{ref}} \ ,$$
$$I_{\text{ref}} = \mathcal{I}_0\omega + \mathcal{I}_1\omega^3 \ .$$

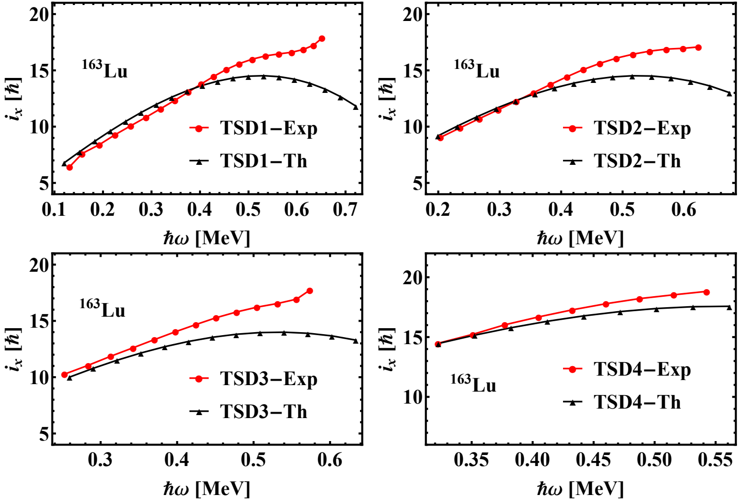




Harris parameters (*Harris, 1965*): $\mathcal{I}_0 = 30\ \hbar^2\text{MeV}^{-1}$, $\mathcal{I}_1 = 40\ \hbar^2\text{MeV}^{-3}$

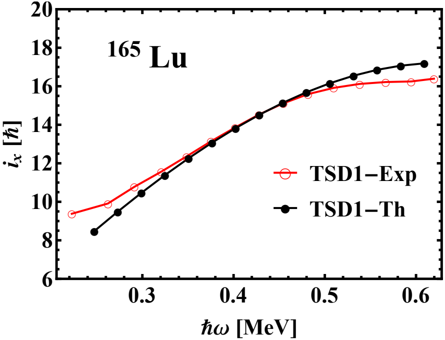
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

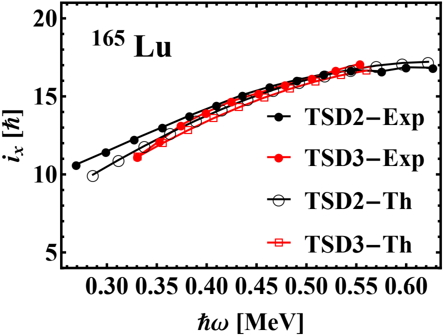
Alignment — ^{163}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Alignment — ^{165}Lu

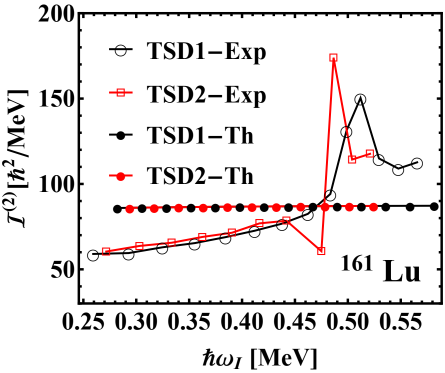


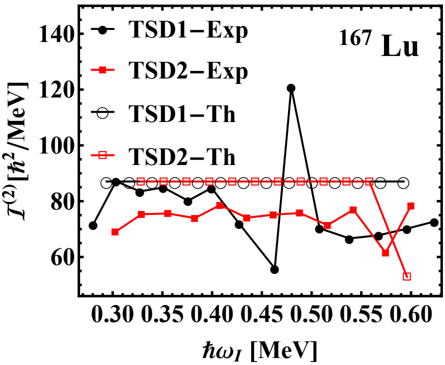


A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{161,167}Lu

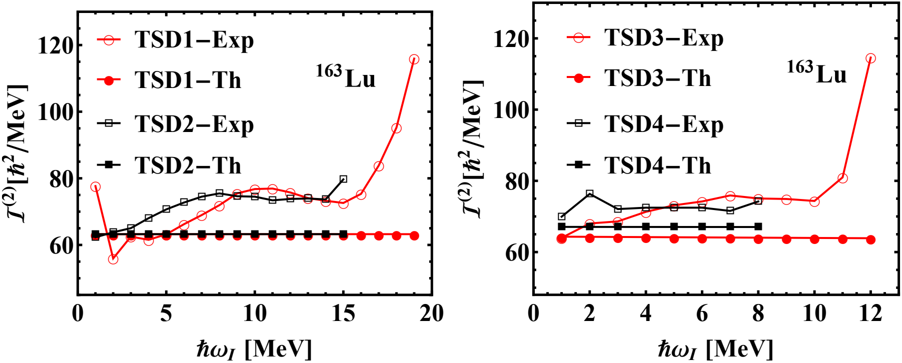
$$\mathcal{I}^{(2)}(I) = \hbar \frac{dI_x}{d\omega} = \hbar^2 \left(\frac{d^2 E}{dI_x^2} \right)^{-1}$$





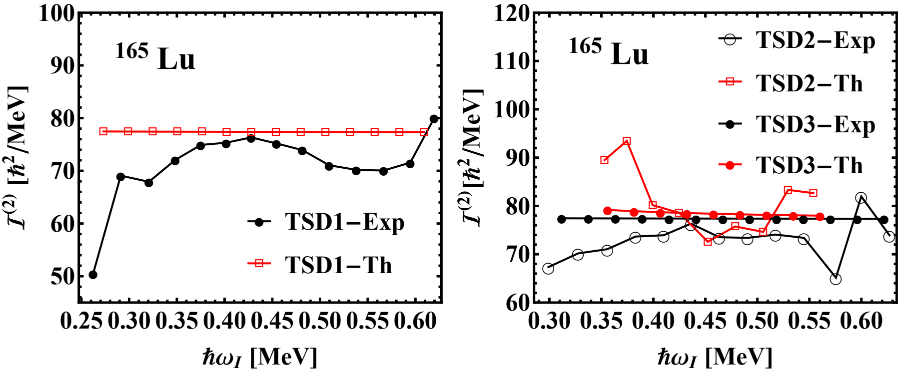
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{163}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{165}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Electromagnetic Calculations

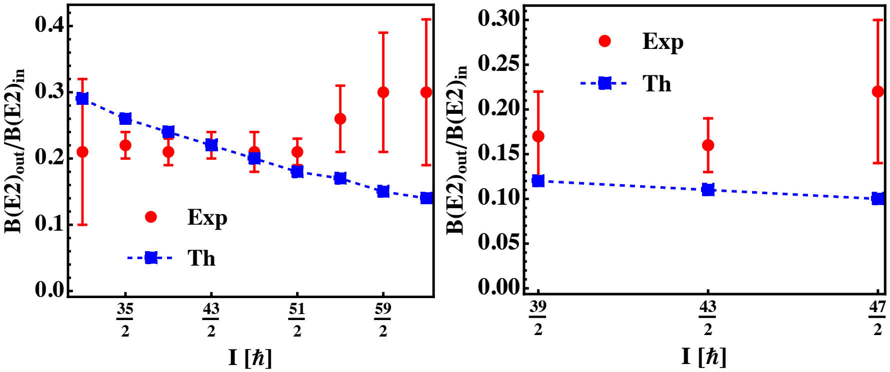


Figure: E2 Branching ratio. **Left**: ^{163}Lu (TSD2) **Right**: ^{165}Lu (TSD2).

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Electromagnetic Calculations II

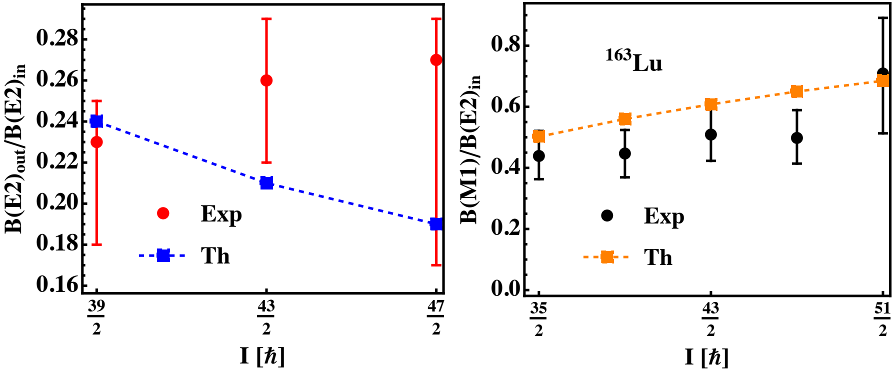


Figure: **Left**: E2 Branching ratio in ^{167}Lu (TSD2). **Right**: The ratio $B(M1)/B(E2)_{\text{in}}$ for states $TSD2 \to TSD1$ (in units of $\mu_N^2/(\mathrm{e}^2 b^2)$).

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

$\mathbf{W}_1$ — Remarks

Characteristics

- + Full semi-classical description (TDVE) with good numerical results
- + Deformation parameters are self-consistent (agree with exp. values)
- − separate fit for TSD4 (different nucleonic configuration)
- − Two sets of MOIs for ^{163}Lu

Onset of a redesign

Start of $\mathbf{W}_2$ formalism in Chapter 5

Fresh-Up 2: $\mathbf{W}_2$

Novel description of ^{163}Lu

- All four bands in ^{163}Lu described by the same triaxial core + odd-particle coupling $\longrightarrow \mathcal{Q}_1 = \pi(i_{13/2})$
- The adopted wave-function admits solutions of both **positive and negative parity**. Parity operator: $\mathcal{P} = \mathrm{e}^{-i\pi\mathbf{J}_2}C$:

$$\mathcal{P}\Psi(r,\varphi;t,\psi) = \Psi(r,\varphi+\pi;t,\psi+\pi)\ ,$$
$$\mathcal{H}(r,\varphi+\pi;t,\psi+\pi) = \mathcal{H}(r,\varphi;t,\psi)\ ,$$
$$\Psi(r,\varphi+\pi;t,\psi+\pi) = \pm\Psi(r,\varphi;t,\psi).$$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New band structure in ^{163}Lu

$$
\begin{aligned}
E_{I,0,0}^{\text{TSD1}} &= \epsilon_{13/2} + \mathcal{H}_{\text{min}}^{I} + \mathcal{F}_{00}^{I}\ ,\ \ I^{\pi} = 13/2^{+}, 17/2^{+}, 21/2^{+} \ldots\ , \\
E_{I,0,0}^{\text{TSD2}} &= \epsilon_{13/2}^{1} + \mathcal{H}_{\text{min}}^{I} + \mathcal{F}_{00}^{I}\ ,\ \ I^{\pi} = 27/2^{+}, 31/2^{+}, 35/2^{+} \ldots\ , \\
E_{I,1,0}^{\text{TSD3}} &= \epsilon_{13/2} + \mathcal{H}_{\text{min}}^{I-1} + \mathcal{F}_{10}^{I-1}\ ,\ \ I^{\pi} = 33/2^{+}, 37/2^{+}, 41/2^{+} \ldots\ , \\
E_{I,0,0}^{\text{TSD4}} &= \epsilon_{13/2}^{2} + \mathcal{H}_{\text{min}}^{I} + \mathcal{F}_{00}^{I}\ ,\ \ I^{\pi} = 47/2^{-}, 51/2^{-}, 55/2^{-} \ldots .
\end{aligned}
$$

Band	n_s	$\mathbf{j}_{\mathcal{Q}}$	$\mathbf{R}_{\mathscr{C}}$ - Sequence	$\mathbf{I}$ - Sequence	Coupling
TSD1	21	$\mathcal{Q}_1$	$\mathscr{C}_1 = 0^+, 2^+, 4^+, \ldots$	$13/2^+, 17/2^+, 21/2^+, \ldots$	$\mathscr{C}_1 + \mathcal{Q}_1$
TSD2	17	$\mathcal{Q}_1$	$\mathscr{C}_2^+ = 1^+, 3^+, 5^+, \ldots$	$27/2^+, 31/2^+, 35/2^+, \ldots$	$\mathscr{C}_2^+ + \mathcal{Q}_1$
TSD3	14	$\mathcal{Q}_1$	1-phonon exc.	$33/2^+, 37/2^+, 41/2^+, \ldots$	
TSD4	11	$\mathcal{Q}_1$	$\mathscr{C}_2^- = 1^-, 3^-, 5^-, \ldots$	$47/2^-, 51/2^-, 55/2^-, \ldots$	$\mathscr{C}_2^- + \mathcal{Q}_1$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New results for ^{163}Lu

Model requires **a unique set of parameters**: $\mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

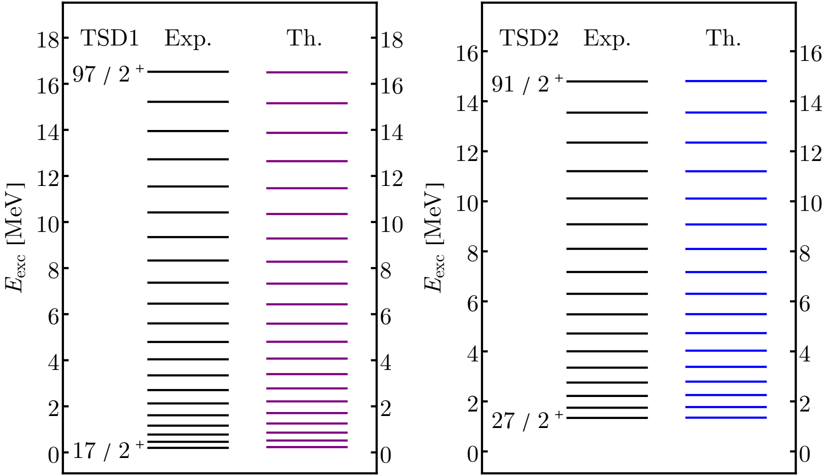
$\mathcal{I}_1$ [$\hbar^2$/MeV]	$\mathcal{I}_2$ [$\hbar^2$/MeV]	$\mathcal{I}_3$ [$\hbar^2$/MeV]	γ [deg.]	V [MeV]
72	15	7	22	2.1

Remarks

- **+** overall $E_{RMS} \approx 79$ keV: **first semi-classical description for a nucleus with deviations smaller than** 100 **keV**.
- **+** γ in agreement with exp. value ($\gamma_{exp} = 20^\circ$, *Jensen, 2004*)
- **?** slight *decrease* of V (breaking of parity symmetry quenches the quadrupole deformation)
- **−** $\epsilon^{\mathbf{1}}_{\mathbf{13/2}}$ and $\epsilon^{\mathbf{2}}_{\mathbf{13/2}}$ agree with microscopic calculations

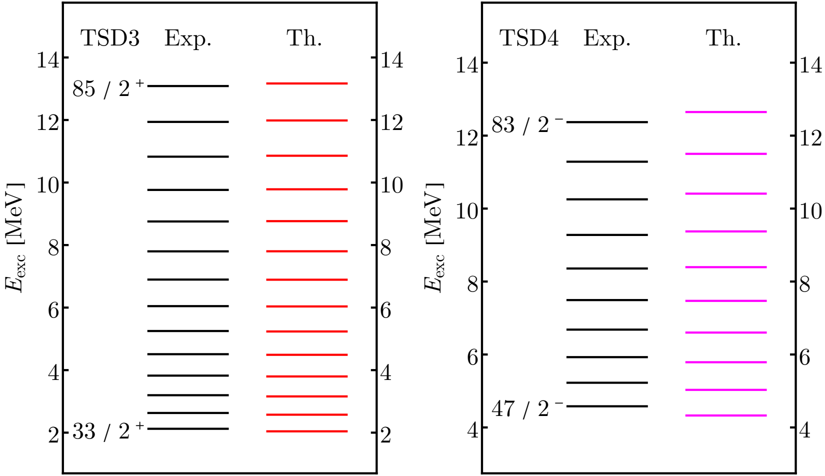
R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

Energy spectrum



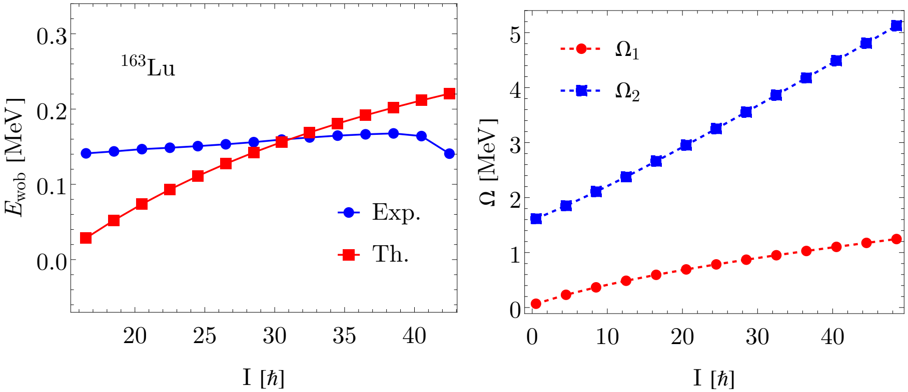
R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Energy spectrum II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

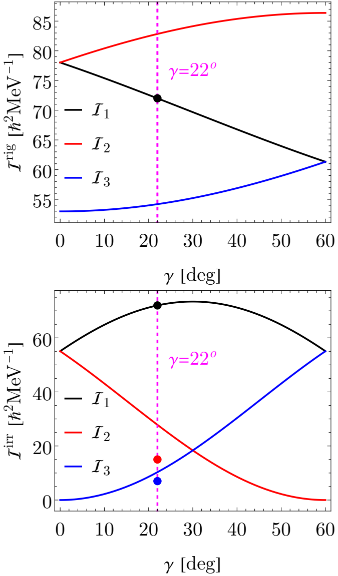
Wobbling Energies

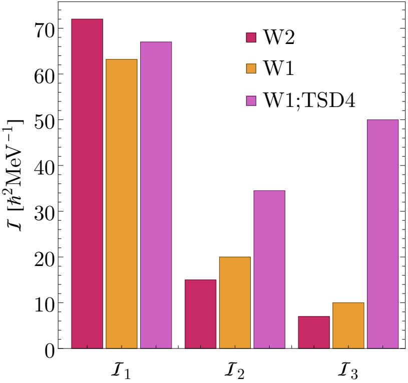


The wobbling energy (**left**) and the two wobbling frequencies (**right**) for ^{163}Lu. **Decreasing trend of E_{wob} in agreement with arguments of Frauendorf 2014.**

R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Moments of inertia for $\mathbf{W}_2$





$\mathbf{W}_2$: *hydrodynamical character* of the triaxial nucleus.

Results presented at the International Conference NSP, 2023, Turkey.

Classical Energy Function

Angular momentum

Polar representation of the angular momentum and $\mathcal{H}$.

$$\mathbf{I} = \{I_1, I_2, I_3\} \equiv \{x_1, x_2, x_3\} \ ,$$
$$x_1 = I\sin\theta\cos\varphi \ , \ x_2 = I\sin\theta\sin\varphi \ , \ x_3 = I\cos\theta .$$

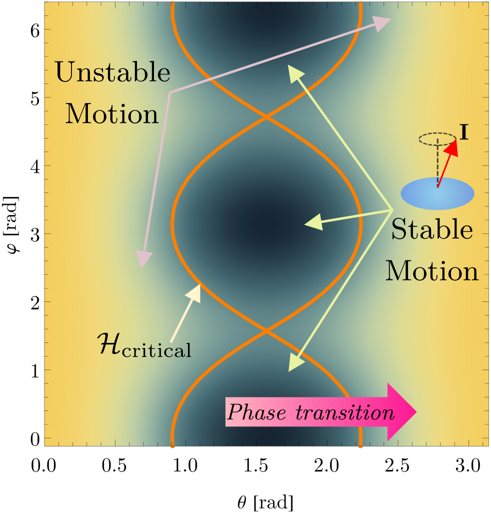
$$\mathcal{H}\ \Big|_{p_0} = I\left(I-\frac{1}{2}\right)\sin^2\theta\cdot\mathcal{A}_\varphi - 2A_1 Ij\sin\theta + T_{\text{core}} + T_{\text{sp}} \ ,$$
$$\mathcal{A}_\varphi = A_1\cos^2\varphi + A_2\sin^2\varphi - A_3 \ ,$$
$$T_{\text{core}} = \frac{I}{2}(A_1+A_2) + A_3 I^2 \ ,$$
$$T_{\text{s.p.}} = \frac{j}{2}(A_2+A_3) + A_1 j^2 - V\frac{2j-1}{j+1}\sin\left(\gamma+\frac{\pi}{6}\right) .$$

CEF — Stability Regions

Minimal point	θ [rad]	φ [rad]	A_k ordering
m_1	$\pi/2$	0	$A_3 > A_2 > A_1$
m_2	$\pi/2$	π	$A_3 > A_2 > A_1$
m_3	$\pi/2$	2π	$A_3 > A_2 > A_1$

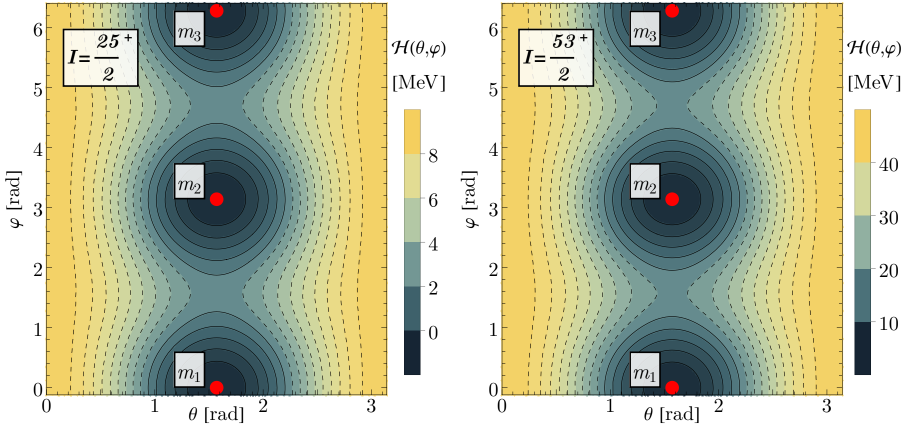
Semi-classical feature

This is the first geometrical description of the *wobbling stability* for an odd-mass nucleus.



Polar representation of $\mathcal{H}$

Figure: ^{163}Lu TSD1



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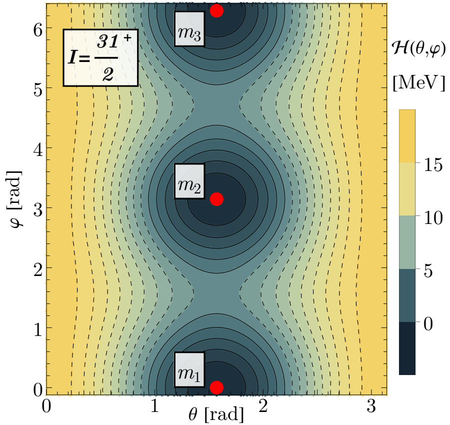
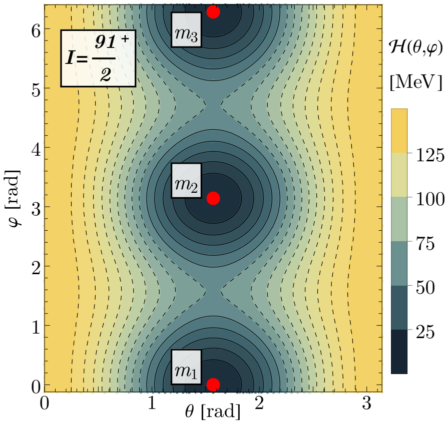
Polar representation of $\mathcal{H}$ II

Figure: ^{163}Lu TSD2





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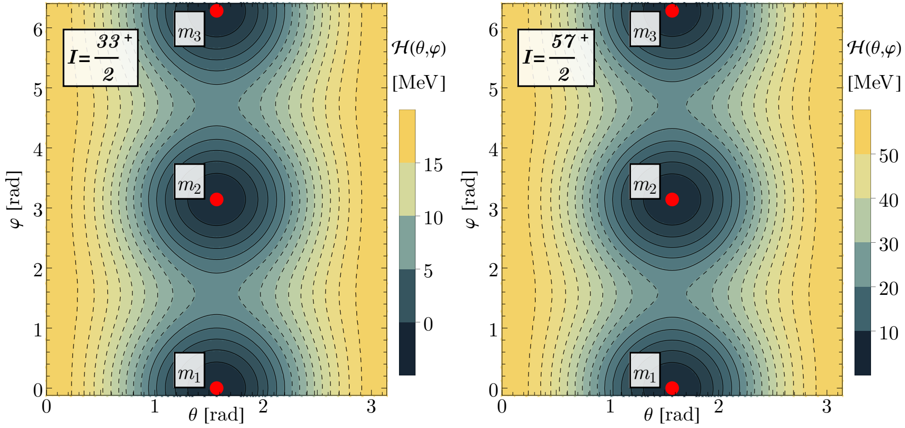
Polar representation of $\mathcal{H}$ III

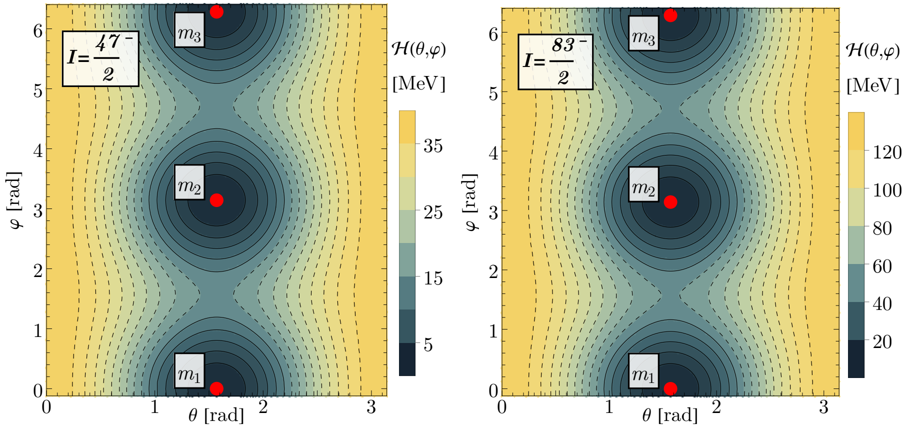
Figure: ^{163}Lu TSD3



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of $\mathcal{H}$ IV

Figure: ^{163}Lu TSD4



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

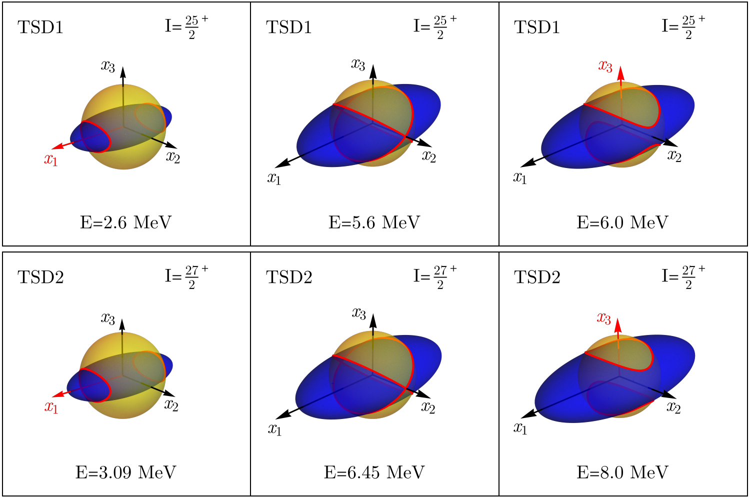
3D interpretation of the WM

- Formalism $\mathbf{W}_2$ gives a 3D interpretation of the nuclear wobbling motion
- **Classical Trajectories**: intersection curves between the **triaxial energy** and the **total angular momentum**

$$I^2 = x_1^2 + x_2^2 + x_3^2,$$

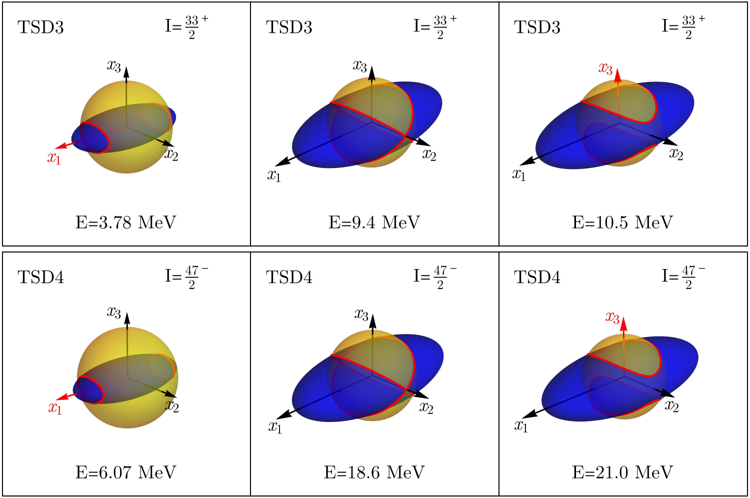
$$E = \left(1 - \frac{1}{2I}\right) A_1 x_1^2 + \left(1 - \frac{1}{2I}\right) A_2 x_2^2 + \left[\left(1 - \frac{1}{2I}\right) A_3 + A_1 \frac{j}{I}\right] x_3^2 - \\ - I\left(I - \frac{1}{2}\right) A_3 - 2A_1 Ij + {\color{red}T_{\text{rot}}} + {\color{blue}T_{\text{sp}}}.$$

^{163}Lu — Classical trajectories



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

^{163}Lu — Classical trajectories II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Outline

New Boson Method for odd-mass nuclei

Rotational Hamiltonian

$$\hat{H}_{\text{rot}} = AH' + H_{sp} + H_{\text{crank}},$$

$$H' = a_1\left(\hat{I}_+^2 + \hat{I}_-^2\right) + a_2\left(\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+\right) + a_3\hat{I}_1 \ ,$$

$$H_{sp} = \sum_{k=1}^{2} A_k\hat{j}_k^2 \ , \ H_{\text{crank}} = A_1 I^2 - A_2 j_2 I \ .$$

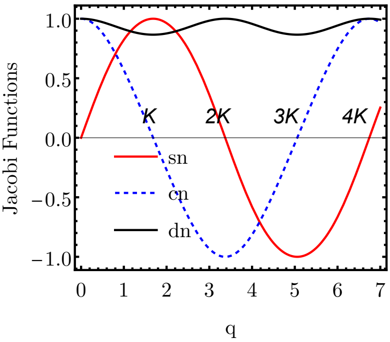
- the triaxial rigid rotor is constrained to move around the 1-axis.
- adopted Frozen-Alignment approximation: $\mathbf{j} = (j\cos\theta, j\sin\theta, 0)$ (*Frauendorf, 2014*)
- a_1, a_2, a_3 inertial properties of the nucleus (i.e., $A_{1,2,3}$)

New angular momentum representation

First boson expansion of this kind in literature:

$$\hat{I}_+ = i\,\frac{cb^\dagger - db^\dagger}{sb^\dagger}\left(I + Icb^\dagger db^\dagger - sb^\dagger b\right)\ ,$$
$$\hat{I}_- = i\,\frac{cb^\dagger + db^\dagger}{sb^\dagger}\left(I - Icb^\dagger db^\dagger + sb^\dagger b\right)\ ,$$
$$\hat{I}_1 = Icb^\dagger db^\dagger - sb^\dagger b\ .$$

- s, c, d : **Jacobi Elliptic Functions** (*Jacobi, 1829*)
- $b, b^\dagger, [b, b^\dagger] = 1$: boson operators (*Bargmann, 1962*)



Elliptic Potential

New Hamiltonian

$$H' = -\frac{d^2}{dq^2} - 2v_0 s\frac{d}{dq} + I(I+1)s^2k^2 + 2v_0 cdI \ ,$$

separated Kinetic and Potential terms in the *Schrodinger Equation*:

$$\left[\frac{d^2}{dq^2} + V(q)\right]\Psi = E\Psi.$$

$$V(q) = \left[I(I+1)k^2 + v_0^2\right]s^2 + (2I+1)v_0 cd = V(-q) \ .$$

Results were presented at the **International Conference TIM-22** (Timisoara) and **World Quantum Day 2023** (IFIN-HH)

Elliptic potential

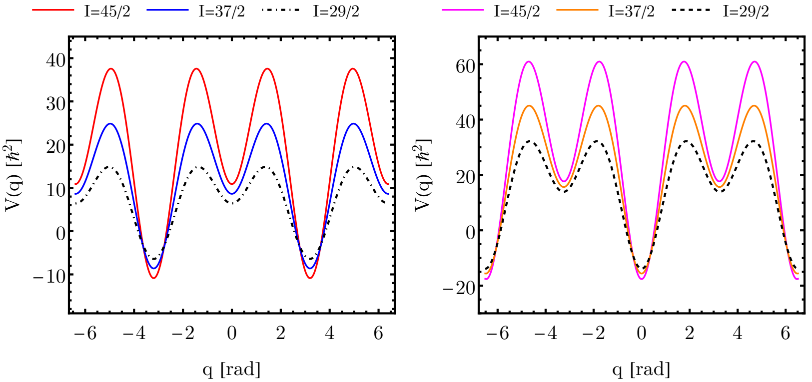


Figure: The elliptic potential as function of the coordinate q with $\theta = -119^\circ$ (**left**) and $\theta = 61^\circ$ (**right**).

A.A. Raduta, C.M. Raduta, **R Poenaru**, Journal of Physics G 48, 2020.

Results for ^{135}Pr

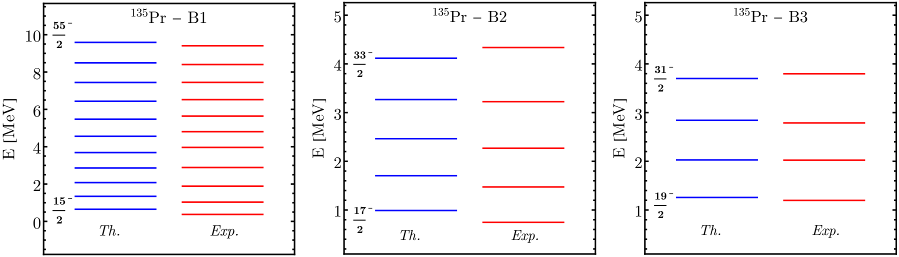


Figure: The excitation energies in ^{135}Pr. Exp data: *Sensharma, 2019*.

$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	θ [degrees]	N.o. states	RMS [MeV]
91	9	51	-119	20	0.174

A.A. Raduta, C.M. Raduta, **R Poenaru**, Journal of Physics G 48, 2020.

Outline

General Conclusions

- Developed **three** semi-classical models that describe wobbling motion in odd-A nuclei ($\mathbf{W}_1$, $\mathbf{W}_2$, and the boson method applied to ^{135}Pr).
- Showed that it is possible to treat the motion of the core and the odd nucleon separately.
- Obtained realistic results concerning wobbling energies and other quantities.
- Special attention to the geometrical interpretation of the wobbling motion was given for ^{163}Lu.

Original Contributions

- Research period: 2018-2023
- **7** ISI papers (2 RJP, 1 IJMPE, 2 PRC, 2 JPG)
- **39** Citations
- **Total IF:** 17.225 ; **Total AIS:** 4.684
- **6** Oral and **2** Poster presentations at international conferences

Thank you for your attention