A Systematic Description of the Wobbling Motion in Odd-Mass Nuclei Within a Semi-Classical Formalism

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A presentation for the degree of Doctor of Philosophy

May 15, 2023

Outline

Aim

Research Objectives

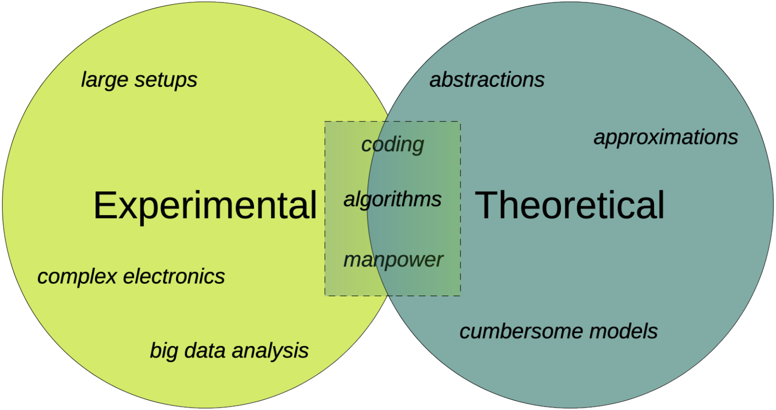
- Extend the current interpretation of the **Nuclear Triaxiality** in the context of its unique fingerprint: **Wobbling Motion**.
- Adopt a framework that is as close as possible to a **classical picture**.

Objectives exclusive to the thesis

- Enough context towards a better understanding of the underlying concepts, methods, and results.
- create a completely *open-source* project.

Motivation

- **Nuclear Triaxiality** recently became a *hot topic* within the scientific community.
- Identifying nuclei with triaxial deformations represents a real **experimental** and **theoretical** challenge.



Outline

Nuclear Deformation

Nuclear shapes

Most generally described in terms of the **nuclear radius**:

$$R(\theta, \varphi) = R_0 \left(1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} Y_{\lambda}^{\mu}(\theta, \varphi) \right)$$

Quadrupole deformations $\lambda = 2$

- For us: Most relevant modes are the **quadrupole vibrations** $\lambda = 2$ $\Longrightarrow$ *Play a crucial role in the rotational spectra of nuclei:*
- $\alpha_{2\mu}$ reduced to only two *deformation parameters*: β_2 (**eccentricity**) and γ (**triaxiality**) (*Bohr and Mottelson, 1969*).

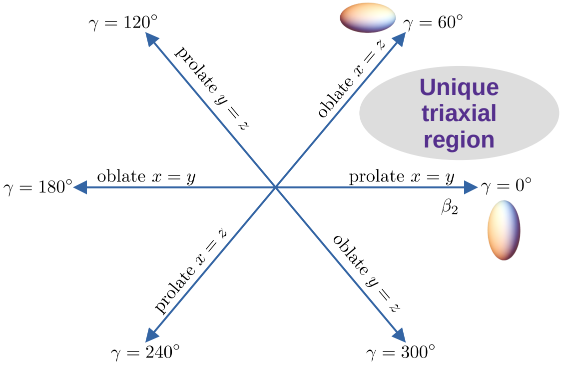
Axial shapes

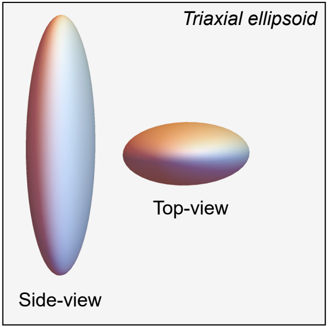
- Most of the nuclei are either **spherical** or **axially symmetric** in their ground-state *(Budaca, 2018)*.
- Moments of inertia: $\mathcal{I}_{1,2,3}$: two are equal, one is different.

Figure: **spherical:** $\beta_2 = 0$ **prolate:** $\beta_2 > 0$ **oblate:** $\beta_2 < 0$. $(\gamma = 0°)$.

Non-axial shapes

- The triaxiality parameter $\gamma \neq 0°$: departure from axial symmetry.
- Moments of inertia: $\mathcal{I}_1 \neq \mathcal{I}_2 \neq \mathcal{I}_3$.





Outline

Fingerprints of Triaxiality

Evidence

- Currently, there are **only two** well-established phenomena uniquely attributed to triaxial deformation.
  1. **Wobbling Motion** - WM (*Bohr and Mottelson, 1970s*)
  2. Chiral Motion - χM (*Frauendorf, 1997*)
- These two can be measured/detected experimentally.

Goal

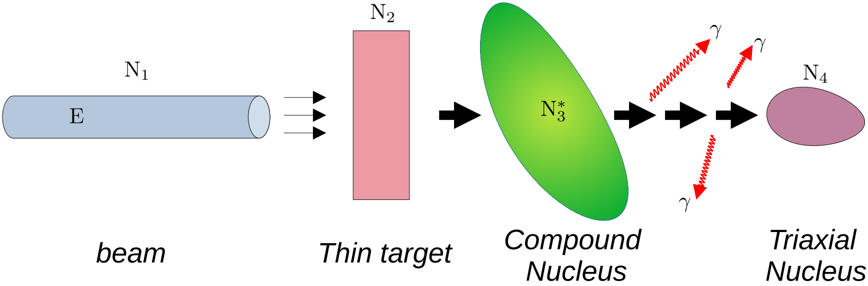
Describe the elusive character of Wobbling Motion in the context of nuclear triaxiality.

Probing triaxiality in nuclei

Triaxial nuclei can be observed/obtained in several experiments:

- Nuclear fission: $A \rightarrow B + C$
- Nuclear fusion: $X + Y \rightarrow Z$
- **Fusion-evaporation reactions**: Long-lived + enhanced deformation

$$\mathit{Beam}(N_1, E) + \mathit{Target}(N_2) \longrightarrow N_3^* \rightarrow \cdots \rightarrow \mathit{triaxial}(N_4)$$



Nuclear facilities

Figure: Gammasphere detector, ANL-ATLAS USA. *Source: aps.org*

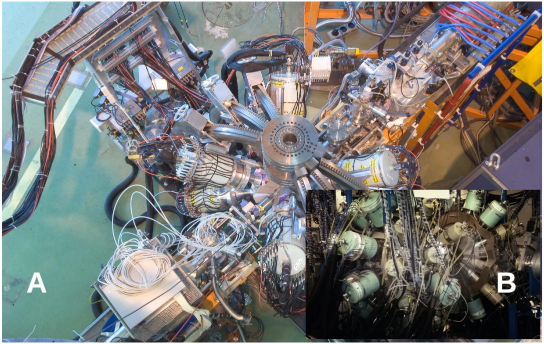
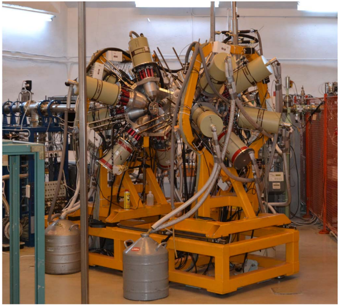


Figure: a) IDS detector, CERN. *Source: isolde.web.cern.ch* b) JUROGAM II, Finland. *Source: twitter.com*

High-Spin Physics @ IFIN-HH



Contents lists available at ScienceDirect

Nuclear Instruments and Methods in Physics Research A

journal homepage: www.elsevier.com/locate/nima

The ROSPHERE γ-ray spectroscopy array

D. Bucurescu[a], I. Căta-Danil[a], G. Ciocan[a], C. Costache[a], D. Deleanu[a], R. Dima[a], D. Filipescu[a,c], N. Florea[a], D.G. Ghiţă[a], T. Glodariu[a], M. Ivaşcu[a], R. Lică[a], N. Mărginean[a], R. Mărginean[a], C. Mihai[a,*], A. Negret[a], C.R. Niţă[a], A. Olăcel[a], S. Pascu[a], T. Sava[a], L. Stroe[a], A. Şerban[a,i], R. Şuvăilă[a], S. Toma[a], N.V. Zamfir[a,c], G. Căta-Danil[b], I. Gheorghe[c], I.O. Mitu[c], G. Suliman[c], C.A. Ur[c], T. Braunroth[d], A. Dewald[d], C. Fransen[d], A.M. Bruce[e], Zs. Podolyák[f], P.H. Regan[f,g], O.J. Roberts[h]

Figure: ROSPHERE, IFIN-HH. *Source: tandem.nipne.ro*

Wobbling Motion

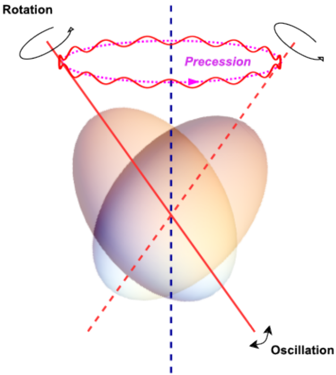
$\gamma \neq 0^\circ$

↓

MOI anisotropy

↓

main rotation around $\mathcal{J}_{\text{max}}$ is disturbed by the other two axes



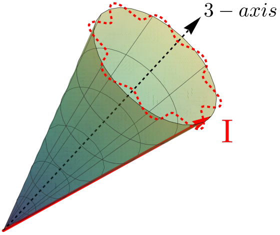
Wobbling Effect

- The **total angular momentum** of the nucleus **precesses** and **oscillates** around $\mathcal{J}_{\text{max}}$.

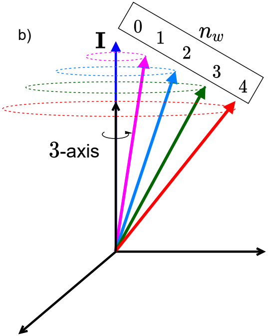
Wobbling Motion

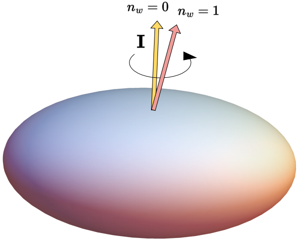
Harmonic oscillation

- Precession of **I** is affected by **rotational frequency** and/or **tilting**
- Tilting only by "specific" amount → **harmonic character** → **wobbling phonon**: $n_w = 0, 1, 2, \ldots$.

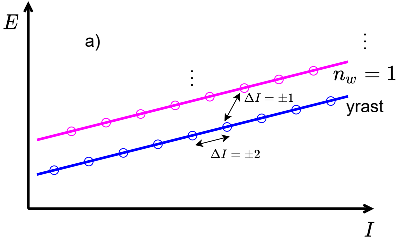


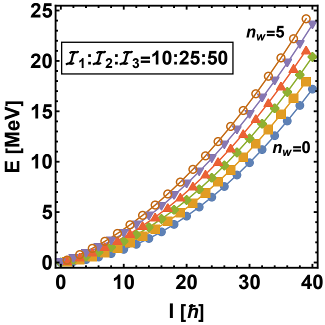
b)

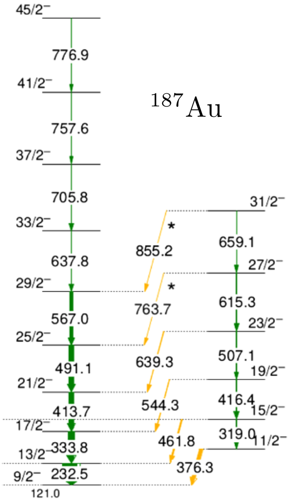




Wobbling Motion II



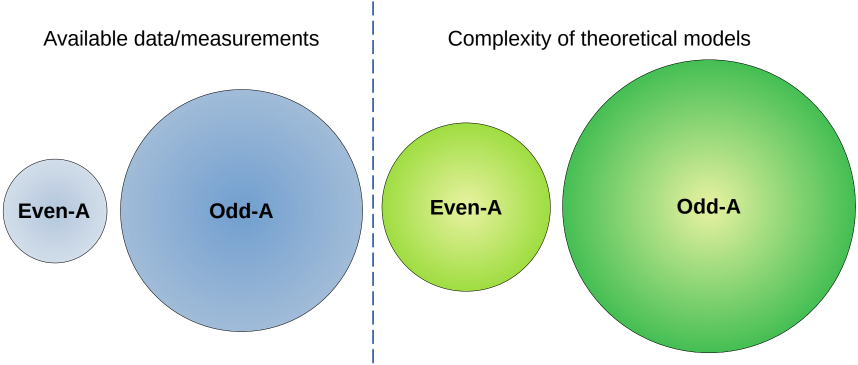




Sensharma, 2020.

Even-A vs. Odd-A Picture

- Predicted for even-A nuclei more than 50 years ago.
- First experimental evidence: ^{163}Lu (*Ødegård, 2001*).
- Current mass-regions for wobblers: $A \approx [130, 160, 180]$.



Wobbling Motion in ^{130}Ba

Experimental measurements show **two** wobbling bands (*Petrache et. al. 2019*).

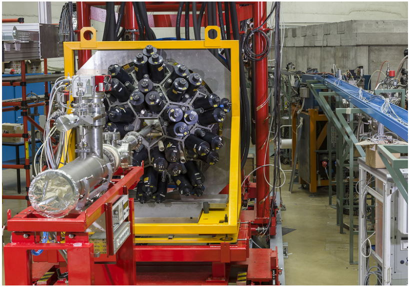
Harmonic formalism

Harmonic Approximation (*Bohr & Mottelson, 1969*):

$$E_{I,n_w} = A_3 I(I+1) + \hbar\omega_w \left(n_w + \frac{1}{2}\right),$$

$$A_3 = (2\mathcal{I}_3)^{-1}.$$

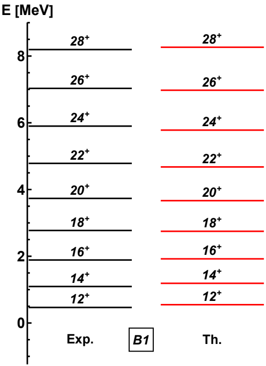
(rotational term + wobbling frequency)

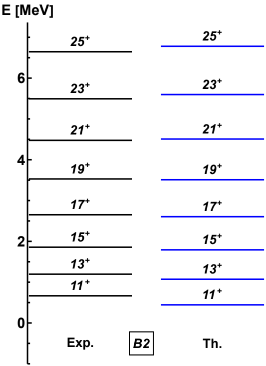


GALILEO, LNL, *Source: lnl.infn.it*

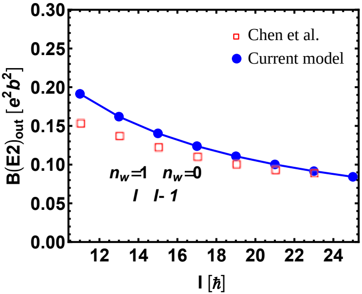
Fusion evaporation: ^{13}C beam of $E = 65$ MeV and ^{122}Sn target.

Results for ^{130}Ba





$\mathcal{P}_{\text{fit}}$			
$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	Unit
27	22	**43**	$\hbar^2\text{MeV}^{-1}$



Full description: Chapter 3 (Section 3.1.2)

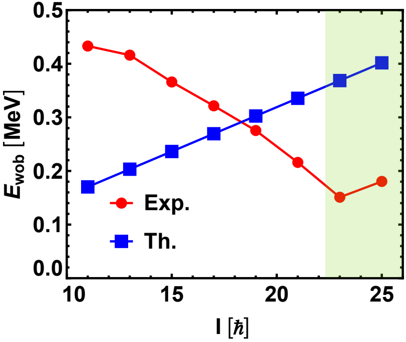
Results presented at the international conference NSP-2022, Turkey.

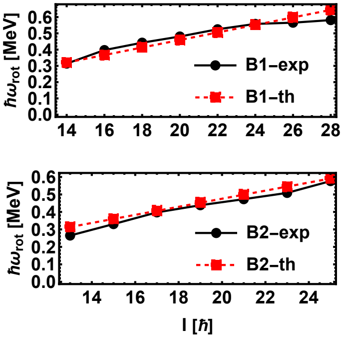
Results for ^{130}Ba II

Excitation energies vs. **Wobbling Energies**:

$$E_{\text{wob}}(I_{\text{even}}) = E_{I,n} - E_{I,0} \ ,$$

$$E_{\text{wob}}(I_{\text{odd}}) = E_{I,n} - \frac{1}{2}\left(E_{I-1,0} + E_{I+1,0}\right)$$





Results presented at the international conference NSP-2022, Turkey.

Outline

Starting Point

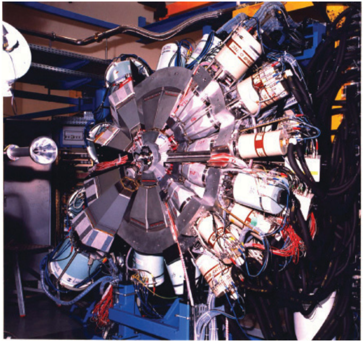
A. A. Raduta, **R. Poenaru**, L. Gr. Ixaru, PRC, 2017 + A. A. Raduta, **R. Poenaru**, Al. H. Raduta, JPG, 2018→$\mathbf{W}_0$ in the thesis.

Framework

- **First semi-classical description** for the ^{163}Lu, using the **Particle-Rotor-Model** (*Hamamoto, 2002.*) for an odd-mass nucleus in the $A \approx 160$ region.

PRM

An odd-nucleon moving in a quadrupole deformed mean field generated by an even-even triaxial core.



Euroball IV, Strasbourg, *Source: technologysi.stfc.ac.uk*

Fusion evaporation: ^{29}Si beam of $E = 152$ MeV and ^{139}La target.

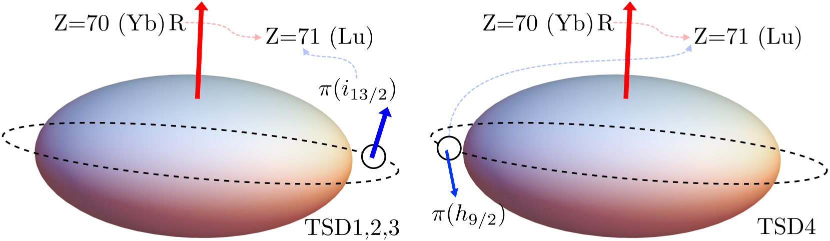
Fresh-Up 1: $\mathbf{W}_1$

Particle-Rotor Model Hamiltonian for an odd-A nucleus:

$$\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{sp}}\ ,\ \ \hat{H}_{\text{rot}} = \sum_{k=1}^{3} A_k(\hat{I}_k - \hat{j}_k)^2,$$

$$\hat{H}_{\text{sp}} = \epsilon_j + \frac{V}{j(j+1)}\left[\cos\gamma\left(3\hat{j}_3^2 - \mathbf{j}^2\right) - \sqrt{3}\sin\gamma\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right].$$

V - single-particle potential strength $\propto \beta_2$ (*Tanabe, 2017*)



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Variational Principle + Eqs. of Motion

Time-Dependent Variational Equation

$$\delta \int_0^t \langle \Psi_{IM;j} | \hat{H} - i\frac{\partial}{\partial t'} | \Psi_{IM;j} \rangle \, dt' = 0$$

$$\Psi_{\text{trial}} \equiv |\Psi_{IM;j}\rangle = \mathcal{N} e^{z\hat{I}_-} \cdot e^{s\hat{j}_-} |IMI\rangle \otimes |jj\rangle$$

- $(z, r, \varphi, \hat{I}, |IMI\rangle)$ - core (rotor **R**)
- $(s, f, \psi, \hat{j}, |jj\rangle)$ - single-particle (**j**)
- $\{z, s\}$ → **phase space coordinates**

Constant of motion:

$$\mathcal{H} \equiv \langle \Psi_{IM;j} | \hat{H} | \Psi_{IM;j} \rangle$$

Canonical equations of motion:

$$\mathcal{S}_1 : \frac{\partial \mathcal{H}}{\partial r} = \dot{\varphi} \, , \quad \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r}$$

$$\mathcal{S}_2 : \frac{\partial \mathcal{H}}{\partial f} = \dot{\psi} \, , \quad \frac{\partial \mathcal{H}}{\partial \psi} = -\dot{f}$$

The two sets of Hamilton equations are **the semi-classical description** of the initial quantal $\hat{H}$.

Wobbling frequency

Solving $\mathcal{S}_1$ and $\mathcal{S}_2$ leads to the algebraic equation:

$$\Omega^4 + B\Omega^2 + C = 0$$

four solutions $\longrightarrow$ **only two are real**:

$$\Omega_{1,2} = \left[\frac{1}{2}\left(-B \mp \sqrt{B^2 - 4C}\right)\right]^{1/2}$$

- Ω_1: wobbling frequency of the even-A core **R**
- Ω_2: wobbling frequency of the odd-nucleon **j**
- **Two wobbling phonon numbers:** n_{w_1} **and** n_{w_2}

A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Energy spectrum

Spectra of odd-A nuclei within $\mathbf{W}_1$

$$E_{I,n_1,n_2} = \epsilon_j + \mathcal{H}^I_{\text{min}} + \mathcal{F}^I_{n_{w_1} n_{w_2}}$$

- Phonon factor:

$$\mathcal{F}^I_{n_{w_1} n_{w_2}} = \hbar\Omega^I_1 \left(n_{w_1} + \frac{1}{2} \right) + \hbar\Omega^I_2 \left(n_{w_2} + \frac{1}{2} \right)$$

- $\mathcal{H}^I_{\text{min}}$ - Classical Energy Function taken in its minimum point: $p_0 = (0, I; 0, j)$.
- ϵ_j - single-particle energy

A new interpretation for TSD1 and TSD2

Previous models

$TSD1 =$ zero-phonon wobbling band
$TSD2 =$ one-phonon wobbling band...

Redefinition

$TSD1$ and $TSD2$ are **Signature Partner Bands** (in favor of Ultimate Cranker calculations, *Jensen, 2004*).

$$\left(\alpha_{fav} = +\frac{1}{2}\right) : \{TSD1\} \equiv \left\{\left[0^+, 2^+, 4^+, \ldots\right] \otimes [j^\pi = 13/2^+]\right\}$$
$$\left(\alpha_{unfav} = -\frac{1}{2}\right) : \{TSD2\} \equiv \left\{\left[1^+, 3^+, 5^+, \ldots\right] \otimes [j^\pi = 13/2^+]\right\}$$

$TSD4$: **ground-state wobbling band**, $\pi(h_{9/2})$ configuration.

A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

A new band structure for ^{163}Lu

Band	Spins	π	α	$\pi(I_j)$	$\mathbf{W_0}$: $\mathscr{C}+\mathscr{Q}_p$	$\mathbf{W_1}$: $\mathscr{C}+\mathscr{Q}_p$
TSD1	$13/2, 17/2\dots 97/2$	$+$	$+1/2$	$\pi(i_{13/2})$	$0^+, 2^+, 4^+, \dots$	$0^+, 2^+, 4^+, \dots$
TSD2	$27/2, 31/2\dots 91/2$	$+$	$-1/2$	$\pi(i_{13/2})$	$\mathrm{TSD1}+1\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$
TSD3	$33/2, 37/2\dots 85/2$	$+$	$+1/2$	$\pi(i_{13/2})$	$\mathrm{TSD1}+2\Gamma^\dagger$	$\mathrm{TSD2}+\Gamma^\dagger$
TSD4	$47/2, 51/2\dots 83/2$	$-$	$-1/2$	$\pi(h_{9/2})$	$\mathrm{TSD1}+3\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$

Bands	n_{w_1}	n_{w_2}	$\mathcal{F}^I_{n_{w_1} n_{w_2}}$	I_0	I_t	$\mathcal{Q}$
TSD1	0	0	$\mathcal{F}^I_{00}=\frac{1}{2}\left(\Omega^I_1+\Omega^I_2\right)$	$13/2^+$	$97/2^+$	$j^\pi=13/2^+ \overset{not}{\equiv} \mathcal{Q}_1$
TSD2	**0**	**0**	$\mathcal{F}^I_{00}=\frac{1}{2}\left(\Omega^I_1+\Omega^I_2\right)$	$27/2^+$	$91/2^+$	$j^\pi=13/2^+ \overset{not}{\equiv} \mathcal{Q}_1$
TSD3	1	0	$\mathcal{F}^{I-1}_{10}=\frac{3}{2}\Omega^{I-1}_1+\frac{1}{2}\Omega^{I-1}_2$	$33/2^+$	$85/2^+$	$j^\pi=13/2^+ \overset{not}{\equiv} \mathcal{Q}_1$
TSD4	**0**	**0**	$\mathcal{F}^I_{00}=\frac{1}{2}\left(\Omega^I_1+\Omega^I_2\right)$	$47/2^-$	$83/2^-$	$j^\pi=9/2^- \overset{not}{\equiv} \mathcal{Q}_2$

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Extension to $A \approx 160$ mass region

The model was further applied to the other Lu isotopes where wobbling motion has been observed.

^{161}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$21/2^+, 25/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$4^+, 6^+, 8^+ \ldots$	$(0,0)$	$21/2$
TSD2	$31/2^+, 35/2^+, \ldots, 79/2^+$	$j^\pi = 13/2^+$	$9^+, 11^+, 13^+ \ldots$	$(0,0)$	

^{165}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \ldots$	$(0,0)$	$25/2$
TSD2	$35/2^+, 39/2^+, \ldots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \ldots$	$(0,0)$	
TSD3	$41/2^+, 45/2^+, \ldots, 81/2^+$	$j^\pi = 13/2^+$	$\text{TSD2} + \Gamma^\dagger$	$(1,0)$	

^{167}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \ldots$	$(0,0)$	$25/2$
TSD2	$35/2^+, 39/2^+, \ldots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \ldots$	$(0,0)$	

A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

$\mathbf{W}_1$ — Numerical Results

Free parameters in the model $\rightarrow \mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

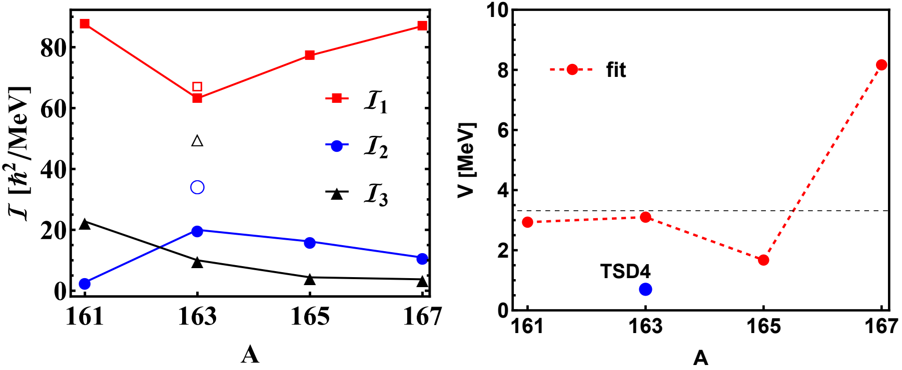
Fitting procedure

$$\chi^2 = \frac{1}{N_T} \sum_i \frac{\left(E_{\text{exp}}^{(i)} - E_{\text{th}}^{(i)}\right)^2}{E_{\text{exp}}^{(i)}}$$

^{163}Lu-TSD4: separate fitting procedure (different nucleon configuration)

Isotope	Bands	$\mathcal{I}_1$ $[\hbar^2/\text{MeV}]$	$\mathcal{I}_2$ $[\hbar^2/\text{MeV}]$	$\mathcal{I}_3$ $[\hbar^2/\text{MeV}]$	V [MeV]	γ [°]	n.o.s	E_{rms} [MeV]
^{161}Lu	TSD1-2	87.555	2.773	22.744	2.933	20	29	0.168
^{163}Lu	TSD1-3	63.2	20	10	3.1	17	52	0.264
	TSD4	67	34.5	50	0.7	17	10	0.057
^{165}Lu	TSD1-3	77.295	16.184	4.399	1.673	20	42	0.125
^{167}Lu	TSD1-2	87.032	10.895	3.758	8.167	19.48	30	0.165

Graphical representation for MOIs and V



$V \approx 3$ MeV $\rightarrow$ agreement with other calculations (*Tanabe, 2017*)

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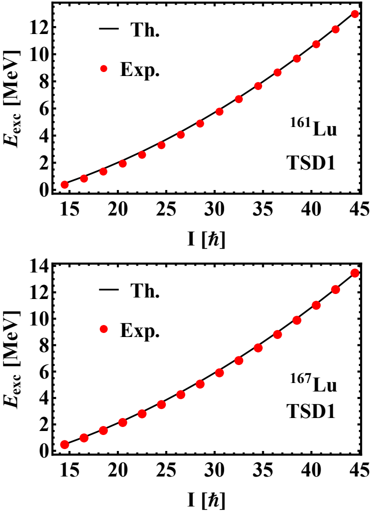
Excitation Energies — ^{161,167}Lu



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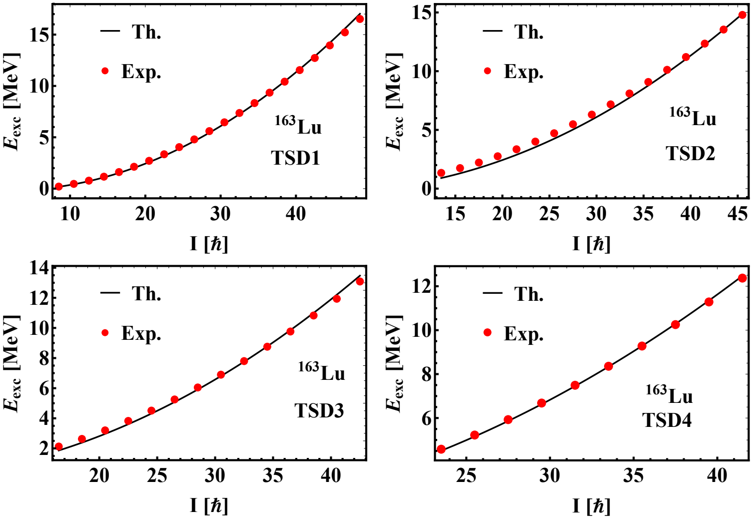
Excitation Energies — ^{163}Lu



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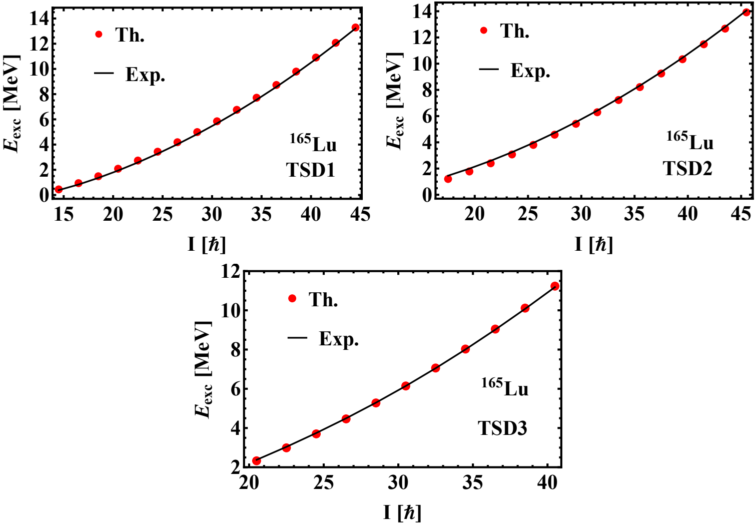
Excitation Energies — ^{165}Lu



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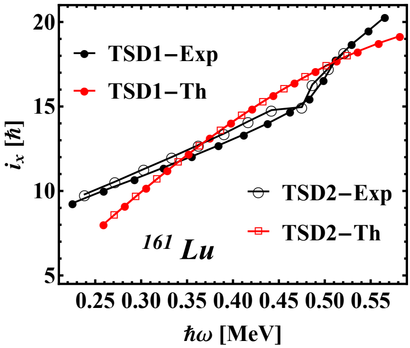
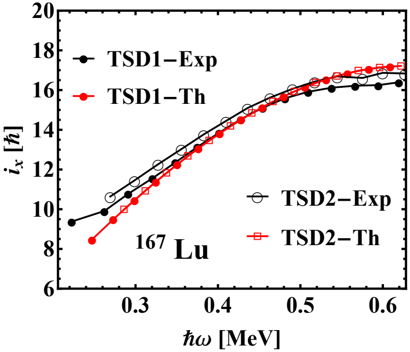
Alignment — ^{161,167}Lu

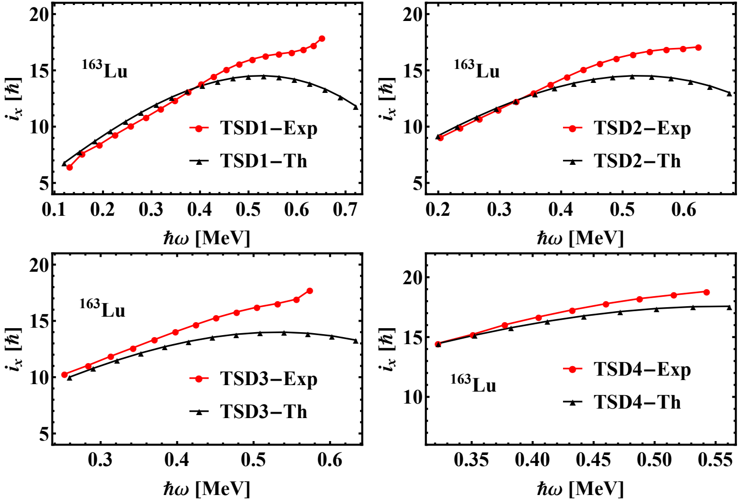
$$i_x = I - I_{\text{ref}} \ ,$$
$$I_{\text{ref}} = \mathcal{I}_0\omega + \mathcal{I}_1\omega^3 \ .$$





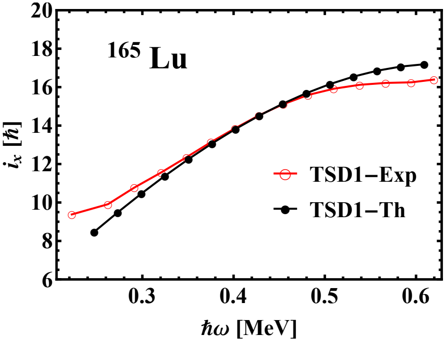
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

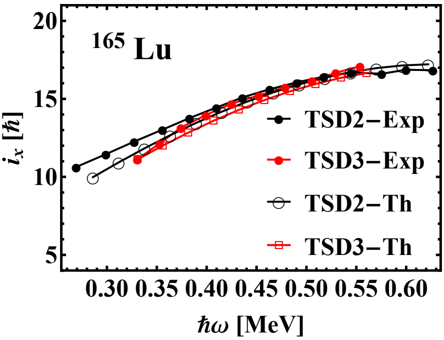
Alignment — ^{163}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Alignment — ^{165}Lu

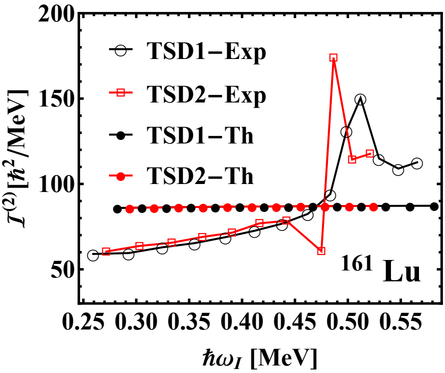


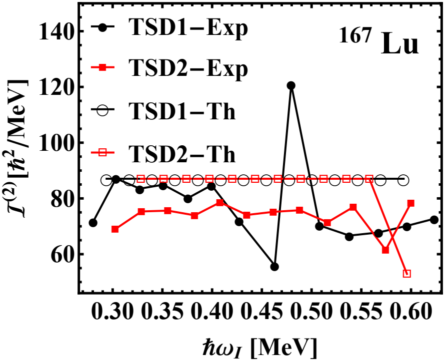


A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{161,167}Lu

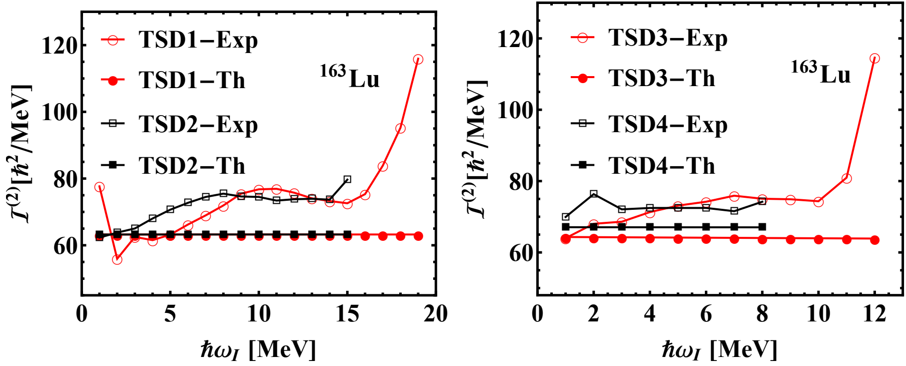
$$\mathcal{I}^{(2)}(I) = \hbar \frac{\mathrm{d}I_x}{\mathrm{d}\omega} = \hbar^2 \left(\frac{\mathrm{d}^2 E}{\mathrm{d}I_x^2} \right)^{-1}$$





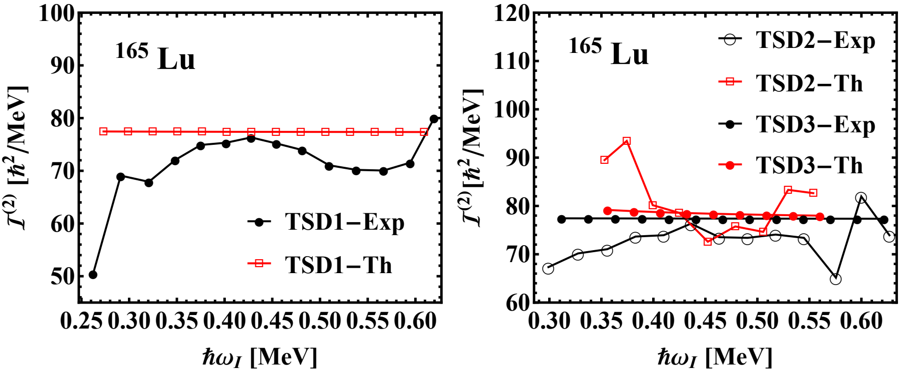
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{163}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{165}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Electromagnetic Calculations

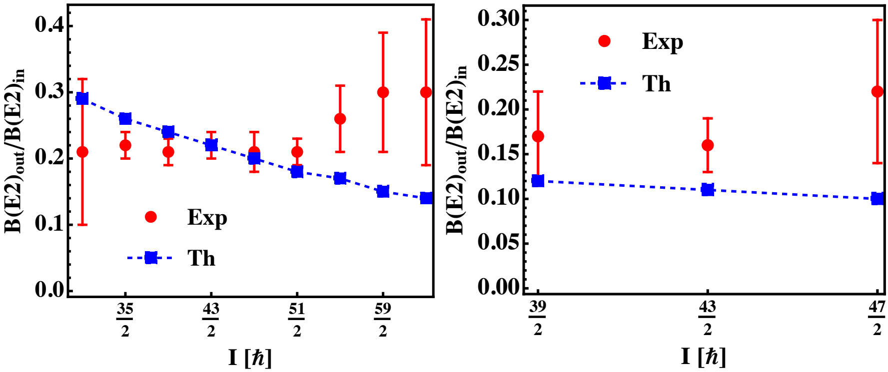


Figure: E2 Branching ratio. **Left**: ^{163}Lu (TSD2) **Right**: ^{165}Lu (TSD2).

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Electromagnetic Calculations II

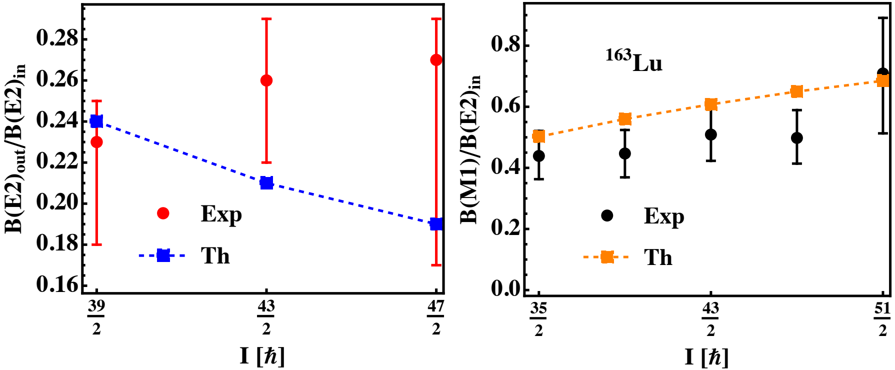


Figure: **Left**: E2 Branching ratio in ^{167}Lu (TSD2). **Right**: The ratio $B(M1)/B(E2)_{\text{in}}$ for states $TSD2 \rightarrow TSD1$ (in units of $\mu_N^2/(\mathrm{e}^2 b^2)$).

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

$\mathbf{W}_1$ — Remarks

Characteristics

- **+** Full semi-classical description (TDVE) with good numerical results
- **+** Deformation parameters are self-consistent (agree with exp. values)
- **−** separate fit for TSD4 (different nucleonic configuration)
- **−** Two sets of MOIs for ^{163}Lu

Onset of another redesign

Start of $\mathbf{W}_2$ formalism in Chapter 5

Fresh-Up 2: $\mathbf{W}_2$

Novel description of ^{163}Lu

- All four bands in ^{163}Lu described by the same triaxial core + odd-particle coupling $\longrightarrow \mathcal{Q}_1 = \pi(i_{13/2})$
- The adopted wave-function admits solutions of both **positive and negative parity**. Parity operator: $\mathcal{P} = \mathrm{e}^{-i\pi \mathbf{J}_2} C$:

$$\begin{aligned} \mathcal{P}\Psi(r,\varphi;t,\psi) &= \Psi(r,\varphi+\pi;t,\psi+\pi) \ , \\ \mathcal{H}(r,\varphi+\pi;t,\psi+\pi) &= \mathcal{H}(r,\varphi;t,\psi) \ , \\ \Psi(r,\varphi+\pi;t,\psi+\pi) &= \pm\Psi(r,\varphi;t,\psi). \end{aligned}$$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New band structure in ^{163}Lu

$$
\begin{aligned}
E_{I,0,0}^{\text{TSD1}} &= \epsilon_{13/2} + \mathcal{H}_{\text{min}}^{I} + \mathcal{F}_{00}^{I}\ , \quad I^{\pi} = 13/2^{+}, 17/2^{+}, 21/2^{+} \ldots\ , \\
E_{I,0,0}^{\text{TSD2}} &= \epsilon_{13/2}^{1} + \mathcal{H}_{\text{min}}^{I} + \mathcal{F}_{00}^{I}\ , \quad I^{\pi} = 27/2^{+}, 31/2^{+}, 35/2^{+} \ldots\ , \\
E_{I,1,0}^{\text{TSD3}} &= \epsilon_{13/2} + \mathcal{H}_{\text{min}}^{I-1} + \mathcal{F}_{10}^{I-1}\ , \quad I^{\pi} = 33/2^{+}, 37/2^{+}, 41/2^{+} \ldots\ , \\
E_{I,0,0}^{\text{TSD4}} &= \epsilon_{13/2}^{2} + \mathcal{H}_{\text{min}}^{I} + \mathcal{F}_{00}^{I}\ , \quad I^{\pi} = 47/2^{-}, 51/2^{-}, 55/2^{-} \ldots .
\end{aligned}
$$

Band	n_s	$\mathbf{j}_{\mathcal{Q}}$	$\mathbf{R}_{\mathscr{C}}$ - Sequence	$\mathbf{I}$ - Sequence	Coupling
TSD1	21	$\mathcal{Q}_1$	$\mathscr{C}_1 = 0^+, 2^+, 4^+, \ldots$	$13/2^+, 17/2^+, 21/2^+, \ldots$	$\mathscr{C}_1 + \mathcal{Q}_1$
TSD2	17	$\mathcal{Q}_1$	$\mathscr{C}_2^+ = 1^+, 3^+, 5^+, \ldots$	$27/2^+, 31/2^+, 35/2^+, \ldots$	$\mathscr{C}_2^+ + \mathcal{Q}_1$
TSD3	14	$\mathcal{Q}_1$	1-phonon exc.	$33/2^+, 37/2^+, 41/2^+, \ldots$	
TSD4	11	$\mathcal{Q}_1$	$\mathscr{C}_2^- = 1^-, 3^-, 5^-, \ldots$	$47/2^-, 51/2^-, 55/2^-, \ldots$	$\mathscr{C}_2^- + \mathcal{Q}_1$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New results for ^{163}Lu

Model requires **a unique set of parameters**: $\mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

$\mathcal{I}_1$ [$\hbar^2$/MeV]	$\mathcal{I}_2$ [$\hbar^2$/MeV]	$\mathcal{I}_3$ [$\hbar^2$/MeV]	γ [deg.]	V [MeV]
72	15	7	22	2.1

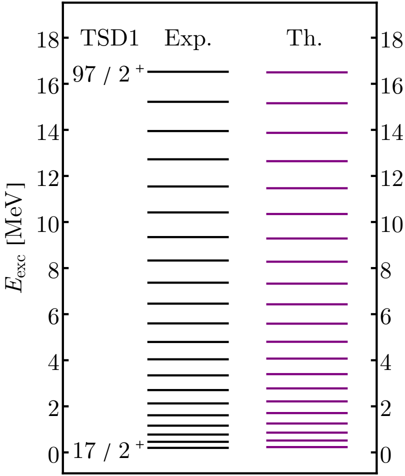
Remarks

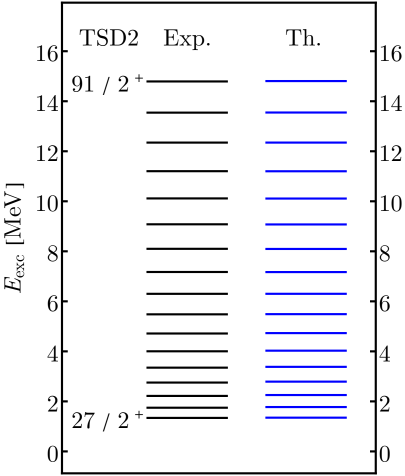
- γ in agreement with exp. value $\gamma_{exp} = 20^\circ$ (*Jensen, 2004*)
- Slight *decrease* of V (? breaking of parity symmetry quenches the quadrupole deformation)
- overall $E_{RMS} \approx 79$ keV: **first semi-classical description for a nucleus with deviations smaller than** 100 **keV**.

First model to describe ^{163}Lu wobbling structure.

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

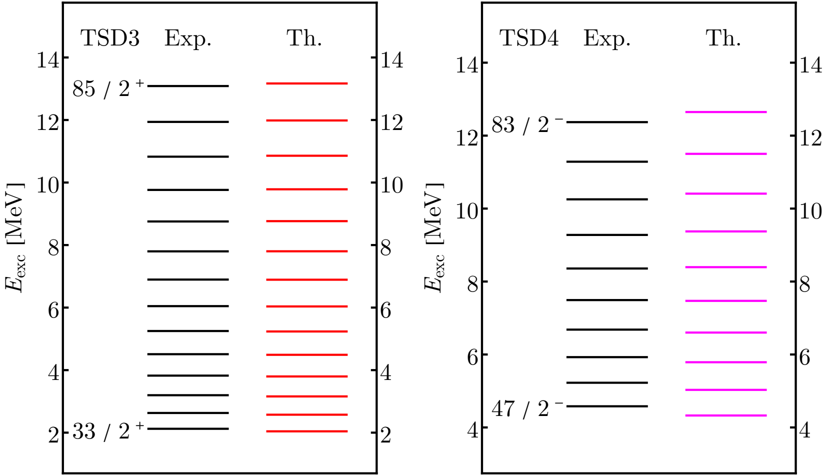
Energy spectrum





R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Energy spectrum II



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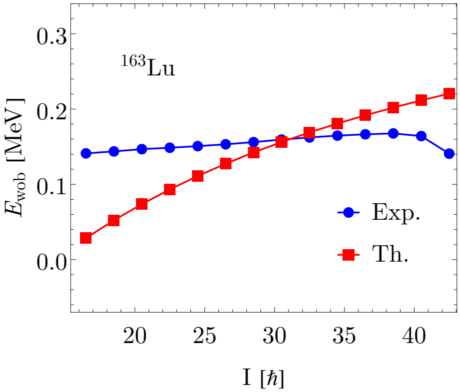
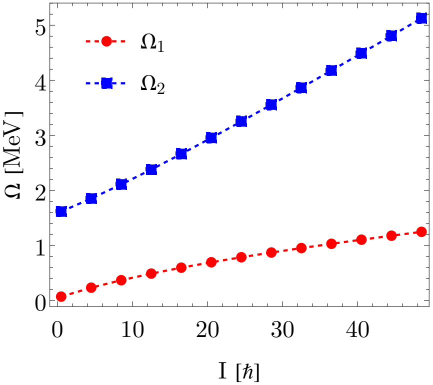
Wobbling Energies

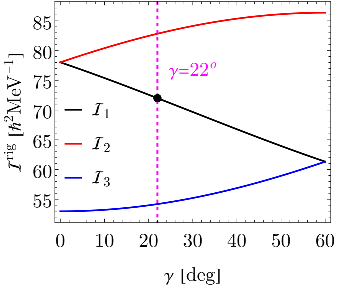


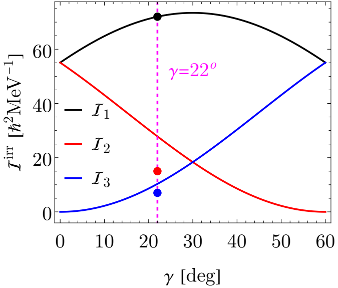


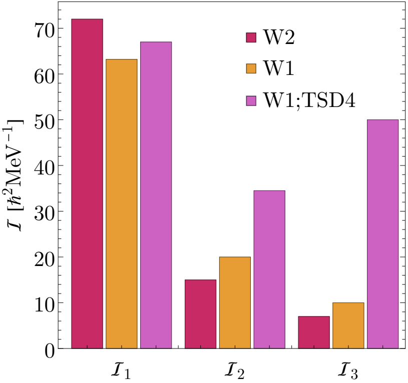
The wobbling energy (**left**) and the two wobbling frequencies (**right**) for ^{163}Lu. **Decreasing trend of E_{wob} in agreement with arguments of Frauendorf 2014.**

R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Moments of inertia for **W**$_2$







W$_2$: *hydrodynamical character* of the triaxial nucleus.

Results presented at the International Conference NSP, 2023, Turkey.

Classical Energy Function

Angular momentum

Polar representation of the angular momentum and $\mathcal{H}$.

$$\mathbf{I} = \{I_1, I_2, I_3\} \equiv \{x_1, x_2, x_3\} \ ,$$

$$x_1 = I\sin\theta\cos\varphi \ , \ x_2 = I\sin\theta\sin\varphi \ , \ x_3 = I\cos\theta .$$

$$\mathcal{H}\,\big|_{p_0} = I\left(I-\frac{1}{2}\right)\sin^2\theta\cdot\mathcal{A}_\varphi - 2A_1 Ij\sin\theta + T_{\text{core}} + T_{\text{sp}} \ ,$$

$$\mathcal{A}_\varphi = A_1\cos^2\varphi + A_2\sin^2\varphi - A_3 \ ,$$

$$T_{\text{core}} = \frac{I}{2}(A_1+A_2) + A_3 I^2 \ ,$$

$$T_{\text{s.p.}} = \frac{j}{2}(A_2+A_3) + A_1 j^2 - V\frac{2j-1}{j+1}\sin\left(\gamma+\frac{\pi}{6}\right) .$$

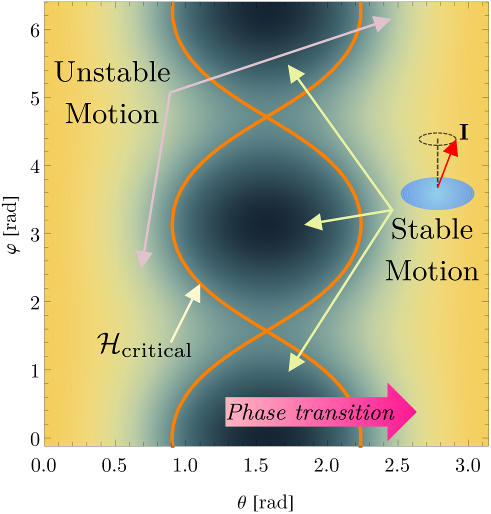
CEF — Stability Regions

Table: The minimum points of $\mathcal{H}$. Using the MOIs from the fitting procedure

Minimal point	θ [rad]	φ [rad]	A_k ordering
m_1	$\pi/2$	0	$A_3 > A_2 > A_1$
m_2	$\pi/2$	π	$A_3 > A_2 > A_1$
m_3	$\pi/2$	2π	$A_3 > A_2 > A_1$

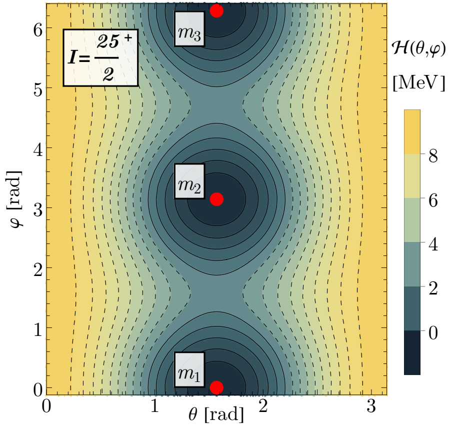
Semi-classical feature

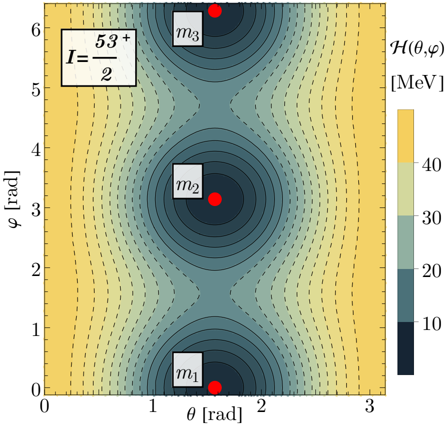
This is the first classical description of the *wobbling stability* for an odd-mass nucleus.



Polar representation of $\mathcal{H}$

Figure: ^{163}Lu TSD1

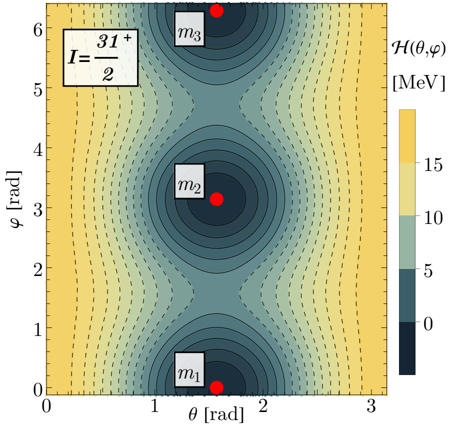


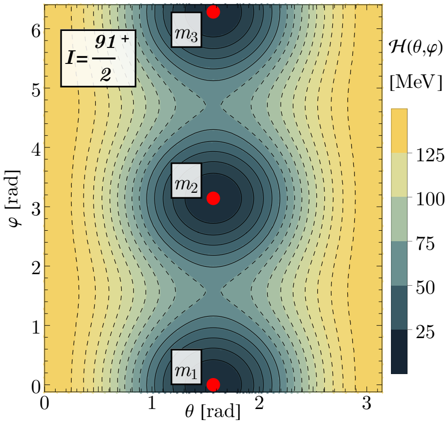


R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of $\mathcal{H}$ II

Figure: ^{163}Lu TSD2





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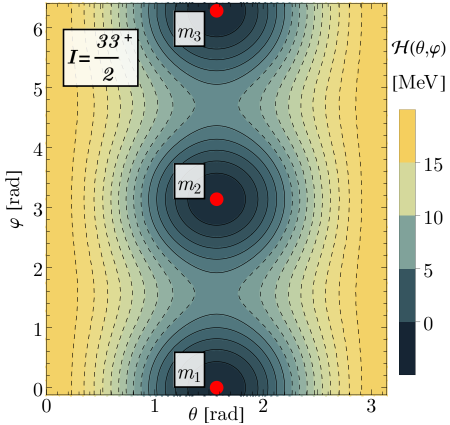
Polar representation of $\mathcal{H}$ III

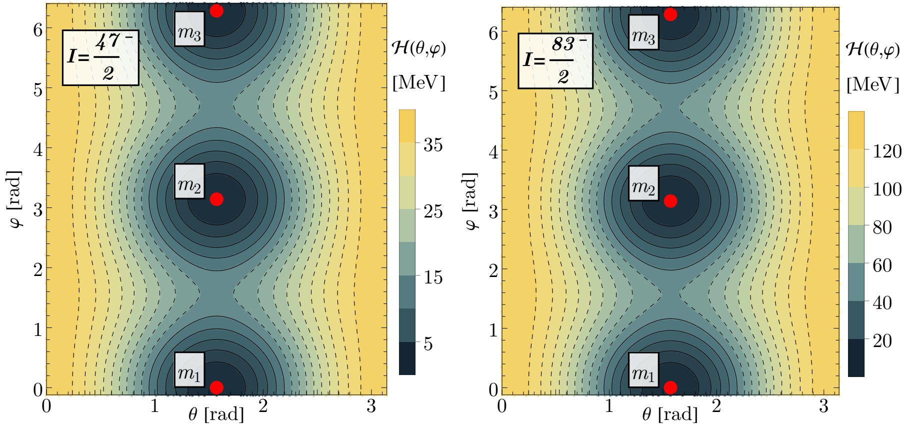
Figure: ^{163}Lu TSD3



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of $\mathcal{H}$ IV

Figure: ^{163}Lu TSD4



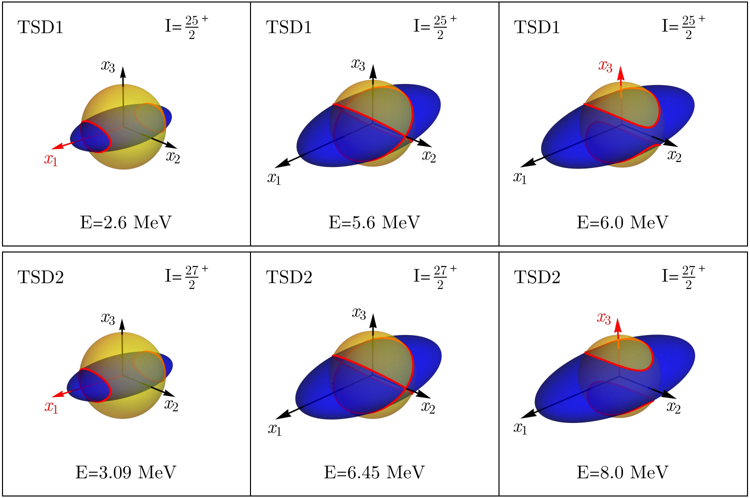
R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

3D interpretation of the WM

- Formalism $\mathbf{W}_2$ gives a 3D interpretation of the nuclear wobbling motion
- **Classical Trajectories**: intersection curves between the **triaxial energy** and the **total angular momentum**

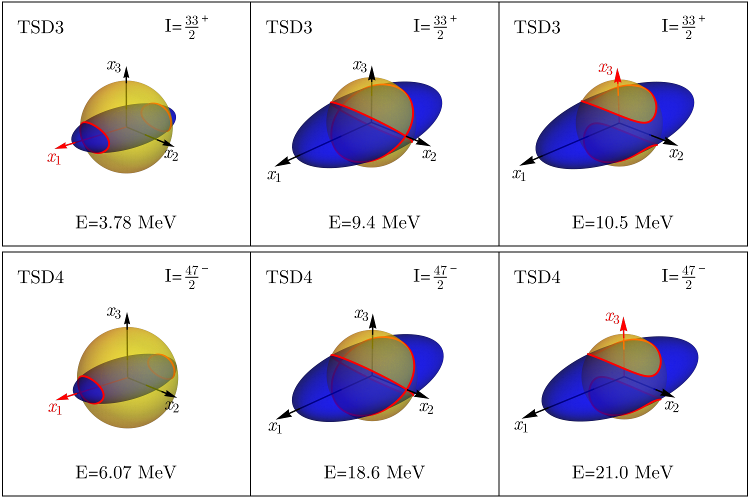
$$
\begin{aligned}
I^2 &= x_1^2 + x_2^2 + x_3^2 ,\\
E &= \left(1-\frac{1}{2I}\right) A_1 x_1^2 + \left(1-\frac{1}{2I}\right) A_2 x_2^2 + \left[\left(1-\frac{1}{2I}\right) A_3 + A_1 \frac{j}{I}\right] x_3^2 - \\
&- I\left(I-\frac{1}{2}\right) A_3 - 2A_1 Ij + \color{red}{T_{\mathrm{rot}}} + \color{blue}{T_{\mathrm{sp}}} .
\end{aligned}
$$

^{163}Lu — Classical trajectories



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

^{163}Lu — Classical trajectories II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Outline

New Boson Method for odd-mass nuclei

Rotational Hamiltonian

$$\hat{H}_{\text{rot}} = AH' + H_{sp} + \text{s.t.},$$

$$H' = a_1\left(\hat{I}_+^2 + \hat{I}_-^2\right) + a_2\left(\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+\right) + a_3\hat{I}_1 \ ,$$

$$H_{sp} = \sum_{k=1}^{2} A_k\hat{j}_k^2 \ , \ \text{s.t.} = A_1 I^2 - A_2 j_2 I \ .$$

- the triaxial rigid rotor is constrained to move around the 1-axis.
- adopted Frozen-Alignment approximation: $\mathbf{j} = (j\cos\theta, j\sin\theta, 0)$ (*Frauendorf, 2014*)
- a_1, a_2, a_3 inertial properties of the nucleus

New angular momentum representation

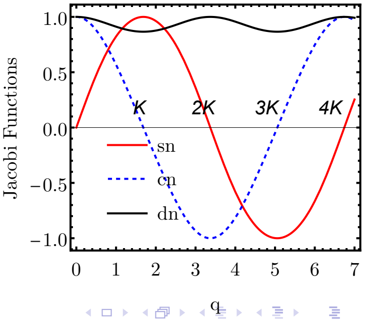
First boson expansion of this kind in literature:

$$\hat{I}_+ = i\frac{cb^\dagger - db^\dagger}{sb^\dagger}\left(I + Icb^\dagger db^\dagger - sb^\dagger b\right) \ ,$$

$$\hat{I}_- = i\frac{cb^\dagger + db^\dagger}{sb^\dagger}\left(I - Icb^\dagger db^\dagger + sb^\dagger b\right) \ ,$$

$$\hat{I}_1 = Icb^\dagger db^\dagger - sb^\dagger b \ .$$

- s, c, d: Jacobi Elliptic Functions
  $s = \mathrm{sn}(q,k)$, $c = \mathrm{cn}(q,k)$,
  $d = \mathrm{dn}(q,k)$.
- boson operators $b, b^\dagger, [b, b^\dagger] = 1$



Elliptic Potential

New Hamiltonian

$$H' = -\frac{d^2}{dq^2} - 2v_0 s \frac{d}{dq} + I(I+1)s^2k^2 + 2v_0 cdI \ ,$$

with the associated *Schrodinger Equation* (fully separated Kinetic and Potential terms):

$$\left[\frac{d^2}{dq^2} + V(q)\right]\Psi = E\Psi .$$

$$V(q) = \left[I(I+1)k^2 + v_0^2\right]s^2 + (2I+1)v_0 cd = V(-q) \ . \tag{1}$$

Results were presented at the **International Conference TIM-22** (Timisoara) and **World Quantum Day 2023** (IFIN-HH)

Elliptic potential

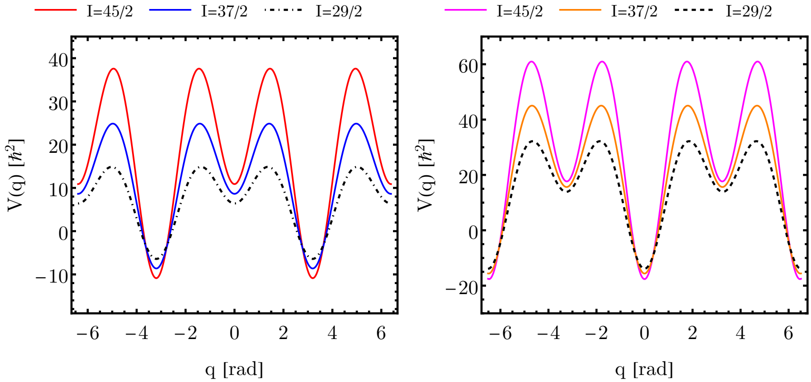


Figure: The elliptic potential as function of the coordinate q with $\theta = -119^\circ$ (**left**) and $\theta = 61^\circ$ (**right**).

A.A. Raduta, C.M. Raduta, **R Poenaru**, Journal of Physics G 48, 2020.

Results for ^{135}Pr

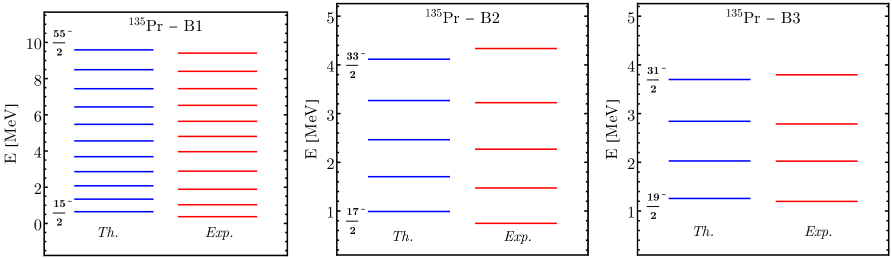


Figure: The excitation energies in ^{135}Pr. Exp data: *Sensharma, 2019*.

$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	θ [degrees]	N.o. states	RMS [MeV]
91	9	51	-119	20	0.174

A.A. Raduta, C.M. Raduta, **R Poenaru**, Journal of Physics G 48, 2020.

Outline

General Conclusions

- Developed **three** semi-classical models that describe wobbling motion in odd-A nuclei ($\mathbf{W}_1$, $\mathbf{W}_2$, and the boson method applied to ^{135}Pr).
- Showed that it is possible to treat the motion of the core and the odd nucleon separately.
- Obtained realistic results concerning wobbling energies and other quantities.
- Special attention to the geometrical interpretation of the wobbling motion was given for ^{163}Lu.

Original Contributions

- Research period: 2018-2022
- **7** ISI papers (2 RJP, 1 IJMPE, 2 PRC, 2 JPG)
- **38** Citations
- **Total IF:** 17.225; **Total AIS:** 4.684
- **5** Oral and **2** Poster presentations at international conferences

Thank you for your attention ❤