BOSON REPRESENTATION OF ANGULAR MOMENTUM. I

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By analogy with Schwinger's two-boson representation, a formalism is developed for the angular momentum that uses an arbitrary number of Bose operators. Operators, eigenfunctions, and also coherent states of the angular momentum are constructed. Their main properties are investigated: for example, completeness, and transformation under rotation, and for coherent states uncertainty relations and relation to the classical limit are also investigated.

It is well known that the operators and eigenstates of the angular momentum and other related quantities can be constructed with two pairs of boson creation and annihilation operators [1]. On the basis of these results, it has been possible to introduce coherent states of the angular momentum [2-4], which to a certain degree are analogous to the coherent states of an harmonic oscillator [5] and have a number of specific and useful properties.

In the present paper, we carry out such constructions for the case of an arbitrary number 2s + 1 of pairs of boson operators. In the special case s = 1/2, the formulas of this paper are transformed into the corresponding formulas of [1-4].

1. Definition of the Operators and States of the Angular Momentum

We take 2s + 1 operators of creation a_μ^+ ($\mu = -s, -s+1, \ldots, s$, where s is an integral or half-integral positive number), and as many annihilation operators a_μ, which satisfy the usual boson commutation rules:

$$[a_\mu, a_{\mu'}^+] = \delta_{\mu\mu'}, \quad [a_\mu, a_{\mu'}] = [a_\mu^+, a_{\mu'}^+] = 0.$$

From this set we form the quadratic combinations

$$J_3 = \sum_{\mu,\mu'} a_\mu^+ \langle s\mu|\hat{J}_3|s\mu'\rangle a_{\mu'} = \sum_\mu \mu a_\mu^+ a_\mu,$$
$$J_+ = \sum_{\mu,\mu'} a_\mu^+ \langle s\mu|\hat{J}_+|s\mu'\rangle a_{\mu'} = \sum_\mu \sqrt{(s-\mu)(s+\mu+1)}\, a_{\mu+1}^+ a_\mu. \tag{1}$$

In addition, $\mathbf{J}_- = (\mathbf{J}_+)^+$. Here $\langle s\mu|\hat{\mathbf{J}}_i|s\mu'\rangle$ are the matrices of the operators of the angular momentum components. The operators $\mathbf{J}_i$ in (1) satisfy the commutation relations for angular momentum operators:

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3.$$

In the case of arbitrary s we do not have a function quadratic in the operators a_μ^+ and a_μ that could play the role of the operator $\mathbf{J}$, which is possible for s = 1/2 when $\mathbf{J} = (1/2)(a_+^+ a_+ + a_-^+ a_-)$ and $\mathbf{J}^2 = \mathbf{J}(\mathbf{J}+1)$, since there does not exist a matrix of arbitrary dimensionality representing $\mathbf{J}$ except for the dimensionality 2 = 2s + 1.

Further, suppose there is a zeroth state $|0\rangle$ with the property $a_\mu|0\rangle = 0$. Using the creation operators, we form the following set of states:

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$$|jms\rangle = \sqrt{l \Big/ \binom{2j}{j-m}} \sum_{\xi} \prod_{\mu=-s}^{s} (1/\xi_\mu!) \left(\sqrt{\binom{2s}{s-\mu}}\, a_\mu^+\right)^{\xi_\mu} |0\rangle, \tag{2}$$

where $\binom{2s}{s-\mu}$ are the binomial coefficients, $j = ls$, and l is a positive integer. Here and in all that follows, the sum over ξ means that the integers ξ_μ take all values compatible with the rules

$$0 \leqslant \xi_\mu \leqslant l, \quad \sum_{\mu=-s}^{s} \xi_\mu = l, \quad \sum_{\mu} \mu \xi_\mu = m, \tag{3}$$

$$j + m = \sum_{\mu} (s+\mu)\xi_\mu, \quad j - m = \sum_{\mu} (s-\mu)\xi_\mu. \tag{4}$$

The last is a consequence of the conditions (3). From two arbitrary complex numbers α_+ and α_- we form the quantities

$$\alpha_{jm}(\alpha_+, \alpha_-) = \alpha_+^{j+m} \alpha_-^{j-m} \sqrt{\binom{2j}{j-m}} = \prod_{\mu} (\alpha_+^{s+\mu} \alpha_-^{s-\mu})^{\xi_\mu} \sqrt{\binom{2j}{j-m}}. \tag{5}$$

Here we have used the properties (4). Contracting the states $|jms\rangle$ from (2) with α_{jm}, we obtain the generating function for these states:

$$\sum_{j=0,s,2s\ldots}^{\infty} \sum_{m=-j}^{j} \frac{1}{\sqrt{l!}} \alpha_{jm} |jms\rangle = \sum_{j} \sum_{m,\xi} \prod_{\mu} \frac{1}{\xi_\mu!} (\alpha_{s\mu} a_\mu^+)^{\xi_\mu} |0\rangle = \sum_{l} \frac{1}{l!} \left(\sum_{\mu} \alpha_{s\mu} a_\mu^+\right)^l |0\rangle = \exp\left(\sum_{\mu} \alpha_{s\mu} a_\mu^+\right) |0\rangle. \tag{6}$$

In deriving (6) we have used the expansion

$$\frac{1}{l!} \left(\sum_{\mu} x_\mu\right)^l = \sum_{m\xi} \prod_{\mu} \frac{1}{\xi_\mu!} x_\mu^{\xi_\mu}. \tag{7}$$

We now turn to the properties of the states $|jms\rangle$. Since $\langle 0|a^m a^{+n}|0\rangle = n!\delta_{mn}$, in the scalar product $\langle jms|j'm's\rangle$ only identical exponent sets $\{\xi_\mu\}$ and $\{\xi_\mu'\}$ give nonvanishing expressions. Because of (3) there are no such sets if j and m and j' and m' are different; therefore, these states are orthogonal. It is easy to prove normalization to unity ($\langle jms|jms\rangle = 1$) using the relation

$$l! \sum_{\xi} \prod_{\mu} \frac{1}{\xi_\mu!} \binom{2s}{s-\mu}^{\xi_\mu} = \binom{2j}{j-m}.$$

And this relation is obtained by comparing the coefficients of $\alpha_+^{j+m}\alpha_-^{j-m}$ in the first and last expressions in the chain of equations [see (7)]

$$l! \sum_{m} \alpha_+^{j+m} \alpha_-^{j-m} \sum_{\xi} \prod_{\mu} \frac{1}{\xi_\mu!} \binom{2s}{s-\mu}^{\xi_\mu} = l! \sum_{m,\xi} \prod_{\mu} \frac{1}{\xi_\mu!} (\alpha_{s\mu})^{\xi_\mu} = \left(\sum_{\mu} \alpha_{s\mu}\right)^l = (\alpha_+ + \alpha_-)^{2sl} = \sum_{m} \alpha_+^{j+m} \alpha_-^{j-m} \binom{2j}{j-m}.$$

Then, applying the operators $\mathbf{J}_3$ and $\mathbf{J}_+$ in (1) to the states (2) and making some simple but in the second case rather tedious calculations, we can show that

$$J_3|jms\rangle = m|jms\rangle,$$

$$J_+|jms\rangle = \sqrt{(j-m)(j+m+1)}\,|j, m+1, s\rangle.$$

Thus, the states $|jms\rangle$ for $j = 0, s, 2s, \ldots$ are orthonormalized basis states of the angular momentum or, in other words, they form a basis of an irreducible representation of $2j+1$ dimensions of the group SU(2). The number $2s+1$ can be called the dimensionality of the generating representation. For each j, one can construct a certain number p of bases that differ by the number of creation operators used to construct the basis. Obviously, p is equal to the number of different integral divisors of the number 2j.

The states $|jms\rangle$ in (2) generalize the already known expressions for bases of irreducible representations of SU(2) and SO(3). In particular, for $s = 1/2$ we have Schwinger's spinor bases [1]

$$|jm\tfrac{1}{2}\rangle = [(j+m)!(j-m)!]^{-1/2} (a_+^+)^{j+m} (a_-^+)^{j-m} |0\rangle.$$

For $s = 1$ the creation operator parametrization

$$a_{+1}^+ = -\frac{1}{2} \sin\theta\, e^{i\varphi}, \quad a_0^+ = \frac{1}{\sqrt{2}} \cos\theta, \quad a_{-1}^+ = \frac{1}{2} \sin\theta\, e^{-i\varphi}$$

gives us the relationship between the states $|jm1\rangle$ and ordinary spherical functions [6]:

$$|jm1\rangle|_{a^+_{+1}=\ldots}=\sqrt{\frac{j!}{(2j+1)!}}\,Y_{jm}(\theta\varphi).$$

For the states (2) there exists this property: if

$$|jms\rangle=\Psi_{jms}(a_s^+,\ldots,a_{-s}^+)\,|0\rangle,\ \text{то}\ |j,-m,s\rangle=\Psi_{jms}(a_{-s}^+,\ldots,a_s^+)\,|0\rangle.$$

We now multiply the generating function (6) by a normalization constant, and obtain

$$|\alpha s\rangle\equiv\prod_\mu|\alpha_{s\mu}\rangle=\exp\left(-\frac{n^{2s}}{2}\right)\prod_\mu\exp(\alpha_{s\mu}a_\mu^+)\,|0\rangle=\exp\left(-\frac{n^{2s}}{2}\right)\sum_{jm}\frac{1}{\sqrt{l!}}\alpha_{jm}|jms\rangle=\sum_{jm}|jms\rangle\langle jms|\alpha s\rangle, \tag{8}$$

$$n^{2s}=n^{2s}(\alpha)=(|\alpha_+|^2+|\alpha_-|^2)^{2s}=\sum_\mu|\alpha_{s\mu}|^2$$

which is a coherent state of the angular momentum in the representation with dimensionality 2s + 1. As we can see from (8), it is a product of the coherent states of 2s + 1 harmonic oscillators [5]. The arguments $\alpha_{s\mu}$ are expressed [see (5)] in terms of the two complex numbers α_+ and α_-, whose 2s-th powers give the arguments of the "outside" oscillators with numbers s and –s. The states $|\alpha s\rangle$ have the properties

$$a_\mu|\alpha s\rangle=\alpha_{s\mu}|\alpha s\rangle, \tag{9}$$

$$\langle\alpha s|\beta s\rangle=\exp\left\{-\frac{1}{2}[n^{2s}(\alpha)+n^{2s}(\beta)]+(\alpha_+^*\beta_++\alpha_-^*\beta_-)^{2s}\right\}. \tag{10}$$

The last equation shows that the coherent states $|\alpha s\rangle$ are nonorthogonal but are normalized to unity. Using (1), (9), and (10), we can obtain the expected values of the components and the square of the angular momentum and the mean square deviations of the angular momentum components in the coherent states $|\alpha s\rangle$:

$$\langle\alpha s|J_i|\alpha s\rangle=\langle J_i\rangle=2sn^{2s-1}j_i, \tag{11}$$

$$\langle\alpha s|J^2|\alpha s\rangle=sn^{2s}(sn^{2s}+s+1), \tag{12}$$

$$\sum_{i=1}^{3}\langle J_i\rangle^2=s^2n^{4s}, \tag{13}$$

$$\langle(\Delta J_i)^2\rangle=sn^{2s}\left[2(2s-1)\left(\frac{j_i}{n(\alpha)}\right)^2+\frac{1}{2}\right], \tag{14}$$

$$j_i=\left\langle\alpha\frac{1}{2}|J_i|\alpha\frac{1}{2}\right\rangle,$$

$$j_1=\mathrm{Re}\,(\alpha_+^*\alpha_-),\quad j_2=\mathrm{Im}\,(\alpha_+^*\alpha_-), \tag{15}$$

$$j_3=\frac{1}{2}(|\alpha_+|^2-|\alpha_-|^2),\quad j_\pm=\alpha_\pm^*\alpha_\mp.$$

The expressions (11) are quadratic in α_+ and α_-, and therefore the states $|\alpha s\rangle$ and $|-\alpha s\rangle$ describe one and the same physical angular momentum vector. The quantity sn^{2S} for large $n(\alpha)$ (and therefore for large angular momentum) represents the length of the angular momentum vector. If $n(\alpha)$ is increased, the ratio of the square root of the mean square deviation of any component (14) to sn^{2S} tends to zero. Therefore, for arbitrary s the assertion drawn in [3] for s = 1/2 stands: for large $n(\alpha)$ the coherent states give a classical description of the angular momentum.

Further, it is well known that if three operators satisfy the commutation relation $[\mathbf{J}_3,\mathbf{J}_1]=i\mathbf{J}_2$, the uncertainty relation for the observables corresponding to these operators has the form $\langle(\Delta\mathbf{J}_3)^2\rangle\langle(\Delta\mathbf{J}_1)^2\rangle\geq\langle\mathbf{J}_2\rangle^2/4$. Commuting the indices, one can obtain two similar inequalities for the three angular momentum components. Then, adding the three inequalities, we obtain

$$M=4\frac{\langle(\Delta J_3)^2\rangle\langle(\Delta J_1)^2\rangle+\langle(\Delta J_2)^2\rangle\langle(\Delta J_1)^2\rangle+\langle(\Delta J_2)^2\rangle\langle(\Delta J_3)^2\rangle}{\langle J_1\rangle^2+\langle J_2\rangle^2+\langle J_3\rangle^2}\geqslant1,$$

which is a measure of the indeterminacy in a simultaneous measurement of $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$. In the states $|\alpha s\rangle$, using (11)-(15), we have

$$M(\alpha)=\frac{16(2s-1)^2}{n^4}[j_3^2j_1^2+j_2^2j_1^2+j_2^2j_3^2]+4s+1.$$

This quantity has the following properties. For $s = 1/2$, we have $M = 3$. If $s > 1/2$, then M does not depend on the orientation of the angular momentum vector in the three-dimensional real space. When two of the angular momentum components vanish, M takes its smallest value (six points on a sphere described by the end of the angular momentum vector in three-dimensional real space). When all three components have the same modulus, M takes its largest value: $M_{max} = ((2s-1)^2/3) + 4s + 1$ (eight points on the sphere). In addition, M has local minima, $M = ((2s-1)^2/4) + 4s + 1$, when one of the components vanishes, and the other two are equal (12 points on the sphere). We see that the indeterminacy for simultaneous measurements of the angular momentum components in coherent states depends strongly on the dimensionality $2s + 1$ of the generating representation and for small s is fairly close to the smallest possible.

Something should be said concerning the completeness property of the states $|jms\rangle$ and $|\alpha s\rangle$. The states $|jms\rangle$ are complete in the sense that one can expand with respect to them an arbitrary function $F(a^+)$ of the $2s + 1$ operators a_μ^+ on which $2s-1$ conditions are imposed, so that only two operators are independent:

$$F(a^+)|0\rangle = \sum_{jm} |jms\rangle\langle jms|F(a^+)|0\rangle. \tag{16}$$

All these $2s-1$ conditions must be invariants of rotations of the coordinates of the three-dimensional real space. For example, for $s = 1$ we have one condition $2a_{+1}^+a_{-1}^+ - a_0^{+2}$. Expansion of the coherent states $|\alpha s\rangle$ with respect to the states $|jms\rangle$ in (8) gives an example of the use of the completeness condition (16). In its turn, an arbitrary function $F(a^+)$ can be expanded with respect to coherent states:

$$F(a^+)|0\rangle = \int |\alpha s\rangle\langle\alpha s|F(a^+)|0\rangle\Phi_s(\alpha)\,d\alpha, \tag{17}$$

$$\Phi_s(\alpha) \equiv \Phi_s(\alpha_+, \alpha_-), \quad d\alpha = d^2\alpha_+ d^2\alpha_-, \quad d^2\alpha_\pm = d(\mathrm{Re}\,\alpha_\pm)\,d(\mathrm{Im}\,\alpha_\pm).$$

Here the integration is over the complete complex α_+ and α_- planes. The weight functions $\Phi_s(\alpha)$ are determined by means of their moments. Namely, using (17) for the expansion of the scalar product $\langle jms | j'm's\rangle$, we find

$$\int \exp(-n^{2s})\,\alpha_+^{j+m}\,\alpha_-^{j-m}\,\alpha_+^{*j'+m'}\,\alpha_-^{*j'-m'}\,\Phi_s(\alpha)\,d\alpha = \frac{l!}{\binom{2j}{j-m}}\,\delta_{jj'}\delta_{mm'}. \tag{18}$$

For $s = 1/2$ we know from [3, 4] that $\Phi_{1/2}(\alpha) = 1/\pi^2$. For the calculation of certain quantities we do not need to know the actual form of the functions; it is sufficient to use the conditions (18), as we shall show below when finding the explicit form of the generalized spherical functions.

2. Transformations under Coordinate Rotation

We consider transformations of $2s + 1$ oscillator states of the angular momentum under a rotation of the coordinate axes of three-dimensional real space. Suppose that such a transformation is described by the unitary operator

$$U(\psi_1, \chi, \psi_2) = \exp(-i\psi_1 J_3)\exp(-i\chi J_2)\exp(-i\psi_2 J_3).$$

Here ψ_1, χ, ψ_2 are the Eulerian angles that describe an arbitrary rotation. Let $\langle s\lambda|U|s\mu\rangle = U_{\lambda\mu}^s$ be the matrix that realizes such a rotation in the space of $2s + 1$ dimensions. Then the transformation of the operators a_μ^+ can be written as

$$Ua_\mu^+U^{-1} = a_\mu'^+ = \sum_\lambda a_\lambda^+ U_{\lambda\mu}^{(s)}.$$

We apply the operator U to a coherent angular momentum state:

$$U|\alpha s\rangle = \exp\left(-\frac{n^{2s}}{2} + \sum_\mu \alpha_{s\mu} a_\mu'^+\right)|0\rangle = \exp\left(-\frac{n^{2s}}{2} + \sum_\mu \alpha_{s\mu}'' a_\mu^+\right)|0\rangle = |\alpha'', s\rangle.$$

Here, the double prime denotes transformation by means of the inverse operator U^{-1}. Recalling the transformation property of the spinors $\alpha_{s\mu}$ obtained in [1], we find

$$\sum_\mu (U^{-1})_{\lambda\mu}^{(s)}\alpha_{s\mu} = \alpha_{s\lambda}'' = \left(\alpha_+^{s+\lambda}\,\alpha_-^{s-\lambda}\sqrt{\binom{2s}{s-\lambda}}\right)'' = \alpha_+''^{\,s+\lambda}\,\alpha_-''^{\,s-\lambda}\sqrt{\binom{2s}{s-\lambda}},$$

where α_+'' and α_-'' are obtained from α_+ and α_- by a transformation in the two-dimensional complex space by means of the matrices $(U^{-1})^{(1/2)}_{\lambda\mu} = V^{-1}$:

$$\begin{pmatrix}\alpha_+'\\ \alpha_-'\end{pmatrix} = \begin{pmatrix}U\alpha_+ U^{-1}\\ U\alpha_- U^{-1}\end{pmatrix} = V\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = \begin{pmatrix}v & w\\ -w^* & v^*\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}.$$
$$v = \exp\left[\frac{i}{2}(\psi_1+\psi_2)\right]\cos\frac{\chi}{2}, \quad w = \exp\left[\frac{i}{2}(\psi_2-\psi_1)\right]\sin\frac{\chi}{2}. \tag{19}$$

Thus, to transform coherent angular momentum states of any dimensionality we use only a two-dimensional matrix.

Returning to the expectation values of the angular momentum components in the coherent states in (11), we see that under rotation they transform as

$$\langle\alpha s|UJ_iU^{-1}|\alpha s\rangle = \langle\alpha s|J_i'|\alpha s\rangle = \langle\alpha' s|J_i|\alpha' s\rangle = 2sn^{2s}j_i(\alpha_+', \alpha_-').$$

It follows from the transformation of α_+ and α_- by means of the matrix $V^{-1}(\psi_1, \chi, \psi_2)$ that the expectation values of J_i in coherent states transform as vectors in the three-dimensional real space and the expressions (11) can be called a vector representation of the angular momentum, which is noted in [3, 4] for $s = 1/2$.

It is well known that the eigenstates $|jm\rangle$ of the angular momentum with half-integral and integral j behave differently under a rotation of the coordinates through an angle 2π: the former reverse their sign, the latter do not. A similar conclusion is true for coherent states. Under a rotation through 2π, the arguments α_+ and α_- reverse sign. Reverting to (5), we see that the amplitude of a coherent state of any of the $2s+1$ oscillators then takes on the factor $(-1)^{2s}$. It follows that the coherent angular momentum states with half-integral s (generating representation of even dimensionality $2s+1$) change under a rotation through 2π ($\alpha_{s\mu} \to -\alpha_{s\mu}$) and those with integral s (odd dimensionality $2s+1$) do not change.

From the expansion of the matrix element $\langle\alpha s|U|\beta s\rangle$ with respect to the states $|jms\rangle$ we can readily obtain, using the definition (8) and (10), generating functions for the matrix elements $\langle jms|U|jm_1s\rangle$:

$$\sum_{jmm_1}\frac{1}{l!}\alpha^*_{jm}\langle jms|U|jm_1s\rangle\beta_{jm_1} = \exp[(\alpha_+^*\beta_+'' + \alpha_-^*\beta_-'')^{2s}]. \tag{20}$$

A direct calculation also gives

$$\exp(-i\varphi J_3)|jms\rangle = \exp(-im\varphi)|jms\rangle. \tag{21}$$

Either from (21) or from (20) we can obtain directly (as in [1]) the character of the representation of $2j+1$ dimensions of the group SU(2) by the matrices $\langle jms|U|jm_1s\rangle$:

$$\chi_s^j = \sum_m\langle jms|U|jms\rangle = \frac{\sin[(j+{}^1/_2)\varphi]}{\sin\varphi/2} = \chi^j.$$

The character does not depend on s. Hence and from (21) it follows that

$$\langle jms|U|jm_1s\rangle = \langle jm|U|jm_1\rangle = U^j_{mm_1}.$$

Thus, although the bases of the representations of SU(2) in the $(2s+1)$-boson construction are new, the matrices of these representations are identical with the well-known generalized spherical functions. The explicit form of the matrices $U^j_{mm_1}$ can be obtained by means of the completeness condition (17) for coherent states:

$$\langle jms|U|j_1m_1s\rangle = \int\langle jms|\alpha s\rangle\langle\alpha s|U|j_1m_1s\rangle\Phi_s(\alpha)\,d\alpha = \int\langle jms|\alpha s\rangle\langle\alpha' s|j_1m_1s\rangle\Phi_s(\alpha)\,d\alpha.$$

Here α_+ in α_- in $\langle\alpha' s|$ are transformed as in (19). Using the definition of the quantities $\langle jms|\alpha s\rangle$ from (8), expanding the binomial powers in the usual manner, and taking into account the condition on the weight function $\Phi_s(\alpha)$ (18), we arrive at the usual formula (see, for example, [6]):

$$U^j_{mm_1} = \sqrt{(j+m_1)!(j-m_1)!(j+m)!(j-m)!}\,(-1)^{m-m_1}\sum_k(-1)^k\frac{v^{*j+m_1-k}w^{*k}v^{j-m-k}w^{m-m_1+k}}{(j+m_1-k)!k!(j-m-k)!(m-m_1+k)!}.$$

Thus, the properties of the states introduced in this paper show that they can be successfully used to solve some quantum-mechanical problems. In particular, the interaction of an ensemble of particles with arbitrary spin s with an external field can be expressed in the language of 2s + 1 harmonic oscillators by means of this formalism.

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