A Systematic Description of the Wobbling Motion in Odd-Mass Nuclei Within a Semi-Classical Formalism

PhD Candidate
Robert Poenaru[1,2]

Scientific Supervisor
Prof. Dr. Em. A. A. Raduta[2]

[1]Doctoral School of Physics, UB
[2]Department of Theoretical Physics, IFIN-HH

A presentation for the degree of Doctor of Philosophy

May 14, 2023

TOC

1. Aim and Motivation
2. Nuclear Shapes
3. Triaxiality and Wobbling Motion
   - Even-A case study
4. Wobbling Motion in Odd-A
   - Old formalism
   - Redefinition of the Odd-A framework
     - Fresh-Up 1
     - Fresh-Up 2
5. Boson Description of Wobbling Motion
   - Formalism
   - Case-Study

Aim

Research Objectives

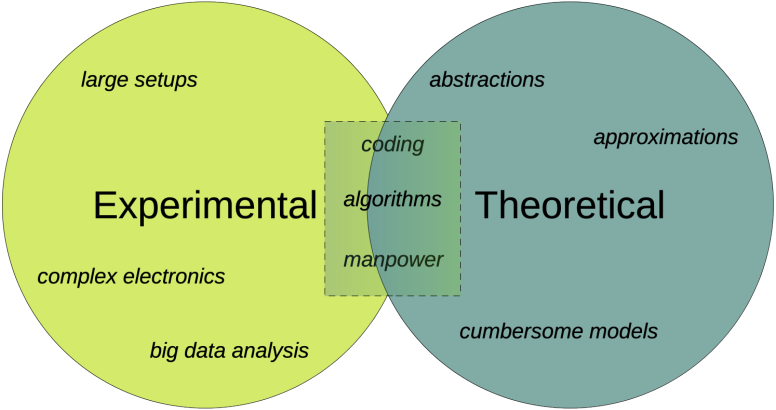
- Extend the current interpretation of the **nuclear triaxiality** in the context of its unique fingerprint: **Wobbling Motion**
- Adopt a framework that is as close as possible to **classical physics**.
- Provide new formalisms for the phenomena related to **nuclear deformation**.

Objectives exclusive to the thesis

- Give the reader enough context towards a better understanding of the underlying concepts, methods, and results.
- create a completely *open-source* project.

Motivation

- **Nuclear Triaxiality** has become a *hot topic* within the scientific community.
- Identifying nuclei with triaxial deformations represents a real **experimental** and **theoretical** challenge.



Nuclear Deformation

Nuclear Radius

The **shape** of the nucleus is most generally described in terms of the *nuclear radius*:

$$R(\theta,\varphi) = R_0\left(1+\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\alpha_{\lambda\mu}Y_{\lambda}^{\mu}(\theta,\varphi)\right)$$

Quadrupole deformations $\lambda = 2$

- For us: Most relevant modes are the **quadrupole vibrations** $\lambda = 2$ $\Longrightarrow$ *Play a crucial role in the rotational spectra of nuclei:*
- *Bohr, 1969*: Coordinates $\alpha_{2\mu}$ can be reduced to only two *deformation parameters*: β_2 (**eccentricity**) and γ (**triaxiality**).

Axial shapes

Collective coordinates

- Most of the nuclei are either **spherical** or **axially symmetric** in their ground-state *(Budaca, 2018)*.
- Moments of inertia: $\mathcal{I}_{1,2,3}$: two are equal, one is different.

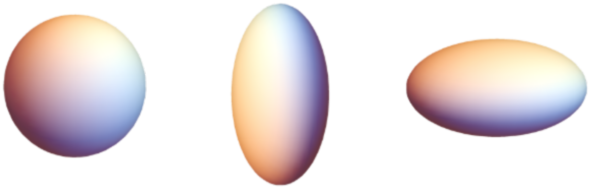
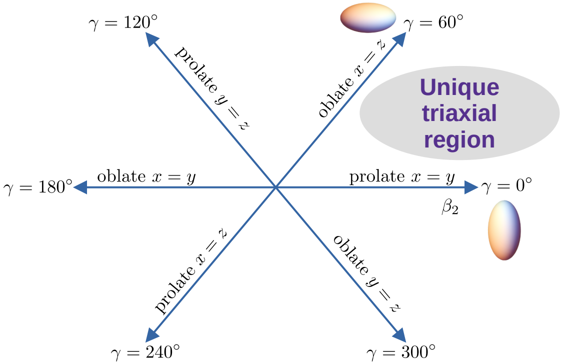
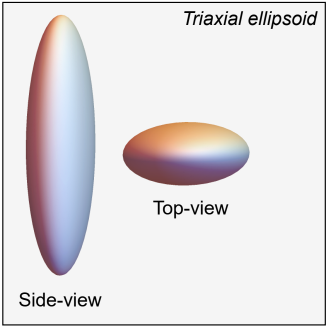


Figure: **spherical:** $\beta_2 = 0$ **prolate:** $\beta_2 > 0$ **oblate:** $\beta_2 < 0$. $(\gamma = 0^\circ)$.

Non-axial shapes

- The triaxiality parameter $\gamma \neq 0^\circ$: departure from axial symmetry.
- Moments of inertia: $\mathcal{I}_1 \neq \mathcal{I}_2 \neq \mathcal{I}_3$.





Fingerprints of Triaxiality

Evidence

- Currently, there are **only two** well-established phenomena uniquely attributed to triaxial deformation.
  1. **Wobbling Motion** - WM (*Bohr and Mottelson, 1970s*)
  2. Chiral Motion - χM (*Frauendorf, 1997*)
- These two can be measured/detected experimentally.

Goal

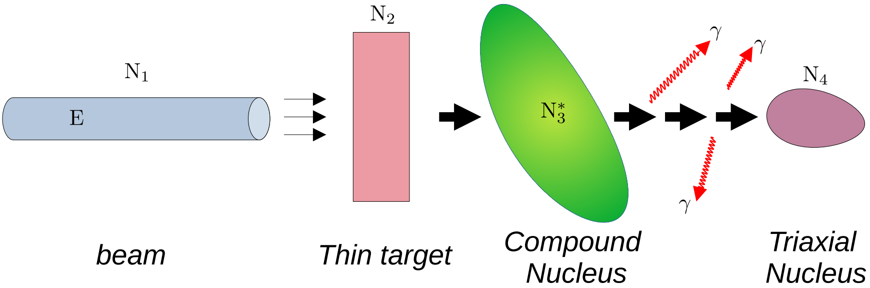
Describe the elusive character of Wobbling Motion in the context of nuclear triaxiality.

Probing triaxiality in nuclei

Triaxial nuclei can be observed/obtained in several experiments:

- Nuclear fission: $A \rightarrow B + C$
- Nuclear fusion: $X + Y \rightarrow Z$
- **Fusion-evaporation reactions**: Long-lived + enhanced deformation

$$Beam(N_1, E) + Target(N_2) \longrightarrow N_3^* \rightarrow \cdots \rightarrow triaxial(N_4)$$



Nuclear facilities

Figure: Gammasphere detector, ANL-ATLAS USA. *Source: aps.org*

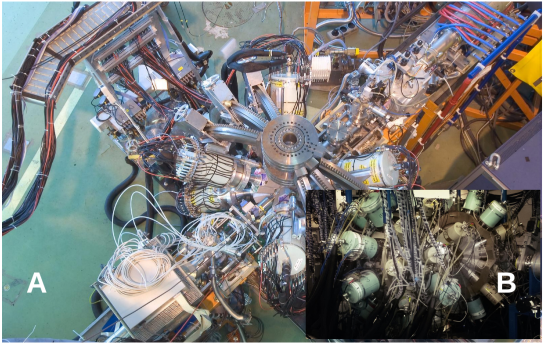


Figure: a) IDS detector, CERN. *Source: isolde.web.cern.ch* b) JUROGAM II, Finland. *Source: twitter.com*

Wobbling Motion

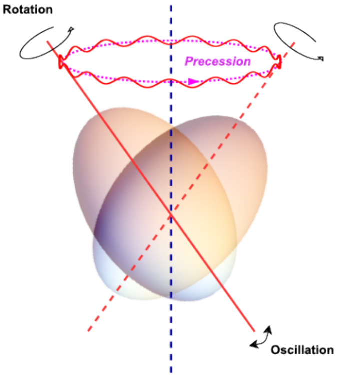
$\gamma \neq 0^\circ$

↓

MOI anisotropy

↓

main rotation around $\mathcal{J}_{\max}$ is disturbed by the other two axes



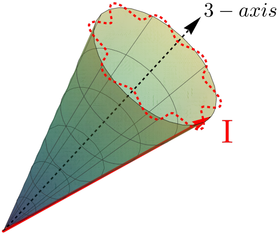
Wobbling Effect

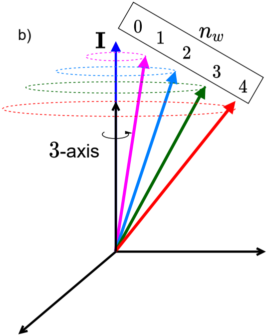
- The **total angular momentum** of the nucleus **precesses** and **oscillates** around $\mathcal{J}_{\max}$.

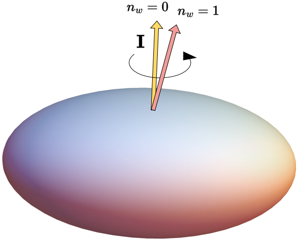
Wobbling Motion

Harmonic oscillation

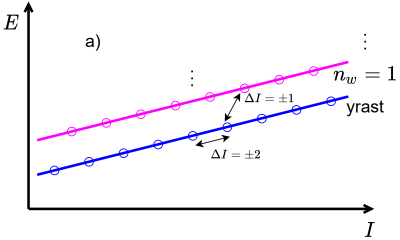
- Precession of **I** is affected by **rotational frequency** and/or **tilting**
- Tilting only by "specific" amount $\rightarrow$ **harmonic character** $\rightarrow$ **wobbling phonon**: $n_w = 0, 1, 2, \ldots$.

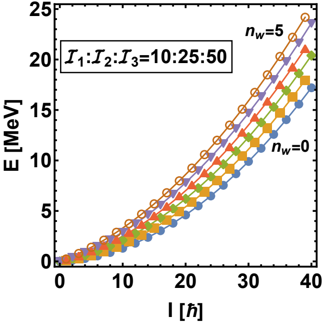


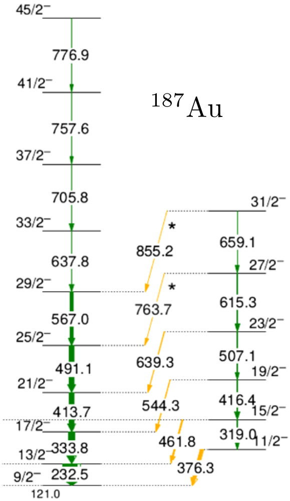




Wobbling Motion II



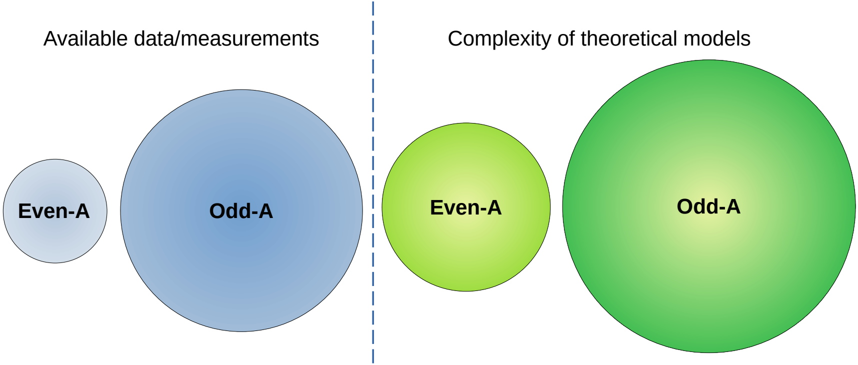




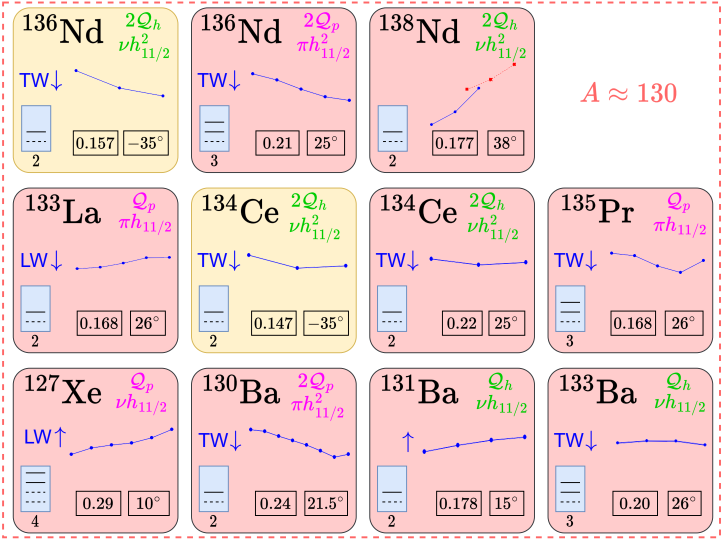
Sensharma, 2020.

Even-A vs. Odd-A Picture

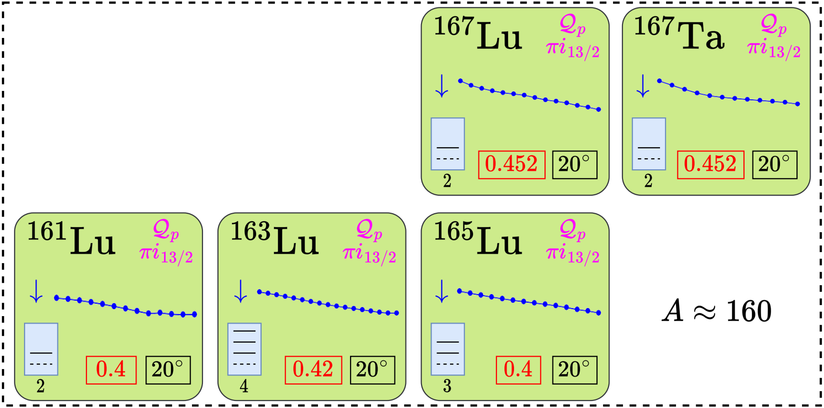
- Predicted for even-A nuclei more than 50 years ago.
- First experimental evidence for **nuclear wobbling motion**: ^{163}Lu (*Ødegård, 2001*).
- Current mass-regions for wobblers: $A = 130, 160, 180$.



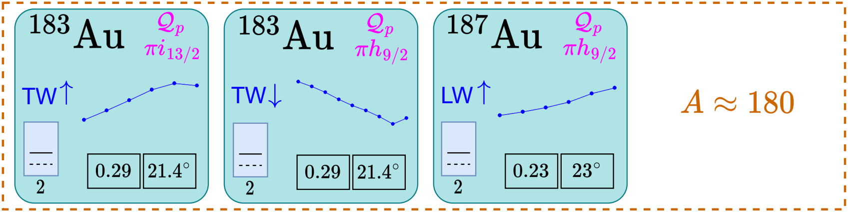
Wobblers in the A=130 mass region



Wobblers in the A=160 mass region



Wobblers in the A=180 mass region



All diagrams and their data sources available in Chapter 3, Section 3.3.5

Presented at the Annual Meeting, FFUB, 2022.

Wobbling Motion in ^{130}Ba

Experimental measurements show **two** wobbling bands (*Petrache et. al. 2019*).

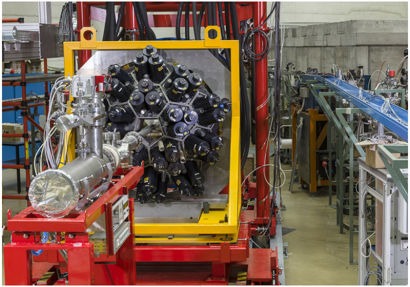
Harmonic formalism

Harmonic Approximation (*Bohr & Mottelson, 1969*):

$$E_{I,n_w} = A_3 I(I+1) + \hbar\omega_w \left(n_w + \frac{1}{2}\right),$$

$$A_3 = (2\mathcal{I}_3)^{-1}.$$

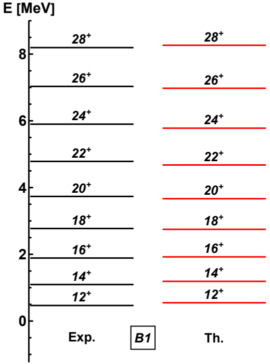
(rotational term + wobbling frequency)

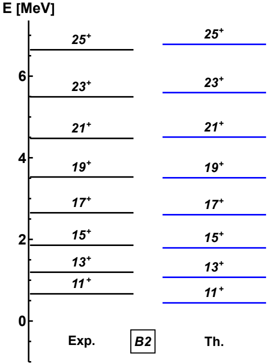


GALILEO, LNL, *Source: lnl.infn.it*

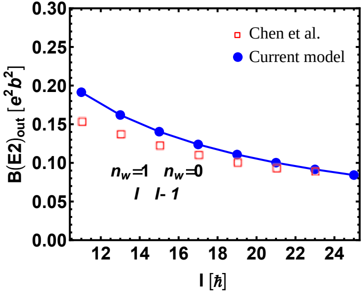
Fusion evaporation: ^{13}C beam of $E = 65$ MeV and ^{122}Sn target.

Results for ^{130}Ba





$\mathcal{P}_{\text{fit}}$			
$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	Unit
27	22	**43**	$\hbar^2\text{MeV}^{-1}$



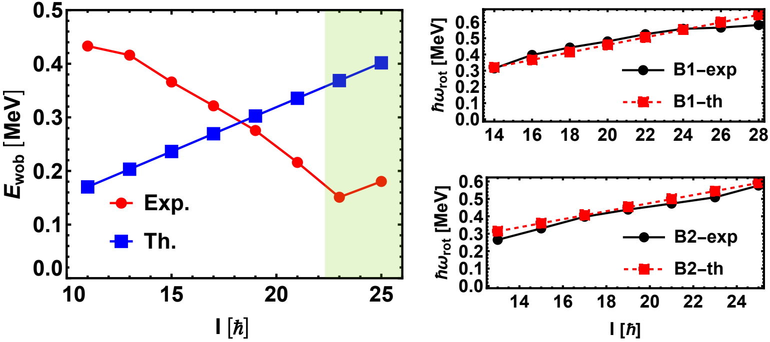
Full description: Chapter 3 (Section 3.1.2)

Results presented at the international conference NSP-2022, Turkey.

Results for ^{130}Ba II

Excitation energies vs. **Wobbling Energies**:

$$E_{\text{wob}}(I_{\text{even}}) = E_{I,n} - E_{I,0} \ ,$$
$$E_{\text{wob}}(I_{\text{odd}}) = E_{I,n} - \frac{1}{2}\left(E_{I-1,0} + E_{I+1,0}\right)$$



Results presented at the international conference NSP-2022, Turkey.

Starting Point

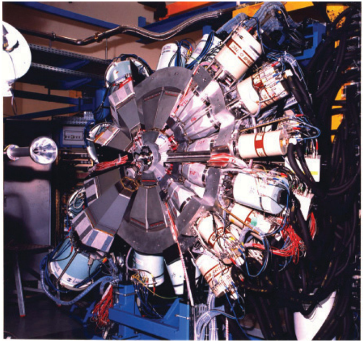
A. A. Raduta, **R. Poenaru**, L. Gr. Ixaru, PRC, 2017 + A. A. Raduta, **R. Poenaru**, Al. H. Raduta, JPG, 2018→$\mathbf{W}_0$ in the thesis.

Framework

- **First semi-classical description** for the ^{163}Lu, using the **Particle-Rotor-Model** (*Hamamoto, 2002.*) for an odd-mass nucleus in the $A \approx 160$ region.

PRM

An odd-nucleon moving in a quadrupole deformed mean field generated by an even-even triaxial core.

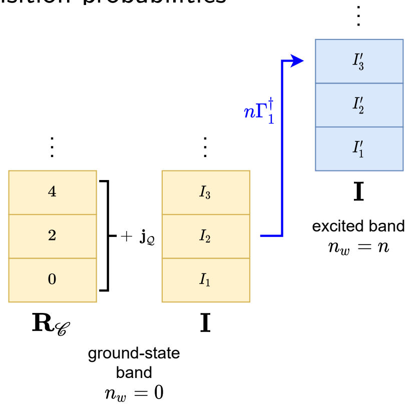


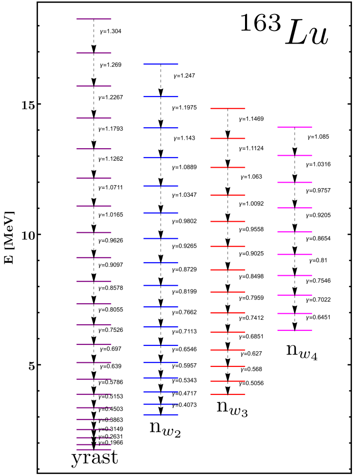
Euroball IV, Strasbourg, *Source: technologysi.stfc.ac.uk*

Fusion evaporation: ^{29}Si beam of $E = 152$ MeV and ^{139}La target.

Overview of $\mathbf{W}_0$

- Time-Dependent Variational Principle applied on the PRM Hamiltonian
- **Phonon operators** → energies + transition probabilities





Overview of $\mathbf{W}_0$ II

Model characteristics

- **+** Numerical data consistent with other work
- **−** TSD4: three-phonon wobbling band (disagreement with Jensen et al.)
- **−** Adopted rigid-body MOIs ("unpleasant" choice to the referees)
- **−** Deformation parameters β and γ taken from literature

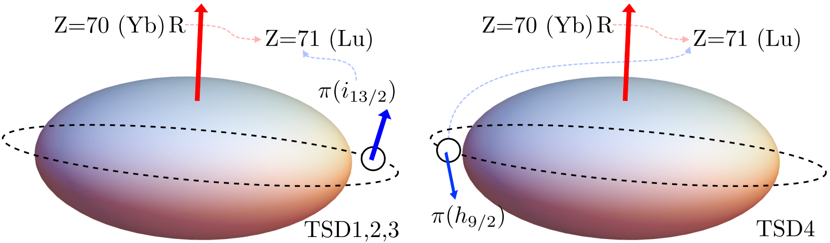
Onset of a redesign $\longrightarrow$ start of a new research project
Two new models developed presented in Chapter 4 ($\mathbf{W}_1$) and 5 ($\mathbf{W}_2$)

Fresh-Up 1: $\mathbf{W}_1$

Particle-Rotor Model Hamiltonian for an odd-A nucleus:

$$\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{sp}}\ ,\ \ \hat{H}_{\text{rot}} = \sum_{k=1}^{3} A_k(\hat{I}_k - \hat{j}_k)^2,$$

$$\hat{H}_{\text{sp}} = \epsilon_j + \frac{V}{j(j+1)}\left[\cos\gamma\left(3\hat{j}_3^2 - \mathbf{j}^2\right) - \sqrt{3}\sin\gamma\left(\hat{j}_1^2 - \hat{j}_2^2\right)\right]$$



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Variational Principle + Eqs. of Motion

Time-Dependent Varational Equation

$$\delta \int_0^t \langle \Psi_{IM;j} | \hat{H} - i\frac{\partial}{\partial t'} | \Psi_{IM;j} \rangle \, dt' = 0$$

$$\Psi_{\text{trial}} \equiv |\Psi_{IM;j}\rangle = \mathcal{N} e^{z\hat{I}_-} \cdot e^{s\hat{j}_-} |IMI\rangle \otimes |jj\rangle$$

- $(z, r, \varphi, \hat{I}, |IMI\rangle)$ - core (**R**)
- $(s, f, \psi, \hat{j}, |jj\rangle)$ - single-particle (**j**)
- $(s, f, \psi, \hat{j}, |jj\rangle)$ - single-particle (**j**)
- $\{z, s\}$ $\rightarrow$ **phase space coordinates**

Constant of motion:

$$\mathcal{H} \equiv \langle \Psi_{IM;j} | \hat{H} | \Psi_{IM;j} \rangle$$

Canonical equations of motion:

$$\mathcal{S}_1 : \frac{\partial \mathcal{H}}{\partial r} = \dot{\varphi} \; , \; \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r}$$

$$\mathcal{S}_2 : \frac{\partial \mathcal{H}}{\partial f} = \dot{\psi} \; , \; \frac{\partial \mathcal{H}}{\partial \psi} = -\dot{f}$$

The two sets of Hamilton equations are **the semi-classical description** of the initial quantal $\hat{H}$.

Wobbling frequency

Solving $\mathcal{S}_1$ and $\mathcal{S}_2$ leads to the algebraic equation:

$$\Omega^4 + B\Omega^2 + C = 0$$

four solutions $\longrightarrow$ **only two are real**:

$$\Omega_{1,2} = \left[\frac{1}{2}\left(-B \mp \sqrt{B^2 - 4C}\right)\right]^{1/2}$$

A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

- Ω_1: wobbling frequency of the even-A core **R**
- Ω_2: wobbling frequency of the odd-nucleon **j**
- **Two wobbling phonon numbers:** n_{w_1} **and** n_{w_2}

Energy spectrum

Spectra of odd-A nuclei within $\mathbf{W}_1$

$$E_{I,n_1,n_2} = \epsilon_j + \mathcal{H}^I_{\text{min}} + \hbar\Omega^I_1\left(n_{w_1} + \frac{1}{2}\right) + \hbar\Omega^I_2\left(n_{w_2} + \frac{1}{2}\right)$$

- Phonon factor:

$$\mathcal{F}^I_{n_{w_1} n_{w_2}} = \hbar\Omega^I_1\left(n_{w_1} + \frac{1}{2}\right) + \hbar\Omega^I_2\left(n_{w_2} + \frac{1}{2}\right)$$

- $\mathcal{H}^I_{\text{min}}$ is the Classical Energy Function taken in its minimum point: $p_0 = (0, I; 0, j)$.

A new interpretation for TSD1 and TSD2

Previous models

$TSD1 =$ zero-phonon wobbling band
$TSD2 =$ one-phonon wobbling band...

Redefinition

$TSD1$ and $TSD2$ are **Signature Partner Bands** (In favor of Ultimate Cranker calculations, *Jensen, 2004*).

$$\left(\alpha_{fav} = +\frac{1}{2}\right) : \{TSD1\} \equiv \left\{\left[0^+, 2^+, 4^+, \dots\right] \otimes [j^\pi = 13/2^+]\right\}$$
$$\left(\alpha_{unfav} = -\frac{1}{2}\right) : \{TSD2\} \equiv \left\{\left[1^+, 3^+, 5^+, \dots\right] \otimes [j^\pi = 13/2^+]\right\}$$

$TSD4$: **ground-state wobbling band**, $\pi(h_{9/2})$ configuration.

A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

A new band structure for ^{163}Lu

Band	Spins	π	α	$\pi(I_j)$	$\mathbf{W_0}$: $\mathscr{C}+\mathscr{Q}_p$	$\mathbf{W_1}$: $\mathscr{C}+\mathscr{Q}_p$
TSD1	$13/2, 17/2 \ldots 97/2$	$+$	$+1/2$	$\pi(i_{13/2})$	$0^+, 2^+, 4^+, \ldots$	$0^+, 2^+, 4^+, \ldots$
TSD2	$27/2, 31/2 \ldots 91/2$	$+$	$-1/2$	$\pi(i_{13/2})$	$\mathrm{TSD1} + 1\Gamma^\dagger$	$1^+, 3^+, 5^+, \ldots$
TSD3	$33/2, 37/2 \ldots 85/2$	$+$	$+1/2$	$\pi(i_{13/2})$	$\mathrm{TSD1} + 2\Gamma^\dagger$	$\mathrm{TSD2} + \Gamma^\dagger$
TSD4	$47/2, 51/2 \ldots 83/2$	$-$	$-1/2$	$\pi(h_{9/2})$	$\mathrm{TSD1} + 3\Gamma^\dagger$	$1^+, 3^+, 5^+, \ldots$

Bands	n_{w_1}	n_{w_2}	$\mathcal{F}^I_{n_{w_1} n_{w_2}}$	I_0	I_t	$\mathcal{Q}$
TSD1	0	0	$\mathcal{F}^I_{00} = \frac{1}{2}\left(\Omega^I_1 + \Omega^I_2\right)$	$13/2^+$	$97/2^+$	$j^\pi = 13/2^+ \overset{not}{\equiv} \mathcal{Q}_1$
TSD2	**0**	**0**	$\mathcal{F}^I_{00} = \frac{1}{2}\left(\Omega^I_1 + \Omega^I_2\right)$	$27/2^+$	$91/2^+$	$j^\pi = 13/2^+ \overset{not}{\equiv} \mathcal{Q}_1$
TSD3	1	0	$\mathcal{F}^{I-1}_{10} = \frac{3}{2}\Omega^{I-1}_1 + \frac{1}{2}\Omega^{I-1}_2$	$33/2^+$	$85/2^+$	$j^\pi = 13/2^+ \overset{not}{\equiv} \mathcal{Q}_1$
TSD4	**0**	**0**	$\mathcal{F}^I_{00} = \frac{1}{2}\left(\Omega^I_1 + \Omega^I_2\right)$	$47/2^-$	$83/2^-$	$j^\pi = 9/2^- \overset{not}{\equiv} \mathcal{Q}_2$

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Extension to $A \approx 160$ mass region

The model was further applied to the other Lu isotopes where wobbling motion has been observed.

^{161}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$21/2^+, 25/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$4^+, 6^+, 8^+ \ldots$	$(0,0)$	$21/2$
TSD2	$31/2^+, 35/2^+, \ldots, 79/2^+$	$j^\pi = 13/2^+$	$9^+, 11^+, 13^+ \ldots$	$(0,0)$	

^{165}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \ldots$	$(0,0)$	$25/2$
TSD2	$35/2^+, 39/2^+, \ldots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \ldots$	$(0,0)$	
TSD3	$41/2^+, 45/2^+, \ldots, 81/2^+$	$j^\pi = 13/2^+$	$\text{TSD2} + \Gamma^\dagger$	$(1,0)$	

^{167}Lu Bands	Spins	$\mathcal{Q}$	$\mathscr{C}$	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \ldots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \ldots$	$(0,0)$	$25/2$
TSD2	$35/2^+, 39/2^+, \ldots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \ldots$	$(0,0)$	

A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

$\mathbf{W}_1$ — Numerical Results

- Free parameters in the model $\rightarrow \mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.
- V: single-particle potential strength $\propto \beta_2$ (*Tanabe, 2017*)

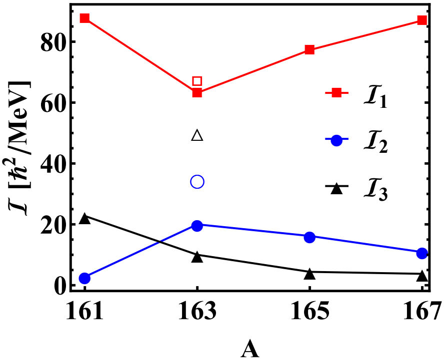
Fitting procedure

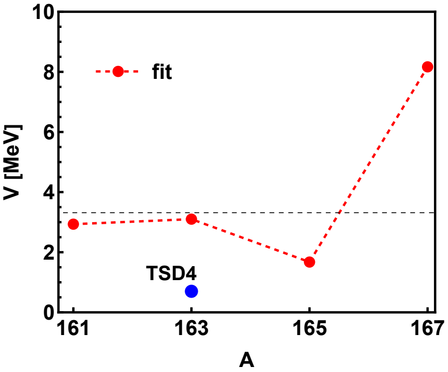
$$\chi^2 = \frac{1}{N_T} \sum_i \frac{\left(E_{\text{exp}}^{(i)} - E_{\text{th}}^{(i)}\right)^2}{E_{\text{exp}}^{(i)}}$$

^{163}Lu-TSD4: separate fitting procedure (different nucleon configuration)

Isotope	Bands	$\mathcal{I}_1$ $[\hbar^2/\text{MeV}]$	$\mathcal{I}_2$ $[\hbar^2/\text{MeV}]$	$\mathcal{I}_3$ $[\hbar^2/\text{MeV}]$	V [MeV]	γ [°]	n.o.s	E_{rms} [MeV]
^{161}Lu	TSD1-2	87.555	2.773	22.744	2.933	20	29	0.168
^{163}Lu	TSD1-3	63.2	20	10	3.1	17	52	0.264
	TSD4	67	34.5	50	0.7	17	10	0.057
^{165}Lu	TSD1-3	77.295	16.184	4.399	1.673	20	42	0.125
^{167}Lu	TSD1-2	87.032	10.895	3.758	8.167	19.48	30	0.165

Graphical representation for MOIs and V

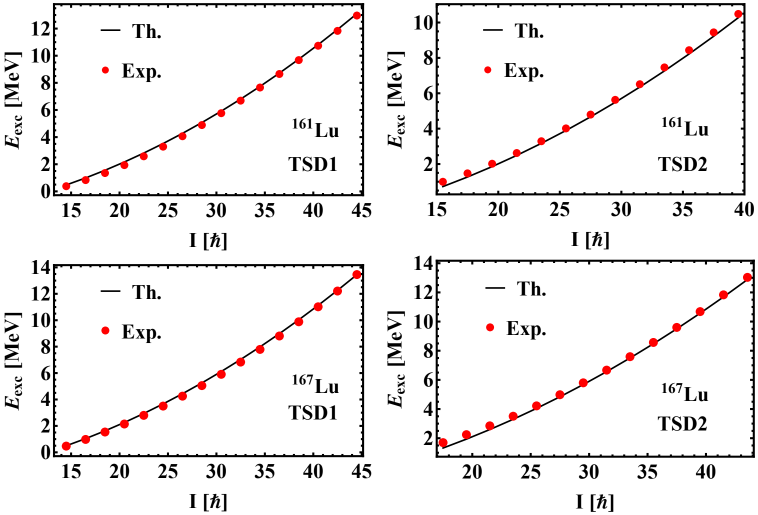




$V \approx 3$ MeV $\rightarrow$ agreement with other calculations (*Tanabe, 2017*)

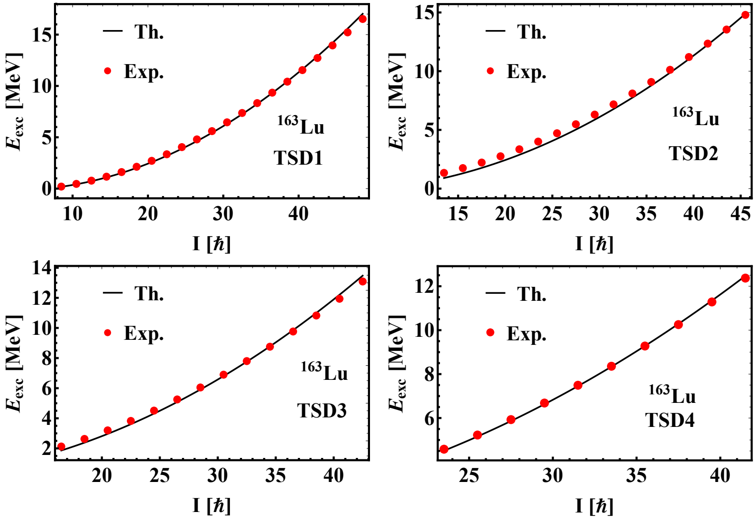
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Excitation Energies — ^{161,167}Lu



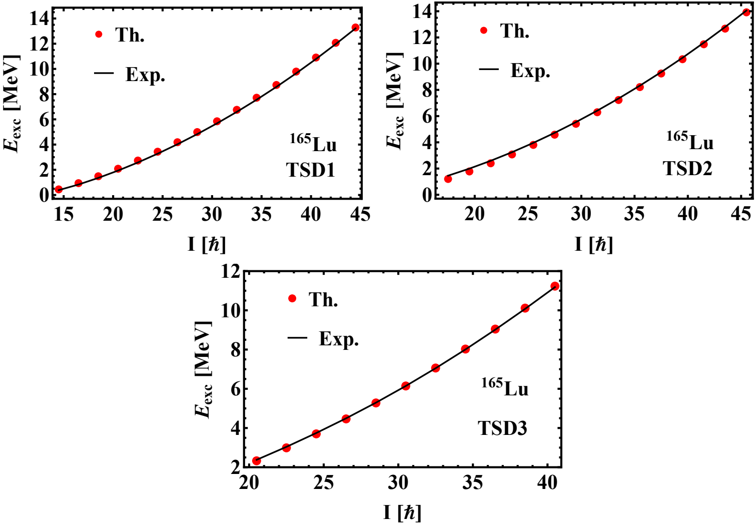
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Excitation Energies — ^{163}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

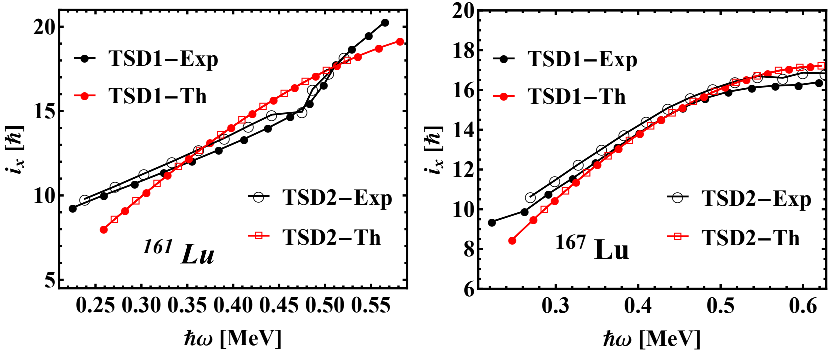
Excitation Energies — ^{165}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

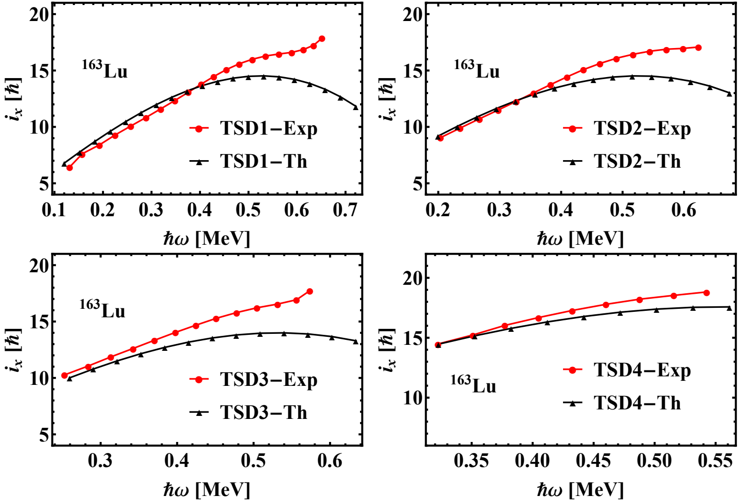
Alignment — ^{161,167}Lu

$$i_x = I - I_{ref} \ ,$$
$$I_{ref} = \mathcal{I}_0\omega + \mathcal{I}_1\omega^3 \ .$$



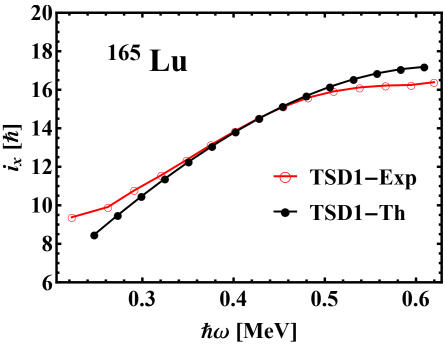
A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

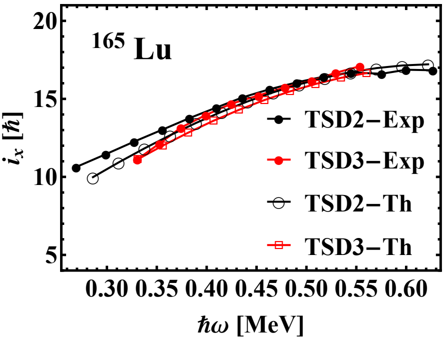
Alignment — ^{163}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Alignment — ^{165}Lu

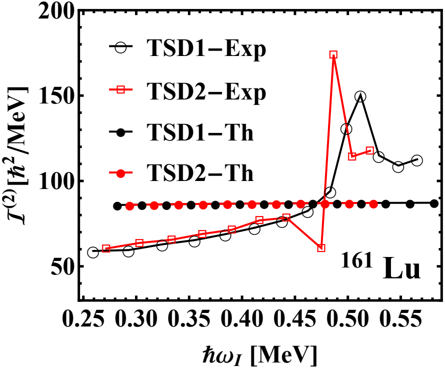


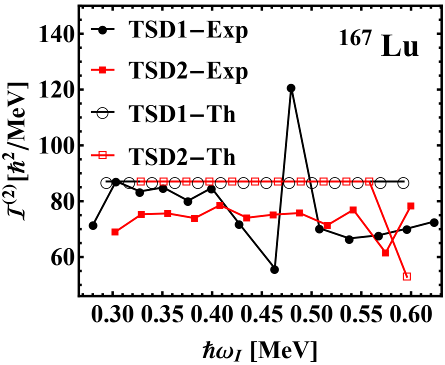


A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{161,167}Lu

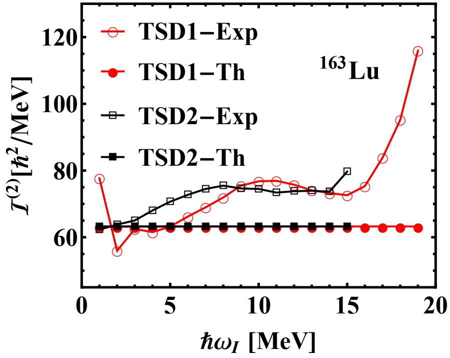
$$\mathcal{I}^{(2)}(I) = \hbar \frac{\mathrm{d}I_x}{\mathrm{d}\omega} = \hbar^2 \left(\frac{\mathrm{d}^2 E}{\mathrm{d}I_x^2}\right)^{-1}$$

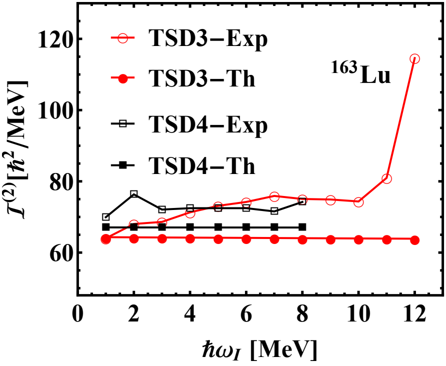




A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

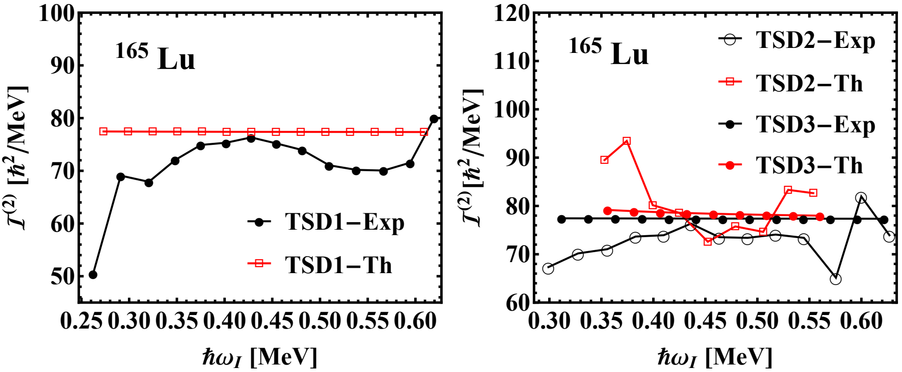
Dynamic Moment of Inertia — ^{163}Lu





A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{165}Lu



A. A. Raduta, **R. Poenaru**, C. M. Raduta, Phys. Rev. C 101, 2020.

Electromagnetic Calculations

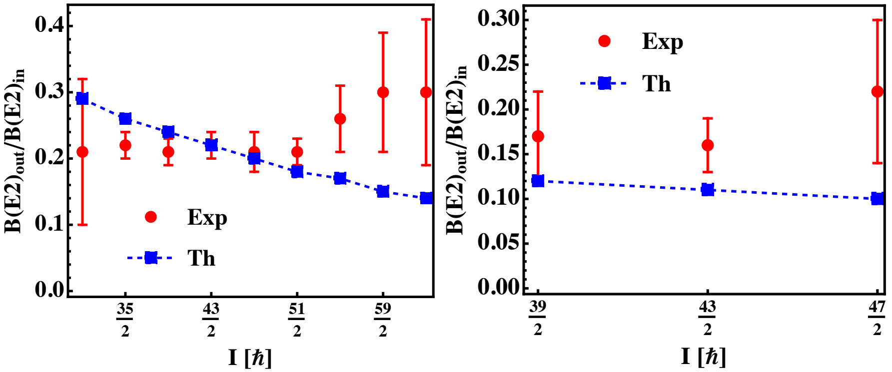


Figure: E2 Branching ratio. **Left**: ^{163}Lu (TSD2) **Right**: ^{165}Lu (TSD2).

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

Electromagnetic Calculations II

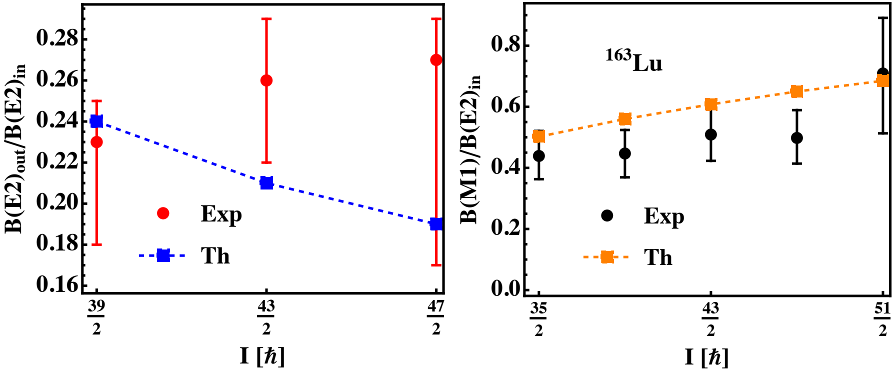


Figure: **Left**: E2 Branching ratio in ^{163}Lu (TSD3). **Right**: The ratio $B(M1)/B(E2)_{\text{in}}$ for states $TSD2 \rightarrow TSD1$ (in units of $\mu_N^2/(\mathrm{e}^2 b^2)$).

A.A. Raduta, **R. Poenaru**, C.M. Raduta, Journal of Physics G 47, 2020.

$\mathbf{W}_1$ — Remarks

Characteristics

- **+** Full semi-classical description (TDVE) with good numerical results
- **+** Deformation parameters are self-consistent (agree with exp. values)
- **−** separate fit for TSD4 (different nucleonic configuration)
- **−** Two sets of MOIs for ^{163}Lu

Onset of another redesign

Start of $\mathbf{W}_2$ formalism in Chapter 5

Fresh-Up 2: $\mathbf{W}_2$

Novel description

- All four bands in ^{163}Lu described by the same triaxial core + odd-particle coupling $\longrightarrow \mathcal{Q}_1 = \pi^+(i_{13/2})$
- The adopted wave-function admits states of both **positive and negative parity**.

$$\bar{\Psi} = p\Psi \ ,$$
$$\bar{\Psi}(r,\varphi;f,\psi) = \pm\,\Psi(r,\varphi;f,\psi)$$

- p eigenvalues of $\hat{P}_T = \hat{P}_{\text{core}} \otimes \hat{P}_{\text{sp}}$.

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New band structure in ^{163}Lu

$$E_{I,0,0}^{\mathrm{TSD1}} = \epsilon_{13/2} + \mathcal{H}_{\mathrm{min}}^{I} + \mathcal{F}_{00}^{I}\ ,\ \ I^{\pi} = 13/2^{+}, 17/2^{+}, 21/2^{+} \ldots\ ,$$
$$E_{I,0,0}^{\mathrm{TSD2}} = \epsilon_{13/2}^{1} + \mathcal{H}_{\mathrm{min}}^{I} + \mathcal{F}_{00}^{I}\ ,\ \ I^{\pi} = 27/2^{+}, 31/2^{+}, 35/2^{+} \ldots\ ,$$
$$E_{I,1,0}^{\mathrm{TSD3}} = \epsilon_{13/2} + \mathcal{H}_{\mathrm{min}}^{I-1} + \mathcal{F}_{10}^{I-1}\ ,\ \ I^{\pi} = 33/2^{+}, 37/2^{+}, 41/2^{+} \ldots\ ,$$
$$E_{I,0,0}^{\mathrm{TSD4}} = \epsilon_{13/2}^{2} + \mathcal{H}_{\mathrm{min}}^{I} + \mathcal{F}_{00}^{I}\ ,\ \ I^{\pi} = 47/2^{-}, 51/2^{-}, 55/2^{-} \ldots .$$

Band	n_s	$\mathbf{j}_{\mathcal{Q}}$	$\mathbf{R}_{\mathscr{C}}$ - Sequence	$\mathbf{I}$ - Sequence	Coupling
TSD1	21	$\mathcal{Q}_1$	$\mathscr{C}_1 = 0^{+}, 2^{+}, 4^{+}, \ldots$	$13/2^{+}, 17/2^{+}, 21/2^{+}, \ldots$	$\mathscr{C}_1 + \mathcal{Q}_1$
TSD2	17	$\mathcal{Q}_1$	$\mathscr{C}_2^{+} = 1^{+}, 3^{+}, 5^{+}, \ldots$	$27/2^{+}, 31/2^{+}, 35/2^{+}, \ldots$	$\mathscr{C}_2^{+} + \mathcal{Q}_1$
TSD3	14	$\mathcal{Q}_1$	1-phonon exc.	$33/2^{+}, 37/2^{+}, 41/2^{+}, \ldots$	
TSD4	11	$\mathcal{Q}_1$	$\mathscr{C}_2^{-} = 1^{-}, 3^{-}, 5^{-}, \ldots$	$47/2^{-}, 51/2^{-}, 55/2^{-}, \ldots$	$\mathscr{C}_2^{-} + \mathcal{Q}_1$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New results for ^{163}Lu

Model requires **a unique set of parameters**: $\mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

$\mathcal{I}_1$ [$\hbar^2$/MeV]	$\mathcal{I}_2$ [$\hbar^2$/MeV]	$\mathcal{I}_3$ [$\hbar^2$/MeV]	γ [deg.]	V [MeV]
72	15	7	22	2.1

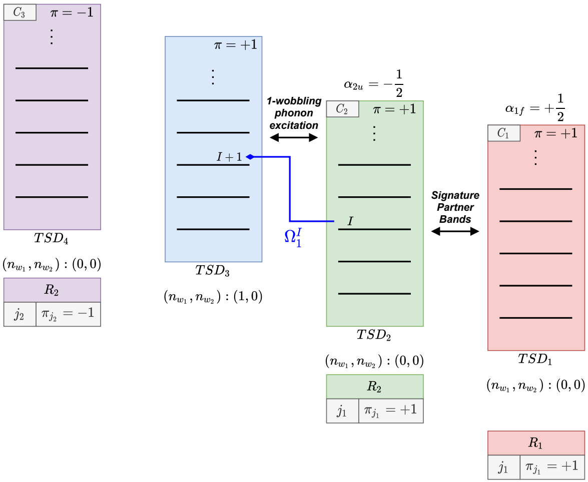
Remarks

- γ in agreement with exp. value $\gamma_{exp} = 20^\circ$ (*Jensen, 2004*)
- Slight *decrease* of V (? breaking of parity symmetry quenches the quadrupole deformation)
- overall $E_{RMS} \approx 79$ keV: **first semi-classical description for a nucleus with deviations smaller than** 100 **keV**.

First model to describe ^{163}Lu wobbling structure.

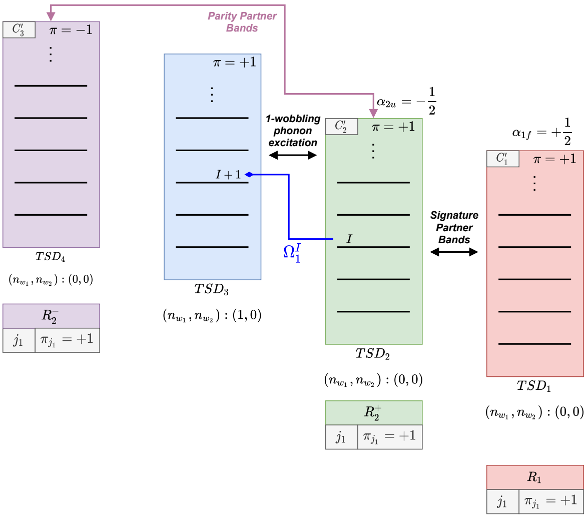
R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

W_1 picture



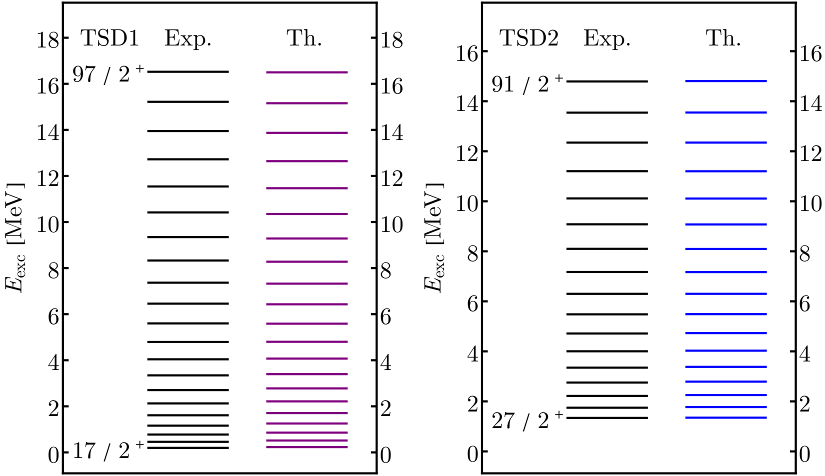
R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

W_2 picture



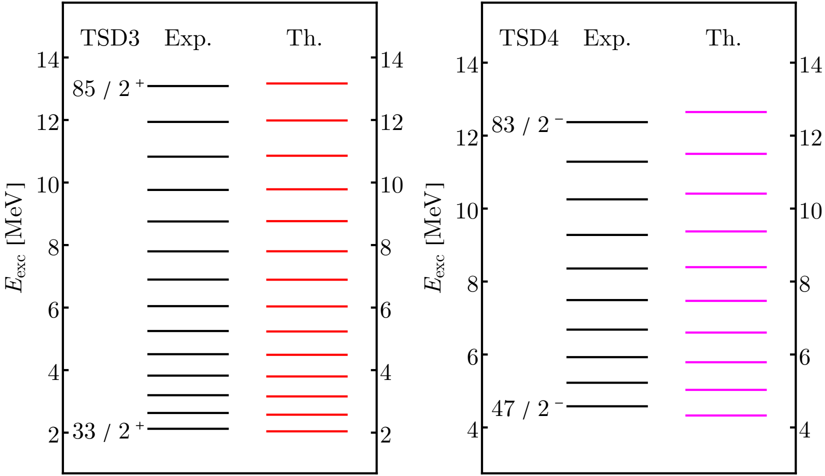
R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

Energy spectrum



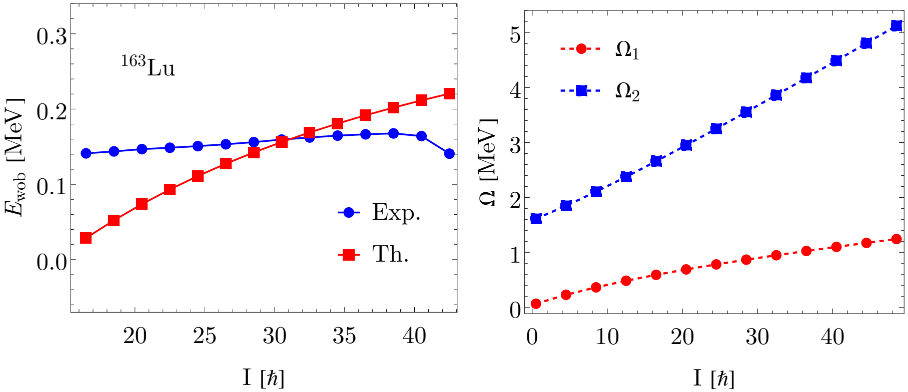
R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Energy spectrum II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

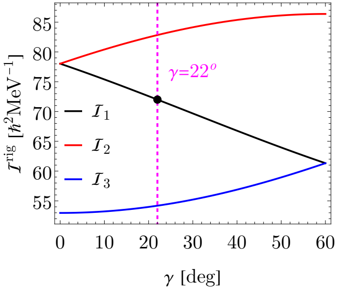
Wobbling Energies

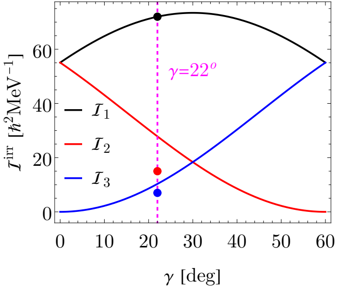


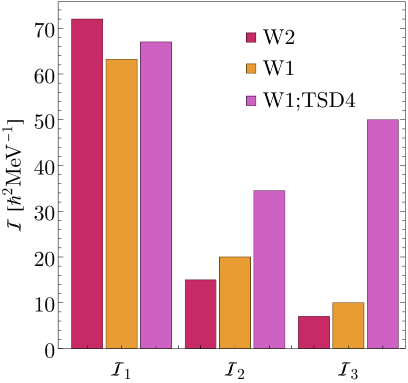
The wobbling energy (**left**) and the two wobbling frequencies (**right**) for ^{163}Lu. **Decreasing trend of $E_{\rm wob}$ in agreement with arguments of Frauendorf 2014.**

R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Moments of inertia for $\mathbf{W}_2$







$\mathbf{W}_2$: *hydrodynamical character* of the triaxial nucleus.

Results presented at the International Conference NSP, 2023, Turkey.

Classical Energy Function

Angular momentum

Polar representation of the angular momentum and $\mathcal{H}$.

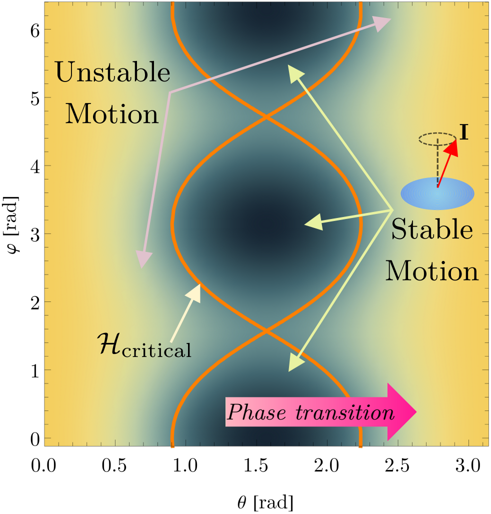
$$\mathbf{I} = \{I_1, I_2, I_3\} \equiv \{x_1, x_2, x_3\} \ ,$$
$$x_1 = I \sin\theta \cos\varphi \ , \ x_2 = I \sin\theta \sin\varphi \ , \ x_3 = I \cos\theta .$$

$$\mathcal{H}\left.\right|_{p_0} = I\left(I - \frac{1}{2}\right)\sin^2\theta \cdot \mathcal{A}_\varphi - 2A_1 Ij \sin\theta + T_{\text{core}} + T_{\text{sp}} \ ,$$
$$\mathcal{A}_\varphi = A_1 \cos^2\varphi + A_2 \sin^2\varphi - A_3 .$$
$$T_{\text{core}} = \frac{I}{2}(A_1 + A_2) + A_3 I^2 \ ,$$
$$T_{\text{s.p.}} = \frac{j}{2}(A_2 + A_3) + A_1 j^2 - V\frac{2j-1}{j+1}\sin\left(\gamma + \frac{\pi}{6}\right)$$

CEF — Stability Regions

Table: The minimum points of $\mathcal{H}$. Using the MOIs from the fitting procedure

Minimal point	θ [rad]	φ [rad]	A_k ordering
m_1	$\pi/2$	0	$A_3 > A_2 > A_1$
m_2	$\pi/2$	π	$A_3 > A_2 > A_1$
m_3	$\pi/2$	2π	$A_3 > A_2 > A_1$

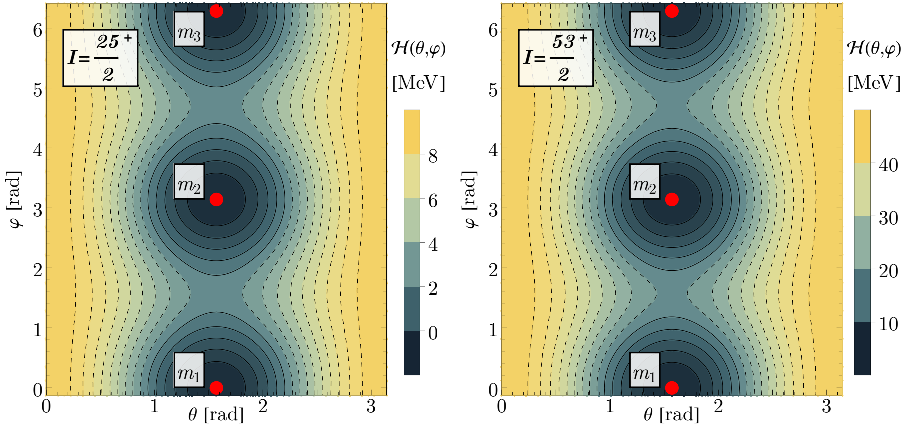


Semi-classical feature

This is the first classical description of the *wobbling stability* for an odd-mass nucleus.

Polar representation of $\mathcal{H}$

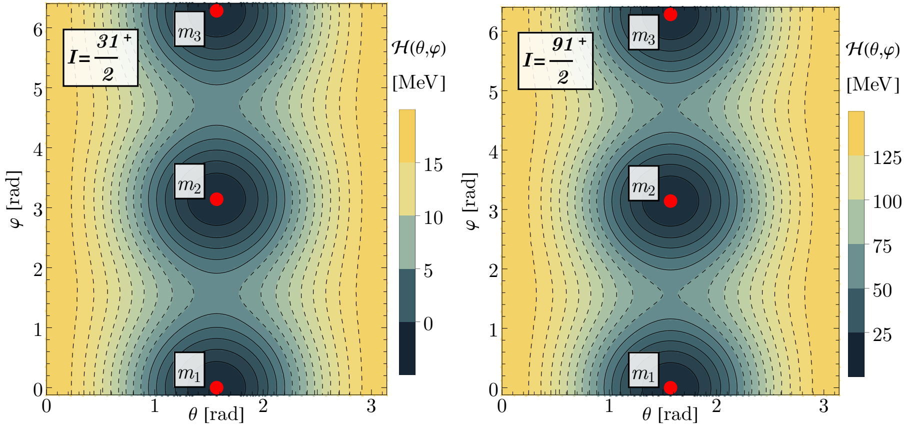
Figure: ^{163}Lu TSD1



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of $\mathcal{H}$ II

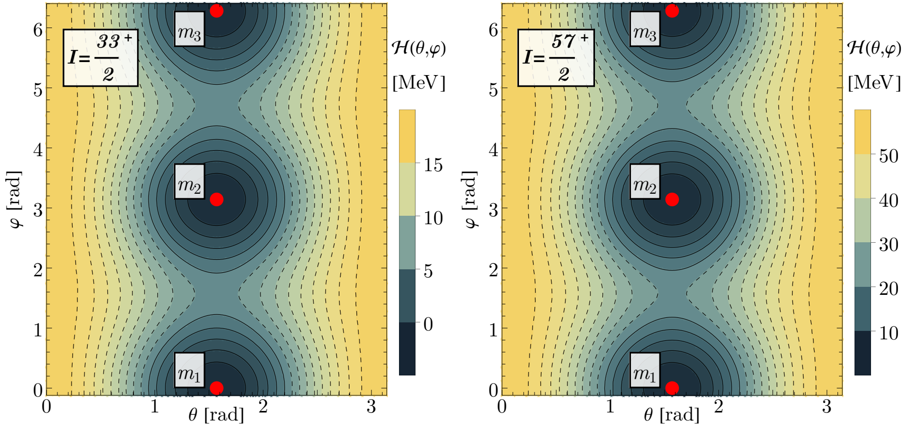
Figure: ^{163}Lu TSD2



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of $\mathcal{H}$ III

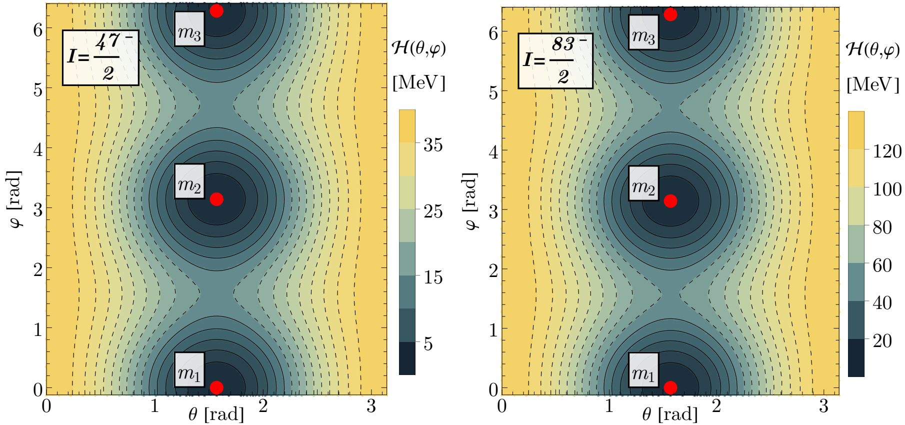
Figure: ^{163}Lu TSD3



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of $\mathcal{H}$ IV

Figure: ^{163}Lu TSD4



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

3D interpretation of the WM

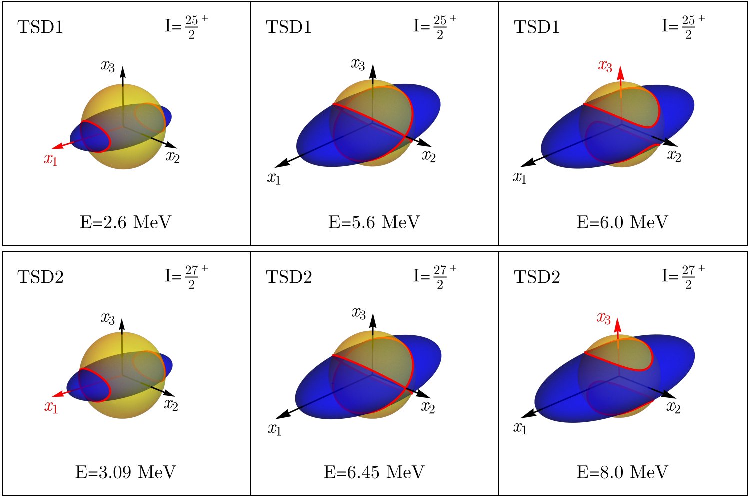
- Formalism $\mathbf{W}_2$ gives a 3D interpretation of the nuclear wobbling motion
- **Classical Trajectories**: intersection curves between the **triaxial energy** and the **total angular momentum**

$$I^2 = x_1^2 + x_2^2 + x_3^2,$$

$$E = \left(1 - \frac{1}{2I}\right) A_1 x_1^2 + \left(1 - \frac{1}{2I}\right) A_2 x_2^2 + \left[\left(1 - \frac{1}{2I}\right) A_3 + A_1 \frac{j}{I}\right] x_3^2 - \\ - I\left(I - \frac{1}{2}\right) A_3 - 2A_1 I j + T_{\text{rot}} + T_{\text{sp}}.$$

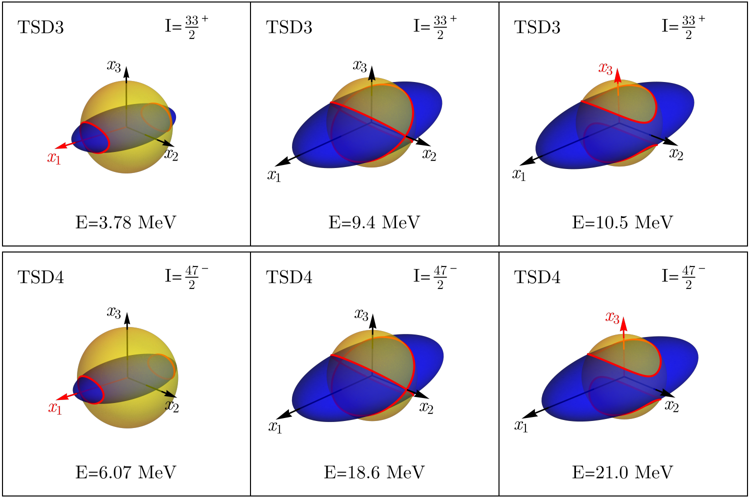
Energy surface: $E' = \frac{x_1^2}{s_1} + \frac{x_2^2}{s_2} + \frac{x_3^2}{s_3}$.

^{163}Lu — Classical trajectories



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

^{163}Lu — Classical trajectories II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

^{163}Lu — Classical trajectories III

Model characteristics

- From the figures:
  - the state $I = 25/2^+$ from TSD1 has a real energy of about 2.6 MeV
  - the **critical value** requires twice that amount.
- unstable trajectories can be identified for each state (i.e., the middle figs)
- **phase transitions** between rotational modes can be identified (i.e., right figs)

Results were presented at the International Conference TIM-21, Timisoara.

New Boson Method for odd-mass nuclei

Chapter 6

Study of the Wobbling Motion via a Boson Description

Model Features

- Extend the boson description for the even-even nuclei made by the team in 2017 to odd-mass nuclei.
- Use the *Cranked* Particle-Rotor Model to study the wobbling spectrum of ^{135}Pr.

$$\hat{H}_{\text{rot}} = AH' + H_{sp} + \text{s.t.} \tag{1}$$

Rotational Hamiltonian

Rigid coupling Hamiltonian

$$H' = a_1\left(\hat{I}_+^2 + \hat{I}_-^2\right) + a_2\left(\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+\right) + a_3\hat{I}_0 \ ,$$

$$H_{sp} = \sum_{k=1}^{2} A_k\hat{j}_k^2 \ , \ \text{s.t.} = A_1 I^2 - A_2 j_2 I \ .$$

- the triaxial rigid rotor is constrained to move around the 1-axis.
- adopted Frozen-Alignment approximation: $\mathbf{j} = (j\cos\theta, j\sin\theta, 0)$ (*Frauendorf, 2014*)

Thank you for your attention ❤