

# A Systematic Description of the Wobbling Motion in Odd-Mass Nuclei Within a Semi-Classical Formalism

**PhD Candidate**

Robert Poenaru<sup>1,2</sup>

**Scientific Supervisor**

Prof. Dr. Em. A. A. Raduta<sup>2</sup>

<sup>1</sup>Doctoral School of Physics, UB

<sup>2</sup>Department of Theoretical Physics, IFIN-HH

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# Aim



## Research Objectives

- Extend the current interpretation of the **Nuclear Triaxiality** in the context of its unique fingerprint: **Wobbling Motion**.
- Adopt a framework that is as close as possible to a **classical picture**.

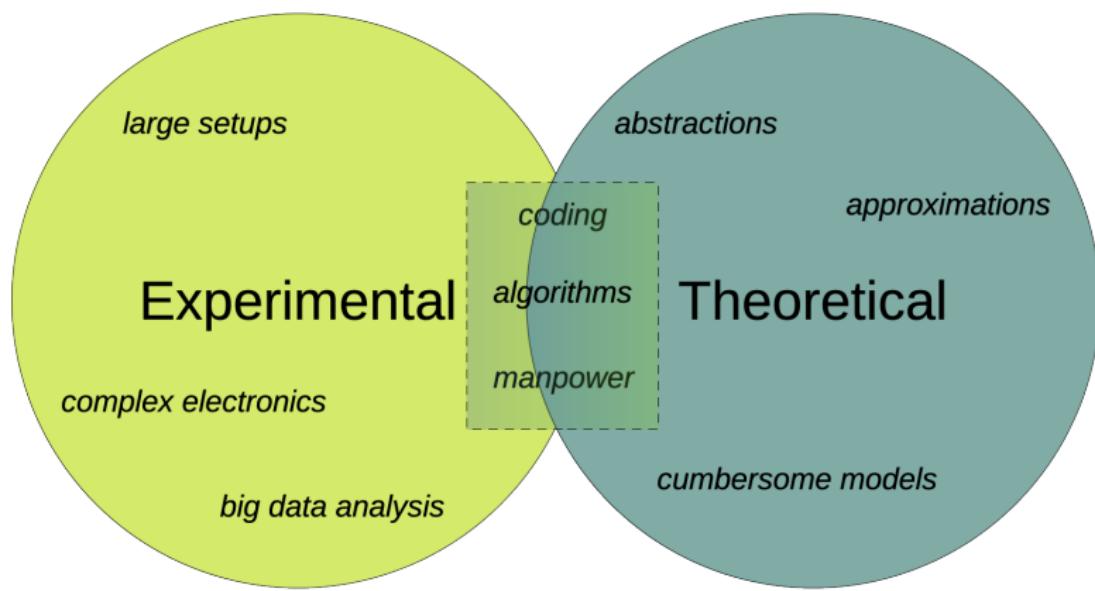


## Objectives exclusive to the thesis

- Enough context towards a better understanding of the underlying concepts, methods, and results.
- create a completely *open-source* project.

# Motivation

- **Nuclear Triaxiality** recently became a *hot topic* within the scientific community.
- Identifying nuclei with triaxial deformations represents a real **experimental** and **theoretical** challenge.



# Nuclear Deformation

## Nuclear shapes

Most generally described in terms of the **nuclear radius**:

$$R(\theta, \varphi) = R_0 \left( 1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} Y_{\lambda}^{\mu}(\theta, \varphi) \right)$$

## Quadrupole deformations $\lambda = 2$

- **For us:** Most relevant modes are the **quadrupole vibrations**  $\lambda = 2$   
 $\Rightarrow$  *Play a crucial role in the rotational spectra of nuclei:*
- $\alpha_{2\mu}$  reduced to only two *deformation parameters*:  $\beta_2$  (**eccentricity**) and  $\gamma$  (**triaxiality**) (Bohr and Mottelson, 1969).

# Axial shapes

- Most of the nuclei are either **spherical** or **axially symmetric** in their ground-state (*Budaca, 2018*).
- Moments of inertia:  $\mathcal{I}_{1,2,3}$ : two are equal, one is different.

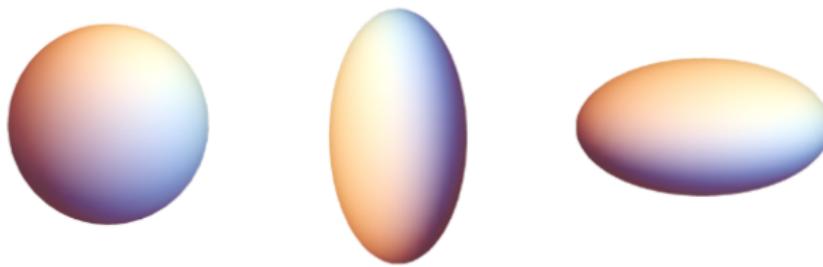
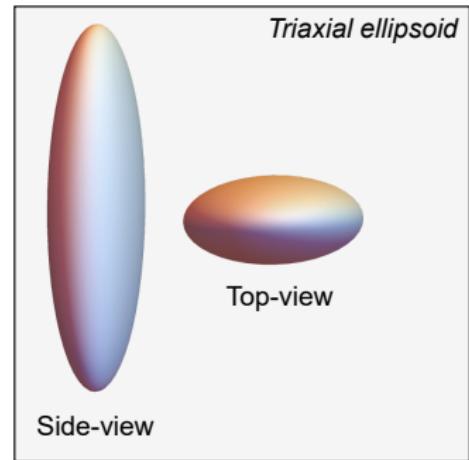
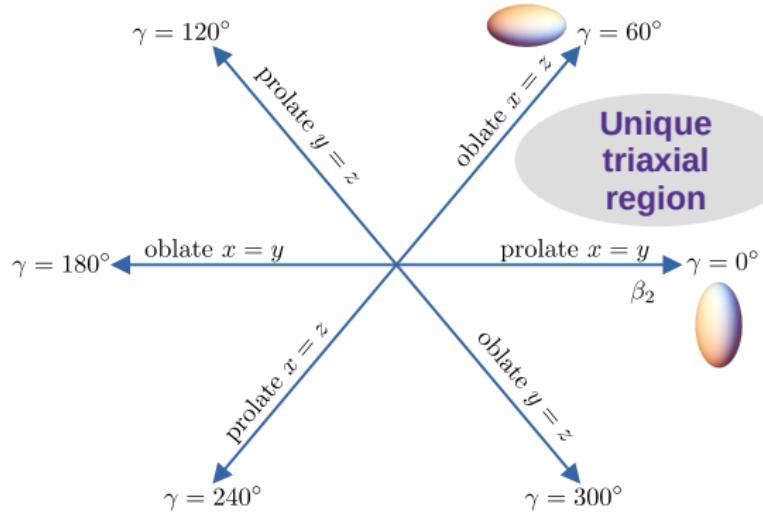


Figure: **spherical**:  $\beta_2 = 0$  **prolate**:  $\beta_2 > 0$  **oblate**:  $\beta_2 < 0$ . ( $\gamma = 0^\circ$ ).

# Non-axial shapes

- The triaxiality parameter  $\gamma \neq 0^\circ$ : departure from axial symmetry.
- Moments of inertia:  $I_1 \neq I_2 \neq I_3$ .



# Fingerprints of Triaxiality

## Evidence

- Currently, there are **only two** well-established phenomena uniquely attributed to triaxial deformation.
  - ① **Wobbling Motion** - WM (*Bohr and Mottelson, 1970s*)
  - ② Chiral Motion -  $\chi$ M (*Frauendorf, 1997*)
- These two can be measured/detected experimentally.

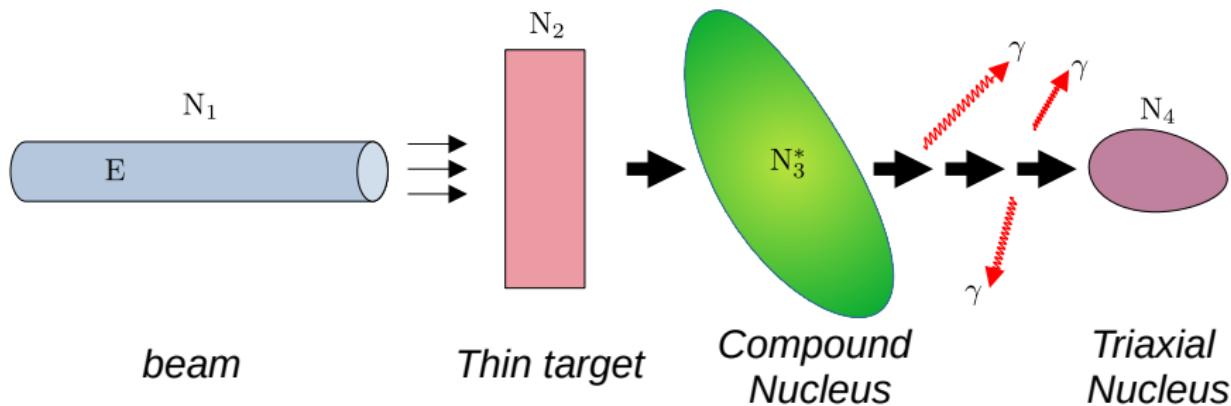
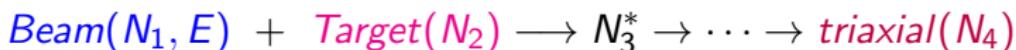
## Goal

**Describe the elusive character of Wobbling Motion in the context of nuclear triaxiality.**

# Q Probing triaxiality in nuclei

Triaxial nuclei can be observed/obtained in several experiments:

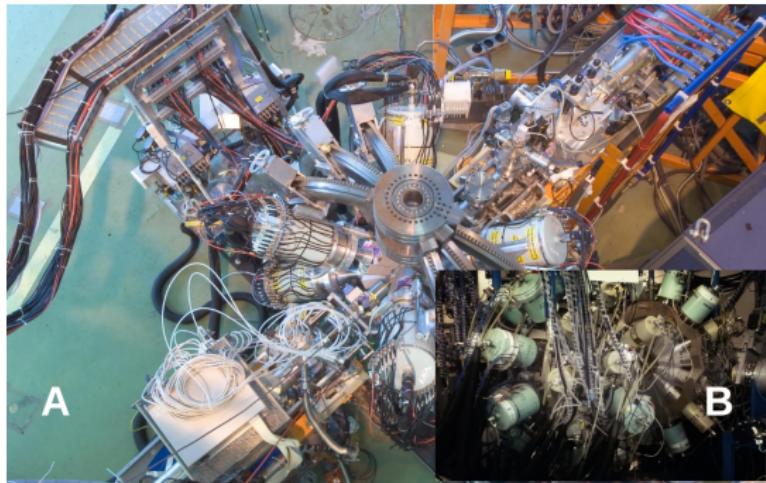
- Nuclear fission:  $A \rightarrow B + C$
- Nuclear fusion:  $X + Y \rightarrow Z$
- **Fusion-evaporation reactions:** Long-lived + enhanced deformation



# Q Nuclear facilities

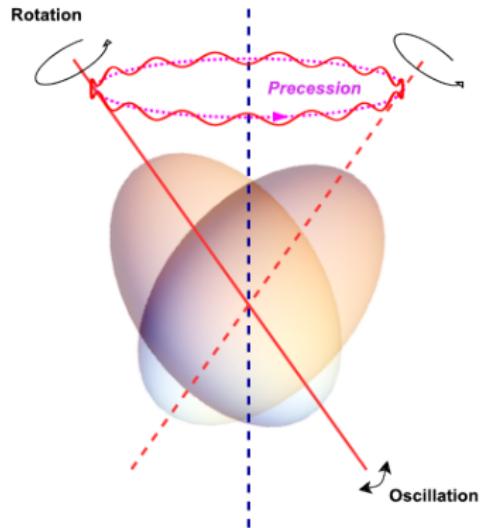
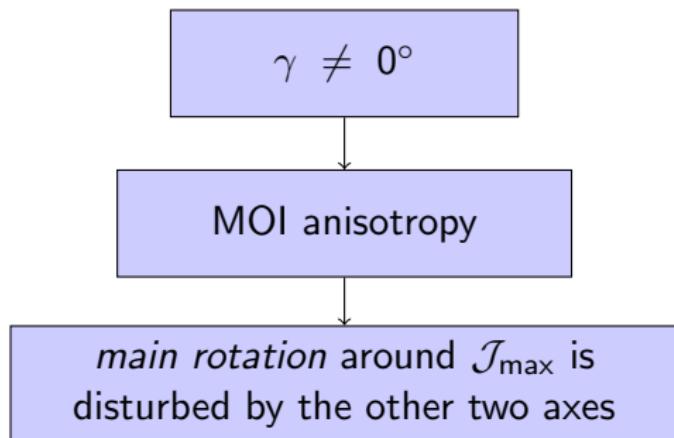


**Figure:** Gammasphere detector,  
ANL-ATLAS USA. *Source:*  
[aps.org](http://aps.org)



**Figure:** a) IDS detector, CERN. *Source:*  
[isodel.web.cern.ch](http://isodel.web.cern.ch) b) JUROGAM II, Finland.  
*Source:* [twitter.com](https://twitter.com)

# Wobbling Motion



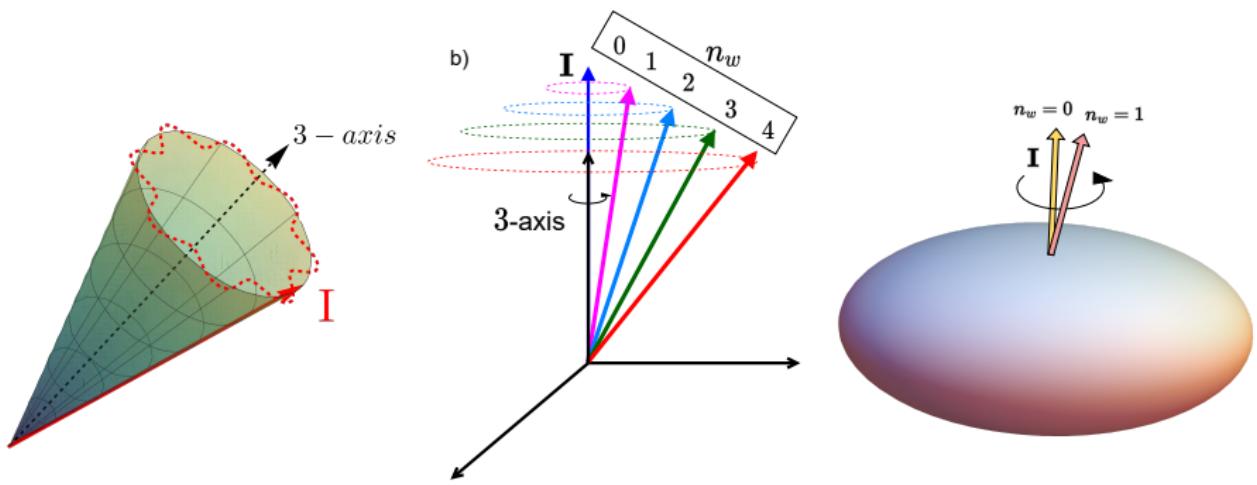
## Wobbling Effect

- The **total angular momentum** of the nucleus **precesses** and **oscillates** around  $\mathcal{J}_{\max}$ .

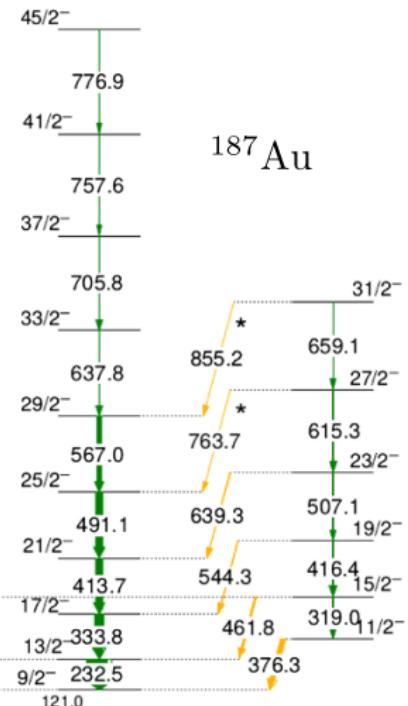
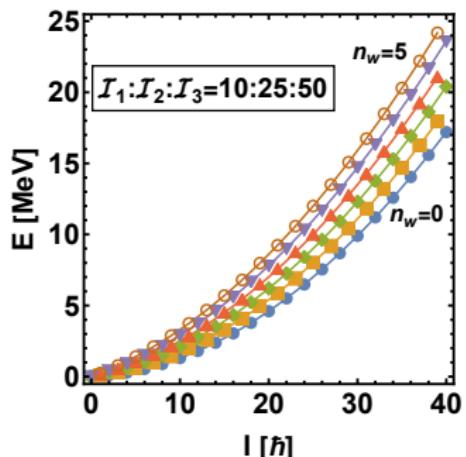
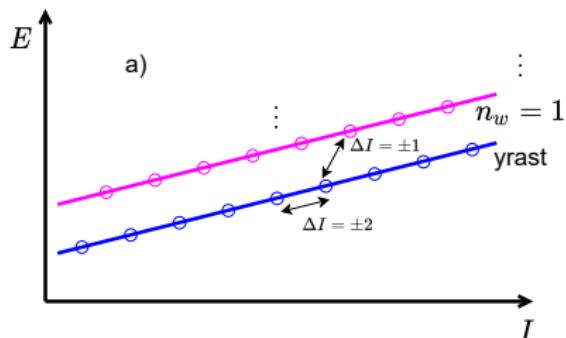
# Wobbling Motion

## Harmonic oscillation

- Precession of  $\mathbf{I}$  is affected by **rotational frequency** and/or **tilting**
- Tilting only by "specific" amount  $\rightarrow$  **harmonic character**  $\rightarrow$  **wobbling phonon**:  $n_w = 0, 1, 2, \dots$



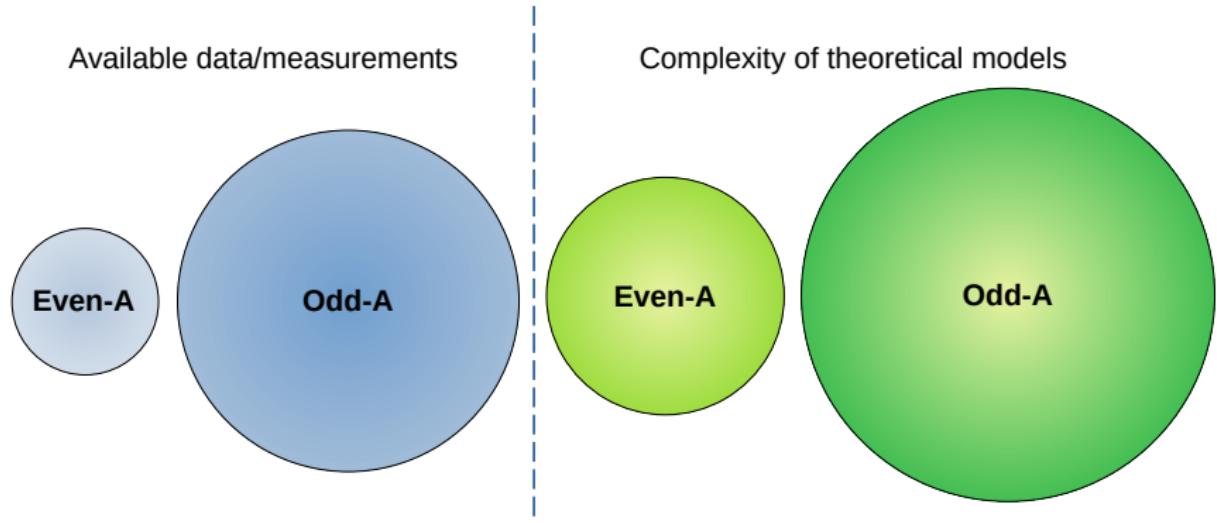
# Wobbling Motion II



Sensharma, 2020.

# Even- $A$ vs. Odd- $A$ Picture

- Predicted for even- $A$  nuclei more than 50 years ago.
- First experimental evidence:  $^{163}\text{Lu}$  (*Ødegård, 2001*).
- Current mass-regions for wobblers:  $A \approx [130, 160, 180]$ .



# Wobbling Motion in $^{130}\text{Ba}$

**Q** Experimental measurements show **two** wobbling bands (*Petrache et. al. 2019*).

## Harmonic formalism

**Harmonic Approximation** (*Bohr & Mottelson, 1969*):

$$E_{I,n_w} = A_3 I(I+1) + \hbar\omega_w \left( n_w + \frac{1}{2} \right),$$
$$A_3 = (2\mathcal{I}_3)^{-1}.$$

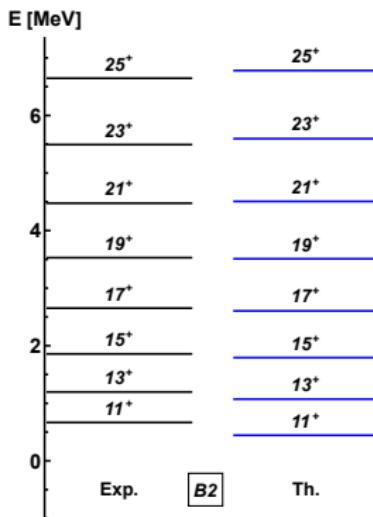
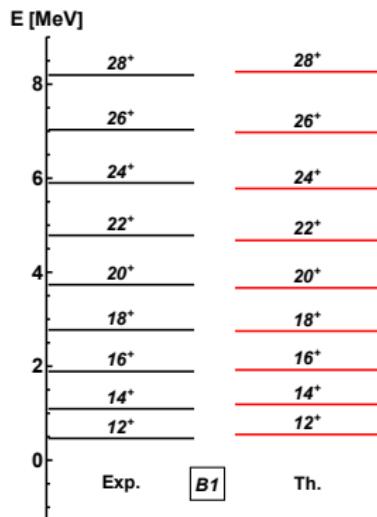
(rotational term + wobbling frequency)



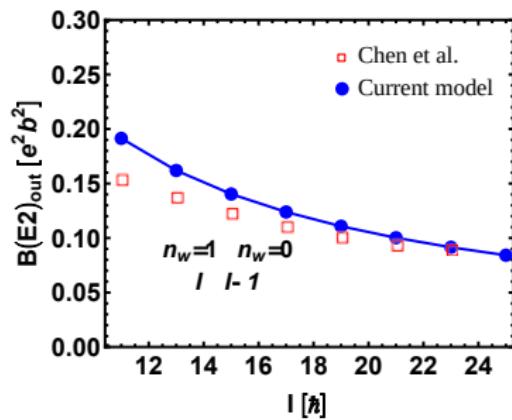
GALILEO, LNL, Source: [lnl.infn.it](http://lnl.infn.it)

Fusion evaporation:  $^{13}\text{C}$  beam of  
 $E = 65$  MeV and  $^{122}\text{Sn}$  target.

# Results for $^{130}\text{Ba}$



$\mathcal{P}_{\text{fit}}$			
$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	Unit
27	22	<b>43</b>	$\hbar^2 \text{MeV}^{-1}$



Full description: Chapter 3 (Section 3.1.2)

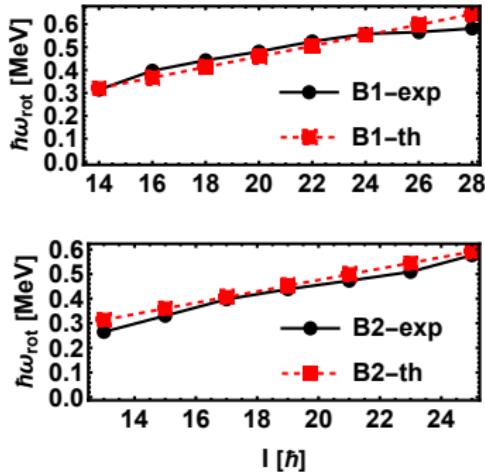
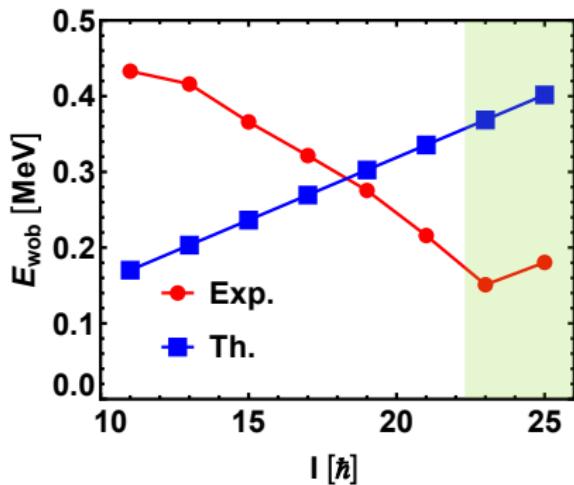
Results presented at the international conference NSP-2022, Turkey.

# Results for $^{130}\text{Ba II}$

## Excitation energies vs. Wobbling Energies:

$$E_{\text{wob}}(I_{\text{even}}) = E_{I,n} - E_{I,0} ,$$

$$E_{\text{wob}}(I_{\text{odd}}) = E_{I,n} - \frac{1}{2} (E_{I-1,0} + E_{I+1,0})$$



Results presented at the international conference NSP-2022, Turkey.

# Starting Point

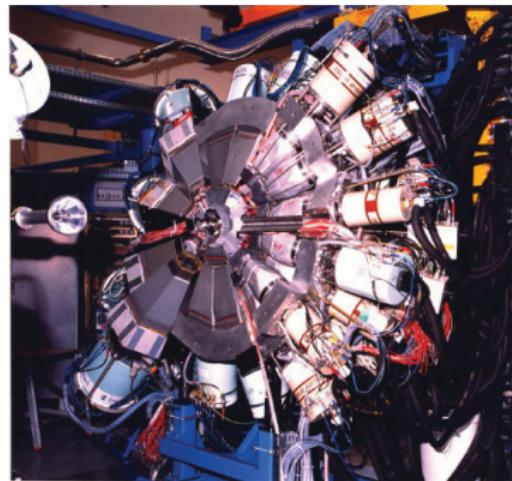
- A. A. Raduta, R. Poenaru, L. Gr. Ixaru, PRC, 2017 + ■ A. A. Raduta, R. Poenaru, Al. H. Raduta, JPG, 2018 →  $W_0$  in the thesis.

## Framework

- First semi-classical description for the  $^{163}\text{Lu}$ , using the **Particle-Rotor-Model** (*Hamamoto, 2002.*) for an odd-mass nucleus in the  $A \approx 160$  region.

## PRM

An odd-nucleon moving in a quadrupole deformed mean field generated by an even-even triaxial core.



Euroball IV, Strasbourg, Source:  
[technology.i.stfc.ac.uk](http://technology.i.stfc.ac.uk)

Fusion evaporation:  $^{29}\text{Si}$  beam of  $E = 152$  MeV and  $^{139}\text{La}$  target.

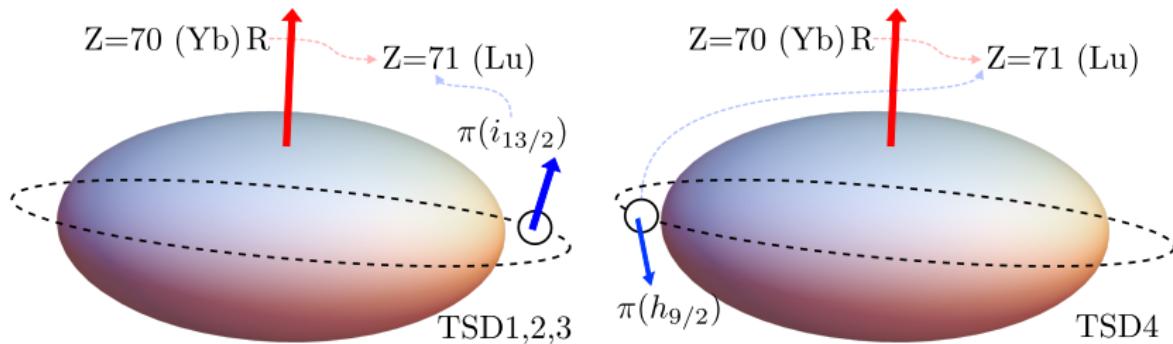
# Fresh-Up 1: $\mathbf{W}_1$

Particle-Rotor Model Hamiltonian for an odd- $A$  nucleus:

$$\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{sp}}, \quad \hat{H}_{\text{rot}} = \sum_{k=1}^3 A_k (\hat{I}_k - \hat{j}_k)^2,$$

$$\hat{H}_{\text{sp}} = \epsilon_j + \frac{V}{j(j+1)} \left[ \cos \gamma (3\hat{j}_3^2 - \mathbf{j}^2) - \sqrt{3} \sin \gamma (\hat{j}_1^2 - \hat{j}_2^2) \right].$$

$V$  - single-particle potential strength  $\propto \beta_2$  (Tanabe, 2017)



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Variational Principle + Eqs. of Motion

## Time-Dependent Variational Equation

$$\delta \int_0^t \langle \Psi_{IM;j} | \hat{H} - i \frac{\partial}{\partial t'} | \Psi_{IM;j} \rangle dt' = 0$$

$$\Psi_{\text{trial}} \equiv |\Psi_{IM;j}\rangle = \mathcal{N} e^{z\hat{I}_-} \cdot e^{s\hat{j}_-} |IMI\rangle \otimes |jj\rangle$$

- $(z, r, \varphi, \hat{I}, |IMI\rangle)$  - core (rotor **R**)
- $(s, f, \psi, \hat{j}, |jj\rangle)$  - single-particle (**j**)
- $\{z, s\}$  → **phase space coordinates**

## Constant of motion:

$$\mathcal{H} \equiv \langle \Psi_{IM;j} | \hat{H} | \Psi_{IM;j} \rangle$$

## Canonical equations of motion:

$$\mathcal{S}_1 : \frac{\partial \mathcal{H}}{\partial r} = \dot{\varphi}, \quad \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r}$$

$$\mathcal{S}_2 : \frac{\partial \mathcal{H}}{\partial f} = \dot{\psi}, \quad \frac{\partial \mathcal{H}}{\partial \psi} = -\dot{f}$$

The two sets of Hamilton equations are **the semi-classical description** of the initial quantal  $\hat{H}$ .

# Wobbling frequency

Solving  $\mathcal{S}_1$  and  $\mathcal{S}_2$  leads to the algebraic equation:

$$\Omega^4 + B\Omega^2 + C = 0$$

four solutions  $\longrightarrow$  **only two are real**:

$$\Omega_{1,2} = \left[ \frac{1}{2} \left( -B \mp \sqrt{B^2 - 4C} \right) \right]^{1/2}$$

- $\Omega_1$ : wobbling frequency of the even-*A* core **R**
- $\Omega_2$ : wobbling frequency of the odd-nucleon **j**
- **Two wobbling phonon numbers:**  $n_{w_1}$  and  $n_{w_2}$

A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Energy spectrum

## Spectra of odd-A nuclei within $W_1$

$$E_{I,n_1,n_2} = \epsilon_j + \mathcal{H}_{\min}^I + \mathcal{F}_{n_{w_1} n_{w_2}}^I$$

- Phonon factor:

$$\mathcal{F}_{n_{w_1} n_{w_2}}^I = \hbar \Omega_1^I \left( n_{w_1} + \frac{1}{2} \right) + \hbar \Omega_2^I \left( n_{w_2} + \frac{1}{2} \right)$$

- $\mathcal{H}_{\min}^I$  - Classical Energy Function taken in its minimum point:  
 $p_0 = (0, I; 0, j)$ .
- $\epsilon_j$  - single-particle energy

# A new interpretation for TSD1 and TSD2

## Previous models

$TSD1$  = zero-phonon wobbling band

$TSD2$  = one-phonon wobbling band...

## Redefinition

$TSD1$  and  $TSD2$  are **Signature Partner Bands** (in favor of Ultimate Cranker calculations, *Jensen, 2004*).

$$\left( \alpha_{fav} = +\frac{1}{2} \right) : \{ TSD1 \} \equiv \{ [0^+, 2^+, 4^+, \dots] \otimes [J^\pi = 13/2^+] \}$$

$$\left( \alpha_{unfav} = -\frac{1}{2} \right) : \{ TSD2 \} \equiv \{ [1^+, 3^+, 5^+, \dots] \otimes [J^\pi = 13/2^+] \}$$

$TSD4$ : **ground-state wobbling band**,  $\pi(h_{9/2})$  configuration.

A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# A new band structure for $^{163}\text{Lu}$

Band	Spins	$\pi$	$\alpha$	$\pi(I_j)$	$\mathbf{W}_0: \mathcal{C} + \mathcal{Q}_p$	$\mathbf{W}_1: \mathcal{C} + \mathcal{Q}_p$
TSD1	$13/2, 17/2 \dots 97/2$	+	+1/2	$\pi(i_{13/2})$	$0^+, 2^+, 4^+, \dots$	$0^+, 2^+, 4^+, \dots$
TSD2	$27/2, 31/2 \dots 91/2$	+	-1/2	$\pi(i_{13/2})$	$\text{TSD1} + 1\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$
TSD3	$33/2, 37/2 \dots 85/2$	+	+1/2	$\pi(i_{13/2})$	$\text{TSD1} + 2\Gamma^\dagger$	$\text{TSD2} + \Gamma^\dagger$
TSD4	$47/2, 51/2 \dots 83/2$	-	-1/2	$\pi(h_{9/2})$	$\text{TSD1} + 3\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$

Bands	$n_{w_1}$	$n_{w_2}$	$\mathcal{F}_{n_{w_1} n_{w_2}}^I$	$I_0$	$I_t$	$\mathcal{Q}$
TSD1	0	0	$\mathcal{F}_{00}^I = \frac{1}{2} (\Omega_1^I + \Omega_2^I)$	$13/2^+$	$97/2^+$	$j^\pi = 13/2^+ \stackrel{\text{not}}{\equiv} \mathcal{Q}_1$
TSD2	<b>0</b>	<b>0</b>	$\mathcal{F}_{00}^I = \frac{1}{2} (\Omega_1^I + \Omega_2^I)$	$27/2^+$	$91/2^+$	$j^\pi = 13/2^+ \stackrel{\text{not}}{\equiv} \mathcal{Q}_1$
TSD3	1	0	$\mathcal{F}_{10}^{I-1} = \frac{3}{2} \Omega_1^{I-1} + \frac{1}{2} \Omega_2^{I-1}$	$33/2^+$	$85/2^+$	$j^\pi = 13/2^+ \stackrel{\text{not}}{\equiv} \mathcal{Q}_1$
<b>TSD4</b>	<b>0</b>	<b>0</b>	$\mathcal{F}_{00}^I = \frac{1}{2} (\Omega_1^I + \Omega_2^I)$	$47/2^-$	$83/2^-$	$j^\pi = 9/2^- \stackrel{\text{not}}{\equiv} \mathcal{Q}_2$

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

# Extension to $A \approx 160$ mass region

The model was further applied to the other Lu isotopes where wobbling motion has been observed.

$^{161}\text{Lu}$ Bands	Spins	$\mathcal{Q}$	$\mathcal{C}$	$(n_{w_1}, n_{w_2})$	$I_b$
TSD1	$21/2^+, 25/2^+, \dots, 89/2^+$	$j^\pi = 13/2^+$	$4^+, 6^+, 8^+ \dots$	$(0, 0)$	
TSD2	$31/2^+, 35/2^+, \dots, 79/2^+$	$j^\pi = 13/2^+$	$9^+, 11^+, 13^+ \dots$	$(0, 0)$	$21/2$

$^{165}\text{Lu}$ Bands	Spins	$\mathcal{Q}$	$\mathcal{C}$	$(n_{w_1}, n_{w_2})$	$I_b$
TSD1	$25/2^+, 29/2^+, \dots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \dots$	$(0, 0)$	
TSD2	$35/2^+, 39/2^+, \dots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \dots$	$(0, 0)$	$25/2$
TSD3	$41/2^+, 45/2^+, \dots, 81/2^+$	$j^\pi = 13/2^+$	$\text{TSD2} + \Gamma^\dagger$	$(1, 0)$	

$^{167}\text{Lu}$ Bands	Spins	$\mathcal{Q}$	$\mathcal{C}$	$(n_{w_1}, n_{w_2})$	$I_b$
TSD1	$25/2^+, 29/2^+, \dots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \dots$	$(0, 0)$	
TSD2	$35/2^+, 39/2^+, \dots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \dots$	$(0, 0)$	$25/2$

A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# W<sub>1</sub> — Numerical Results

Free parameters in the model  $\rightarrow \mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$ .

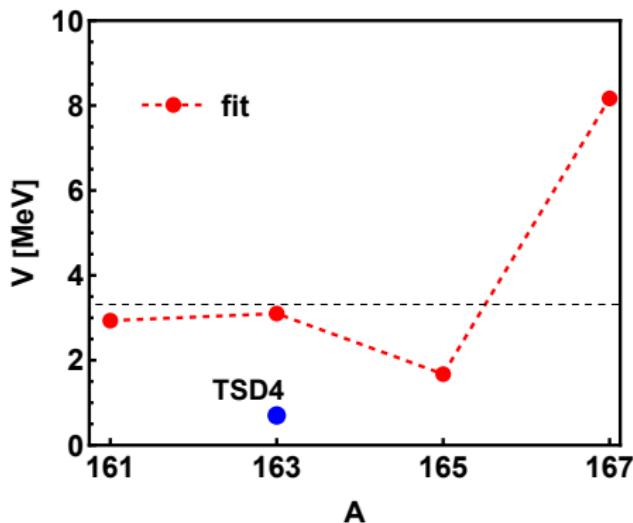
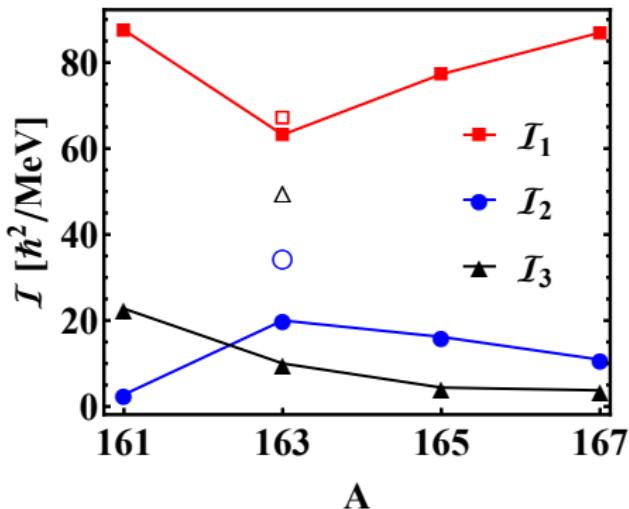
## Fitting procedure

$$\chi^2 = \frac{1}{N_T} \sum_i \frac{\left( E_{\text{exp}}^{(i)} - E_{\text{th}}^{(i)} \right)^2}{E_{\text{exp}}^{(i)}}$$

<sup>163</sup>Lu-TSD4: separate fitting procedure (different nucleon configuration)

Isotope	Bands	$\mathcal{I}_1$ [ $\hbar^2/\text{MeV}$ ]	$\mathcal{I}_2$ [ $\hbar^2/\text{MeV}$ ]	$\mathcal{I}_3$ [ $\hbar^2/\text{MeV}$ ]	$V$ [MeV]	$\gamma$ [°]	n.o.s	$E_{\text{rms}}$ [MeV]
<sup>161</sup> Lu	TSD1-2	87.555	2.773	22.744	2.933	20	29	0.168
<sup>163</sup> Lu	TSD1-3	63.2	20	10	3.1	17	52	0.264
	TSD4	67	34.5	50	0.7	17	10	0.057
<sup>165</sup> Lu	TSD1-3	77.295	16.184	4.399	1.673	20	42	0.125
<sup>167</sup> Lu	TSD1-2	87.032	10.895	3.758	8.167	19.48	30	0.165

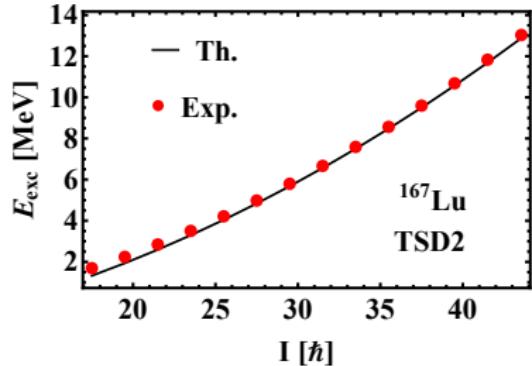
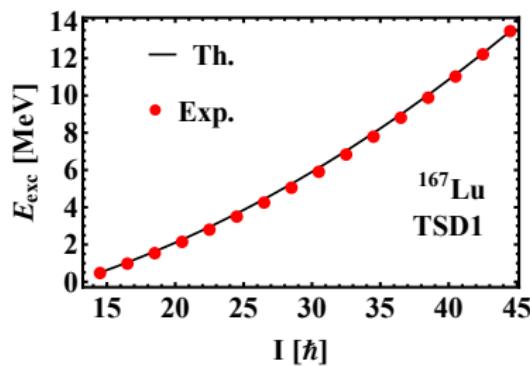
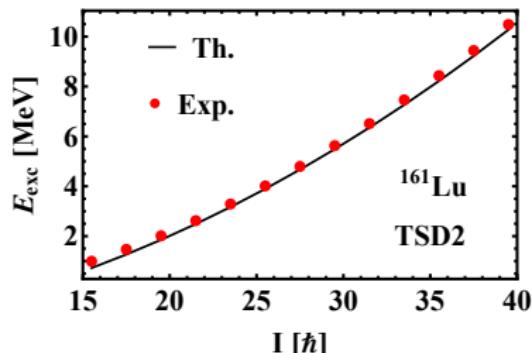
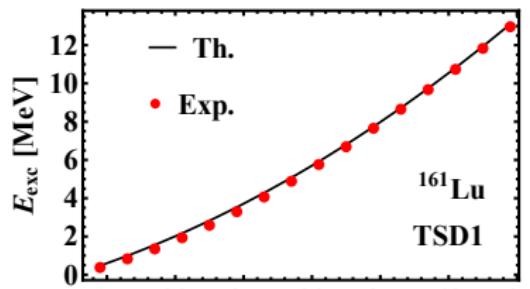
# Graphical representation for MOIs and V



$V \approx 3 \text{ MeV} \rightarrow$  agreement with other calculations (Tanabe, 2017)

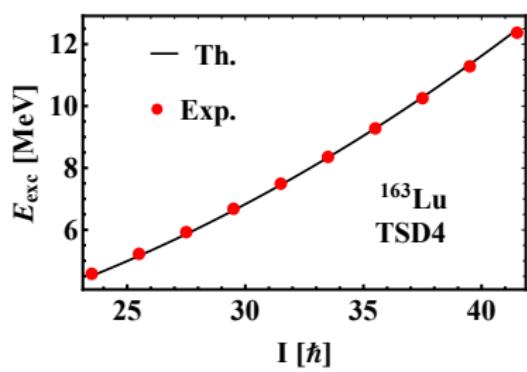
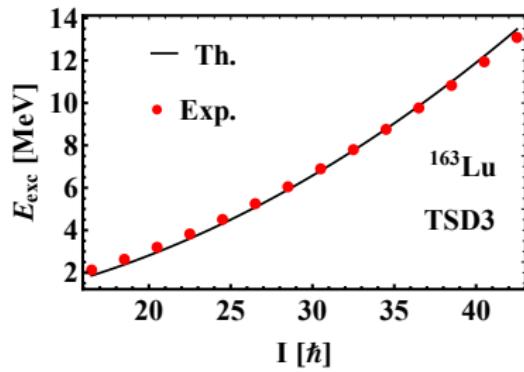
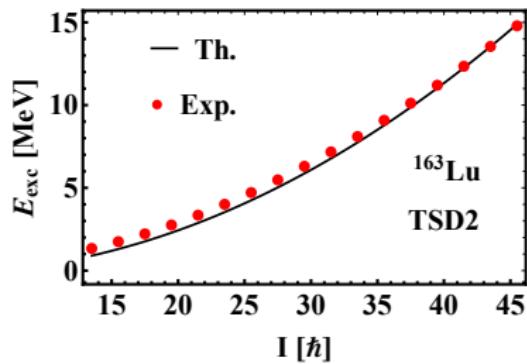
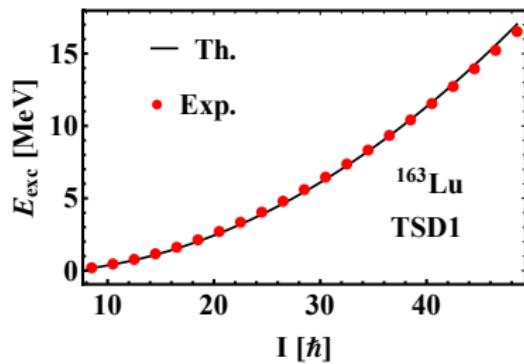
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Excitation Energies — $^{161,167}\text{Lu}$

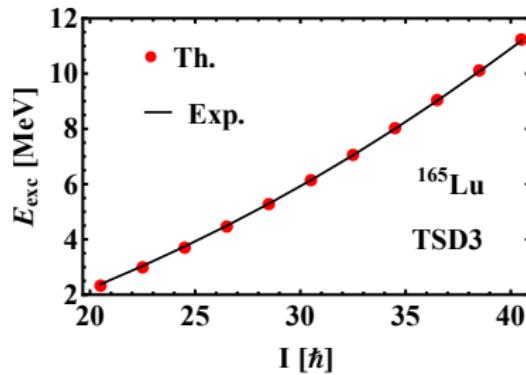
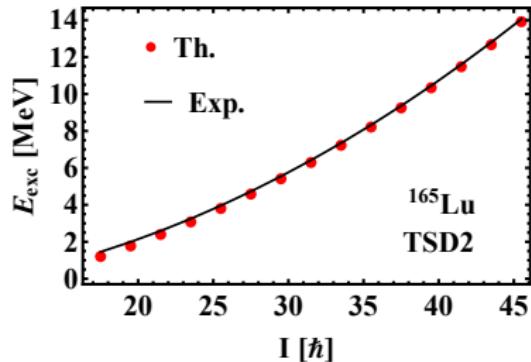
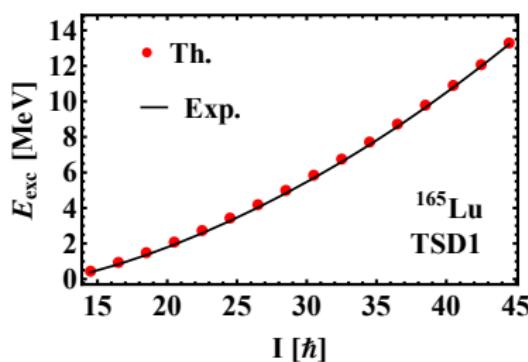


A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Excitation Energies — $^{163}\text{Lu}$



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

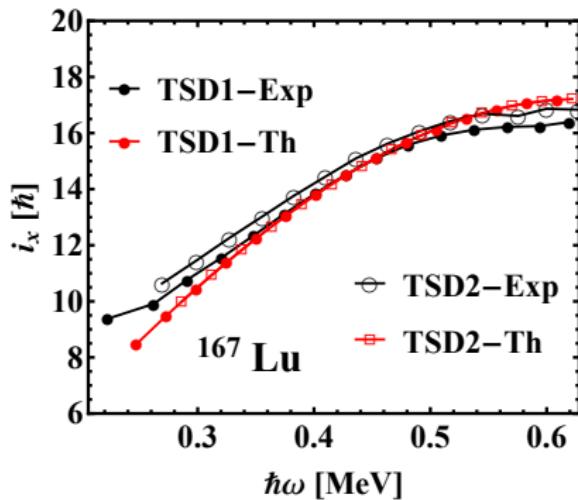
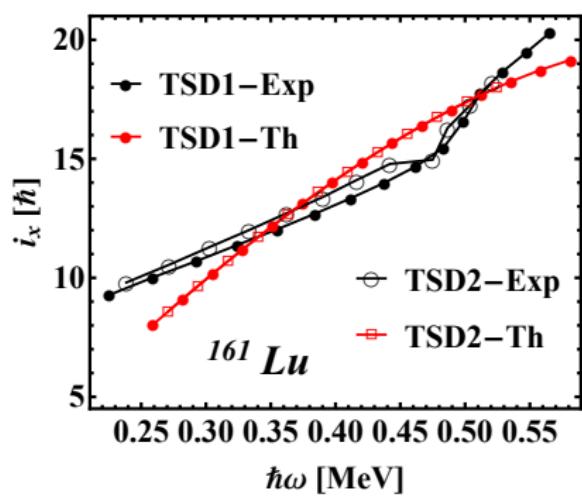
Excitation Energies —  $^{165}\text{Lu}$ 

A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Alignment — $^{161,167}\text{Lu}$

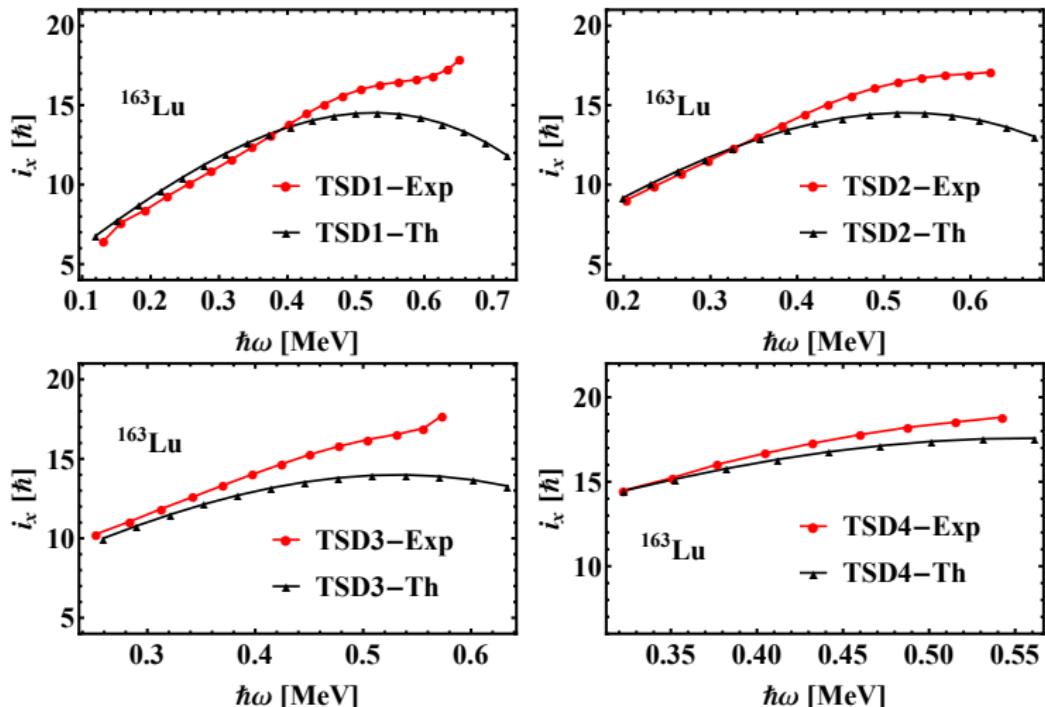
$$i_x = I - I_{\text{ref}} ,$$

$$I_{\text{ref}} = \mathcal{I}_0 \omega + \mathcal{I}_1 \omega^3 .$$



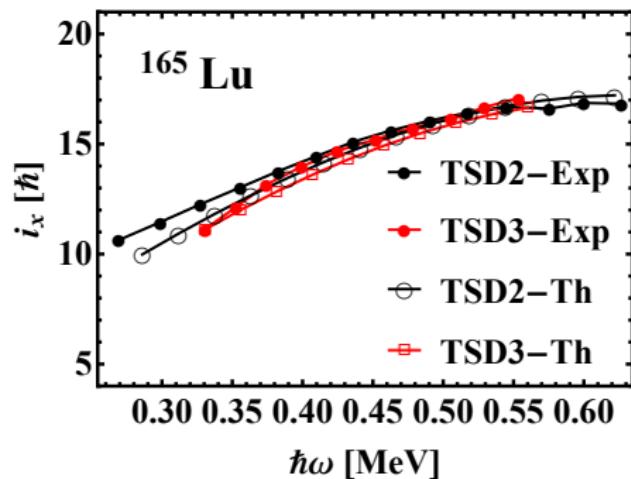
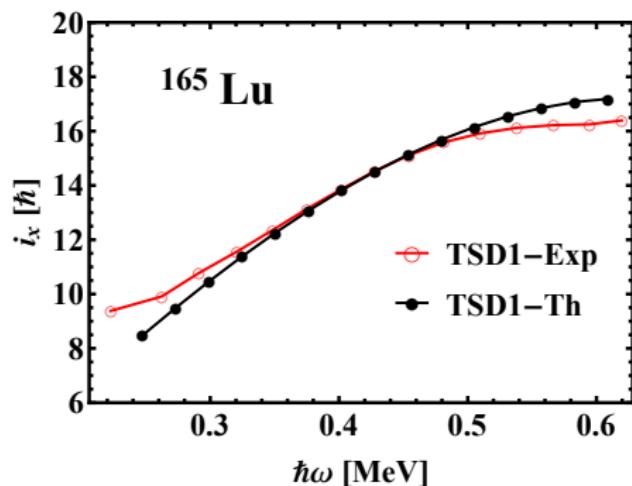
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Alignment — $^{163}\text{Lu}$



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

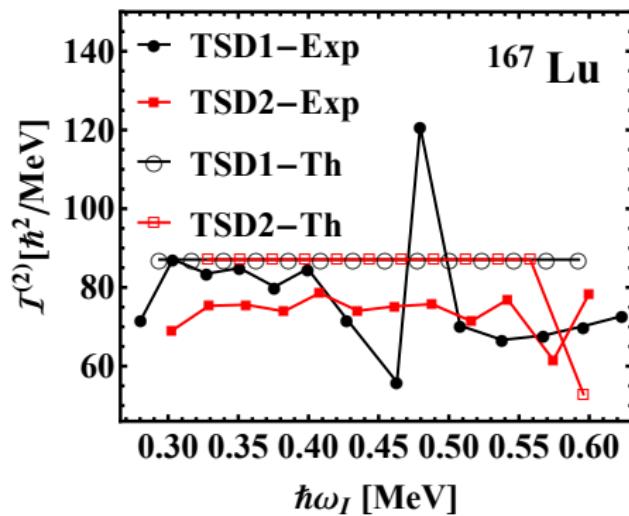
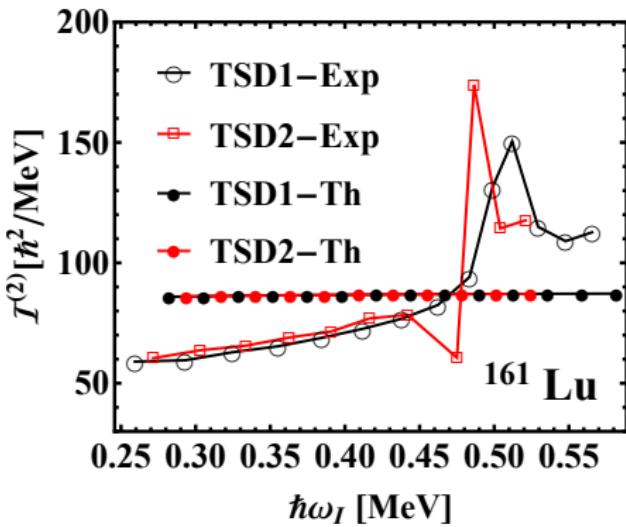
# Alignment — $^{165}\text{Lu}$



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

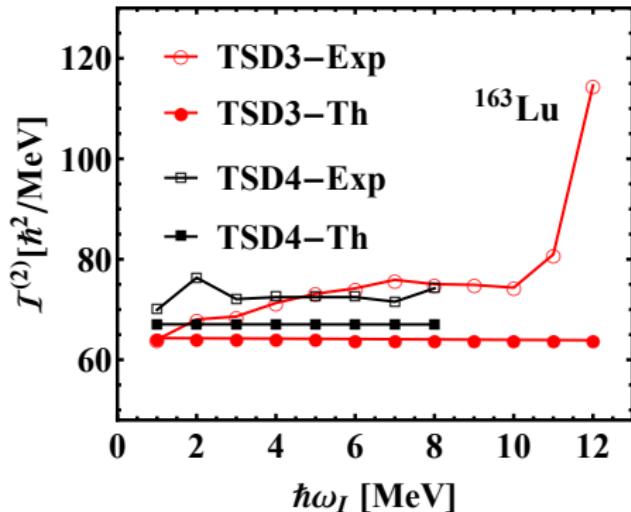
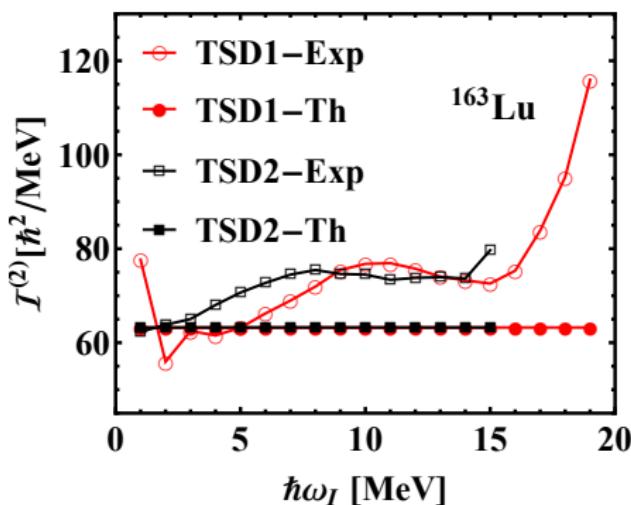
# Dynamic Moment of Inertia — $^{161,167}\text{Lu}$

$$\mathcal{I}^{(2)}(I) = \hbar \frac{dI_x}{d\omega} = \hbar^2 \left( \frac{d^2 E}{dI_x^2} \right)^{-1}$$



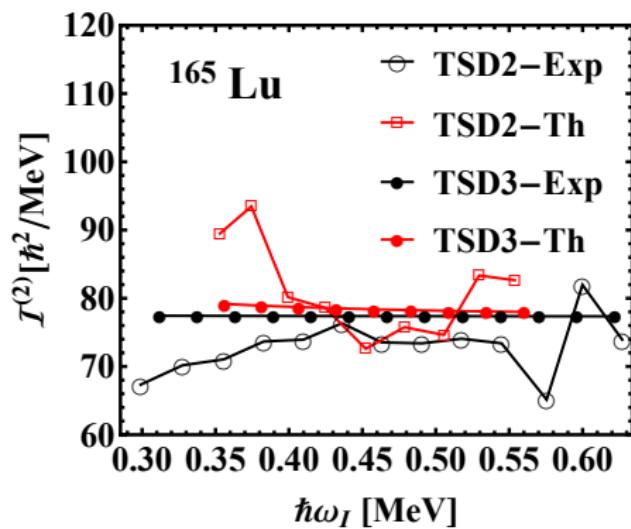
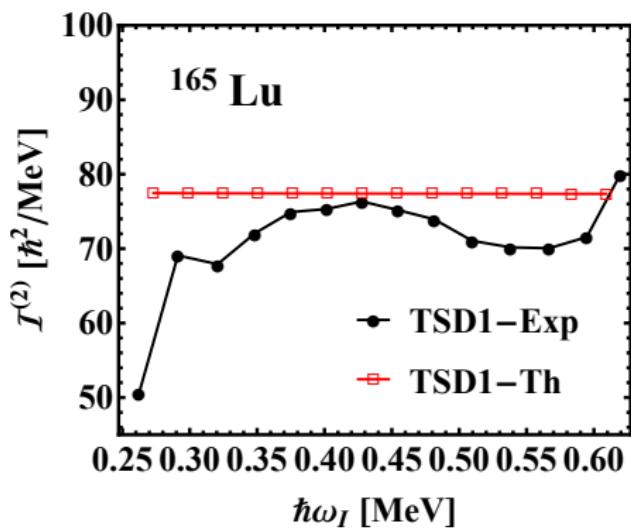
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Dynamic Moment of Inertia — $^{163}\text{Lu}$



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Dynamic Moment of Inertia — $^{165}\text{Lu}$



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

# Electromagnetic Calculations

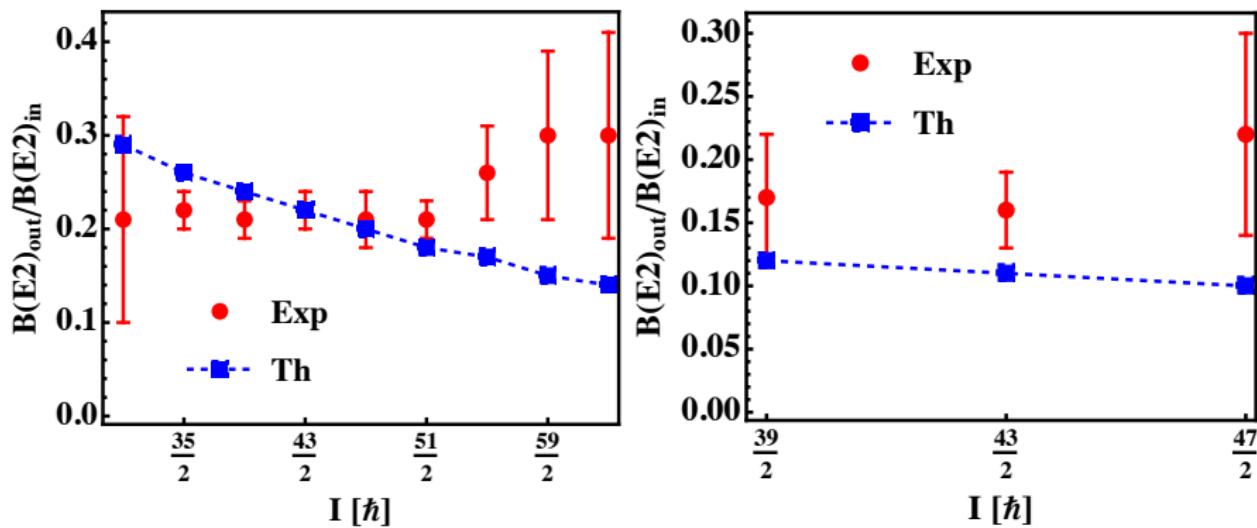


Figure: E2 Branching ratio. **Left:**  $^{163}\text{Lu}$  (TSD2) **Right:**  $^{165}\text{Lu}$  (TSD2).

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

# Electromagnetic Calculations II

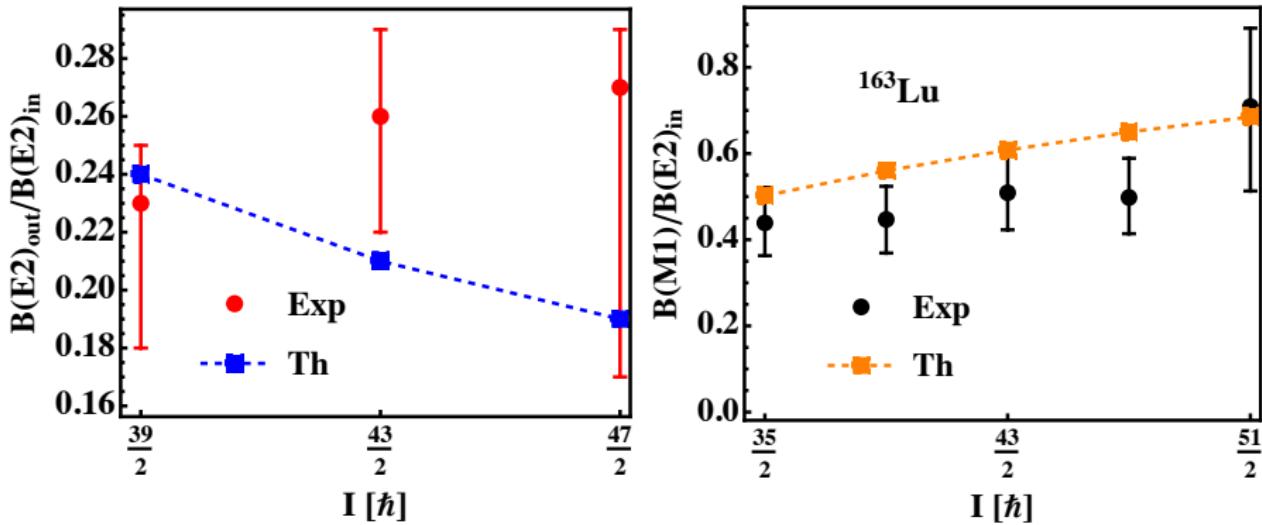


Figure: **Left:** E2 Branching ratio in  $^{167}\text{Lu}$  (TSD2). **Right:** The ratio  $B(M1)/B(E2)_{\text{in}}$  for states  $TSD2 \rightarrow TSD1$  (in units of  $\mu_N^2/(e^2 b^2)$ ).

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

# W<sub>1</sub> — Remarks

## Characteristics

- + Full semi-classical description (TDVE) with good numerical results
- + Deformation parameters are self-consistent (agree with exp. values)
- separate fit for TSD4 (different nucleonic configuration)
- Two sets of MOIs for  $^{163}\text{Lu}$

**Onset of another redesign  
Start of W<sub>2</sub> formalism in Chapter 5**

# Fresh-Up 2: $W_2$

## Novel description of $^{163}\text{Lu}$

- All four bands in  $^{163}\text{Lu}$  described by the same triaxial core + odd-particle coupling  $\rightarrow Q_1 = \pi(i_{13/2})$
- The adopted wave-function admits solutions of both **positive and negative parity**. Parity operator:  $\mathcal{P} = e^{-i\pi J_2} C$ :

$$\mathcal{P}\Psi(r, \varphi; t, \psi) = \Psi(r, \varphi + \pi; t, \psi + \pi),$$

$$\mathcal{H}(r, \varphi + \pi; t, \psi + \pi) = \mathcal{H}(r, \varphi; t, \psi),$$

$$\Psi(r, \varphi + \pi; t, \psi + \pi) = \pm \Psi(r, \varphi; t, \psi).$$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

# New band structure in $^{163}\text{Lu}$

$$E_{I,0,0}^{\text{TSD1}} = \epsilon_{13/2} + \mathcal{H}_{\min}^I + \mathcal{F}_{00}^I, \quad I^\pi = 13/2^+, 17/2^+, 21/2^+ \dots,$$

$$E_{I,0,0}^{\text{TSD2}} = \epsilon_{13/2}^1 + \mathcal{H}_{\min}^I + \mathcal{F}_{00}^I, \quad I^\pi = 27/2^+, 31/2^+, 35/2^+ \dots,$$

$$E_{I,1,0}^{\text{TSD3}} = \epsilon_{13/2} + \mathcal{H}_{\min}^{I-1} + \mathcal{F}_{10}^{I-1}, \quad I^\pi = 33/2^+, 37/2^+, 41/2^+ \dots,$$

$$E_{I,0,0}^{\text{TSD4}} = \epsilon_{13/2}^2 + \mathcal{H}_{\min}^I + \mathcal{F}_{00}^I, \quad I^\pi = 47/2^-, 51/2^-, 55/2^- \dots.$$

Band	$n_s$	$\mathbf{j}_Q$	$\mathbf{R}_{\mathcal{C}}$ - Sequence	I - Sequence	Coupling
TSD1	21	$\mathcal{Q}_1$	$\mathcal{C}_1 = 0^+, 2^+, 4^+, \dots$	$13/2^+, 17/2^+, 21/2^+, \dots$	$\mathcal{C}_1 + \mathcal{Q}_1$
TSD2	17	$\mathcal{Q}_1$	$\mathcal{C}_2^+ = 1^+, 3^+, 5^+, \dots$	$27/2^+, 31/2^+, 35/2^+, \dots$	$\mathcal{C}_2^+ + \mathcal{Q}_1$
TSD3	14	$\mathcal{Q}_1$	1-phonon exc.	$33/2^+, 37/2^+, 41/2^+, \dots$	
TSD4	11	$\mathcal{Q}_1$	$\mathcal{C}_2^- = 1^-, 3^-, 5^-, \dots$	$47/2^-, 51/2^-, 55/2^-, \dots$	$\mathcal{C}_2^- + \mathcal{Q}_1$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

# New results for $^{163}\text{Lu}$

Model requires a **unique set of parameters**:  $\mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$ .

$\mathcal{I}_1$ [ $\hbar^2/\text{MeV}$ ]	$\mathcal{I}_2$ [ $\hbar^2/\text{MeV}$ ]	$\mathcal{I}_3$ [ $\hbar^2/\text{MeV}$ ]	$\gamma$ [deg.]	$V$ [MeV]
72	15	7	22	2.1

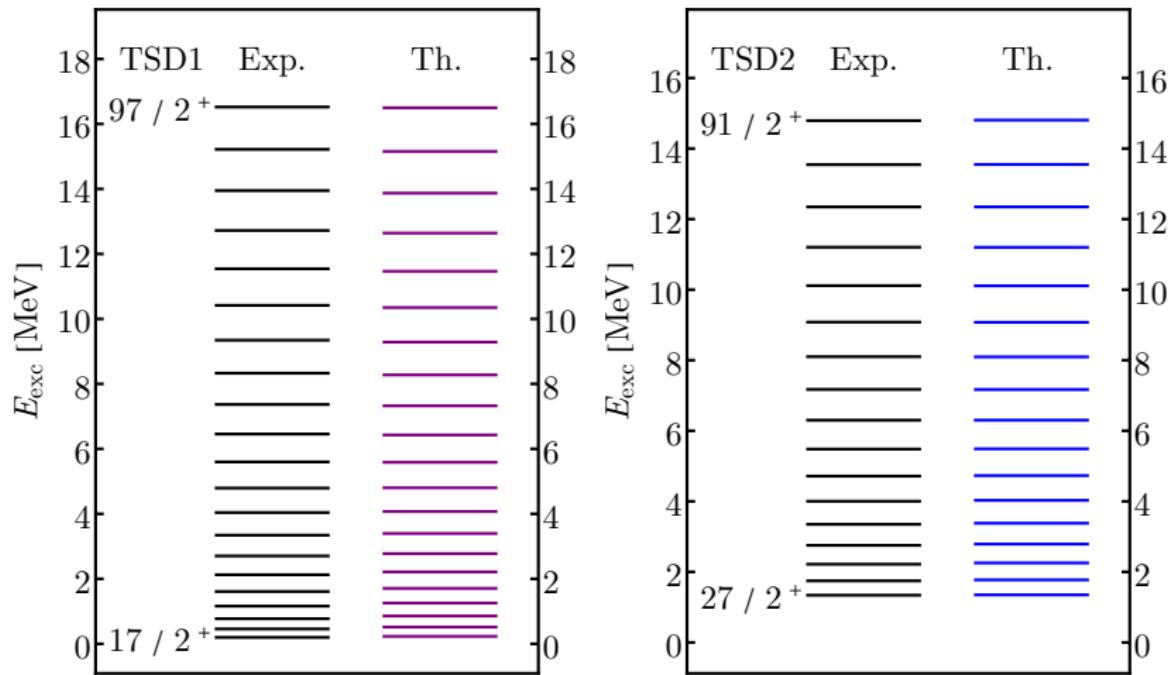
## Remarks

- $\gamma$  in agreement with exp. value  $\gamma_{\text{exp}} = 20^\circ$  (*Jensen, 2004*)
- Slight *decrease* of  $V$  (? breaking of parity symmetry quenches the quadrupole deformation)
- overall  $E_{\text{RMS}} \approx 79$  keV: **first semi-classical description for a nucleus with deviations smaller than 100 keV.**

First model to describe  $^{163}\text{Lu}$  wobbling structure.

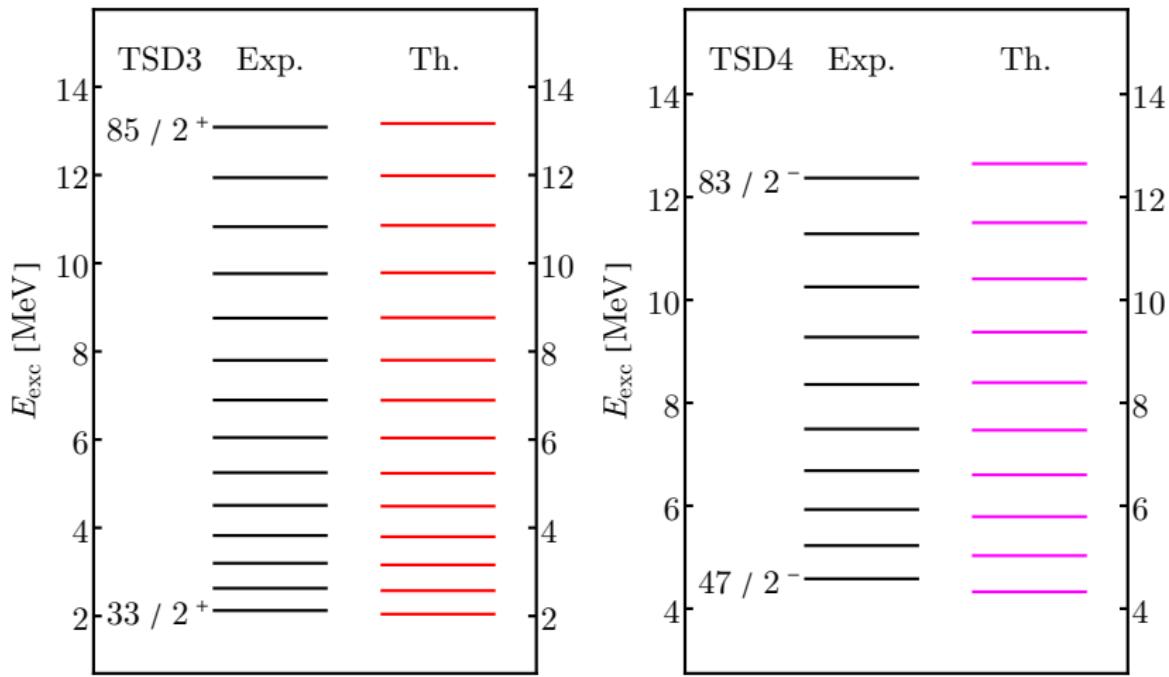
R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

# Energy spectrum



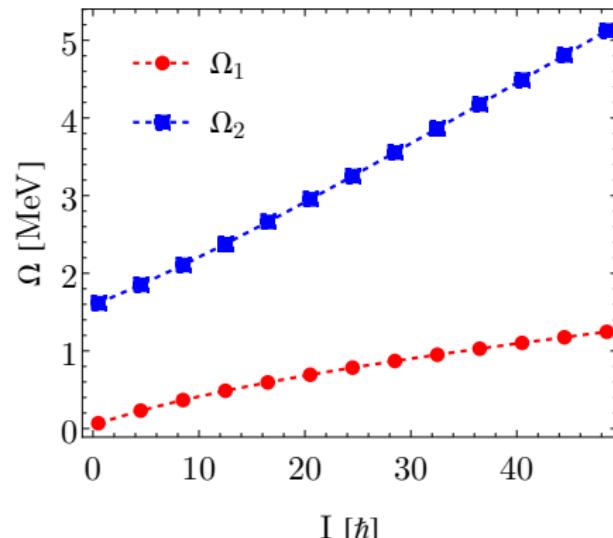
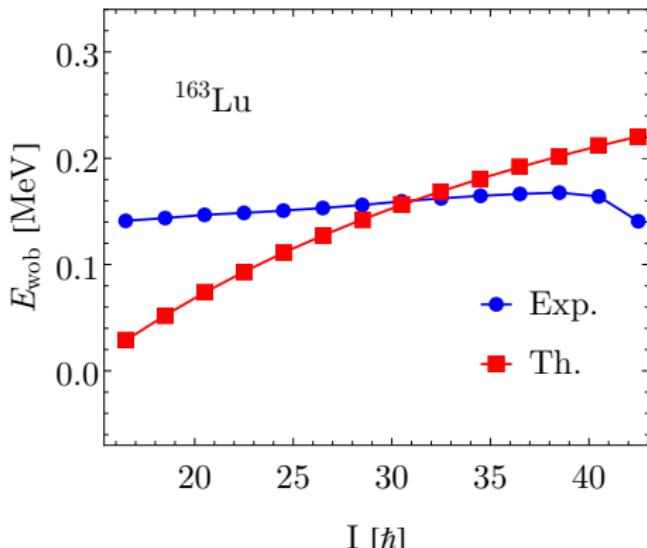
R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

# Energy spectrum II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

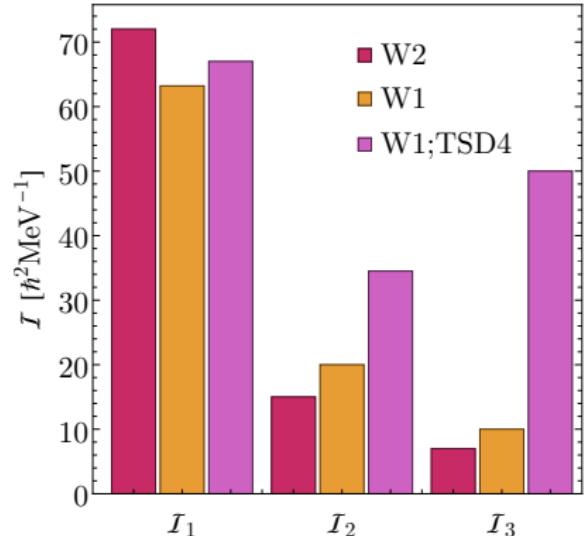
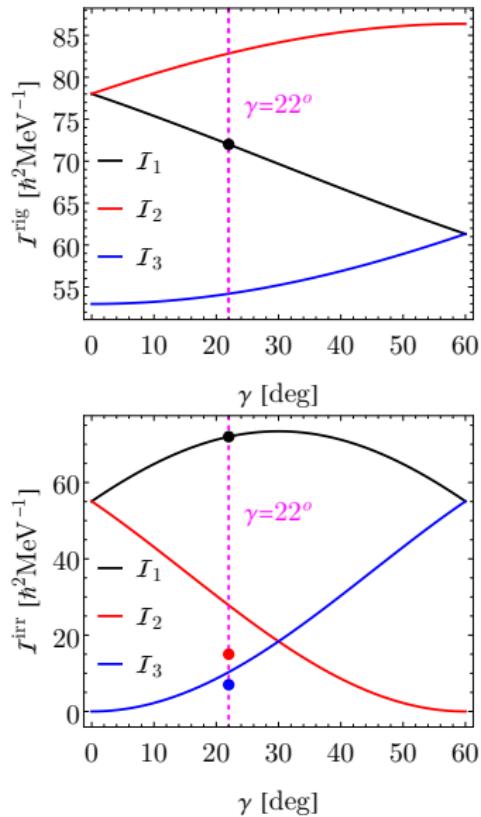
# Wobbling Energies



The wobbling energy (**left**) and the two wobbling frequencies (**right**) for  $^{163}\text{Lu}$ . **Decreasing trend of  $E_{\text{wob}}$  in agreement with arguments of Frauendorf 2014.**

R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

# Moments of inertia for $\mathbf{W}_2$



$\mathbf{W}_2$ : hydrodynamical character of the triaxial nucleus.

Results presented at the International Conference NSP, 2023, Turkey.

# Classical Energy Function

## Angular momentum

Polar representation of the angular momentum and  $\mathcal{H}$ .

$$\mathbf{l} = \{l_1, l_2, l_3\} \equiv \{x_1, x_2, x_3\} ,$$

$$x_1 = l \sin \theta \cos \varphi , \quad x_2 = l \sin \theta \sin \varphi , \quad x_3 = l \cos \theta .$$

$$\mathcal{H} |_{p_0} = l \left( l - \frac{1}{2} \right) \sin^2 \theta \cdot \mathcal{A}_\varphi - 2A_1 l j \sin \theta + T_{\text{core}} + T_{\text{sp}} ,$$

$$\mathcal{A}_\varphi = A_1 \cos^2 \varphi + A_2 \sin^2 \varphi - A_3 ,$$

$$T_{\text{core}} = \frac{l}{2} (A_1 + A_2) + A_3 l^2 ,$$

$$T_{\text{s.p.}} = \frac{j}{2} (A_2 + A_3) + A_1 j^2 - V \frac{2j-1}{j+1} \sin \left( \gamma + \frac{\pi}{6} \right) .$$

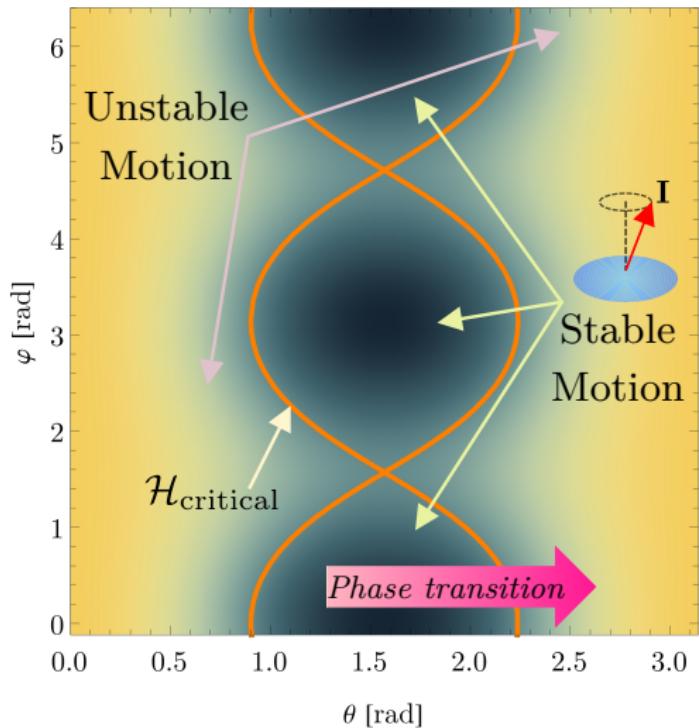
# CEF — Stability Regions

**Table:** The minimum points of  $\mathcal{H}$ .  
Using the MOIs from the fitting procedure

Minimal point	$\theta$ [rad]	$\varphi$ [rad]	$A_k$ ordering
$m_1$	$\pi/2$	0	$A_3 > A_2 > A_1$
$m_2$	$\pi/2$	$\pi$	$A_3 > A_2 > A_1$
$m_3$	$\pi/2$	$2\pi$	$A_3 > A_2 > A_1$

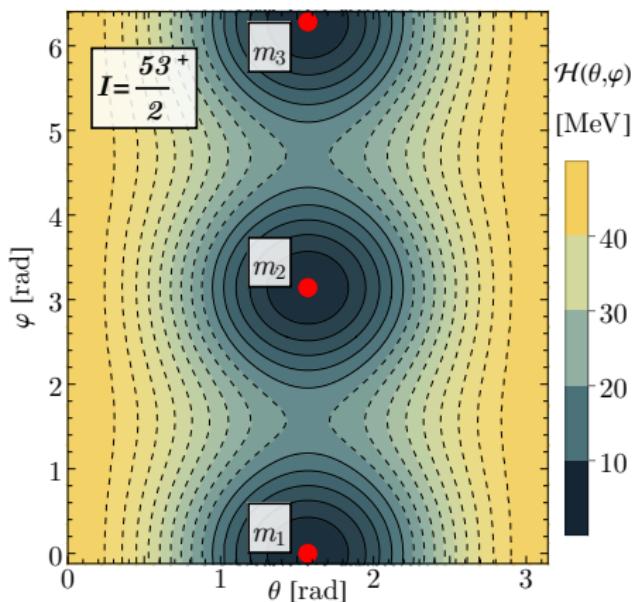
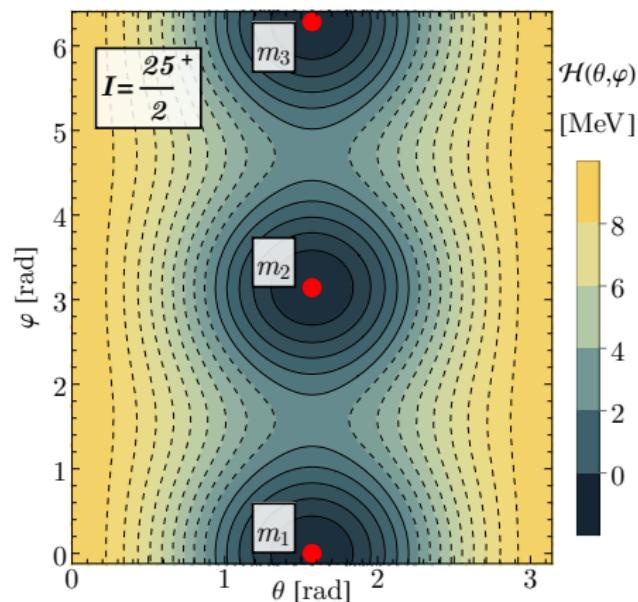
## Semi-classical feature

This is the first classical description of the *wobbling stability* for an odd-mass nucleus.



# Polar representation of $\mathcal{H}$

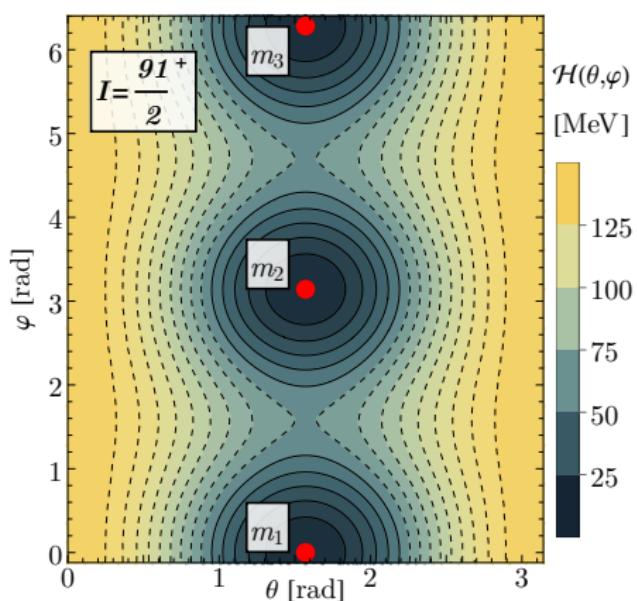
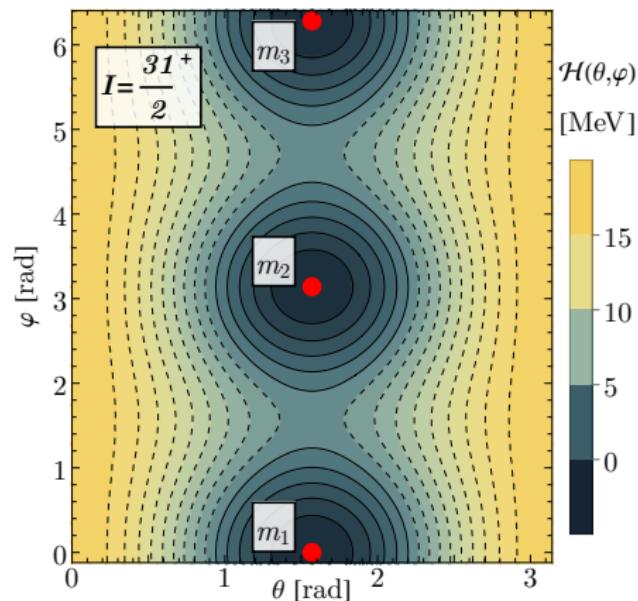
Figure:  $^{163}\text{Lu}$  TSD1



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

# Polar representation of $\mathcal{H}$ II

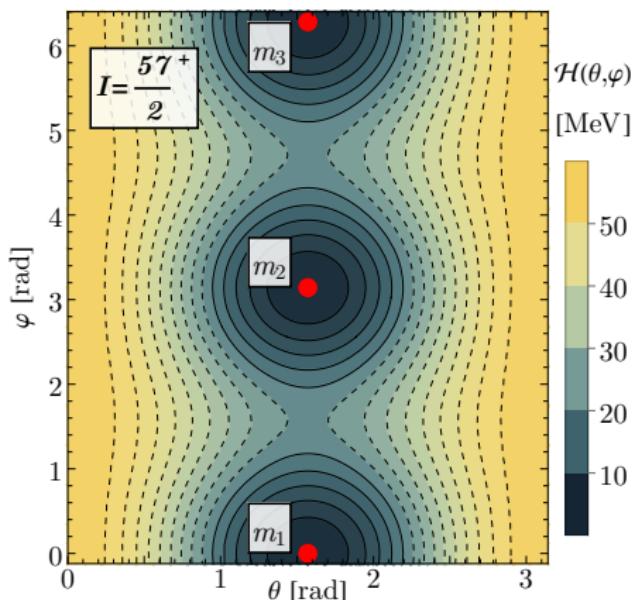
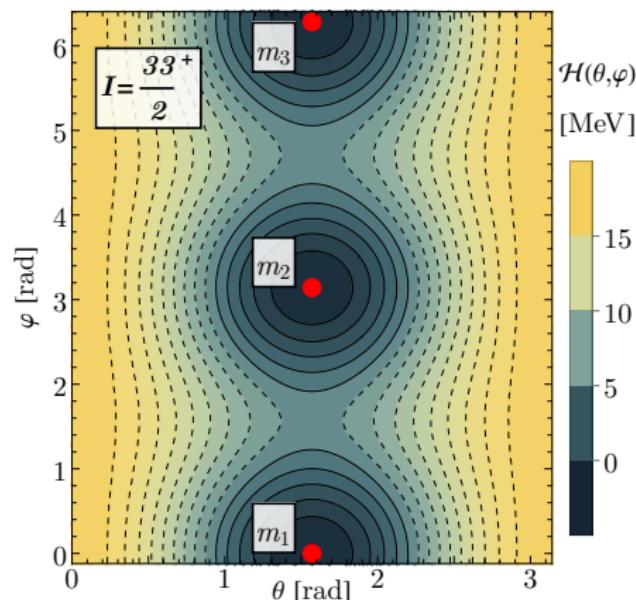
Figure:  $^{163}\text{Lu}$  TSD2



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

# Polar representation of $\mathcal{H}$ III

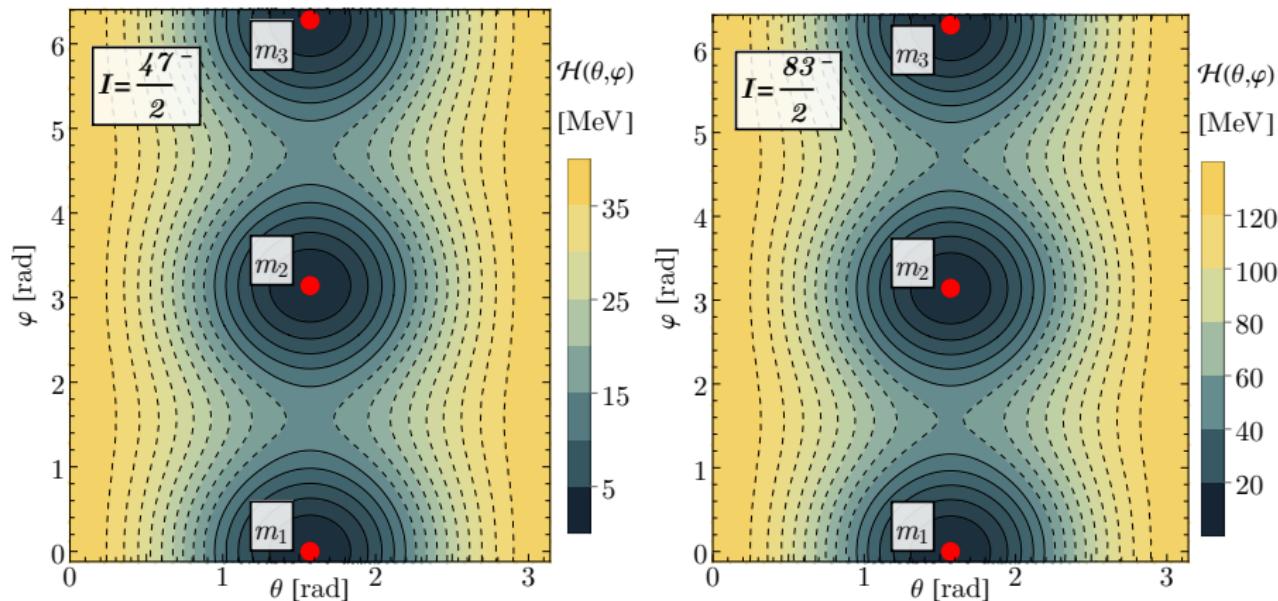
Figure:  $^{163}\text{Lu}$  TSD3



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

# Polar representation of $\mathcal{H}$ IV

Figure:  $^{163}\text{Lu}$  TSD4



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

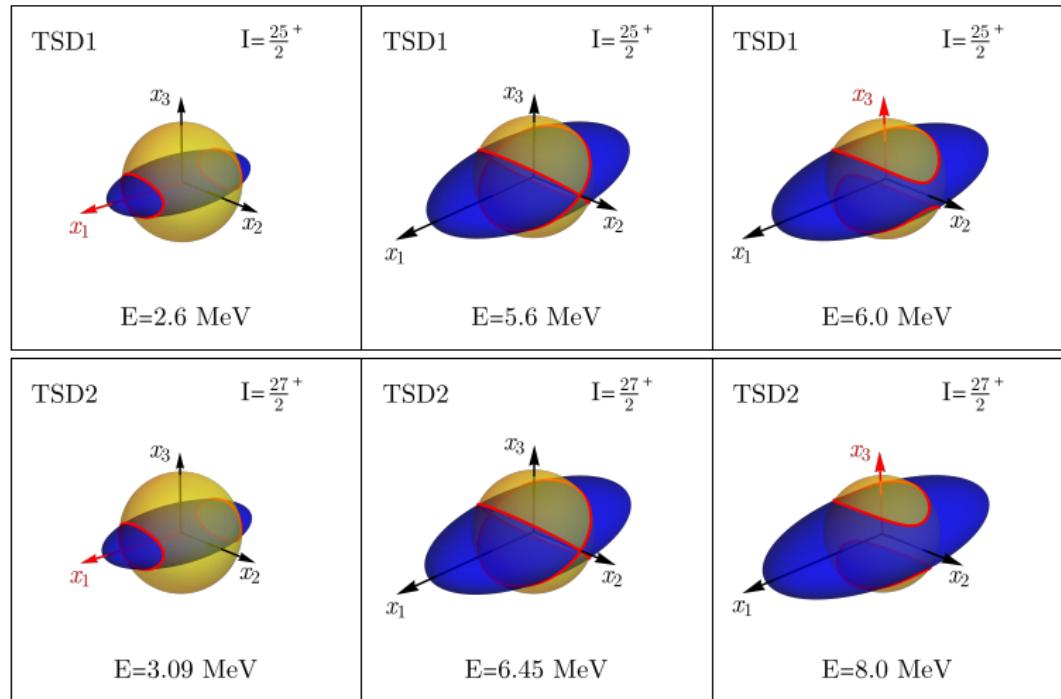
# 3D interpretation of the WM

- Formalism  $\mathbf{W}_2$  gives a 3D interpretation of the nuclear wobbling motion
- **Classical Trajectories:** intersection curves between the **triaxial energy** and the **total angular momentum**

$$I^2 = x_1^2 + x_2^2 + x_3^2,$$

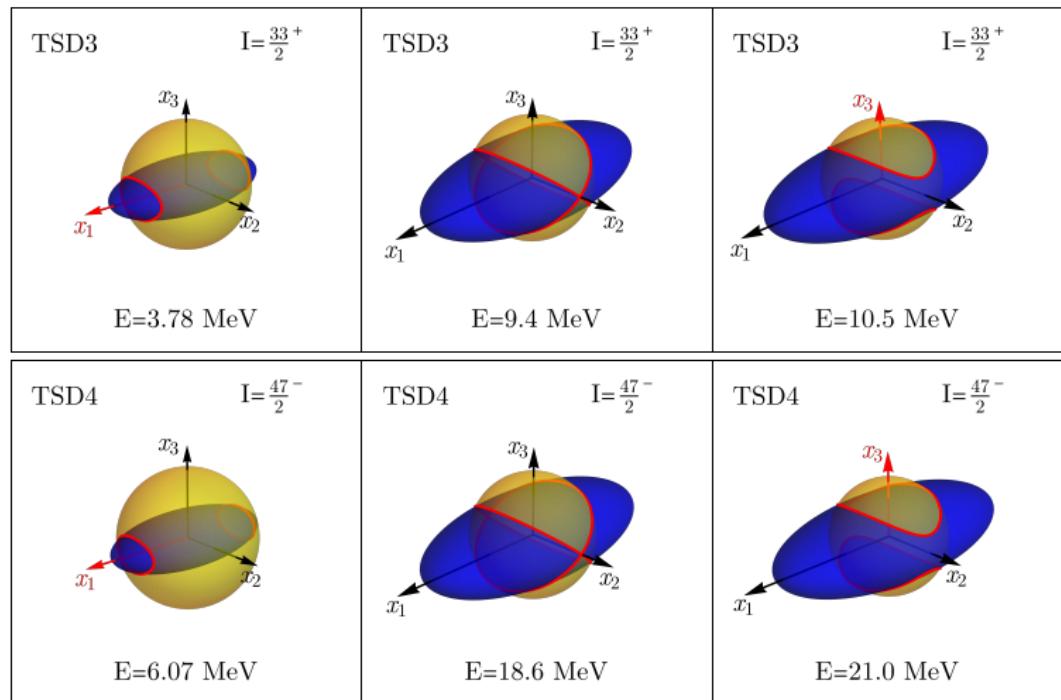
$$\begin{aligned} E = & \left(1 - \frac{1}{2I}\right) A_1 x_1^2 + \left(1 - \frac{1}{2I}\right) A_2 x_2^2 + \left[\left(1 - \frac{1}{2I}\right) A_3 + A_1 \frac{j}{I}\right] x_3^2 - \\ & - I \left(I - \frac{1}{2}\right) A_3 - 2A_1 I j + T_{\text{rot}} + T_{\text{sp}}. \end{aligned}$$

# $^{163}\text{Lu}$ — Classical trajectories



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

# $^{163}\text{Lu}$ — Classical trajectories II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

# $^{163}\text{Lu}$ — Classical trajectories III

## Model characteristics

- From the figures:
  - the state  $I = 25/2^+$  from TSD1 has a real energy of about 2.6 MeV
  - the **critical value** requires twice that amount.
- unstable trajectories can be identified for each state (i.e., the middle figs)
- **phase transitions** between rotational modes can be identified (i.e., right figs)

Results were presented at the International Conference TIM-21, Timisoara.

# New Boson Method for odd-mass nuclei

## Chapter 6

### Study of the Wobbling Motion via a Boson Description

#### Model Features

- Extend the boson description for the even-even nuclei made by the team in 2017 to odd-mass nuclei.
- Use the *Cranked Particle-Rotor Model* to study the wobbling spectrum of  $^{135}\text{Pr}$ .

$$\hat{H}_{\text{rot}} = \textcolor{red}{AH'} + \textcolor{blue}{H_{sp}} + \text{s.t.}$$

# Rotational Hamiltonian

## Rigid coupling Hamiltonian

$$\textcolor{red}{H'} = a_1 \left( \hat{I}_+^2 + \hat{I}_-^2 \right) + a_2 \left( \hat{I}_+ \hat{I}_- + \hat{I}_- \hat{I}_+ \right) + a_3 \hat{I}_1 ,$$

$$\textcolor{blue}{H_{sp}} = \sum_{k=1}^2 A_k \hat{j}_k^2 , \text{ s.t. } = A_1 I^2 - A_2 j_2 I .$$

- the triaxial rigid rotor is constrained to move around the 1-axis.
- adopted Frozen-Alignment approximation:  $\mathbf{j} = (j \cos \theta, j \sin \theta, 0)$   
(*Frauendorf, 2014*)

# New angular momentum representation

**First boson expansion of this kind in literature:**

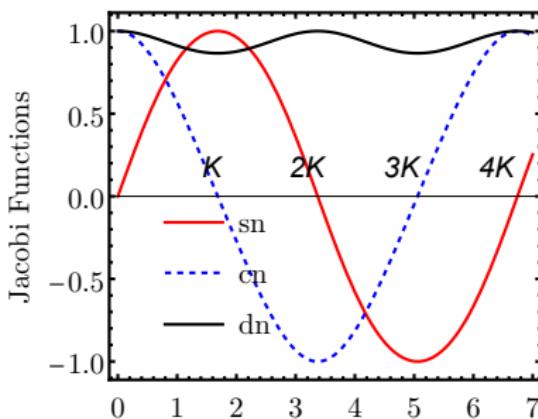
$$\begin{aligned}\hat{l}_+ &= i \frac{cb^\dagger - db^\dagger}{sb^\dagger} \left( I + lcb^\dagger db^\dagger - sb^\dagger b \right) , \\ \hat{l}_- &= i \frac{cb^\dagger + db^\dagger}{sb^\dagger} \left( I - lcb^\dagger db^\dagger + sb^\dagger b \right) , \\ \hat{l}_1 &= lcb^\dagger db^\dagger - sb^\dagger b .\end{aligned}$$

$s, c, d$  are the **Jacobi Elliptic Functions**:

$$s = \text{sn}(q, k),$$

$$c = \text{cn}(q, k),$$

$$d = \text{dn}(q, k).$$



# Elliptic Potential

## New Hamiltonian

$$H' = -\frac{d^2}{dq^2} - 2v_0 s \frac{d}{dq} + I(I+1)s^2 k^2 + 2v_0 c d l ,$$

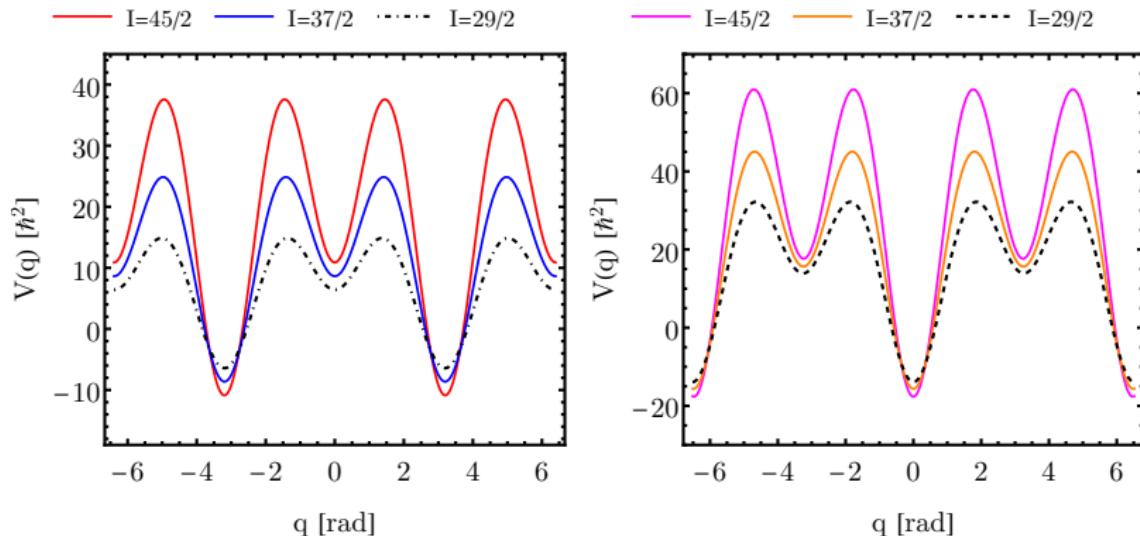
with the associated *Schrodinger Equation* (fully separated Kinetic and Potential terms):

$$\left[ \frac{d^2}{dq^2} + V(q) \right] \Psi = E \Psi .$$

$$V(q) = [I(I+1)k^2 + v_0^2] s^2 + (2I+1)v_0 c d = V(-q) . \quad (1)$$

Results were presented at the International Conference TIM-22 (Timisoara) and  
World Quantum Day 2023 (IFIN-HH)

# Elliptic potential



**Figure:** The elliptic potential as function of the coordinate  $q$  with  $\theta = -119^\circ$  (**left**) and  $\theta = 61^\circ$  (**right**).

A.A. Raduta, C.M. Raduta, R Poenaru, Journal of Physics G 48, 2020.

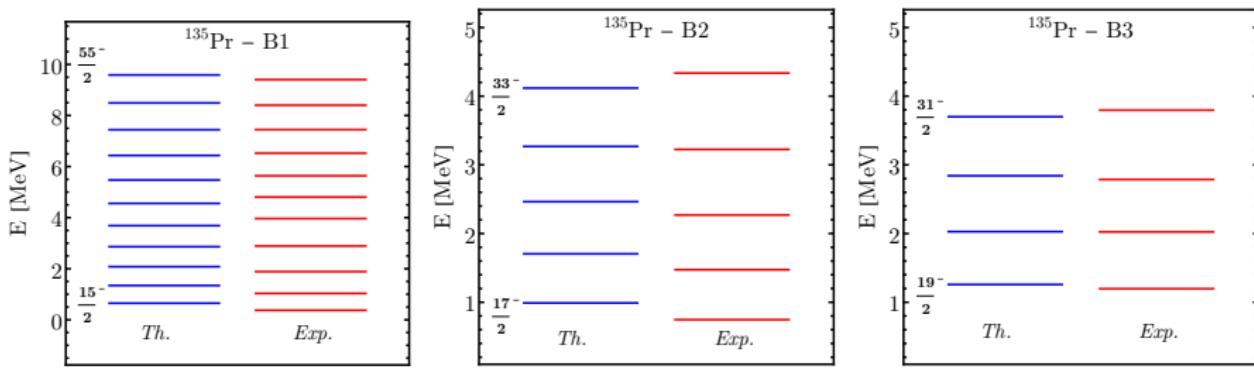
Results for  $^{135}\text{Pr}$ 

Figure: The excitation energies in  $^{135}\text{Pr}$ . Exp data: *Sensharma, 2019*.

A.A. Raduta, C.M. Raduta, R Poenaru, Journal of Physics G 48, 2020.

Thank you for your attention ❤