

A Systematic Description of the Wobbling Motion in Odd-Mass Nuclei Within a Semi-Classical Formalism

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A presentation for the degree of Doctor of Philosophy

May 15, 2023

Outline

- 1 Aim and Motivation
- 2 Nuclear Shapes
- 3 Triaxiality and Wobbling Motion
- 4 Wobbling Motion in Odd-A
- 5 Boson Description of Wobbling Motion
- 6 Conclusions

Aim



Research Objectives

- Extend the current interpretation of the **Nuclear Triaxiality** in the context of its unique fingerprint: **Wobbling Motion**.
- Adopt a framework that is as close as possible to a **classical picture**.

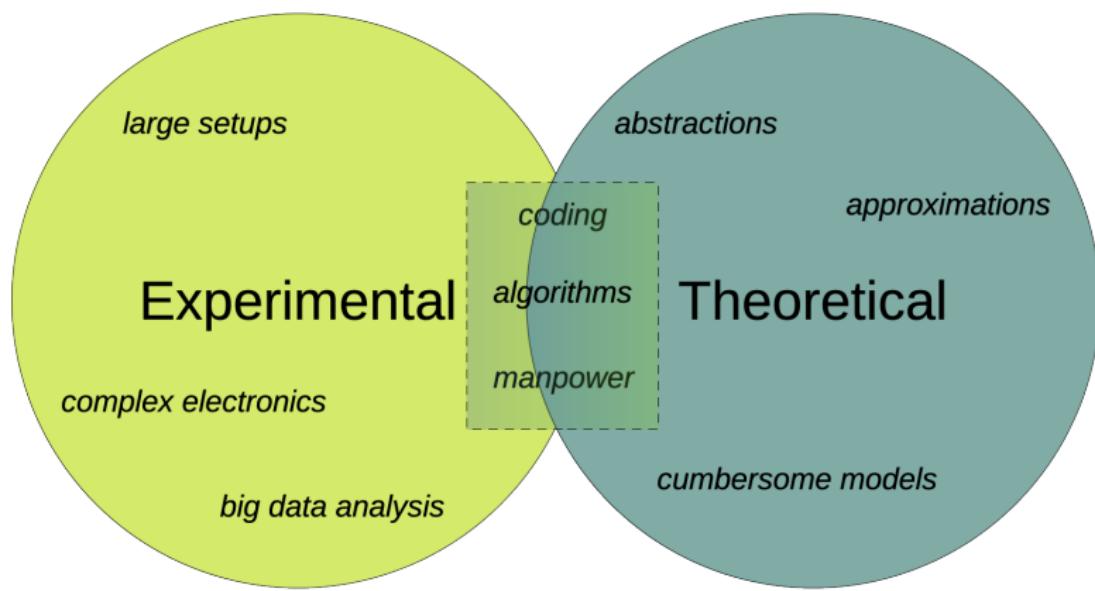


Objectives exclusive to the thesis

- Enough context towards a better understanding of the underlying concepts, methods, and results.
- create a completely *open-source* project.

Motivation

- **Nuclear Triaxiality** recently became a *hot topic* within the scientific community.
- Identifying nuclei with triaxial deformations represents a real **experimental** and **theoretical** challenge.



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Nuclear Deformation

Nuclear shapes

Most generally described in terms of the **nuclear radius**:

$$R(\theta, \varphi) = R_0 \left(1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu} Y_{\lambda}^{\mu}(\theta, \varphi) \right)$$

Quadrupole deformations $\lambda = 2$

- **For us:** Most relevant modes are the **quadrupole vibrations** $\lambda = 2$
 \implies *Play a crucial role in the rotational spectra of nuclei:*
- $\alpha_{2\mu}$ reduced to only two *deformation parameters*: β_2 (**eccentricity**) and γ (**triaxiality**) (Bohr and Mottelson, 1969).

Axial shapes

- Most of the nuclei are either **spherical** or **axially symmetric** in their ground-state (*Budaca, 2018*).
- Moments of inertia: $\mathcal{I}_{1,2,3}$: two are equal, one is different.

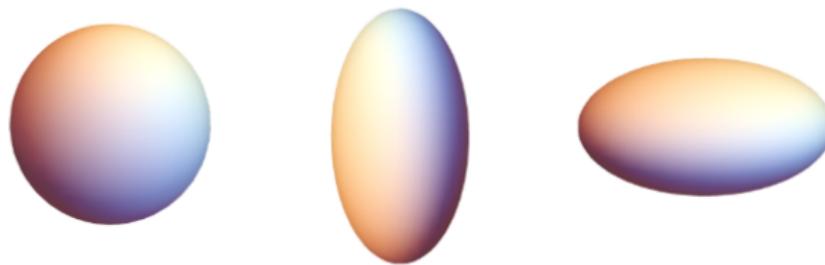
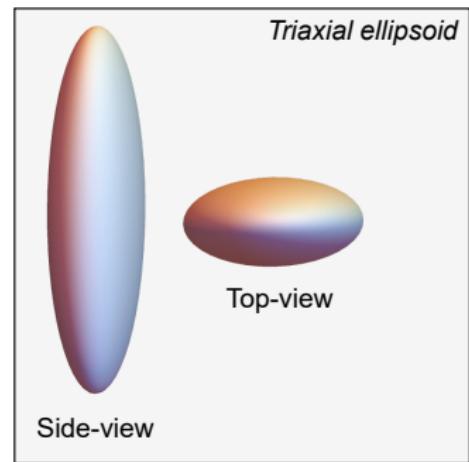
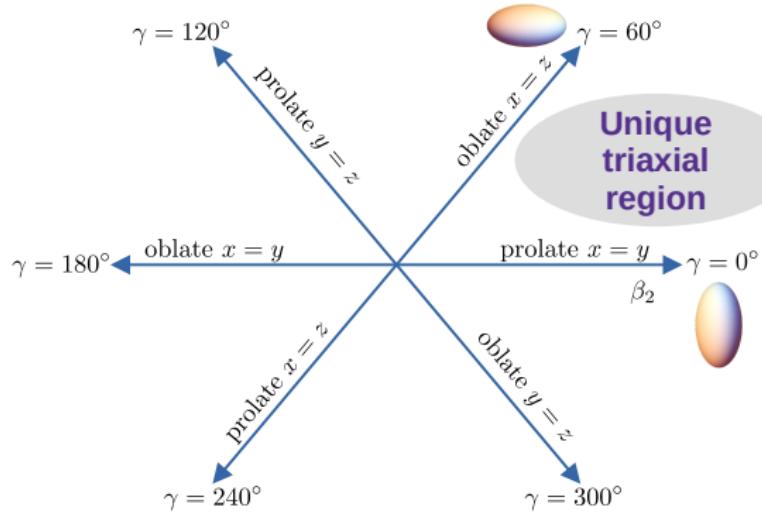


Figure: **spherical**: $\beta_2 = 0$ **prolate**: $\beta_2 > 0$ **oblate**: $\beta_2 < 0$. ($\gamma = 0^\circ$).

Non-axial shapes

- The triaxiality parameter $\gamma \neq 0^\circ$: departure from axial symmetry.
- Moments of inertia: $I_1 \neq I_2 \neq I_3$.



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Fingerprints of Triaxiality

Evidence

- Currently, there are **only two** well-established phenomena uniquely attributed to triaxial deformation.
 - ① **Wobbling Motion** - WM (*Bohr and Mottelson, 1970s*)
 - ② Chiral Motion - χ M (*Frauendorf, 1997*)
- These two can be measured/detected experimentally.

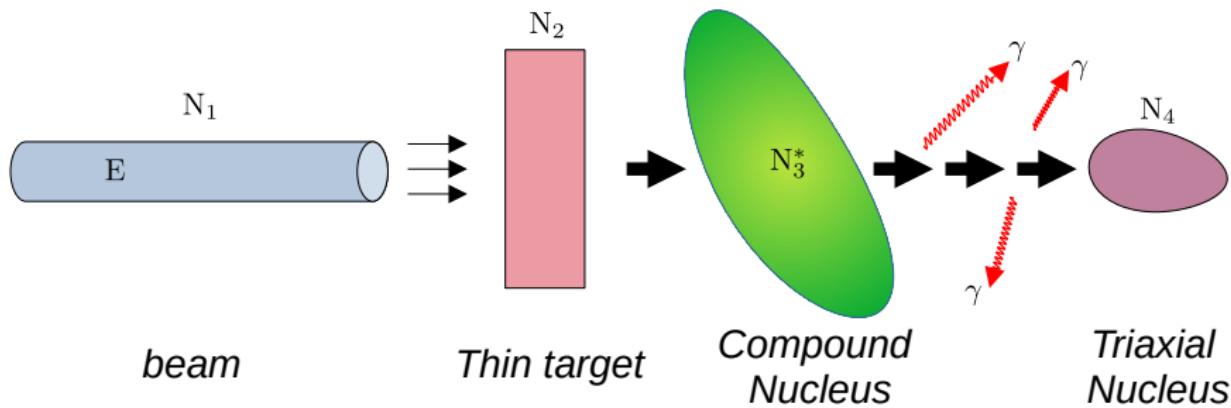
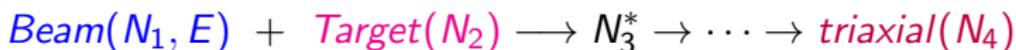
Goal

Describe the elusive character of Wobbling Motion in the context of nuclear triaxiality.

Q Probing triaxiality in nuclei

Triaxial nuclei can be observed/obtained in several experiments:

- Nuclear fission: $A \rightarrow B + C$
- Nuclear fusion: $X + Y \rightarrow Z$
- **Fusion-evaporation reactions:** Long-lived + enhanced deformation



Q Nuclear facilities



Figure: Gammasphere detector,
ANL-ATLAS USA. *Source:*
aps.org

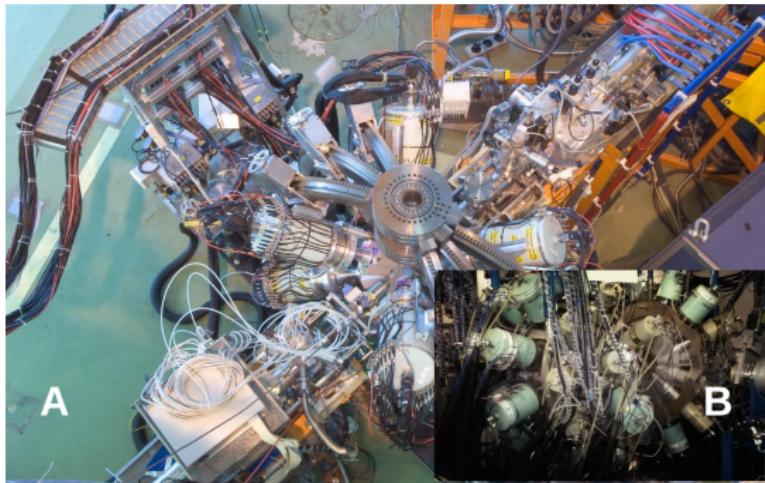


Figure: a) IDS detector, CERN. *Source:*
isodel.web.cern.ch b) JUROGAM II, Finland.
Source: twitter.com

Q High-Spin Physics @ IFIN-HH



Contents lists available at ScienceDirect
 Nuclear Instruments and Methods in Physics Research A
journal homepage: www.elsevier.com/locate/nima

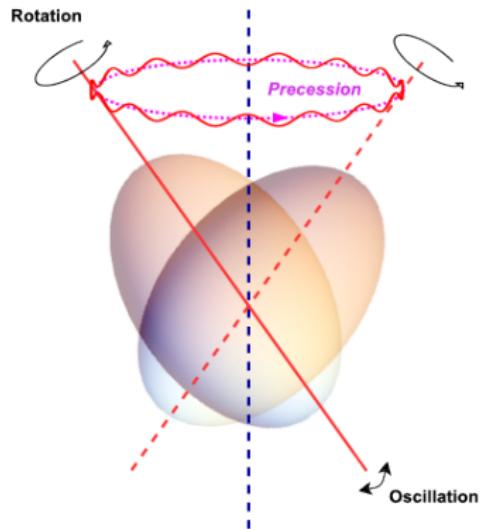
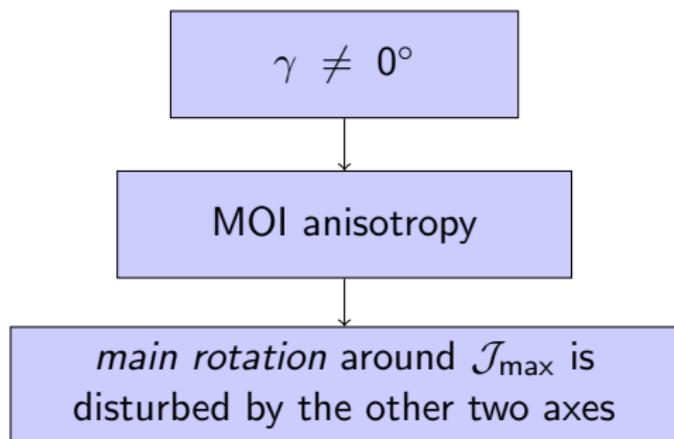

The ROSPHERE γ -ray spectroscopy array

D. Bucurescu^a, I. Căta-Danil^a, G. Ciocan^a, C. Costache^a, D. Deleanu^a, R. Dima^a, D. Filipescu^{a,c}, N. Florea^a, D.G. Ghiță^a, T. Glodariu^a, M. Ivașcu^a, R. Lică^a, N. Mărginean^a, R. Mărginean^a, C. Mihai^{a,b}, A. Negret^a, C.R. Nită^a, A. Olăcel^a, S. Pascu^a, T. Sava^a, L. Stroe^a, A. Ţerban^{a,d}, R. Suvală^a, S. Toma^a, N.V. Zamfir^{a,c}, G. Căta-Danil^b, I. Gheorghe^c, I.O. Mitu^c, G. Suliman^c, C.A. Ur^c, T. Braurnoth^d, A. Dewald^d, C. Fransen^d, A.M. Bruce^e, Zs. Podolyák^f, P.H. Regan^{f,g}, O.J. Roberts^h



Figure: ROSPHERE, IFIN-HH. Source: tandem.nipne.ro

Wobbling Motion



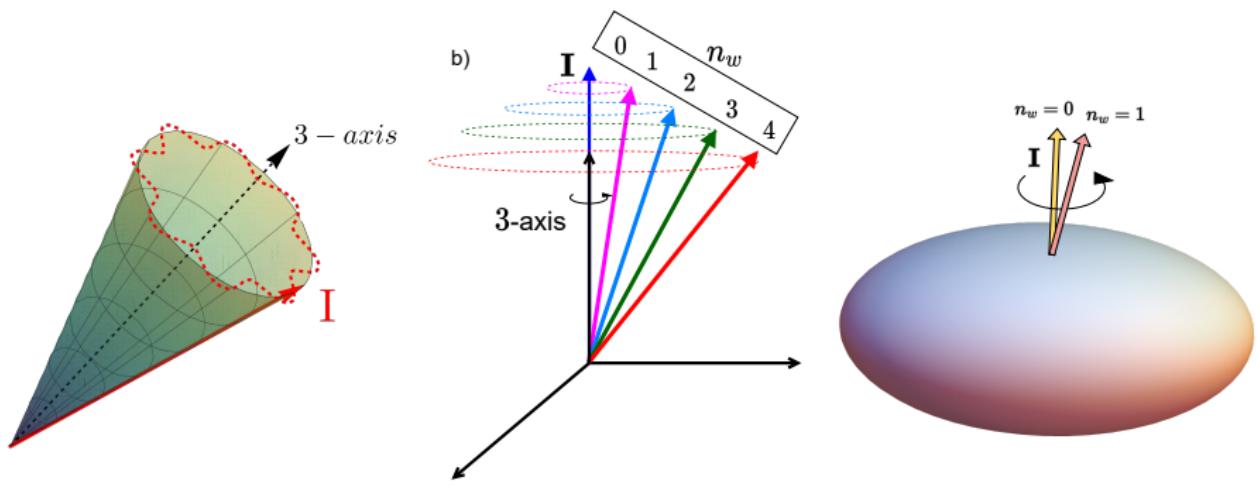
Wobbling Effect

- The **total angular momentum** of the nucleus **precesses** and **oscillates** around \mathcal{J}_{\max} .

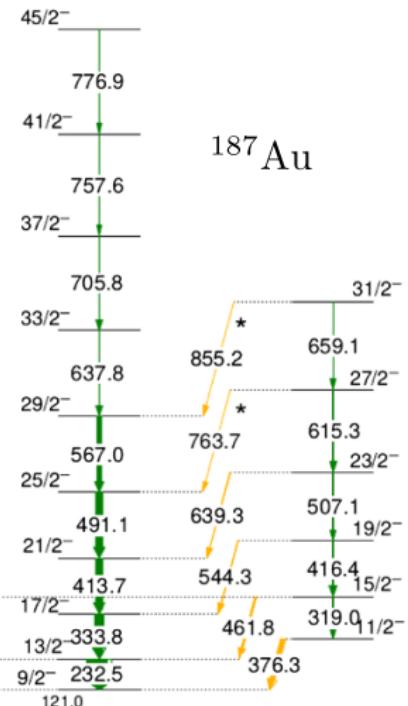
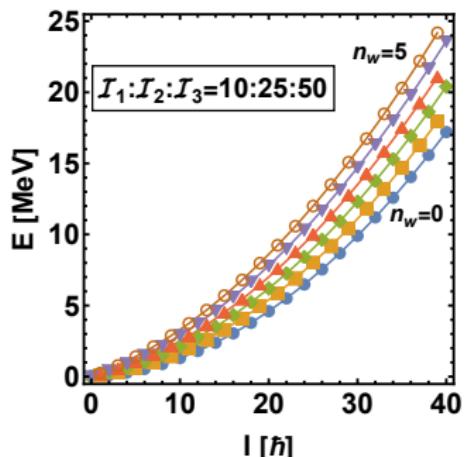
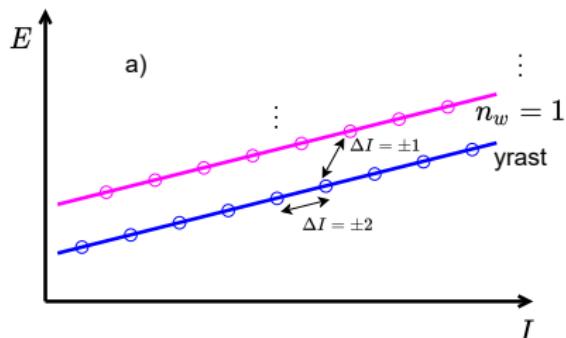
Wobbling Motion

Harmonic oscillation

- Precession of \mathbf{I} is affected by **rotational frequency** and/or **tilting**
 - Tilting only by "specific" amount \rightarrow **harmonic character** \rightarrow **wobbling phonon**: $n_w = 0, 1, 2, \dots$



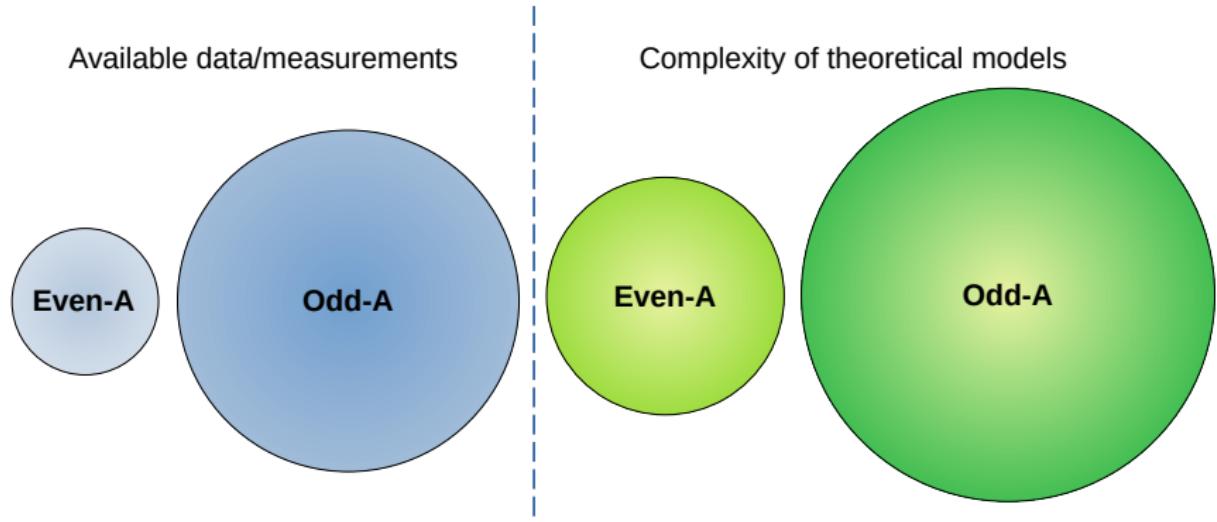
Wobbling Motion II



Sensharma, 2020.

Even- A vs. Odd- A Picture

- Predicted for even- A nuclei more than 50 years ago.
- First experimental evidence: ^{163}Lu (*Ødegård, 2001*).
- Current mass-regions for wobblers: $A \approx [130, 160, 180]$.



Wobbling Motion in ^{130}Ba

Q Experimental measurements show **two** wobbling bands (*Petrache et. al. 2019*).

Harmonic formalism

Harmonic Approximation (*Bohr & Mottelson, 1969*):

$$E_{I,n_w} = A_3 I(I+1) + \hbar\omega_w \left(n_w + \frac{1}{2} \right),$$

$$A_3 = (2\mathcal{I}_3)^{-1}.$$

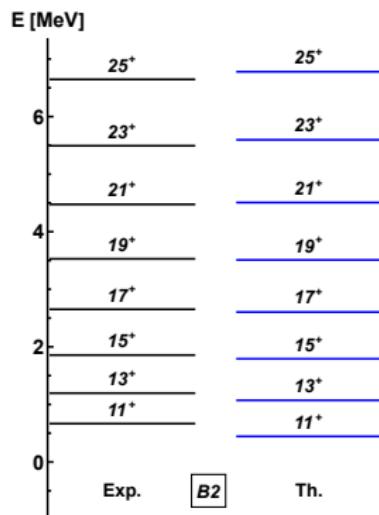
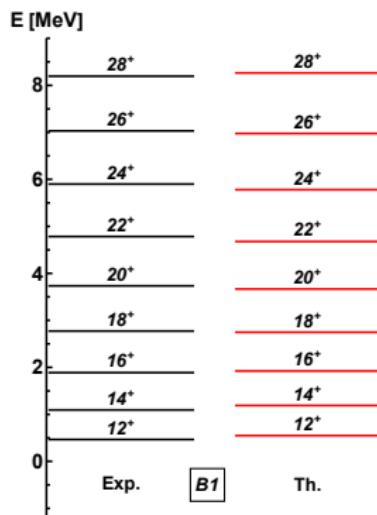
(rotational term + wobbling frequency)



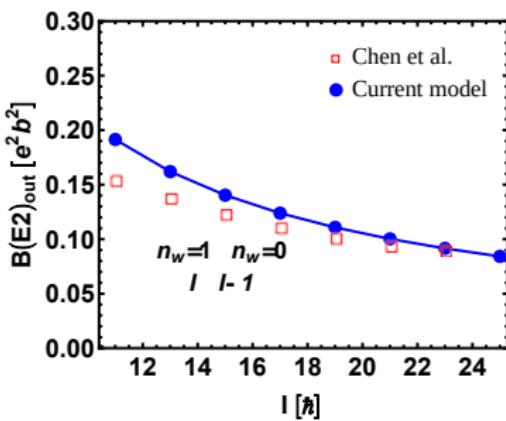
GALILEO, LNL, Source: lnl.infn.it

Fusion evaporation: ^{13}C beam of
 $E = 65$ MeV and ^{122}Sn target.

Results for ^{130}Ba



\mathcal{P}_{fit}			
\mathcal{I}_1	\mathcal{I}_2	\mathcal{I}_3	Unit
27	22	43	$\hbar^2 \text{MeV}^{-1}$



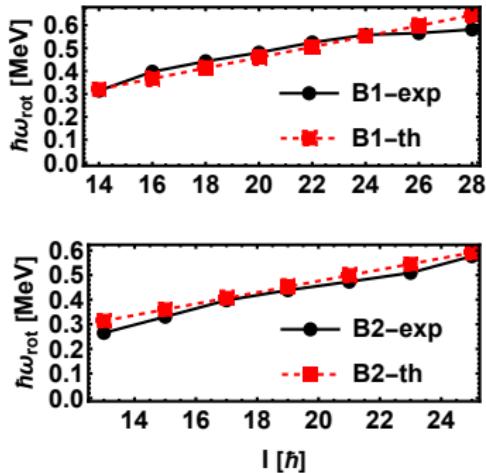
Results presented at the international conference NSP-2022, Turkey.

Results for $^{130}\text{Ba II}$

Excitation energies vs. Wobbling Energies:

$$E_{\text{wob}}(I_{\text{even}}) = E_{I,n} - E_{I,0} ,$$

$$E_{\text{wob}}(I_{\text{odd}}) = E_{I,n} - \frac{1}{2} (E_{I-1,0} + E_{I+1,0})$$



Results presented at the international conference NSP-2022, Turkey.

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 - Fresh-Up 2
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Starting Point

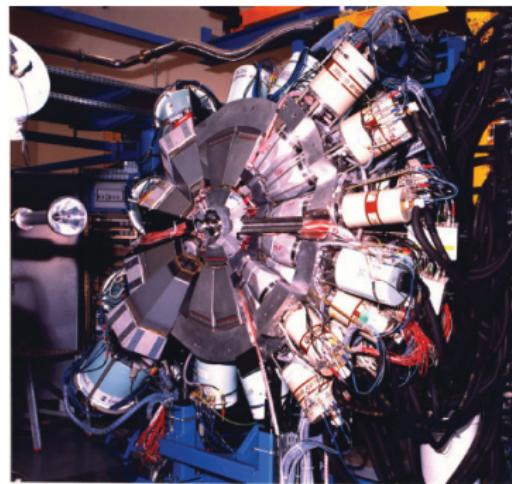
■ A. A. Raduta, **R. Poenaru**, L. Gr. Ixaru, PRC, 2017 + ■ A. A. Raduta, **R. Poenaru**, Al. H. Raduta, JPG, 2018 → W_0 in the thesis.

Framework

- **First semi-classical description** for the ^{163}Lu , using the **Particle-Rotor-Model** (*Hamamoto, 2002*).

PRM

An odd-nucleon moving in a quadrupole deformed mean field generated by an even-even triaxial core.



Euroball IV, Strasbourg, Source:
technology.i.stfc.ac.uk

Fusion evaporation: ^{29}Si beam of $E = 152$ MeV and ^{139}La target.

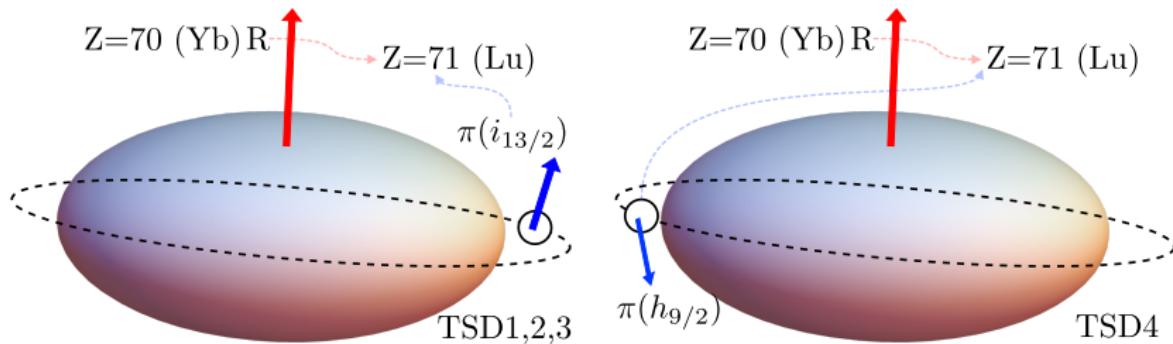
Fresh-Up 1: \mathbf{W}_1

Particle-Rotor Model Hamiltonian for an odd- A nucleus:

$$\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{sp}}, \quad \hat{H}_{\text{rot}} = \sum_{k=1}^3 A_k (\hat{I}_k - \hat{j}_k)^2,$$

$$\hat{H}_{\text{sp}} = \epsilon_j + \frac{V}{j(j+1)} \left[\cos \gamma (3\hat{j}_3^2 - \mathbf{j}^2) - \sqrt{3} \sin \gamma (\hat{j}_1^2 - \hat{j}_2^2) \right].$$

V - single-particle potential strength $\propto \beta_2$ (Tanabe, 2017)



A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

Variational Principle + Eqs. of Motion

Time-Dependent Variational Equation

$$\delta \int_0^t \langle \Psi_{IM;j} | \hat{H} - i \frac{\partial}{\partial t'} | \Psi_{IM;j} \rangle dt' = 0$$

$$\Psi_{\text{trial}} \equiv |\Psi_{IM;j}\rangle = \mathcal{N} e^{z\hat{I}_-} \cdot e^{s\hat{j}_-} |IMI\rangle \otimes |jj\rangle$$

- $(z, r, \varphi, \hat{I}, |IMI\rangle)$ - core (rotor **R**)
- $(s, f, \psi, \hat{j}, |jj\rangle)$ - single-particle (**j**)
- $\{z, s\}$ → **phase space coordinates**

Constant of motion:

$$\mathcal{H} \equiv \langle \Psi_{IM;j} | \hat{H} | \Psi_{IM;j} \rangle$$

Canonical equations of motion:

$$\mathcal{S}_1 : \frac{\partial \mathcal{H}}{\partial r} = \dot{\varphi}, \quad \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r}$$

$$\mathcal{S}_2 : \frac{\partial \mathcal{H}}{\partial f} = \dot{\psi}, \quad \frac{\partial \mathcal{H}}{\partial \psi} = -\dot{f}$$

The two sets of Hamilton equations are **the semi-classical description** of the initial quantal \hat{H} .

Wobbling frequency

Solving \mathcal{S}_1 and \mathcal{S}_2 leads to the algebraic equation:

$$\Omega^4 + B\Omega^2 + C = 0$$

four solutions \longrightarrow **only two are real**:

$$\Omega_{1,2} = \left[\frac{1}{2} \left(-B \mp \sqrt{B^2 - 4C} \right) \right]^{1/2}$$

- Ω_1 : wobbling frequency of the even-*A* core **R**
- Ω_2 : wobbling frequency of the odd-nucleon **j**
- **Two wobbling phonon numbers: n_{w_1} and n_{w_2}**

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

Energy spectrum

Spectra of odd-A nuclei within W_1

$$E_{I,n_1,n_2} = \epsilon_j + \mathcal{H}_{\min}^I + \mathcal{F}_{n_{w_1} n_{w_2}}^I$$

- Phonon factor:

$$\mathcal{F}_{n_{w_1} n_{w_2}}^I = \hbar \Omega_1^I \left(n_{w_1} + \frac{1}{2} \right) + \hbar \Omega_2^I \left(n_{w_2} + \frac{1}{2} \right)$$

- \mathcal{H}_{\min}^I - Classical Energy Function taken in its minimum point:
 $p_0 = (0, I; 0, j)$.
- ϵ_j - single-particle energy

A new interpretation for TSD1 and TSD2

Previous models

$TSD1$ = zero-phonon wobbling band

$TSD2$ = one-phonon wobbling band...

Redefinition

$TSD1$ and $TSD2$ are **Signature Partner Bands** (in favor of Ultimate Cranker calculations, *Jensen, 2004*).

$$\left(\alpha_{fav} = +\frac{1}{2} \right) : \{ TSD1 \} \equiv \{ [0^+, 2^+, 4^+, \dots] \otimes [J^\pi = 13/2^+] \}$$

$$\left(\alpha_{unfav} = -\frac{1}{2} \right) : \{ TSD2 \} \equiv \{ [1^+, 3^+, 5^+, \dots] \otimes [J^\pi = 13/2^+] \}$$

$TSD4$ is a **ground-state wobbling band**, $\pi(h_{9/2})$ configuration.

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

A new band structure for ^{163}Lu

Band	Spins	π	α	$\pi(I_j)$	$\mathbf{W}_0: \mathcal{C} + \mathcal{Q}_p$	$\mathbf{W}_1: \mathcal{C} + \mathcal{Q}_p$
TSD1	$13/2, 17/2 \dots 97/2$	+	+1/2	$\pi(i_{13/2})$	$0^+, 2^+, 4^+, \dots$	$0^+, 2^+, 4^+, \dots$
TSD2	$27/2, 31/2 \dots 91/2$	+	-1/2	$\pi(i_{13/2})$	$\text{TSD1} + 1\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$
TSD3	$33/2, 37/2 \dots 85/2$	+	+1/2	$\pi(i_{13/2})$	$\text{TSD1} + 2\Gamma^\dagger$	$\text{TSD2} + \Gamma^\dagger$
TSD4	$47/2, 51/2 \dots 83/2$	-	-1/2	$\pi(h_{9/2})$	$\text{TSD1} + 3\Gamma^\dagger$	$1^+, 3^+, 5^+, \dots$

Bands	n_{w_1}	n_{w_2}	$\mathcal{F}_{n_{w_1} n_{w_2}}^I$	I_0	I_t	\mathcal{Q}
TSD1	0	0	$\mathcal{F}_{00}^I = \frac{1}{2} (\Omega_1^I + \Omega_2^I)$	$13/2^+$	$97/2^+$	$j^\pi = 13/2^+ \stackrel{\text{not}}{\equiv} \mathcal{Q}_1$
TSD2	0	0	$\mathcal{F}_{00}^I = \frac{1}{2} (\Omega_1^I + \Omega_2^I)$	$27/2^+$	$91/2^+$	$j^\pi = 13/2^+ \stackrel{\text{not}}{\equiv} \mathcal{Q}_1$
TSD3	1	0	$\mathcal{F}_{10}^{I-1} = \frac{3}{2} \Omega_1^{I-1} + \frac{1}{2} \Omega_2^{I-1}$	$33/2^+$	$85/2^+$	$j^\pi = 13/2^+ \stackrel{\text{not}}{\equiv} \mathcal{Q}_1$
TSD4	0	0	$\mathcal{F}_{00}^I = \frac{1}{2} (\Omega_1^I + \Omega_2^I)$	$47/2^-$	$83/2^-$	$j^\pi = 9/2^- \stackrel{\text{not}}{\equiv} \mathcal{Q}_2$

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

Extension to $A \approx 160$ mass region

The model was further applied to the other Lu isotopes.

^{161}Lu Bands	Spins	\mathcal{Q}	\mathcal{C}	(n_{w_1}, n_{w_2})	I_b
TSD1	$21/2^+, 25/2^+, \dots, 89/2^+$	$j^\pi = 13/2^+$	$4^+, 6^+, 8^+ \dots$	$(0, 0)$	
TSD2	$31/2^+, 35/2^+, \dots, 79/2^+$	$j^\pi = 13/2^+$	$9^+, 11^+, 13^+ \dots$	$(0, 0)$	$21/2$

^{165}Lu Bands	Spins	\mathcal{Q}	\mathcal{C}	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \dots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \dots$	$(0, 0)$	
TSD2	$35/2^+, 39/2^+, \dots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \dots$	$(0, 0)$	$25/2$
TSD3	$41/2^+, 45/2^+, \dots, 81/2^+$	$j^\pi = 13/2^+$	$\text{TSD2} + \Gamma^\dagger$	$(1, 0)$	

^{167}Lu Bands	Spins	\mathcal{Q}	\mathcal{C}	(n_{w_1}, n_{w_2})	I_b
TSD1	$25/2^+, 29/2^+, \dots, 89/2^+$	$j^\pi = 13/2^+$	$6^+, 8^+, 10^+ \dots$	$(0, 0)$	
TSD2	$35/2^+, 39/2^+, \dots, 91/2^+$	$j^\pi = 13/2^+$	$11^+, 13^+, 15^+ \dots$	$(0, 0)$	$25/2$

A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

W₁ — Numerical Results

Free parameters in the model $\rightarrow \mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

Fitting procedure

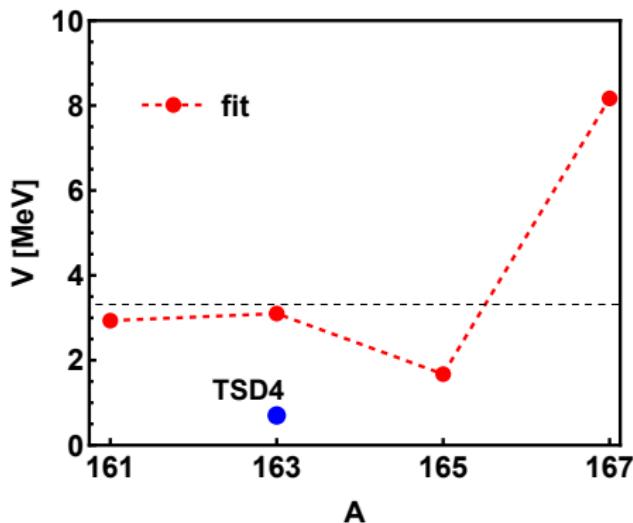
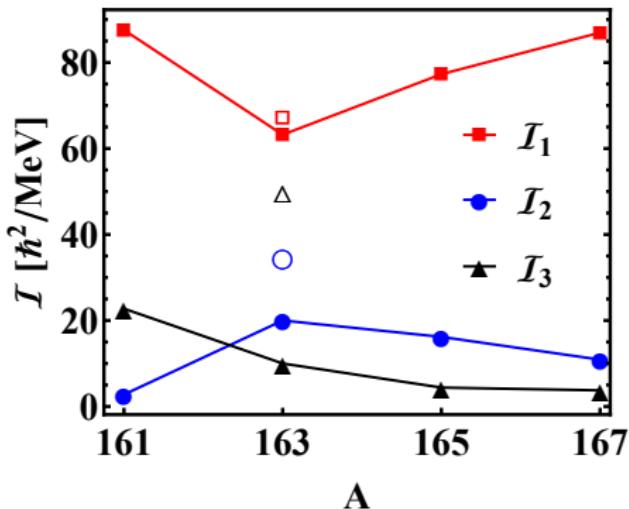
$$\chi^2 = \frac{1}{N_T} \sum_i \frac{(E_{\text{exp}}^{(i)} - E_{\text{th}}^{(i)})^2}{E_{\text{exp}}^{(i)}}$$

¹⁶³Lu-TSD4: separate fitting procedure (different nucleon configuration)

Isotope	Bands	\mathcal{I}_1 [\hbar^2/MeV]	\mathcal{I}_2 [\hbar^2/MeV]	\mathcal{I}_3 [\hbar^2/MeV]	V [MeV]	γ [°]	n.o.s	E_{rms} [MeV]
¹⁶¹ Lu	TSD1-2	87.555	2.773	22.744	2.933	20	29	0.168
¹⁶³ Lu	TSD1-3	63.2	20	10	3.1	17	52	0.264
	TSD4	67	34.5	50	0.7	17	10	0.057
¹⁶⁵ Lu	TSD1-3	77.295	16.184	4.399	1.673	20	42	0.125
¹⁶⁷ Lu	TSD1-2	87.032	10.895	3.758	8.167	19.48	30	0.165

A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

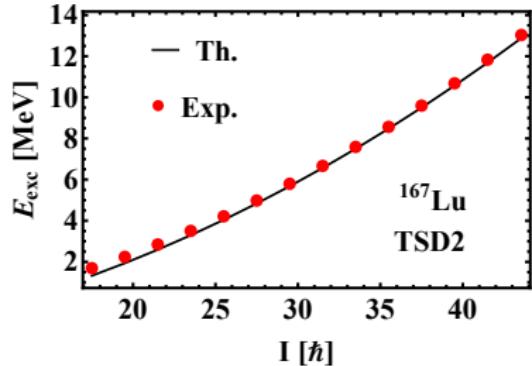
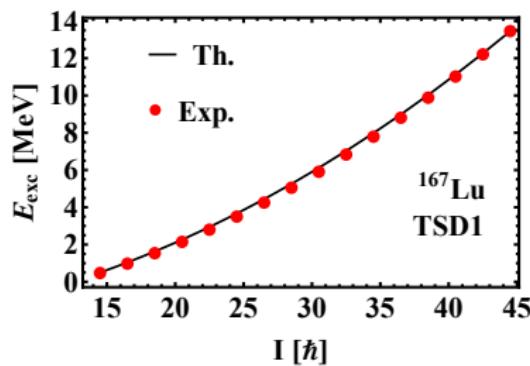
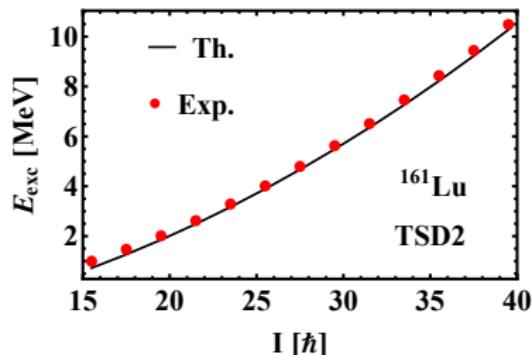
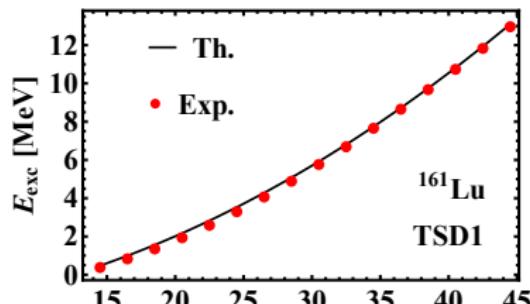
Graphical representation for MOIs and V



$V \approx 3 \text{ MeV} \rightarrow$ agreement with other calculations (Tanabe, 2017)

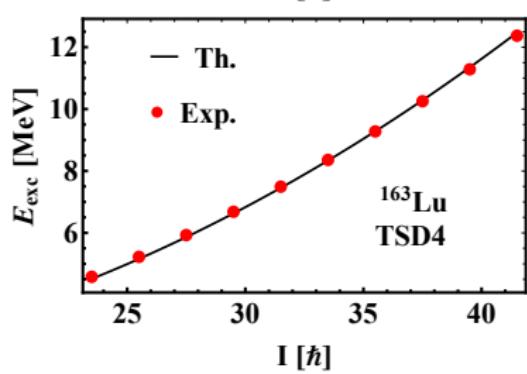
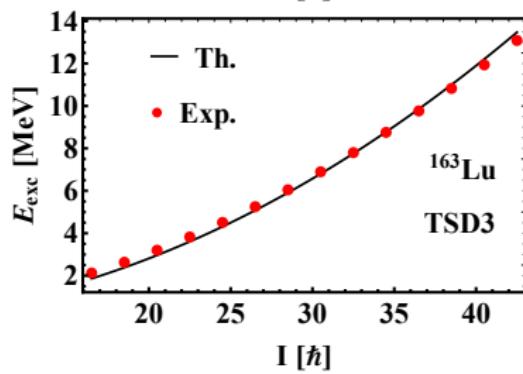
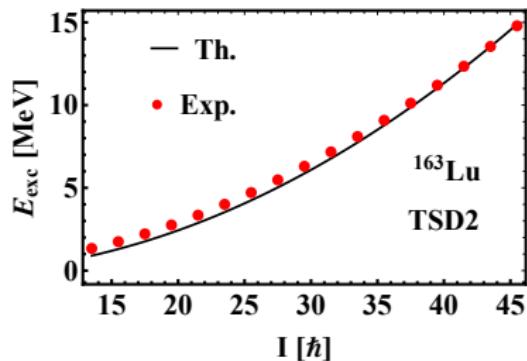
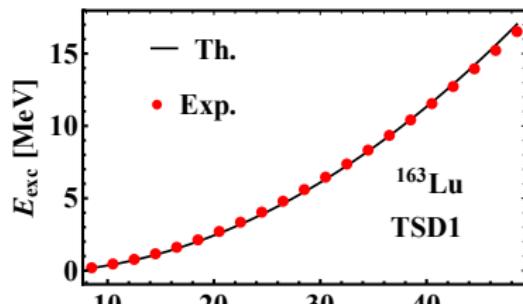
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Excitation Energies — $^{161,167}\text{Lu}$



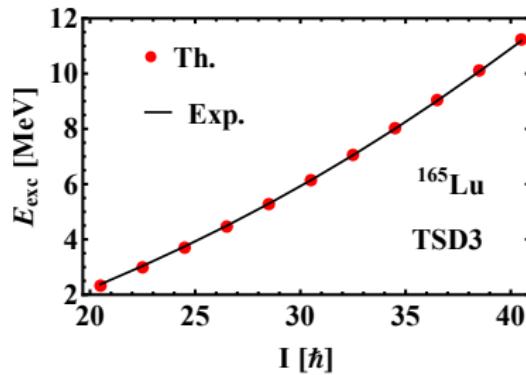
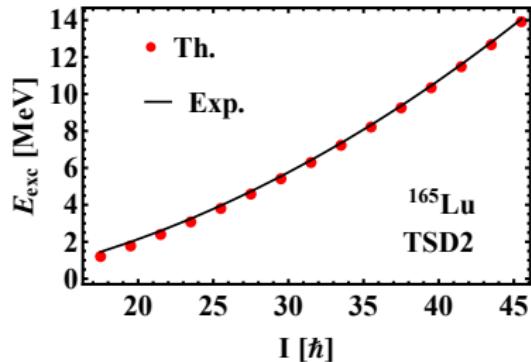
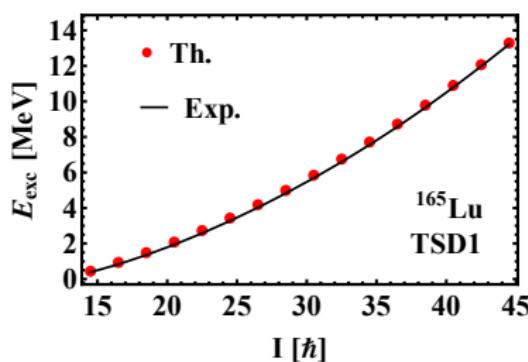
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Excitation Energies — ^{163}Lu



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Excitation Energies — ^{165}Lu

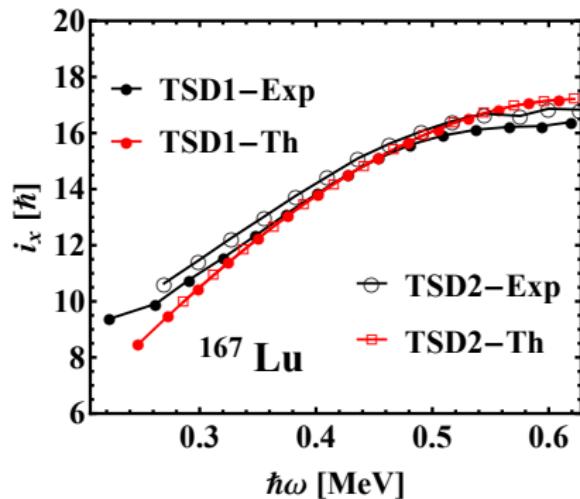
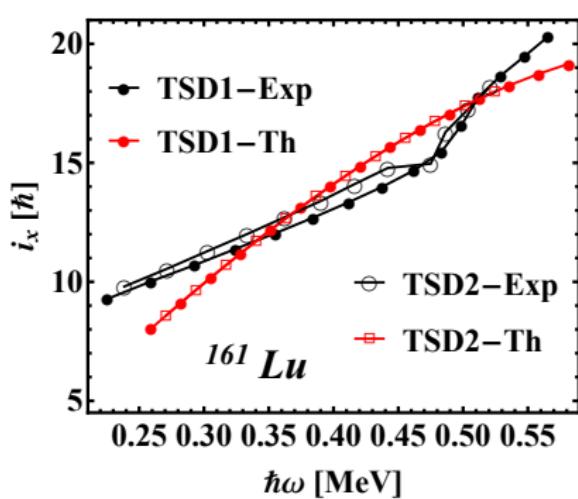


A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Alignment — $^{161,167}\text{Lu}$

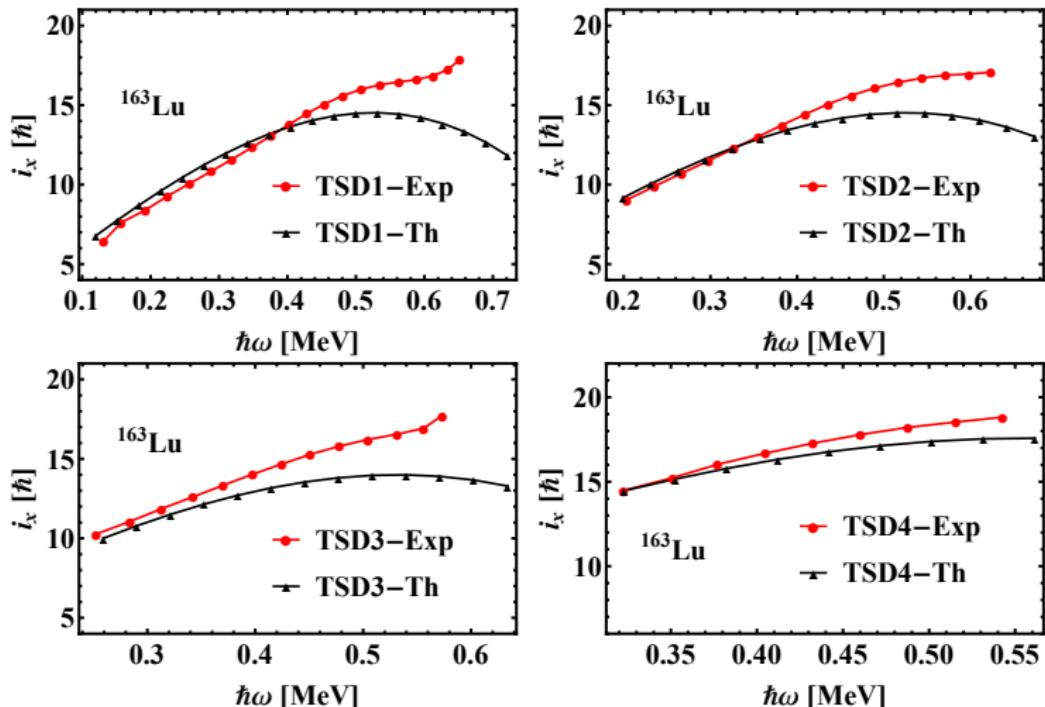
$$i_x = I - I_{\text{ref}},$$

$$I_{\text{ref}} = \mathcal{I}_0 \omega + \mathcal{I}_1 \omega^3.$$



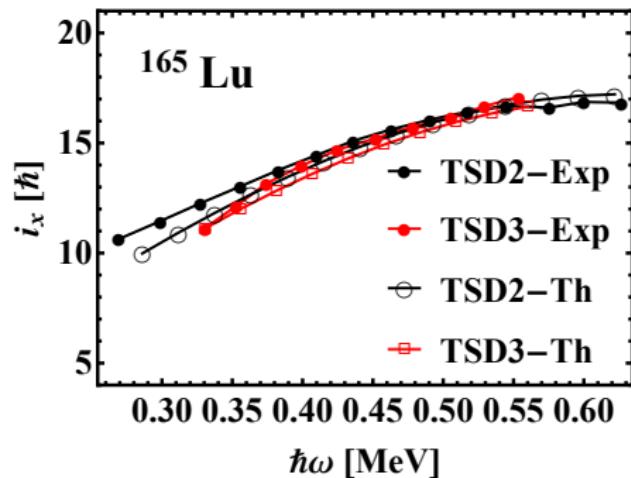
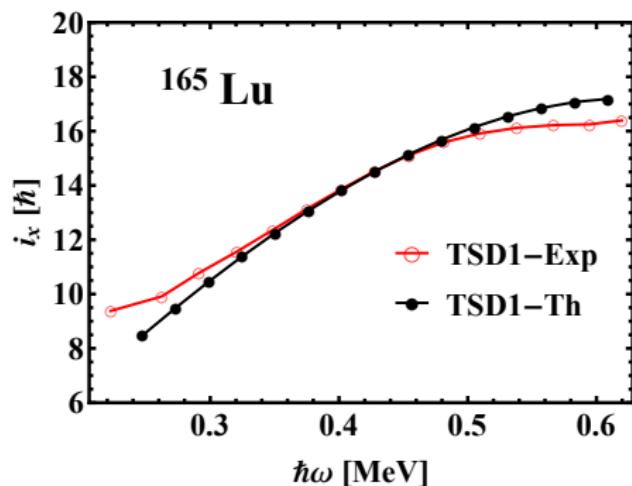
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Alignment — ^{163}Lu



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

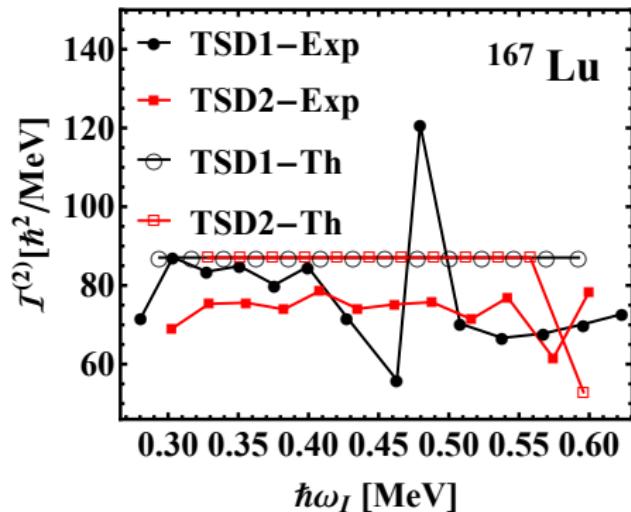
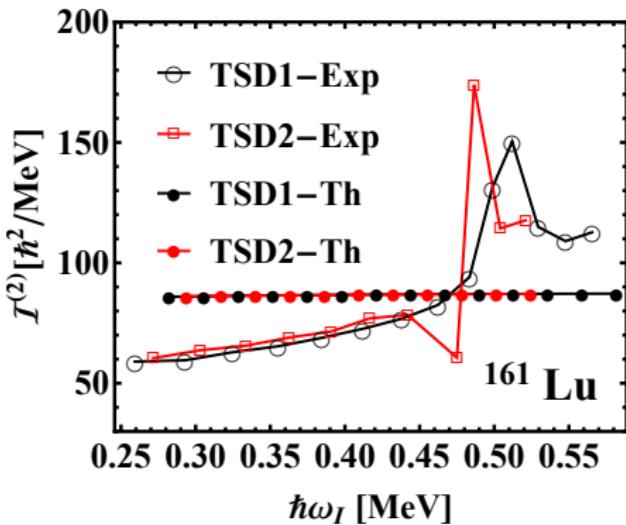
Alignment — ^{165}Lu



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

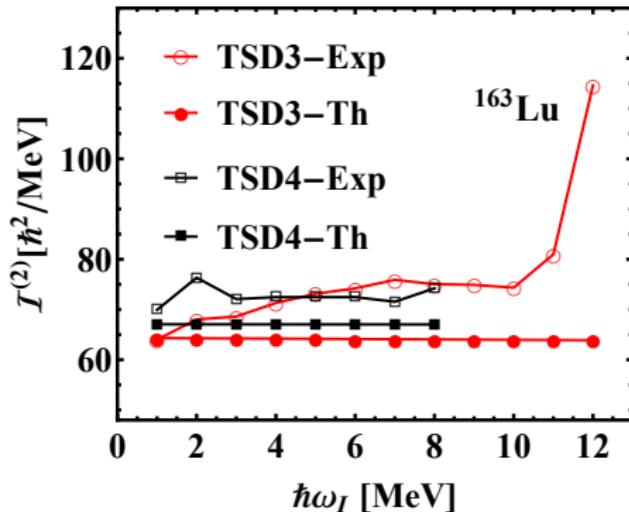
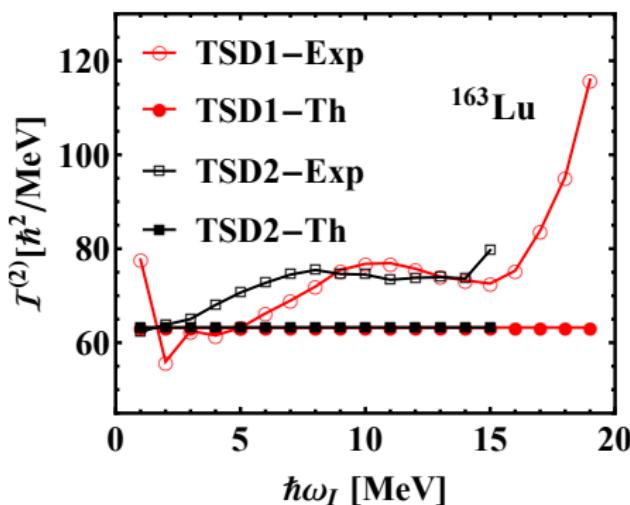
Dynamic Moment of Inertia — $^{161,167}\text{Lu}$

$$\mathcal{I}^{(2)}(I) = \hbar \frac{dI_x}{d\omega} = \hbar^2 \left(\frac{d^2 E}{dI_x^2} \right)^{-1}$$



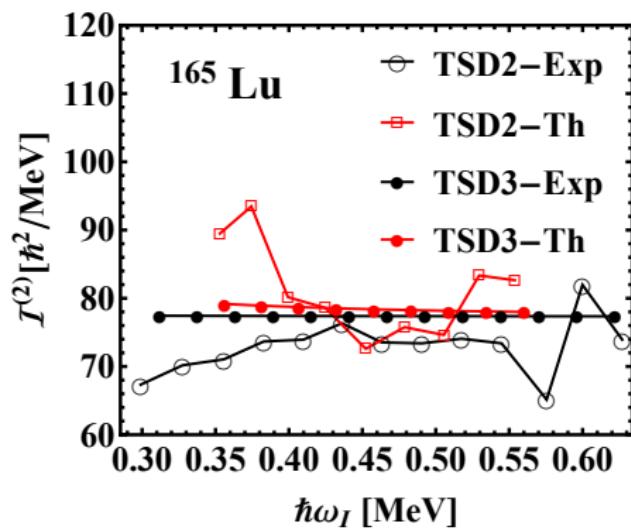
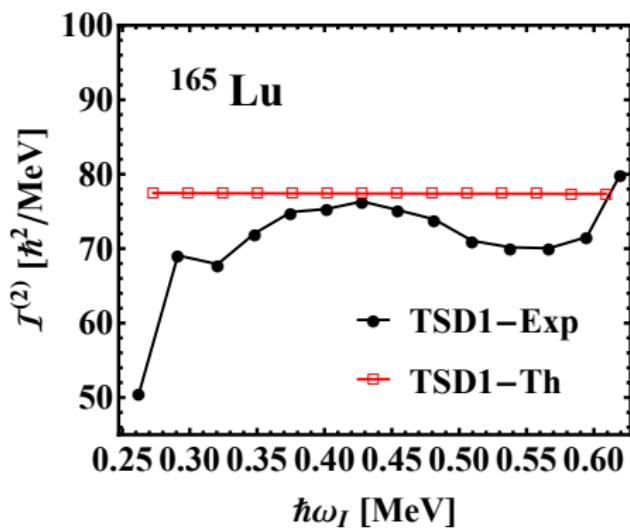
A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{163}Lu



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Dynamic Moment of Inertia — ^{165}Lu



A. A. Raduta, R. Poenaru, C. M. Raduta, Phys. Rev. C 101, 2020.

Electromagnetic Calculations

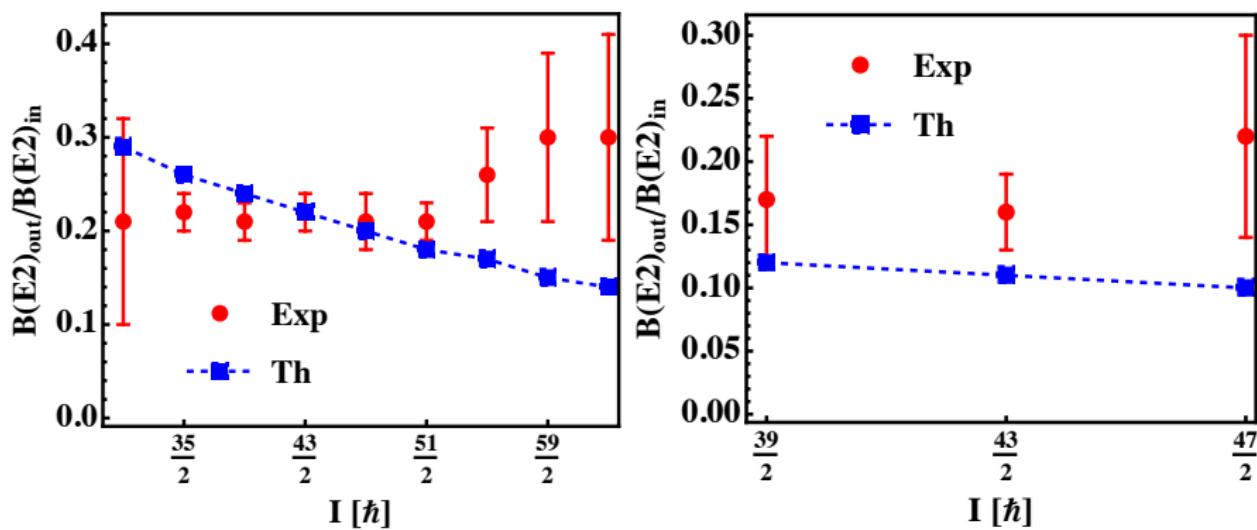


Figure: E2 Branching ratio. **Left:** ^{163}Lu (TSD2) **Right:** ^{165}Lu (TSD2).

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

Electromagnetic Calculations II

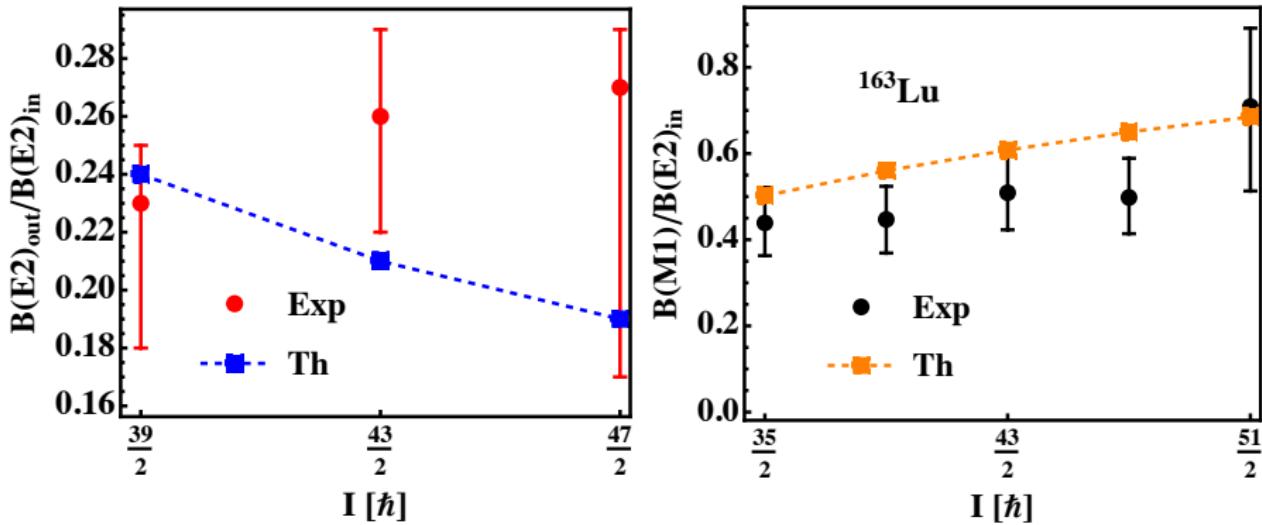


Figure: **Left:** E2 Branching ratio in ^{167}Lu (TSD2). **Right:** The ratio $B(M1)/B(E2)_{\text{in}}$ for states $TSD2 \rightarrow TSD1$ (in units of $\mu_N^2/(e^2 b^2)$).

A.A. Raduta, R. Poenaru, C.M. Raduta, Journal of Physics G 47, 2020.

W₁ — Remarks

Characteristics

- + Full semi-classical description (TDVE) with good numerical results
- + Deformation parameters are self-consistent (agree with exp. values)
- separate fit for TSD4 (different nucleonic configuration)
- Two sets of MOIs for ^{163}Lu

**Onset of another redesign
Start of W₂ formalism in Chapter 5**

Fresh-Up 2: W_2

Novel description of ^{163}Lu

- All four bands in ^{163}Lu described by the same triaxial core + odd-particle coupling $\rightarrow Q_1 = \pi(i_{13/2})$
- The adopted wave-function admits solutions of both **positive and negative parity**. Parity operator: $\mathcal{P} = e^{-i\pi J_2} C$:

$$\mathcal{P}\Psi(r, \varphi; t, \psi) = \Psi(r, \varphi + \pi; t, \psi + \pi),$$

$$\mathcal{H}(r, \varphi + \pi; t, \psi + \pi) = \mathcal{H}(r, \varphi; t, \psi),$$

$$\Psi(r, \varphi + \pi; t, \psi + \pi) = \pm \Psi(r, \varphi; t, \psi).$$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New band structure in ^{163}Lu

$$E_{I,0,0}^{\text{TSD1}} = \epsilon_{13/2} + \mathcal{H}_{\min}^I + \mathcal{F}_{00}^I, \quad I^\pi = 13/2^+, 17/2^+, 21/2^+ \dots,$$

$$E_{I,0,0}^{\text{TSD2}} = \epsilon_{13/2}^1 + \mathcal{H}_{\min}^I + \mathcal{F}_{00}^I, \quad I^\pi = 27/2^+, 31/2^+, 35/2^+ \dots,$$

$$E_{I,1,0}^{\text{TSD3}} = \epsilon_{13/2} + \mathcal{H}_{\min}^{I-1} + \mathcal{F}_{10}^{I-1}, \quad I^\pi = 33/2^+, 37/2^+, 41/2^+ \dots,$$

$$E_{I,0,0}^{\text{TSD4}} = \epsilon_{13/2}^2 + \mathcal{H}_{\min}^I + \mathcal{F}_{00}^I, \quad I^\pi = 47/2^-, 51/2^-, 55/2^- \dots.$$

Band	n_s	\mathbf{j}_Q	$\mathbf{R}_{\mathcal{C}}$ - Sequence	I - Sequence	Coupling
TSD1	21	\mathcal{Q}_1	$\mathcal{C}_1 = 0^+, 2^+, 4^+, \dots$	$13/2^+, 17/2^+, 21/2^+, \dots$	$\mathcal{C}_1 + \mathcal{Q}_1$
TSD2	17	\mathcal{Q}_1	$\mathcal{C}_2^+ = 1^+, 3^+, 5^+, \dots$	$27/2^+, 31/2^+, 35/2^+, \dots$	$\mathcal{C}_2^+ + \mathcal{Q}_1$
TSD3	14	\mathcal{Q}_1	1-phonon exc.	$33/2^+, 37/2^+, 41/2^+, \dots$	
TSD4	11	\mathcal{Q}_1	$\mathcal{C}_2^- = 1^-, 3^-, 5^-, \dots$	$47/2^-, 51/2^-, 55/2^-, \dots$	$\mathcal{C}_2^- + \mathcal{Q}_1$

R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

New results for ^{163}Lu

Model requires a unique set of parameters: $\mathcal{P}_{\text{fit}} = [\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, V, \gamma]$.

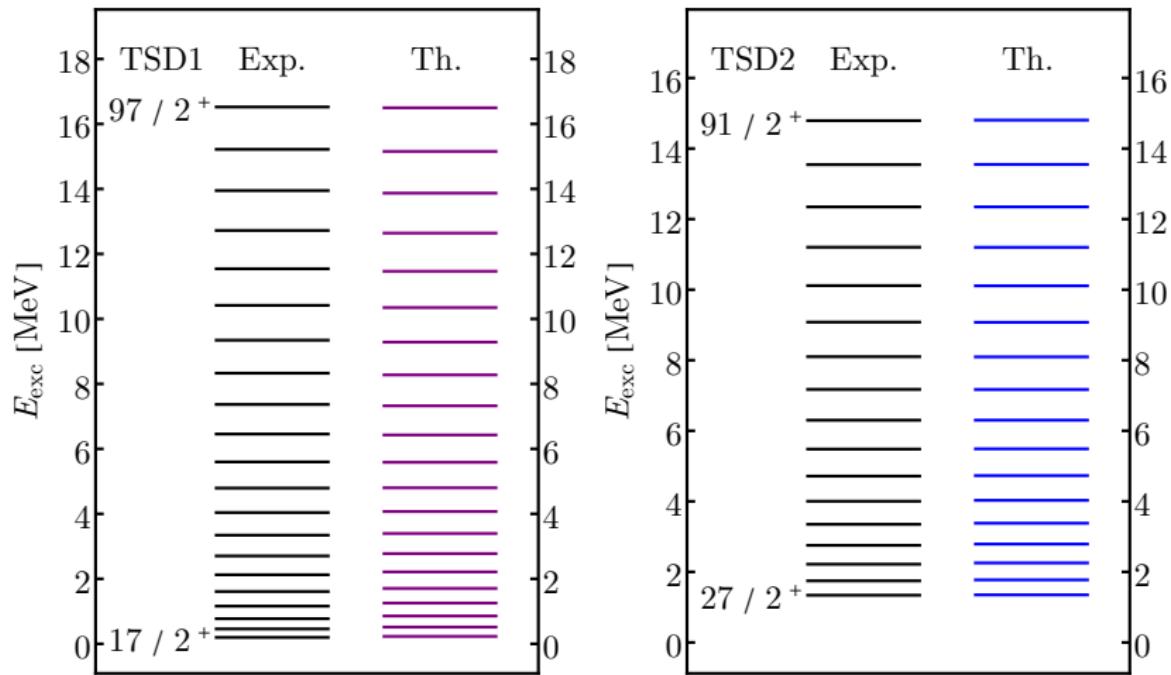
$\mathcal{I}_1 [\hbar^2/\text{MeV}]$	$\mathcal{I}_2 [\hbar^2/\text{MeV}]$	$\mathcal{I}_3 [\hbar^2/\text{MeV}]$	$\gamma [\text{deg.}]$	$V [\text{MeV}]$
72	15	7	22	2.1

Remarks

- γ in agreement with exp. value ($\gamma_{\text{exp}} = 20^\circ$) (Jensen, 2004)
- Slight decrease of V (?) breaking of parity symmetry quenches the quadrupole deformation)
- overall $E_{\text{RMS}} \approx 79 \text{ keV}$: first semi-classical description for a nucleus with deviations smaller than 100 keV.

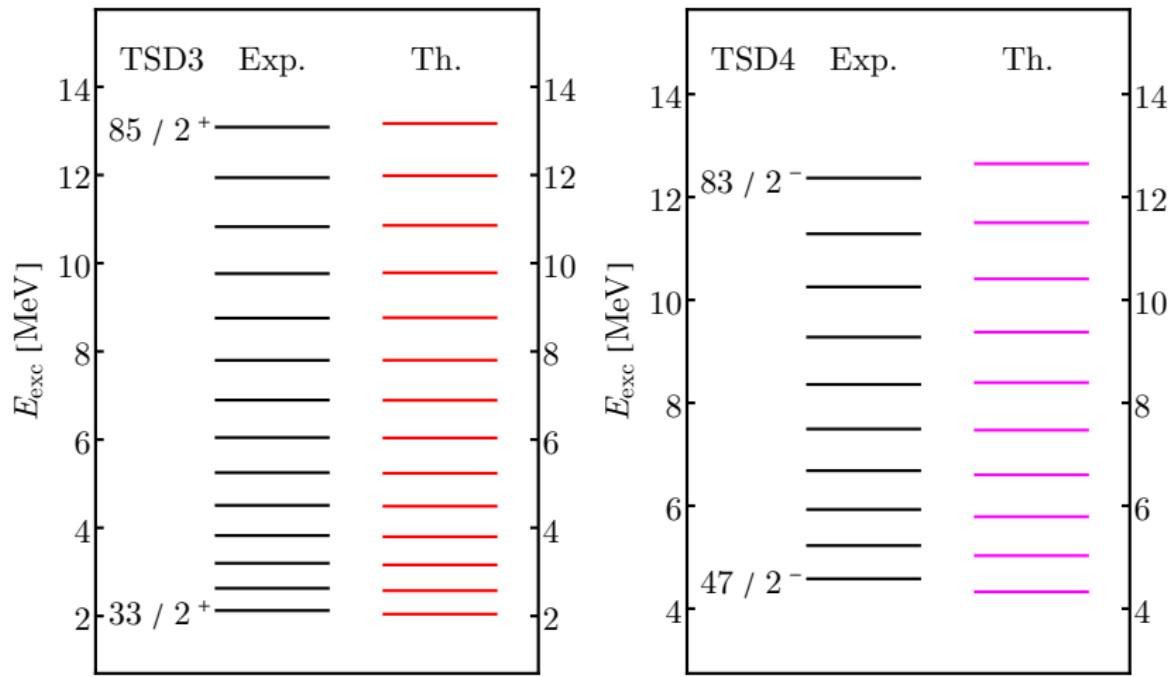
R. Poenaru, A.A. Raduta, International Journal of Modern Physics E 30, 2021.

Energy spectrum



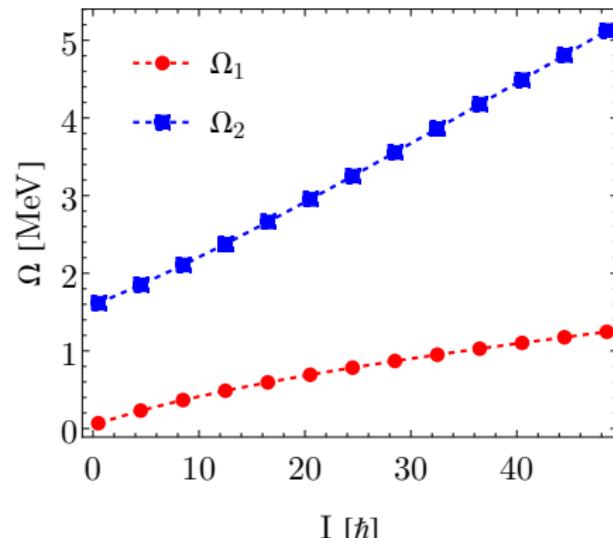
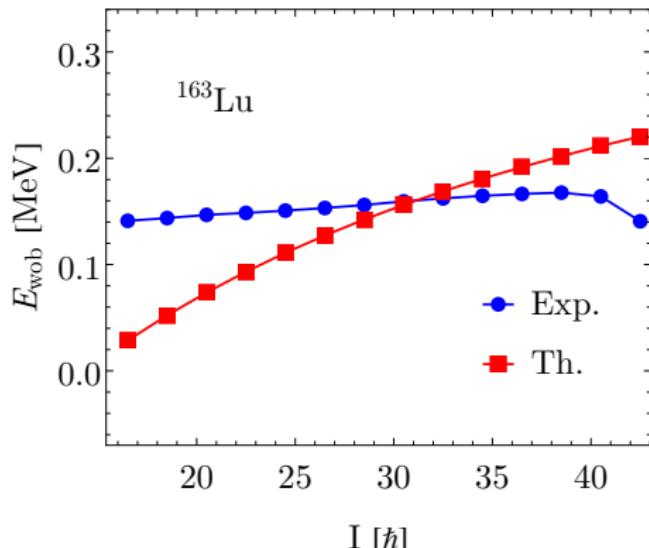
R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Energy spectrum II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

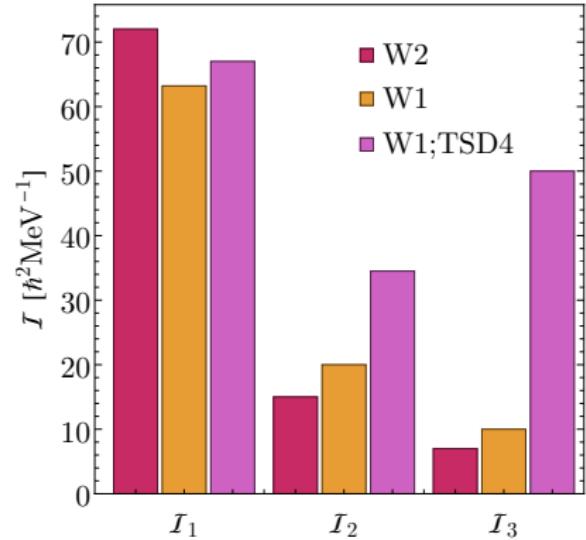
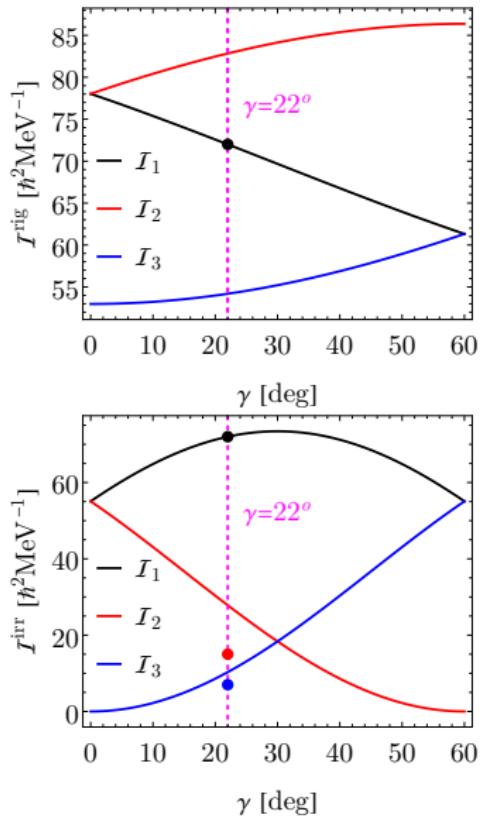
Wobbling Energies



The wobbling energy (**left**) and the two wobbling frequencies (**right**) for ^{163}Lu . **Decreasing trend of E_{wob} in agreement with arguments of Frauendorf 2014.**

R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 308, 2021.

Moments of inertia for \mathbf{W}_2



\mathbf{W}_2 : hydrodynamical character of the triaxial nucleus.

Results presented at the International Conference NSP, 2023, Turkey.

Classical Energy Function

Angular momentum

Polar representation of the angular momentum and \mathcal{H} .

$$\mathbf{l} = \{l_1, l_2, l_3\} \equiv \{x_1, x_2, x_3\} ,$$

$$x_1 = l \sin \theta \cos \varphi , \quad x_2 = l \sin \theta \sin \varphi , \quad x_3 = l \cos \theta .$$

$$\mathcal{H} |_{p_0} = l \left(l - \frac{1}{2} \right) \sin^2 \theta \cdot \mathcal{A}_\varphi - 2A_1 l j \sin \theta + T_{\text{core}} + T_{\text{sp}} ,$$

$$\mathcal{A}_\varphi = A_1 \cos^2 \varphi + A_2 \sin^2 \varphi - A_3 ,$$

$$T_{\text{core}} = \frac{l}{2} (A_1 + A_2) + A_3 l^2 ,$$

$$T_{\text{s.p.}} = \frac{j}{2} (A_2 + A_3) + A_1 j^2 - V \frac{2j-1}{j+1} \sin \left(\gamma + \frac{\pi}{6} \right) .$$

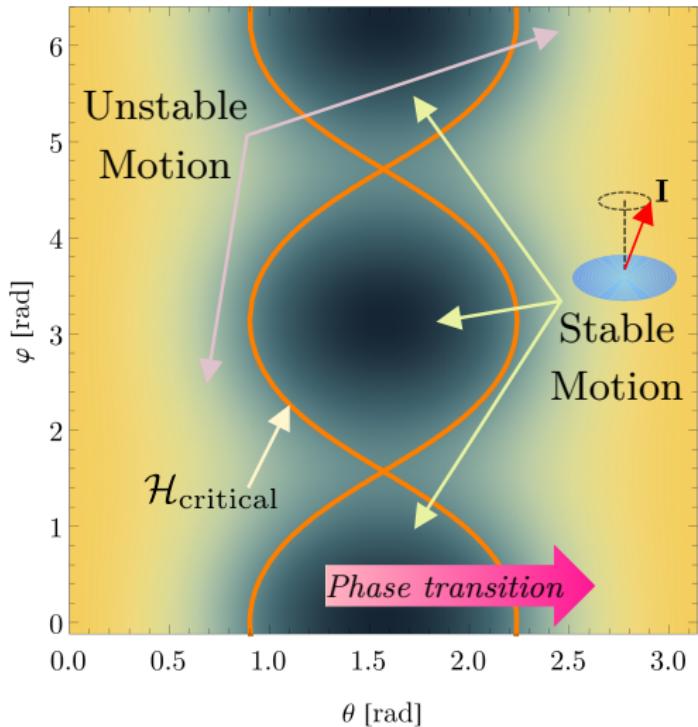
CEF — Stability Regions

Table: The minimum points of \mathcal{H} .
Using the MOIs from the fitting procedure

Minimal point	θ [rad]	φ [rad]	A_k ordering
m_1	$\pi/2$	0	$A_3 > A_2 > A_1$
m_2	$\pi/2$	π	$A_3 > A_2 > A_1$
m_3	$\pi/2$	2π	$A_3 > A_2 > A_1$

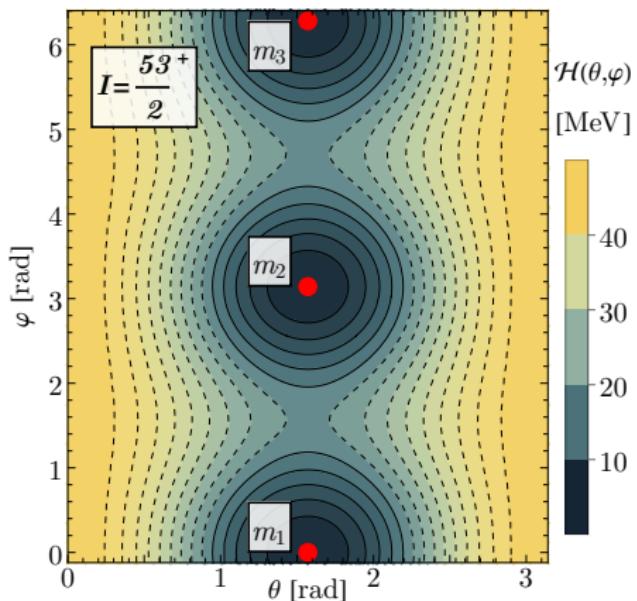
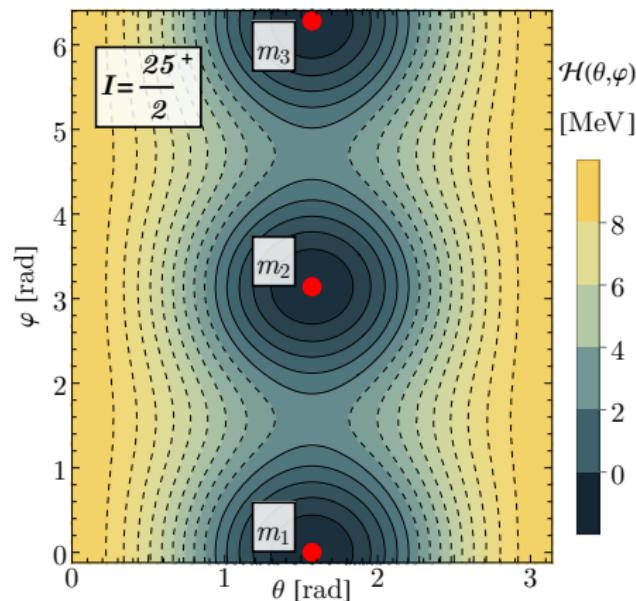
Semi-classical feature

This is the first classical geometrical description of the *wobbling stability* for an odd-mass nucleus.



Polar representation of \mathcal{H}

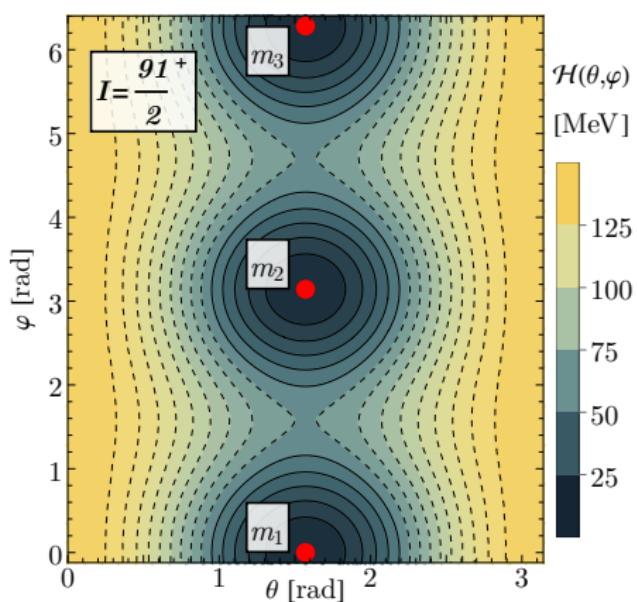
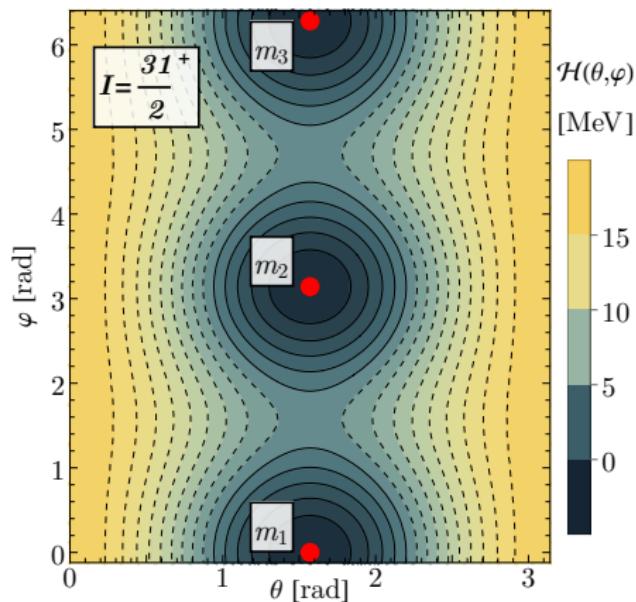
Figure: ^{163}Lu TSD1



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of \mathcal{H} II

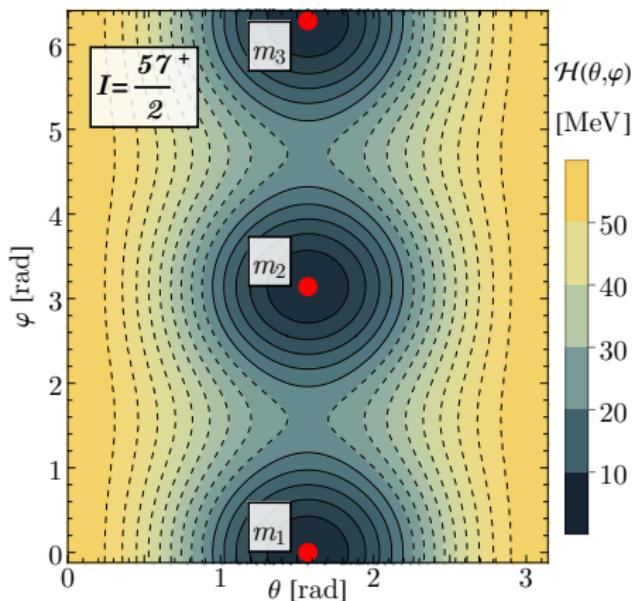
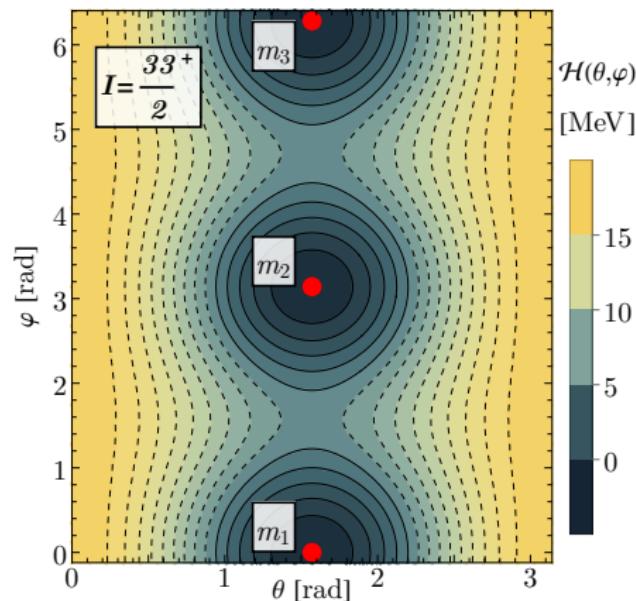
Figure: ^{163}Lu TSD2



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of \mathcal{H} III

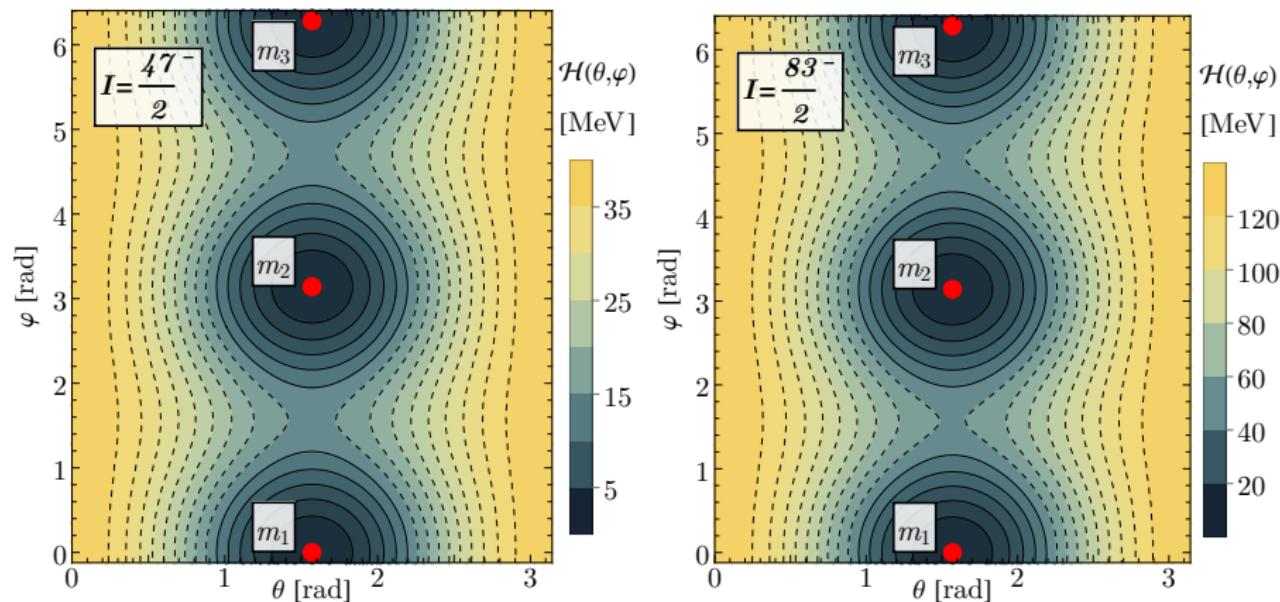
Figure: ^{163}Lu TSD3



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Polar representation of \mathcal{H} IV

Figure: ^{163}Lu TSD4



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

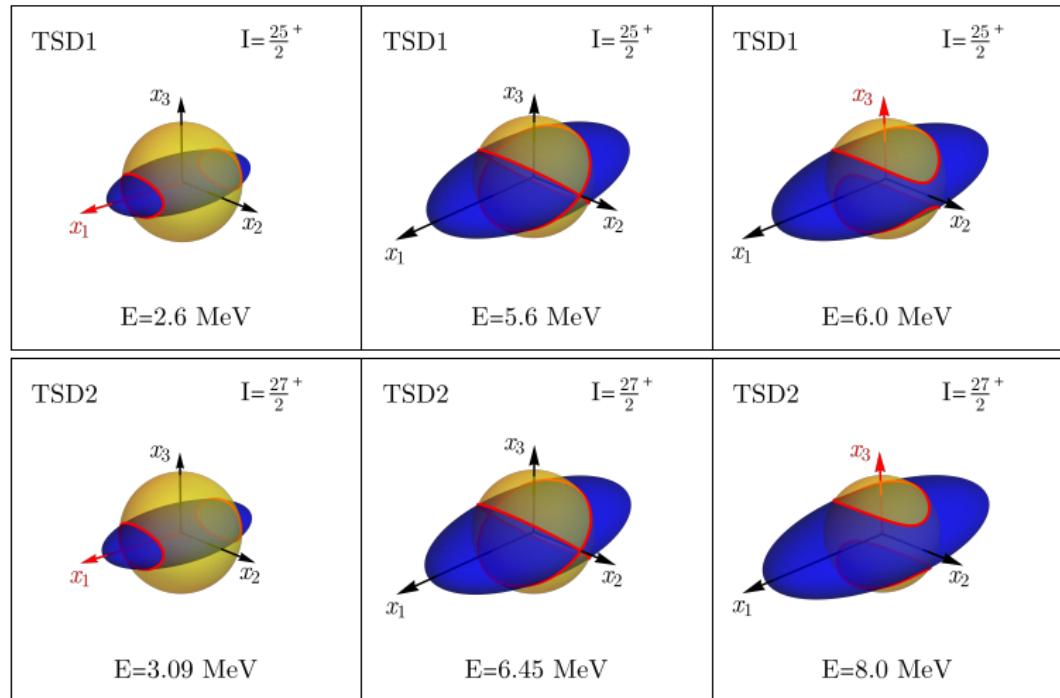
3D interpretation of the WM

- Formalism \mathbf{W}_2 gives a 3D interpretation of the nuclear wobbling motion
- **Classical Trajectories:** intersection curves between the **triaxial energy** and the **total angular momentum**

$$I^2 = x_1^2 + x_2^2 + x_3^2,$$

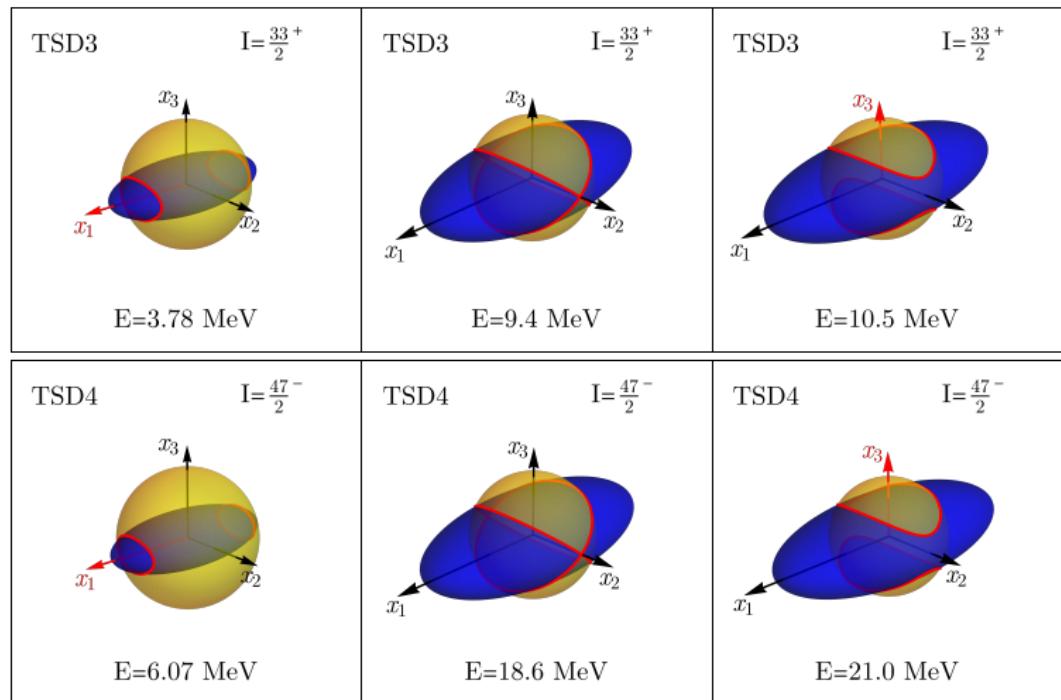
$$\begin{aligned} E = & \left(1 - \frac{1}{2I}\right) A_1 x_1^2 + \left(1 - \frac{1}{2I}\right) A_2 x_2^2 + \left[\left(1 - \frac{1}{2I}\right) A_3 + A_1 \frac{j}{I}\right] x_3^2 - \\ & - I \left(I - \frac{1}{2}\right) A_3 - 2A_1 I j + T_{\text{rot}} + T_{\text{sp}}. \end{aligned}$$

^{163}Lu — Classical trajectories



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

^{163}Lu — Classical trajectories II



R. Poenaru, A.A. Raduta, Romanian Journal of Physics 66, 309, 2021.

Outline

- 1 Aim and Motivation
- 2 Nuclear Shapes
- 3 Triaxiality and Wobbling Motion
- 4 Wobbling Motion in Odd-A
- 5 Boson Description of Wobbling Motion
 - Case-Study
- 6 Conclusions

New Boson Method for odd-mass nuclei

Rotational Hamiltonian

$$\hat{H}_{\text{rot}} = \textcolor{red}{AH'} + \textcolor{blue}{H_{sp}} + \textcolor{magenta}{\text{s.t.}},$$

$$\textcolor{red}{H'} = a_1 (\hat{I}_+^2 + \hat{I}_-^2) + a_2 (\hat{I}_+ \hat{I}_- + \hat{I}_- \hat{I}_+) + a_3 \hat{I}_1 ,$$

$$\textcolor{blue}{H_{sp}} = \sum_{k=1}^2 A_k \hat{j}_k^2 , \textcolor{magenta}{\text{s.t.}} = A_1 I^2 - A_2 j_2 I .$$

- the triaxial rigid rotor is constrained to move around the 1-axis.
- adopted Frozen-Alignment approximation: $\mathbf{j} = (j \cos \theta, j \sin \theta, 0)$ (*Frauendorf, 2014*)
- a_1, a_2, a_3 inertial properties of the nucleus (i.e., $\mathcal{I}_{1,2,3}$)

New angular momentum representation

First boson expansion of this kind in literature:

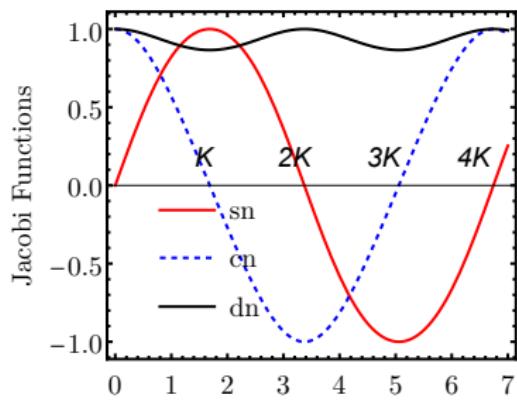
$$\begin{aligned}\hat{l}_+ &= i \frac{cb^\dagger - db^\dagger}{sb^\dagger} \left(I + lcb^\dagger db^\dagger - sb^\dagger b \right) , \\ \hat{l}_- &= i \frac{cb^\dagger + db^\dagger}{sb^\dagger} \left(I - lcb^\dagger db^\dagger + sb^\dagger b \right) , \\ \hat{l}_1 &= lcb^\dagger db^\dagger - sb^\dagger b .\end{aligned}$$

- s, c, d : Jacobi Elliptic Functions (Jacobi, 1829):

$$s = \text{sn}(q, k), \quad c = \text{cn}(q, k),$$

$$d = \text{dn}(q, k).$$

- boson operators $b, b^\dagger, [b, b^\dagger] = 1$



Elliptic Potential

New Hamiltonian

$$H' = -\frac{d^2}{dq^2} - 2v_0 s \frac{d}{dq} + I(I+1)s^2 k^2 + 2v_0 c d l ,$$

with the associated *Schrodinger Equation* (fully separated Kinetic and Potential terms):

$$\left[\frac{d^2}{dq^2} + V(q) \right] \Psi = E \Psi .$$

$$V(q) = [I(I+1)k^2 + v_0^2] s^2 + (2I+1)v_0 c d = V(-q) . \quad (1)$$

Results were presented at the International Conference TIM-22 (Timisoara) and
World Quantum Day 2023 (IFIN-HH)

Elliptic potential

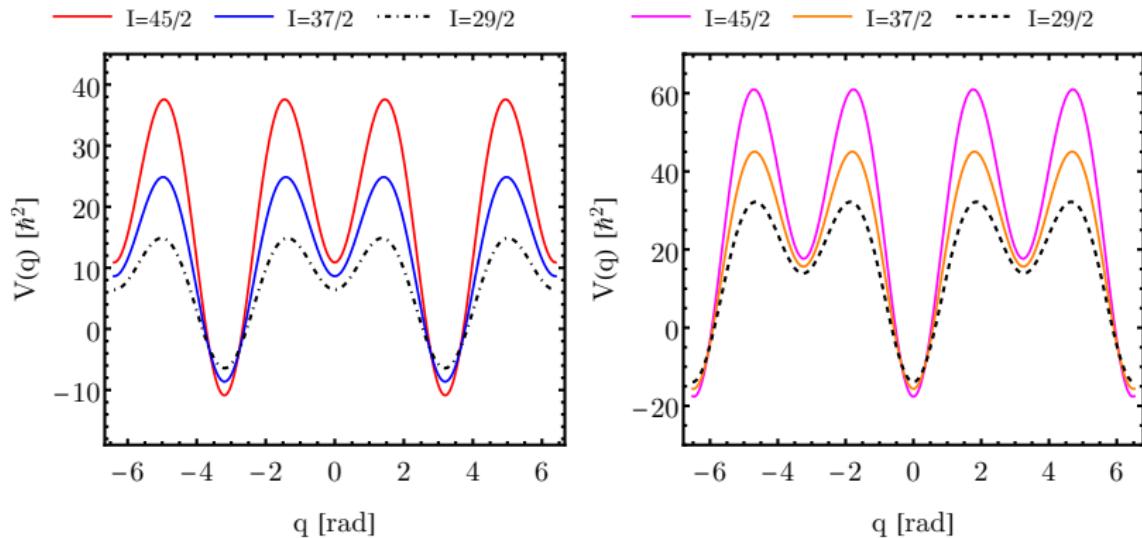


Figure: The elliptic potential as function of the coordinate q with $\theta = -119^\circ$ (**left**) and $\theta = 61^\circ$ (**right**).

A.A. Raduta, C.M. Raduta, R Poenaru, Journal of Physics G 48, 2020.

Results for ^{135}Pr

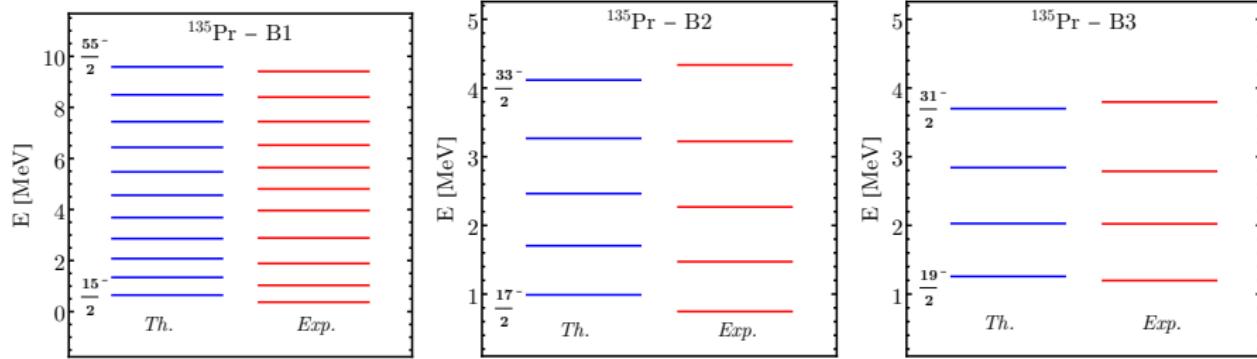


Figure: The excitation energies in ^{135}Pr . Exp data: *Sensharma, 2019*.

\mathcal{I}_1	\mathcal{I}_2	\mathcal{I}_3	θ [degrees]	N.o. states	RMS [MeV]
91	9	51	-119	20	0.174

A.A. Raduta, C.M. Raduta, R Poenaru, Journal of Physics G 48, 2020.

Outline

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General Conclusions

- Developed **three** semi-classical models that describe wobbling motion in odd- A nuclei (W_1 , W_2 , and the boson method applied to ^{135}Pr).
- Showed that it is possible to treat the motion of the core and the odd nucleon separately.
- Obtained realistic results concerning wobbling energies and other quantities.
- Special attention to the geometrical interpretation of the wobbling motion was given for ^{163}Lu .

Original Contributions

- Research period: 2018-2022
- 7 ISI papers (2 RJP, 1 IJMPE, 2 PRC, 2 JPG)
- **38** Citations
- **Total IF:** 17.225; **Total AIS:** 4.684
- **5** Oral and **2** Poster presentations at international conferences

Thank you for your attention ❤