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Semiclassical and Boson Descriptions of the Wobbling Motion in Odd-A Nuclei

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Abstract

Nova.

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Contents

Chapter 1

Introduction

Ground-state nuclear shapes with spherical symmetry or axial symmetry are predominant across the char of nuclides. Near closed shells, the deformation is indeed sufficient that models based on spherical symmetries can be used to describe nuclear properties (e.g., energies, quadrupole moments, and so on). Besides the spherical and axially-symmetric shapes, the existence of triaxial nuclear deformation was theoretically predicted a long time ago [1]. The rigid triaxiality of nuclei is defined by the asymmetry parameter γ, giving rise to unique quantum phenomena (this parameter will be characterized later on). The quantum mechanical properties of the rigid triaxial shapes drew a lot of attention within the nuclear physics community lately, since the description of nuclear properties for the deformed nuclei represents a great challenge from both an experimental and a theoretical standpoint (e.g., great progress for the experimental evidence of strong nuclear deformation has only been possible after the 2000s). It is worth mentioning that some experiments concerning alpha-α particle reactions induced in heavy nuclei in the early 1960s (e.g., [2]) helped to produce decent amount of data related to the rotational in the high-spin region ($\geq 20\hbar$). Within the experimental studies made by Morinaga et. al., the alpha reactions which were induced in the nuclei were generated by the formation of a so-called *compound nucleus*. This system may exist at a large spin value due to the absorption of the angular momentum from the incident particle (i.e., spin values up to $\approx 25\ \hbar$ can be obtained from a 50 MeV alpha particle energy - relative to the target nucleus [2]).

The physics of *high-spin* states have been studied from the early 1950s, with the major breakthrough on the theoretical side made by Bohr and Mottelson [1].

The elusive properties of nuclear rotation were described in terms of the rotational degrees of freedom associated with other nuclear degrees of freedom (e.g., particle-vibration, quadrupole-quadrupole, parity, and so on).

Chapter 2

Deformed Nuclei

2.1 Nuclear deformation

Most of the nuclei across the nuclide chart are spherical or symmetric in their ground state. Moreover, for the axially symmetric nuclei (i.e, either *oblate* or *prolate*), there is a prolate over oblate dominance.

The spherical shell model only describes nuclei near the closed shells. On the other side, for the nuclei that lie far from closed shells, a deformed potential must be employed.

In the case of even-even nuclei, unique band structures resulting from the vibrations and rotations of the nuclear surface (as proposed by Bohr and Mottelson [1] in the *Geometric Collective Model* - GCM) appear in the energy range 0-2 MeV.

Within the GCM, the nucleus is described as a classical charged liquid drop. For the low-lying energy spectrum, usually, the compression of nuclear matter and the nuclear skin thickness are neglected. This results in the final picture of a liquid drop with a constant nuclear density and a sharp surface [3].

2.1.1 Collective coordinates

The nuclear surface can be described via an expansion of the spherical harmonic functions with some time-dependent parameters as *expansion coefficients*. The expression of the nuclear shape is shown below [3]:

$$R(\theta, \varphi, t) = R_0 \left(1 + \sum_{\lambda=0}^{\infty} \sum_{-\lambda}^{\lambda} \alpha_{\lambda\mu}(t) Y_{\lambda}^{\nu}(\theta, \varphi) \right) \ . \tag{2.1}$$

In 2.1, R denotes the nuclear radius as a function of the spherical coordinates $\theta, \varphi)$ expressing the direction, and the time t, while R_0 is the radius of the spherical nucleus when all the expansion coefficients vanish. It is worth mentioning that the expansion coefficients $\alpha_{\lambda\mu}$ act as *collective coordinates* since the time-dependent amplitudes describe the vibrations of the nuclear surface.

2.1.2 Nuclear radius under rotation

To get a grasp at the physical meaning behind the deformation parameters that are used to describe the nuclear surface, it is instructive to see what happens when the system undergoes a rotation transformation.

The function $R(\theta, \varphi)$ describes the original (non-rotated) nuclear shape. Rotating the system will result in the change of the angular coordinates (θ, φ) to (θ', φ'), which will correspond to a new function $R'(\theta', \varphi')$. Moreover, both nuclear surfaces (i.e., the non-rotated and the rotated one) must hold the equality:

$$R'(\theta', \varphi') = R(\theta, \varphi) \tag{2.2}$$

The rotational invariance of R employs that $R'(\theta, \varphi)$ must have the same functional form, but the expansion coefficients $\alpha_{\lambda\mu}$ must be rotated, meaning:

$$\sum_{\lambda\mu} \alpha'_{\lambda\mu} Y'_{\lambda\mu}(\theta, \varphi) = \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \varphi) \ . \tag{2.3}$$

Note that in Eq. 2.3, the spherical harmonics $Y'_{\lambda\mu}$ are obtained via the usual rotation matrices. Finally, the invariance of Eq. 2.1 is achieved if the set of parameters $\alpha_{\lambda\mu}$ transform similarly to a *spherical tensor with angular momentum* λ [4], that is:

$$\alpha'_{\lambda\mu} = \sum_{\mu'} \mathcal{D}^{(\lambda)}_{\mu\mu'} \alpha_{\lambda\mu'} \ . \tag{2.4}$$

Besides the spherical tensor character, the collective coordinates also have the following properties (emerging from Eq. 2.1):

- Complex Conjugation.

$$Y^*_{\lambda\mu}(\theta,\varphi) = (-1)^\mu Y_{\lambda-\mu}(\theta,\varphi), \tag{2.5}$$
$$\alpha^*_{\lambda\mu} = (-1)^\mu \alpha_{\lambda-\mu}\ . \tag{2.6}$$

- Parity - the coordinates $\alpha_{\lambda\mu}$ must undergo the same change of sign under a parity transformation as the spherical harmonics, in order to keep the invariance of the nuclear surface.

$$(r,\theta,\varphi) \underset{P}{\rightarrow} (r,\pi-\theta,\pi+\varphi)\ ,$$
$$Y_{\lambda\mu}(\theta,\varphi) \underset{P}{\rightarrow} Y_{\lambda\mu}(\pi-\theta,\pi+\varphi) = (-1)^\lambda Y_{\lambda\mu}(\theta,\varphi)\ .$$

  Therefore, the parity of the expansion coefficients are:

$$\pi(\alpha_{\lambda\mu}) = (-1)^\lambda\ . \tag{2.7}$$

2.1.3 Multipole deformations

In the expansion of the nuclear surface defined by Eq. 2.1, the different values for λ will determine different effects regarding the physical aspects of the nucleus. As such, the first values of λ will be examined in terms of the physical meaning.

Monopole mode This corresponds to the first value of $\lambda = 0$. This is the simplest mode of *deformation* of a nuclear surface. Within this approximation, the spherical harmonic Y^0_0 is constant, which would imply that any non-vanishing values for α_{00} will correspond to the change in radius of the nucleus. This kind of excitation is also called *breathing mode* of the nucleus [1, 3]. The energy required for this kind of excitation mode is very large, since it implies a compression of the nuclear matter. As a result, this mode is irrelevant in the low-lying excited spectra of atomic nuclei.

Dipole mode Corresponds to $\lambda = 1$. In reality, this type of mode does not manifest itself as a deformation of the nucleus, but rather as a shift of the nuclear center of mass. In the lowest order $\lambda = 1$, the shift is in fact a translation of the entire nucleus, and it does not represent an actual nuclear excitation.

Quadrupole mode Excited modes that correspond to $\lambda = 2$. These are the most important collective excitations that take place inside the nucleus. The loss of axial symmetry, triaxial deformations, and other shape-specific transitions that happen within the nucleus are mostly described (and very accurately) via the quadrupole effects.

Octupole mode This corresponds to the next increasing value of $\lambda = 3$, representing the main asymmetric excitations of a nucleus with states of negative-parity. The specific shape of a nuclear system governed by octuple deformations is similar to that of a pear.

Hexadecupole deformations Excitations which correspond to $\lambda = 4$. Within the nuclear theory, this is considered the highest angular momentum which can still provide relevant information for the nuclear phenomena that are studied. Currently, there is no clear evidence for pure excitations with hexadecupole nature, however, these excitations seem to have a major role in the admixture to quadrupole excitations for the ground-state shape of heavy nuclei [3].

The multipole deformations for the cases $\lambda = 1, 2, 3$ and $\lambda = 4$ discussed above are pictorially shown in Fig. 2.1. Excitations with higher angular momentum than the mentioned ones have practically no application within the study of atomic nuclei. Moreover, one can also see that there is an intrinsic limitation on the maximal value of λ, which dictates the smallness of the individual bumps of the surface (see Fig. 2.1). These bumps are described by the spherical harmonics Y_{λ}^{μ}, and they decrease in size with increasing values of λ, but with the physical limitation given by the size of the nucleon diameter.

2.1.4 Quadrupole Deformation

One of the most important excitation modes (vibrational degrees of freedom) is the quadrupole deformation, corresponding to $\lambda = 2$. In the case of pure quadrupole

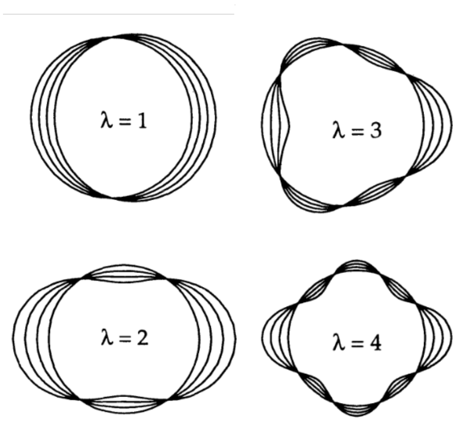

Figure 2.1: Graphical representation for the first few modes of excitations of the nuclear surface. The figure is taken from Ref. [3].

deformation, the nuclear surface will be given by the following expression:

$$R(\theta,\varphi) = R\left(1 + \sum_{\mu} \alpha_{2\mu} Y_2^{\mu}(\theta,\varphi)\right) \ . \tag{2.8}$$

From this expression, the term α_{00} is of second order in $\alpha_{2\mu}$ and it can be neglected further on. This term also reflects the conservation of volume [3, 4]. The real and independent degrees of freedom from the above expression are: α_{20}, the real and imaginary parts of α_{21}, and the real and imaginary parts of α_{22}, respectively.

More insight with regards to the quadrupole shape of the nucleus can be achieved if one expresses R in terms of cartesian coordinates. The spherical harmonics will attain a new form, depending on the cartesian components of the unit vector pointing in a direction defined by (θ,φ):

$$\xi = \sin\theta\cos\varphi \ , \ \eta = \sin\theta\sin\varphi \ , \ \zeta = \cos\theta \ , \tag{2.9}$$

with the condition $\xi^2+\eta^2+\zeta^2 = 1$. With the expressions of the spherical harmonics as functions of (ξ,η,ζ), the nuclear radius will change accordingly (cartesian expression $R = R(\xi,\eta,\zeta)$). A relationship between the cartesian components and the spherical ones for the deformation can be also obtained if one writes all coefficients $\alpha_{2\mu}$ as functions of α_{ij} (with $i,j = \xi,\eta,\zeta$). Since the cartesian deformations can be regarded as closely related to a stretch/contraction of the nucleus in a given direction, a first interpretation of the physical meaning behind the parameters $\alpha_{2\mu}$ can be established:

- α_{20}: describes the stretching of the z axis with respect to the y and x axes.
- α_{2-2} and α_{22}: give the relative length of the x axis compared to the y axis. Moreover, it also gives the oblique deformation in the $x-y$ plane.
- α_{2-1} and α_{21}: describe an oblique deformation, but with respect to the z axis.

With the set of parameters defined above, the shape and orientation of the nucleus can have arbitrary values (the coefficients $\alpha_{2\mu}$ are mixing the shape and orientation), thus making the parametrization somewhat problematic. In order to fix that, the geometry can be changed if one considers the *principal axis system* (the PA reference system is a coordinate system in which the moments of inertia associated with the nucleus are diagonal). When using this reference frame, the number of parameters is still unchanged, however their physical significance becomes clearer. By denoting the new coordinate system with primed letters, nuclear radius will be described as a function $R = R(\xi', \eta', \zeta')$, with the conditions that $\alpha'_{ij} = 0 \ , \ i \neq j$. The condition will further imply that the newly expressed parameters ($\alpha'_{2\mu}$) have the following form:

$$\begin{aligned} \alpha'_{2\pm1} &= 0 \ , \\ \alpha'_{2\pm2} &\equiv a_2 \ , \\ \alpha'_{20} &\equiv a_0 \ . \end{aligned} \tag{2.10}$$

, where the conveniently denoted terms a_2 and a_0 are some functions that depend on the cartesian components $\alpha_{\xi,\xi} \ , \ \alpha_{\eta,\eta} \ , \ \alpha_{\zeta,\zeta}$. From this set of equations the physical significance of the five real and independent parameters is clearer:

- a_0 is indicating the stretch of z' axis w.r.t. the x' and y' axes.
- a_2 is indicating the asymmetry between the lengths of x' and y' axes respectively.
- the Three *Euler angles* $\theta = (\theta_1, \theta_2, \theta_3)$. These angles will determine the orientation of the PA system (x', y', z') with respect to the laboratory-fixed frame (x, y, z).

One can now clearly see the advantage of working within the PA system: rotation and shape vibration degrees of freedom can be completely separated. A change in the Euler angles will result in a pure rotation of the nucleus (without changing its shape), while a change in shape will be affected exclusively by the a_0 and a_2 parameters. If $a_2 = 0$, then the nucleus has a shape with axial symmetry around the z axis (equal axis lengths along the x and y directions).

Another way of describing the excitations of quadrupole type is to adopt the parameters introduced by A. Bohr [5]. These two parameters can be viewed as a set of polar coordinates in the space generated by (a_0, a_2) and they are defined as:

$$\begin{aligned} a_0 &= \beta_2 \cos\gamma \ , \\ a_2 &= \frac{1}{\sqrt{2}}\beta_2 \sin\gamma \ , \end{aligned} \tag{2.11}$$

where the numeric factor $\frac{1}{2}$ was added such that the following relation holds true:

$$\sum_{\mu} |\alpha_{2\mu}|^2 = \sum_{\mu} |\alpha'_{2\mu}|^2 = a_0^2 + 2a_2^2 = \beta_2^2 \ . \tag{2.12}$$

It is worth mentioning that the Eq. 2.12 is rotationally invariant, having the same value in the laboratory and the principal axes systems.

Now that the shape of the nucleus (i.e., the nuclear surface radius R) can be described consistently with via the parameters defined in Eq. 2.11, one can calculate the stretching of the nuclear radius along any of the directions is given in terms of (β, γ) as follows:

$$\delta R_k = \sqrt{\frac{5}{4\pi}}\beta \cos\left(\gamma - \frac{2\pi k}{3}\right) \ . \tag{2.13}$$

2.1.4.1 Axial quadrupole deformations

Using this set of new coordinates, the expression of the nuclear radius for axially quadrupole-deformed nuclei is given as:

$$R(\theta, \varphi) = R_0 \left(1 + \beta_2 Y_2^0(\theta, \varphi)\right) \ . \tag{2.14}$$

In Eq. 2.14, the parameter β_2 is called the *quadrupole deformation parameter*, and its value dictates wether the nucleus is *oblate* - $\beta_2 < 0$ (i.e., a flattened sphere), *prolate* - $\beta_2 > 0$ (i.e., elongated sphere, like a rugby ball), or *spherical* - β_2. The

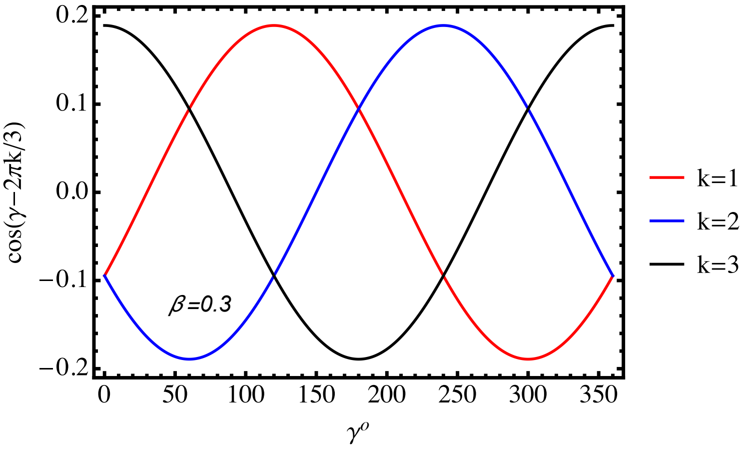

FIGURE 2.2: A graphical representation with the stretching of the nuclear axis δR_k for $k = 1, 2, 3$, corresponding to the increase in axis lengths along the x, y, and the z directions, respectively. The representation used an arbitrary value for the quadrupole deformation $\beta_2 = 0.3$. Figure was reproduced according to the calculations done in [3].

nuclear shapes that are characterized only by β_2 (i.e., $\gamma = 0$) have shapes that correspond to spheroids. These shapes are axially symmetric, meaning that they only have one deformed axis. For the spherical case $\beta_2 = 0$, all three axes have the same lengths, meaning that the shape of the nucleus is in fact a sphere.

For the axially-symmetric quadrupole deformations, the parameter β_2 can be related to the axes of the spheroid via the formula [3]:

$$\delta R_k = \sqrt{\frac{5}{4\pi}}\beta_2 \cos\left(\gamma - \frac{2\pi k}{3}\right) \ , \tag{2.15}$$

with $k = 1, 2, 3$ indices that correspond to each of the three principal axes x', y', and z', respectively. The stretching of the nuclear axis in a particular direction (denoted by k in the above formula) varies according to the change in γ, for a fixed value of β_2.

Taking a look at Fig. 2.2, one can see the variations of the three axes with γ. When $\gamma = 0°$ the nucleus is elongated along the z' axis, but the x' and y' axes are identical (the prolate case) - axial shape. As γ increases, the x' axis grows, while the other two axes decrease in size, making a region with *triaxial shapes* - all three axes are unequal in magnitude. Symmetry is reached again at $\gamma = 60°$, however the x' and z' axes are equal this time but longer than y' axis, making the nucleus look like a flattened shape (the oblate case) - axial shape. This pattern is repeated every $\gamma = 60°$, where axial shapes emerge, with alternating prolate/oblate shapes.

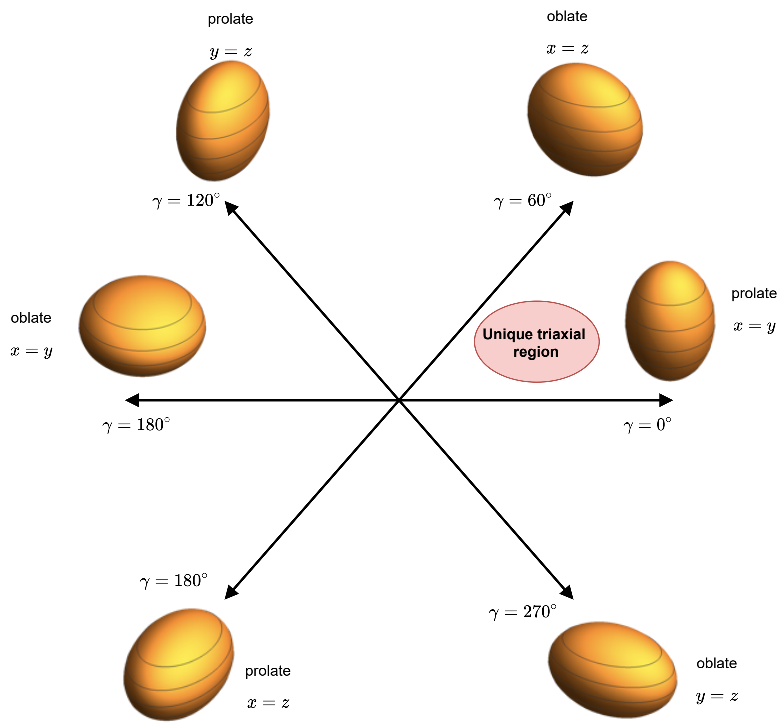

FIGURE 2.3: Beta gamma plane divided into six regions. The first part, delimited from $\gamma = 0^\circ$ to $\gamma = 60^\circ$ can be considered as the representative one, while the others can be reproduced from this interval.

It is possible to summarize the various nuclear shapes that can occur with the help of a diagram within in the (β, γ) plane. The repeating pattern of the nuclear shapes is graphically represented in Fig. 2.3. One can see that the oblate axially symmetric shapes that occur at $\gamma = 60^\circ, 180^\circ$ and 300° are identical and only the axes naming scheme differs. The triaxial shapes are also repeated each 60°.

Regarding the characteristics of Fig. 2.3, the triaxial regions have basically identical shapes, only the axes orientations are different. Moreover, the associated Euler angles are also different, leading to the conclusion that identical physical shapes - including the space orientation - can be represented by different sets of deformation parameters (β, γ) and Euler angles.

2.1.4.2 Non-Axial quadrupole deformations

Besides the nuclei characterized by a *spheroidal* shape, where two of the three principal axes have the same length and the quadrupole deformation parameter

β_2 is the key parameter that describes these kind of shapes, there are also *triaxial* nuclei (or non-axial deformed nuclei).

The triaxial shapes are defined by the γ degree of freedom: the parameter which describes the asymmetry between the length of the three axis of the nucleus (e.g., it describes a stretching along an axis that is perpendicular to the symmetry axis). The nuclear radius for the axially-asymmetric quadrupole deformations is given by:

$$R(\theta,\varphi) = R_0 \left(1 + \beta_2 \cos\gamma Y_2^0(\theta,\varphi) + \frac{1}{\sqrt{2}} \sin\gamma (Y_2^2(\theta,\varphi) + Y_2^{-2}(\theta,\varphi))\right) \ , \quad (2.16)$$

which is different than Eq. 2.14. As it was already mentioned, the values $\gamma = 0°$ and $\gamma = 60°$ correspond to symmetric prolate and oblate shapes, respectively. Between these values, the triaxial region exist, with *maximal triaxiality* reached at $\gamma = 30°$. The deformation parameters (β, γ) is also called the Hill-Wheeler set **CITATION REQUIRED**.

In Eq. 2.16, the spherical harmonics are expressed as follows:

$$\begin{aligned} Y_2^0(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos(\theta)^2 - 1\right) \ , \\ Y_2^2(\theta,\varphi) &= \frac{1}{4}e^{2i\varphi}\sqrt{\frac{15}{2\pi}}\sin(\theta)^2 \ , \\ Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}e^{-2i\varphi}\sqrt{\frac{15}{2\pi}}\sin(\theta)^2 \ , \end{aligned} \quad (2.17)$$

and substituting these terms in $R(\theta,\varphi)$, Eq. 2.16 will become:

$$R(\theta,\varphi) = R_0 \left[1 + \sqrt{\frac{5}{16\pi}}\beta\left(\cos\gamma(3\cos\left(\theta^2\right) - 1) + \sqrt{3}\sin\gamma\sin\left(\theta^2\right)\cos 2\varphi\right)\right] \ . \quad (2.18)$$

Shape	# of deformed axes	i-l plane	i-s plane
Prolate	1		
Oblate	1		
Triaxial	2		
Figure			

Table 2.1: This is a table

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