- finite nuclear mass → small nuclear recoil energy

$\Downarrow$

"mass shift" between atomic transitions

- finite nuclear size (not a point charge) → isotope shift

$\Updownarrow$

extended charge distribution

- nucleus → liquid drop → protons homogeneously distributed over the sphere of the nucleus
- R_p → radius of the proton distribution

R_p = nuclear mass distribution

$$R = R_0 A^{\frac{1}{3}} \qquad R_0 \simeq 1.2\,fm.$$

- the isotope shift is approximated very well by:

↳ second radial moment of the nuclear charge distribution

$$\langle r^2 \rangle = \frac{\int_0^R \rho(r) r^2 dr}{\int_0^R \rho(r) dr}$$

→ nuclear charge Ze

nuclear mean square charge radius

(monopole moment of the nucleus)

- Liquid-Drop: $\langle r^2 \rangle_{LD} = \frac{3}{5} R^2 = \frac{3}{5} R_0^{\,2} A^{\frac{2}{3}}$

↳ deviation from L-D of $\langle r^2 \rangle$ ⟨ proton/neutron skins; shell closures; pairing; deformation

- Nuclear deformation ⟶ Liquid-Drop
  (collective phenomenon)

- Spherical and non-spherical nuclei
  ↳ nuclear deformation → described by the "quadrupole deformation parameter" β

β: defined by the angular dependence of the length of the radius vector to the nuclear surface expressed in spherical harmonics
- rotational symmetry
  ↳ expansion of the nuclear shape in spherical harmonics Y_{kq}

⇓

$$R(\theta) = R_1 \left[1 + \beta Y_{20}(\theta) \right]$$

R_1: chosen such that nuclear volume is constant

R_1 → independent on β

$$\langle r^2 \rangle = \underbrace{\frac{3}{5} R^2}_{R_{sph}} + \frac{3}{4\pi} R^2 \beta^2$$

no restriction to a sharp nuclear surface

+

R_{sph} → can be expressed by means of the mean squared radius $\langle r^2 \rangle_{sph}$

⇓

- a spherical nucleus which has the same volume

$\frac{3}{5} R^3 \to \langle r^2 \rangle_{sph}$

$$\langle r^2 \rangle = \langle r^2 \rangle_{sph} + \frac{5}{4\pi} \langle r^2 \rangle_{sph} \beta^2$$

↓ differential effect

$$\delta\langle r^2\rangle = \delta\langle r^2\rangle_{sph} + \frac{5}{4\pi}\,\delta\langle r^2\rangle_{sph}\,\delta\beta^2$$

- change in nuclear deformation affects the change in nuclear mean square radius
- all quantities refer to charge (proton) distribution in the nucleus
- β → charge deformation related to this charge distribution

The nuclear electric quadrupole moment

- non-spherical distribution of nucleon charges

⇓

- electric quadrupole moment

- in Cartesian coordinate system

$$\hat{Q}_z = \sum_{i=1}^{A} \hat{Q}_z^i = \sum_{i=1}^{A} e_i\left(3z_i^2 - r_i^2\right)$$

e_i ↓ charge of a particular nucleon

(x_i, y_i, z_i) → nucleon coordinates

- In a spherical tensor basis: z-component of Quadrupole operator

↓ expressed as the zero-order tensor component of T_q^2

$$Q_2^0 = Q_z = \sqrt{\frac{16\pi}{5}}\sum_{i=1}^{A} e_i r_i^2\, Y_2^0(\theta_i, \varphi_i)$$

- nucleus → quantum mechanical system

↳ described by a Nuclear Wave-Function

↳ characterized by nuclear spin I

• in experiments → observation of Spectroscopic Quadrupole Moment

$$Q_s(I) = \langle I, m=I \,|\, Q_2^0 \,|\, I, m=I \rangle = \sqrt{\frac{I(2I-1)}{(2I+1)(2I+3)(I+1)}} \cdot \langle I \| Q \| I \rangle$$

if $I < 1$
↳ cannot measure Q_s

• Q_s is related to the intrinsic quadrupole moment Q_0
⇕ reflects the nuclear deformation β
certain assumptions about nuclear surface

1) Nuclear deformation → axially symmetric with nuclear spin I
2) Nuclear deformation has a well-defined direction w.r.t. symmetry axis of deformation
⇓
Strong coupling

$$(1) + (2) \Longrightarrow Q_s = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0$$

spectroscopic Q. mom.

intrinsic Q. moment

K = projection of the nuclear spin on the deformation axis

Q_0 → induced by non-spherical charge distribution of protons
↓ related to nuclear charge deformation β

$$Q_0 = \frac{3}{\sqrt{5\pi}} Z R^2 \beta (1 + 0.36\beta)$$

$R = R_0 A^{\frac{1}{3}}$

$$\langle r^2 \rangle = \frac{3}{5} R^2 + \frac{3}{4\pi} R^2 \beta^2$$

• Shell-model → nuclei are described
- nuclear mean-field (A)
- valence nucleons (B)

(B) → move in that (A)-mean-field
↓
interact with each-other through a residual interaction

(A)+(B) → $\boxed{\text{Shell-model}}$ → (Q_s) → the sum $\sum_{i=1}^{A}$

reduced ↓

sum over valence nucleon

• To take into account the interaction with (A) (mean-field)
⇓
• effective charge (Z_{eff}) is attributed to the nucleons
↓
Z_{eff} < protons, neutrons

• Light nuclei : $e_n^{eff} \simeq 1.3e$

$e_p^{eff} \simeq 0.35e$

• Heavy-nuclei : $e_n^{eff} \simeq 1.6e$

$e_p^{eff} \simeq 0.95e$

• Quadrupole moment → 1-body operator
⇓
• deduce additivity rules for quadrupole moments
→ tensor algebra
→ angular momentum coupling

- decomposing the single-particle wave function

  → radial + spin + orbital parts

- the single-particle quadrupole moment for unpaired nucleon in an orbit with a.m. j

$$Q_{sp} = -e_j \frac{2j-1}{2j+2} \langle r_j^2 \rangle$$

e_j → effective charge of the nucleon in the orbital j

$\langle r_j^2 \rangle$ → mean square radius of the nucleon in orbital j

- free neutrons have no charge $e_\nu = 0$

  $\Downarrow$

  do not induce a single-particle quadrupole moment

- neutrons in a nucleus interact with the nucleons of the core

$$\left(\text{valence neutrons}\right) \xrightarrow{\text{interaction}} \left(\text{core nucleons}\right)$$

polarization of the core

effective charge

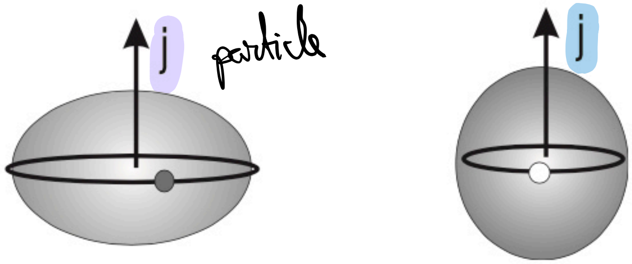
- (b) nuclear energy is minimized

(a) core nucleons ⟷ maximal overlap ⟷ valence particle / hole

(a)
↓
(b)

$\Downarrow$

a particle (hole) will polarize the core towards $\downarrow$ oblate (prolate) deformation



particle hole

Plate 1. Graphical representation of a particle in a orbital $\boldsymbol{j}$, polarizing the core towards oblate deformation with a negative spectroscopic quadrupole moment (*left*), and a hole in an orbital giving rise to a prolate core polarization (*right*)

Barret - 1974:

- parameterized expression of $\rho(r)$:

$$\rho(r) = \frac{\rho_0\left[1 + w\left(\frac{r}{c}\right)^2\right]}{1 + e^{\left(\frac{r^n - c^n}{a^n}\right)}}$$

$[w, a, c]$ $\downarrow$ tabulated values

- Motion in a fixed symmetric field

$\hookrightarrow$ Consider $V(r)$: electrostatic force between
- charge distribution $Ze\rho(r)$
- negatively charged lepton

$$V(r) = -Ze^2 \int \frac{\rho(r')\,d^3r'}{|r - r'|}$$

$\rho(\vec{r}) \rightarrow$ normalized such that $\int \rho(r)\,d^3r = 1$

$\rho(\vec{r}) \rightarrow$ sherical symmetry $\Rightarrow$ Solutions to Dirac equation

$\downarrow$ linear combinations of terms

$\swarrow$ total a.m. j - term $\searrow$ parity - term

Cross-sections $\longrightarrow$ elastic scattering of electrons with a fixed nucleus

$\downarrow$

Solve Dirac equation in the first-order in the potential $V(r)$

$\Downarrow$ Born Approximation

- Born approximation for a point charge $\longrightarrow$ Mott scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_M = \frac{4Z^2e^4E^2}{q^4c^4}\left(1-\frac{q^2c^2}{4E^2}\right)$$

differential cross-section

- non-relativistic $\rightarrow$ Rutherford-scattering
  - extended nucleus

$\Downarrow$ Born Approximation

$$Ze^2\int \rho(r')\, e^{i\vec{q}\cdot\vec{r}'} d^3r' \int \frac{e^{iq(\vec{r}-\vec{r}')}}{|\vec{r}-\vec{r}'|} d^3r = \frac{Ze^2}{q^2} F(\vec{q})$$

$$F(\vec{q}) = \int e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r})\, d^3$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_M |F(\vec{q})|^2$$

- $\rho(\vec{r}) \rightarrow$ spherical symmetry $\Rightarrow$ $F(\vec{q}) = \frac{4\pi}{q}\int_0^\infty r\sin(qr)\rho(r)dr$

$$F(\vec{q}) = 1 - \frac{q^2\langle r^2\rangle}{3!} + \frac{q^4\langle r^4\rangle}{5!} - \dots$$

$$\langle r^n \rangle = \int \rho(\vec{r})\, r^n\, d^3r$$

mean n-th power moment (n-th power radius) of the charge distribution

- Electron scattering by deformed nuclei

→ Born approximation : • scattering amplitude for a fixed deformed nucleus → proportional to

$$F(\vec{q}) = \int e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r})\, d^3r$$

$$= \sum_L F_L(q^2)\, P_L(\theta')$$

$$F_L(q^2) = \int j_L(qr)\, \rho_L(r)\, d^3r$$

θ' → angle between $\vec{q}$ and the symmetry axis