Torque on a Current Loop

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1 Definition of Torque

1.1 Basic definition

Torque is defined as the cross product of a displacement and a force. The displacement is from the center for taking torques, which is arbitarily defined, to the point of application of the force to the body experiencing the torque:

$$\overline{T} = \overline{r} \times \overline{F}. \tag{1}$$

1.2 Torque when total force vanishes

It can be shown that when the total force applied to a body is zero, then the total torque on the body is independent of the choice of the center for taking torques:

$$\sum_i (\overline{r}_i - \overline{r}_o) \times \overline{F}_i = \sum_i \overline{r}_i \times \overline{F}_i - \sum_i \overline{r}_o \times \overline{F}_i = \sum_i \overline{r}_i \times \overline{F}_i - \overline{r}_o \times \sum_i \overline{F}_i = \sum_i \overline{r}_i \times \overline{F}_i \tag{2}$$

1.3 Torque with continuous force distribution

When the force on a body is actually a force field, $\overline{f}(\overline{r})$, where $\overline{f}$ is the force per unit volume, then the formula for torque becomes an integral:

$$\overline{T} = \iiint \overline{r} \times \overline{f} dv. \tag{3}$$

When the force acts on a filamentary loop of material then the equation becomes

$$\overline{T} = \oint \overline{r} \times d\overline{F}, \tag{4}$$

where $d\overline{F}$ is the force acting at $\overline{r}$ on increment of length $d\bar{l}$.

2 Torque on a current loop

2.1 General case

When the body subject to torque is a filamentary loop carrying a current I, and the force producing the torque is the magnetic force on the current, then the $d\overline{F}$ of eq.(4) is

$$d\overline{F} = I d\bar{l} \times \overline{B}, \tag{5}$$

where $\overline{B}$ is the magnetic flux density. Thus the torque on a current loop, under general conditions, is

$$\overline{T} = I \oint \overline{r} \times (d\bar{l} \times \overline{B}). \tag{6}$$

2.2 Case of constant magnetic flux density

When $\overline{B}$ is constant with position, then the formula of eq.(6) can be simplified by removing the constant term from the integration and applying appropriate vector identies.

2.2.1 Identity for $\overline{A} \times (\overline{B} \times \overline{C})$:

$$\overline{A} \times (\overline{B} \times \overline{C}) \equiv (\overline{A} \cdot \overline{C})\overline{B} - (\overline{A} \cdot \overline{B})\overline{C}. \tag{7}$$

2.2.2 Identity for closed loop integral of a gradient

$$\oint \nabla u \cdot d\bar{l} \equiv \oint du \equiv 0 \tag{8}$$

2.2.3 Identity for $\nabla \times (f\overline{C})$

$$\nabla \times (f\overline{C}) \equiv \nabla(f) \times \overline{C} \tag{9}$$

2.2.4 Identity for $\oint f d\bar{l}$

Let $\overline{C}$ be a constant vector. Then apply Stoke's theorem to $f\overline{C}$ where f is a scalar field. (This is problem number 11 on page 47 of Shadowitz, *The Electromagnetic Field.*) Thus

$$\oint f\overline{C} \cdot d\bar{l} \equiv \iint \nabla \times (f\overline{C}) \cdot d\overline{S}. \tag{10}$$

From identity 9,

$$\oint f\overline{C} \cdot d\bar{l} \equiv \iint \nabla(f) \times \overline{C} \cdot d\overline{S}. \tag{11}$$

In this last equation we interchange the dot and cross in the right hand side integral, and then interchange f and $\overline{C}$, and then bring the dot product of the constant $\overline{C}$ with the remainder of the integrand in both the left and right hand integrals outside of the integral sign to obtain:

$$\overline{C} \cdot \oint f d\bar{l} \equiv -\overline{C} \cdot \iint \nabla(f) \times d\overline{S}. \tag{12}$$

Since this result must be true for any choice of the constant vector $\overline{C}$ we have the desired result

$$\oint f d\bar{l} \equiv - \iint \nabla(f) \times d\overline{S}. \tag{13}$$

2.2.5 Application to torque equation

First, we apply identity 7 to eq.(6) to obtain

$$\overline{T} = I \oint (\overline{r} \cdot \overline{B}) d\bar{l} - I \oint (\overline{r} \cdot d\bar{l}) \overline{B}. \tag{14}$$

Next, recognize that the second integral in eq.(14) is a case of identity 8 with $\overline{r} = \frac{1}{2}\nabla(r^2)$; thus it is zero. The first integral in eq.(14) is a case of identity 13 with $\overline{r} \cdot \overline{B}$ being the scalar field. Thus

$$\overline{T} = -I \iint \nabla(\overline{r} \cdot \overline{B}) \times d\overline{S}. \tag{15}$$

But, since $\overline{B}$ is a constant, $\nabla(\overline{r} \cdot \overline{B}) = \overline{B}$. Thus, eq.(15) yields

$$\overline{T} = -I \iint \overline{B} \times d\overline{S} = I \left(\iint d\overline{S} \right) \times \overline{B}. \tag{16}$$

2.2.6 Magnetic moment defined

We define the magnetic moment, $\overline{m}$, of a current loop as

$$\overline{m} = I \iint d\overline{S}. \tag{17}$$

2.2.7 Final result

Thus for cases where the magnetic flux density is constant over the area of a filamentary loop carrying a current,

$$\overline{T} = \overline{m} \times \overline{B}. \tag{18}$$

Note that the value of $\overline{S} = \iint d\overline{S}$ is not the scalar area of a particular surface having the loop as its edge. Instead $\overline{S}$ is a vector, and will be the same for any surface having the loop as its edge. When the loop lies in a plane, then the magnitude of $\overline{S}$ is the area of the plane enclosed by the loop.