

# Theory of Nucleus

## Nuclear Structure and Nuclear Interaction

by

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## CHAPTER 1

### NUCLEONS AND NUCLEAR FORCES

#### 1.1. Constitution of Nuclei and Nucleon Properties

**Main characteristics of nuclei.** Atomic nuclei are characterized by definite masses and electric charges. The charge of an atomic nucleus  $q$  is equal to minus the multiple of the absolute value of the electron charge  $-e$ , i.e.,

$$q = Ze.$$

The integer  $Z$  determines the position of the atom in the periodic table, it is called the *atomic number*. Nuclei with the same charge but different masses are called *isotopes*. Taking 12 to be the mass of the most abundant carbon isotope, one finds all the nuclear masses to be nearly integers. The integer  $A$  that is the closest to the mass value is referred to as the *mass number* of the nucleus.

According to the present-day conceptions, the atomic nucleus consists of protons and neutrons – elementary particles with approximately equal masses. These particles interact via nuclear forces. The proton possesses a positive electric charge whose value is equal in magnitude to that of the electron; the neutron is electrically neutral. In spite of differing electrical properties, protons and neutrons manifest similar behaviour in nuclear interactions and therefore are often unified by the term *nucleons*.

An atomic nucleus with the charge  $q = Ze$  and mass number  $A$  consists of  $Z$  protons and  $N = A - Z$  neutrons. The mass number  $A$  determines the total number of nucleons contained in the nucleus. *Isotopes* are nuclei with equal proton numbers  $Z$  and different neutron numbers  $N$ . Nuclei with equal neutron numbers  $N$  and different proton numbers  $Z$  are called *isotones*. Nuclei with equal nucleon numbers (equal mass numbers  $A$ ) are called *isobars*.

The most important peculiarity of all nuclei, with exception the lightest nuclei (nuclei of heavy hydrogen isotopes and  ${}^3\text{He}$ ) and nuclei with an extra neutron numbers is the existence of their sharp border. Experiment shows that the nuclear density is almost constant inside the nucleus, and falls sharply to zero over a distance that is small compared with the dimensions of the nucleus. If we assume that the nucleus possesses spherical symmetry,

the dimensions of the nucleus will be characterized by a single parameter, namely, the *nuclear radius*  $R$ .

The nuclear radii  $R$  are proportional to the cube root of the mass number  $A$ ,

$$R = R_0 A^{1/3},$$

where the quantity  $R_0$  decreases slowly with increasing  $A$ , from a value  $R_0 = 1.3 \times 10^{-13}$  cm for light nuclei to a value  $R_0 = 1.2 \times 10^{-13}$  cm for heavier nuclei. For most nuclei we can put, approximately,

$$R_0 = 1.2 \times 10^{-13} \text{ cm.}$$

The proportionality coefficient  $R_0$  does not depend on the mass number  $A$  that means that the nuclear density is almost the same for all nuclei.

**Neutron and proton properties.** Individual nucleon properties have been well studied. The proton mass is equal to 1836.15 electron masses or 1.007276 atomic mass units (*amu*); the neutron mass is greater by approximately 2.5 electron masses, it is equal to 1.008665 *amu*. The proton and the neutron have spins of the same magnitude, equal to  $1/2$ , and are governed by the *Fermi-Dirac statistics*. Their *magnetic moments* (in terms of nuclear magnetons) are equal respectively to

$$\mu_p = 2.7928, \quad \mu_n = -1.9130. \quad (1.1)$$

These numbers considerably differ from 1 and 0 which follow from the Dirac equation for particles with spin  $1/2$ . Anomalous values of both proton and neutron magnetic moments indicate that Dirac's description of nucleons is incomplete.

A free neutron is unstable and by virtue of weak interaction decays into a proton, an electron, and an electronic antineutrino:  $n \rightarrow p + e^- + \bar{\nu}_e$ . The excess mass, approximately equivalent 1.3 MeV, is consumed to the electron production (0.5 MeV) and kinetic energy of decay products (0.8 MeV). The *mean neutron life-time* is about 15 min. Nevertheless, in spite of being unstable in the free state, neutrons can form stable nuclei together with protons. At the same time, a free proton is stable, whereas being bound in the nucleus it can, due to the weak interaction, decay into a neutron, a positron, and an electronic neutrino:  $p \rightarrow n + e^+ + \nu_e$ . The nuclei are stable only for certain proportions of neutron and proton numbers. If these proportions are violated, either neutron or proton decay occurs in the nucleus until the latter becomes stable.

Small difference of masses, equal spins, and similarity of some other properties, as well as the possibility of mutual transformations, make it reasonable to treat the neutron and the proton as two different states of

the same particle – the *nucleon*. In such approaches, an additional intrinsic degree of freedom, which is referred to as *charge of isospin* coordinate, is attributed to the nucleon along with the space and spin coordinates.

For nonrelativistic energies, both neutrons and protons may be treated as point particles. For high energies, however, one must involve into consideration their space structure which is manifested, e.g., in the high energy scattering experiments. An adequate quantitative description of nucleon space structure and anomalous magnetic moments is provided only by quantum chromodynamics.

## 1.2. Nuclear Interaction between Nucleons

**Short-range nature of nuclear forces.** The nucleus constituents, neutrons and protons, interact via specific *nuclear forces*. These forces cannot be reduced to electromagnetic interactions since they are manifested irrespective of whether the nucleon possesses an electric charge or is electrically neutral.

A detailed information on the nature of nuclear forces can be obtained by the study of two-nucleon systems since the two-particle problem is rather simple and can be solved exactly. The two-nucleon nuclear forces are responsible for the nucleon-nucleon scattering and the formation of a composite particle – *deuteron* which is the simplest nucleus consisting of a neutron and a proton. The existence of the neutron-proton bound state indicates that the neutron-proton nuclear interaction is associated with the particle attraction.

It is highly peculiar of the nuclear interaction that its range is very short whereas the interaction is very strong. Wigner was the first who assumed the nuclear interaction to be short-range. He analyzed the values of the deuteron binding energy (which is small, 2.23 MeV) and binding energy of the  ${}^4\text{He}$  nucleus (which is large, 28 MeV), and thus found the nuclear-force range to be of the order of magnitude of  $2 \times 10^{-13}$  cm. Assuming the neutron-proton interaction to be produced by a square well of width  $2 \times 10^{-13}$  cm and taking into account the small value of the deuteron binding energy, one finds the potential well depth to be about 30 MeV. For comparison we note that the Coulomb interaction energy for two protons spaced to  $2 \times 10^{-13}$  cm is as small as 0.7 MeV.

**Other peculiar features of nuclear forces.** In the nonrelativistic limiting case, with particle velocities being much smaller than the light velocity, the nuclear interaction does not depend on the velocities of interacting particles and can be described by a potential. It is peculiar of nuclear forces that nuclear interaction potential depends both on the interparticle distance and on the mutual orientation of particle spins. The particle- spin-dependence

of nuclear interaction immediately follows from the experiments on slow neutron scattering by molecular hydrogen. The existence of the deuteron electric quadrupole moment indicates that the nuclear interaction is non-central, i.e., depends on the mutual orientation of the total spin and the relative radius-vector of the interacting particles. At last, the observation that particle scattering produces particle polarization shows that spin-orbit forces considerably contribute to the nuclear interaction.

Insofar as the stable nuclei exist, the nucleon-nucleon nuclear interaction must be mainly attractive. The experimental data on the high-energy nucleon-nucleon scattering show, however, that nuclear forces must be repulsive at very small distances (shorter than  $0.4 \times 10^{-13}$  cm). High energy nucleon scattering experiments also suggest the partially exchange nature of the nuclear interaction, i.e., the interacting nucleons can exchange some parameters (space coordinates, spins, charges).

It follows from the properties of *mirror nuclei* (nuclei in which the neutrons are replaced by protons and the protons by neutrons) that the forces acting between two protons are equivalent to the forces acting between two neutrons (if we exclude the *Coulomb interaction* from consideration). This property was given the name of the *charge symmetry of nuclear forces*. Later studies have shown that the charge symmetry is a manifestation of a deeper symmetry of the nuclear interaction which is called the *isotopic invariance*, or *charge independence of nuclear forces*. The isotopic invariance of nuclear forces means that the interaction in any two pairs of nucleons is the same if these pairs of nucleons are in the same states. All the presently available experimental data on the interaction of nucleons (at both low and high energies) are in agreement with the postulate of the isotopic invariance of nuclear forces.

In 1935 Yukawa employed the short-range nature of the nuclear interaction to propose a field theory of nuclear forces which predicted the existence of particles with medium masses – *mesons*. Later on, these particles have been observed experimentally. The early meson theory, however, could not explain some peculiarities of the nuclear interaction. Though the recent theory – quantum chromodynamics – provides a more consistent treatment of strong interactions, at the moment it is unsufficiently advanced to develop a systematic description of nuclei and nuclear processes. That is why we give here a phenomenological study of the problem which is essentially based on the peculiarities of nuclear interactions which follow from the experimental observations.

### 1.3. Ground State of the Deuteron

**Main characteristics of the deuteron.** As we have already mentioned, a system consisting of a neutron and a proton can be found in a bound state. The bound state of the neutron-proton system is called the *deuteron* and is the simplest compound atomic nucleus. Despite the fact that one of the particles (the neutron) constituting the deuteron is unstable, the deuteron is characterized by an infinite lifetime, that is, is a stable system.

The most important characteristic of the deuteron is its *binding energy*, which is determined by the difference in mass between the deuteron and the particles forming it. The deuteron binding energy can be measured from the threshold of the photodisintegration reaction. The value of the *deuteron binding energy* found by this method is

$$\varepsilon = 2.22457 \pm 0.00001 \text{ MeV.} \quad (1.2)$$

Other important characteristics of the deuteron are its *spin*, *magnetic moment* and *electric quadrupole moment*. The *spin of the deuteron* is equal to 1. The *magnitude of the deuteron spin* can be determined from the band intensities of the spectrum of molecular deuterium.

The magnetic moment of the deuteron is close to the sum of the magnetic moments of the neutron and proton. The magnitude of the deuteron magnetic moment, expressed in nuclear magnetons, is

$$\mu_d = 0.8574. \quad (1.3)$$

The magnetic moment of the deuteron is measured most accurately in experiments on the deflection of a molecular beam in an inhomogeneous magnetic field.

The *electric quadrupole moment of the deuteron* is

$$Q = 2.860 \times 10^{-27} \text{ cm}^2. \quad (1.4)$$

This quantity was found from the fine structure in the radio frequency magnetic resonance spectrum of deuterium. Note the relative smallness of the deuteron electric quadrupole moment. In fact, the magnitude of the electric quadrupole moment must be compared with the spatial dimensions of the system. If we assume that the dimensions of the deuteron coincide with the range of the nuclear forces, then even in this case the quadrupole moment is approximately 50 times smaller than the estimated cross-sectional area of the deuteron. The small magnitude of the quadrupole moment of the deuteron and the approximate additivity of the magnetic moments of the neutron and proton in the deuteron imply that the ground state of the deuteron is almost spherically symmetric. But the ground state of a

system is characterized by spherical symmetry in cases where the interaction is central. Therefore, it follows from the spherical symmetry of the deuteron ground state that the nuclear interaction between the neutron and the proton is also almost central.

**Ground state of the deuteron in the case of central forces.**<sup>1</sup> We shall study the properties of the *ground state of the deuteron*, assuming that the nuclear interaction between the neutron and proton is described by a potential  $V(r)$ , where  $r$  is the magnitude of the distance between the particles. The Schrödinger equation for the wave function  $\psi(\mathbf{r})$  describing the relative motion of the neutron and proton has the form

$$\left\{ -\frac{\hbar^2}{2\mu} \Delta + V(r) - E \right\} \psi(\mathbf{r}) = 0, \quad (1.5)$$

where  $\mu = M/2$  is the reduced mass (for simplicity, we assume that the masses of the neutron and proton are the same and are equal to  $M$ ) and  $E$  is the energy of the relative motion. For the ground state of the system, the energy  $E$  is negative and equal in magnitude to the binding energy,  $E = -\varepsilon$ . The wave function  $\psi(\mathbf{r})$  should vanish at infinity and be finite at  $r = 0$ .

In the case of a central interaction, the ground state of the system is characterized by a zero value of the relative orbital angular momentum,  $l = 0$  (*S*-state). The wave function for this state is spherically symmetric and can be written in the form

$$\psi(\mathbf{r}) = u(r)/r, \quad (1.6)$$

where  $u(r)$  depends only on the magnitude of the distance between the particles. Substituting (1.6) into (1.5), we obtain the following equation for the function  $u(r)$ :

$$u'' - (M/\hbar^2)[V(r) + \varepsilon]u = 0, \quad (1.7)$$

where the function  $u(r)$  must satisfy the boundary conditions  $u(0) = u(\infty) = 0$ . The nuclear interaction is characterized by a finite range  $r_0$ , so that outside the range of the nuclear forces, eqn. (1.7) is simplified:

$$u'' - \alpha^2 u = 0, \quad r > r_0, \quad (1.8)$$

Here we have introduced the notation  $\alpha^2 = M\varepsilon/\hbar^2$ .

The solution of (1.8) that vanishes at infinity has the form

$$u = C e^{-\alpha r}, \quad r > r_0, \quad (1.9)$$

<sup>1</sup>The interaction of two nucleons at low energies is treated in detail in the review by Hulthen and Sugawara (1957), which contains a detailed bibliography.

where  $C$  is a normalization constant. Since the wave function and its derivative must be continuous, the solution of eqn. (1.8) in the outer region ( $r \geq r_0$ ) must match with the solution of eqn. (1.7) in the inner region ( $r \leq r_0$ ), which means that the logarithmic derivatives of the functions  $u(r)$  for the outer and inner regions must be equal at  $r = r_0$ . It follows from the expression (1.9) that the logarithmic derivative of the function  $u(r)$  in the outer region ( $r \geq r_0$ ) does not depend on  $r$  and is equal to  $-\alpha$ :

$$u'/u = -\alpha.$$

The logarithmic derivative of the function  $u(r)$  in the inner region ( $r \leq r_0$ ) must take the same value at  $r = r_0$ :

$$\left. \frac{u'_{\text{inner}}}{u_{\text{inner}}} \right|_{r=r_0} = -\alpha. \quad (1.10)$$

The magnitude of the logarithmic derivative of the inner function at  $r = r_0$  is determined by the potential energy  $V(r)$  and is a fundamental parameter characterizing the properties of the neutron-proton system. (Such an approach to the description of the neutron-proton system was first proposed by Bethe and Peierls (1935).)

According to the relation (1.9), outside the range of the nuclear forces the wave function of the deuteron decreases exponentially with increasing distance, so that the quantity  $1/\alpha$  can be considered as a parameter characterizing the spatial dimensions of the deuteron (the *deuteron radius*  $R_d$ ). Making use of the value (1.2) for the deuteron binding energy, we have

$$R_d \equiv \frac{1}{\alpha} = 4.3177 \times 10^{-13} \text{ cm.}$$

The magnitude of  $R_d$  is appreciably greater than the range  $r_0$  of the nuclear forces, and this means that the deuteron is a weakly bound system in which the neutron and proton spend a significant part of their time outside the range of the nuclear forces.

We note that the radial dependence of the wave function of the deuteron ground state outside the range of the nuclear forces is completely determined by specifying the binding energy. This circumstance permits us, in the limit of zero range of the nuclear forces, to construct an approximate wave function for the deuteron, completely independently of the form of the nuclear potential:

$$\psi_0(\mathbf{r}) = \sqrt{\alpha/2\pi} (e^{-\alpha r}/r), \quad r_0 \rightarrow 0, \quad (1.11)$$

where the normalization constant is determined from the condition

$$\int d^3r \psi_0^2(\mathbf{r}) = 1. \quad (1.12)$$

The formula (1.11) can be used in those cases when the behavior of the wave function at distances greater than the range of the nuclear forces is important.

**Relation between the width and depth of the potential well.** If we assume that the nuclear potential has the form of a square well of radius  $r_0$  and depth  $V_0$ , then the inner function  $u(r)$  that is a solution of (1.7) will have the form

$$u = C' \sin K_0 r, \quad r < r_0, \quad (1.13)$$

where  $C'$  is a normalization constant and

$$K_0 = (1/\hbar) \sqrt{M(V_0 - \varepsilon)}. \quad (1.14)$$

Substituting (1.13) into the expression (1.10) for the logarithmic derivative, we can obtain a relation connecting the quantities  $r_0$  and  $V_0$ . The relation between the width and depth of the potential well can be obtained more intuitively, however, from the following considerations. Since in the ground state the function  $u(r)$  has no zeros (except for the point  $r = 0$ ), not more than half of a wavelength can be fitted into the inner region ( $r < r_0$ );  $r_0 < \frac{1}{2}(2\pi/K_0)$ . However, to make it possible to match the solution (1.13) with the exponentially decreasing solution (1.9), it is necessary that there be not less than a quarter of a wavelength in the inner region. Consequently, the inequalities

$$\frac{1}{4}(2\pi/K_0) \leq r_0 < \frac{1}{2}(2\pi/K_0). \quad (1.15)$$

must be fulfilled.

Using (1.14) and neglecting the binding energy of the deuteron in comparison with the depth of the well, from (1.15) it is not difficult to determine limits within which the depth of the potential well must lie:

$$(\pi^2/4)(\hbar^2/M) \leq V_0 r_0^2 < \pi^2(\hbar^2/M). \quad (1.16)$$

Putting  $r_0 = 1.7 \times 10^{-13}$  cm, we obtain a quantity of the order of 35 MeV for the lower bound on the depth  $V_0$  of the potential well; this appreciably exceeds the binding energy of the deuteron.

**Absence of excited states of the deuteron.** It is not difficult to show that, if the nuclear forces are assumed to be centrally symmetric, the deuteron has no excited bound states.

For the existence of an excited bound state with zero orbital angular momentum to be possible, it is necessary that the radial function  $u(r)$  in the inner region have an additional zero, that is, that more than three-quarters of a wavelength fit into the inner region. Neglecting the binding

energy of the excited state, we can write this condition in the form

$$r_0 \geq \frac{3}{4}(2\pi\hbar/\sqrt{MV_0}), \quad \text{whence} \quad V_0 r_0^2 \geq \frac{9}{4}\pi^2(\hbar^2/M).$$

But this condition contradicts the inequality (1.16).

The fact that the deuteron can also have no excited bound states with non-zero values of the relative orbital angular momentum is connected with the fact that levels with  $l \neq 0$  lie appreciably above the level with  $l = 0$ , since in a state with non-zero orbital angular momentum there is additional (centrifugal) energy. The mean value of this energy, equal to  $\hbar^2 l(l+1)/Mr^2$ , can be estimated if we substitute in place of  $r$  a quantity of the order of the effective dimensions of the deuteron, that is,  $1/\alpha$ . For  $l = 1$ , we obtain for the centrifugal energy a quantity equal to  $2\varepsilon$ ; that is, even a  $P$ -state should lie in the region of the continuous spectrum.

We note that in our proof of the absence of excited bound  $S$ -states we neglected the spin dependence of the nuclear forces. In reality, even if the nuclear forces are central they can depend on the mutual orientation of the spins of the neutron and proton. Therefore, in principle, two bound  $S$ -states of the neutron-proton system can exist: one with parallel orientation of the spins of the particles (a triplet state, with the total spin of the particles equal to unity) and one with the anti-parallel orientation of the spins of the particles (a singlet state, with the total spin of the particles equal to zero). However, it follows from the experimental data on the scattering of slow neutrons in molecular hydrogen that the singlet state in the neutron-proton system is not realized.

#### 1.4. Scattering of Slow Neutrons by Protons

**Scattering amplitude and phase shifts.** Along with the study of the bound state of the neutron-proton system, the study of the scattering of neutrons by protons is a second very important source of information about the nuclear interaction between neutrons and protons. Unlike the case of the bound state of the neutron-proton system, when the energy is negative, in the case of scattering of a neutron by a proton the energy of the relative motion of the particles is positive. The wave function  $\psi(r)$  describing the scattering of a neutron by a proton is given by the solution of the Schrödinger equation (1.5) for a positive value of the energy  $E$ . In this case, the wave function  $\psi(r)$  at large distances must have the form of a sum of the incident plane wave and a spherical outgoing scattered wave:

$$\psi(\mathbf{r}) \rightarrow e^{ikz} + f(\vartheta)(e^{ikr}/r), \quad (1.17)$$

where  $k = \sqrt{ME}/\hbar$  is the magnitude of the wave-vector of the relative motion of the particles, and the  $z$ -axis is chosen to lie along the direction

of the incident beam of neutrons. The coefficient  $f(\vartheta)$  of the outgoing wave depends on the angle  $\vartheta$  between the directions of motion of the incident and scattered particles and is called the *scattering amplitude*; angle  $\vartheta$  is the *scattering angle*.

In the case of a central interaction, the scattering amplitude can be expressed directly in terms of the *phase shifts*  $\delta_l$ :

$$f(\vartheta) = (i/2k) \sum_{l=0}^{\infty} (2l+1)(1 - e^{2i\delta_l}) P_l(\cos \vartheta). \quad (1.18)$$

The phases  $\delta_l$  characterize the interaction of a particle with different values of the orbital angular momentum  $l$  and depend on the energy of the particle and on the form of the potential  $V(r)$ . In general, the phase shifts are complex. If only elastic scattering occurs, then the phase shifts  $\delta_l$  are real functions of energy.

The *scattering cross-section* is connected with the scattering amplitude by the relation

$$d\sigma = \sigma(\vartheta) d\omega = |f(\vartheta)|^2 d\omega, \quad (1.19)$$

where  $d\omega$  is an element of solid angle, defining the direction of motion of the scattered particle. Substituting (1.18) into (1.19) and integrating (1.19) over the whole solid angle, we easily find the integral scattering cross-section:

$$\sigma = (4\pi/k^2) \sum_l (2l+1) \sin^2 \delta_l. \quad (1.20)$$

**Energy dependence of the S-scattering phase shift.** We shall consider the scattering of slow neutrons by protons in the case when the wavelength  $\lambda = k^{-1}$  of the relative motion is greater than the range  $r_0$  of the nuclear forces. The condition  $\lambda > r_0$  is fulfilled if the energy of the relative motion of the neutron and proton does not exceed 10 MeV. (In the laboratory frame, in which the proton is at rest before the scattering, the energy of the incident neutron is then less than 20 MeV.) Since the effective impact parameter for the collision of two particles with relative angular momentum  $l$  is equal to  $l\lambda$ , if  $\lambda > r_0$  interaction of the particles is possible only in an *S*-state ( $l = 0$ ). Then all the phase shifts  $\delta_l$  (except the phase shift in the *S*-state,  $\delta_0 \equiv \delta$ ) are equal to zero, and the scattering amplitude takes the form

$$f = (1/k)e^{i\delta} \sin \delta. \quad (1.21)$$

In this case, the scattering is spherically symmetric. The integral scattering cross-section is equal to

$$\sigma = (4\pi/k^2) \sin^2 \delta. \quad (1.22)$$

We shall find the dependence of the phase shift  $\delta$  on the energy in the limiting case of infinitesimally small range of the nuclear forces ( $r_0 \rightarrow 0$ ). For  $l = 0$ , the radial function  $u$  outside the range of the nuclear forces is determined by the equation

$$u'' + k^2 u = 0, \quad r > r_0,$$

the solution of which has the form

$$u = C \sin(kr + \delta), \quad r > r_0. \quad (1.23)$$

The phase  $\delta$  can be found from the conditions that the logarithmic derivatives of the wave function (1.23) in the outer region and of the wave function in the inner region be equal at  $r = r_0$ :

$$k \cot(kr_0 + \delta) = \frac{u'_{\text{inner}}}{u_{\text{inner}}} \Big|_{r=r_0} \quad (1.24)$$

If the energy of the relative motion of the particles is small compared with the depth of the potential well the wave function in the inner region for the scattering problem will not differ from the wave function in the inner region for the bound-state problem. Therefore, we replace the right-hand side of the equality (1.24) by the expression (1.10) and thus obtain the following relation

$$k \cot(kr_0 + \delta) = -\alpha, \quad (1.25)$$

or, in the limiting case of *zero range of the nuclear forces*,

$$k \cot \delta = -\alpha, \quad (1.26)$$

We emphasize that the possibility of introducing the same parameter  $\alpha$  to describe both the scattering and the asymptotic form of the bound-state wave function is a direct consequence of the smallness of the deuteron binding energy and of the energy of the relative motion of the neutron and proton compared with the effective potential energy of the interaction of the neutron and proton.

Assuming zero range of the nuclear forces and using (1.26), we can write the cross-section for the scattering of a neutron by a proton in the form

$$\sigma = 4\pi/(\alpha^2 + k^2), \quad r_0 \rightarrow 0. \quad (1.27)$$

**Spin dependence of the nuclear interaction.** According to (1.27), the scattering cross-section in the limit of zero energy of relative motion is completely determined by the magnitude of the deuteron binding energy:

$$\sigma = 4\pi/\alpha^2, \quad k \rightarrow 0. \quad (1.28)$$

If we make use of the numerical value (1.2) of the deuteron binding energy, we obtain, according to (1.28), the value  $2.33 \times 10^{-24} \text{ cm}^2$  for the scattering cross-section. The experimental value of the cross-section for the scattering of a neutron by a proton at zero energy is found to be considerably greater and is equal to

$$\sigma_{\text{exp}} = (20.49 \pm 0.01) \times 10^{-24} \text{ cm}^2. \quad (1.29)$$

Such a significant difference between the experimental value of the scattering cross-section and the expression (1.28) points directly to the important dependence of the nuclear interaction on the spin state of the system.

The neutron-proton system can be found in two different spin states: the *triplet state* (with total spin equal to unity) and the *singlet state* (with total spin equal to zero). Since the total angular momentum (spin) of the deuteron is equal to unity and the ground state of the deuteron is even, the triplet spin state is realized in the deuteron. In the case of scattering of an unpolarized neutron by an unpolarized proton, the system can be found either in a triplet state (with weight  $\frac{3}{4}$ ) or in a singlet state (with weight  $\frac{1}{4}$ ). If the nuclear interaction depends substantially on the spin state, then to describe the interaction between a neutron and proton it is necessary to introduce two different parameters  $\delta_t$  and  $\delta_s$ , for the triplet and singlet spin states. The phase shifts  $\delta_t$  and  $\delta_s$  corresponding to the triplet and singlet states are connected with  $\alpha_t$  and  $\alpha_s$  for  $r_0 \rightarrow 0$  by the relations

$$k \cot \delta_t = -\alpha_t, \quad k \cot \delta_s = -\alpha_s. \quad (1.30)$$

The quantity  $\alpha_t$  is related directly to the deuteron binding energy

$$\varepsilon = \hbar^2 \alpha_t^2 / M. \quad (1.31)$$

The cross-sections for scattering of neutrons by protons in the triplet and singlet states, according to the relations (1.30) and (1.22), are

$$\sigma_t = 4\pi / (\alpha_t^2 + k^2), \quad \sigma_s = 4\pi / (\alpha_s^2 + k^2), \quad r_0 \rightarrow 0. \quad (1.32)$$

In the case of scattering of unpolarized neutrons by unpolarized protons, the scattering cross-section is given by the formula

$$\bar{\sigma} = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s, \quad (1.33)$$

where the coefficients  $\frac{3}{4}$  and  $\frac{1}{4}$  are the statistical weights of the triplet and singlet spin states respectively. Using the experimental value (1.29) of the averaged cross-section for the scattering of a neutron by a proton at zero energy and knowing the binding energy of the deuteron, we can determine the quantity  $\alpha_s^2$ , or the energy formally associated with it:

$$\varepsilon' = \hbar^2 \alpha_s^2 / M = 0.069 \text{ MeV}. \quad (1.34)$$

This energy is sometimes called the *virtual level of the deuteron*. (But it must be kept in mind that a bound state of the neutron-proton system with energy  $-\varepsilon'$  does not exist.)

**Scattering lengths.** We now consider the scattering of neutrons by protons in the limiting case of zero energy of the neutrons, assuming the range of the nuclear forces to be non-zero. The equation for the radial function outside the range of the nuclear forces in the case of zero energy has the form

$$u'' = 0, \quad r > r_0, \quad (1.35)$$

whence

$$u = C'(r - a), \quad r > r_0, \quad (1.36)$$

where  $C'$  and  $a$  are constants. The solution (1.36) is the equation of a straight line intersecting the  $r$ -axis at the point  $a$ . The quantity  $a$  is called the *scattering length*.

The scattering length  $a$  is easily related to the phase shift  $\delta$ . In fact, the solution (1.36) must coincide with the function  $u$  given by (1.23), for  $k \rightarrow 0$ . Comparison of the two formulae shows that

$$a = -\lim_{k \rightarrow 0} \left( \frac{\tan \delta}{k} \right) \quad (1.37)$$

that is, for small  $k$ , when the inequality  $kr_0 \ll 1$  is fulfilled, the phase shift  $\delta$  can be chosen in the form

$$\delta = n\pi - ak, \quad kr_0 \ll 1, \quad (1.38)$$

where  $n$  is an arbitrary integer<sup>2</sup>. Hence it follows, on the basis of formula (1.22), that the scattering cross-section in the limiting case of zero energy of the neutrons is

$$\sigma = 4\pi a^2, \quad k \rightarrow 0. \quad (1.39)$$

The scattering length depends on the spin state of the system and, therefore, we must distinguish scattering lengths  $a_t$  and  $a_s$  in the triplet and singlet states.

Measurement of the scattering cross-section at zero energy enables us to determine only the magnitude of the scattering length, and not its sign. The scattering length can be taken to be either positive or negative. The dependence of the wave function on  $r$  at zero energy for positive and negative scattering lengths is depicted schematically in Fig. 1.1. The behaviour of the wave function within the range of the nuclear forces ( $r < r_0$ ) is almost independent of the energy and, therefore, is the same as for the bound

<sup>2</sup>If we demand that the phase shift goes to zero as  $k \rightarrow \infty$ , then it follows from Levinson's theorem that  $n = 1$ .

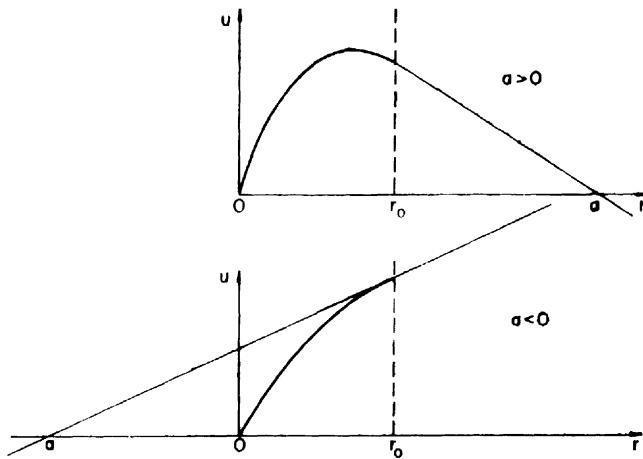


Figure 1.1. Dependence on  $r$  of the radial wave function  $u$  at zero energy, in the cases of positive and negative scattering lengths.

state, if the binding energy is not very large. From Fig. 1.1 we see that, for positive scattering lengths, the wave function for the inner region can be matched to the exponentially decreasing solution corresponding to the bound state in the outer region. For negative scattering lengths, the wave function in the inner region behaves in such a way that it is impossible to match it with the exponentially decreasing solution in the outer region, that is, a bound state of the system is impossible in this case. Thus, if  $a > 0$ , a bound state of the system can exist; but if  $a < 0$ , a bound state is impossible.

**Effective range of the nuclear interaction.** The formulae (1.32) determining the cross-section for the scattering of slow neutrons by protons for  $k \neq 0$  were derived under the assumption that the range of the nuclear forces is equal to zero ( $r_0 \rightarrow 0$ ). We now show how the scattering problem can be formulated in such a way that the fact that the range of the forces is finite can be taken into account consistently, irrespective of the form of the potential (Bethe, 1949). We write the exact equation for the radial function  $u(r)$  that describes the motion in a field  $V(r)$  and corresponds to the value  $E$  of the energy of the relative motion:

$$u'' - (M/\hbar^2)V(r)u = -k^2u; \quad k^2 = ME/\hbar^2. \quad (1.40)$$

We denote the radial function corresponding to zero energy by  $u_0(r)$ . This function satisfies the equation

$$u_0'' - (M/\hbar^2)V(r)u_0 = 0. \quad (1.41)$$

Multiplying (1.40) by  $u_0$  and (1.41) by  $u$  and subtracting the first equality from the second, we obtain the following relation:

$$\frac{d}{dr}(u'_0 u - u_0 u') = k^2 u_0 u. \quad (1.42)$$

We introduce the auxiliary functions  $v(r)$  and  $v_0(r)$  satisfying the equations for free motion ( $v'' = -k^2 v$ ,  $v_0'' = 0$ ), these functions being defined in such a way that they coincide with  $u$  and  $u_0$  for  $r > r_0$ . Obviously, a relation analogous to (1.42) holds for the functions  $v$  and  $v_0$ :

$$\frac{d}{dr}(v'_0 v - v_0 v') = k^2 v_0 v. \quad (1.43)$$

Subtracting the relation (1.42) from (1.43), integrating the resulting equality over  $r$  from 0 to infinity and noting that  $u(0) = u_0(0) = 0$ , we obtain

$$v_0(0)v'(0) - v'_0(0)v(0) = k^2 \int_0^\infty dr(v_0 v - u_0 u). \quad (1.44)$$

We now choose the normalization of the functions  $u$  and  $u_0$  in such a way that the functions  $v$  and  $v_0$  go to unity at  $r = 0$ . Obviously, with this normalization, the functions  $v$  and  $v_0$  have the following form:

$$v(r) = \sin(kr + \delta) / \sin \delta, \quad v_0(r) = 1 - r/a, \quad (1.45)$$

where  $\delta$  and  $a$  have been defined previously.

Substituting the relations (1.45) into (1.44), we obtain the general expression

$$k \cot \delta = -(1/a) + \frac{1}{2} k^2 r(0, E), \quad (1.46)$$

relating the phase shift  $\delta$  to the scattering length  $a$  and the effective range

$$r(0, E) \equiv 2 \int_0^\infty dr(v_0 v - u_0 u). \quad (1.47)$$

For the bound state, we obtain a relation analogous to (1.46) by replacing  $k^2$  by  $-\alpha^2$  and  $k \cot \delta$  by  $-\alpha$  in (1.46). As a result of the replacement, we have

$$\alpha = (1/a) + \frac{1}{2} \alpha^2 r(0, -\varepsilon), \quad (1.48)$$

where

$$r(0, -\varepsilon) \equiv 2 \int_0^\infty dr(v_0 v_{-\varepsilon} - u_0 u_{-\varepsilon}). \quad (1.49)$$

Since the functions  $v_0$  and  $v$  coincide with  $u_0$  and  $u$  outside the range of the nuclear forces, in the formulae for  $r(0, E)$  and  $r(0, -\varepsilon)$  the integration is performed only over the region of the nuclear interaction. We note that the relations (1.46) and (1.48) are completely independent of the form of the nuclear potential, if by the effective ranges we mean the quantities (1.47) and (1.49). (The concept of the effective range was introduced by Schwinger (1947).)

For energies  $E$  that are small compared with the effective depth of the potential well, the functions  $u$  and  $v$  within the region of the nuclear interaction are practically the same as the functions  $u_0$  and  $v_0$ , and the quantities  $r(0, E)$  and  $r(0, -\varepsilon)$  can be considered as independent of the energy to a first approximation. These approximate values of  $r(0, E)$  and  $r(0, -\varepsilon)$  are usually denoted by  $r_0$ :

$$r_0 = 2 \int_0^\infty dr (v_0^2 - u_0^2). \quad (1.50)$$

The quantity  $r_0$ , which has the dimensions of length, is determined by the character of the nuclear interaction and may be called the *effective range of the nuclear forces*. Indeed, the integrand in (1.50) is non-zero only within the range of the nuclear forces and, in order of magnitude, is equal to unity. Therefore, the integral (1.50) characterizes the distance over which the nuclear interaction occurs.

**Lengths and effective ranges for triplet and singlet states.** In the low-energy region, the relation (1.46) can be represented in the form

$$k \cot \delta = -(1/a) + \frac{1}{2} k^2 r_0. \quad (1.51)$$

The effective range of the nuclear forces, like the phase shift and scattering lengths, depends on the spin state of the neutron-proton system. Denoting the effective ranges in the triplet and singlet states by  $r_{0t}$  and  $r_{0s}$ , we can write the following relations:

$$k \cot \delta_t = -(1/a_t) + \frac{1}{2} k^2 r_{0t}, \quad k \cot \delta_s = -(1/a_s) + \frac{1}{2} k^2 r_{0s}. \quad (1.52)$$

We note that for the triplet state an additional relation holds, connecting the scattering length  $a_t$  and the effective range  $r_{0t}$ :

$$1/a_t = \alpha_t - \frac{1}{2} \alpha_t^2 r_{0t}. \quad (1.53)$$

The right-hand sides in the formulae (1.52) are the first two terms of the expansion of  $k \cot \delta$  in a series in powers of  $k^2$ . It can be seen that, to

within terms of order  $k^2$ , the quantity  $k \cot \delta$  is determined by only two parameters (for each spin state) and does not depend on the actual form of the nuclear potential. Using the formulae (1.52), it is not difficult to obtain the following expressions for the cross-sections for the scattering of slow neutrons by protons in the triplet and singlet states:

$$\sigma_t = \frac{4\pi}{\left(\frac{1}{a_t} - \frac{1}{2}k^2r_{0t}\right)^2 + k^2}, \quad \sigma_s = \frac{4\pi}{\left(\frac{1}{a_s} - \frac{1}{2}k^2r_{0s}\right)^2 + k^2}. \quad (1.54)$$

The averaged cross-section for the scattering of unpolarized slow neutrons by free unpolarized protons is given by the formula

$$\sigma_t = \frac{3\pi}{\left(\frac{1}{a_t} - \frac{1}{2}k^2r_{0t}\right)^2 + k^2} + \frac{\pi}{\left(\frac{1}{a_s} - \frac{1}{2}k^2r_{0s}\right)^2 + k^2}. \quad (1.55)$$

As can be seen, the scattering of neutrons by protons in the region of low energies of the incident neutrons is characterized by four parameters  $a_t$ ,  $a_s$ ,  $r_{0t}$  and  $r_{0s}$  and does not depend on the form of the potential; that is, the scattering cross-section is the same for different potentials  $V(r)$  that lead to the same values of the scattering lengths and effective ranges.

Using the experimental value for the averaged cross-section for the scattering of neutrons by free protons at zero energy, and also the data on the scattering of neutrons by para-hydrogen at zero energy, we find the scattering lengths  $a_t$  and  $a_s$ :

$$a_t = (5.424 \pm 0.003) \times 10^{-13} \text{ cm}; \quad a_s = -(23.749 \pm 0.008) \times 10^{-13} \text{ cm}. \quad (1.56)$$

Note that the negative sign of the scattering length  $a_s$  indicates the absence of a bound singlet state in the neutron-proton system. Knowing the scattering length  $a_t$  in the triplet state and the deuteron binding energy and taking the equality (1.53) into account, we can determine the *effective range in the triplet state* with a high degree of accuracy:

$$r_{0t} = (1.760 \pm 0.005) \times 10^{-13} \text{ cm}. \quad (1.57)$$

Using the experimental dependence on the neutron energy of the averaged cross-section for scattering of slow neutrons by free protons, we find the *effective range in the singlet state*:

$$r_{0s} = (2.77 \pm 0.05) \times 10^{-13} \text{ cm}. \quad (1.58)$$

The error in the determination of  $r_{0s}$  turns out to be much greater than the error in the determination of  $r_{0t}$ .

**The shape parameter.** As we have already observed, the relation (1.51) represents the first two terms of the expansion of  $k \cot \delta$  in a series in powers of  $k^2$ , the coefficients of this expansion being independent of the detailed form of the nuclear potential. The next terms of the expansion of  $k \cot \delta$  in powers of  $k^2$  depend on the form of the potential; therefore, by comparing the experimental data on the scattering of slow neutrons by protons with the theoretical formula for the cross-section in which the subsequent terms of the expansion of (1.46) are taken into account, we can obtain information about the form of the potential. With the inclusion of the terms proportional to  $k^4$ , the expansion of  $k \cot \delta$  has the form

$$k \cot \delta = -(1/a) + \frac{1}{2} k^2 r_0 - P k^4 r_0^3 + \dots, \quad (1.59)$$

where

$$P \equiv -\frac{1}{r_0^3} \int_0^\infty dr \left\{ v_0 \frac{\partial v}{\partial k^2} \Big|_{k=0} - u_0 \frac{\partial u}{\partial k^2} \Big|_{k=0} \right\}. \quad (1.60)$$

The dimensionless quantity  $P$  is called the *shape parameter*, since it is connected with the shape of the nuclear potential  $V(r)$ .

To characterize potentials of different shapes, we usually introduce the potential-depth parameter  $s$  and the characteristic width  $b$ ; by means of these parameters, potentials of different shapes can be compared. The *potential-depth parameter*  $s$  is defined as the number by which the potential  $V(r)$  must be divided in order to reduce the binding energy of the ground state of the system to zero. If the depth parameter  $s > 1$ , the potential  $V(r)$  admits a bound state; but if  $s < 1$ , the potential  $V(r)$  does not lead to a bound state. The *characteristic width*  $b$  of the potential is defined as the effective range of the reduced potential  $V(r)/s$  (corresponding to zero binding energy), that is,

$$b = \lim_{s \rightarrow 1} r_0, \quad (1.61)$$

where  $r_0$  is given by the expression (1.50). For a square well, the characteristic width coincides with the ordinary width of the well,  $b = r_0$ , and the depth parameter is given by the expression  $s = 4MV_0r_0^2/\pi\hbar^2$ .

## 1.5. Scattering of Slow Neutrons in Molecular Hydrogen

**The pseudo-potential.** The scattering amplitude for scattering of a slow neutron by a proton depends on the spin state of the neutron-proton system. Since in the cross-section for elastic scattering of a neutron by a free proton the squares of the scattering lengths  $a_t$  and  $a_s$  appear, we can find only the absolute values of the quantities  $a_t$  and  $a_s$  from experiments on the scattering of neutrons by free protons. In practice one is usually concerned

with the scattering of neutrons in substances containing hydrogen. Obviously, the protons can be regarded as free only if the energy of the neutrons appreciably exceeds the binding energy of the protons in the molecules of the substance. But if the energy of the neutrons is comparable with, or less than, the binding energy of the protons, the chemical binding of the protons will have an appreciable influence on the scattering.

The sign of the scattering length  $a_s$  in the singlet state can be determined by studying the scattering of slow neutrons in molecular hydrogen (Schwinger and Teller, 1937). If the wavelength of the neutron is comparable with the distance between the protons forming the molecule, or is large compared with this distance, then in the scattering of the neutron by both protons interference effects occur; a study of these effects makes it possible to determine both the absolute values and the signs of the scattering lengths.

The scattering of slow neutrons by free protons is spherically symmetric and, for  $k \rightarrow 0$ , is determined by a constant scattering length  $a$ . These properties of the scattering of slow neutrons by protons can be described formally by means of a special form of perturbation theory, if we introduce a certain fictitious potential that leads to the correct value of the scattering amplitude when the perturbation theory is used.

In the case of sufficiently slow neutrons (when  $\lambda \gg r_0$ ) we can make use of the *zero-range approximation*, that is, assume that interaction between the neutron and protons exists only when their coordinates coincide (contact interaction). This approximation corresponds to a choice of the potential in the form of a delta-function (Breit, 1947):

$$V(\mathbf{r}) = (2\pi\hbar^2/\mu)a\delta(\mathbf{r}), \quad \mathbf{r} = \mathbf{r}_n - \mathbf{r}_p, \quad (1.62)$$

where  $\mathbf{r}_n$  and  $\mathbf{r}_p$  are the position vectors of the neutron and proton, and the coefficient of the delta-function is found from the requirement that the scattering amplitude in the first order of the perturbation theory in (1.62) coincide in magnitude with the scattering length,  $f = -a$  ( $\mu$  is the reduced mass of the neutron and proton). When we take the effects of the chemical binding into account, different powers of the ratio of the scattering length  $a$  to the molecular dimensions  $d$  or to the neutron wavelength  $\lambda$  correspond to different orders of perturbation theory in (1.62). Therefore, we write the condition for the applicability of (1.62) in the form  $a \ll d$  and  $a \ll \lambda$ . Moreover, in order that we can disregard the finite range of the nuclear forces, the following conditions must be fulfilled:  $r_0 \ll d$  and  $r_0 \ll \lambda$ .

The scattering of slow neutrons in molecular hydrogen can be studied by means of the pseudo-potential (1.62), if we previously include in it the dependence of  $a$  on the spin state of the interacting particles. We introduce the Pauli matrices  $\sigma_n$  and  $\sigma_p$  to describe the spins of the neutron and

proton. The total spin of the neutron-proton system is equal to

$$S = \frac{1}{2}(\sigma_n + \sigma_p).$$

As is easily verified, the eigenvalues of the operator  $\sigma_n \cdot \sigma_p$  in the triplet and singlet states are equal to 1 and  $-3$  respectively. Therefore, the quantities

$$\pi_t = \frac{1}{4}(3 + \sigma_n \cdot \sigma_p), \quad \pi_s = \frac{1}{4}(1 - \sigma_n \cdot \sigma_p), \quad (1.63)$$

can be regarded as projection operators, projecting out the triplet and singlet states. Indeed, the eigenvalues of these operators are:

$$\pi_t = \begin{cases} 1, & S = 1, \\ 0, & S = 0; \end{cases} \quad \pi_s = \begin{cases} 0, & S = 1, \\ 1, & S = 0. \end{cases}$$

The scattering length with the spin dependence of the nuclear forces taken into account can be represented by means of the projection operators in the form

$$a = a_t \pi_t + a_s \pi_s = \frac{1}{4}(3a_t + a_s) + \frac{1}{4}(a_t - a_s)\sigma_n \cdot \sigma_p. \quad (1.64)$$

If we are considering the scattering of slow neutrons by a system of protons, then for the potential we must take a sum of expressions of the type (1.62)

$$V = \sum_i (2\pi\hbar^2/\mu) a^{(i)} \delta(\mathbf{r}_n - \mathbf{r}_i), \quad (1.65)$$

where  $\mathbf{r}_i$  is the position vector of the  $i$ th proton and  $a^{(i)}$  is the scattering length for the scattering of a neutron by the  $i$ th proton. This quantity depends on the spin state of the scattered neutron and  $i$ th proton. In the case of scattering of slow neutrons in molecular hydrogen, the energy of the interaction of the neutron with a molecule can be written in the form

$$\begin{aligned} V &= (\pi\hbar^2/2\mu)[3a_t + a_s + (a_t - a_s)\sigma_n \cdot \sigma_1]\delta(\mathbf{r}_n - \mathbf{r}_1) \\ &+ (\pi\hbar^2/2\mu)[3a_t + a_s + (a_t - a_s)\sigma_n \cdot \sigma_2]\delta(\mathbf{r}_n - \mathbf{r}_2), \end{aligned} \quad (1.66)$$

where the indices 1 and 2 correspond to the two protons of the molecule.

**Elastic scattering of slow neutrons in para- and ortho-hydrogen.** We introduce the total spin of the molecule,  $S = \frac{1}{2}(\sigma_1 + \sigma_2)$ . The quantum number  $S$  of the square of the spin angular momentum of the molecule can take two values:  $S = 0$  and  $S = 1$ . If  $S = 0$  the corresponding state is called *para-hydrogen*; if  $S = 1$ , the state is called *ortho-hydrogen*. In the

interaction energy (1.66) of the neutron with the molecule, we separate the symmetric and anti-symmetric parts:  $V = V_s + V_a$ , where

$$\begin{aligned} V_s &= (\pi\hbar^2/2\mu)[3a_t + a_s + (a_t - a_s)\boldsymbol{\sigma}_n \cdot \mathbf{S}][\delta(\mathbf{r}_n - \mathbf{r}_1) + \delta(\mathbf{r}_n - \mathbf{r}_2)], \\ V_a &= (\pi\hbar^2/4\mu)(a_t - a_s)\boldsymbol{\sigma}_n \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)[\delta(\mathbf{r}_n - \mathbf{r}_1) - \delta(\mathbf{r}_n - \mathbf{r}_2)]. \end{aligned} \quad (1.67)$$

Clearly, the symmetric part  $V_s$  of the interaction energy is responsible for transitions in which the spin symmetry of the molecule does not change; in other words,  $V_s$  causes transitions from para-hydrogen to para-hydrogen, and from ortho-hydrogen to ortho-hydrogen. The anti-symmetric part  $V_a$  induces transitions in which the symmetry of the molecule changes, that is, transitions from ortho-hydrogen to para-hydrogen and vice versa.

We shall consider the scattering of slow neutrons in the limiting case when the wavelength  $\lambda$  of the neutron is appreciably greater than the mean distance  $d$  between the protons in the hydrogen molecule. (In the equilibrium state, the mean distance between the protons in the hydrogen molecule is of the order of  $0.75 \times 10^{-8}$  cm.) In this case, we can neglect the difference in the positions of the two protons and assume that they are at the same point. Therefore, for  $\lambda \gg d$ , the anti-symmetric part  $V_a$  of the interaction energy vanishes and only the symmetric part  $V_s$  remains. In this case, clearly, only elastic scattering of neutrons is possible, since  $V_s$  does not depend on the relative coordinates of the protons, and the matrix elements of  $V_s$  corresponding to inelastic processes will be equal to zero by virtue of the orthogonality of the wave functions.

Comparing the relation (1.67) with (1.62), we can easily establish that the amplitude for elastic scattering of a slow neutron by a hydrogen molecule is, for  $\lambda \gg d$ ,

$$A = -\frac{1}{2}[3a_t + a_s + (a_t - a_s)\boldsymbol{\sigma}_n \cdot \mathbf{S}]. \quad (1.68)$$

In this case, the cross-section for scattering of slow neutrons by a hydrogen molecule is given by the formula

$$\sigma = 4\pi(\tilde{\mu}/\mu)^2 \overline{|A|^2}, \quad (1.69)$$

where  $\tilde{\mu} = 2M/3$  is the reduced mass of the neutron and hydrogen molecule,  $\mu = M/2$  is the reduced mass of the neutron and proton, and the line over the square of the modulus of the scattering amplitude denotes averaging over the spin orientations of the scattered neutron.

Noting that  $\boldsymbol{\sigma}_n \cdot \mathbf{S} = 0$  and  $(\boldsymbol{\sigma}_n \cdot \mathbf{S})^2 = S(S+1)$ , we find

$$\overline{|A|^2} = \frac{1}{4}\{(3a_t + a_s)^2 + S(S+1)(a_t - a_s)^2\}.$$

Consequently, the cross-section for elastic scattering of a slow neutron is

$$\sigma = \frac{16}{9}\pi\{(3a_t + a_s)^2 + S(S+1)(a_t - a_s)^2\}. \quad (1.70)$$

Putting  $S = 0$  and  $S = 1$  in this formula, we obtain values for the cross-sections for elastic scattering of slow neutrons in para- and ortho-hydrogen:

$$\left. \begin{aligned} \sigma_{\text{para}} &= \frac{16}{9}\pi(3a_t + a_s)^2, \\ \sigma_{\text{ortho}} &= \frac{16}{9}\pi\{(3a_t + a_s)^2 + 2(a_t - a_s)^2\}. \end{aligned} \right\} \quad (1.71)$$

The elastic scattering cross-section  $\sigma_{\text{para}}$  in para-hydrogen contains the combination  $(3a_t + a_s)^2$ ; in the cross-section  $\sigma_{\text{ortho}}$ , along with this quantity the quantity  $(a_t - a_s)^2$  also appears. Consequently, from experiments on the scattering of slow neutrons in para- and ortho-hydrogen, it is possible to determine the quantities  $|3a_t + a_s|$  and  $|a_t - a_s|$ , that is, to determine the absolute values of  $a_t$  and  $a_s$  and the sign of the ratio  $a_t/a_s$ .

The ground state of the hydrogen molecule is the para-state. By studying the scattering of zero-energy neutrons by para-hydrogen molecules, we can determine, according to the relation (1.71), the quantity  $|3a_t + a_s|$ . Using the experimental value of the averaged cross-section for scattering of zero-energy neutrons by free protons and the fact that the bound state of the neutron-proton system is the triplet state, it is not difficult to find values for  $a_t$  and  $a_s$  (cf. §1.4.).

## 1.6. Proton-proton Scattering

**Coulomb scattering.** Important data on the nature of nuclear forces are given by the study of proton proton scattering. Since protons possess electric charge, in studying proton proton scattering we must take into account not only the nuclear interaction but also the *Coulomb interaction between the protons*.

The wave function  $\psi(\mathbf{r})$  describing the relative motion of two protons satisfies the Schrödinger equation

$$\{-(\hbar^2/M)\Delta + (e^2/r) + V(r) - E\}\psi(\mathbf{r}) = 0, \quad (1.72)$$

where  $e^2/r$  is the Coulomb interaction energy,  $V(r)$  is the potential of the nuclear interaction, and  $E$  is the energy of the relative motion of the protons ( $E > 0$ ). In the case of scattering, the problem is to determine the asymptotic form of the function  $\psi(\mathbf{r})$  for large values of  $\mathbf{r}$ . Expanding the wave function  $\psi(\mathbf{r})$  in Legendre polynomials  $P_l(\cos \vartheta)$  ( $\vartheta$  is the angle between the wave vector  $\mathbf{k}$  of the incident proton and the position vector  $\mathbf{r}$ )

$$\psi(\mathbf{r}) = \sum_l \frac{u_l(r)}{r} P_l(\cos \vartheta), \quad (1.73)$$

we obtain the following equation for the radial function  $u_l(\mathbf{r})$  corresponding to angular momentum  $l$  of the relative motion:

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{2\xi k}{r} - \frac{M}{\hbar^2} V(r) \right\} u_l = 0, \quad (1.74)$$

where  $\xi = e^2/\hbar v$  is the so-called *Coulomb parameter* ( $v$  is the relative speed of the protons). If the nuclear interaction of the proton were absent, that is,  $V(r) = 0$ , the asymptotic form of the solution of eqn. (1.74) would be

$$u_l^C \rightarrow i^l (2l+1) \frac{e^{i\eta_l}}{k} \sin \left( kr - \xi \ln 2kr - \frac{l\pi}{2} + \eta_l \right), \quad r \rightarrow \infty, \quad (1.75)$$

where  $\eta_l$  is the *Coulomb phase shift*, given by the equality

$$e^{2i\eta_l} = \frac{\Gamma(l+1+i\xi)}{\Gamma(l+1-i\xi)}.$$

Using the relation (1.75), one can show that, with neglect of the nuclear interaction, the wave function  $\psi(\mathbf{r})$  for  $r \rightarrow \infty$  has the form

$$\psi^C(\mathbf{r}) \rightarrow e^{ikz+i\xi \ln[k(r-z)]} + f^C(\vartheta) \frac{e^{ikr-i\xi \ln kr}}{r}, \quad r \rightarrow \infty, \quad (1.76)$$

where  $f^C(\vartheta)$  is the *Coulomb scattering amplitude*

$$\begin{aligned} f^C(\vartheta) &= \frac{i}{2k} \sum_l (2l+1)(1-e^{2i\eta_l}) P_l(\cos \vartheta) \\ &= \frac{e^2}{Mv^2 \sin^2(\vartheta/2)} e^{-i\xi \ln \sin^2(\vartheta/2) + 2i\eta_0 + i\pi} \end{aligned} \quad (1.77)$$

( $\vartheta$  is the scattering angle in the centre-of-mass frame of the protons).

The expression (1.77) for the scattering amplitude does not take into account the indistinguishability of the protons. Since the protons have a spin equal to  $\frac{1}{2}$ , by virtue of the Pauli principle the complete wave function of a system of two protons must be anti-symmetric. When the total spin of the protons  $S = 0$ , the spatial part of the complete wave function is connected with the function  $\psi(\mathbf{r})$ , given by the expansion (1.73), by the relation  $\psi_s(\mathbf{r}) = \psi(\mathbf{r}) + \psi(-\mathbf{r})$ , and when the total spin  $S = 1$ , by the relation  $\psi_a(\mathbf{r}) = \psi(\mathbf{r}) - \psi(-\mathbf{r})$ . Since the replacement  $\vartheta \rightarrow \pi - \vartheta$ , the scattering amplitudes in the singlet ( $S = 0$ ) and triplet ( $S = 1$ ) states are connected with the amplitude  $f(\vartheta)$  calculated without taking account of the *Pauli principle* by the relations

$$f_s(\vartheta) = f(\vartheta) + f(\pi - \vartheta), \quad f_t(\vartheta) = f(\vartheta) - f(\pi - \vartheta). \quad (1.78)$$

The averaged differential cross-section for proton-proton scattering is in this case given by the formula

$$\sigma(\vartheta) = \frac{1}{4}|f_s(\vartheta)|^2 + \frac{3}{4}|f_t(\vartheta)|^2, \quad (1.79)$$

where the coefficients  $\frac{1}{4}$  and  $\frac{3}{4}$  characterize the weights of the singlet and triplet states.

Using the formula (1.77) for the amplitude  $f^C(\vartheta)$  and taking (1.78) and (1.79) into account, we obtain the following expression for the differential cross-section for proton-proton scattering with neglect of the nuclear interaction:

$$\sigma_C(\vartheta) = \left( \frac{e^2}{Mv^2} \right)^2 \left\{ \frac{1}{\sin^4(\vartheta/2)} + \frac{1}{\cos^4(\vartheta/2)} - \frac{\cos[\xi \ln \tan^2(\vartheta/2)]}{\sin^2(\vartheta/2) \cos^2(\vartheta/2)} \right\}. \quad (1.80)$$

The first term in (1.80) gives the number of protons scattered through an angle  $\vartheta$ ; the second determines the number of recoiling protons moving at an angle  $\vartheta$  to the initial beam; the last term describes the quantum effect of exchange of the colliding protons.

Experiments on proton-proton scattering have shown that the formula (1.80) gives an incorrect description of the scattering of protons. The experimentally observed number of protons scattered through an angle of  $90^\circ$  at energies of the order of 1 MeV is several times greater than the number given by the formula (1.80). Consequently, in addition to the Coulomb forces between protons, nuclear forces act, and their magnitude appreciably exceeds that of the electromagnetic interaction forces.

**Proton-proton scattering with allowance for the nuclear interaction.** We shall assume that the nuclear forces acting between protons, like the forces acting between neutrons and protons, are characterized by very short range. In this case, the nuclear interaction is manifested only in states with very small values of the relative angular momentum  $l$ . If the wavelength exceeds the range of the nuclear forces, then the nuclear interaction occurs only in an  $S$ -state ( $l = 0$ ).

The proton-proton scattering amplitude in the presence of the nuclear interaction is given by the formula (if the Pauli principle is not taken into account)

$$f(\vartheta) = (i/2k) \sum_l (2l+1) \{1 - e^{2i(\eta_l + \delta_l)}\} P_l(\cos \vartheta), \quad (1.81)$$

where  $\delta_l$  is the additional phase shift due to the nuclear interaction of the protons in a state with angular momentum  $l$ . Using the expression (1.77)

for the *Coulomb scattering* amplitude  $f(\vartheta)$ , we rewrite the formula (1.81) in the form

$$f(\vartheta) = f^C(\vartheta) + (i/2k) \sum_l (2l+1) e^{2in_l} (1 - e^{2i\delta_l}) P_l(\cos \vartheta). \quad (1.82)$$

If the energy of the incident proton is less than 10 MeV, all the phase shifts  $\delta_l$ , apart from  $\delta_0$ , can be assumed equal to zero. In this case, the scattering amplitude in which the nuclear interaction is taken into account only in the *S*-state is given by the formula

$$f(\vartheta) = f^C(\vartheta) + (1/k)(\sin \delta) e^{i\delta + 2i\eta_0} \quad (1.83)$$

(in the following we shall drop the index 0 from  $\delta_0$ ).

Clearly, the relations (1.78) are valid both for purely Coulomb scattering and for scattering in which the nuclear interaction is taken into account. In accordance with (1.78), it is not difficult to obtain the following expressions for the proton scattering amplitudes in the singlet and triplet states:

$$\left. \begin{aligned} f_s(\vartheta) &= \frac{e^2}{Mv^2} \left\{ \frac{e^{-i\xi \ln \sin^2(\vartheta/2)}}{\sin^2(\vartheta/2)} + \frac{e^{-i\xi \ln \cos^2(\vartheta/2)}}{\cos^2(\vartheta/2)} - \frac{4}{\xi} (\sin \delta) e^{i\delta} \right\} e^{2i\eta_0 + i\pi}, \\ f_t(\vartheta) &= \frac{e^2}{Mv^2} \left\{ \frac{e^{-i\xi \ln \sin^2(\vartheta/2)}}{\sin^2(\vartheta/2)} - \frac{e^{-i\xi \ln \cos^2(\vartheta/2)}}{\cos^2(\vartheta/2)} \right\} e^{2i\eta_0 + i\pi}. \end{aligned} \right\} \quad (1.84)$$

It can be seen that, for low proton energies, the nuclear interaction is manifested only in the singlet state.

With the nuclear interaction in the *S*-state taken into account, the proton-proton scattering cross-section, averaged over the spin states, has the form

$$\begin{aligned} \sigma(\vartheta) &= \left( \frac{e^2}{Mv^2} \right)^2 \left\{ \frac{1}{\sin^4(\vartheta/2)} + \frac{1}{\cos^4(\vartheta/2)} - \frac{\cos[\xi \ln \tan^2(\vartheta/2)]}{\sin^2(\vartheta/2) \cos^2(\vartheta/2)} \right. \\ &\quad - \frac{2}{\xi} \sin \delta \left[ \frac{\cos[\delta + \xi \ln \sin^2(\vartheta/2)]}{\sin^2(\vartheta/2)} + \frac{\cos[\delta + \xi \ln \cos^2(\vartheta/2)]}{\cos^2(\vartheta/2)} \right] \\ &\quad \left. + \frac{4}{\xi^2} \sin^2 \delta \right\}. \end{aligned} \quad (1.85)$$

In this expression, the interference term proportional to  $\sin \delta$  appears. Therefore, a comparison of the experimental data on proton-proton scattering with the theoretical formula (1.85) makes it possible to determine not only the magnitude but also the sign of the phase shift  $\delta$ . A positive sign of the phase shift corresponds to forces of attraction, and a negative sign to forces of repulsion. The experimental results show that the phase

shift  $\delta$  is positive, that is, attractive nuclear forces act between the protons in the  $S$ -state ( $l = 0$ ).

The proton-proton differential scattering cross-section (1.85) is symmetric about the angle  $90^\circ$  in the centre-of-mass frame. In the small-angle region, the Coulomb scattering is dominant. The interference between the nuclear and Coulomb scattering has a negative character, since the nuclear forces are attractive and the Coulomb forces repulsive. Therefore, in the range of angles in which the nuclear scattering begins to manifest itself, a characteristic interference minimum appears. At large angles, the nuclear scattering is dominant.

**Scattering length and effective range.** As in the case of scattering of a neutron by a proton, in the case of proton-proton scattering we can define a scattering length and an effective range. We write the equation for the radial function  $u \equiv u_0$  of two protons with the nuclear and Coulomb interactions taken into account:

$$u'' - (2\xi k/r)u - (M/\hbar^2)V(r)u = -k^2u. \quad (1.86)$$

Along with the function  $u$ , we introduce into the treatment a function  $v$  which coincides asymptotically with  $u$  for large  $r$  and is a solution of the equation

$$v'' - (2\xi k/r)v = -k^2v. \quad (1.87)$$

The equation (1.87) has two linearly independent solutions: one regular (vanishing at  $r = 0$ ) and the other irregular at the point  $r = 0$ . We shall denote these solutions by  $F$  and  $G$ ; at large distances  $r \rightarrow \infty$  they have the following form:

$$\left. \begin{aligned} F(r) &\sim \sin(kr - \xi \ln 2kr + \eta_0), \\ G(r) &\sim -\cos(kr - \xi \ln 2kr + \eta_0). \end{aligned} \right\} \quad (1.88)$$

For small values of  $r$ , the expansions

$$\begin{aligned} F(r) &\sim Ckr, \\ G(r) &\sim -(1/C) \left\{ 1 + \frac{r}{\rho} [\ln(r/\rho) + h(\xi) + 2\gamma - 1] \right\}, \end{aligned} \quad (1.89)$$

are valid, where

$$C^2 = \frac{2\pi\xi}{e^{2\pi\xi} - 1}, \quad \rho = \frac{\hbar^2}{Me^2}, \quad h(\xi) = \xi^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu^2 + \xi^2)} - \ln \xi - \gamma,$$

and  $\gamma = 0.57722\dots$  is Euler's constant.

We shall find a solution of eqn. (1.87) having the same asymptotic form at large  $r$  as the function  $u$  which is the solution of eqn. (1.86), that is,  $v \sim \sin(kr - \xi \ln 2kr + \eta_0 + \delta)$ , where  $\delta$  is the phase shift due to the nuclear interaction. We normalize the functions  $u$  and  $v$  in such a way that  $v(0) = 1$ . It is not difficult to show that the required solution  $v$  can be expressed in terms of the functions  $F$  and  $G$  in the following way:

$$v(r) = C\{F(r)\cot\delta - G(r)\}. \quad (1.90)$$

Proceeding then as in the derivation of (1.44), we obtain the analogous relation

$$[v'(r) - v'_0(r)]_{r \rightarrow 0} = k^2 \int_0^\infty dr(v_0 v - u_0 u), \quad (1.91)$$

where the function  $u_0$  and  $v_0$  correspond to zero energy.

According to the relations (1.90) and (1.89), at small values of  $r$  we have

$$v'(r) = (1/\rho)\{\ln(r/\rho) + 2\gamma + K(k)\}, \quad (1.92)$$

where we have introduced the notation

$$K(k) \equiv \pi \cot\delta / (e^{2\pi\xi} - 1) + h(\xi). \quad (1.93)$$

The quantity  $(1/\rho)K(k)$  plays the same role in the case of the interaction of two protons as  $k \cot\delta$  in the case of the interaction of a neutron with a proton. This quantity has a simple physical meaning. The first term appearing in the function  $(1/\rho)K(k)$  is the quantity  $k \cot\delta$  multiplied by the penetration factor  $C^2$  of the Coulomb barrier. The second term in  $(1/\rho)K(k)$  takes into account the infinite range of the Coulomb interaction.

Using (1.92), we represent the relation (1.91) in the form

$$(1/\rho)K(k) = -(1/a_p) + \frac{1}{2}k^2 r(0, E), \quad (1.94)$$

where

$$-1/a_p \equiv (1/\rho) \lim_{k \rightarrow 0} K(k), \quad (1.95)$$

$$r(0, E) \equiv 2 \int_0^\infty dr(v_0 v - u_0 u). \quad (1.96)$$

The quantity  $a_p$ , which does not depend on the proton energy, may be called the *proton-proton scattering length*. The quantity  $r(0, E)$  takes account of the fact that the range of the nuclear forces between two protons is finite and is introduced in exactly the same way as in the case of the interaction of

a neutron and a proton. The value of  $r(0, E)$  at zero energy of the protons is usually called the *effective range of the nuclear forces* and is denoted by  $r_{0p}$ :

$$r_{0p} = 2 \int_0^\infty dr (v_0^2 - u_0^2). \quad (1.97)$$

Replacing  $r(0, E)$  by  $r_{0p}$  in the formula (1.94) in the case of low energies, we obtain the relation (Landau and Smorodinskii, 1944)

$$(1/\rho)K(k) = -(1/a_p) + \frac{1}{2}k^2 r_{0p}, \quad (1.98)$$

which gives the dependence of the phase shift  $\delta$  on the proton energy in an approximation which is independent of the shape of the potential. (The function  $K(k)$  is connected with the phase shift  $\delta$  by the relation (1.93).) It follows from the formula (1.98) that if we plot the dependence of the function  $K(k)$  on the energy or  $k^2$ , we obtain a straight line defining two parameters: the scattering length  $a_p$  and the effective range  $r_{0p}$ . These parameters, which characterize the nuclear interaction of the protons in the low-energy region, can be found from the experimental data on the phase shift  $\delta$  as a function of energy.

The cross-section for scattering of protons by protons has been measured over a broad range of energies. Experimentally determined values of the quantity  $(1/\rho)K(k)$  as a function of the proton energy are shown in Fig. 1.2. We see that in the low-energy region there is indeed a linear dependence of  $K(k)$  on the energy, in accordance with the formula (1.98). Comparison of the experimental data with the formula (1.98) leads to the following values for the scattering length  $a_p$  and the effective range  $r_{0p}$ :

$$a_p = -(7.806 \pm 0.003) \times 10^{-13} \text{ cm}, \quad r_{0p} = (2.79 \pm 0.01) \times 10^{-13} \text{ cm}. \quad (1.99)$$

The negative sign of the proton-proton scattering length  $a_p$  implies that a bound state of a system of two protons is impossible.

It is interesting to compare the parameters  $r_{0p}$  and  $a_p$  characterizing the nuclear interaction of two protons with the parameters  $r_{0s}$  and  $a_s$  characterizing the nuclear interaction between a neutron and a proton in the singlet state. The values of the effective ranges of the nuclear forces between two protons  $r_{0p}$  and between a neutron and a proton in the singlet state  $r_{0s}$  coincide within the limits of the experimental errors.

A direct comparison of the proton-proton scattering length  $a_p$  and the singlet-state neutron-proton scattering length  $a_s$  is not possible. To compare them, it is necessary to exclude the effect of the Coulomb interaction on the scattering length  $a_p$ . Since, within the range of the nuclear forces, the

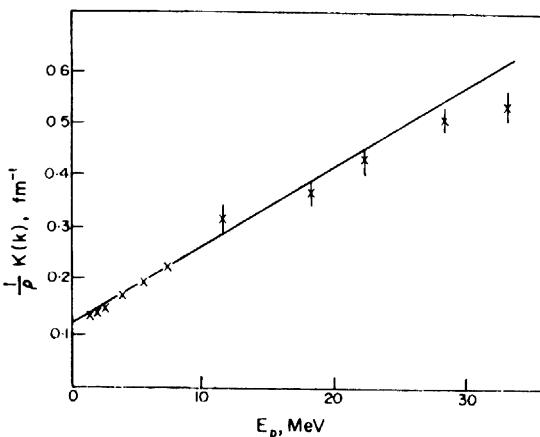


Figure 1.2. Experimental dependence of the values of  $(1/\rho)K(k)$  on the proton energy. The straight line corresponds to the approximation independent of the shape of the potential.

Coulomb forces are a small perturbation, it is possible to obtain (Jackson and Blatt, 1950) an approximate relation between the proton-proton scattering length  $a_p$  and the equivalent length  $a'_p$  for proton-proton scattering in the absence of the Coulomb force:

$$1/a'_p = (1/a_p) + (1/\rho)[\ln(\rho/r_{0p}) - \frac{1}{3}].$$

By substituting the experimental values of the quantities  $a_p$  and  $r_{0p}$ , we find

$$a'_p = -(17.1 \pm 0.1) \times 10^{-13} \text{ cm}. \quad (1.99')$$

In the case of neutron-neutron scattering, the experimental values of the scattering length  $a_n$  and the effective radius  $r_{on}$  are given by

$$a_n = -(18.5 \pm 0.4) \times 10^{-13} \text{ cm}, \quad r_{on} = (2.8 \pm 0.1) \times 10^{-13} \text{ cm}. \quad (1.100)$$

Within the limits of the experimental error, the quantities  $a'_p$  and  $a_n$  correspond to the singlet-state neutron-proton scattering length (1.56). It can be assumed, therefore, that the nuclear interaction between a neutron and a proton in the singlet state is the same as that between two protons and two neutrons.

### 1.7. Non-central Nuclear Forces

**The potential of non-central forces.** Our preceding treatment has been based on the assumption that the nuclear forces are central in character;

this enabled us to explain certain properties of a two-nucleon system, in particular, the binding energy of the deuteron and the scattering of neutrons and protons at low energies. The existence of an electric quadrupole moment and the non-additivity of the magnetic moments of the neutron and proton in the deuteron, however, point to the approximate nature of this assumption. Indeed, in the case of central forces, the ground state of the deuteron should be an *S*-state, described by a spherically symmetric wave function. The presence of the electric quadrupole moment and the non-additivity of the magnetic moments of the particles in the deuteron mean that the ground state of the deuteron is not a pure *S*-state, but is a superposition of an *S*-state and states with non-zero angular momenta that lead to a violation of the spherical symmetry in the charge distribution and to the appearance of an additional magnetic moment. Such a complex ground state is possible only if the interaction between the particles is not purely central. This means that the nuclear interaction between a neutron and a proton must be described by a potential which depends not only on the distance between the particles but also on the orientation of the spins of the particles with respect to the relative position vector. Allowance for these *non-central or tensor* forces enables us to explain the existence of the electric quadrupole moment and the extra magnetic moment in the deuteron.

First we shall elucidate the form that the nuclear interaction potential has in the presence of non-central forces. In the case of low nucleon velocities, to which we shall confine ourselves here, the nuclear interaction potential of two nucleons depends, clearly, on the relative position vector  $\mathbf{r}$  between the nucleons and the two spin vectors  $\boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma}_2$  of the nucleons. The nuclear interaction potential  $V$  is invariant with respect to spatial rotations and reflections; therefore, only the invariant combinations that can be formed from the quantities  $\mathbf{r}$ ,  $\boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma}_2$  can appear in the potential  $V$ . Since the Pauli matrices  $\boldsymbol{\sigma}$  satisfy the relations

$$\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k, \quad (1.101)$$

where  $i, j, k = x, y, z$  and  $\varepsilon_{ijk}$  is the completely antisymmetric unit tensor, these invariant combinations are linear in  $\boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma}_2$ . However, the quantities  $\mathbf{n} \cdot \boldsymbol{\sigma}_1$  and  $\mathbf{n} \cdot \boldsymbol{\sigma}_2$  where  $\mathbf{n}$  is the unit vector in the direction of  $\mathbf{r}$ , cannot occur separately in the expression for the potential, since  $\boldsymbol{\sigma}$  is a pseudo-vector and the products  $\mathbf{n} \cdot \boldsymbol{\sigma}_1$  and  $\mathbf{n} \cdot \boldsymbol{\sigma}_2$  are therefore not invariants and change their sign on the reflection of the coordinates. From this it follows that only the quantities  $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$  and  $(\mathbf{n} \cdot \boldsymbol{\sigma}_1)(\mathbf{n} \cdot \boldsymbol{\sigma}_2)$  can occur in the potential  $V$ , and these must occur linearly. In place of  $(\mathbf{n} \cdot \boldsymbol{\sigma}_1)(\mathbf{n} \cdot \boldsymbol{\sigma}_2)$ , it is convenient to introduce the quantity

$$S_{12} = 3(\mathbf{n} \cdot \boldsymbol{\sigma}_1)(\mathbf{n} \cdot \boldsymbol{\sigma}_2) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \quad (1.102)$$

whose value averaged over the directions of the vector  $\mathbf{n}$  is equal to zero.

Thus, the nuclear interaction potential between two nucleons can be represented in the form (Eisenbud and Wigner, 1941)

$$V = V_1(r) + V_2(r)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_3(r)S_{12}, \quad (1.103)$$

where  $V_1(r)$ ,  $V_2(r)$  and  $V_3(r)$  are certain functions depending on the distance between the particles. The expression (1.103) is the most general expression if we disregard the exchange character of the nuclear forces. In the formula (1.103), the first term describes the central forces that do not depend on the spins, the second term describes the central forces depending on the spins, and the third term describes the tensor forces.

**Classification of the states of the neutron-proton system.** We shall consider the question of the classification of the states of the neutron-proton system when the interaction between the neutron and proton is described by the potential (1.103). We note that the potential energy of the neutron-proton interaction, given by the formula (1.103), is invariant with respect to transformations of spatial rotation and reflection, and also with respect to interchange of the spins of the two particles.

The invariance of the interaction energy with respect to rotations implies the conservation of the total angular momentum, composed of the orbital and spin angular momenta, of the whole system. Therefore, the neutron-proton system can be characterized by the quantum numbers  $J$  and  $M$ , defining the *square of the total angular momentum* and the *projection of the total angular momentum* respectively. We note that since, in the general case, the orbital and spin angular momenta are not separately conserved, it is no longer possible to classify the states of the system by the values of the orbital angular momentum.

The invariance of the interaction energy with respect to reflection implies the conservation of the parity  $w$  of the neutron-proton system.

The invariance of the interaction energy with respect to interchange of the spins of the particles makes it possible to divide the states of the neutron-proton system into states that are symmetric and states that are anti-symmetric in the spin variables. The symmetric states are triplet states and correspond to the value  $S = 1$  of the spin of the system. The anti-symmetric states are singlet states and correspond to the spin value  $S = 0$ . Consequently, in a neutron-proton system described by the potential (1.103), the square of the total spin of the particles is conserved. One can also convince oneself of this by verifying directly that the operator  $S^2$  of the square of the spin commutes with the operators  $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1$  and  $S_{12}$ . For this, we need only observe that the operators  $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1$  and  $S_{12}$  can be represented in the form

TABLE 1.1.

J	Singlets		Triplets	
	Even	Odd	Even	Odd
0	$^1S_0$			$^3P_0$
1		$^1P_1$	$^3S_1 + ^3D_1$	$^3P_1$
2	$^1D_2$		$^3D_2$	$^3P_2 + ^3F_2$

$$\left. \begin{aligned} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 &= 2\mathbf{S}^2 - 3; \\ S_{12} &= 6(\mathbf{n} \cdot \mathbf{S})^2 - 2\mathbf{S}^2, \end{aligned} \right\} \quad (1.104)$$

where  $\mathbf{S}$  is the total spin of the system:  $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$ . Note, however, that the projection of the total spin on the  $z$ -axis is not a constant of the motion.

Thus, we see that a system consisting of a neutron and a proton can be characterized by specifying the quantum numbers  $J$ ,  $S$ ,  $w$  and  $M$ .

It follows from (1.104) that  $S_{12} = 0$  in a singlet state; this means that for  $S = 0$  the nuclear forces are central. Therefore, in a singlet state the orbital angular momentum  $l$  is conserved and states with  $S = 0$  can be classified by the quantity  $l$ . The triplet state, which can be characterized by determining the value of the total angular momentum  $J$ , is a superposition of states with values of  $l$  determined by the rules for the addition of angular momenta. For example, the triplet state with  $J = 1$  is a superposition of the states  $^3S_1$ ,  $^3P_1$  and  $^3D_1$ . The *parity of the state* is connected with the value of the orbital angular momentum by the relation  $w = (-1)^l$ . Therefore, the states  $^3S_1$  and  $^3D_1$  with even values of  $l$  and the state  $^3P_1$  with an odd value of  $l$  are characterized by opposite parity.

In Table 1.1. we give the classification of the first few states of the neutron proton system

**Ground state of the deuteron with inclusion of non-central forces.** We shall consider the question of the ground state of the deuteron when the non-central character of the nuclear forces is taken into account. It follows from the experimental data that the total angular momentum of the deuteron  $J = 1$ . On the other hand, in the ground state an  $S$ -state is undoubtedly present, since the quadrupole moment is comparatively small and, consequently, the asymmetry in the charge distribution is slight. Therefore, the deuteron ground state is the triplet state  $^3S_1 + ^3D_1$  (see Table 1.1). The parity of this state is equal to unity. Thus, the ground state

of the deuteron is characterized by the quantum numbers  $J = 1$ ,  $S = 1$  and  $w = 1$ , and is a superposition of  $S$ - and  $D$ -states.

The ground-state wave function of the deuteron can be presented in the form

$$\Psi_0 = \Psi_S + \Psi_D, \quad (1.105)$$

where  $\Psi_S$  and  $\Psi_D$  describe the  $S$ - and  $D$ -states and are products of radial functions and functions depending on the angles and spin variables.

The wave function  $\mathcal{Y}$ , which depends on the angles and spin variables and describes the state of the system with total angular momentum  $J$  and angular momentum component  $M$ , can be expressed in terms of the eigenfunctions  $Y_{lm}$  and  $\chi_{s\mu}$  of the orbital and spin angular momenta:

$$\mathcal{Y}_{lSJM} = \sum_{m+\mu=M} (lmS\mu|JM) Y_{lm} \chi_{s\mu}, \quad (1.106)$$

where  $l$  and  $S$  are the quantum numbers of the squares of the orbital and spin angular momenta,  $m$  and  $\mu$  are their components, and  $(lmS\mu|JM)$  are the *Clebsch-Gordan coefficients*. Therefore, the functions  $\Psi_S$  and  $\Psi_D$  can be presented in the form

$$\Psi_S = \frac{u(r)}{r} \mathcal{Y}_{011M}, \quad \Psi_D = \frac{w(r)}{r} \mathcal{Y}_{211M}, \quad (1.107)$$

where  $u$  and  $w$  are the radial functions of the  $S$ - and  $D$ -states. The functions  $\mathcal{Y}_{011M}$  and  $\mathcal{Y}_{211M}$  have the form

$$\left. \begin{aligned} \mathcal{Y}_{011M} &= Y_{00} \chi_{1M}; \\ \mathcal{Y}_{211M} &= \left. \begin{aligned} &\sqrt{\frac{(3+M)(2+M)}{4 \times 5}} Y_{2,M+1} \chi_{1-1} \\ &- \sqrt{\frac{(2-M)(2+M)}{2 \times 5}} Y_{2M} \chi_{10} \\ &+ \sqrt{\frac{(2-M)(3-M)}{4 \times 5}} Y_{2,M-1} \chi_{11} \end{aligned} \right\} \end{aligned} \right\} \quad (1.108)$$

Since the  $\mathcal{Y}_{lSJM}$  are orthonormal functions, it follows from the normalization condition for the *wave function of the deuteron ground state*

$$\int d^3r |\Psi_0|^2 = 1 \quad (1.109)$$

that the relation  $\int_0^\infty dr (u^2 + w^2) = 1$  holds. Clearly, the quantities  $p_S = \int_0^\infty dr u^2$  and  $p_D = \int_0^\infty dr w^2$  can be interpreted as the probabilities of

finding the deuteron in the  $S$ - and  $D$ -states ( $p_S + p_D = 1$ ). It can be verified directly without difficulty that the relations

$$\left. \begin{aligned} S_{12}\mathcal{Y}_{011M} &= \sqrt{8}\mathcal{Y}_{211M}; \\ S_{12}\mathcal{Y}_{211M} &= \sqrt{8}\mathcal{Y}_{011M} - 2\mathcal{Y}_{211M}, \end{aligned} \right\} \quad (1.110)$$

hold; therefore, the wave function of the deuteron can be written in the form (*the function of Rarita and Schwinger (1941)*)

$$\Psi_0 = \frac{1}{\sqrt{4\pi}} \left\{ \frac{u(r)}{r} + \frac{1}{\sqrt{8}} \frac{w(r)}{r} S_{12} \right\} \chi_{1M}, \quad (1.111)$$

where  $\chi_{1M}$  is the spin function of the system with the spin component coinciding with the component  $M$  of the total angular momentum of the deuteron.

The Schrödinger equation for a neutron-proton system in a triplet state with energy  $E$  has the form

$$\{-(\hbar^2/M)\Delta + V_C + V_T S_{12} - E\}\Psi = 0, \quad (1.112)$$

where the central part  $V_C$  and the tensor part  $V_T$  of the potential, according to (1.103), are  $V_C = V_1 + V_2$  and  $V_T = V_3$ . Substituting the function  $\Psi$  in the form (1.105) into the Schrödinger equation (1.112) and using the relations (1.110), it is not difficult to obtain the following system of coupled differential equations for the radial functions  $u$  and  $w$ :

$$\left. \begin{aligned} \frac{\hbar^2}{M} \frac{d^2u}{dr^2} - V_C u + Eu &= \sqrt{8}V_T w; \\ \frac{\hbar^2}{M} \left( \frac{d^2w}{dr^2} - \frac{6}{r^2}w \right) - V_C w + 2V_T w + Ew &= \sqrt{8}V_T u. \end{aligned} \right\} \quad (1.113)$$

In eqns. (1.113), two potentials  $V_C(r)$  and  $V_T(r)$  appear; to characterize these in the low-energy region, it is necessary to specify four parameters (the effective depth of the central potential  $V_C$ , the effective depth of the tensor potential  $V_T$ , and the ranges  $r_0$  and  $r_T$  of these potentials). However, even in the simplest case when the potentials are chosen in the form of square wells, it is not possible to find a solution of the system (1.113) in explicit form.

Outside the range of the nuclear forces, the equations for the radial functions  $u$  and  $w$  are uncoupled. In this case, for the neutron proton bound state ( $E = -\varepsilon$ ) the solutions have the form

$$\left. \begin{aligned} u &= A_S e^{-\alpha r}; \\ w &= A_D \left\{ 1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2} \right\} e^{-\alpha r}, \end{aligned} \right\} \quad (1.114)$$

where  $\alpha = \sqrt{M\varepsilon/\hbar}$ , and  $A_S$  and  $A_D$  are normalization coefficients.

In order to find the *ratio of D- to S-wave normalization coefficients* of the radial wave functions,

$$\eta = \frac{A_D}{A_S},$$

we regard the *S*-matrix as an analytic function of the complex wave number and find the value of this function near the pole (to be more exact, its residue at the pole), associated with the  $n - p$  bound state, by analyzing the angular dependence of the proton-neutron scattering cross-section. According to (Stoks, van Campen, Spit and de Swart, 1988), we have

$$\eta = 0.0271 \pm 0.0002.$$

The relevant numerical value of the *S*-wave normalization coefficient is given by

$$A_S = (0.8838 \pm 0.0004) \times 10^{-13} \text{ cm}^{-1/2}.$$

The asymptotic expression for the radial function  $u$  of the *S*-state is valid at distances  $r$  greater than the ranges of both the central and the tensor forces; the asymptotic expression for the radial function  $w$  of the *D*-state is valid for  $r > r_T$ . In fact, in the first equation we can discard the terms  $V_C u$  and  $V_T w$  only when  $r > r_0$  and  $r > r_T$ . In the second equation, the term  $V_C w$  can be neglected as soon as  $r > r_T$ , since it is small compared with the centrifugal energy. At small  $r$ , the radial functions  $u$  and  $w$  behave as  $u \sim r$  and  $w \sim r^3$  ( $r \rightarrow 0$ ). The function  $w$  has a sharp maximum at  $r \approx r_T$ , since the expression (1.114) for  $w$  falls off like  $1/r^2$  for  $r_T < r < 1/\alpha$ . Therefore, the principal contribution to the integral  $p_D = \int_0^\infty dr w^2$  determining the weight of the *D*-state is made by distances  $r \approx r_T$ .

**Magnetic moment of the deuteron.** The presence of a *D*-wave in the ground state of the deuteron leads to the existence of the deuteron quadrupole moment and to the deviation of the deuteron magnetic moment from the sum of the magnetic moments of the neutron and proton. In fact, because of the presence of the *D*-wave in the ground state, the spherical symmetry in the charge distribution is disturbed and this leads to the appearance of the quadrupole moment. There is a magnetic moment associated with the orbital angular momentum, and it is this magnetic moment which is responsible for the deviation of the deuteron magnetic moment from the sum of the neutron and proton magnetic moments.

The *deuteron magnetic moment operator*  $\mu$  (in nuclear magnetons) can be represented in the form

$$\mu = \mu_n \sigma_n + \mu_p \sigma_p + \frac{1}{2} l, \quad (1.115)$$

where  $\mu_n$  and  $\mu_p$  are the magnetic moments of the neutron and proton, and  $\mathbf{l}$  is the angular momentum of the relative motion. (The factor  $\frac{1}{2}$  multiplying  $\mathbf{l}$  takes into account the fact that only the proton has charge.) Introducing the total spin  $\mathbf{S}$  of the system, we rewrite the magnetic moment  $\boldsymbol{\mu}$  of the deuteron in the form

$$\boldsymbol{\mu} = (\mu_n + \mu_p) \mathbf{S} + \frac{1}{2}(\mu_n - \mu_p)(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_p) + \frac{1}{2}\mathbf{l}.$$

Noting that the mean value of the operator  $\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_p$  in the ground state is equal to zero, we obtain the following expression for the effective value of  $\boldsymbol{\mu}$ :

$$\boldsymbol{\mu} = (\mu_n + \mu_p) \mathbf{J} - (\mu_n + \mu_p - \frac{1}{2})\mathbf{l}, \quad (1.116)$$

where  $\mathbf{J} = \mathbf{l} + \mathbf{S}$  is the operator of the total angular momentum of the system.

Since the orbital angular momentum  $\mathbf{l}$  is not conserved, to determine the magnetic moment of a deuteron in a state with a definite value of  $\mathbf{J}$  we must replace the vector  $\mathbf{l}$  in the formula (1.116) by its component along the direction of the *total angular momentum  $\mathbf{J}$* :

$$\mathbf{l} \rightarrow \frac{\mathbf{l} \cdot \mathbf{J}}{\mathbf{J}^2} \mathbf{J} = \frac{\mathbf{J}^2 + \mathbf{l}^2 - \mathbf{S}^2}{2\mathbf{J}^2} \mathbf{J}.$$

As a result, we obtain the following expression for the component of the magnetic moment of the deuteron in the direction of the total angular momentum  $\mathbf{J}$ :

$$\boldsymbol{\mu} = \left\{ \mu_n + \mu_p - \left( \mu_n + \mu_p - \frac{1}{2} \right) \frac{\mathbf{J}^2 + \mathbf{l}^2 - \mathbf{S}^2}{2\mathbf{J}^2} \right\} \mathbf{J}. \quad (1.117)$$

We shall average this expression over the ground state of the deuteron. Since  $\mathbf{J}^2$  and  $\mathbf{S}^2$  are constants of the motion, we have

$$\langle \mathbf{J}^2 \rangle = J(J+1), \quad \langle \mathbf{S}^2 \rangle = S(S+1).$$

Taking into account that the ground state of the deuteron is a superposition of *S*- and *D-waves* with *weights*  $p_S$  and  $p_D$  we find for the mean value of the square of the orbital angular momentum  $\mathbf{l}^2$

$$\langle \mathbf{l}^2 \rangle = 0 \times p_S + 2 \times 3p_D = 6p_D.$$

Projecting the vector (1.117) on to the  $z$ -axis and determining the mean value of this quantity for the ground state of the deuteron ( $J = S = 1$ ) with  $M = J = 1$ , we obtain the *magnetic moment of the deuteron*

$$\mu_d = \mu_n + \mu_p - \frac{3}{2}(\mu_n - \mu_p - \frac{1}{2})p_D. \quad (1.118)$$

The formula (1.118) enables us to determine the weight of the  $D$ -state in the ground state of the deuteron from the experimental values of the magnetic moments  $\mu_n$ ,  $\mu_p$  and  $\mu_d$ . Using the values (1.1) and (1.3), we obtain for the *weight of the  $D$ -wave in the ground state of the deuteron* the value

$$p_D = 0.04. \quad (1.119)$$

In deriving the formula (1.118), we have neglected relativistic effects and also effects associated with exchange currents. Allowance for these corrections can increase the weight  $p_D$  of the  $D$ -wave by an amount of the order of 0.03. We note that the above mentioned effects can be allowed for only under certain assumptions concerning the interaction nature. Therefore, the quantity  $p_D$  turns out to be dependent on the assumption. Unlike the parameter  $\eta$ , the quantity  $p_D$  cannot be obtained immediately from the experiment.

**Quadrupole moment of the deuteron.** To conclude this section, we consider the *electric quadrupole moment of the deuteron*. The quadrupole-moment tensor of a system of equal charges is given by the formula

$$Q_{ij} = \sum (3x_i x_j - \mathbf{r}^2 \delta_{ij}), \quad (1.120)$$

where  $\mathbf{r}$  is the position vector of a charge; the summation is taken over all the charges. The quadrupole moment  $Q$  is defined as the mean value of the component  $Q_{zz}$  of the tensor in the state with  $J_z = J$ , where  $J$  is the total angular momentum of the system. In the case of the deuteron, this component of the quadrupole moment is

$$Q_{zz} = \frac{1}{4}(3z^2 - \mathbf{r}^2) = \sqrt{\pi/5} \mathbf{r}^2 Y_{20}(\vartheta). \quad (1.121)$$

The factor  $\frac{1}{4}$  being connected with the fact that the position vector  $\mathbf{r}_p$  of the proton is equal to half the relative position vector  $\mathbf{r}$ . The quadrupole moment  $Q$  of the deuteron is defined as the mean value of (1.121) in the ground state (1.105) of the deuteron:

$$Q = (\Psi_S, Q_{zz} \Psi_S) + 2(\Psi_S, Q_{zz} \Psi_D) + (\Psi_D, Q_{zz} \Psi_D).$$

Noting that the first term in this expression vanishes because of the spherical symmetry of the function  $\Psi_S$  and neglecting the third term because of the small weight of the  $D$ -wave, we obtain for the quadrupole moment of the deuteron the expression

$$Q \approx (\sqrt{2}/10) \int_0^\infty dr r^2 uw. \quad (1.122)$$

Since the integrand in (1.122) contains a factor  $r^2$ , the region of small values of  $r$  makes a small contribution to the integral; therefore, in estimating the magnitude of  $Q$ , we can make use of the asymptotic expressions (1.114), valid outside the range of the nuclear forces, for the radial functions.

Consequently,

$$Q \approx \frac{\sqrt{2}}{10} A_S A_D \int_0^\infty dr r^2 \left(1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2}\right) e^{-2\alpha r} = \frac{1}{\sqrt{8}} \frac{A_S A_D}{\alpha^3}. \quad (1.123)$$

We note that  $A_D = \eta A_S$  and obtain

$$Q = \frac{1}{\sqrt{2}} \eta R_d^2.$$

We substitute the numerical value for  $\eta$  and thus obtain for  $Q$  the number that is in agreement with (1.4). If we take into account the small weight of the  $D$ -wave, the normalization factor of the  $S$ -wave can be assumed to be the same as in the case of central forces,  $A_S = \sqrt{4\pi}C \approx \sqrt{2}\alpha$ . By using this value of  $A_S$ , we can relate the normalization factor of the  $D$ -wave to the magnitude of the quadrupole moment of the deuteron:

$$A_D \approx 2\alpha^{5/2} Q. \quad (1.124)$$

Knowing the weight  $p_D$  of the  $D$ -wave and the magnitude of the quadrupole moment  $Q$ , we can estimate the magnitude of the range of the tensor forces. Indeed, the principal contribution to the integral  $p_D = \int_0^\infty dr w^2$  determining the weight of the  $D$ -wave is made by values of  $r$  close to  $r_T$ . Assuming that the contributions of the inner and outer regions are the same and using the asymptotic expression (1.114) for  $w$ , it is not difficult to estimate the magnitude of  $p_D$  for  $r_T\alpha < 1$ :

$$p_D \approx 2A_D^2 \int_{r_T}^\infty dr \left(1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2}\right)^2 e^{-2\alpha r} \approx 24 \frac{\alpha Q^2}{r_T^3}. \quad (1.125)$$

Putting  $p_D = 0.04$  and  $Q = 2.86 \times 10^{-27} \text{ cm}^2$  in this relation, we obtain for the range of the tensor nuclear forces

$$r_T \approx 3 \times 10^{-13} \text{ cm}. \quad (1.126)$$

According to (1.125), for a given value of the quadrupole moment  $Q$ , a small weight of the  $D$ -wave is incompatible with a very small value of the range  $r_T$  of the tensor forces.

### 1.8. Exchange Forces

**Scattering of neutrons by protons at high energies and the exchange nature of nuclear forces.** In treating the scattering of slow neutrons by protons, we assumed that the nuclear forces are described by a potential depending on the relative position vector between the particles and on the spins of the particles. Starting from such an assumption, we can completely explain the principal features of the scattering of neutrons by protons in the low-energy region. It has been found that, if the wavelength of the particles is of the order of or greater than the range of the nuclear forces, the scattering is entirely insensitive to the dependence of the interaction on the distance. We might expect that by increasing the energy of the interacting particles we should be able to obtain more detailed information on the radial dependence of the nuclear potential. But in fact, using a potential of the form of (1.103) it is completely impossible to explain the angular dependence of the scattering of neutrons by protons at high energies.

In fact, if the only forces acting between a neutron and a proton could be described by the potential (1.103), then, at sufficiently high energies, a large number of neutrons would be scattered through small angles in the centre-of-mass frame, and the recoiling protons would move in the opposite direction (in the laboratory frame, the protons would move principally at right angles to the incident neutron beam). But experiments on the scattering of neutrons by protons at energies greater than a few tens of MeV show that the differential cross-section for scattering of neutrons by protons has, in the centre-of-mass frame, a maximum both in the region of small scattering angles and in the region of angles close to  $180^\circ$ . In order to explain this property of the scattering, it is necessary to assume that, along with forces describable by the potential (1.103), additional forces, associated with exchange of the particles, act between the neutron and the proton. These forces may be called *exchange forces*, to distinguish them from the ordinary non-exchange forces. We shall show that, by introducing the exchange forces, we can explain the increase in the scattering cross-section at high energies in the large-angle region. If the energy of the neutron exceeds the effective energy of the nuclear interaction between the proton and neutron, the differential cross-section for scattering of a neutron by proton in the centre-of-mass frame can be determined by means of perturbation theory:

$$d\sigma = \frac{M^2}{16\pi^2\hbar^4} \left| \int d^3r \psi'^*(\mathbf{r}) V \psi(\mathbf{r}) \right|^2 do, \quad (1.127)$$

where  $\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$  and  $\psi'(\mathbf{r}) = e^{i\mathbf{k}'\cdot\mathbf{r}}$  are the wave functions of the relative motion before and after the scattering ( $\mathbf{k}$  and  $\mathbf{k}'$  are the wave-vectors),

$V$  is the nuclear interaction potential between the particles, and  $do$  is the element of solid angle containing the wave-vector after the scattering.

In the case of ordinary forces, the potential  $V$  is a function of the position vector  $\mathbf{r}$ , so that the *differential scattering cross-section has the form*

$$d\sigma = \frac{M^2}{16\pi^2\hbar^4} \left| \int d^3\mathbf{r} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) \right|^2 do. \quad (1.128)$$

The integral occurring in this expression differs appreciably from zero if  $|\mathbf{k} - \mathbf{k}'|r_0 \equiv 2k \sin(\vartheta/2)r_0 \lesssim 2\pi$ , where  $\vartheta$  is the scattering angle and  $r_0$  is the range of the nuclear forces. Thus, in the case of ordinary forces, the scattering is mainly through angles  $\vartheta \lesssim \lambda/r_0$ , where  $\lambda = 2\pi/k$  is the wavelength of the relative motion. For sufficiently high energies,  $\lambda \ll r_0$  so that  $\vartheta_{\text{eff}} \ll 1$ .

We now consider scattering in the case of exchange forces. We represent the *potential  $V$  for exchange forces* in the form

$$V = V(\mathbf{r})P_r, \quad (1.129)$$

where  $P_r$  is the particle-exchange operator. The action of this operator on the function  $\psi(\mathbf{r})$  reduces to replacing  $\mathbf{r}$  by  $-\mathbf{r}$ :

$$P_r \psi(\mathbf{r}) = \psi(-\mathbf{r}). \quad (1.130)$$

Therefore, the differential cross-section for scattering due to exchange forces characterized by the potential (1.129) is given by the formula

$$d\sigma = \frac{M^2}{16\pi^2\hbar^4} \left| \int d^3\mathbf{r} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) \right|^2 do. \quad (1.131)$$

This expression differs appreciably from zero when  $|\mathbf{k} + \mathbf{k}'|r_0 \lesssim 2\pi$ . Consequently, in the case of scattering under the influence of exchange forces, the main role is played by scattering angles for which the quantity  $|\mathbf{k} + \mathbf{k}'|$  is small, that is, by scattering angles close to  $180^\circ$ . This result is easily understood if we assume that the incident particle is deflected through a small angle in the case of the exchange forces too, but that the nature of the colliding particles changes in the scattering process, that is, the neutron is transformed into a proton, and the proton into a neutron. Consequently, the distribution of neutrons in exchange scattering should be the same as the distribution of recoiling protons in ordinary scattering, and vice versa.

In the general case of scattering under the influence of ordinary and exchange forces, the differential cross-section has two maxima, corresponding to the forward and backward directions; the relative magnitudes of these maxima are determined by the weights of the ordinary and exchange forces

in the interaction between the particles. The experimental data on the scattering of neutrons by protons at energies of a few tens of MeV show that the ordinary and exchange forces have approximately equal weight.

According to the formulae (1.128) and (1.131), which were obtained using perturbation theory, the total cross-section for scattering of a neutron by a proton at high energies should be inversely proportional to the energy, irrespective of the nature of the nuclear forces.

**Exchange operators.** In the general case, the potential characterizing the exchange interaction of two particles can be represented in the form of a product of some function depending on the coordinates of the particles with the exchange operator, which interchanges the coordinates of the interacting particles. Since the neutrons and protons are characterized not only by their spatial coordinates but also by spin coordinates, there exist different possibilities of exchange, corresponding to the different forms of exchange forces. Along with forces that lead to a complete exchange of the coordinates of the interacting particles, forces may exist which are associated with the exchange of the spatial or spin coordinates separately. Usually, the different exchange forces are known by the names of the investigators who first studied them (Heisenberg, 1932; Majorana, 1933; Bartlett, 1936).

Let  $\psi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2)$  be the wave function of a two-particle system, depending on the spatial and spin coordinates  $\mathbf{r}_1, \sigma_1$  and  $\mathbf{r}_2, \sigma_2$  of the particles. Clearly, we can introduce three different exchange operators,  $\mathbf{P}_r$ ,  $\mathbf{P}_\sigma$  and  $\mathbf{P}_{r\sigma}$ , the action of which on the wave function reduces respectively to the interchange of the spatial coordinates, the interchange of the spin coordinates and the simultaneous interchange of the spatial and spin coordinates:

$$\left. \begin{aligned} \mathbf{P}_r \psi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2) &= \psi(\mathbf{r}_2, \sigma_1; \mathbf{r}_1, \sigma_2); \\ \mathbf{P}_\sigma \psi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2) &= \psi(\mathbf{r}_1, \sigma_2; \mathbf{r}_2, \sigma_1); \\ \mathbf{P}_{r\sigma} \psi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2) &= \psi(\mathbf{r}_2, \sigma_2; \mathbf{r}_1, \sigma_1). \end{aligned} \right\} \quad (1.132)$$

The operator  $\mathbf{P}_r$  interchanging the spatial coordinates of the particles is usually called the *Majorana exchange operator*, the operator  $\mathbf{P}_\sigma$  interchanging the spin coordinates of the particles is called the *Bartlett exchange operator*, and the operator  $\mathbf{P}_{r\sigma}$  interchanging both the spatial and the spin coordinates of the particles is called the *Heisenberg exchange operator*. Obviously,

$$\mathbf{P}_{r\sigma} = \mathbf{P}_r \mathbf{P}_\sigma; \quad (1.133)$$

$$\mathbf{P}_r^2 = \mathbf{P}_\sigma^2 = \mathbf{P}_{r\sigma}^2 = 1. \quad (1.134)$$

According to (1.134), the exchange operators  $\mathbf{P}_r$ ,  $\mathbf{P}_\sigma$ , and  $\mathbf{P}_{r\sigma}$  have the eigenvalues  $\pm 1$  only. Interchange of the coordinates  $\mathbf{r}_1$  and  $\mathbf{r}_2$  is equivalent to a change of sign of the relative coordinate  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ . Therefore, for a system consisting of two particles, the operator  $\mathbf{P}_r$  coincides with the spatial

inversion operator and, consequently, the eigenvalues of the operator  $\mathbf{P}_r$  are determined by the magnitude  $l$  of the relative angular momentum:

$$\mathbf{P}_r = (-1)^l. \quad (1.135)$$

The eigenvalues of the operator  $\mathbf{P}_\sigma$  are determined by the symmetry of the spin part of the wave function. Since a system of two nucleons can be in either a triplet state ( $S = 1$ ), with a corresponding symmetric spin function, or in a singlet state ( $S = 0$ ), with a corresponding anti-symmetric spin function, the eigenvalues of  $\mathbf{P}_\sigma$  can be related to the magnitude  $S$  of the total spin of the system:

$$\mathbf{P}_\sigma = (-1)^{S+1}. \quad (1.136)$$

The eigenvalues of the operator  $\mathbf{P}_{r\sigma}$  are determined by the quantities  $l$  and  $S$ :

$$\mathbf{P}_{r\sigma} = (-1)^{l+S+1}. \quad (1.137)$$

**Nuclear interaction potential with inclusion of exchange forces.** Using the exchange operators  $\mathbf{P}_r$ ,  $\mathbf{P}_\sigma$  and  $\mathbf{P}_{r\sigma}$  introduced above, it is not difficult to find a general expression for the nuclear interaction potential with the exchange forces taken into account. According to (1.103), the nuclear interaction potential in the general case consists of two parts, describing the central and non-central forces. The potentials for the central exchange forces can be represented in the form of products of the operators  $\mathbf{P}_r$ ,  $\mathbf{P}_\sigma$  and  $\mathbf{P}_{r\sigma}$  with certain functions depending on the distance between the particles. The corresponding exchange forces are usually called the Majorana, Bartlett and Heisenberg forces. The non-exchange nuclear forces are often called Wigner forces.

The non-central nuclear forces can be of the exchange or non-exchange type. Since the tensor operator  $S_{12}$  is equal to zero in the singlet state, and the operator  $\mathbf{P}_\sigma$  is equal to +1 in the triplet state, we have

$$S_{12}\mathbf{P}_\sigma = \mathbf{P}_\sigma S_{12} = S_{12}. \quad (1.138)$$

Therefore, only one form of exchange tensor force is possible – the Majorana tensor force. (The Heisenberg tensor forces, by virtue of (1.138), reduce to Majorana tensor forces.)

Thus, in the most general case, the nuclear interaction potential between two nucleons, with the exchange character and non-central nature of the nuclear forces taken into account, can be written in the form

$$\begin{aligned} V &= V_W(\mathbf{r}) + V_M(\mathbf{r})\mathbf{P}_r + V_B(\mathbf{r})\mathbf{P}_\sigma + V_H(\mathbf{r})\mathbf{P}_{r\sigma} \\ &\quad + V_{TW}(\mathbf{r})S_{12} + V_{TM}(\mathbf{r})S_{12}\mathbf{P}_r, \end{aligned} \quad (1.139)$$

where  $V_W(\mathbf{r})$ ,  $V_M(\mathbf{r})$ ,  $V_B(\mathbf{r})$ ,  $V_H(\mathbf{r})$ ,  $V_{TW}(\mathbf{r})$  and  $V_{TM}(\mathbf{r})$  are functions depending on the distance between the particles. These functions can differ both in magnitude and in the nature of their dependence on the distance. Note that there is no need to introduce separately into the relation (1.139) a term taking the spin dependence of the central forces into account, since this dependence is taken into account by the exchange terms.

The expression (1.139) for the potential, which was obtained in accordance with the requirements of invariance under spatial rotations and reflections and of the symmetry with respect to interchange of particles, contains, in the general case, six functions  $V(r)$ , the form of which can be determined only from experimental data. In practice, however, the question of the determination of the form of the functions  $V(r)$  remains open at the present time. To a considerable extent, this is connected with the fact that nuclear phenomena in the low-energy region depend very weakly on the shape of the nuclear potential. The original hopes that it might be possible to determine the form of  $V(r)$  from experiments on the scattering of two nucleons at high energies have also proved unfounded, inasmuch as the expression (1.139) was obtained in the static limit.

**The Serber potential.** One of the simplest potentials, and one which is often used in calculations, is the so-called *Serber potential*, incorporating a mixture of the Wigner and Majorana forces with equal weights:

$$V_S = V_S(r)(1 + P_r)/2. \quad (1.140)$$

Since the value of  $P_r$  is  $-1$  in states with odd values of  $l$ , the interaction described by the potential (1.140) is manifested only in even states. Because of this, the Serber potential leads to scattering that is symmetric about the angle  $90^\circ$ . In reality, the angular distribution in the scattering of neutrons by protons at energies of the order of 100 MeV has an asymmetric character, indicating interaction in the odd states too. However, this interaction is considerably weaker than the interaction in the even states.

### 1.9. Isotopic Spin and Charge Independence of the Nuclear Interaction

**Isotopic spin of the nucleon.** As we have already observed, despite the fact that the proton has an electric charge and the neutron does not, neutrons and protons possess very similar physical properties. This similarity is manifested in the closeness of the neutron and proton masses; moreover, neutrons and protons have the same spin  $\frac{1}{2}$  and obey the same statistics (Fermi-Dirac statistics); the neutron and proton can be transformed into each other, by absorbing a meson; neutrons and protons are transformed

into each other in  $\beta$ -decay, and so on. However, the most striking manifestation of the physical similarity of the neutron and proton is the *charge independence* of their nuclear interaction. The charge independence, or, as it is more often called, the *isotopic invariance*, appears in the fact that the nuclear interaction between any two particles (neutrons or protons) does not depend on the nature of the particles and is determined only by the state in which the system is found.

The similarity of the physical properties of the neutron and proton enables us to regard them as two different states of one and the same particle – the nucleon. (These states differ in the magnitude of the charge.) The nucleon is characterized not only by its spatial and spin coordinates, but also by an additional degree of freedom – the *charge coordinate*. Since the charge coordinate (which distinguishes the neutron and proton states of the nucleon) assumes only two values in all, it is convenient to associate with it a vector  $\mathbf{t}$ , similar in its properties to the vector of an ordinary spin  $\frac{1}{2}$ , in some fictitious space. The vector  $\mathbf{t}$  is called the *isotopic spin*, or *isospin* (Heisenberg, 1932). The introduction of the isotopic spin enables us to describe the isotopic invariance of the nuclear interaction very simply.

The absolute value of the isotopic-spin vector  $\mathbf{t}$  for a nucleon is equal to  $\frac{1}{2}$ ; therefore, the components of the vector  $\mathbf{t}$  can be expressed directly in terms of the Pauli matrices  $\tau$ :

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.141)$$

namely, by

$$\mathbf{t} = \frac{1}{2}\boldsymbol{\tau}. \quad (1.142)$$

The projection of the isotopic spin of a nucleon on to the  $z$ -axis can take only two values:  $\nu = \pm\frac{1}{2}$ . The value  $+\frac{1}{2}$  is conventionally associated with the proton and the value  $-\frac{1}{2}$  with the neutron. We denote the charge or isospin function of the nucleon by  $\zeta(\tau)$ , where  $\tau$  is the charge coordinate or isospin coordinate. For the charge coordinate  $\tau$ , we can choose the value of the  $z$ -component of the isospin. Wave functions for states with well-defined values of the isospin component are diagonal,  $\zeta_\nu(\tau) = \delta_{\nu\tau}$ , in this representation. Usually, the isospin functions corresponding to the proton and neutron states are represented in the following form:

$$\zeta_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.143)$$

Since the spatial, spin and isospin coordinates are independent, the complete wave function  $\psi$  for a free nucleon can be represented in the form of

a product of a spatial function  $\varphi$ , a spin function  $\chi$  and an isospin function  $\zeta$ :

$$\psi(\mathbf{r}, \sigma, \tau) = \varphi(\mathbf{r})\chi(\sigma)\zeta(\tau). \quad (1.144)$$

We introduce projection operators for the neutron and proton states:

$$\Lambda_n = \frac{1}{2}(1 - \tau_z), \quad \Lambda_p = \frac{1}{2}(1 + \tau_z). \quad (1.145)$$

The operator  $\Lambda_n$  leaves the neutron function unchanged and makes the proton function vanish; the operator  $\Lambda_p$  has the opposite effect. Using the projection operators (1.145), it is not difficult to construct operators for the charge, mass and magnetic moment of the nucleon:

$$\left. \begin{aligned} q &= \frac{1}{2}(1 + \tau_z)e; \\ M &= \frac{1}{2}(1 - \tau_z)M_n + \frac{1}{2}(1 + \tau_z)M_p; \\ \mu &= \frac{1}{2}(1 - \tau_z)\mu_n + \frac{1}{2}(1 + \tau_z)\mu_p. \end{aligned} \right\} \quad (1.146)$$

The charge creation and destruction operators  $\tau_+$  and  $\tau_-$  have the form

$$\tau_+ = \frac{1}{2}(\tau_x + i\tau_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \frac{1}{2}(\tau_x - i\tau_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (1.147)$$

The operator  $\tau_+$  converts a neutron into a proton, and the operator  $\tau_-$  converts a proton into a neutron.

**Isotopic spin of a system of nucleons.** The isotopic spin of a system of nucleons is composed of the isotopic spins of the separate particles, in accordance with the rules for vector addition of angular momenta. The isotopic spin operator of a system of  $A$  nucleons is equal to

$$\mathbf{T} = \sum_{i=1}^A \mathbf{t}_i, \quad (1.148)$$

where  $\mathbf{t}_i$  is the isotopic spin operator of an individual nucleon. The eigenvalues of the square of the isotopic spin (1.148) are equal to  $T(T+1)$ , where  $T$  is the *total isotopic spin of system*.

If  $A$  is even, then  $T$  can take values in the range from 0 to  $A/2$ ; if  $A$  is odd, then  $T$  takes values in the range from  $\frac{1}{2}$  to  $A/2$ . For each  $T$ , there exist  $2T+1$  eigenvalues of  $T_z$  which are integers or odd multiples of half-integers in the interval  $-T \leq T_z \leq T$ .

For a nucleus with atomic number  $Z$  and mass number  $A$ , containing  $Z$  protons and  $N = A - Z$  neutrons, the projection of the total isotopic spin is

$$T_z = \frac{1}{2}(Z - N) = Z - A/2. \quad (1.149)$$

In this case, the absolute value  $T$  of the total isotopic spin of the system can be equal to one of the values in the range  $|T_z| \leq T \leq A/2$ . For a given number  $A$  of nucleons, according to (1.149), the magnitude of  $T_z$  determines the total charge of the nucleus. Since the charge is conserved, a conservation law holds for the projection  $T_z$  of the isotopic spin of the system.

States of a system of  $A$  nucleons with the same value  $T$  of the isotopic spin form an *isotopic multiplet*. For a given value of  $T$ , the isotopic multiplet contains  $2T + 1$  components. These components differ in their values of the isotopic-spin projection  $T_z$ , that is, they correspond to different charges  $Z$  of the system. Clearly, the limiting values of the charge for the components of the multiplet are equal to  $A/2 \pm T$ . When the electromagnetic interaction is neglected, the components of the multiplet have the same structure and are characterized by the same binding energy.

Since the nucleons possess spin equal to  $\frac{1}{2}$ , they satisfy the Pauli principle. We write the complete wave function of a system  $A$  nucleons in the form

$$\Psi(\mathbf{r}_1, \sigma_1, \tau_1; \mathbf{r}_2, \sigma_2, \tau_2; \dots; \mathbf{r}_A, \sigma_A, \tau_A),$$

where  $\mathbf{r}_i$ ,  $\sigma_i$  and  $\tau_i$  are, respectively, the spatial, spin and charge coordinates of the  $i$ th nucleon. Then the Pauli principle for nucleus can be formulated as follows: the wave function of a system of nucleons must be anti-symmetric with respect to interchange of the spatial, spin and isospin coordinates of any pair of nucleons. Because the isospin coordinates of the nucleons are independent, the *complete wave function*  $\Psi$  of a system of nucleons can always be represented in the form of a product of a *space-spin function*  $\Phi$  with an *isospin function*  $Z$ :

$$\Psi = \Phi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2; \dots; \mathbf{r}_A, \sigma_A)Z(\tau_1, \tau_2, \dots, \tau_A). \quad (1.150)$$

The symmetry of the isospin function  $Z$  is uniquely determined by the magnitude  $T$  of the isotopic spin of the system. By virtue of the Pauli principle, the complete wave function  $\Psi$  is anti-symmetric with respect to interchanges of the coordinates of any pair of nucleons; therefore, the functions  $Z$  and  $\Phi$  must possess opposite (dual) symmetry.

**Classification of the states of a two-nucleon system.** We shall consider a system consisting of two nucleons. Since the isotopic spin of each nucleon is equal to  $\frac{1}{2}$ , the total isotopic spin of the system can take two values:  $T = 1$  and  $T = 0$ . Therefore, all the states of the system can be divided into two groups – one corresponding to the isotopic spin  $T = 1$  and the other to  $T = 0$ .

In the case  $T = 1$ , three different values of the isospin projection are possible:  $T_z = 1, 0$  and  $-1$ . The corresponding states are characterized by charges  $2e$ ,  $e$  and  $0$  and correspond to a system consisting of two protons,

a neutron and a proton, and two neutrons. The wave functions describing these states are symmetric and can be expressed in terms of the one-nucleon isospin functions (1.143):

$$\left. \begin{aligned} Z_{1,1}(\tau_1, \tau_2) &= \zeta_{1/2}(\tau_1)\zeta_{1/2}(\tau_2); \\ Z_{1,0}(\tau_1, \tau_2) &= (1/\sqrt{2})\{\zeta_{1/2}(\tau_1)\zeta_{-1/2}(\tau_2) + \zeta_{-1/2}(\tau_1)\zeta_{1/2}(\tau_2)\}; \\ Z_{1,-1}(\tau_1, \tau_2) &= \zeta_{-1/2}(\tau_1)\zeta_{-1/2}(\tau_2). \end{aligned} \right\} \quad (1.151)$$

The functions (1.151) are normalized to unity.

In the case  $T = 0$ , the isospin projection  $T_z = 0$ . The corresponding singlet state is described by an anti-symmetric isospin function:

$$Z_{00}(\tau_1, \tau_2) = (1/\sqrt{2})\{\zeta_{1/2}(\tau_1)\zeta_{-1/2}(\tau_2) - \zeta_{-1/2}(\tau_1)\zeta_{1/2}(\tau_2)\}. \quad (1.152)$$

The anti-symmetric state (1.152) can be realized only in a system consisting of a neutron and a proton. The function (1.152) is orthogonal to the functions (1.151).

The complete wave function of a system of two nucleons can be represented in the form of a product of a space-spin function with an isospin function:

$$\Psi_{wSJM}(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2) Z_{TM}(\tau_1, \tau_2). \quad (1.153)$$

Thus, the state of the system is determined if the following complete set of quantum numbers is specified: *the parity w*, *the total spin S*, *the total angular momentum J*, *the projection M of the total angular momentum*, *the total isospin T* and *the projection M<sub>T</sub> of the isospin*. By virtue of the Pauli principle, the complete wave function (1.153) must be anti-symmetric with respect to an interchange of the nucleons, that is, with respect to a simultaneous interchange of the spatial, spin and isospin coordinates of the nucleons. The symmetry of the wave function  $\Phi$  with respect to interchange of the spatial coordinates is determined by the parity  $w$  of the state, and the symmetry with respect to interchange of the spin coordinates is determined by the spin magnitude  $S$ ; the symmetry of the isospin function  $Z$  with respect to interchange of the isospin coordinates is determined by the magnitude  $T$  of the isotopic spin of the system. Therefore, in a system of two nucleons, even singlet states and odd triplet states are always symmetric in the isospin variables ( $T = 1$ ), and odd singlet states and even triplet states are always anti-symmetric in the isospin variables ( $T = 0$ ). Hence it follows that only even singlet states and odd triplet states can be realized in a system of two identical nucleons (since the isospin function must be symmetric), whereas states of any parity and spin can be realized in systems of two different nucleons.

TABLE 1.2.

J	S = 0		S = 1	
	l-even	l-odd	l-even	l-odd
	T = 1	T = 0	T = 0	T = 1
0	$^1S_0$			$^3P_0$
1		$^1P_1$	$^3S_1 + ^3D_1$	$^3P_1$
2	$^1D_2$		$^3D_2$	$^3P_2 + ^3F_2$
3		$^1F_3$	$^3D_3 + ^3G_3$	$^3F_3$
4	$^1G_4$		$^3G_4$	$^3F_4 + ^3H_4$
5		$^1H_5$	$^3G_5 + ^2I_5$	$^3H_5$

In Table 1.2 we give a classification of the states of a system of two nucleons for  $J = 0, 1, 2, 3, 4$  and  $5$ . In particular, it follows from the Table that the deuteron isotopic spin is equal to zero,  $T = 0$ .

**Explicit form of the exchange operators.** The Majorana, Bartlett and Heisenberg exchange operators  $\mathbf{P}_r$ ,  $\mathbf{P}_\sigma$  and  $\mathbf{P}_{r\sigma}$  introduced in the study of nuclear exchange forces can be expressed simply in terms of the operators of the spins and isospins of the particles. The operator  $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$  is characterized by the eigenvalues  $+1$  and  $-3$  for the triplet ( $S = 1$ ) and singlet ( $S = 0$ ) states respectively. Therefore, the operator  $\mathbf{P}_\sigma$  interchanging the spin coordinates of two nucleons can be represented in the form

$$\mathbf{P}_\sigma = \frac{1}{2}(\mathbf{1} + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2). \quad (1.154)$$

In fact, the operator (1.154) is characterized by the eigenvalues  $+1$  and  $-1$  for the triplet and singlet states respectively; therefore,  $\mathbf{P}_\sigma \chi(\sigma_1, \sigma_2) = \chi(\sigma_2, \sigma_1)$ . The formula (1.154) gives a convenient representation for the Bartlett exchange operator.

Analogously, we can introduce the operator  $\mathbf{P}_\tau$  interchanging the isospin coordinates of two nucleons:

$$\mathbf{P}_\tau = \frac{1}{2}(\mathbf{1} + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2). \quad (1.155)$$

The operator  $\mathbf{P}_\tau$  acts on the isospin function of the nucleons:  $\mathbf{P}_\tau \mathbf{Z}(\tau_1, \tau_2) = \mathbf{Z}(\tau_2, \tau_1)$ . Since the complete wave function of a system of two nucleons is anti-symmetric with respect to interchange of the nucleons, which reduces to interchange of the spatial, spin and isospin coordinates of the nucleons, we have

$$\mathbf{P}_r \mathbf{P}_\sigma \mathbf{P}_\tau = -1. \quad (1.156)$$

Using this equality and noting that  $\mathbf{P}_\tau^2 = 1$ , it is not difficult to obtain the following representation for the Majorana exchange operator:

$$\mathbf{P}_r = -\frac{1}{4}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2). \quad (1.157)$$

Recalling the definition (1.133) of the Heisenberg exchange operator, we easily find that

$$\mathbf{P}_{r\sigma} = -\frac{1}{2}(1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2), \quad (1.158)$$

that is, the Heisenberg exchange operator  $\mathbf{P}_{r\sigma}$  coincides in magnitude with the operator  $\mathbf{P}_r$  interchanging the charge coordinates of the particles.

**Isotopic invariance of the nuclear interaction.** The *isotopic-spin formalism* is very convenient for the description of the charge independence or, as it is sometimes called, the isotopic invariance of the nuclear interaction (Breit, Condon and Present, 1936; Cassen and Condon, 1936). We shall formulate the concept of the isotopic invariance of the interaction by using the concept of the isotopic spin of the particles.

According to the definition, the isotopic spin is a vector in a certain fictitious space. The absolute value of the isotopic spin of a nucleon is equal to  $\frac{1}{2}$ . The state with isospin projection  $+\frac{1}{2}$  corresponds to a proton, and the state with isospin projection  $-\frac{1}{2}$  corresponds to a neutron. A transition from a proton to a neutron implies a change of the isospin projection from  $+\frac{1}{2}$  to  $-\frac{1}{2}$ . Therefore, by analogy with rotations in ordinary space, such a transition can be regarded as a rotation in the isotopic space through an angle  $\pi$  about an axis perpendicular to the axis of quantization. If the system consists of several nucleons, their isotopic spins are combined into the total isotopic spin of the system in accordance with the rules of vector addition. In this case, states of the system corresponding to a definite value of the total isotopic spin but to different values of the isospin projection can be associated with rotations in the isotopic space. For example, a system consisting of two protons differs from a system consisting of a proton and a neutron with  $T = 1$  by a rotation in isotopic space through an angle  $\pi/2$ ; a system consisting of two protons differs from a system consisting of two neutrons by a rotation through an angle  $\pi$  in the isotopic space, and so on. Clearly, with this approach, the isotopic invariance of the nuclear interaction can be regarded as an invariance property with respect to rotations of the system in the isotopic space.

We note that the charge symmetry of the nuclear interaction (the invariance with respect to replacement of all the neutrons by protons and vice versa), which is a particular case of isotopic invariance, can be regarded as invariance with respect to a rotation in isotopic space through an angle  $\pi$  about an axis perpendicular to the quantization axis.

Thus, the isotopic invariance of the nuclear interaction means that the interaction should not depend on the orientation of the isotopic-spin vector  $T$  of the system. Consequently, the isotopic space should be isotopic as far as the nuclear interaction is concerned.

The isotopic invariance does not hold for the electromagnetic interaction. In fact, the electric charge and magnetic moment of the system are determined by the isotopic-spin projection; therefore, the electromagnetic interaction also depends on the isotopic-spin projection, which is not conserved in rotations in the isotopic space.

Since the isotopic spin possesses properties similar to those of an ordinary angular momentum, it follows from the isotropy of the isotopic space with respect to nuclear interactions that the isotopic spin  $T$  is conserved. Unlike the conservation law for the isotopic-spin projection  $T_z$  which expresses the law of conservation of the charge of the system, the conservation law for the isotopic spin  $T$  of the system is approximate, since it is valid only when the electromagnetic interaction is neglected.

We show now that the potential found earlier from considerations of the invariance with respect to spatial rotations and spatial inversion, which takes into account the non-central and exchange character of the nuclear interaction, possesses the property of isotopic invariance. In fact, using the expressions (1.154), (1.157) and (1.158) for the exchange operators, we can represent the nuclear interaction potential (1.139) of two nucleons in the form

$$\begin{aligned} V = & V_C(\mathbf{r}) + V_\sigma(\mathbf{r})(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + V_\tau(\mathbf{r})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + V_{\sigma\tau}(\mathbf{r})(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \\ & + V_T(\mathbf{r})S_{12} + V_{T\tau}(\mathbf{r})S_{12}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2), \end{aligned} \quad (1.159)$$

where

$$\left. \begin{aligned} V_C &= V_W + \frac{1}{2}V_B - \frac{1}{2}V_H - \frac{1}{4}V_M; \\ V_\sigma &= \frac{1}{2}V_B - \frac{1}{4}V_M; \\ V_\tau &= -\frac{1}{2}V_H - \frac{1}{4}V_M; \\ V_{\sigma\tau} &= -\frac{1}{4}V_M; \\ V_T &= V_{TW} - \frac{1}{2}V_{TM}; \\ V_{T\tau} &= -\frac{1}{2}V_{TM}. \end{aligned} \right\} \quad (1.160)$$

The expression (1.159) is invariant with respect to rotations in the isotopic space. Consequently, a nuclear interaction describable by the potential (1.159) is characterized by the property of isotopic invariance, and this means that the magnitude  $T$  of the isotopic spin of the system is conserved. According to (1.159), the nuclear interaction depends on the quantum numbers  $S$  and  $T$ , that is, on the state in which the system is found. The fact

that (1.159) is independent of the projection of the isospin of the system implies that the interaction is independent of the nature of the nucleons (that is, of whether they are protons or neutrons).

The *hypothesis of the isotopic invariance of the nuclear interaction* agrees with the results of experiments on nucleon-nucleon scattering at low energies. The fact that the scattering lengths and effective ranges are of the same order of magnitude for proton-proton scattering, neutron-neutron scattering and singlet-state neutron-proton scattering is evidence that the interaction between two protons and the interaction between a neutron and a proton in the singlet state are identical.

### 1.10. Nucleon-nucleon Scattering at High Energies

**Principal features of nucleon-nucleon scattering at high energies.** As we have seen, the scattering of neutrons by protons and of protons by protons at low energies is completely insensitive to the shape of the nuclear interaction potential. This is associated with the fact that, at low energies, the wavelength of the particles exceeds the range of the nuclear forces and the interaction is manifested only in the  $S$ -state. To discover the dependence of the potential on the distance between the particles, it is necessary that the wavelength be less than the range of the forces. In this case, however, the interaction will appear not only in the  $S$ -state, but also in states with  $l \neq 0$ . It is obvious that the greater the number of partial waves participating in the scattering, the more detailed will be the information that can be obtained on the radial dependence of the potential.

It is necessary to take into account, however, that at sufficiently high nucleon energies, alongside the elastic scattering, inelastic processes can occur. Thus, if the energy of the incident nucleon exceeds 500 MeV, it is necessary to take into account a scattering process with the formation of  $\pi$ -mesons. In the following, we shall consider the region of energies in which only elastic scattering of nucleons is important.

At the present time, a sufficient amount of experimental data on proton-proton and neutron-neutron scattering in this energy range has been accumulated<sup>3</sup>. Figures 1.3 and 1.4 show the angular dependences of the experimental differential cross-sections and the polarization for proton-proton scattering (the curves are drawn through the experimental points).

As can be seen from Fig. 1.3, the differential cross-section for proton-proton elastic scattering in the (approximate) energy range from 150 to 500 MeV is practically constant and depends neither on the scattering angle nor

<sup>3</sup>The phenomenological theory of nucleon-nucleon scattering is described in the monograph by Wilson (1965).

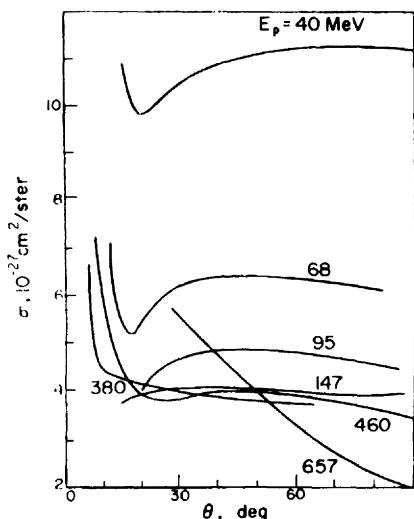


Figure 1.3. Angular dependence of the experimental differential cross-sections for scattering of protons of different energies by protons.

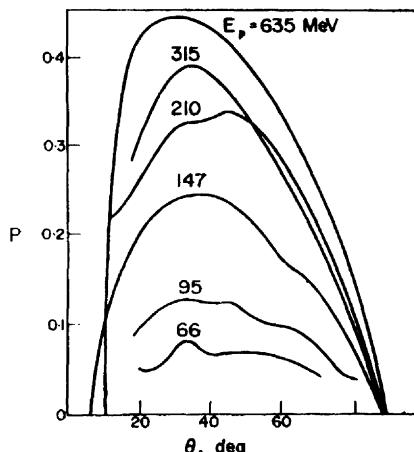


Figure 1.4. Angular dependence of the polarization in the case of scattering of protons of different energies by protons.

on the energy. The magnitude of the cross-section per unit solid angle is

$$\sigma(\vartheta) \approx 4 \times 10^{-27} \text{cm}^2/\text{ster} \quad (1.161)$$

within the experimental error bars. Only in the region of very small angles does the cross-section increase, because of the Coulomb scattering. At energies above 500 MeV, the isotropy is violated: a maximum appears in the angular distribution in the forward direction. This behaviour of the scattering cross-section is in sharp contradiction with the predictions of perturbation theory. This means that even at high nucleon energies we cannot assume that the nuclear interaction is weak, that is, we cannot use perturbation theory to describe it.

The experimental study of the scattering of neutrons by protons at high energies has led to a picture which is substantially different from that for the proton-proton case. Figures 1.5 and 1.6 show the angular dependences of the experimental differential cross-sections and polarization in the case of scattering of neutrons by protons.

It can be seen from Fig. 1.5 that the differential cross-section for scattering of neutrons by protons depends strongly on the scattering angle, increasing to a value at an angle  $180^\circ$  that is several times greater than its value at  $90^\circ$ . Thus, for an energy of 300 MeV, the cross-section at  $90^\circ$  is equal to  $2 \times 10^{-27} \text{cm}^2/\text{ster}$ , and at  $180^\circ$  is  $9.2 \times 10^{-27} \text{cm}^2/\text{ster}$ .

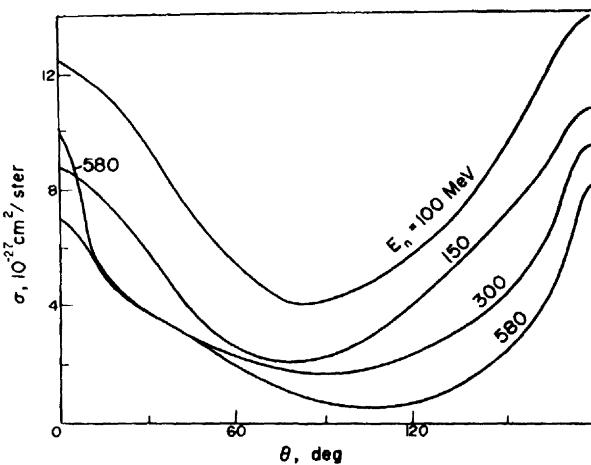


Figure 1.5. Angular dependence of the experimental differential cross-sections for scattering of neutrons of different energies by protons.

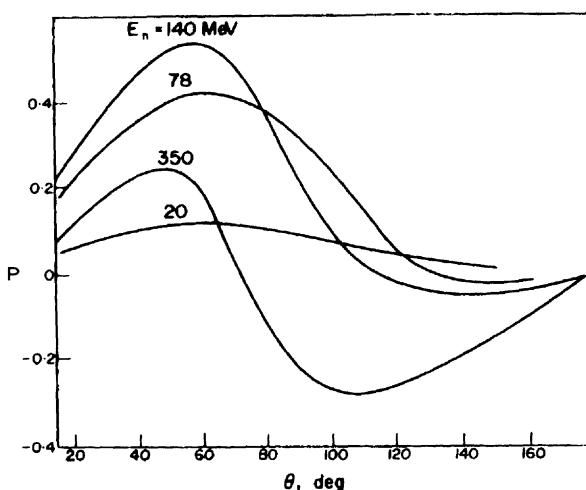


Figure 1.6. Angular dependence of the polarization in the case of scattering of neutrons of various energies by protons.

With increasing energy, the cross-section at  $90^\circ$  falls, reaching the value  $1 \times 10^{-27} \text{ cm}^2/\text{ster}$  at 580 MeV, while the cross-section at  $180^\circ$  remains almost constant. As we move from  $90^\circ$  towards smaller angles the cross-section does not increase so rapidly and, therefore, the angular distribution has an asymmetric character.

The total cross-sections for proton-proton and neutron-proton scat-

tering also differ appreciably. Whereas the total cross-section for elastic proton-proton scattering in the range of energies from 150 to 500 MeV is practically constant and equal to  $25 \times 10^{-27} \text{ cm}^2$ , the total cross-section for elastic scattering of neutrons by protons decreases like  $1/E$  up to the energies at which inelastic processes are found to be important. The small deviations from the  $1/E$  law are due to the energy dependence of the phase shifts.

**Inelastic scattering of nucleons.** In collisions of nucleons with energies greater than 500 MeV an important role is played by *inelastic scattering*, accompanied by the creation of  $\pi$ -mesons. Allowance for this effect leads to an increase in the total cross-section. The *total interaction cross-section* for two protons (the sum of the cross-section for elastic scattering due to the nuclear interaction and the cross-section for creation of  $\pi$ -mesons) is approximately equal to 50 mbarn at energy 800 MeV and then remains almost constant up to energy 3000 MeV. The elastic scattering cross-section is practically independent of the energy and equal to 25 mbarn.

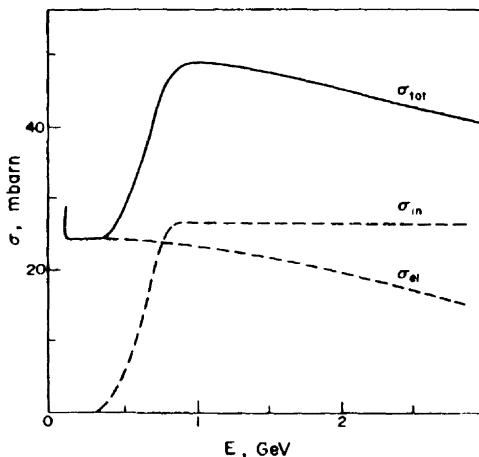


Figure 1.7. Energy dependence of the total interaction cross-section  $\sigma_{\text{tot}}$  for two protons, the proton-proton elastic scattering cross-section  $\sigma_{\text{el}}$  and the proton-proton inelastic scattering cross-section  $\sigma_{\text{in}}$ .

Thus, in the range of energies from 500 to 3000 MeV, the cross-sections for elastic and inelastic scattering of two protons are found to be practically equal to each other, and constant (Fig. 1.7). The angular dependence of the elastic scattering cross-section is characterized by a sharp tendency to the forward direction. Such properties characterize the diffractional scattering of particles in the presence of a perfectly absorbing sphere. In this case, the absorption cross-section and elastic scattering cross-section are equal

to  $\pi R^2$ , where  $R$  is the radius of the sphere. If we describe the interaction of two protons with the aid of this diffraction model, we find, by putting  $R$  equal to the sum of the radii of the two protons, the magnitude of the *proton radius*  $R_p$ :

$$R_p = 0.45 \times 10^{-13} \text{ cm.} \quad (1.162)$$

We note that the results of experiments on the scattering of electrons by protons are in agreement with this estimate.

**High-energy nucleon scattering and isotopic invariance.** Despite the fact that the scattering cross-section in the proton-proton and neutron-proton systems behave so differently, the data on *high-energy scattering* do not contradict the hypothesis of isotopic invariance of the nuclear interaction. If the assumption of isotopic invariance is correct, the forces acting between two protons also act between a neutron and a proton in the state  $T = 1$ . The more complicated behaviour of the neutron-proton scattering cross-section is due to the interaction in the state  $T = 0$ , which is not present for a system of two protons.

It is not difficult to establish a simple criterion which the cross-sections must satisfy if the interaction between the nucleons is isotopically invariant. In fact, for scattering through an angle of  $90^\circ$ , states with odd values of  $l$  make no contribution to the cross-section, since for odd values of  $l$   $P_l(\cos 90^\circ) = 0$ . For even values of  $l$ , a system of two protons ( $T = 1$ ) can only be in a singlet spin state, while a neutron-proton system can be in either a singlet spin state (with  $T = 1$ ) or in a triplet spin state (with  $T = 0$ ). Therefore, if the interaction is isotopically invariant, the *differential scattering cross-sections* in the above cases can be written in the form

$$\left. \begin{aligned} \sigma_{pp}(90^\circ) &= \sigma_{T=1}(90^\circ); \\ \sigma_{np}(90^\circ) &= \frac{1}{4}\sigma_{T=1}(90^\circ) + \frac{3}{4}\sigma_{T=0}(90^\circ), \end{aligned} \right\} \quad (1.163)$$

where  $\frac{1}{4}$  and  $\frac{3}{4}$  are the weights of the singlet and triplet spin states respectively. It follows directly from the expressions (1.163) that, if the interaction is isotopically invariant, the inequality

$$\sigma_{np}(90^\circ) \geq \frac{1}{4}\sigma_{pp}(90^\circ) \quad (1.164)$$

must be fulfilled for all energies.

As we have already observed the cross-section for scattering of a neutron by a proton decreases rapidly with increasing energy, whereas the proton-proton scattering cross-section remains practically unchanged. Therefore, the condition (1.164) is found to be more critical in the high-energy region. All the available experimental data on the scattering cross-sections are in agreement with the condition (1.164).

It follows from a comparison of the formulae (1.163) that, in the case of sufficiently high energies, the cross-section for scattering through an angle of  $90^\circ$  is considerably smaller for  $T = 0$  than for  $T = 1$ . At the same time, the differential cross-section for scattering of a neutron by a proton through an angle of  $180^\circ$  is considerably greater than the proton-proton scattering cross-section. This difference is obviously connected with the interaction for  $T = 0$ . Thus, in the case  $T = 0$ , in contrast to the case  $T = 1$ , the scattering depends very strongly on the angle.

Using the experimental values for the cross-sections  $\sigma_{pp}(\vartheta)$  and  $\sigma_{np}(\vartheta)$ , it is not difficult to find the scattering cross-sections in the states with  $T = 1$  and  $T = 0$ . Clearly, the cross-section for scattering in the state  $T = 1$  is the proton-proton scattering cross-section:

$$\sigma_{T=1}(\vartheta) = \sigma_{pp}(\vartheta). \quad (1.165)$$

To determine the cross-section for scattering in the state  $T = 0$ , we note that the isospin state of the neutron-proton system is a superposition of the states  $T = 1$  and  $T = 0$ . According to the relations (1.151) and (1.152), we have

$$\zeta_{1/2}(1)\zeta_{-1/2}(2) = (1/\sqrt{2})\{Z_{10}(1, 2) + Z_{00}(1, 2)\}.$$

Therefore, the neutron proton scattering amplitude can be expressed in an analogous manner in terms of the *scattering amplitudes* in the cases  $T = 1$  and  $T = 0$ :

$$f_{np}(\vartheta) = (1/\sqrt{2})\{f_{T=1}(\vartheta) + f_{T=0}(\vartheta)\}. \quad (1.166)$$

The neutron-proton scattering cross-section is defined as the square of the modulus of (1.166):

$$\sigma_{np}(\vartheta) = \frac{1}{2}\{\sigma_{T=1}(\vartheta) + \sigma_{T=0}(\vartheta) + 2\text{Re}[f_{T=1}(\vartheta)f_{T=0}^*(\vartheta)]\}. \quad (1.167)$$

By virtue of the Pauli principle, the amplitudes  $f_{T=1}(\vartheta)$  and  $f_{T=0}(\vartheta)$  are characterized by opposite parity for a given spin value  $S$ . Therefore, in the expression  $\sigma_{np}(\pi - \vartheta)$  for the cross-section for scattering through an angle  $\pi - \vartheta$ , the interference term occurs with the opposite sign (cf. formula (1.167)), and the sum of the cross-sections  $\sigma_{np}(\vartheta)$  and  $\sigma_{np}(\pi - \vartheta)$  will be equal to the sum of the scattering cross-sections in the states with  $T = 1$  and  $T = 0$ :

$$\sigma_{np}(\vartheta) + \sigma_{np}(\pi - \vartheta) = \sigma_{T=1}(\vartheta) + \sigma_{T=0}(\vartheta) \quad (1.168)$$

The expression obtained has a simple physical meaning: it gives the total number of neutrons and protons scattered through a given angle  $\vartheta$ .

Using the relations (1.168) and (1.165), we find the cross-section for scattering of nucleons in the state  $T = 0$ :

$$\sigma_{T=0}(\vartheta) = \sigma_{np}(\vartheta) + \sigma_{np}(\pi - \vartheta) - \sigma_{pp}(\vartheta). \quad (1.169)$$

We note that this cross-section  $\sigma_{T=0}(\vartheta)$  is always found to be positive, and this gives experimental confirmation of the hypothesis of the isotopic invariance of the nuclear interaction.

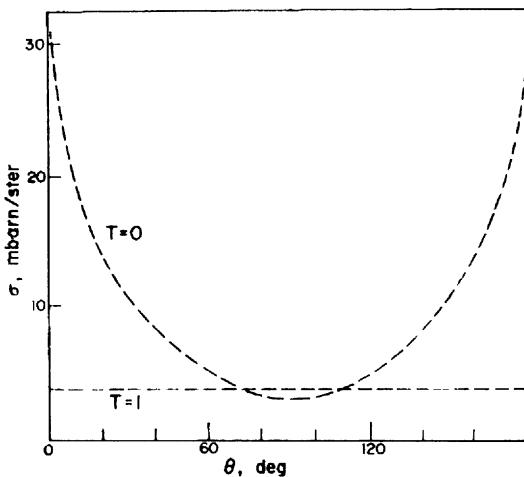


Figure 1.8. Angular dependence of the cross-sections for scattering of nucleons in states with different isospins.

The cross-sections  $\sigma_{T=1}(\vartheta)$  and  $\sigma_{T=0}(\vartheta)$  for a nucleon energy of the order of 400 MeV are shown schematically in Fig. 1.8. As can be seen from the figure, the angular dependence of the scattering is completely different in the cases  $T = 1$  and  $T = 0$ . Whereas for  $T = 1$  the scattering is isotopic, in the case  $T = 0$  the angular distribution of the scattering is characterized by maxima in the forward and backward directions. The character of this angular dependence of the scattering points to the possibility of describing the interaction between nucleons in the state  $T = 0$  by means of perturbation theory.

**Determination of nucleon-nucleon scattering amplitude from experimental data.** If the nuclear forces had no spin dependence, experiments on nucleon-nucleon scattering would enable us to determine, for a fixed value of the energy, only one quantity – the differential scattering cross-section  $\sigma(\vartheta)$ . The differential cross-section  $\sigma(\vartheta)$  and the scattering amplitude  $f(\vartheta)$  are connected by the relation  $\sigma(\vartheta) = |f(\vartheta)|^2$ . Since, in the

absence of spin dependence, the scattering amplitude  $f(\vartheta)$  is a scalar quantity, it follows that, if we know the differential scattering cross-section at all angles and use the optical theorem, we could in principle completely determine the scattering amplitude.

Because of the spin dependence of the nuclear interaction between the nucleons, scattering experiments enable us to determine not only the differential scattering cross-section but also the polarization of the scattered nucleon, the polarization of the recoiling nucleon, and also the correlation between the directions of the polarizations of the nucleons. If the interaction is invariant with respect to rotations in space and inversion in space and time, the scattering amplitude for scattering of two particles, each having spin  $\frac{1}{2}$ , can be represented in the most general case in the form (Wolfenstein, 1956)

$$\begin{aligned} f(\mathbf{k}, \mathbf{k}') = & \alpha + \beta(\mathbf{n} \cdot \boldsymbol{\sigma}_1)(\mathbf{n} \cdot \boldsymbol{\sigma}_2) + i\gamma\mathbf{n} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + i\gamma'\mathbf{n} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \\ & + \delta(\mathbf{m} \cdot \boldsymbol{\sigma}_1)(\mathbf{m} \cdot \boldsymbol{\sigma}_2) + \varepsilon(\mathbf{l} \cdot \boldsymbol{\sigma}_1)(\mathbf{l} \cdot \boldsymbol{\sigma}_2), \end{aligned} \quad (1.170)$$

where  $\mathbf{k}$  and  $\mathbf{k}'$  are the relative of the particles before and after the scattering,  $\boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma}_2$  are the Pauli matrices corresponding to the spin of the interacting particles, and  $\mathbf{n}$ ,  $\mathbf{l}$  and  $\mathbf{m}$  are a set of three mutually orthogonal unit vectors:

$$\mathbf{n} = \frac{\mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|}, \quad \mathbf{l} = \frac{\mathbf{k} + \mathbf{k}'}{|\mathbf{k} + \mathbf{k}'|}, \quad \mathbf{m} = \frac{\mathbf{k} - \mathbf{k}'}{|\mathbf{k} - \mathbf{k}'|}. \quad (1.171)$$

The coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\gamma'$ ,  $\delta$  and  $\varepsilon$  are complex functions depending on the angle and the energy. If we take the charge invariance of the nuclear interaction between the nucleons into account, the coefficient  $\gamma'$  in (1.170) must be put equal to zero:

$$\gamma' = 0. \quad (1.172)$$

Thus, to describe the elastic scattering of two nucleons it is necessary, for a given energy, to find five complex functions ( $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  and  $\varepsilon$ ) of the scattering angle. The problem is simplified somewhat if we take into account the unitarity condition on the scattering amplitude (1.170) (the generalized optical theorem). From this condition, five different integral relations between the complex functions  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  and  $\varepsilon$  follow. Therefore, for a complete determination of the scattering amplitude (1.170), it is necessary to measure the angular dependence, at a given energy, of five independent quantities. Obviously, the required information can be obtained only from polarization experiments.

If the nucleons are initially unpolarized, scattering experiments enable us to determine only the angular dependence of the differential cross-section  $\sigma(\vartheta)$ . Experiments on *double scattering* enable us to measure the angular

dependence of the *polarization*  $P(\vartheta)$ . (In the case of coplanar double scattering, the polarization  $P(\vartheta)$  is measured by the *asymmetry of the scattering* at the second scatterer.) Experiments on *triple scattering* make it possible to measure the angular dependence of three other quantities. These are usually chosen to be the *depolarization*  $D(\vartheta)$  caused by the second scatterer in the case of triple coplanar scattering, the *rotation*  $R(\vartheta)$  of the *polarization vector* at the second scattering in the case of triple scattering in planes successively at right angles to each other, and the *longitudinal polarization*  $A(\vartheta)$  resulting when the spin of the particle being scattered is flipped with the help of the magnetic field due to the second scatterer.

The measurable quantities  $\sigma(\vartheta)$ ,  $P(\vartheta)$ ,  $D(\vartheta)$ ,  $R(\vartheta)$  and  $A(\vartheta)$  can be expressed in the following way in terms of the coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  and  $\varepsilon$  determining the scattering amplitude (Puzikov, Ryndin and Smorodinskii, 1957):

$$\left. \begin{aligned} \sigma(\vartheta) &= |\alpha|^2 + |\beta|^2 + 2|\gamma|^2 + |\delta|^2 + |\varepsilon|^2; \\ \sigma(\vartheta)P(\vartheta) &= 2\text{Im}[(\alpha + \beta)^*\gamma]; \\ \sigma(\vartheta)\{1 - D(\vartheta)\} &= 2|\delta|^2 + 2|\varepsilon|^2; \\ \sigma(\vartheta)R(\vartheta) &= \{|\alpha|^2 - |\beta|^2 + \text{Re}[(\delta + \varepsilon)^*(\delta - \varepsilon)]\} \cos(\vartheta/2) \\ &\quad - 2\text{Re}[(\alpha - \beta)^*\gamma] \sin(\vartheta/2); \\ \sigma(\vartheta)A(\vartheta) &= -\{|\alpha|^2 - |\beta|^2 + \text{Re}[(\delta + \varepsilon)^*(\delta - \varepsilon)]\} \sin(\vartheta/2) \\ &\quad - 2\text{Re}[(\alpha - \beta)^*\gamma] \cos(\vartheta/2). \end{aligned} \right\} \quad (1.173)$$

The relations (1.173), along with the unitarity relations, enable us in principle to find the quantities  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  and  $\varepsilon$ , that is, to determine completely the scattering amplitude (1.170). The realization of this, however, is extremely difficult in practice and, therefore, the experimental data are usually analyzed on the basis of the representation of the scattering amplitude in terms of the phase shifts.

**Isotropic character of proton-proton scattering and interaction in states with  $l \neq 0$ .** It would be possible to explain the principal feature of the nuclear scattering of protons by protons, namely, the fact that the cross-section is independent of the scattering angle, by assuming that, up to energies of the order of 500 MeV, the protons interact only in the *S*-state. However, the maximum possible magnitude of the scattering cross-section per unit solid angle in the *S*-state (equal to  $\chi^2$ ) at energy of 500 MeV is found to be approximately half of the observed cross-section (1.161). Therefore, it is necessary to take into account the interaction of the protons in states with  $l \neq 0$ , as is also confirmed by the large magnitude of the experimentally observed polarization arising in proton-proton scattering. We remark that polarization is possible only when the spin of the system

has the value  $S = 1$ , which is associated with states with odd  $l$ . Therefore, the large magnitude of the polarization in the scattering implies that there is strong interaction corresponding to states with odd  $l$ , and in particular to  $P$ -states.

The interaction in the  ${}^3P_0$  state, like that in the  ${}^1S_0$  state, leads to isotropic scattering, and the maximum possible magnitude of the cross-section per unit solid angle is also  $\propto^2$ . Therefore, allowance for the interaction of the protons in the  ${}^1S_0$  and  ${}^3P_0$  states makes it possible in principle to obtain for the isotropic cross-section a value in agreement with the experimental value (1.161). In doing this, however, it is necessary to assume that the phase shifts in the other states with  $l = 1$  ( ${}^3P_1$  and  ${}^3P_2$ ) are small, since otherwise the cross-section will depend on the angle. The marked difference between the phase shifts for  ${}^3P_0$ ,  ${}^3P_1$  and  ${}^3P_2$  indicates the important role of the non-central or *spin-orbit interaction*.

The isotropic nature of the scattering can also be explained if we assume that the interaction occurs in  $S$ - and  $D$ -states, but that the corresponding phase shifts have opposite signs. In fact, if the interaction occurs only in states with  $l = 0$  and  $l = 2$ , the scattering amplitude can be written:

$$f(\vartheta) = (1/k)\{e^{i\delta_0} \sin \delta_0 + 5e^{i\delta_2} (\sin \delta_2) P_2(\cos \vartheta)\}, \quad (1.174)$$

where  $\delta_0$  and  $\delta_2$  are the phase shifts in the states  ${}^1S_0$  and  ${}^1D_2$ . (Since  $T = 1$  and the parity is positive, we have  $S = 0$  for the states considered.)

The differential scattering cross-section is equal to

$$\begin{aligned} \sigma(\vartheta) &= (1/k^2)\{\sin^2 \delta_0 + 25(\sin^2 \delta_2) P_2^2(\cos \vartheta) \\ &\quad + 10 \sin \delta_0 \sin \delta_2 \cos(\delta_0 - \delta_2) P_2(\cos \vartheta)\}. \end{aligned} \quad (1.175)$$

The angular dependence of the cross-section is determined by the second and third terms in the relation (1.175). The second term is smaller by a factor of 4 at a scattering angle of  $90^\circ$  than at angle  $0^\circ$ . The sign of the third (interference) term is determined by the relative signs of the phase shifts  $\delta_0$  and  $\delta_2$  and by the value of the angle  $\vartheta$ . Since the quantity  $P_2(\cos \vartheta)$  is negative for  $\vartheta = 90^\circ$ , allowance for the interference can lead to an increase of the cross-section at  $90^\circ$ , and consequently to a smoothing of the angular dependence of the cross-section, only when the phase shifts  $\delta_0$  and  $\delta_2$  have opposite signs.

**Repulsion between nucleons at short distances.** The change of sign of the phase shift on going from the  $S$ - to the  $D$ -state can be obtained by assuming that strong repulsive forces act between nucleons at small distances, in contrast to the attractive forces which appear at large distances (Jastrow, 1951). At sufficiently high energies, the phase shift in the  $S$ -state is determined by this repulsion, whereas the phase shifts in states with

$l \neq 0$ , because of the effect of the centrifugal forces, are determined by the longer-range attractive forces.

It is not difficult to find the phase shift  $\delta_0$  in the  $S$ -state in the limiting case of infinite repulsive forces (the hard-sphere model or repulsive-core model) which appear at distances less than a certain distance  $\mathbf{r}_c$ , called the *repulsive-core radius*:

$$\delta_0 = -kr_c. \quad (1.176)$$

The nuclear forces of attraction appearing at distances greater than  $\mathbf{r}_c$  lead to a positive correction to the right-hand side of eqn. (1.176). If the nuclear potential is chosen to be of the form of repulsive core surrounded by a square potential well,

$$V(r) = \begin{cases} +\infty, & r < r_c; \\ -V_0, & r_c < r < r_0; \\ 0, & r_0 < r, \end{cases} \quad (1.177)$$

the phase shift  $\delta_0$  will be given by the expression

$$\delta_0 = -kr_c + \Delta, \quad (1.178)$$

where

$$\begin{aligned} \Delta &\equiv \arctan\{(k/k_0)\tan[k_0(r_0 - r_c)]\} - k(r_0 - r_c), \\ k_0 &= \sqrt{k^2 + (2\mu V_0/\hbar^2)}. \end{aligned} \quad (1.179)$$

It is easily verified that the phase shift  $\Delta$  given by the equality (1.179) is always positive. For this, we need use only the facts that  $k < k_0$  and that  $\alpha \tan x > \tan \alpha x$  for any value of  $x$  ( $0 < x < \pi/2$ ) if  $\alpha < 1$ .

At low energies ( $kr_c \rightarrow 0$ ), the main role is played by the attractive forces, and the negative term in (1.178) can be neglected. With increasing energy, the positive term in (1.178) decreases rapidly, while the phase shift due to the repulsive forces increases in magnitude, so that the phase shift  $\delta_0$  will change in sign from a positive value at low energies to a negative value at energies greater than a certain critical energy, the magnitude of which is determined by the parameters of the potential. In order that the change of sign of the phase shift occur at an energy in the region of 250 MeV we must choose  $r_c \approx 0.4 \times 10^{-13}$  cm. Generally speaking, the magnitude of  $r_c$  depends on the shape of the attractive potential.

The conclusion that strong repulsive forces are present at short distances also follows from the energy dependence of the total scattering cross-section. In fact, if the dependence of the potential on the distance between the

particles were monotonic, the phase shifts should increase monotonically with the energy. Therefore, the product

$$k^2 \sigma = 4\pi \sum_l (2l+1) \sin^2 \delta_l \quad (1.180)$$

should also increase monotonically with increasing energy. Figure 1.9 shows the dependence of the quantity  $k^2 d\sigma/d\theta$  at  $\vartheta = 90^\circ$  on the energy in the laboratory coordinate frame in the case of proton-proton scattering, and Fig. 1.10 shows the energy dependence of  $k^2 \sigma$  in the case of neutron-proton scattering. According to the experimental dependences (see Figs. 1.9 and 1.10) the quantity (1.180) decreases in the energy range from 10 to 100 MeV, in which the interaction in the *S*-state is the most important. This means that, in this energy range, the phase shift in the *S*-state decreases, which is possible only in the case of strong repulsion between the nucleons at short distances.

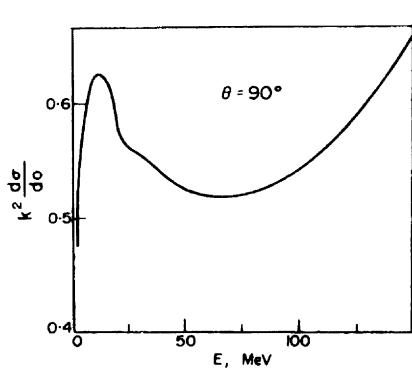


Figure 1.9. Energy dependence of the quantity  $k^2 d\sigma/d\theta$  at  $\vartheta = 90^\circ$  in the case of proton-proton scattering.

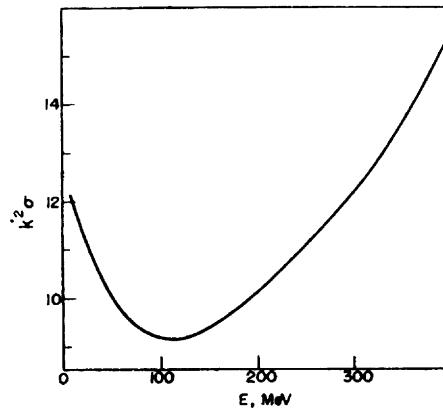


Figure 1.10. Energy dependence of the quantity  $k^2 \sigma$  in the case of neutron-proton scattering.

**Phase-shift analysis.** The most important problem in the analysis of experimental data on scattering is the determination of the phase shifts. Specifying the phase shift completely characterizes the interaction between the particles in a given state. Therefore, the extraction of all the phase shifts from the experimental data is equivalent to obtaining complete information on the nuclear interaction between the nucleons.

A knowledge of the phase shifts makes it possible in principle to find the dependence of the forces on distance, that is, to determine the nuclear interaction potential. It is true that this problem (the inverse problem of

scattering theory) requires not only information on the energy dependence of the phase shifts but also additional information on the discrete levels in the system and on the asymptotic forms of the wave functions corresponding to these levels. Because of this, at the present time it has still not been possible to solve such a problem completely. We remark that the determination of the phase shifts from the experimental data on the angular dependences of the cross-section, the polarization, etc., is a laborious task which requires the use of a computer to solve it.

The energy dependence of the phase shifts can be found by using the experimental data on scattering at different energies. Clearly, the phase shifts are real only when inelastic processes can be neglected. Therefore, phase-shift analysis is meaningful up to energies of 500 MeV.

The *phase-shift analysis* of the scattering of neutrons by protons is more complicated than that of the scattering of protons by protons, since the neutron-proton system can be found in the isospin states  $T=1$  and  $T=0$ , whereas a system of two protons can be found only in the state  $T=1$ . With the assumption of isotopic invariance of the nuclear interaction, the phase shifts corresponding to  $T=1$  should be the same for a system of two protons and for a neutron-proton system. The phase shifts corresponding to  $T=1$  can be found, therefore, from the proton-proton scattering data, and from the analysis of the scattering of neutrons by protons we need to determine only the phase shifts corresponding to  $T=0$ .

We shall make use of the classification of the states of a system of two nucleons (see §1.9) and characterize the different states by the quantum numbers of the isospin  $T$ , the total angular momentum  $J$ , the spin  $S$  and the parity  $w$ . If  $S=0$  (singlet spin state), the total angular momentum  $J$  coincides with the orbital angular momentum  $l$ , and the scattering amplitude has the form

$$f^{S=0}(\vartheta) = (i/2k) \sum_l (2l+1)(1-S_l) P_l(\cos \vartheta), \quad (1.181)$$

where  $S_l$  is the unitary scattering matrix, connected with the phase shift  $2^{S+1}\delta_l^J$  by the usual relation

$$S_l = e^{2i\delta_l^J}. \quad (1.182)$$

If  $S=1$  (the triplet spin state), the orbital angular momentum is not conserved, and the scattering amplitude can be represented in the form of a three-row matrix

$$\begin{aligned} f_{\mu'\mu}^{S=1}(\vartheta, \varphi) &= (i/2k) \sum_{ll'J} \sqrt{4\pi(2l+1)} Y_{l',\mu-\mu'}(\vartheta, \varphi) (l01\mu|J\mu) \\ &\times (l'\mu - \mu'1\mu'|J\mu) (\delta_{ll'} - S_{ll'}^J), \end{aligned} \quad (1.183)$$

where  $\mu$  and  $\mu'$  are the components of the spin of the system along a certain direction in the initial and final states,  $\varphi$  is the angle between the plane defined by this direction and by the direction of motion of the incident particle and the scattering plane, and  $S_{\mu l}^J$  is the scattering matrix. Since  $l$  is not conserved, the matrix  $S_{\mu l}^J$ , generally speaking, is not diagonal. For a fixed value of  $J$ , the quantities  $l$  and  $l'$  can take the values  $J - 1$ ,  $J$  and  $J + 1$ . If  $l = J$ , then  $l' = J$  because of conservation of parity. Taking into account the unitarity of the scattering matrix, in this case we can write

$$S_{JJ}^J = e^{2i^3\delta_J^J}, \quad l = l' = J. \quad (1.184)$$

If  $l$  is equal to  $J + 1$  or  $J - 1$ , the possible values of  $l'$  are also equal to  $J + 1$  or  $J - 1$ . Therefore,  $S_{\mu l}^J$  is a two-row matrix. But since the scattering matrix is a unitary symmetric matrix, it can be diagonalized by means of the orthogonal matrix

$$U^J = \begin{pmatrix} \cos \varepsilon^J & \sin \varepsilon^J \\ -\sin \varepsilon^J & \cos \varepsilon^J \end{pmatrix}, \quad (1.185)$$

where  $\varepsilon^J$  is some real parameter. Thus, in the case  $l, l' = J \pm 1$ , the scattering matrix can be represented in the form

$$S_{l'l}^J = \begin{pmatrix} \cos \varepsilon^J & -\sin \varepsilon^J \\ \sin \varepsilon^J & \cos \varepsilon^J \end{pmatrix} \begin{pmatrix} e^{2i^3\delta_{J+1}^J} & 0 \\ 0 & e^{2i^3\delta_{J-1}^J} \end{pmatrix} \begin{pmatrix} \cos \varepsilon^J & \sin \varepsilon^J \\ -\sin \varepsilon^J & \cos \varepsilon^J \end{pmatrix} \quad (1.186)$$

(the *Blatt-Biedenharn representation* (1952)). According to (1.186), the scattering matrix  $S_{\mu l}^J$  is determined by three real parameters: the two scattering eigenphases  ${}^3\delta_{J+1}^J$  and  ${}^3\delta_{J-1}^J$  in the states with  $l = J + 1$  and  $l = J - 1$ , and the *mixture parameter*  $\varepsilon^J$ .

Sometimes another representation for the scattering matrix  $S_{\mu l}^J$  is used (the Stapp representation (1957)):

$$S_{l'l}^J = \begin{pmatrix} e^{i^3\bar{\delta}_{J+1}^J} & 0 \\ 0 & e^{i^3\bar{\delta}_{J-1}^J} \end{pmatrix} \begin{pmatrix} \cos 2\bar{\varepsilon}^J & i \sin 2\bar{\varepsilon}^J \\ i \sin 2\bar{\varepsilon}^J & \cos 2\bar{\varepsilon}^J \end{pmatrix} \begin{pmatrix} e^{i^3\bar{\delta}_{J+1}^J} & 0 \\ 0 & e^{i^3\bar{\delta}_{J-1}^J} \end{pmatrix}. \quad (1.187)$$

We give the relations connecting the phase shifts and the mixture parameters for the above representations:

$$\left. \begin{aligned} \sin(\delta_{J+1} - \delta_{J-1}) &= \sin 2\bar{\varepsilon}^J / \sin 2\varepsilon^J; \\ \sin(\bar{\delta}_{J+1} - \bar{\delta}_{J-1}) &= \tan 2\bar{\varepsilon}^J / \tan 2\varepsilon^J; \\ \delta_{J+1} + \delta_{J-1} &= \bar{\delta}_{J+1} + \bar{\delta}_{J-1}. \end{aligned} \right\} \quad (1.188)$$

In place of the parameter  $\bar{\varepsilon}^J$ , one often uses the quantity  $\varrho^J$ :

$$\varrho^J \equiv \sin 2\bar{\varepsilon}^J. \quad (1.189)$$

Because of the small range of the nuclear forces, the elastic scattering of nucleons can be described by means of a small number of phase shifts. In the case  $T=1$ , the phase-shift analysis of the scattering is performed with inclusion of the phase shifts in the states  ${}^1S_0$ ,  ${}^3P_{0,1,2}$ ,  ${}^1D_2$ ,  ${}^3F_{2,3,4}$ ,  ${}^1G_4$ ,  ${}^3H_{4,5,6}$ , and the mixture parameters  $\varepsilon^2$  and  $\varepsilon^4$ . In the case  $T=0$ , the phase-shift analysis is performed with inclusion of the phase shifts in the states  ${}^3S_1$ ,  ${}^1P_1$ ,  ${}^3D_{1,2,3}$ ,  ${}^1F_3$ ,  ${}^3G_{3,4,5}$ , and the mixture parameters  $\varepsilon^1$  and  $\varepsilon^3$ . Figures 1.11 and 1.12 show the *energy dependences of the phase shifts* and the mixture parameters (MacGregor, Arndt and Wright, 1968a,b).

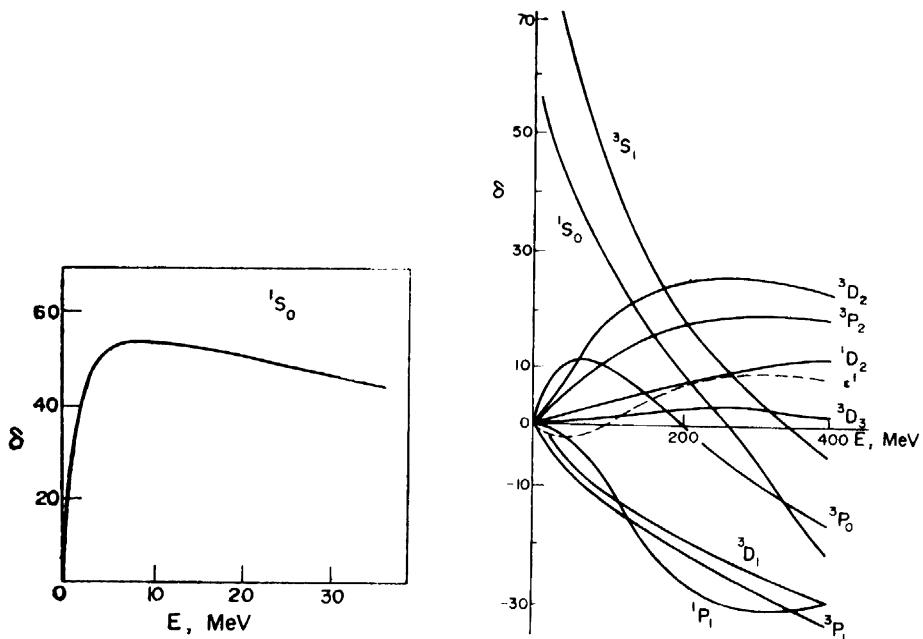


Figure 1.11. Energy dependence of phase shifts and of the mixture parameter  $\varepsilon^1$ .

The  ${}^1S_0$  phase shift vanishes at low energies, indicating the absence of a singlet bound state in a system of two nucleons. With increasing energy, the  ${}^1S_0$  phase shift increases, reaching values in the region of  $60^\circ$  at 3 MeV, and then decreases and becomes negative at energies greater than 250 MeV. As we have already observed, the change of sign of the  ${}^1S_0$  phase shift indicates the existence of strong repulsion between the nucleons at short distances. The  ${}^3S_1$  phase shift at zero energy is equal to  $180^\circ$ , and this is directly

connected with the existence of a triplet bound state in a system of two nucleons, namely, the deuteron. At an energy of 325 MeV, the  $^3S_1$  phase shift goes to zero, and then takes negative values. The phase shifts in the states  $^3P_{0,1,2}$  differ markedly, indicating the presence of strong non-central or spin-orbit interaction between the nucleons. We note that the phase shifts in the  $S$ -,  $P$ - and  $D$ -states are quantities of the same order. The phase shifts in the  $F$ -,  $G$ - and  $H$ -states do not exceed a few degrees.

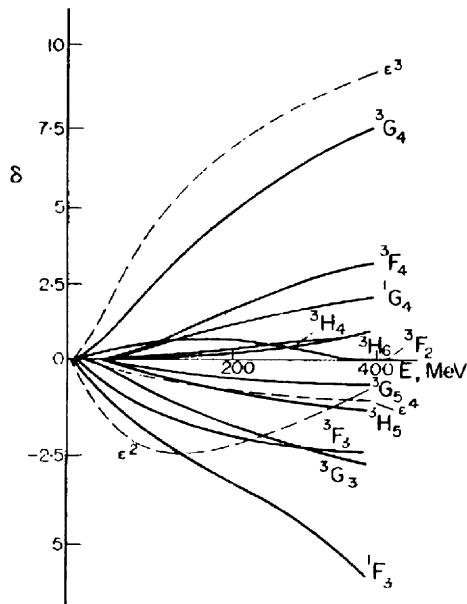


Figure 1.12. Energy dependence of phase shifts and of the mixture parameters.

**Phenomenological potentials.** So long as inelastic processes do not play an important role, the interaction between nucleons can be described by means of a potential. The determination of the form of the nuclear interaction potential between nucleons from scattering data is the most important problem of the non-relativistic theory. It follows from the data on nucleon-nucleon scattering at high energies that the nuclear potential should describe the exchange interaction, the strong repulsion at short distances, and the important tensor and spin-orbit interactions (in addition to the attractive forces). The numerous attempts to derive the potential of the nucleon-nucleon interaction from meson field theory have not led to the desired result. Therefore, at the present time, to describe the interaction between nucleons there is wide-spread use of potentials that are chosen phenomenologically in such a way as to give the best explanation of the

accumulated experimental scattering data. Several variants of potentials giving a good description of the scattering data in the energy range from 0 to 400 MeV have been proposed. As illustrations, we shall give the expressions for the Gammel-Thaler and Hamada-Johnston potentials, which are those most often used in practice.

The *Gammel-Thaler potential* contains central, tensor and spin-orbit interactions:

$$V_{GT} = V_C(\mathbf{r}) + V_T(\mathbf{r})S_{12} + V_{LS}\mathbf{L} \cdot \mathbf{S}. \quad (1.190)$$

The radial dependence of all the terms in (1.190) is chosen in the form of a perfectly hard sphere of radius  $r_c$  and a *potential of the Yukawa type*:

$$V(r) = \begin{cases} \infty, & r < r_c; \\ -Ve^{-\mu r}/\mu r, & r > r_c. \end{cases} \quad (1.191)$$

The parameters  $V$ ,  $\mu$  and  $r_c$  are chosen independently for each state, that is, definite values are chosen for the even singlet states and the even triplet states (for the central, tensor and spin-orbit forces separately), and analogously for the odd states. In the general case, the potential (1.190) is characterized by 24 parameters. Since, in practice, the same value of  $r_c$  has been used for the central, tensor and spin-orbit forces, the number of independent parameters has been reduced to 20. The parameter values ensuring the best agreement with the experimental data are given in Table 1.3 (Gammel and Thaler, 1957, 1959).

The Hamada-Johnston potential takes into account not only the central, tensor and spin-orbit interactions but also the squared spin-orbital interaction:

$$V_{HJ} = V_C(\mathbf{r}) + V_T(\mathbf{r})S_{12} + V_{LS}(\mathbf{r})\mathbf{L} \cdot \mathbf{S} + V_{LL}(\mathbf{r})L_{12}, \quad (1.192)$$

where

$$L_{12} = [\delta_{LJ} + 2S(S+1) - 3]\mathbf{L}^2 - (\mathbf{L} \cdot \mathbf{S})^2. \quad (1.193)$$

The radial dependence in (1.192) corresponds at short distances to the presence of a perfectly hard sphere of radius  $r_c$ ; at large distances, the dependences of the individual terms in (1.192) are chosen in the form

$$\left. \begin{aligned} V_C(r) &= V_C Y(x) \{1 + a_C Y(x) + b_C Y^2(x)\}; \\ V_T(r) &= V_T Z(x) \{1 + a_T Y(x) + b_T Y^2(x)\}; \\ V_{LS}(r) &= V_{LS} Y^2(x) \{1 + a_{LS} Y(x)\}; \\ V_{LL}(r) &= V_{LL} (Z(x)/x^2) \{1 + a_{LL} Y(x) + b_{LL} Y^2(x)\}, \end{aligned} \right\} \quad (1.194)$$

where  $Y(x) = e^{-x}/x$ ,  $Z(x) = (1 + 3/x + 3/x^2)Y(x)$ ,  $x \equiv \mu cr/\hbar$  and  $\mu$  is the meson mass ( $\hbar/\mu c = 1.415 \times 10^{-13}$  cm,  $\mu c^2 = 139.4$  MeV and  $x_c \equiv \mu cr_c/\hbar = 0.343$ ).

TABLE 1.3.

Potential parameters	$S = 0$		$S = 1$	
	$T = 1$ (l-even)	$T = 0$ (l-odd)	$T = 0$ (l-even)	$T = 1$ (l-odd)
$V_C$ , MeV	425.5	-100 for $E_{np} \leq 90$ and $E_{np} = 156$ . -150 for $E_{np} = 310$	100.7 for $E_{np} \leq 90$ and $E_{np} = 156.60$ for $E_{np} = 310$	0
$\mu_C$ , $\text{cm}^{-1}$	$1.45 \times 10^{13}$	$1.0 \times 10^{13}$	$1.23 \times 10^{13}$	-
$V_r$ , MeV	-	-	257 for $E_{np} \leq 90$ 230 for $E_{np} = 156$ 175 for $E_{np} = 310$	-22.5
$\mu_T$ , $\text{cm}^{-1}$	-	-	$1.203 \times 10^{13}$	$0.8 \times 10^{13}$
$V_{LS}$ , MeV	-	-	5000	7317.5
$\mu_{LS}$ , $\text{cm}^{-1}$	-	-	$3.7 \times 10^{13}$	$3.7 \times 10^{13}$
$r_c$ , cm	$0.4 \times 10^{-13}$	$0.5 \times 10^{-13}$	$0.4 \times 10^{-13}$	$0.4125 \times 10^{-13}$

The potentials  $V_C(r)$  and  $V_T(r)$  are chosen in such a way that at large distances they coincide with the meson-theory potentials taking one-pion exchange into account:

$$V_C = 0.08(\mu c^2/3)[2S(S+1) - 3][2T(T+1) - 3];$$

$$V_T = 0.08(\mu c^2/3)[2T(T+1) - 3].$$

The values of the parameters  $V_C$ ,  $V_T$ ,  $V_{LS}$ ,  $V_{LL}$ ,  $a$  and  $b$  are given in Table 1.4 (Hamada and Johnston, 1962).

The tensor spin-orbit interaction is most important in the triplet and singlet spin states with  $J = L$ . The *Hamada-Johnston potential* (1.192) agrees well with the data on the interaction of two nucleons at low energies (the binding energy of the deuteron, the scattering lengths, the effective ranges, etc.) and reproduces well the energy dependence of the phase shifts up to energies of the order of 400 MeV.

We note the following features of the interaction, which are characteristic of both the *Gammel-Thaler potential* and the Hamada-Johnston potential. In all states, strong repulsive forces act between the nucleons at short distances, and these are described by a core radius of  $0.4 \times 10^{-13}$  cm. Outside the core region, the potential for even states corresponds to attractive forces. In the odd states the interaction is considerably weaker than

TABLE 1.4.

Potential parameters	$S = 0$		$S = 1$	
	$T = 1$ (l-even)	$T = 0$ (l-odd)	$T = 0$ (l-even)	$T = 1$ (l-odd)
$V_C$ MeV	-11.2	33.5	-11.2	3.71
$a_C$	8.7	-8.0	6.0	-9.07
$b_C$	10.6	12.0	-1.0	3.48
$V_T$ , MeV	-	-	-11.2	3.71
$a_T$	-	-	-0.5	-1.29
$b_T$	-	-	0.2	0.55
$V_{LS}$ , MeV	-	-	10.4	27.4
$a_{LS}$	-	-	-0.1	-7.12
$V_{LL}$ , MeV	-0.124	-0.371	0.371	-0.124
$a_{LL}$	0.2	2.0	1.8	-7.26
$b_{LL}$	-0.2	6.0	-0.4	6.92

in the even states. In the odd singlet states, the interaction is repulsive. At large distances the tensor forces make a substantial contribution to the interaction. The tensor forces are attractive in the  $^3S_1$  and  $^3P_0$  states.

Along with the Gammel-Thaler and Hamada-Johnston potentials, other phenomenological potentials are employed. Almost all these potentials are constructed in a similar manner, so we shall restrict the consideration to brief descriptions. The first phenomenological nucleon-nucleon potentials taking into account short-distance repulsion were introduced by Gartenhaus (Gartenhaus, 1955) and Signell and Marshak (Signell and Marshak, 1957). Both these were derived in terms of the approximate statical meson theory of nuclear forces. These potentials contain both central forces with repulsive cores and tensor forces. The *Gartenhaus potential*, however, is proposed for nucleons at rest and therefore ignores the spin-orbit interaction. The Signell-Marshak potential is equal to the sum of the Gartenhaus potential and a short-range spin-orbit potential which is employed to explain the polarization phenomena and thus to improve the agreement of the theory and experiment. However, similarly to the Gammel-Thaler potential, the Signell-Marshak potential overestimates the contribution of the spin-orbit interaction. Gartenhaus and Signell-Marshak potentials are very close at long distances between nucleons,  $r > 10^{13}\text{cm} \equiv 1\text{ fm}$ .

When calculating the nuclear structure, many authors make use of the phenomenological nucleon-nucleon potential proposed by the Yale group (Lassila e.a., 1962). It contains terms responsible for central, tensor, spin-orbit, and quadratic spin-orbit interaction (similarly to the Hamada-Johnston potential) and, besides that, includes the one-pion exchange potential

that was obtained using the simplest version of the meson field theory. The contribution of the last term is assumed to be correct and considerable for sufficiently large ( $> 2 \text{ fm}$ ) distances between the nucleons. The parameters of the *Yale-group potential* were found by comparing calculated (using this potential) and experimental values of nucleon-nucleon scattering phases. The potential contains a *hard repulsive core*; it was parametrized with the use of sums of simple functions  $a_n x^{-n} e^{-2x}$  ( $n$  – positive integers,  $x = \frac{uc}{\hbar} r$ ).

Many authors employ the phenomenological *Reid potential* (Reid, 1968) in spite of its complexity. This potential is described by different analytical expressions for different nucleon-nucleon scattering partial waves for central, tensor, and spin-orbit interaction; their dependences on the internucleon distances are described by superpositions of terms of the form  $x^{-m} e^{-nx}$ , where  $m$  and  $n$  are positive integers. Theoretical and experimental results were compared to find the sets of parameters of the Reid potential for the  $S$ ,  $P$ ,  $D$  and  $F$  scattering phases for both  $T = 0$  and  $T = 1$  values of the total two-nucleon isotopic spin. Both hard repulsive core, when the potential tends to infinity for  $r < r_c$ , and soft core, when the repulsive potential is finite though large for  $r < r_c$ , were considered. The sets of parameters were estimated numerically involving the data obtained in terms of the above mentioned theory of one-pion exchange potential.

Lately, many authors employ as well the Paris (Lacombe e.a., 1980) and Bonn (Macleidt e.a., 1987) phenomenological potentials. We consider the first one in brief, since modern approaches to the construction of phenomenological potentials have much in common. Usually, nucleon-nucleon potentials are constructed by fitting  $np$  data for  $T = 0$  states and either  $np$  or  $pp$  data for  $T = 1$  states. The *Bonn potential* was fit to  $np$  data in all states. In contrast, the *Paris potential* was fit to  $pp$  data for  $T = 1$  channels. The Paris potential contains central, tensor, spin-spin, spin-orbit, and quadratic spin-orbit interactions for two isotopic states with  $T = 0$  and  $T = 1$ . At medium and long internucleon distances ( $> 0.8 \text{ fm}$ ), the behavior of this potential is well described by the meson theory of nuclear forces which allows for the exchange of both light  $\pi$ -mesons and heavy  $\omega$ -mesons. At short distances ( $< 0.8 \text{ fm}$ ), including the repulsive core range where the intrinsic quark-gluon structure of the nucleon makes its effect, the potential is described by a model (phenomenological) dependence. (We shall consider dependence of nuclear forces on the internucleon distances and other factors in Chapter 7). The analytic form of the Paris potential is rather simple, it is described by a finite sum of Yukawa-type terms  $g_j \frac{e^{-\gamma_j r}}{r}$  and their derivatives. The parameters  $g_j$  and  $\gamma_j$ , where  $j = 1, 2, 3, \dots$  (in the case  $j = 1$  data of the one-pion-exchange theory are employed), satisfy conditions which provide that the potential is finite for  $r \rightarrow 0$  (i.e., contains a soft core). The potential provides a good description of the phase shift for

$J \leq 7$ , the deuteron ground state, and is successfully applied in the theory of nuclear matter (see Chapter 5) to obtain a satisfactory value of binding energy per nucleon ( $\sim 10$  MeV) for the majority of stable nuclei.

A new high-quality nucleon-nucleon potential with explicit charge dependence and charge asymmetry (an updated version of the *Argonne potential*) was presented by R. Wiringa, V. Stoks and R. Schiavilla (1995). That potential fits both  $pp$  and  $nn$  data, as well as low-energy  $nn$  scattering parameters and deuteron properties.

**Deuteron ground state parameters in terms of a phenomenological potential.** To conclude this section, we consider the deuteron-ground-state wave function and its main characteristics calculated in terms of a modern phenomenological nucleon-nucleon potential which has been constructed using the data on nucleon-nucleon scattering in a wide energy range from very low energies ( $\leq 1$  MeV) up to energies about 400 MeV. For higher energies, inelastic processes become very important. To find the deuteron wave function and to describe its properties by solving the Schrödinger equation with the above potential is an interesting problem since main deuteron characteristics have been obtained with high accuracy by various experiments and it seems to be promising to describe them theoretically making use of independent data on nucleon-nucleon scattering.

The description of both deuteron and low-energy nucleon-nucleon parameters is the best in the case of the Paris phenomenological potential which includes noncentral (tensor) forces. Figure 1.13 gives the calculated deuteron radial wave functions  $u(r)$  and  $w(r)$  in the  $S$ - and  $D$ -states, respectively (in terms of  $Fm^{\frac{1}{2}}$  units). These functions attain their maximum values 0.52394 and 0.17366 approximately for  $r = 1.6$  Fm and  $r = 1.4$  Fm, respectively. The calculations with the Paris potential yield the following values of the deuteron binding energy  $\varepsilon = 2.225$  MeV, probability to find the deuteron in the  $D$ -state  $p_D = 0.058$ , and deuteron magnetic dipole and electric quadrupole moments:  $\mu_d = 0.853 \frac{e\hbar}{2M_p c}$  and  $Q = 2.79 \times 10^{-13}$  cm. The relevant nucleon-nucleon scattering lengths are given by  $a_t \equiv a_{np} = 5.427 \times 10^{-13}$  cm,  $a_p \equiv a_{pp} = -7.810 \times 10^{-13}$  cm,  $a_n \equiv a_{nn} = -17.612 \times 10^{-13}$  cm and the effective radii are equal to  $r_{ot} = 1.766 \times 10^{-13}$  cm,  $r(0, -\varepsilon) = 1.765 \times 10^{-13}$  cm,  $r_{op} \equiv r_{pp} = 2.797 \times 10^{-13}$  cm,  $r_{on} = 2.881 \times 10^{-13}$  cm.

We note that the cross-section of elastic electron-deuteron scattering (*deuteron formfactor*  $A(q^2)$ ), calculated with the use of the deuteron wave function that corresponds to the Paris potential, is in a good accordance with experimental data for a wide range of momentum transfer ( $0 \leq q \leq 5 \text{ fm}^{-1}$ ). The  $D$ -wave contribution to scattering is important only for sufficiently large values of the momentum transfer (for  $q \geq 4 \text{ fm}^{-1}$ ).

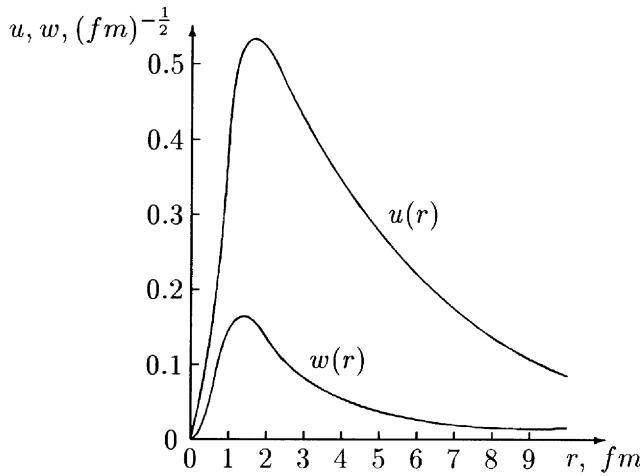


Figure 1.13. The deuteron radial functions of the  $S$ - and  $D$ -states in the case of the Paris potential.

**Non-relativistic theory of the nucleus.** The principal problem of the *non-relativistic theory of the nucleus* is to explain the properties of nuclei, which are complex systems consisting of nucleons, on the basis of information on the properties of nucleons and on the nature of the interaction between them. At the present time, the properties of free nucleons and the character of the interaction between them in the non-relativistic energy region have been sufficiently well studied, both experimentally and theoretically.

The non-relativistic theory of the nucleus is based on the assumption that only two-body forces act between the simplest constituent parts of nuclei, that is, between the nucleons. In the non-relativistic theory of the nucleus, it is also assumed that the properties of the nucleons themselves (the mass, charge, magnetic moment, etc.) are not changed inside the nuclei, despite the presence of the strong interaction between the nucleons. Therefore, the Hamiltonian of a nucleus consisting of  $A$  nucleons can be represented in the form

$$H = \sum_i [-(\hbar^2/2M)\nabla_i^2] + \sum_{i<j} V_{ij}, \quad (1.195)$$

where  $-(\hbar^2/2M)\nabla_i^2$  is the kinetic energy operator of an individual nucleon and  $V_{ij}$  is the two-body interaction potential between the nucleons  $i$  and  $j$ .

In contrast to a system of charged particles with Coulomb interaction, for which the total energy is made up of the interaction energies of the sep-

arate pairs of particles (this additivity being a consequence of the linearity of the equations of electrodynamics), in the case of nuclear systems there is no direct proof that the nuclear interaction is also additive, that is, that the interaction of two nucleons does not depend on the presence of a third nucleon. Therefore, at the present time, the only justification for the hypothesis of the *two-body nature of the nuclear forces* between the nucleons in nuclei is the absence of effects which would indicate the violation of this hypothesis.

According to the Hamiltonian (1.195), the interaction between nucleons is described by a potential in the non-relativistic theory. However, at sufficiently high energies inelastic processes associated with the formation of mesons become important and, therefore, the possibility of describing the interaction between nucleons by means of a potential is limited in the high-energy region. This restriction on the energy means that an unambiguous determination of the potential at short distances is completely impossible. However, since in nuclei the maximum possible values of the momenta of the nucleons correspond to energies less than 150 MeV, it may be hoped that relativistic effects can be neglected in the study of the structure of nuclei.

The study of the properties of nuclei containing more than two nucleons can serve as an additional source of information on the interaction between nucleons. For example, the properties of a system of two nucleons with energies close to zero can be explained by forces with zero range. However, in the case of forces of zero range, the binding energy of a system of three nucleons is found to be infinitely large. Therefore, the very fact of the existence of the nuclei  $^3H$  and  $^3He$  indicates that the nuclear forces are characterized by finite range. The data on nucleon-nucleon scattering at low energies ( $\leq 10$  MeV) make it possible to determine only the effective parameters characterizing the magnitude and range of the interaction potential. As we have seen, these data are insufficient to determine the shape of the potential. Unlike the two-nucleon problem, the problem of the motion of a system consisting of three or more nucleons turns out to be very sensitive to the shape of the two-nucleon potential. This is connected with the fact that two-nucleon scattering is completely determined by the scattering matrix on the energy shell, whereas the properties of systems consisting of a larger number of nucleons depend also on the scattering matrix off the energy shell. Therefore, the indeterminacy of a potential obtained from experiments on two-nucleon scattering can in principle be eliminated if we take into account the properties of more complex nuclei.

### 1.11. Problems

**1.1.** Show that the wave function  $\varphi_0(r)$  of the ground state of the neutron-proton system and the functions  $\varphi_{\mathbf{k}}(r)$  the continuous spectrum form a complete set.

The energy of the neutron-proton system consists of a discrete level, corresponding to the bound state of a system, and the continuous spectrum of positive energy values, corresponding to the unbound states. In the approximation that the nuclear forces have zero range, the functions  $\varphi_0(r)$  and  $\varphi_{\mathbf{k}}(r)$  have the form

$$\varphi_0(\mathbf{r}) = \sqrt{\frac{\alpha}{2\pi}} \frac{e^{-\alpha r}}{r}, \quad \varphi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} - \frac{1}{\alpha + ik} \frac{e^{ikr}}{r}. \quad (1.196)$$

The functions  $\varphi_0(r)$  and  $\varphi_{\mathbf{k}}(r)$  correspond to the values  $E_0 = -\hbar^2 \alpha^2/M$  and  $E_k = \hbar^2 k^2/M$  of the energy of the system. In the approximation that the nuclear forces have zero range, the interaction is characterized by a single parameter  $\alpha$ , which determines both the energy of the bound state of the system and the scattering amplitude in the unbound states.

Functions  $\varphi_{\mathbf{k}}(r)$  and  $\varphi'_{\mathbf{k}}(r)$  corresponding to different values  $\mathbf{k}$  and  $\mathbf{k}'$  of the wave vector are orthogonal:

$$\int d^3\mathbf{r} \varphi_{\mathbf{k}}(\mathbf{r}) \varphi_{\mathbf{k}'}^*(\mathbf{r}) = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}').$$

The functions  $\varphi_{\mathbf{k}}(r)$  are also orthogonal to the function  $\varphi_0(r)$  describing the bound state of the system:

$$\int d^3\mathbf{r} \varphi_{\mathbf{k}}(\mathbf{r}) \varphi_0(\mathbf{r}) = 0.$$

It can easily be verified directly that the set of functions (1.196) satisfies the completeness condition

$$\varphi_0(\mathbf{r}) \varphi_0(\mathbf{r}') + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \varphi_{\mathbf{k}}(\mathbf{r}) \varphi_{\mathbf{k}}^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (1.197)$$

As an example of a system of functions that take into account the finite range of the nuclear forces, we can choose the bound state wave function for the *Hulthén potential*

$$\varphi_0(\mathbf{r}) = \sqrt{\frac{\alpha(\alpha + \beta)\beta}{2\pi(\beta - \alpha)^2}} \frac{e^{-\alpha r} - e^{-\beta r}}{r} \quad (1.198)$$

and complete it by the scattering functions which are orthogonal to the bound state function (1.198)

$$\varphi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} + f(k) \frac{e^{ikr} - e^{-\beta r}}{r}, \quad (1.199)$$

where

$$f(k) = -\left\{ \beta - \frac{\beta^2 + k^2}{2\beta} - \frac{(\beta^2 + k^2)^2}{2(\alpha + \beta)^2 \beta} + ik \right\}^{-1}, \quad (1.200)$$

and  $\beta$  is a parameter usually chosen to have the value  $\beta = 7\alpha$ . The functions (1.198) and (1.199), as well as functions (1.196), satisfy the completeness condition (1.197).

**1.2.** Find the normalization factor for the wave function of the ground state of the deuteron with allowance for the finite range of the nuclear forces (Smorodinskii, 1957).

The formula (1.11) for the wave function of the ground state of the deuteron gives the correct dependence on  $r$  outside the range of the nuclear forces. For many problems, it is necessary to know the wave function of the deuteron inside this region. But the normalization coefficient in (1.11) was obtained with the assumption of zero range of the nuclear forces. We shall show how the normalization coefficient in (1.11) must be changed to take account of the finite range of the nuclear forces.

We shall denote the exact wave function of the deuteron ground state by  $\tilde{\varphi}_0(r) \equiv u(r)/r$ . This function should be normalized in accordance with the condition

$$\int d^3r \tilde{\varphi}_0^2(r) = 1. \quad (1.201)$$

The radial function  $u$  satisfies the equation

$$u'' - \frac{M}{\hbar^2} V(r) u = \alpha^2 u. \quad (1.202)$$

We shall regard  $\alpha^2$  in this equation as some parameter. Differentiating eqn. (1.120) with respect to this parameter, we find

$$\frac{\partial}{\partial \alpha^2} u'' - \frac{M}{\hbar^2} V(r) \frac{\partial}{\partial \alpha^2} u = u + \alpha^2 \frac{\partial}{\partial \alpha^2} u. \quad (1.203)$$

Multiplying eqn. (1.202) by  $\partial u / \partial \alpha^2$  and the equality (1.203) by  $u$  and subtracting the first equality from the second, we obtain

$$u^2 = u \frac{\partial u''}{\partial \alpha^2} - u'' \frac{\partial u}{\partial \alpha^2} \equiv \frac{\partial}{\partial r} \left( u^2 \frac{\partial}{\partial \alpha^2} \frac{u'}{u} \right),$$

whence

$$\int_0^\infty dr u^2 = u^2 \frac{\partial}{\partial \alpha^2} \left( \frac{u'}{u} \right) \Big|_0^\infty. \quad (1.204)$$

We introduce the approximate deuteron wave function  $\varphi_0(r)$ :

$$\varphi_0(r) = \frac{v(r)}{r},$$

where  $v(r)$  is the approximate radial function, which coincides with the exact function  $u(r)$  outside the range of the nuclear forces ( $r > r_0$ ):

$$v(r) = u(r), \quad r > r_0.$$

For the approximate function  $v(r)$ , we can write a relation analogous to (1.204):

$$\int_0^\infty dr v^2 = v^2 \frac{\partial}{\partial \alpha^2} \left( \frac{v'}{v} \right) \Big|_0^\infty. \quad (1.205)$$

Subtracting the relation (1.205) from the relation (1.204) and taking into account that  $u(0) = 0$  and that the functions  $u$  and  $v$  coincide outside the range of the nuclear forces, we find

$$\int_0^\infty dr u^2 - \int_0^\infty dr v^2 = v^2(0) \frac{\partial}{\partial \alpha^2} \left( \frac{v'(0)}{v(0)} \right). \quad (1.206)$$

We now make use of the relation (1.48) from the theory of the effective range, from which it follows that the ratio  $v'(0)/v(0)$  is equal to

$$\frac{v'(0)}{v(0)} = -\frac{1}{a} - \frac{1}{2} \alpha^2 r_0,$$

and, therefore,

$$\frac{\partial}{\partial \alpha^2} \left( \frac{v'(0)}{v(0)} \right) = -\frac{1}{2} r_0. \quad (1.207)$$

The approximate radial function  $v(r)$  has the form

$$v(r) = C e^{-\alpha r}. \quad (1.208)$$

Substituting  $v(r)$  into (1.206) and using the relation (1.207), we obtain

$$\int_0^\infty dr u^2 - C^2 \int_0^\infty dr e^{-2\alpha r} = -\frac{1}{2} C^2 r_0.$$

Using the normalization condition (1.201) for the exact wave function, we find the following value for the normalization constant in the relation (1.208):

$$C = \sqrt{\frac{\alpha}{2\pi(1-\alpha r_0)}}. \quad (1.209)$$

Thus, the approximate wave function of the deuteron ground state, corresponding to the normalization condition (1.201) for the exact wave function, has the form

$$\varphi_0(r) = \sqrt{\frac{\alpha}{2\pi(1-\alpha r_0)}} \frac{e^{-\alpha r}}{r}. \quad (1.210)$$

For  $r_0 \rightarrow 0$ , this formula goes over into (1.11).

**1.3.** Find the probability of a reversal of spins in the scattering of a slow neutron by a proton (Fermi, 1951).

Since the nuclear interaction between the neutron and proton depends on their spins, a change in the orientation of the spins is possible in the scattering, although the total spin of the system is conserved.

We denote the spin wave functions of the neutron and proton by  $\chi_\mu(n)$  and  $\chi_\mu(p)$ , where  $\mu$  is the  $z$ -component of the spin ( $\mu = \pm \frac{1}{2}$ ). The spin wave function of the neutron-proton system is a product of the wave functions  $\chi_\mu(n)$  and  $\chi_\mu(p)$ . Before the scattering, the neutron-proton system can be in one of the following spin states:

$$\chi(n, p) = \begin{cases} \chi_{1/2}(n)\chi_{1/2}(p); \\ \chi_{-1/2}(n)\chi_{-1/2}(p); \\ \chi_{1/2}(n)\chi_{-1/2}(p); \\ \chi_{-1/2}(n)\chi_{1/2}(p). \end{cases} \quad (1.211)$$

We shall consider the eigenfunctions of the operator of the square of the total spin of the system. The spin functions corresponding to a total spin equal to unity (the triplet state) are

$$\left. \begin{aligned} \chi_{1,1}(n, p) &= \chi_{1/2}(n)\chi_{1/2}(p); \\ \chi_{1,0}(n, p) &= (1/\sqrt{2})\{\chi_{1/2}(n)\chi_{-1/2}(p) + \chi_{-1/2}(n)\chi_{1/2}(p)\}; \\ \chi_{1,-1}(n, p) &= \chi_{-1/2}(n)\chi_{-1/2}(p). \end{aligned} \right\} \quad (1.212)$$

The spin function corresponding to a total spin equal to zero (the singlet state) is

$$\chi_{0,0}(n, p) = (1/\sqrt{2})\{\chi_{1/2}(n)\chi_{-1/2}(p) - \chi_{-1/2}(n)\chi_{1/2}(p)\}. \quad (1.213)$$

The first two functions of (1.211),  $\chi_{1/2}(n)\chi_{1/2}(p)$  and  $\chi_{-1/2}(n)\chi_{-1/2}(p)$  coincide with the triplet-state eigenfunctions  $\chi_{1,1}(n, p)$  and  $\chi_{1,-1}(n, p)$ , respectively, while the third and fourth functions of (1.211) are superpositions of the triplet and singlet states  $\chi_{1,0}(n, p)$  and  $\chi_{0,0}(n, p)$ :

$$\left. \begin{aligned} \chi_{1/2}(n)\chi_{-1/2}(p) &= (1/\sqrt{2})\{\chi_{1,0}(n, p) + \chi_{0,0}(n, p)\}; \\ \chi_{-1/2}(n)\chi_{1/2}(p) &= (1/\sqrt{2})\{\chi_{1,0}(n, p) - \chi_{0,0}(n, p)\}. \end{aligned} \right\} \quad (1.214)$$

If the neutron and proton are in the spin state  $\chi_{1/2}(n)\chi_{1/2}(p)$  or  $\chi_{-1/2}(n)\chi_{-1/2}(p)$ , a change of orientation of the spins obviously does not occur. Then the scattering cross-section is equal to

$$\sigma_{1,2} = 4\pi a_t^2. \quad (1.215)$$

We now consider neutron-proton scattering in the spin state  $\chi_{1/2}(n)\chi_{-1/2}(p)$ . Since the scattering lengths are different in the triplet and singlet states, the coefficient of the outgoing spherical wave in the wave function of the neutron-proton system in the case under consideration has the form

$$\begin{aligned} (1/\sqrt{2})(a_t\chi_{1,0}(n, p) + a_s\chi_{0,0}(n, p)) &= ((a_t + a_s)/2)\chi_{1/2}(n)\chi_{-1/2}(p) \\ &\quad + ((a_t - a_s)/2)\chi_{-1/2}(n)\chi_{1/2}(p). \end{aligned}$$

The first term on the right in this expression corresponds to scattering without change of the spin orientation, and the second corresponds to scattering with

change of the spin orientation. We see that the scattering lengths  $(a_t + a_s)/2$  and  $(a_t - a_s)/2$  correspond to these two processes respectively. Therefore, the cross-section for scattering without change of the spin orientation in the case of the spin state  $\chi_{1/2}(n)\chi_{-1/2}(p)$  is equal to

$$\sigma_3 = \pi(a_t + a_s)^2. \quad (1.216)$$

If the system was in the state  $\chi_{1/2}(n)\chi_{-1/2}(p)$  before the scattering, the cross-section for scattering of a neutron with change of the spin orientation is equal to

$$\sigma'_3 = \pi(a_t - a_s)^2. \quad (1.217)$$

Scattering in the spin state  $\chi_{-1/2}(n)\chi_{1/2}(p)$ , like that in the state  $\chi_{1/2}(n)\chi_{-1/2}(p)$ , can occur either with or without change of the spin orientations. The corresponding cross-sections are equal to

$$\left. \begin{aligned} \sigma_4 &= \pi(a_t + a_s)^2; \\ \sigma'_4 &= \pi(a_t - a_s)^2. \end{aligned} \right\} \quad (1.218)$$

In the case of scattering of unpolarized neutrons by protons, all four spin states (1.211) possess equal weight. Therefore, in this case, the probability of a change of the neutron spin orientation is equal to

$$w = \frac{(a_t - a_s)^2}{4a_t^2 + (a_t + a_s)^2 + (a_t - a_s)^2}. \quad (1.219)$$

Substituting the value of the scattering lengths into eqn.(1.219), we obtain  $w = 0.65$ .

**1.4.** Show that the cross-section for scattering of a very slow neutron by a proton contained in a heavy molecule is four times greater than the cross-section for scattering of a neutron by a free proton (Fermi, 1936).

In studying the scattering of slow neutrons in a substance containing hydrogen, we can only neglect the chemical bonding of the protons, that is, assume the protons to be free, if the neutron energy substantially exceeds the bonding energy of the protons in the molecules of the substance. The bonding energy is equal to  $\hbar\omega$  in order of magnitude, where  $\omega$  is the frequency of the proton vibrations in the molecule of the substance. For example, for a paraffin the magnitude of the bonding energy is about 0.4 eV. For neutron energies less than the proton bonding energy  $\hbar\omega$  or of the same order, it proves to be extremely important to take account of the chemical bonding of the protons.

Using the definition of the pseudo-potential (1.62), it is not difficult to obtain a general formula, taking into account the bonding of the protons in the molecules, for the cross-section for scattering of the neutrons. We denote the wave functions of the proton in the initial and final states by  $\varphi_0(\mathbf{r}_p)$  and  $\varphi_f(\mathbf{r}_p)$ . We choose the neutron wave functions in the form of plane waves. Thus, the initial and final functions of the neutron-proton system can be represented in the form

$$\left. \begin{aligned} \Phi_0 &= e^{i\mathbf{k} \cdot \mathbf{r}_n} \varphi_0(\mathbf{r}_p); \\ \Phi_f &= e^{i\mathbf{k}' \cdot \mathbf{r}_n} \varphi_f(\mathbf{r}_p), \end{aligned} \right\} \quad (1.220)$$

where  $\mathbf{k}$  and  $\mathbf{k}'$  are the initial and final wave vectors of the relative motion of the neutron and the molecule. According to perturbation theory, the scattering cross-section is given by the formula

$$d\sigma_{0f} = \frac{\tilde{\mu}^2}{4\pi^2\hbar^4} \frac{k'}{k} |V_{0f}|^2 do, \quad (1.221)$$

where  $\tilde{\mu} = MM_{\text{mol}}/(M + M_{\text{mol}})$  is the reduced mass for the relative motion of the neutron and the molecule;  $V_{0f}$  is the matrix element of the pseudo-potential (1.62). In this case, the energy conservation law

$$E = E' + E_f \quad (E = \hbar^2 k^2 / 2\tilde{\mu}, \quad E' = \hbar^2 k'^2 / 2\tilde{\mu}) \quad (1.222)$$

holds. Here,  $E_f$  is the excitation energy of the molecule. (It is assumed that the molecule is in the ground state before the scattering.) Integrating the matrix element  $V_{0f}$  over  $\mathbf{r}_n$  with the aid of the delta-function, we obtain

$$d\sigma_{0f} = \left(\frac{\tilde{\mu}}{\mu}\right)^2 \frac{k'}{k} a^2 |\mathcal{V}_{0f}|^2 do, \quad (1.223)$$

where  $\mathcal{V}_{0f}$  is the form factor defined by the integral

$$\mathcal{V}_{0f} = \int d^3\mathbf{r} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \varphi_f^*(\mathbf{r}) \varphi_0(\mathbf{r}) \quad (1.224)$$

( $\mu$  is the reduced mass of the neutron-proton system). If the wavelength of the neutron is large compared with the dimensions of the molecule, then the factor  $e^{i(k-k')\cdot r} \approx 1$ . In this case, only elastic scattering of the neutrons occurs and the state of the proton in the molecule is unchanged. The elastic scattering cross-section is equal to

$$d\sigma_{00} = (\tilde{\mu}/\mu)^2 a^2 do. \quad (1.225)$$

The scattering is characterized by spherical symmetry in the centre-of-mass system of the neutron and the molecule. The total cross-section for elastic scattering of the neutron by a bound proton is equal to

$$\sigma_{00} = (\tilde{\mu}/\mu)^2 \sigma_0, \quad (1.226)$$

where  $\sigma_0$  is the total cross-section for scattering of neutrons by free protons. In the limiting case of heavy molecules ( $M_{\text{mol}} \gg M$ ), the ratio  $\tilde{\mu}/\mu = 2$  and, consequently,

$$\sigma_{00} = 4\sigma_0, \quad (1.227)$$

that is, because of the effect of the chemical bonding of the proton the cross-section for scattering of a very slow neutron is four times greater than the cross-section for scattering for a neutron by a free proton.

**1.5.** Calculate the cross-section for inelastic scattering of a neutron with excitation of a vibrational level of molecule (Bethe, 1947).

If the neutron energy exceeds an excitation energy of the molecule, the scattering of a neutron by a bound proton may be accompanied by excitation of the molecule. To determine the cross-section for such a process, it is necessary to know

the wave functions of the proton in the excited states of the molecule. We shall make use of the harmonic-oscillator model to describe the vibrations of the proton in the molecule. In this case, the proton wave function satisfies the equation

$$\{-(\hbar^2/2M)\nabla_p^2 + V(r_p) - E_f\}\varphi(r_p) = 0, \quad (1.228)$$

in which the potential energy of the proton is equal to

$$V(r_p) = \frac{1}{2}M\omega^2 r_p^2$$

( $\omega$  is the vibrational frequency of the proton in the molecule).

The proton wave function can be represented in the form of a product of three linear-oscillator wave functions:

$$\varphi(r_p) = u_{n_x}(\xi)u_{n_y}(\eta)u_{n_z}(\zeta). \quad (1.229)$$

Here  $\xi = \sqrt{\frac{2M\omega}{\hbar}}x$ ,  $\eta = \sqrt{\frac{2M\omega}{\hbar}}y$  and  $\zeta = \sqrt{\frac{2M\omega}{\hbar}}z$  are dimensionless coordinates, and  $n_x$ ,  $n_y$ , and  $n_z$  are the vibrational quantum numbers of the oscillator. The functions  $u_{n_x}(\xi)$  satisfy the equation

$$\left\{-\frac{d^2}{d\xi^2} + \frac{1}{4}\xi^2 - n_x\right\}u_{n_x}(\xi) = 0 \quad (1.230)$$

and are expressed in terms of Hermite polynomials

$$H_{n_x}(\xi) = e^{(1/2)\xi^2} \frac{d^{n_x}}{d\alpha^{n_x}} e^{(-1/2)(\xi-\alpha)^2} \Big|_{\alpha=0}$$

as follows:

$$u_{n_x}(\xi) = (2\pi)^{-1/4}(n_x!)^{-1/2}e^{(-1/4)\xi^2} H_{n_x}(\xi). \quad (1.231)$$

These functions are normalized in accordance with the condition

$$\int_{-\infty}^{\infty} d\xi u_{n_x}^2(\xi) = 1.$$

The matrix element appearing in the cross-section (1.223) breaks down into a product of three integrals of the type

$$\mathcal{V}_{0n_x} = (2\pi)^{-1/2}(n_x!)^{-1/2} \int_{-\infty}^{\infty} d\xi e^{iq_x \xi} \frac{d^{n_x}}{d\alpha^{n_x}} e^{(-1/2)(\xi-\alpha)^2} \Big|_{\alpha=0}. \quad (1.232)$$

where

$$q = \sqrt{\frac{\hbar}{2M\omega}}(k - k').$$

(It is assumed that the proton was in the ground state before the scattering.) Integrating (1.232) by parts, we find

$$\mathcal{V}_{0n_x} = (iq_x)^{n_x} (n_x!)^{-1/2} e^{(-1/2)q_x^2}.$$

Therefore, the cross-section for scattering of a neutron with excitation of the  $n$ -th vibrational level of the molecule with energy

$$E_f = E_n = n\hbar\omega \quad (n = n_x + n_y + n_z) \quad (1.233)$$

is given by the formula

$$d\sigma_{0n} = 4a^2 \frac{k'}{k} \sum \frac{q_x^{2n_x} q_y^{2n_y} q_z^{2n_z}}{n_x! n_y! n_z!} e^{-q^2} do,$$

where the summation is performed over all values of the quantum numbers  $n_x$ ,  $n_y$ , and  $n_z$  satisfying the condition  $n_x + n_y + n_z = n$ . As a result of the summation, we obtain

$$d\sigma_{0n} = 4a^2 \frac{k'}{k} \cdot \frac{q^{2n}}{n!} e^{-q^2} do. \quad (1.234)$$

Noting that  $do = 2\pi \frac{\hbar\omega}{2\sqrt{EE'}} dq^2$  and introducing the dimensionless energy  $\varepsilon = E/\hbar\omega$ , we rewrite this formula in the form

$$d\sigma_{0n} = \frac{\sigma_0}{\varepsilon} \frac{q^{2n}}{n!} e^{-q^2} dq^2. \quad (1.235)$$

Using the energy conservation law  $E = E' + n\hbar\omega$ , we see that  $q$  is confined within the limits

$$q_{\min} = \sqrt{\varepsilon - \sqrt{\varepsilon - n}}, \quad q_{\max} = \sqrt{\varepsilon + \sqrt{\varepsilon - n}}.$$

Integrating the expression (1.235) within these limits, we obtain the following formula for the total cross-section for scattering of a neutron with excitation of the  $n$ -th vibrational level of the molecule:

$$\sigma_{0n} = \frac{\sigma_0}{\varepsilon} \{f_n(q_{\min}^2) - f_n(q_{\max}^2)\}, \quad (1.236)$$

where

$$f_n(x) = \left(1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!}\right) e^{-x}.$$

In the case of elastic scattering, the formula (1.236) acquires the form

$$\sigma_{00} = (\sigma_0/\varepsilon)(1 - e^{-4\varepsilon}). \quad (1.237)$$

In the limiting case  $\varepsilon \ll 1$ , we obtain from the relation (1.237) the result (1.227) found previously:

$$\sigma_{00} = 4\sigma_0, \quad \varepsilon \ll 1. \quad (1.238)$$

For  $\varepsilon = 1$ , the elastic scattering cross-section is equal to  $\sigma_{00} \approx 0.98\sigma_0$ . For  $\varepsilon \gg 1$ , the elastic scattering cross-section is given by the expression

$$\sigma_{00} = \sigma_0/\varepsilon, \quad \varepsilon \gg 1. \quad (1.239)$$

For large values of  $\varepsilon$  and for any  $n$  that is not too close to  $\varepsilon$ , the second term in the relation (1.236) is very small, and the first differs little from unity. Therefore, the cross-section for scattering with excitation of any allowed level will

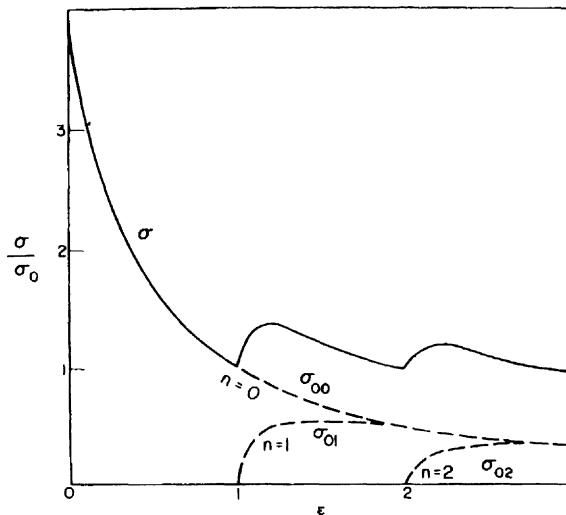


Figure 1.14. Dependence of the cross-sections  $\sigma_{00}$ ,  $\sigma_{01}$ ,  $\sigma_{02}$  and the total scattering cross-section  $\sigma$  on the dimensionless neutron energy  $\varepsilon = E/\hbar\omega$ .

be determined in this case by the same expression as that for the elastic scattering cross-section:

$$\sigma_{0n} = \sigma_0/\varepsilon, \quad \varepsilon \gg 1. \quad (1.240)$$

According to the energy conservation law, the maximum value of  $n$  is equal to  $\varepsilon$ ; therefore, the total cross-section for large  $\varepsilon$  will be equal to  $\sigma_0$ , that is, for high neutron energies the bonding of the proton turns out to be unimportant.

Figure 1.14 shows the dependence of the cross-section  $\sigma_{00}$ ,  $\sigma_{01}$  and  $\sigma_{02}$  and the total scattering cross-section  $\sigma = \sum_n \sigma_{0n}$  on the neutron energy  $\varepsilon$ . If the neutron energy is less than the energy  $\hbar\omega$  of a vibrational quantum, it cannot excite vibrations in the molecule; in this case, only elastic scattering is possible. For neutron energy values that are multiples of  $\hbar\omega$ , there occurs a sharp increase in the scattering cross-section, due to the possibility of transfer of the neutron energy to the vibrational motion of the protons. If the neutron energy is large compared with the chemical-bond energy, the protons can be regarded as free. A small fraction of the neutron energy, spent on breaking the chemical bond of the proton in the molecule, is transferred to the molecule. The scattering in this energy region is practically elastic and is characterized by spherical symmetry in the centre-of-mass frame of the neutron and proton.

**1.6.** If the wavelength  $\lambda$  of an incident neutron is comparable with the distance between the protons in the hydrogen molecule, then both elastic scattering of the neutron and inelastic scattering accompanied by a change of state of the molecule are possible. Determine the cross-section for elastic scattering of a neutron in a hydrogen molecule with allowance for the effects of retardation, and also the cross-sections of the inelastic scattering processes accompanied by excitation of the hydrogen molecule (Schwinger and Teller, 1937).

We shall consider the collision of a neutron of wave vector  $\mathbf{k}$  with a hydrogen molecule in a state characterized by the vibrational quantum number  $v$ , rotational quantum number  $J$  and spin  $S$ . We shall determine the probability of a process in which the neutron acquires the wave vector  $\mathbf{k}'$  and the molecule undergoes a transition to a state characterized by the quantum numbers  $v'$ ,  $J'$  and  $S'$ .

We choose the wave function of the initial state of the neutron and the molecule in the form

$$\Phi_i = e^{i\mathbf{k} \cdot \mathbf{r}_n} \Phi_{vJM_J}(\mathbf{r}) \chi_\mu \chi_{S\mu_S}, \quad (1.241)$$

where  $e^{i\mathbf{k} \cdot \mathbf{r}_n}$  is the wave function of the relative motion of the neutron and the molecule,  $\Phi_{vJM_J}(\mathbf{r})$  is the wave function describing the motion of the protons in a molecule in a state characterized by the quantum numbers  $v$ ,  $J$  and  $M_J$ ,  $\chi_\mu$  is the spin function of the neutron ( $\mu$  is the  $z$ -component of the neutron spin), and, finally,  $\chi_{S\mu_S}$  is the spin function of the protons in a state with resultant spin  $S$  (the  $z$ -component of the spin is equal to  $\mu_S$ ). In an analogous way, we choose the final-state wave function in the form

$$\Phi_f = e^{i\mathbf{k}' \cdot \mathbf{r}_n} \Phi_{v'J'M'_J}(\mathbf{r}) \chi_{\mu'} \chi_{S'\mu'_S}. \quad (1.242)$$

The differential scattering cross-section, averaged over the initial and summed over the final magnetic quantum numbers, is given by the general formula

$$d\sigma_{if} = \frac{16}{9} \frac{k'}{k} \frac{1}{2(2S+1)(2J+1)} \sum_{\mu\mu_S M_J} \sum_{\mu'\mu'_S M'_J} \left| \left( \Phi_f, \frac{\mu}{2\pi\hbar^2} V \Phi_i \right) \right|^2 d\omega, \quad (1.243)$$

where  $V$  is the interaction potential of the neutron with the molecule, given by the expression (1.66), and  $d\omega$  is the solid-angle element in which the wave vector of the scattered neutron lies. In the scattering, the energy conservation law

$$\frac{\hbar^2 k^2}{2\tilde{\mu}} + E_{vJ} = \frac{\hbar^2 k'^2}{2\tilde{\mu}} + E_{v'J'}, \quad (1.244)$$

is fulfilled, where  $E_{vJ}$  is the energy of the molecule in the state  $\Phi_{vJM_J}$ ;  $\hbar^2 k^2 / 2\tilde{\mu}$  is the energy of the relative motion of the neutron and the molecule ( $\tilde{\mu} = 2M/3$  is the reduced mass).

We must distinguish scattering of neutrons without change of the spin state of the hydrogen molecule ( $S \rightarrow S' = S$ ), and transitions in which the spin state of the molecule does change ( $S \rightarrow S' = 1 - S$ ). Transitions with unchanged spin are due to the interaction energy  $V_s$ , and transitions with a change of spin are due to the interaction energy  $V_a$  (cf.(1.67)). Noting that

$$\overline{|3a_t + a_s + (a_t - a_s)\sigma_n \cdot S|^2} = (3a_t + a_s)^2 + S(S+1)(a_t - a_s)^2,$$

it is not difficult to obtain the following expression for the cross-section for scattering of neutrons without change of the spin of the molecule:

$$d\sigma_{SvJ;Sv'J'} = \frac{4}{9} \frac{k'}{k} [(3a_t + a_s)^2 + S(S+1)(a_t - a_s)^2] \times \frac{1}{2J+1} \sum_{M_J M'_J} \left| \int d^3 r \cos \left[ \frac{1}{2} (\mathbf{k} - \mathbf{k}') \cdot \mathbf{r} \right] \Phi_{v'J'M'_J}^*(\mathbf{r}) \Phi_{vJM_J}(\mathbf{r}) \right|^2 d\omega. \quad (1.245)$$

In this expression,  $J$  and  $J'$  should be numbers of the same parity. Actually, since protons obey Fermi-Dirac statistics, the total wave function of the hydrogen molecule should be anti-symmetric. Therefore, in the case of para-hydrogen, when the spin function of the molecule is anti-symmetric, the spatial function should be symmetric and, consequently, the rotational quantum number  $J$  is even. In the case of orthoglydrogen the rotational quantum number  $J$  is odd.

In order to find the cross-section for scattering of a neutron with change of spin of the molecule, we calculate the mean value of the quantity  $|\boldsymbol{\sigma}_n \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)|^2$  over the spin states:

$$\overline{|\boldsymbol{\sigma}_n \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)|^2} = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^2 = 4[3 - S(S + 1)].$$

Therefore, the cross-section for scattering of neutrons with change of the spin of the molecule is given by the formula

$$d\sigma_{SvJ;(1-S)v'J'} = \frac{4}{9} \frac{k'}{k} [3 - S(S + 1)] (a_t - a_s)^2 \times \frac{1}{2J + 1} \sum_{M_J M'_J} \left| \int d^3r \sin \left[ \frac{1}{2}(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r} \right] \Phi_{v'J'M'_J}^*(\mathbf{r}) \Phi_{vJM_J}(\mathbf{r}) \right|^2 do. \quad (1.246)$$

Para-ortho and ortho-para transitions are associated with a change of parity of the rotational quantum number  $J$ .

The minimum neutron energy necessary for a transition of the molecule from the state  $v, J$  to the state  $v', J'$  to be possible is equal to

$$E_{\min} \equiv \frac{3\hbar^2}{4M} (k^2 - k'^2) = E_{v'J'} - E_{vJ}.$$

The initial momentum of the neutron in the laboratory frame,  $\tilde{\mathbf{k}}$ , is equal to  $3\mathbf{k}/2$ ; therefore, the minimum neutron energy in the laboratory frame is equal to

$$\tilde{E}_{\min} = \frac{3}{2} (E_{v'J'} - E_{vJ}). \quad (1.247)$$

For small values of the quantum numbers  $v$  and  $J$ , the excitation energy of the hydrogen molecule is given by the expression ( $\mathcal{J}$  is the moment of inertia):

$$E_{vJ} = v\hbar\omega + \frac{\hbar^2}{2\mathcal{J}} J(J + 1), \quad (1.248)$$

where  $\hbar\omega = 0.533$  eV, and  $\hbar^2/\mathcal{J} = 0.015$  eV. Direct calculation shows that, in order to convert the molecule from the state  $v = 0, J = 1$  to the state  $v' = 0, J' = 2$ , the neutron must possess an energy of 0.045 eV. For a transition from the state  $v = 0, J = 0$  to the state  $v' = 0, J' = 1$ , an energy of 0.023 eV is necessary. Therefore, if the neutron energy is less than 0.045 eV, only the following rotational transitions are possible:  $0 \rightarrow 0, 0 \rightarrow 1, 1 \rightarrow 0$  and  $1 \rightarrow 1$ .

In order to find the transition probabilities for the hydrogen molecule, we make use of the rigid-rotator model. In this case, the wave function of the molecule can be written in the form

$$\Phi_{0JM_J}(r) = (1/r)[\delta(r - d)]^{1/2} Y_{JM_J}(\vartheta_r, \varphi_r), \quad (1.249)$$

where  $d$  is the equilibrium distance between the protons in the molecule ( $d \approx 0.75 \times 10^{-8}$  cm) and  $\vartheta_r$  and  $\varphi_r$  are the angles defining the orientation of the vector  $\mathbf{r}$ . Using (1.245) and (1.249), it is not difficult to obtain the following expression for the differential elastic cross-section ( $0 \rightarrow 0$ ):

$$d\sigma_{0 \rightarrow 0} = \frac{4}{9}(3a_t + a_s)^2 \left[ \frac{\sin[\xi \sin(\vartheta/2)]}{\xi \sin(\vartheta/2)} \right]^2 2\pi \sin \vartheta d\vartheta, \quad (1.250)$$

where  $\xi = kd$  and  $\vartheta$  is the scattering angle in the centre-of-mass frame. The integral cross-section is equal to

$$\begin{aligned} \sigma_{0 \rightarrow 0} &= \frac{32}{9}\pi(3a_t + a_s)^2 \xi^{-2} \int_0^\xi dx \frac{\sin^2 x}{x} \\ &= \frac{16}{9}\pi(3a_t + a_s)^2 \xi^{-2} \{ \ln 2\xi - \text{ci } 2\xi + 0.5772 \}, \end{aligned} \quad (1.251)$$

where

$$\text{ci } x \equiv - \int_x^\infty dt \frac{\cos t}{t}.$$

In an analogous way, it is not difficult to obtain the following expressions for the cross-sections of the transitions  $0 \rightarrow 1$ ,  $1 \rightarrow 0$  and  $1 \rightarrow 1$ :

$$\begin{aligned} \sigma_{0 \rightarrow 1} &= 32\pi(a_t - a_s)^2 \xi^{-2} \\ &\times \left\{ \frac{\sin x}{x} + \frac{\cos x - 1}{x^2} + \frac{1}{2} \ln x - \frac{1}{2} \text{ci } x \right\} \Big|_{\xi-\xi^-}^{\xi+\xi^-}; \end{aligned} \quad (1.252)$$

$$\begin{aligned} \sigma_{1 \rightarrow 0} &= \frac{32}{3}\pi(a_t - a_s)^2 \xi^{-2} \\ &\times \left\{ \frac{\sin x}{x} + \frac{\cos x - 1}{x^2} + \frac{1}{2} \ln x - \frac{1}{2} \text{ci } x \right\} \Big|_{\xi-\xi^+}^{\xi+\xi^+}; \end{aligned} \quad (1.253)$$

$$\begin{aligned} \sigma_{1 \rightarrow 1} &= \frac{16}{3}\pi[(3a_t + a_s)^2 + 2(a_t - a_s)^2] \xi^{-2} \\ &\times \left\{ \frac{\sin^2 \xi}{\xi^2} - \frac{3}{\xi^2} \left( \frac{\sin \xi}{\xi} - \cos \xi \right)^2 + \ln 2\xi - \text{ci } 2\xi - 0.4228 \right\}, \end{aligned} \quad (1.254)$$

where

$$\xi^\pm = \sqrt{\xi^2 \pm \frac{2\tilde{\mu}d^2}{\hbar^2} E_{01}}.$$

In the scattering of very slow neutrons in molecular hydrogen, it turns out to be important to take account of the thermal motion of the molecules. If  $\mathbf{v}$  is the velocity of the incident neutron and  $\mathbf{u}$  is the thermal velocity of the molecule, then the relative wave vector  $\mathbf{k}$  appearing in the cross-section must be assumed equal to  $\mathbf{k} = (\tilde{\mu}/\hbar)(\mathbf{v} - \mathbf{u})$ . The effective values of the scattering cross-sections can be

found by averaging the expressions (1.251)–(1.254) over the thermal motion of the molecules

$$\bar{\sigma} = \frac{1}{v} \int d^3 u |\mathbf{v} - \mathbf{u}| \sigma(\mathbf{v} - \mathbf{u}) N(u), \quad (1.255)$$

where  $N(u)$  is the velocity distribution function of the molecules.

As an example, we give the results of calculations of the cross-sections in the case of scattering of a neutron with energy 0.001723 eV (this energy corresponds to a temperature of 20°K):

$$\left. \begin{aligned} \sigma_{0 \rightarrow 0} &= 6.444(3a_t + a_s)^2; \\ \sigma_{1 \rightarrow 0} &= 1.753(a_t - a_s)^2; \\ \sigma_{1 \rightarrow 1} &= 6.450\{(3a_t + a_s)^2 + 2(a_t - a_s)^2\}. \end{aligned} \right\} \quad (1.256)$$

Experimentally, the following quantities can be measured:

$$\sigma_{\text{para}} = \sigma_{0 \rightarrow 0}, \quad \sigma_{\text{ortho}} = \sigma_{1 \rightarrow 0} + \sigma_{1 \rightarrow 1}. \quad (1.257)$$

The experimental values of these quantities at a temperature of 20°K are equal to

$$\sigma_{\text{para}} = 3.97 \times 10^{-24} \text{ cm}^2, \quad \sigma_{\text{ortho}} = 124 \times 10^{-24} \text{ cm}^2. \quad (1.258)$$

Using the formulae (1.256) and (1.257), one can determine the scattering lengths  $a_t$  and  $a_s$  with a high degree of accuracy from the values (1.258).

**1.7.** Verify that the coefficients  $\alpha, \beta, \gamma, \delta$  and  $\varepsilon$  occurring in the nucleon-nucleon scattering amplitude (1.70) are expressed in terms of the phase shifts  ${}^1\delta_1^J$

and  ${}^3\delta_1^J$  and mixture parameters  $\varepsilon^J$  in the following way:

$$\left. \begin{aligned} \alpha &= \frac{1}{4k} \sum_{l=0}^{\infty} [(l+2)a_l^{l+1} + (2l+1)a_l^l + (l-1)a_l^{l-1} + (l+1)b_l^{l+1} \\ &\quad + lb_l^{l-1} + (2l+1)c_l^l] P_l(\cos \vartheta); \\ \beta &= \frac{1}{4k} \left\{ \sum_{l=0}^{\infty} [(l+1)b_l^{l+1} + lb_l^{l-1} - (2l+1)c_l^l] P_l(\cos \vartheta) \right. \\ &\quad \left. - \sum_{l=2}^{\infty} \left[ \frac{1}{l+1}a_l^{l+1} - \frac{2l+1}{l(l+1)}a_l^l + \frac{1}{l}a_l^{l-1} \right] P_l^2(\cos \vartheta) \right\}; \\ \gamma &= \frac{1}{4k} \sum_{l=1}^{\infty} \left[ \frac{l+2}{l+1}a_l^{l+1} - \frac{2l+1}{l(l+1)}a_l^l - \frac{l-1}{l}a_l^{l-1} + b_l^{l+1} - b_l^{l-1} \right] P_l^1(\cos \vartheta); \\ \delta &= \frac{1}{4k \cos \vartheta} \left\{ \sum_{l=0}^{\infty} \left[ \frac{1}{2} \{ (l+2)a_l^{l+1} + (2l+1)a_l^l + (l-1)a_l^{l-1} \} (\cos \vartheta - 1) \right. \right. \\ &\quad \left. \left. + (l+1)b_l^{l+1} + lb_l^{l-1} - (2l+1)c_l^l \cos \vartheta \right] P_l(\cos \vartheta) \right. \\ &\quad \left. + \frac{1}{2} \sum_{l=2}^{\infty} \left[ \frac{1}{l+1}a_l^{l+1} - \frac{2l+1}{l(l+1)}a_l^l + \frac{1}{l}a_l^{l-1} \right] (1 + \cos \vartheta) P_l^2(\cos \vartheta) \right\}; \\ \varepsilon &= \frac{1}{4k \cos \vartheta} \left\{ \sum_{l=0}^{\infty} \left[ \frac{1}{2} \{ (l+2)a_l^{l+1} + (2l+1)a_l^l + (l-1)a_l^{l-1} \} (1 + \cos \vartheta) \right. \right. \\ &\quad \left. \left. - (l+1)b_l^{l+1} - lb_l^{l-1} - (2l+1)c_l^l \cos \vartheta \right] P_l(\cos \vartheta) \right. \\ &\quad \left. + \frac{1}{2} \sum_{l=2}^{\infty} \left[ \frac{1}{l+1}a_l^{l+1} - \frac{2l+1}{l(l+1)}a_l^l + \frac{1}{l}a_l^{l-1} \right] (\cos \vartheta - 1) P_l^2(\cos \vartheta) \right\}, \end{aligned} \right\} \quad (1.259)$$

where we have introduced the notation:

$$\left. \begin{aligned} a_l^{J=1} &\equiv \sin^3 \delta_J^J e^{i^3 \delta_J^J}; \\ a_l^{J=l+1} &\equiv \alpha^J \cos^2 \varepsilon^J + \beta^J \sin^2 \varepsilon^J + \frac{1}{2} \sqrt{\frac{J}{J+1}} (\alpha^J - \beta^J) \sin 2\varepsilon^J; \\ b_l^{J=l+1} &\equiv \alpha^J \cos^2 \varepsilon^J + \beta^J \sin^2 \varepsilon^J - \frac{1}{2} \sqrt{\frac{J+1}{J}} (\alpha^J - \beta^J) \sin 2\varepsilon^J; \\ a_l^{J=l-1} &\equiv \alpha^J \sin^2 \varepsilon^J + \beta^J \cos^2 \varepsilon^J + \frac{1}{2} \sqrt{\frac{J+1}{J}} (\alpha^J - \beta^J) \sin 2\varepsilon^J; \\ b_l^{J=l-1} &\equiv \alpha^J \sin^2 \varepsilon^J + \beta^J \cos^2 \varepsilon^J - \frac{1}{2} \sqrt{\frac{J}{J+1}} (\alpha^J - \beta^J) \sin 2\varepsilon^J; \\ \alpha^J &\equiv \sin^3 \delta_{J-1}^J e^{i^3 \delta_{J-1}^J}; \\ \beta^J &\equiv \sin^3 \delta_{J+1}^J e^{i^3 \delta_{J+1}^J}; \\ c^{J=l} &\equiv \sin^1 \delta_J^J e^{i^1 \delta_J^J}. \end{aligned} \right\} \quad (1.260)$$

**1.8.** Derive the integral unitarity relations for the two-nucleon elastic scattering amplitude (Puzikov, Ryndin and Smorodinskii, 1957).

If we assume that only elastic scattering is possible in a two-nucleon system, from the unitarity of the  $S$ -matrix it is not difficult to obtain the following integral relation for the matrix scattering amplitude  $f(\mathbf{k}, \mathbf{k}')$ :

$$f(\mathbf{k}, \mathbf{k}') - f^+(\mathbf{k}' \mathbf{k}) = \frac{i\mathbf{k}}{2\pi} \int d\omega'' f(\mathbf{k}, \mathbf{k}'') f^+(\mathbf{k}' \mathbf{k}''), \quad (1.261)$$

where the integration is performed over the directions of the vector  $\mathbf{k}''$ .

For the two-nucleon elastic scattering amplitude, we make use of the Wolfenstein formula (1.170). Multiplying the left- and right-hand sides of the equality (1.261) successively by the matrices  $\mathbf{1}$ ,  $(\mathbf{n} \cdot \boldsymbol{\sigma}_1)(\mathbf{n} \cdot \boldsymbol{\sigma}_2)$ ,  $\mathbf{n} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$ ,  $(\mathbf{m} \cdot \boldsymbol{\sigma}_1)(\mathbf{m} \cdot \boldsymbol{\sigma}_2)$  and  $(\mathbf{l} \cdot \boldsymbol{\sigma}_1)(\mathbf{l} \cdot \boldsymbol{\sigma}_2)$  and then calculating the traces in the left-hand sides of the equalities, we obtain five integral unitarity relations for the coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  and  $\varepsilon$ :

$$\left. \begin{aligned} (4\pi/\mathbf{k}) \text{Im } \alpha(\vartheta) &= \left( \frac{1}{4} \right) \int d\omega'' \text{Tr} \{ f(\mathbf{k}, \mathbf{k}'') f^+(\mathbf{k}', \mathbf{k}'') \}; \\ (4\pi/\mathbf{k}) \text{Im } \beta(\vartheta) &= \left( \frac{1}{4} \right) \int d\omega'' \text{Tr} \{ (\mathbf{n} \cdot \boldsymbol{\sigma}_1)(\mathbf{n} \cdot \boldsymbol{\sigma}_2) f(\mathbf{k}, \mathbf{k}'') f^+(\mathbf{k}', \mathbf{k}'') \}; \\ (4\pi/\mathbf{k}) \text{Im } \gamma(\vartheta) &= \left( \frac{1}{8} i \right) \int d\omega'' \text{Tr} \{ \mathbf{n} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) f(\mathbf{k}, \mathbf{k}'') f^+(\mathbf{k}', \mathbf{k}'') \}; \\ (4\pi/\mathbf{k}) \text{Im } \delta(\vartheta) &= \left( \frac{1}{4} \right) \int d\omega'' \text{Tr} \{ (\mathbf{m} \cdot \boldsymbol{\sigma}_1)(\mathbf{m} \cdot \boldsymbol{\sigma}_2) f(\mathbf{k}, \mathbf{k}'') f^+(\mathbf{k}', \mathbf{k}'') \}; \\ (4\pi/\mathbf{k}) \text{Im } \varepsilon(\vartheta) &= \left( \frac{1}{4} \right) \int d\omega'' \text{Tr} \{ (\mathbf{l} \cdot \boldsymbol{\sigma}_1)(\mathbf{l} \cdot \boldsymbol{\sigma}_2) f(\mathbf{k}, \mathbf{k}'') f^+(\mathbf{k}', \mathbf{k}'') \}, \end{aligned} \right\} \quad (1.262)$$

where  $\vartheta$  is the angle between the vectors  $\mathbf{k}'$  and  $\mathbf{k}$ . The calculation of the traces in the right-hand sides of the equalities (1.262) is elementary but leads to cumbersome expressions, which we shall not give.

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## CHAPTER 2

### FEW-NUCLEON SYSTEMS

#### 2.1. Bound States and Scattering in a System of Two Particles

**The scattering amplitude.** In the previous Chapter, we considered the main peculiarities of the nuclear interaction between nucleons. It was shown that such can cause either formation of a two-nucleon bound state or *scattering of nucleons*. At the same time, in the center-of-inertia coordinate system, the energy spectrum is nondegenerate and consists of one discrete level and a continuous part. In the case of greater number of particles, when the system contains three or more interacting nucleons, the description in terms of quantum mechanics becomes more complicated. The reason is that, unlike the two-particle system, the energy spectrum of a system of three or more particles is always degenerate, i.e., discrete levels overlap with the continuous part of the spectrum. This results in the possible redistribution of particles in the interacting subsystems. This must be taken into account when one formulates the boundary conditions in the complex configuration space of the system.

Before passing to the analysis of systems consisting of three or more particles, let us consider the description of a system of two interacting particles in terms of quantum mechanics.

First of all, we consider the elastic scattering which does not change intrinsic states of the particles involved in the collision. In the center-of-mass system, the two-particle scattering problem is treated as scattering of a particle with *reduced mass*  $\mu$  in the field  $V(\mathbf{r})$  of a stationary force center. The Hamiltonian of the system is

$$H = H_0 + V,$$

where  $H_0 = (-\hbar^2/2\mu)\Delta$  is the kinetic energy of relative motion. The scattering wave function is governed by the Schrödinger equation

$$H\psi = E\psi \tag{2.1}$$

for positive energies ( $E > 0$ ), with the boundary conditions which require that the solution at large distances from the scatterer must be a sum of the incident and scattered waves.

In order to find the solution with the required asymptotics, we rewrite the Schrödinger equation in the form

$$(E - H_0)\psi = V\psi \quad (2.2)$$

and treat the right-hand part as a given function. Then the general solution of (2.2) may be written as

$$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \int d\mathbf{r}' G_0(E; \mathbf{r} - \mathbf{r}') V(\mathbf{r}') \psi(\mathbf{r}'), \quad (2.3)$$

where  $\varphi$  is the general solution of (2.2) with vanishing right-hand part, i.e.,

$$(E - H_0)\varphi = 0, \quad (2.4)$$

and  $G_0(E; \mathbf{r} - \mathbf{r}')$  is the *Green function* that satisfies the inhomogeneous equation with a point source

$$(E - H_0)G_0(E; \mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (2.5)$$

The first term of (2.3) may be regarded as the incident wave that describes the free motion of the particle. The second term is associated with the scattered wave which can be either converging or diverging (it depends on the choice of the Green function).

If the eigenfunctions of the equation (2.4) are plane waves

$$\varphi_{\mathbf{k}'}(\mathbf{r}) = e^{i\mathbf{k}'\cdot\mathbf{r}}, \quad E' = \frac{\hbar^2 k'^2}{2\mu}, \quad (2.6)$$

then the Green function is described by the integral

$$G_0(E; \mathbf{r} - \mathbf{r}') = \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{\exp[i\mathbf{k}(\mathbf{r} - \mathbf{r}')]}{E - E'}. \quad (2.7)$$

Integrating (2.7) over the angle variables yields

$$G_0(E; \mathbf{r} - \mathbf{r}') = \frac{\mu}{2\pi^2 i \hbar^2} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \int_{-\infty}^{\infty} dk' k' \frac{\exp(ik' |\mathbf{r} - \mathbf{r}'|)}{k^2 - k'^2}, \quad (2.8)$$

where  $E = \hbar^2 k^2 / 2\mu$ . Integration over  $k'$  may be carried out by means of the residue method; the boundary conditions determine the procedure of encircling the poles  $k' = \pm k$  of the integrand. In order to obtain a diverging wave at large distances, we take the integration contour in the plane of the complex variable  $k'$  as shown in Fig. 2.1a. Since the integral over the circle of infinitely large radius vanishes, the integral (2.8) is equal to  $2\pi i$  times

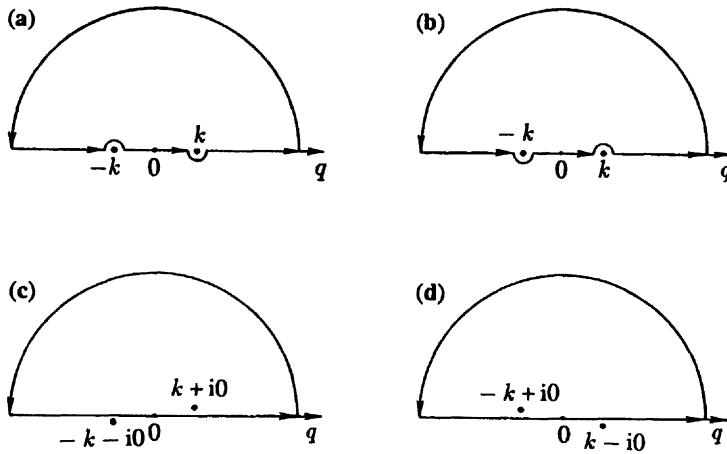


Figure 2.1. Integration contours for the Green function  $G_0^{(+)}(E; \mathbf{r} - \mathbf{r}')$  and  $G_0^{(-)}(E; \mathbf{r} - \mathbf{r}')$

to residue at the only pole enclosed by the integration contour. Thus we obtain

$$G_0^{(+)}(E; \mathbf{r} - \mathbf{r}') = -\frac{\mu}{2\pi\hbar^2} \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}. \quad (2.9)$$

In order to calculate the Green function associated with the converging wave, we take the integration contour as given in Fig. 2.1b. Then

$$G_0^{(-)}(E; \mathbf{r} - \mathbf{r}') = -\frac{\mu}{2\pi\hbar^2} \frac{\exp(-ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}. \quad (2.10)$$

Another way to find how to encircle the poles when calculating the Green functions, is substitute  $E + i\mathcal{O}$  and  $E - i\mathcal{O}$  for  $E$  in the denominator of (2.7) for the Green function associated with diverging and converging waves, respectively (Fig. 2.1c and Fig. 2.1d). Then we have

$$G_0^{(\pm)}(E; \mathbf{r} - \mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\exp[i\mathbf{k}(\mathbf{r} - \mathbf{r}')]}{E - E' \pm i\mathcal{O}}, \quad (2.11)$$

and the integration contour which consists of the real axis and the semicircle of infinite radius in the upper half-plane, contains either the pole  $k' = k + i\mathcal{O}$  in the case of  $G_0^{(+)}$ , or the pole  $k' = -k + i\mathcal{O}$  in the case of  $G_0^{(-)}$ .

Taking  $\varphi$  in (2.3) to be the plane wave corresponding to the incident particle with momentum  $\mathbf{k}$  and the Green function to be the converging wave (2.9), we rewrite (2.3) as

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \varphi_{\mathbf{k}}(\mathbf{r}) - \frac{\mu}{2\pi\hbar^2} \int d\mathbf{r}' \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} V(\mathbf{r}') \psi_{\mathbf{k}}^{(+)}(\mathbf{r}'). \quad (2.12)$$

This *integral equation* determines the wave function of the scattering problem. It is equivalent to the Schrödinger equation with the boundary conditions.

At large distances, the Green function (2.9) may be approximated by the expression

$$G_0^{(+)}(E; \mathbf{r} - \mathbf{r}') \rightarrow -\frac{\mu}{2\pi\hbar^2} \frac{\exp(ikr)}{r} e^{-i\mathbf{k}\mathbf{r}'} \quad (r \rightarrow \infty), \quad (2.13)$$

where  $\mathbf{k}' = (\mathbf{r}/r)k$ . Therefore, if the potential is nonzero within a finite region, then the *wave function asymptotic* is given by

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) \rightarrow \varphi_{\mathbf{k}}(\mathbf{r}) + f(\mathbf{k}, \mathbf{k}') \frac{\exp(ikr)}{r} \quad (r \rightarrow \infty), \quad (2.14)$$

where

$$f(\mathbf{k}, \mathbf{k}') = -\frac{\mu}{2\pi\hbar^2} \int d\mathbf{r} e^{-i\mathbf{k}'\mathbf{r}} V(\mathbf{r}) \psi_{\mathbf{k}}^{(+)}(\mathbf{r}). \quad (2.15)$$

The coefficient  $f(\mathbf{k}, \mathbf{k}')$  before the diverging wave is usually referred to as the scattering amplitude. Thus,  $\mathbf{k}'$  must be regarded as the final particle momentum. According to (2.15), the *scattering amplitude*  $f(\mathbf{k}, \mathbf{k}')$  depends on the energy of relative motion, the angle between the vectors  $\mathbf{k}$  and  $\mathbf{k}'$ , and the scattering potential.

**Partial wave expansion.** Let us consider in detail the problem of particle scattering in the central field, assuming that the potential  $V$  depends only on the modulus of the distance  $r$ . Since  $H_0 = -\frac{\hbar^2}{2\mu}\Delta$ , (2.1) may be rewritten as

$$\{\Delta - v(r) + k^2\} \psi_{\mathbf{k}} = 0, \quad (2.16)$$

where  $k^2 = \frac{2\mu}{\hbar^2} E$  and  $v(r) = \frac{2\mu}{\hbar^2} V(r)$  will be referred to as the *reduced potential*. The Laplace operator in the spherical coordinate system is

$$\Delta = \frac{1}{r^2} \left( \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \right) - \frac{\mathbf{L}^2}{r^2},$$

where  $\mathbf{L}^2$  is the operator of the squared angular momentum,

$$\mathbf{L}^2 = - \left\{ \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \right\}.$$

Because central potentials are spherically symmetric, the angular momentum is an integral of motion. The states associated with different angular momenta take part in the scattering process independently. Therefore, it is convenient to write the solution  $\psi_{\mathbf{k}}(\mathbf{r})$  of (2.16) as a superposition of partial waves associated with definite values of the angular momentum. Taking the  $z$  axis to be directed along the incident particle momentum  $\mathbf{k}$ , we can write the *partial wave expansion of the wave function*  $\psi_{\mathbf{k}}(\mathbf{r})$  as

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{4\pi} \sum_l (2l+1) i^l \psi_{l,k}(r) P_l(\cos \vartheta), \quad (2.17)$$

where  $P_l(\cos \vartheta)$  is the eigenfunction of the operator of squared angular momentum  $\mathbf{L}^2$ , i.e.,

$$\mathbf{L}^2 P_l(\cos \vartheta) = l(l+1) P_l(\cos \vartheta). \quad (2.18)$$

The coefficient in expansion (2.17) is chosen so that the function  $\psi_{\mathbf{k}}(\mathbf{r})$  transforms into the plane wave  $\psi_{\mathbf{k}}^0(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}}$ , when there is no interaction. By using the expansion of the plane wave in terms of the partial waves, we obtain

$$\psi_{l,k}^0(r) = 4\pi j_l(kr). \quad (2.19)$$

Substituting (2.17) into (2.16) and making use of (2.18), we obtain a differential equation for the radial functions  $\psi_{l,k}(r)$ ,

$$\left\{ \frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d}{dr}) - \frac{l(l+1)}{r^2} - v(r) + k^2 \right\} \psi_{l,k}(r) = 0 \quad (2.20)$$

This equation must be supplemented with the boundary condition for  $\psi_{l,k}(r)$  which follows from (2.14). In order to derive it, we make use of the asymptotics of the plane wave radial part expansion

$$\psi_{l,k}^0(r) \equiv 4\pi j_l(kr) \xrightarrow[r \rightarrow \infty]{} 4\pi \frac{\sin(kr - \frac{l\pi}{2})}{kr},$$

or

$$\psi_{l,k}^0(r) \xrightarrow[r \rightarrow \infty]{} \frac{2\pi i}{kr} \left\{ e^{-i(kr - \frac{l\pi}{2})} - e^{i(kr - \frac{l\pi}{2})} \right\}. \quad (2.21)$$

The first term within the curly braces describes the spherical wave that converges to the center, the second term is the diverging spherical wave. Far from the center the radial wave function  $\psi_{l,k}$  similarly to  $\psi_{l,k}^0$ , is a superposition of spherical waves converging to and diverging from the center. The *particle-potential interaction* modifies only the diverging wave amplitude, whereas the converging wave amplitude is governed by the incident flux

density. The amplitude variation of the wave diverging from the scattering center is described by the diagonal element of the *scattering matrix*  $S_l(k)$  corresponding to the angular momentum quantum number  $l$ . Therefore, the required asymptotics of the solution of (2.20) must be

$$\psi_{l,k}(r) \xrightarrow[r \rightarrow \infty]{} \frac{2\pi i}{kr} \left\{ e^{-i(kr - \frac{l\pi}{2})} - S_l(k) e^{i(kr - \frac{l\pi}{2})} \right\}. \quad (2.22)$$

A difference between amplitudes of the divergent waves in (2.21) and (2.22) determines the so-called *partial scattering amplitude*

$$f_l(k) = \frac{i}{2k} \{1 - S_l(k)\}. \quad (2.23)$$

The *total scattering amplitude*  $f(\mathbf{k}, \mathbf{k}')$  can be presented as a sum:

$$f(\mathbf{k}, \mathbf{k}') = \sum_l (2l + 1) f_l(k) P_l(\cos \vartheta). \quad (2.24)$$

In general, the matrix elements  $S_l$  are complex. If only elastic scattering occurs, then  $S_l$  is a function of the real *scattering phase shift*  $\delta_l(k)$ , i.e.,

$$S_l(k) = e^{2i\delta_l(k)}. \quad (2.25)$$

Usually one describes the scattering phase shift by an expression that vanishes as  $k \rightarrow \infty$ .

Substituting (2.25) into (2.22) yields the radial function asymptotic in the form

$$\psi_{l,k}(r) \xrightarrow[r \rightarrow \infty]{} 4\pi \exp(i\delta_l) \frac{\sin(kr - \frac{l\pi}{2} + \delta_l)}{kr}. \quad (2.26)$$

Comparing (2.26) to (2.21), we find that interaction is responsible for the phase shift  $\delta_l$  in the asymptotics of the radial particle wave, which does not occur in the free wave asymptotics.

**Bound states of a two-particle system.** Depending on the interaction type, equation (2.2) may have solutions also for definite negative energies  $E_n < 0$ , ( $n=0,1,2,\dots$ ), which form a discrete set. These solutions correspond to the *bound states of the system*. The bound state wave functions must satisfy the square integrability condition, which, in turn, determines the negative energy spectrum.

Making use of the Green function (2.9) or (2.10), we reduce the Schrödinger equation for the bound states of the system to a *homogeneous integral equation*

$$\psi^{(n)} = G_0^{(\pm)}(E_n) V \psi^{(n)}. \quad (2.27)$$

Let us consider the solutions of equation (2.2) in the general case and introduce the following notation for the energy

$$E = \frac{\hbar^2 k^2}{2\mu}.$$

In what follows we present the solutions as expansions in terms of partial waves. Let the radial function of the state with the angular momentum  $l$  be given by

$$\psi_{l,k}(r) = \frac{u_l(k, r)}{r}. \quad (2.28)$$

The equation for the radial function can be written as

$$\frac{d^2 u_l}{dr^2} + \left[ k^2 - \frac{l(l+1)}{r^2} - v(r) \right] u_l = 0. \quad (2.29)$$

Different solutions of this equation are specified by the boundary conditions at the point  $r = 0$  or at infinity  $r \rightarrow \infty$ . Suppose the quantity  $k$  is complex. Let us consider the dependence of different solutions on  $k$ . Moreover, we assume that the singularity of the potential  $V(r)$  at  $r = 0$  is weaker than  $r^{-2}$ , and  $V(r)$  decreases at infinity faster than  $r^{-3}$ .

The *regular solution* of equation (2.29),  $\Phi_l(k, r)$ , which satisfies the boundary conditions at the point  $r = 0$ ,

$$\Phi_l(k, r) \xrightarrow[r \rightarrow 0]{} r^{l+1}, \quad (2.30)$$

is an analytic function without singularities in the open plane of the complex variable  $k$  for any value of  $r$ . Besides that, we consider the solutions of equation (2.29)  $f_l(k, r)$  and  $f_l(-k, r)$ , which have the asymptotic properties

$$\lim_{r \rightarrow \infty} e^{ikr} f_l(k, r) = 1. \quad (2.31)$$

These solutions are usually called the *Jost solutions*. It is not difficult to prove that the solutions  $f_l(k, r)$  and  $f_l(-k, r)$  are independent since the relevant Wronskian is not equal to zero, i.e.,

$$W[f_l(k, r), f_l(-k, r)] = 2ik. \quad (2.32)$$

The solution  $f_l(k, r)$  is an analytic function with no singularities in the lower half-plane of the complex variable  $k$  ( $Im k < 0$ ); it is continuous along the real axis  $k$ .

We define now the *Jost function*  $f_l(k)$  by the equality

$$f_l(k) \equiv W[f_l(k, r), \Phi_l(k, r)]. \quad (2.33)$$

It is clear that the Jost function  $f_l(k)$ , similarly to the Jost solution  $f_l(k, r)$ , is an analytic function with no singularities in the lower half-plane of the complex variable  $k$ . The regular solution  $\Phi_l(k, r)$  can be expressed in terms of the Jost solutions  $f_l(k, r)$  and  $f_l(-k, r)$  as given by

$$\Phi_l(k, r) = -\frac{1}{2ik} \{ f_l(-k)f_l(k, r) - f_l(k)f_l(-k, r) \}. \quad (2.34)$$

The asymptotics of the expression (2.34) at infinity takes the form

$$\Phi_l(k, r) \xrightarrow[r \rightarrow \infty]{} -\frac{1}{2ik} \left\{ f_l(-k)e^{-ikr} - f_l(k)e^{ikr} \right\}. \quad (2.35)$$

At the same time, in the case of scattering the asymptotics of the radial function  $\psi$ , normalized to the unit amplitude of the incident wave, is given by

$$\psi_{l,k}(r) \xrightarrow[r \rightarrow \infty]{} \frac{2\pi i}{kr} \left\{ \exp[-i(kr - \frac{l\pi}{2})] - S_l(k) \exp[i(kr - \frac{l\pi}{2})] \right\}. \quad (2.36)$$

Comparing this expression to (2.35), we obtain the relation between *the scattering matrix*  $S_l(k)$  and the Jost functions  $f_l(k)$  and  $f_l(-k)$ , i.e.,

$$S_l(k) = (-1)^l \frac{f_l(k)}{f_l(-k)}. \quad (2.37)$$

The relation between the normalized wave function  $\psi_{l,k}(r)$  and the regular solution  $\Phi_l(k, r)$  is given by the formula

$$\psi_{l,k}(r) = (-1)^l \frac{4\pi}{f_l(-k)} \cdot \frac{\Phi_l(k, r)}{r}. \quad (2.38)$$

Zeros of the function  $f_l(k)$  in the lower half-plane of the complex variable  $k$  correspond to the bound states of the system. These zeros lie in the negative part of the imaginary axis, i.e.,

$$f_l(k_n) = 0, \quad k_n = -i\kappa_n, \quad \kappa_n > 0. \quad (2.39)$$

Zeros of the function  $f_l(k)$  correspond to a *discrete set of energy levels*

$$E_n = -\frac{\hbar^2 \kappa_n^2}{2\mu}. \quad (2.40)$$

Each level, in turn, is associated with the wave function  $\Phi_l(-i\kappa_n, r)$  which is a solution of (2.29) and satisfies the square integrability condition. Therefore, it describes a bound state of the system. The functions  $\Phi_l(-i\kappa_n, r)$  vanish at  $r = 0$  and have asymptotics

$$\Phi_l(-i\kappa_n, r) \xrightarrow[r \rightarrow \infty]{} \frac{1}{\gamma_n} e^{-\kappa_n r}. \quad (2.41)$$

Using (2.35) the coefficient  $\gamma_n$  can easily be found to be

$$\gamma_n = -\frac{2\kappa_n}{f_l(i\kappa_n)}.$$

Note that  $\Phi_l(-i\kappa_n, r)$  are real functions.

The zeros of the function  $f_l(k)$  are simple since the derivative  $df_l(k_n)/dk_n$  does not vanish

$$\left. \frac{df_l(k)}{dk} \right|_{k_n} = -2k_n \gamma_n \int_0^\infty dr \Phi_l^2(k_n, r) \neq 0. \quad (2.42)$$

It is clear that  $f_l(-k)$  is analytic in the upper half-plane of complex  $k$  and possesses simple zeros located on the positive imaginary semi-axis. These zeros correspond to the bound states of the system. According to the definition (2.37) the zeros of the function  $f_l(-k)$  correspond to the poles of the scattering matrix  $S_l(k)$ . Thus, each bound state of the system is associated with a zero of the scattering matrix located on the negative imaginary semi-axis, and a symmetric (with respect to the real axis) pole of the scattering matrix on the positive imaginary semi-axis.

We denote the *normalized wave function of the bound state* by  $u_l^{(n)}(r)$ ,

$$u_l^{(n)}(r) = C_n f_l(-i\kappa_n, r). \quad (2.43)$$

Applying (2.42), we have

$$C_n^2 = -\frac{f_l(i\kappa_n)}{\frac{df_l(-i\kappa_n)}{d\kappa_n}}. \quad (2.44)$$

Making use of the scattering matrix definition (2.37), we obtain the normalization constant  $C_n$  in the integral form,

$$C_n^2 = \frac{(-1)^l}{2\pi} \oint dk S_l(k), \quad (2.45)$$

the integration contour being a small circle that goes clockwise around the pole  $k = i\kappa_n$ . The *scattering matrix* in the vicinity of the pole may be written as

$$S_l(k) \simeq \frac{b_n}{k - i\kappa_n} \quad (k \rightarrow i\kappa_n).$$

According to (2.45), the residue  $b_n$  is determined by the normalization constant of the bound state wave function, i.e.,

$$b_n = (-1)^{l+1} i C_n^2. \quad (2.46)$$

According to (2.28) and (2.43), a radial function of the bound states can be presented as

$$\psi_l^{(n)} = C_n \left\{ \frac{e^{-\kappa_n r}}{r} + q_n(r) \right\}, \quad (2.47)$$

where

$$q_n(r) = \frac{1}{r} (f_l(-i\kappa_n, r) - e^{-\kappa_n r})$$

is the short range or interior part of  $\psi_l^{(n)}(r)$  that for large  $r$  falls off more rapidly than  $e^{-\kappa_n r}/r$ . At the same time making use of (2.27), the *wave function of the bound state* in the momentum representation can be presented as

$$\langle \mathbf{k}|nl\rangle = -\frac{\langle \mathbf{k}|v(r)|nl\rangle}{k^2 + \kappa_n^2}, \quad (2.48)$$

where  $\mathbf{k}$  is the appropriate relative momentum. The energy dominator of (2.48) has a zero at the on-shell point  $k^2 = -\kappa_n^2$ , while the numerator, which we call the vertex function, depends on the potential. This form for the bound state of interaction dependent vertex divided by an energy dominator that vanishes at the on-shell point is very general. The pole of (2.48) at  $k^2 = -\kappa_n^2$  is related to the large  $r$  behavior of the coordinate space wave function. The asymptotic part of (2.47) comes only from the pole of (2.48). In terms of the momentum space wave function, the shorter range parts of (2.47),  $q_n(r)$ , translate into singularities of (2.48) that are further from the physical region ( $k^2 > 0$ ), than the pole (Amado, 1979). From (2.48) we see that these come from the vertex function and therefore, must carry the potential range. Knowing the wave function completely is therefore equivalent to knowing  $C_n$  and  $q_n(r)$ . The normalization constant  $C_n$  is easy to determine empirically while information on the interior region, that is knowledge of  $q_n(r)$  is far more difficult to get without knowing a good deal about the dynamics.

**The Lippmann-Schwinger equation.** It is convenient to rewrite the results obtained in a symbolic form that is suitable for extensions to many-particle systems. Let us consider a quantum mechanical system of several interacting particles. We assume the Hamiltonian to be a sum of two parts

$$H = H_0 + V,$$

where  $H_0$  describes the nonperturbed motion of the system and  $V$  is the interaction which vanishes if the distance between the interacting parts of the system is sufficiently large.

In the stationary formulation, the scattering problem reduces to the Schrödinger equation

$$(E - H)\psi = 0 \quad (2.49)$$

with fixed *boundary conditions*, the energy  $E$  of the system being positive. At infinity, the solution  $\psi$  must be a sum of the incident wave  $\varphi$ , governed by the nonperturbed equation

$$(E - H_0)\varphi = 0, \quad (2.50)$$

and the diverging scattered wave.

The solution of (2.49) with the above mentioned boundary condition formally may be written as

$$\psi^{(+)} = \varphi + G_0^{(+)}(E)V\psi^{(+)}, \quad (2.51)$$

where  $G_0^{(+)}(E)$  is the *Green function* of the nonperturbed equation (2.50),

$$G_0^{(+)}(E) = \frac{1}{E - H_0 + i\mathcal{O}}. \quad (2.52)$$

In fact,  $G(E)$  is the integral operator whose kernel is a Green function for the nonperturbed equation (2.50). The choice of the integration contour enclosing the pole of (2.52) follows from the requirement that the asymptotics of  $\psi^{(+)}$  must contain a diverging scattered wave.

The formal solution (2.51) of the Schrödinger equation (2.49) is an integral equation

$$\psi^{(+)} = \varphi + \frac{1}{E - H_0 + i\mathcal{O}}V\psi^{(+)}, \quad (2.53)$$

which is referred to as the *Lippmann-Schwinger equation*. Thus, the Lippmann-Schwinger equation itself consists of the boundary condition (at infinity) for the wave function, which is the most important property of this equation. It is clear, that the divergent wave in (2.51) is connected uniquely with the choice of the positive sign of the infinitesimally small imaginary addition to the energy  $E$ . The equation

$$\psi_\varepsilon = \varphi + \frac{1}{E - H_0 - i\varepsilon}V\psi_\varepsilon \quad (2.54)$$

has a unique solution for any finite  $\varepsilon$ . An ambiguity could occur if the homogeneous equation obtained from (2.54) for  $\varphi = 0$  had a solution. However, the eigenvalues corresponding to the decreasing solutions of the homogeneous equation (in the asymptotic region for which  $V \rightarrow 0$ ) must be real. Therefore, the equation

$$\psi_\varepsilon = \frac{1}{E - H_0 + i\varepsilon} V \psi_\varepsilon \quad (2.55)$$

has no solutions for any finite  $\varepsilon$ .

The scattering wave function whose asymptotics at infinity is a sum of the incident and converging scattered waves, satisfies the equation in which the integration contour must be taken in another way, i.e.,

$$\psi^{(-)} = \varphi + \frac{1}{E - H_0 - i\mathcal{O}} V \psi^{(-)}, \quad (2.56)$$

because it is determined by the Green function

$$G_0^{(-)}(E) = \frac{1}{E - H_0 - i\mathcal{O}}.$$

Equations (2.53) and (2.56) are written in the operator form. To write them explicitly, one has to expand the function  $V\psi$  in terms of the eigenfunctions of the nonperturbed Hamiltonian  $H_0$ . This results in

$$\psi_\alpha^{(\pm)} = \varphi_\alpha + \sum_\beta \frac{\varphi_\beta(\varphi_\beta, V\psi_\alpha^{(\pm)})}{E_\alpha - E_\beta \pm i\mathcal{O}}. \quad (2.57)$$

Thus, the *Green function* is given by

$$G_0^{(\pm)}(E) \dots = \sum_\beta \frac{\varphi_\beta(\varphi_\beta, \dots)}{E - E_\beta \pm i\mathcal{O}}. \quad (2.58)$$

This relation determines the expansion of the Green function  $G_0^{(\pm)}$  in terms of the complete set of eigenfunctions of the operator  $H_0$ . In the coordinate representation, we have

$$G_0^{(\pm)}(E; \mathbf{r}, \mathbf{r}') \equiv \langle \mathbf{r}' | G_0^{(\pm)}(E) | \mathbf{r} \rangle = \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{\varphi_{\mathbf{k}'}(\mathbf{r}) \varphi_{\mathbf{k}'}^*(\mathbf{r}')}{E - E' \pm i\mathcal{O}}, \quad (2.59)$$

which is in agreement with (2.11). It is not difficult to verify that (2.57) in the coordinate representation reproduces (2.12).

Green functions  $G_0^{(+)}$  and  $G_0^{(-)}$  take the especially simple form in the momentum representation

$$\langle \mathbf{k}' | G_0^{(\pm)}(E) | \mathbf{k} \rangle = \frac{\delta(\mathbf{k} - \mathbf{k}')}{E - \frac{\hbar^2 k^2}{2\mu} \pm i\mathcal{O}}. \quad (2.60)$$

If the energies of relative motion are negative, then the solutions of the Schrödinger equation (2.49) are associated with the bound states of the system. In this case the Schrödinger equation reduces to the *homogeneous integral equation*

$$\psi = G_0^{(\pm)}(E)V\psi, \quad E < 0. \quad (2.61)$$

In the general case, the *Lippmann-Schwinger equation* (2.51) has no ambiguous solutions if the system possesses bound states. Indeed, the energy levels

$$\mathcal{E} = \mathbf{P}^2/2M + E$$

( $\mathbf{P}$  is the total momentum,  $M$  is the total mass,  $E$  is the energy of relative motion) are multiply degenerate because each state is associated with a certain energy distribution between the inner motion and motion of the system as a whole. Since existence of a bound state infers that the homogeneous equation (2.61) has a solution for fixed  $\mathcal{E}$ , the solution of the nonhomogeneous equation (2.51) becomes ambiguous. For a two-particle system, the Lippmann-Schwinger equation has an unique solution only in the c.m.s., for which  $\mathbf{P} = 0$  and energy degeneration does not occur ( $\mathcal{E} = E$ ).

**The Green function  $G$  and the transition operator  $t$ .** The solutions of (2.53) may be expressed in terms of the asymptotic function  $\varphi$  by means of the *Green function* of the equation (2.49)

$$G^{(+)}(E) = \frac{1}{E - H + i\mathcal{O}}. \quad (2.62)$$

We define the Green functions  $G_0(z)$  and  $G(z)$  of the complex argument  $z$  as

$$G_0(z) = \frac{1}{z - H_0}, \quad G(z) = \frac{1}{z - H}. \quad (2.63)$$

(These functions are usually called resolvents of the operators  $H_0$  and  $H$ .) Evidently,

$$G_0^{(\pm)}(E) = G_0(E \pm i\mathcal{O}), \quad G^{(\pm)}(E) = G(E \pm i\mathcal{O}).$$

The Green functions  $G(z)$  and  $G_0(z)$  are related by the equation

$$G(z) = G_0(z) + G_0(z)VG(z). \quad (2.64)$$

It may be easily verified that the solution (2.53) may be written in the form

$$\psi^{(+)} = \lim_{\varepsilon \rightarrow 0} i\varepsilon G(E + i\varepsilon)\varphi. \quad (2.65)$$

Indeed, multiplying (2.64) for  $z = E + i\varepsilon$  by  $i\varepsilon$ , applying it to the function  $\varphi$ , and making use of relation  $\lim_{\varepsilon \rightarrow 0} i\varepsilon G_0(E + i\varepsilon)\varphi = \varphi$ , one obtains (2.51).

The *Green functions*  $G^{(\pm)}(E)$  may be expanded in terms of the complete set of eigenfunctions of the operator  $H$  in the same way as  $G_0^{(\pm)}(E)$  were expanded in terms of eigenfunctions of the operator  $H_0$ . We denote the eigenfunctions of  $H$  associated with bound and continuum states as  $\psi_n$  and  $\psi_\beta$ , respectively. Suppose these functions form a complete set, so that

$$\sum_n \psi_n(\psi_n, \dots) + \sum_\beta \psi_\beta(\psi_\beta, \dots) = 1 \dots, \quad (2.66)$$

then, having applied the operator  $G^{(\pm)}$  to this equality, one obtains the expansion

$$G^{(\pm)}(E) \dots = \sum_n \frac{\psi_n(\psi_n, \dots)}{E - E_n} + \sum_\beta \frac{\psi_\beta(\psi_\beta, \dots)}{E - E_\beta \pm i\mathcal{O}}, \quad (2.67)$$

with the eigenvalues  $E_n$  and  $E_\beta$  being negative and positive, respectively.

We rewrite (2.66, 67) in the coordinate representation for the simplest case of a particle in the external field. It is convenient to take  $\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$  or  $\psi_{\mathbf{k}}^{(-)}(\mathbf{r})$  to be the continuum spectrum wave functions. Then the *completeness condition* takes the form

$$\sum_n \psi_n(\mathbf{r}) \psi_n^*(\mathbf{r}') + \int \frac{d\mathbf{q}}{(2\pi)^3} \psi_{\mathbf{q}}(\mathbf{r}) \psi_{\mathbf{q}}^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (2.68)$$

and the Green function expansion is given by

$$\begin{aligned} G^{(\pm)}(E; \mathbf{r}, \mathbf{r}') &\equiv \langle \mathbf{r}' | G^{(\pm)}(E) | \mathbf{r} \rangle \\ &= \sum_n \frac{\psi_n(\mathbf{r}) \psi_n^*(\mathbf{r}')}{E - E_n} + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\psi_{\mathbf{q}}(\mathbf{r}) \psi_{\mathbf{q}}^*(\mathbf{r}')}{E - E_q \pm i\mathcal{O}}. \end{aligned} \quad (2.69)$$

In contrast to the free motion Green function  $G_0^{(\pm)}(E; \mathbf{r}, \mathbf{r}')$ , the complete Green function  $G^{(\pm)}(E; \mathbf{r}, \mathbf{r}')$  describing the motion in the field  $V(\mathbf{r})$  depends on the coordinates  $\mathbf{r}$  and  $\mathbf{r}'$  rather than on their difference  $\mathbf{r} - \mathbf{r}'$ .

It is not difficult to verify within the context of the explicit expression (2.69) that the Green function  $G^{(\pm)}(E; \mathbf{r}, \mathbf{r}')$  indeed satisfies the Schrödinger equation (2.49) with a point source in the right-hand part, i.e.,

$$(E - H)G^{(\pm)}(E; \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (2.70)$$

When considering the scattering process, it is convenient to use the *transition operator*. We introduce this operator  $t$  with the help of the equality

$$VG(z) = t(z)G_0(z). \quad (2.71)$$

Using (2.64) we obtain the following equation of the transition operator

$$t(z) = V + VG_0(z)t(z). \quad (2.72)$$

This equation is usually called the *Lippmann-Schwinger operator equation*

Acting by the both parts of the operator equation (2.72) on the free motion wave function of particles  $\varphi$  and comparing the result with the *Lippmann-Schwinger equation* (2.53) multiplied by the potential  $V$  from the left, we obtain

$$t(E)\varphi_{\mathbf{k}} = V\psi_{\mathbf{k}}. \quad (2.73)$$

Using this relation the solution of the Lippmann-Schwinger equation may be written in the form

$$\psi^{(+)} = \varphi + \frac{1}{E - H_0 + i\mathcal{O}}t\varphi. \quad (2.74)$$

In the coordinate representation, the second term in the right-hand part of (2.74) at large distances is a diverging scattered wave. The coefficient before the latter is just the scattering amplitude

$$f = -\frac{\mu}{2\pi\hbar^2}(\varphi', t\varphi),$$

or

$$f(\mathbf{k}, \mathbf{k}') = -\frac{\mu}{2\pi\hbar^2}\langle \mathbf{k}' | t | \mathbf{k} \rangle. \quad (2.75)$$

Thus, the scattering amplitude is specified directly by a matrix element of the transition operator.

**The scattering amplitude off the energy shell.** Let us calculate the *scattering amplitude off the energy shell* by using the general relation between the scattering amplitude and the transition matrix. The transition

operator  $t$  is governed by the Lippmann-Schwinger equation (2.72). In the momentum representation, this equation may be written as ( $\hbar = 1$ )

$$\langle \mathbf{k}' | t(z) | \mathbf{k} \rangle = \langle \mathbf{k}' | V | \mathbf{k} \rangle + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\langle \mathbf{k}' | V | \mathbf{q} \rangle \langle \mathbf{q} | t(z) | \mathbf{k} \rangle}{z - (q^2/2\mu)}, \quad (2.76)$$

$$z = E + i\mathcal{O},$$

where  $E$  is the energy of two-particle relative motion which is conserved under elastic scattering,  $\mu$  is the reduced mass. Equation (2.76) determines the  $t$  matrix both on the energy shell, when the initial and final momenta are related to energy by

$$\frac{k^2}{2\mu} = \frac{k'^2}{2\mu} = E, \quad (2.77)$$

and off the energy shell, when

$$\frac{k^2}{2\mu} \neq \frac{k'^2}{2\mu} \neq E. \quad (2.78)$$

According to (2.75), the elastic scattering amplitude  $f(\mathbf{k}, \mathbf{k}')$  can be expressed in terms of the  $t$  matrix on the energy shell, so that

$$f(\mathbf{k}, \mathbf{k}') = -\frac{\mu}{2\pi} \langle \mathbf{k}' | t(E + i\mathcal{O}) | \mathbf{k} \rangle. \quad (2.79)$$

The *scattering amplitude off the energy shell*,  $f(\mathbf{k}, \mathbf{k}'; z)$ , can be found under the assumption that (2.79) holds when the condition (2.77) is violated.

Both the scattering amplitude  $f(\mathbf{k}, \mathbf{k}'; z)$  and the transition matrix  $\langle \mathbf{k}' | t(z) | \mathbf{k} \rangle$  possess singularities in the plane of complex energy  $z$ . These are poles associated with the discrete spectrum and the cut along the positive real semi-axis given rise to by the continuum spectrum of the system. The singularities can be explicitly derived from the so-called *spectral representation of the  $t$  matrix*,

$$\langle \mathbf{k}' | t(z) | \mathbf{k} \rangle = \langle \mathbf{k}' | V | \mathbf{k} \rangle + \sum_N \frac{g_N(\mathbf{k}') g_N(\mathbf{k})}{z + (\kappa_N^2/2\mu)} + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\langle \mathbf{k}' | t[(q^2/2\mu) + i\mathcal{O}] | \mathbf{q} \rangle \langle \mathbf{q} | t[(q^2/2\mu) - i\mathcal{O}] | \mathbf{k} \rangle}{z - (q^2/2\mu)}, \quad (2.80)$$

where  $g_N(\mathbf{k}) = [(k^2 + \kappa_N^2)/2]\varphi_N(\mathbf{k})$ ,  $\kappa_N^2/2\mu$  is the binding energy,  $\varphi_N(\mathbf{k})$  is the bound state wave function in the momentum representation.

If the interaction is spherically symmetric, then the partial component expansion of the potential is given by

$$\begin{aligned} \langle \mathbf{k}' | V | \mathbf{k} \rangle &= \sum_l (2l+1) V_l(k, k') P_l(\cos \vartheta) \\ &= 4\pi \sum_{lm} V_l(k', k) Y_{lm}^*(\mathbf{n}') Y_{lm}(\mathbf{n}), \end{aligned} \quad (2.81)$$

where  $Y_{lm}(\mathbf{n})$  and  $Y_{lm}(\mathbf{n}')$  are the spherical functions of the angles describing the directions of the vectors  $\mathbf{k}$  and  $\mathbf{k}'$ ;  $\vartheta$  is the angle between  $\mathbf{k}$  and  $\mathbf{k}'$ . Each term in (2.81) describes the interaction in the state with the angular momentum quantum number  $l$ . The partial component of the potential,  $V_l(k', k)$ , is given by

$$V_l(k', k) = 4\pi \int_0^\infty dr r^2 j_l(k'r) V(r) j_l(kr).$$

The  $t$  matrix and the *scattering amplitude* can be expanded in terms of partial components, too, so that

$$\langle \mathbf{k}' | t(z) | \mathbf{k} \rangle = \sum_l (2l+1) t_l(k', k; z) P_l(\cos \vartheta), \quad (2.82)$$

$$f(\mathbf{k}, \mathbf{k}'; z) = \sum_l (2l+1) f_l(k, k'; z) P_l(\cos \vartheta). \quad (2.83)$$

Here  $t_l(k', k; z)$  is the partial transition matrix and  $f_l(k, k'; z)$  is the partial *scattering amplitude off the energy shell*. The equation for the partial transition matrix readily follows from (2.72), i.e.,

$$t_l(z) = V_l + V_l \frac{1}{z - H_0} t_l(z), \quad (2.84)$$

or

$$\begin{aligned} t_l(k', k; z) &= V_l(k', k) \\ &+ \frac{1}{2\pi^2} \int_0^\infty dq q^2 V_l(k', q) \frac{1}{z - (q^2/2\mu)} t_l(q, k; z). \end{aligned} \quad (2.85)$$

The partial scattering amplitude and the partial transition matrix satisfy a relation analogous to (2.79):

$$f_l(k, k'; z) = -\frac{\mu}{2\pi} t_l(k', k; z). \quad (2.86)$$

The *partial scattering amplitude on the energy shell* is a function of the scattering phase shift  $\delta_l(k)$ , i.e.,

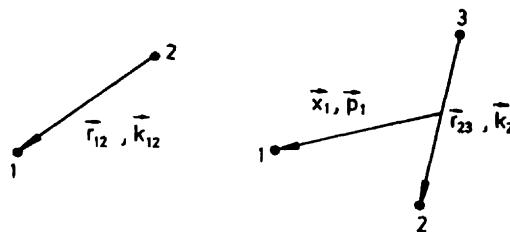


Figure 2.2. Relative coordinates and momenta in the two- and three-particle systems.

$$f_l\left(k, k'; \frac{k^2}{2\mu}\right) \equiv f_l(k) = \frac{1}{k} e^{i\delta_l} \sin \delta_l. \quad (2.87)$$

Two-particle scattering is completely described by the given scattering amplitude on the energy shell because, according (1.20), the scattering cross-section is directly expressed in terms of the phase shifts.

## 2.2. Quantum Mechanical Description of Three-particle Systems

**Peculiarities of the three-particle energy spectrum.** When passing from two-particle to three-particle systems, some difficulties arise, which are associated both with the kinematics and dynamics of systems containing more than two particles. One can always pass to the center-of-mass system and thus reduce the two-particle problem to the one-particle one by considering the particle motion in the field of some potential. If the nuclear range is finite, the asymptotic scattering phase determines the wave function behavior in the whole space (except the finite interaction region). All observable characteristics of the process may be expressed directly in terms of the phases. To describe bound states and scattering in a two-particle system, it is convenient to use the Lippmann-Schwinger equation. Unfortunately, this approach is inappropriate in the case of three-particle systems. In the general case, this is due to the ambiguity of solutions of the Lippmann-Schwinger equation by virtue of peculiarities of the many-particle energy spectrum. In order to reveal the difference, we consider the simplest two- and three-particle systems. Naturally, the two systems differ in the number of independent parameters required for their description. By virtue of the translation invariance, it is convenient to describe both systems in the c.m.s. (Fig. 2.2).

The two-particle system may be treated in terms of a single relative-distance vector or the momentum:

$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{k}_{12} = \frac{m_2 \mathbf{k}_1 - m_1 \mathbf{k}_2}{m_1 + m_2} \quad (2.88)$$

( $\mathbf{r}_1$  and  $\mathbf{r}_2$  are coordinates,  $\mathbf{k}_1$  and  $\mathbf{k}_2$  are momenta of individual particles having masses  $m_1$  and  $m_2$  respectively); the three-particle system is described by two relative-distance vectors

$$\mathbf{x}_1 = \mathbf{r}_1 - \frac{m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_2 + m_3}, \quad \mathbf{r}_{23} = \mathbf{r}_2 - \mathbf{r}_3, \quad (2.89)$$

or two momentum vectors

$$\mathbf{p}_1 = \frac{(m_2 + m_3) \mathbf{k}_1 - m_1 (\mathbf{k}_2 + \mathbf{k}_3)}{m_1 + m_2 + m_3}, \quad \mathbf{k}_{23} = \frac{m_3 \mathbf{k}_2 - m_2 \mathbf{k}_3}{m_2 + m_3} \quad (2.90)$$

( $\mathbf{r}_1$ ,  $\mathbf{r}_2$  and  $\mathbf{r}_3$  are coordinates,  $\mathbf{k}_1$ ,  $\mathbf{k}_2$  and  $\mathbf{k}_3$  are momenta of individual particles with masses  $m_1$ ,  $m_2$  and  $m_3$  respectively). This difference is, however, purely qualitative and not too important. The essential difference between the two systems under consideration is manifested in their energy spectra (Fig. 2.3).

The *two-particle energy spectrum* is very simple. It consists of two clearly distinguished parts: the discrete levels associated with bound states of the system, and the continuum range corresponding to the scattering states.

The *three-particle spectrum* is much more complicate. It consists of:

- a) discrete levels corresponding to the bound states of the three-particle system;
- b) continuous level sequences associated with particle scattering by a bound state of two other particles (each sequence begins from the energy determined by the binding energy of the two-particle subsystem);
- c) continuum spectral range given rise to by the three-particle scattering states (this range begins from energies above the three-particle break up threshold  $E = 0$ ).

An important peculiarity of the three-particle energy spectrum is its degeneracy. The same total energy  $E$  can be associated with different physical states of the system. This makes the Lippmann-Schwinger equation inapplicable for the description of three-particle systems. Indeed, the two-particle bound-states correspond to the solutions of the homogeneous equation  $\psi = G_0 V \psi$  ( $G_0$  is the Green function of the system without interaction), while the *nonhomogeneous Lippmann-Schwinger equation*

$$\psi = \varphi + G_0 V \psi$$

allows, for positive energies, an unique solution that determines the scattering states. The system of three or more particles possesses, along with the

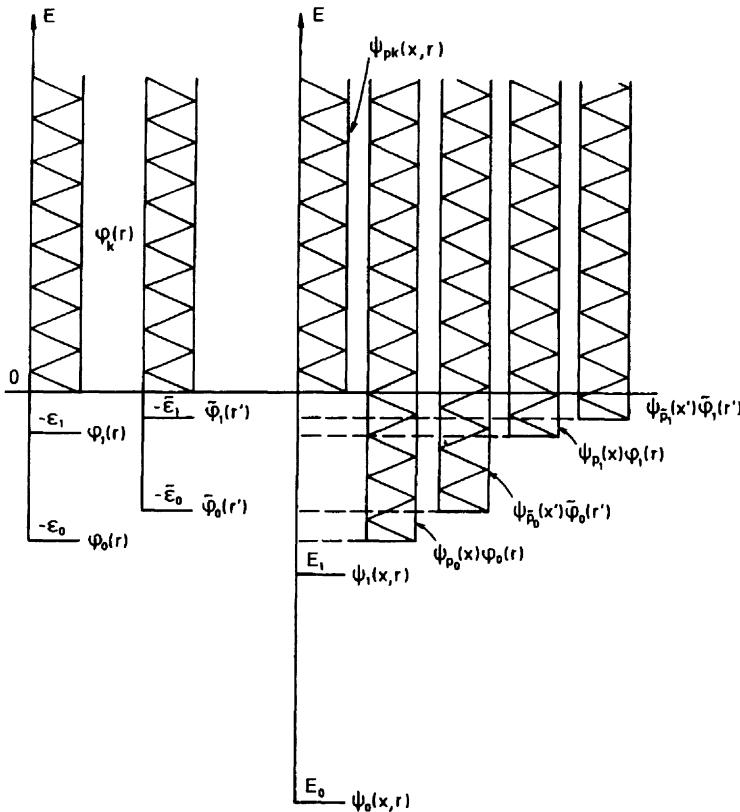


Figure 2.3. Energy spectra of two- and three-particle systems.

bound states of the whole system, also bound states of fewer-particle subsystems and thus the solutions of the inhomogeneous Lippmann-Schwinger equations are always ambiguous, and only the homogeneous equation for the bound state of the whole system in the c.m.s. has unique solutions. Ambiguity can be eliminated by appropriate rearrangement of the Lippmann-Schwinger equations. The resultant equations are usually referred to as the *Faddeev equations* (Faddeev, 1960).

**The Faddeev equations.** Let us derive the integral Faddeev equations for a system of three nonrelativistic spinless particles. The total Hamiltonian of the system may be written as

$$H = H_0 + V, \quad (2.91)$$

where  $H_0$  is the particle kinetic energy operator, and  $V$  is the interaction potential. Taking into account only two-particle forces, we can write  $V$  as

a sum of three terms

$$V = V_{12} + V_{23} + V_{31}, \quad (2.92)$$

where  $V_{ij}$  describes interaction between the particles  $i$  and  $j$  which decreases with increasing distance between the particles.

We consider three particles with masses  $m_1$ ,  $m_2$  and  $m_3$ . To describe three-particle relative motion, we introduce the coordinates  $\mathbf{x}_1$  and  $\mathbf{r}_{23}$ . Instead of relative coordinates  $\mathbf{x}_1$  and  $\mathbf{r}_{23}$ , the pairs  $\mathbf{x}_2$  and  $\mathbf{r}_{31}$  or  $\mathbf{x}_3$  and  $\mathbf{r}_{12}$  can be used:

$$\begin{aligned} \mathbf{x}_2 &= -\frac{m_1}{m_1 + m_3} \mathbf{x}_1 + \frac{m_3(m_1 + m_2 + m_3)}{(m_1 + m_3)(m_2 + m_3)} \mathbf{r}_{23}, \\ \mathbf{r}_{31} &= -\mathbf{x}_1 - \frac{m_2}{m_2 + m_3} \mathbf{r}_{23}; \end{aligned} \quad (2.93)$$

or

$$\begin{aligned} \mathbf{x}_3 &= -\frac{m_1}{m_1 + m_2} \mathbf{x}_1 - \frac{m_2(m_1 + m_2 + m_3)}{(m_1 + m_2)(m_2 + m_3)} \mathbf{r}_{23}, \\ \mathbf{r}_{12} &= \mathbf{x}_1 - \frac{m_3}{m_2 + m_3} \mathbf{r}_{23}. \end{aligned} \quad (2.94)$$

In the c.m.s., the kinetic energy operator of the system can be written in terms of the coordinates  $\mathbf{x}_1$  and  $\mathbf{r}_{23}$ , so that

$$H_0 = -\frac{1}{2\mu_1} \Delta_1 - \frac{1}{2\mu_{23}} \Delta_{23}, \quad (2.95)$$

with the reduced masses being

$$\mu_1 = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3}, \quad \mu_{23} = \frac{m_2 m_3}{m_2 + m_3}.$$

In the momentum representation,  $H_0$  takes the form

$$H_0 = \frac{\mathbf{p}_1^2}{2\mu_1} + \frac{\mathbf{k}_{23}^2}{2\mu_{23}}. \quad (2.96)$$

The *transition operator*  $T$  is determined by the Lippmann-Schwinger equation

$$T(Z) = V + V G_0(Z) T(Z), \quad (2.97)$$

where

$$G_0(Z) = (Z - H_0)^{-1}, \quad Z = E + i\mathcal{O}.$$

Note that the kernel of the integral equation (2.97) is singular by virtue of delta functions which appear due to momentum conservation for the particle that is not involved in interaction with the other particles forming the pair.

In the case of two-particle interaction given by (2.92), the transition operator  $T$  can also be written as a sum

$$T(Z) = T^{(1)}(Z) + T^{(2)}(Z) + T^{(3)}(Z), \quad (2.98)$$

the individual terms of which are defined by

$$T^{(k)}(Z) = V_{ij} + V_{ij}G_0(Z)T(Z), \quad ijl = 123, 231, 312. \quad (2.99)$$

Having substituted (2.98) for  $T(Z)$  in the right-hand part of (2.99), we can treat these relations as a system of coupled operator equations for the individual terms of (2.98). Note that the iteration series  $T^{(k)}(Z)$  in (2.99) contains both singular terms (of the type  $V_{23}G_0(Z)V_{23}$ ,  $V_{23}G_0(Z)V_{23}G_0(Z)V_{23}$ , so on) and the terms in which the delta functions are eliminated by the intermediate integration (of the type  $V_{23}G_0(Z)V_{31}$  etc.). The resultant system (2.99) is equivalent to (2.97) and therefore its solutions are ambiguous, similarly to those of the Lippmann-Schwinger equation.

In order to eliminate ambiguity, we rearrange the system (2.99) regarding the *two-particle transition operators*  $T_{ij}$  as being known. Suppose the operators  $T_{ij}$  are determined by

$$T_{ij}(Z) = V_{ij} + V_{ij}G_0(Z)T_{ij}(Z), \quad (2.100)$$

which follows from (2.97) if interaction between the particles  $i, j$  and the third particle is neglected. Note that the right-hand part of (2.100) contains all the singularities of (2.99) and hence can serve to exclude them. Separating in (2.99) the diagonal part

$$\begin{aligned} [1 - V_{ij}G_0(Z)]T^{(k)}(Z) &= V_{ij} + V_{ij}G_0(Z)[T^{(i)}(Z) + T^{(j)}(Z)], \\ (ijk &= 123, 231, 312) \end{aligned}$$

inverting the two-particle operator  $[1 - V_{ij}G_0(Z)]$ , and then making use of (2.100), we rewrite the system of coupled equations for individual terms of the transition operator as

$$\begin{aligned} T^{(k)}(Z) &= T_{ij}(Z) + T_{ij}(Z)G_0(Z)[T^{(i)}(Z) + T^{(j)}(Z)], \\ (ijk &= 123, 231, 312). \end{aligned} \quad (2.101)$$

In contrast to (2.99), this system of integral equations has a single-valued solution. The iteration series of the second term in the right-hand part of (2.101) contains no singularities. Therefore, the system of integral equations (2.101) can be solved by the Fredholm methods.

Using (2.71) for the relation between the transition operator  $T$  and the *Green function*  $G$  we obtain a system of equations for  $G$  from (2.101). Within the context of (2.98), we have

$$\begin{aligned} G(Z) &= G_0(Z) + G^{(1)}(Z) + G^{(2)}(Z) + G^{(3)}(Z), \\ G^{(i)}(Z) &= G_0(Z)T^{(i)}(Z)G_0(Z), \quad i = 1, 2, 3. \end{aligned} \quad (2.102)$$

The functions  $G^{(i)}$  satisfy the equations

$$\begin{aligned} G^{(k)}(Z) &= G_{ij}(Z) - G_0(Z) + G_0(Z)T_{ij}(Z)[G^{(i)}(Z) + G^{(j)}(Z)], \\ ijk &= 123, 231, 312, \end{aligned} \quad (2.103)$$

where

$$G_{ij}(Z) = G_0(Z) + G_0(Z)T_{ij}(Z)G_0(Z). \quad (2.104)$$

The equations for the corresponding wave function of the system can be derived from (2.103) by using (2.65).

In a three-particle system, the range of motion of all three particles as well as of one particle relative to the bound state of the other two can be infinite. We denote the *asymptotic wave functions* as  $\Phi_{123}$  and  $\Phi_i$ , where  $i=1, 2$ , or  $3$ . (The subscripts label unbound particles. All three particles are assumed to be different.)

If particle 1 is scattered by a bound state of particles 2 and 3, then the asymptotic wave function is  $\Phi_1$ . We apply (2.102), multiplied by  $i\varepsilon$ , to the function  $\Phi_1$  and use (2.65). Then we obtain an equation for the wave function of the system  $\Psi_1$ , i.e.,

$$\Psi_1 = \Psi_1^{(1)} + \Psi_1^{(2)} + \Psi_1^{(3)},$$

$$\begin{aligned} \Psi_1^{(1)} &= \Phi_1 + G_0(Z)T_{23}(Z)[\Psi_1^{(2)} + \Psi_1^{(3)}], \\ \Psi_1^{(2)} &= G_0(Z)T_{31}(Z)[\Psi_1^{(3)} + \Psi_1^{(1)}], \\ \Psi_1^{(3)} &= G_0(Z)T_{12}(Z)[\Psi_1^{(1)} + \Psi_1^{(2)}], \end{aligned} \quad (2.105)$$

$$Z = E + i\mathcal{O}.$$

The functions  $\Psi_2$  and  $\Psi_3$  are governed by analogous systems of equations.

We apply (2.102), multiplied by  $i\varepsilon$ , to the function  $\Phi_{123}$  and use (2.65). Then we obtain an equation for the wave function of the system of three unbound particles  $\Psi_{123}$ , i.e.,

$$\begin{aligned}\Psi_{123} &= \Phi_{123} + \Psi_{123}^{(1)} + \Psi_{123}^{(2)} + \Psi_{123}^{(3)}, \\ \Psi_{123}^{(1)} &= \Phi_{1(23)} - \Phi_{123} + G_0(Z)T_{23}(Z)[\Psi_{123}^{(2)} + \Psi_{123}^{(3)}], \\ \Psi_{123}^{(2)} &= \Phi_{2(31)} - \Phi_{123} + G_0(Z)T_{31}(Z)[\Psi_{123}^{(3)} + \Psi_{123}^{(1)}], \\ \Psi_{123}^{(3)} &= \Phi_{3(12)} - \Phi_{123} + G_0(Z)T_{12}(Z)[\Psi_{123}^{(1)} + \Psi_{123}^{(2)}],\end{aligned}\quad (2.106)$$

$$Z = E + i\mathcal{O},$$

where

$$\Phi_{1(23)} = \lim_{\varepsilon \rightarrow 0} i\varepsilon G_{23}(E + i\varepsilon)\Phi_{123}, \dots$$

The function  $\Phi_{1(23)}$ , in contrast to  $\Phi_{123}$ , takes into account interaction between the particles 2 and 3. It is not difficult to verify that at large distances the difference  $\Phi_{1(23)} - \Phi_{123}$  is a divergent wave with respect to the distance between particles 2 and 3.

Note that both the infinite motion of all three particles and scattering of one particle by the bound state of the other two are described by nonuniform systems of integral equations which have unique solutions.

The wave function  $\Psi_0$  of the bound state of the whole system is determined by a uniform set of integral equations which can be derived in a manner similar to the previous analysis. We obtain

$$\begin{aligned}\Psi_0 &= \Psi_0^{(1)} + \Psi_0^{(2)} + \Psi_0^{(3)}, \\ \Psi_0^{(1)} &= G_0(Z)T_{23}(Z)[\Psi_0^{(2)} + \Psi_0^{(3)}], \\ \Psi_0^{(2)} &= G_0(Z)T_{31}(Z)[\Psi_0^{(3)} + \Psi_0^{(1)}], \\ \Psi_0^{(3)} &= G_0(Z)T_{12}(Z)[\Psi_0^{(1)} + \Psi_0^{(2)}].\end{aligned}\quad (2.107)$$

The uniform set of (2.107) has solutions only for energies associated with bound states of the system. In the c.m.s., these energies are negative ( $E < 0$ ). Note that set (2.107) can be formally derived from (2.105) if we put  $\Phi_1 = 0$ .

**Momentum representation.** The *Faddeev equations* take the simplest form in the momentum representation. Each component  $\Psi^{(i)}$  of the total wave function can be written as a function of the relevant set of coordinates, i.e.,

$$\Psi^{(i)} \equiv \Psi^{(i)}(\mathbf{r}_{jk}, \mathbf{x}_i, \mathbf{R}), \quad ijk = 123, 231, 312 \quad (2.108)$$

( $\mathbf{R}$  is the center-of-mass radius).

In the *momentum representation*,  $\Psi^{(i)}$  is defined by

$$\begin{aligned} \Psi^{(i)}(\mathbf{k}_{jk}, \mathbf{p}_i, \mathbf{K}) &= \int d\mathbf{R} d\mathbf{x}_i d\mathbf{r}_{jk} \exp[-i\mathbf{k}_{jk}\mathbf{r}_{jk} - i\mathbf{p}_i\mathbf{x}_i - i\mathbf{K}\mathbf{R}] \\ &\times \Psi^{(i)}(\mathbf{r}_{jk}, \mathbf{x}_i, \mathbf{R}) \end{aligned} \quad (2.109)$$

( $\mathbf{K}$  is the center-of-mass momentum).

Now let us consider explicit expressions for the asymptotic functions. We label the initial momenta by the superscript zero. The free motion of all particles is described by the asymptotic function  $\Phi_{123}$

$$\Phi_{123} = \exp[i\mathbf{p}_1^0\mathbf{x}_1 + i\mathbf{k}_{23}^0\mathbf{r}_{23} + i\mathbf{K}^0\mathbf{R}] \quad (2.110)$$

$$E_{123} = \frac{p_1^{0^2}}{2\mu_1} + \frac{k_{23}^{0^2}}{2\mu_{23}} + \frac{K^{0^2}}{2M}, \quad M = m_1 + m_2 + m_3.$$

Note that  $\Phi_{123}$  is invariant with respect to interchanges of the particle coordinates.

The *asymptotic function*  $\Phi_1$  is given by

$$\Phi_1 = \exp[i\mathbf{p}_1^0\mathbf{x}_1 + i\mathbf{K}^0\mathbf{R}]\varphi_{\kappa_{23}}(\mathbf{r}_{23}), \quad (2.111)$$

$$E_{123} = \frac{p_1^{0^2}}{2\mu_1} - \frac{\kappa_{23}^2}{2\mu_{23}} + \frac{K^{0^2}}{2M},$$

where  $\varphi_{\kappa_{23}}(\mathbf{r}_{23})$  is the solution of the equation

$$\left(-\frac{1}{2\mu_{23}}\Delta_{23} + V_{23} - E_{23}\right)\varphi_{\kappa_{23}}(\mathbf{r}_{23}) = 0 \quad (2.112)$$

for negative energies of relative motion  $E_{23} = -\kappa_{23}^2/2\mu_{23} < 0$ , i.e., the *bound state wave function* of the particles 2 and 3.

The *asymptotic function*  $\Phi_{1(23)}$  can be written as

$$\Phi_{1(23)} = \exp[i\mathbf{p}_1^0 \mathbf{x}_1 + i\mathbf{K}^0 \mathbf{R}] \varphi_{\mathbf{k}_{23}^0}(\mathbf{r}_{23}), \quad E_{1(23)} = E_{123}, \quad (2.113)$$

where  $\varphi_{\mathbf{k}_{23}^0}(\mathbf{r}_{23})$  is the solution of (2.112) with  $E_{23} = (\mathbf{k}_{23}^0)^2/2\mu_{23} > 0$ , which reduces at infinity to the sum of a plane wave and a diverging spherical wave.

The kinetic energy operator in the momentum representation reduces to the multiplication operator, hence the *Green function*  $G_0(Z)$  is diagonal:

$$\begin{aligned} \langle \mathbf{k}'_{jk} \mathbf{p}'_i \mathbf{K}' | G_0(Z) | \mathbf{K} \mathbf{p}_i \mathbf{k}_{jk} \rangle &= (2\pi)^9 \left( Z - \frac{k_{jk}^2}{2\mu_{jk}} - \frac{p_i^2}{2\mu_i} - \frac{K^2}{2M} \right)^{-1} \\ &\times \delta(\mathbf{k}_{jk} - \mathbf{k}'_{jk}) \delta(\mathbf{p}_i - \mathbf{p}'_i) \delta(\mathbf{K} - \mathbf{K}'). \end{aligned} \quad (2.114)$$

The *two-particle scattering operator*  $T_{jk}(Z)$  is diagonal in the representation of the total momentum of the system and the free particle momentum  $\mathbf{p}_i$ , i.e.,

$$\begin{aligned} &\langle \mathbf{k}'_{jk} \mathbf{p}'_i \mathbf{K}' | T_{jk}(Z) | \mathbf{K} \mathbf{p}_i \mathbf{k}_{jk} \rangle \\ &= (2\pi)^6 \left\langle \mathbf{k}'_{jk} \left| t_{jk} \left( Z - \frac{p_i^2}{2\mu_i} - \frac{K^2}{2M} \right) \right| \mathbf{k}_{jk} \right\rangle \\ &\times \delta(\mathbf{p}_i - \mathbf{p}'_i) \delta(\mathbf{K} - \mathbf{K}'), \end{aligned} \quad (2.115)$$

where  $t_{jk}(Z)$  is the two-particle  $t$  matrix determined by the integral equation (2.76).

Since all the operators entering the Faddeev equations are diagonal in the  $\mathbf{K}$ -representation, the wave function  $\Psi$  must contain the factor  $\delta(\mathbf{K} - \mathbf{K}^0)$  which reflects conservation of the total momentum of the system. Therefore, all  $\mathbf{K}$ -dependencies can be excluded from consideration as soon as we convert to the center-of-mass system.

For example, we write the system of integral *Faddeev equations* (2.105) *in the momentum representation* for the case of particle scattering by the bound state of the other two particles:

$$\Psi_1 = \Psi_1^{(1)}(\mathbf{k}_{23}, \mathbf{p}_1) + \Psi_1^{(2)}(\mathbf{k}_{31}, \mathbf{p}_2) + \Psi_1^{(3)}(\mathbf{k}_{12}, \mathbf{p}_3), \quad (2.116)$$

$$\begin{aligned} \Psi_1^{(1)}(\mathbf{k}_{23}, \mathbf{p}_1) &= \Phi_1(\mathbf{k}_{23}, \mathbf{p}_1) \\ &+ \left( Z - \frac{k_{23}^2}{2\mu_{23}} - \frac{p_1^2}{2\mu_1} \right)^{-1} \int \frac{d\mathbf{k}'_{23}}{(2\pi)^3} \left\langle \mathbf{k}_{23} \left| t_{23} \left( Z - \frac{p_1^2}{2\mu_1} \right) \right| \mathbf{k}'_{23} \right\rangle \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \Psi_1^{(2)}(\mathbf{k}'_{31}, \mathbf{p}'_2) + \Psi_1^{(3)}(\mathbf{k}'_{12}, \mathbf{p}'_3) \right\}, \\
\Psi_1^{(2)}(\mathbf{k}_{31}, \mathbf{p}_2) &= \left( Z - \frac{k_{31}^2}{2\mu_{31}} - \frac{p_2^2}{2\mu_2} \right)^{-1} \\
&\times \int \frac{d\mathbf{k}'_{31}}{(2\pi)^3} \left\langle \mathbf{k}_{31} \left| t_{31} \left( Z - \frac{p_2^2}{2\mu_2} \right) \right| \mathbf{k}'_{31} \right\rangle \\
&\times \left\{ \Psi_1^{(3)}(\mathbf{k}'_{12}, \mathbf{p}'_3) + \Psi_1^{(1)}(\mathbf{k}'_{23}, \mathbf{p}'_1) \right\}, \\
\Psi_1^{(3)}(\mathbf{k}_{12}, \mathbf{p}_3) &= \left( Z - \frac{k_{12}^2}{2\mu_{12}} - \frac{p_3^2}{2\mu_3} \right)^{-1} \\
&\times \int \frac{d\mathbf{k}'_{12}}{(2\pi)^3} \left\langle \mathbf{k}_{12} \left| t_{12} \left( Z - \frac{p_3^2}{2\mu_3} \right) \right| \mathbf{k}'_{31} \right\rangle \\
&\times \left\{ \Psi_1^{(1)}(\mathbf{k}'_{23}, \mathbf{p}'_1) + \Psi_1^{(2)}(\mathbf{k}'_{31}, \mathbf{p}'_2) \right\}, \\
Z &\equiv \frac{p_1^{0^2}}{2\mu_1} - \frac{\kappa_{23}^2}{2\mu_{23}} + i\mathcal{O}.
\end{aligned} \tag{2.117}$$

The quantities  $\mathbf{k}'_{31}$ ,  $\mathbf{p}'_2$  and  $\mathbf{k}'_{12}$ ,  $\mathbf{p}'_3$  entering the intergrand in the first equation must be expressed in terms of  $\mathbf{p}_1$  and  $\mathbf{k}'_{23}$ . The function  $\Phi_1(\mathbf{k}_{23}, \mathbf{p}_1)$ , by virtue of (2.111), is given by

$$\Phi_1(\mathbf{k}_{23}, \mathbf{p}_1) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}_1^0) \varphi_{\kappa_{23}}(\mathbf{k}_{23}), \tag{2.118}$$

where  $\varphi_{\kappa_{23}}(\mathbf{k}_{23})$  is the wave function of the two-particle bound state with the binding energy  $\kappa_{23}^2/2\mu_{23}$ ;  $\mathbf{p}_1^0$  is the initial momentum of relative motion of the system.

If all three particles are identical and have zero spin and isotopic spin, then the total wave function of the system  $\Psi_1$  must be symmetric with respect to permutations of any pairs of particles. In this case we have

$$\Psi_1^{(1)}(\mathbf{k}, \mathbf{p}) = \Psi_1^{(2)}(\mathbf{k}, \mathbf{p}) = \Psi_1^{(3)}(\mathbf{k}, \mathbf{p}) \equiv \psi(\mathbf{k}, \mathbf{p})$$

and the wave function can be written as

$$\Psi_1 = \psi(\mathbf{k}_{23}, \mathbf{p}_1) + \psi(\mathbf{k}_{31}, \mathbf{p}_2) + \psi(\mathbf{k}_{12}, \mathbf{p}_3). \tag{2.119}$$

[The relative momenta  $\mathbf{k}_{ij}$  and  $\mathbf{p}_k$  (where  $ijk=123, 231$ , and  $312$ ) are given by (2.90) with  $m_1 = m_2 = m_3 = m$ .] Then, instead of the system of three equations (2.117), we obtain a single *integral equation for the function  $\psi(\mathbf{k}, \mathbf{p})$* :

$$\begin{aligned} \psi(\mathbf{k}, \mathbf{p}) &= \varphi(\mathbf{k}, \mathbf{p}) + \left( Z - \frac{k^2}{2m} - \frac{3}{4} \frac{p^2}{m} \right)^{-1} \\ &\quad \times \int \frac{d\mathbf{p}'}{(2\pi)^3} \left\{ \left\langle \mathbf{k} \left| t \left( Z - \frac{3}{4} \frac{p^2}{m} \right) \frac{\mathbf{p}}{2} + \mathbf{p}' \right\rangle \right. \right. \\ &\quad \left. \left. + \left\langle \mathbf{k} \left| t \left( Z - \frac{3}{4} \frac{p^2}{m} \right) \left| -\frac{\mathbf{p}}{2} - \mathbf{p}' \right. \right\rangle \right\} \psi(\mathbf{p} + \frac{\mathbf{p}'}{2}, \mathbf{p}') \right), \end{aligned} \quad (2.120)$$

where  $\varphi(\mathbf{k}, \mathbf{p})$  is a function whose symmetrization yields the initial wave function of the system

$$\Phi_1 = \varphi(\mathbf{k}_{23}, \mathbf{p}_1) + \varphi(\mathbf{k}_{31}, \mathbf{p}_2) + \varphi(\mathbf{k}_{12}, \mathbf{p}_3). \quad (2.121)$$

For a three-particle bound state we have  $\varphi = 0$ ,  $z = E_0 < 0$ . In the case of particle scattering by a two-particle bound state,

$$\varphi(\mathbf{k}, \mathbf{p}) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}^0) \varphi_{N_0}(\mathbf{k}), \quad Z = \frac{3}{4} \frac{p^{0^2}}{m} - \frac{\kappa_{N_0}^2}{m} + i\mathcal{O}. \quad (2.122)$$

Note that both initial and final states of the matrix elements  $\langle \mathbf{k}|t(Z - \frac{3}{4}(p^2/m))|\mathbf{k}' \rangle$  entering (2.120) lie off the energy shell, i.e.,  $k^2/m \neq k'^2/m \neq Z - \frac{3}{4}(p^2/m)$ .

In the general case the wave function of the system described by the integral equation (2.120) depends on six variables (two relative vectors). For a spherically symmetric interaction potential, the three-dimensional integral equation (2.120) can be reduced to a system of two-dimensional integral equations by expanding the wave function in terms of angular functions and separating the angle variables. Making use of the separable representation of the two-particle scattering amplitude, one reduces the systems of two-dimensional integral equations to systems of one-dimensional integral equations.

**Impossibility of separating the interaction region in the three-particle configuration space.** If the interaction range is finite, the two-particle interaction is manifested only within a finite configuration region. For three-particle systems, however, one cannot separate in the configuration space an interaction region whose boundaries would be independent

of the particle relative energy. To show this, let us consider the scattering details in the two- and three-particle systems. In the simplest case of a collision of two elementary particles, there exists the single elastic scattering channel, the wave function off the interaction region being the sum of the incident plane wave and the diverging spherical wave. If at least one of the colliding particles is composite, there exist, along with the elastic scattering, other channels associated with inelastic processes. For binary processes, the wave functions in the inelastic scattering or reaction exit channels are diverging spherical waves. It is peculiar of binary processes that the number of open channels is restricted (due to the discrete energy spectrum of scattered particles) and the interaction is manifested within a limited configuration region. The binary processes are described by the  $S$ -matrix which is completely determined by the behaviour of the system in the interaction region. With the three particles being produced in the final state, the description becomes much more involved.

Let us consider the *scattering in a three-particle system*. Suppose the particle 1 is scattered by a bound state of the other two particles 2 and 3. If the incident particle energy is higher than the scatterer binding energy, there may occur, along with the binary processes of elastic scattering and rescattering, also the disintegration after which all the three particles turn out to be free. The wave function asymptotics may be presented in this case in the form

$$\Psi(\mathbf{x}, \mathbf{r}) = \begin{cases} \left\{ e^{i\mathbf{p}_0 \cdot \mathbf{x}} + f(\mathbf{p}_0, \mathbf{p}'_0) \frac{e^{i\mathbf{p}_0 \cdot \mathbf{x}}}{x} \right\} \varphi_0(r), & \text{at } \begin{array}{l} x \rightarrow \infty \\ r = \text{const} \end{array} \\ \tilde{f}(\mathbf{p}_0, \tilde{\mathbf{p}}_0) \frac{e^{i\tilde{\mathbf{p}}_0 \cdot \tilde{\mathbf{x}}}}{\tilde{x}} \tilde{\varphi}_0(\tilde{r}), & \text{at } \begin{array}{l} \tilde{\mathbf{x}} \rightarrow \infty \\ \tilde{\mathbf{r}} = \text{const} \end{array} \\ F(\mathbf{p}_0; \mathbf{p}, \mathbf{k}) \left( \frac{\mu_{23}^2 E}{2\pi^2 \hbar^2 \mu_1} \right)^{3/4} \frac{e^{i\sqrt{\frac{2\mu E}{\hbar^2}(x^2 + \frac{\mu_{23}}{\mu_1} r^2)}}}{(x^2 + \frac{\mu_{23}}{\mu_1} r^2)^{5/4}}, & \text{at } \begin{array}{l} x, r \rightarrow \infty \\ x/r = \text{const} \end{array} \end{cases} \quad (2.123)$$

where  $f$ ,  $\tilde{f}$ , and  $F$  are the elastic scattering, rescattering, and disintegration amplitudes respectively ( $E = \frac{\hbar^2 p_0^2}{2\mu_1} - \varepsilon_0 = \frac{\hbar^2 \tilde{p}_0^2}{2\tilde{\mu}_1} - \tilde{\varepsilon}_0 = \frac{\hbar^2 p^2}{2\mu_1} + \frac{\hbar^2 k^2}{2\mu_{23}}$ ). The coordinate dependence of the three-particle disintegration wave function is governed by the *asymptotic Green function for three noninteracting particles*. Since all particle velocities off the interaction region are fixed, the triangle formed by their radius-vector ends must increase with time remaining of the same shape. Therefore, in the free motion region,  $r = \beta x$  ( $x \rightarrow \infty$ ), where  $\beta = \frac{\mu_1}{\mu_{23}} \frac{k}{p}$ . It follows immediately from the asymptotics (2.123) that if the process produces three particles, one cannot separate a configuration region in which the interaction is manifested (in contrast to the binary

processes). Indeed, suppose the interaction between the particles 2 and 3 is characterized by a finite range  $R$ , and interaction between the particle 1 and the bound state of the particle system 2-3 is characterized by the range  $X$ . Then the wave function describing the three-particle interaction takes asymptotic form only provided  $x \geq X$ ,  $R/\beta$  or  $r \geq \beta X$ ,  $R$ . And since the quantity  $\beta$  depends on the relative momenta  $p$  and  $k$ , one can always find such momenta for which the interaction is manifested in the whole infinite space.

Another distinguishing feature of the processes under consideration is that any fixed total energy  $E$  is associated with an infinite number of states corresponding to various probable energy redistributions over the subsystems. This follows from the fact that the *three-particle disintegration wave function* may be presented as an expansion in terms of the scattering system continuum states:

$$\Psi(\mathbf{x}, \mathbf{r}) = \sum_{\mathbf{p}, \mathbf{k}} a_{\mathbf{p}\mathbf{k}} \psi_{\mathbf{p}}(\mathbf{x}) \varphi_{\mathbf{k}}(\mathbf{r}), \quad (2.124)$$

where the range of allowed  $\mathbf{p}$  and  $\mathbf{k}$  is determined by the condition  $E = \text{const}$  (the energy level  $E$  is infinitely degenerated). An example suitable for illustrating the peculiarities of the three-particle processes under consideration is the deuteron diffraction dissociation by a black nucleus.

Inasmuch as the interaction region cannot be distinguished, there occurs another peculiar feature of the three-particle systems: their properties essentially depend on the off energy behaviour of the two-particle interaction. Indeed, the conservation laws for the two-particle scattering provide the interaction to be manifested only on the energy shell. The scattering amplitude is directly expressed in terms of scattering phases which describe the two-particle wave function asymptotics. Under the three-particle scattering, the particle interaction is manifested off energy shell too. Intermediate state momenta and energies of two interacting particles do not satisfy usual relations due to the third particle influence, and, hence, the three-particle dynamical equations contain two-particle off energy amplitudes. The off energy behaviour of the two-particle amplitude is governed not only by the two-particle scattering state asymptotics, but also by the wave function coordinate-dependence at short distances. That is why the three-particle problem turns out to be much more sensitive to the particle interaction details than the two-particle problem.

With the two-particle interaction being described by some potential, the two-particle amplitude off energy behaviour may be considered in terms of the Schrödinger equation. So, the data of the two-particle problem are sufficient for obtaining the off energy two-particle amplitude. However, the two-nucleon interaction data are unsufficient and restricted and hence the

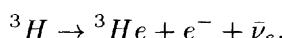
three-particle problem provides an important additional information source concerning nucleon interactions.

### 2.3. Three-Nucleon Systems

**The main characteristics of three-nucleon nuclei.** The study of the *three-nucleon systems* provides additional information on the nucleon-nucleon interaction. The two simplest three-nucleon nuclei are triton,  $^3H$ , and nucleus  $^3He$ . The triton (the nucleus of superheavy hydrogen atom) contains one proton and two neutrons; the nucleus  $^3He$  contains two protons and one neutron. One might assume that, in principle, the three-particle forces could be manifested in the three-nucleon nuclei. However, the study of the properties of the mirror nuclei  $^3H$  and  $^3He$ , nucleon-deuteron interaction, and systems of three unbound nucleons show that the above mentioned systems can be satisfactorily described within the range of experimental errors under the assumption of two-particle nature of the nuclear forces with finite range of action. The latter assumption makes the basis for the further analysis.

Unlike the two-nucleon problem, the problem of three nucleon motion is more sensitive to the shape of the two-nucleon potentials. The other peculiarities of nuclear forces manifest themselves more clearly in the three-nucleon rather than the two-nucleon systems. E.g., in the case of scattering of an incident nucleon by a nucleon that is bound in a deuteron or another complex nucleus, the energies of the colliding nucleons before and after the collision may be different. This is due to the fact that other nucleons of the target nucleus may obtain or transfer some amount of energy in the scattering process so that the energy of the whole system is conserved. Such scattering of an incident nucleon by the bound nucleon is referred to as the process off the energy shell (scattering of two free nucleons occurs on the energy shell). The study of the off-energy-shell scattering in the three-nucleon systems provides additional information on the nucleon-nucleon interaction.

Let us consider the experimental data concerning the main properties of the three-nucleon nuclei  $^3H$  and  $^3He$ . The nucleus  $^3He$  is stable; the mirror nucleus  $^3H$  is unstable and decays with the decay period of 12.3 years. The decay results in the formation of the nucleus  $^3He$ , i.e.,



Therefore, one of neutrons in the nucleus  $^3H$  is transformed into a proton, electron and antineutrino. The process of the electron  $\beta$ -decay of  $^3H$  nucleus is associated with the weak interaction and is rather slow. Other

physical properties of the mirror nuclei  $^3H$  and  $^3He$  are close since they are associated mainly with the strong nuclear interaction.

The binding energies of the nuclei  $^3H$  and  $^3He$  are equal to 8.432 MeV and 7.718 MeV, respectively. The difference in the binding energies,

$$\Delta E = 0.764 \text{ MeV}, \quad (2.125)$$

can be explained by the Coulomb repulsion between two protons in the nucleus  $^3He$ . We assume that the electric charge within the nucleus  $^3He$  is distributed homogeneously over the volume. Then the energy of electrostatic (Coulomb) interaction between the proton provided the constant charge density within the nucleus ( $\rho = e/( \frac{4}{3} \pi R^3 )$ , where  $R$  is the nuclear radius), and another proton, that produces the potential  $\varphi(r) = 2\pi\rho(R^2 - \frac{1}{3}r^2)$ , is given by

$$E_c = 4\pi\rho \int_0^R dr r^2 \varphi(r) = \frac{6e^2}{5R}. \quad (2.126)$$

Substituting the known value  $R \approx 2 \cdot 10^{-13}$  cm, for the radius of  $^3He$  nucleus in (2.126), we find that  $E_c$  is close to 0.764 MeV (slightly larger). On the other hand, for

$$E_c = \frac{6e^2}{5R} = 0.764 \text{ MeV},$$

we obtain a reasonable value for the nucleus  $^3He$  radius,  $R = 2.3 \cdot 10^{-13}$  cm, which is in agreement with the value obtained from the data on the high-energy electron scattering by  $^3He$  nuclei (assuming that the charge density is distributed according to the Gauss law). This approximate correspondence can be regarded as the evidence confirming that the nuclear interaction between two protons and two neutrons is basically the same.

The *binding energy per a nucleon pair* is equal to 2.8 MeV in the  $^3H$  nucleus, and to 2.6 MeV in the  $^3He$ . Both values are considerably greater than the deuteron binding energy. Inasmuch as linear dimensions of a deuteron and a three-nucleon nucleus are slightly different, (apparently, a deuteron is even slightly larger), we can deduce that both  $^3H$  and  $^3He$  nuclei are more compact than the deuteron. Therefore, three nucleons in three-nucleon nuclei are more close to each other (in average) than two nucleons in a more loose deuteron. It is worth noting that nuclei  $^3H$  and  $^3He$ , as well as deuteron and  $^4He$  nucleus, have no excited states.

The measured spin  $J$  of each nuclei  $^3H$  and  $^3He$  is  $1/2$ . The parities  $w$  of the ground states of  $^3H$  and  $^3He$  nuclei and the deuteron are positive ( $w = +1$ ). Let us assume that the ground state of the three-nucleon nucleus

is the  $S$ -state and that all three nucleons are in the states with zero orbital moments. Then, according to the Pauli principle, two identical nucleons in the nucleus ( $nn$  or  $pp$ ) must have opposite spins. Therefore, the total spin of three nucleons is equal to  $1/2$ , i.e., to the spin of the third nucleon that is not identical to the first two. Since the total orbital moment of three nucleons in the  $S$ -state is  $L = 0$ , the total angular momentum of the three-nucleon nucleus, i.e., its spin  $J$  coincides numerically with the total spin  $S$  and equals to  $1/2$ .

It is clear that the properties of nuclei  $^3H$  and  $^3He$  are determined mainly by the central nuclear interaction between nucleons. However, inasmuch as three-nucleon nuclei can contain pairs of nonidentical nucleons, which can be in the triplet states, i.e., the states with the total spin  $1$ , we can expect the manifestation of non-central (tensor) forces in the properties of both  $^3H$  and  $^3He$  nuclei and the deuteron. The comparison binding energies and magnetic moments of the nuclei  $^3H$  and  $^3He$  shows that the ground states of these nuclei are not the pure  $S$ -states but contain the  $D$ -states ( $L = 2$ ) with weights  $p_D \approx 0.04$ .

The measured *magnetic (dipole) moments of nuclei  $^3H$  and  $^3He$*  (in nuclear magnetons) are equal, respectively, to

$$\mu_{^3H} = 2.9788 \text{ and } \mu_{^3He} = -2.1274. \quad (2.127)$$

If the  $S$ -states were pure, the magnetic moments of the nuclei  $^3H$  and  $^3He$  would have numerical values  $\mu_p = 2.7928$  and  $\mu_n = -1.9130$ , respectively, by virtue of the fact that the magnetic moments of two identical neutrons in both nuclei are compensated. The cited numerical values, however, differ from the experimental values of the magnetic moments of the nuclei  $^3H$  and  $^3He$  approximately by  $10\%$ . The addition of the  $D$ -wave (with the weight  $p_D = 0.04$ ) can only slightly improve the situation. There still remains a discrepancy in magnitudes of the nuclear magnetic moments which can be explained taking into account the meson interchange currents between nucleons which induce additional magnetic moments.

**The isotopic spin of the three-nucleon systems.** If in a system of three nucleon there are two or three protons, then along with the nuclear forces the Coulomb forces appear. On taking account only the nuclear forces, the total isotopic spin  $T$  and its projection  $T_z$  are conserved. Three nucleons can be in the states with the total value of the isotopic spin  $T = 3/2$  and  $T = 1/2$ . The states with  $T = 3/2$  are related to the three-nucleon systems  $nnn$ ,  $nnp$ ,  $npp$ ,  $ppp$  with isotopic spin projections  $T_z = -3/2, -1/2, +1/2, +3/2$ , correspondingly. Three identical nucleons  $nnn$  or  $ppp$  can be only in the states with  $T = 3/2$  because  $T \geq |T_z|$ . The states with  $T = 1/2$  are related to the systems  $nnp$  and  $npp$  with the isospin projections  $T_z = -1/2$  and  $T_z = 1/2$ .

As we have seen, the nuclear interaction in the two-particle systems strongly depends on a magnitude of the isotopic spin. Because of this the properties of the three-nucleon systems depend on the isospin magnitude as well. This follows from the general simple considerations. In particular, the spatial structure of the three-nucleon systems strongly depends on the total isospin  $T$ . In fact, in the system of three neutrons  $nnn$  ( $T = 3/2$ ), only two from three identical nucleons can be in the states with zero orbital moments, the third nucleon, accordingly the Pauli principle, has nonzero orbital moment. The same spatial structure is typical for the systems  $nnp$  and  $npp$  in the states with  $T = 3/2$ , whose energy is larger than the energy of  $^3H$  and  $^3He$ , where all three nucleons have zero orbital moments. Since, there are no the bound systems consisting of identical nucleons  $nnn$  and  $ppp$ , then the systems  $nnp$  and  $npp$  cannot be bound in the states with  $T = 3/2$ . From this follows that the nuclei  $^3H$  and  $^3He$ , which are the systems  $nnp$  and  $npp$ , must have the isotopic spin  $T = \frac{1}{2}$  but not  $T = 3/2$ . We will see further, that there are different considerations confirming this conclusion. The mirror nuclei  $^3H$  and  $^3He$  along with a neutron and proton belong to the isotopic doublet with the isospin projections  $T_z = -\frac{1}{2}$  and  $T_z = +\frac{1}{2}$ .

Incorporation of the Coulomb interaction between protons in the three-nucleon systems  $npp$  and  $ppp$  results in violation of the total isotopic spin  $T$  conservation law. However, its projection  $T_z$  is preserved in this case as well. Because of this, the states of nuclei  $^3H$  and the system neutron-deuteron (accordingly Table 1.2, the deuteron isospin equals to zero) are described by the strictly defined value of the total isotopic spin  $T = \frac{1}{2}$ . At the same time, the wave functions of the ground state of nuclei  $^3He$  and the system proton-deuteron are the superpositions of the states with isospins  $T = \frac{1}{2}$  and  $T = 3/2$ . However, the contribution from the terms with  $T = 3/2$  in the total wave functions of these systems turns out to be very small.

**The spin and isospin states of three-nucleon system.** Three nucleons can be in the state with the total spin  $S = \frac{1}{2}$  and  $S = 3/2$ . A normalized spin wave function of the three-nucleon system with the definite total spin  $S$  and its projection  $M_S$  can be presented in the form of superposition of products of the ortho-normalized *one-nucleon spin wave functions*

$$\begin{aligned}\chi_{SM_S}^{(S_{23})}(123) &= \sum_{m_1 M_{23}} \left( \frac{1}{2} m_1 S_{23} M_{23} |SM_S| \right) \chi_{\frac{1}{2}m_1}(1) \chi_{S_{23}M_{23}}(23), \\ \chi_{S_{23}M_{23}}(23) &= \sum_{m_2 m_3} \left( \frac{1}{2} m_2 \frac{1}{2} m_3 |S_{23}M_{23}| \right) \chi_{\frac{1}{2}m_2}(2) \chi_{\frac{1}{2}m_3}(3),\end{aligned}\quad (2.128)$$

where in the spin function instead of arguments we put simply the nucleon numbers. By using the Clebsch-Gordan coefficients, we interrelated the spin states of the second and third nucleons in the two-particle state with the total spin  $S_{23}$  and projection  $M_{23}$ . After this we connected the obtained state with the first nucleon spin state. It could be done in a different order as well.

We consider firstly the spin states of the three-nucleon system, when their total spin  $S = \frac{1}{2}$ . The intermediate total spin of two nucleons  $S_{23}$  at  $S = \frac{1}{2}$  can be 0 or 1. Accordingly, we have introduced the following denotations of two *spin functions of the three-nucleon system* in the case  $S = \frac{1}{2}$ :

$$\chi' \equiv \chi'_{\frac{1}{2}M_S}(1, 23) = \chi_{\frac{1}{2}M_S}^{(S_{23}=0)}(123), \quad \chi'' \equiv \chi''_{\frac{1}{2}M_S}(1, 23) = \chi_{\frac{1}{2}M_S}^{(S_{23}=1)}(123).$$

Using the denotations  $\alpha(j) = \chi_{\frac{1}{2}, +\frac{1}{2}}(j)$  and  $\beta(j) = \chi_{\frac{1}{2}, -\frac{1}{2}}(j)$  for the one-particle spin functions (the functions  $\alpha(j)$  and  $\beta(j)$  related to positive and negative signs of the spin projections) and substituting numerical values of the Clebsh-Gordan coefficients in (2.128), we obtain the following expressions for functions  $\chi'$  and  $\chi''$  with all possible values  $M_S$ :

$$\begin{aligned} \chi'_{\frac{1}{2}, +\frac{1}{2}}(1, 23) &= \frac{\alpha(1)}{\sqrt{2}}[\alpha(2)\beta(3) - \beta(2)\alpha(3)], \\ \chi'_{\frac{1}{2}, -\frac{1}{2}}(1, 23) &= \frac{\beta(1)}{\sqrt{2}}[\alpha(2)\beta(3) - \beta(2)\alpha(3)], \\ \chi''_{\frac{1}{2}, +\frac{1}{2}}(1, 23) &= \frac{\alpha(1)}{\sqrt{6}}[\alpha(2)\beta(3) + \beta(2)\alpha(3)] \\ &\quad - \sqrt{\frac{2}{3}}\beta(1)\alpha(2)\alpha(3), \\ \chi''_{\frac{1}{2}, -\frac{1}{2}}(1, 23) &= \frac{\beta(1)}{\sqrt{6}}[\alpha(2)\beta(3) + \beta(2)\alpha(3)] \\ &\quad + \sqrt{\frac{2}{3}}\alpha(1)\beta(2)\beta(3). \end{aligned} \tag{2.129}$$

As it must be, the functions  $\chi'$  are antisymmetric with respect to permutations of the second and third nucleons, since these two nucleons are in the singlet spin state with the total spin  $S_{23} = 0$ , and the functions  $\chi''$  are symmetric with respect to permutation of the second and third nucleons in the triplet spin state ( $S_{23} = 1$ ).

If the total *spin of the system of three nucleons* is  $S = \frac{3}{2}$ , then the corresponding spin wave function is denoted as follows

$$\chi^s \equiv \chi_{\frac{3}{2}M_S}^s(123) = \chi_{\frac{3}{2}M_S}^{(S_{23}=1)}(123). \quad (2.130)$$

Apparently, in this case the intermediate total spin of two nucleons  $S_{23}$  can be equal only to unity, and the spin wave function  $\chi^s$  is completely symmetric with respect to permutations of three nucleons (a sign  $s$  stands for this purpose in its upper part). The symmetry properties of function (2.130) can be seen also from the explicit expressions  $\chi_{\frac{3}{2}M_S}^s(123)$  in terms  $\alpha(j)$  and  $\beta(j)$ :

$$\begin{aligned}\chi_{\frac{3}{2},+\frac{3}{2}}^s(123) &= \alpha(1)\alpha(2)\alpha(3), \\ \chi_{\frac{3}{2},+\frac{1}{2}}^s(123) &= \frac{1}{\sqrt{3}}[\alpha(1)\alpha(2)\beta(3) + \alpha(1)\beta(2)\alpha(3) + \beta(1)\alpha(2)\alpha(3)], \\ \chi_{\frac{3}{2},-\frac{1}{2}}^s(123) &= \frac{1}{\sqrt{3}}[\alpha(1)\beta(2)\beta(3) + \beta(1)\alpha(2)\beta(3) + \beta(1)\beta(2)\alpha(3)], \\ \chi_{\frac{3}{2},-\frac{3}{2}}^s(123) &= \beta(1)\beta(2)\beta(3).\end{aligned} \quad (2.131)$$

All eight functions (2.129) and (2.131) are ortho-normalized.

The spin functions of different symmetries (2.129) and (2.131) may be constructed by using relevant permutation operators. Let the permutation of two nucleons  $i$  and  $j$  be denoted by  $(ij)$ . We introduce the following *three-nucleon permutation operators*:

$$\begin{aligned}T^s &= (12) + (23) + (31), \\ T' &= \frac{\sqrt{3}}{2} [(12) - (23)], \\ T'' &= -(31) + \frac{1}{2} [(12) + (23)].\end{aligned} \quad (2.132)$$

Applying operators (2.132) to a function symmetric with respect to the coordinates of particles 1 and 2, one obtains three functions. The first one is completely symmetric with respect to the coordinates 1, 2, and 3, the other two are transformed under the coordinate permutations as a two-dimensional irreducible representation of the permutation group  $S_3$ . The *spin wave functions* of the states  $S = \frac{1}{2}$  and  $S = \frac{3}{2}$  (the total spin projection is taken to be equal to  $\frac{1}{2}$ ) are given by

$$S = \frac{1}{2}, \left\{ \begin{array}{l} \chi' = \sqrt{\frac{2}{3}} T' \alpha(1)\alpha(2)\beta(3), \\ \chi'' = \sqrt{\frac{2}{3}} T'' \alpha(1)\alpha(2)\beta(3), \end{array} \right. \quad (2.129')$$

$$S = \frac{3}{2}, \chi^s = \frac{1}{\sqrt{3}} T^s \alpha(1) \alpha(2) \beta(3). \quad (2.131')$$

The factors in (2.129') and (2.131') are chosen to provide the functions to be normalized to unity.

The functions  $\chi'$  and  $\chi''$  are the basis functions of the *two-dimensional irreducible representation of the permutation group* of three particles, i.e., they have the mixed permutation symmetry. The functions  $\chi^s$  are the basis functions of the one-dimensional (symmetric) representation of the same permutation group. Simultaneously, the functions  $\chi'$  and  $\chi''$  are the basis functions of the two-dimensional irreducible representation  $D^{1/2}$ , and the functions  $\chi^s$  are the basis functions of the four-dimensional representation  $D^{3/2}$  of the three-dimensional rotations.

The isospin wave functions of three nucleons  $\zeta'$ ,  $\zeta''$  and  $\zeta^s$  for the total isotopic spins  $T = \frac{1}{2}$  and  $T = \frac{3}{2}$  can be constructed in perfect analogy to the above reported. Introducing denotations of the proton and neutron isospin functions  $x(j) = \zeta_{\frac{1}{2},+\frac{1}{2}}(j)$ ,  $y(j) = x_{\frac{1}{2},-\frac{1}{2}}(j)$ , we can write, for example, the *isospin wave functions of the three-nucleon system*:

$$\begin{aligned} \zeta'_{\frac{1}{2},+\frac{1}{2}}(1, 23) &= \frac{x(1)}{\sqrt{2}} [x(2)y(3) - y(2)x(3)], \\ \zeta''_{\frac{1}{2},+\frac{1}{2}}(1, 23) &= \frac{x(1)}{\sqrt{6}} [x(2)y(3) + y(2)x(3)] - \sqrt{\frac{2}{3}} y(1)x(2)x(3), \\ \zeta^s_{\frac{3}{2},+\frac{3}{2}}(123) &= x(1)x(2)x(3). \end{aligned} \quad (2.133)$$

These isospin functions have the analogous form to the corresponding spin wave functions  $\chi'_{\frac{1}{2},+\frac{1}{2}}(1, 23)$ ,  $\chi''_{\frac{1}{2},+\frac{1}{2}}(1, 23)$ ,  $\chi^s_{\frac{3}{2},+\frac{3}{2}}(123)$ , given in (2.129) and (2.131).

We call attention to the fact that it is impossible to construct the antisymmetric (with respect to permutations of any nucleon pair) spin wave function of the system of three nucleons  $\chi^a$  from the product of one-particle spin wave functions. In fact, the function  $\chi^a$  must be proportional to a determinant of the third order, which is built from the functions  $\chi_{\frac{1}{2}m_j}(j)$ . However, the spin projection  $m_j$  takes only two values  $\pm\frac{1}{2}$  (there are only two different spin functions of a nucleon  $\alpha$  and  $\beta$ ). Because of this the above-mentioned determinant and the function  $\chi^a$  are equal to zero. Due to the same reason, the isospin function of the three nucleon system antisymmetric with respect to permutations of any nucleon pair isospin does not exist  $\zeta^a = 0$ . In this connection it should be noted that in the case of the spatial symmetric wave function of three-nucleon nuclei  ${}^3H$  and  ${}^3He$

with the total spin  $S = \frac{1}{2}$  the isospin of these nuclei cannot be equal to  $\frac{3}{2}$ . At  $T = \frac{3}{2}$  the isospin state is completely symmetric, i.e., the spin state must be completely antisymmetric, but, as we have seen above, this state is impossible since  $\chi^a = 0$ .

We construct now the spin-isospin wave functions of the three-nucleon system with definite values of the total spin  $S$  and isospin  $T$  and different symmetry with respect to the nucleon permutations. First of all, we consider the case  $S = T = \frac{1}{2}$ , which is possible both for bound and unbound states of three nucleons. Omitting the indexes of spin, isospin, and their projections, we can write down the *spin-isospin wave functions of the three-nucleon system* with different permutation symmetry at  $S = \frac{1}{2}$  and  $T = \frac{1}{2}$  in the following form

$$\xi^s = \frac{1}{2}(\chi' \zeta' + \chi'' \zeta''), \quad \xi^a = \frac{1}{\sqrt{2}}(\chi' \zeta'' - \chi'' \zeta'), \quad (2.134)$$

$$\xi' = \frac{1}{\sqrt{2}}(\chi' \zeta'' + \chi'' \zeta'), \quad \xi'' = \frac{1}{\sqrt{2}}(\chi' \xi' - \chi'' \xi''). \quad (2.135)$$

The functions  $\xi^s$  and  $\xi^a$ , respectively, are completely symmetric and anti-symmetric functions with respect to permutations of any pair of nucleons (it can be checked directly). The functions  $\xi^s$  and  $\xi^a$  are the basis functions of two one-dimensional (symmetric and antisymmetric, respectively) representations of the permutation group of three nucleons. And functions  $\xi'$  and  $\xi''$  are the basis functions (with different *Yamanouchi symbols*) of the two-dimensional irreducible representation of the same group and are transformed in terms of each other at permutations of nucleons. They are the functions with the mixed (intermediate) symmetry. All functions (2.134) and (2.135) are normalized by unity and mutually orthogonal.

For a system of three unbound nucleons, we use also the spin-isospin wave functions in the product form  $\chi^s \zeta^s$  at  $S = T = \frac{3}{2}$ , which is a completely symmetric function, and in the forms  $\chi^s \zeta''$  and  $\chi'' \zeta'$  at  $S = \frac{3}{2}, T = \frac{1}{2}$  or  $\chi'' \zeta^s$  and  $\chi' \zeta^a$  at  $S = \frac{1}{2}, T = \frac{3}{2}$ , which have the mixed permutation symmetry. For the nucleon-deuteron system the total spin  $S$  can be equal  $\frac{1}{2}$  or  $\frac{3}{2}$ , and the total isospin  $T$  equals to  $\frac{1}{2}$  (neglecting the small contribution of the term with  $T = \frac{3}{2}$  in the system wave function).

**Total wave functions of three-nucleon systems in the case of the central forces.** We construct now the total wave function of a three-nucleon system, which supposed to be antisymmetric with respect to simultaneous permutations of the spatial, spin and isospin coordinates of any nucleon pair. The antisymmetry of the total wave functions of nucleon systems follows from the Pauli principle according to which a neutron and proton become identical fermions after introduction the isotopic spin

formalism. We take now into account the Coulomb interaction between protons in system  $npp$  and  $ppp$ . This results in an uncertainty of the system total isotopic spin, but the corresponding distorting additions to the wave functions are very small. The central forces give the main contribution in the interactions in three-nucleon systems determining their basic properties, as well as in a deuteron.

Assuming that the interaction between nucleons has a two-particle type, we can present the interaction potential between  $k$ -th and  $j$ -th nucleons as follows

$$\begin{aligned}\hat{V}_{kj} = & V^{ts}(kj)\pi_t^\sigma(kj)\pi_s^\tau(kj) + V^{tt}(kj)\pi_t^\sigma(kj)\pi_t^\tau(kj) \\ & + V^{st}(kj)\pi_s^\tau(kj)\pi_t^\tau(kj) + V^{ss}(kj)\pi_s^\sigma(kj)\pi_s^\tau(kj) \\ & + V_c(kj)\Lambda_p(k)\Lambda_p(j),\end{aligned}\quad (2.136)$$

where the upper signs  $t$  and  $s$  are related to the triplet ( $\sigma$ ) and singlet spin and isospin ( $\tau$ ) states of two nucleons. Here  $V^{ts}(kj) \equiv V^{ts}(|\mathbf{r}_k - \mathbf{r}_j|)$  and so on,  $V_c(kj) = \frac{e^2}{|\mathbf{r}_k - \mathbf{r}_j|}$  is the Coulomb potential,  $\pi_t^\sigma(kj) \equiv \frac{1}{4}(3 + \boldsymbol{\sigma}_k \boldsymbol{\sigma}_j)$ ,  $\pi_s^\sigma = \frac{1}{4}(1 - \boldsymbol{\sigma}_k \boldsymbol{\sigma}_j)$ ,  $\pi_t^\tau(kj) = \frac{1}{4}(3 + \boldsymbol{\tau}_k \boldsymbol{\tau}_j)$ ,  $\pi_s^\tau(kj) = \frac{1}{4}(1 - \boldsymbol{\tau}_k \boldsymbol{\tau}_j)$  are the projection operators into the triplet spin and singlet isospin states,  $\Lambda_p(k) = \frac{1+\tau_{kz}}{2}$  is the project operator into a proton state. Magnitudes of the potentials  $V^{tt}$  and  $V^{ss}$  at small relative energies of two nucleons are much less than  $V^{ts}$  and  $V^{st}$ , which enables the terms with  $V^{tt}$  and  $V^{ss}$  in (2.136) to be neglected.

Since the total Hamiltonian of the three-nucleon system  $\hat{H} = \hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31}$  with the interactions from (2.136) commutes with operators  $\mathbf{S}^2$ ,  $\hat{S}_z$  and  $\hat{T}_z$ , then the total spin  $S$ , spin projection  $S_z$  and isospin projection  $T_z$  have definite values (conserved). The operator  $\mathbf{T}^2$  does not commute with the Hamiltonian  $\hat{H}$  and, that is why, the total isotopic spin  $T$  has a definite value only if there is no the Coulomb interaction. Thus, the total isotopic spin  $T$  is an exact integral of motion only for the systems  $nnn$  and  $nnp$ .

We neglect for the moment a small influence of the electromagnetic interaction on the three-nucleon system structure (or we consider the systems  $nnn$  and  $nnp$ ). In this case the total isospin  $T$  has an exact value. The total antisymmetric wave function of the relative motion is denoted as  $\Psi^{ST}$ . (For brevity sake, we omit arguments of the wave functions and its quantum numbers.) This function, in a general case, can be presented in the form of a finite sum of products of the spatial and spin-isospin wave functions of different symmetry with respect to the nucleon permutation.

We begin with the construction of the total wave function  $\Psi^{\frac{1}{2}\frac{1}{2}}$  with  $S = T = \frac{1}{2}$ . As the possible candidates, that appear in the function  $\Psi^{\frac{1}{2}\frac{1}{2}}$ ,

we can immediately chose two already antisymmetric products  $\psi^a \xi^s$  and  $\psi^s \xi^a$ , where  $\psi^a$  and  $\psi^s$  are the antisymmetric and symmetric with respect to permutations of any nucleon pair spatial wave functions and  $\xi^s$  and  $\xi^a$  are the symmetric and antisymmetric spin-isospin wave functions. Moreover, there are two more spin-isospin wave functions  $\xi'$  and  $\xi''$  of the intermediate symmetry for  $S = T = \frac{1}{2}$ . We can construct two spatial functions  $\psi'$  and  $\psi''$ , which are transformed along with the functions  $\xi'$  and  $\xi''$  according to the two-dimensional irreducible representation of the three-particle permutation group. From the functions of the intermediate symmetry we can construct for  $S = T = \frac{1}{2}$  one more antisymmetric combination, namely,  $\psi' \xi'' - \psi'' \xi'$ , in the same manner as we constructed the antisymmetric spin-isospin wave function. Since we exhausted all spin-isospin wave functions of the three-nucleon system with  $S = T = \frac{1}{2}$ , the total antisymmetric wave function of the system in this case is a superposition of four terms:

$$\Psi^{\frac{1}{2} \frac{1}{2}} = \psi^a \xi^s + \psi^s \xi^a + \psi' \xi'' - \psi'' \xi'. \quad (2.137)$$

This wave function can describe the nucleus  ${}^3H$  state, as well as unbound systems  $nnp$ , in particular, the neutron-deuteron system. Wave function (2.137) can be used for the description of  ${}^3He$  nucleus and the systems  $npp$  and  $ppp$ , if the Coulomb interaction between protons is neglected.

We consider now the state of the *three-nucleon system* with  $S = \frac{3}{2}$  and  $T = \frac{1}{2}$ . A total antisymmetric wave function  $\Psi^{\frac{3}{2} \frac{1}{2}}$ , in this case, can be presented as the product of the completely symmetric spin wave function  $\chi^s$  (as we have seen it exists for  $S = \frac{3}{2}$ ) and the antisymmetric spatial isospin wave function. The latter cannot be presented in the form of the product of the symmetric spatial function  $\psi^s$  and the antisymmetric isospin function  $\xi^a$ , since the function  $\xi^a$  does not exist in systems of three nucleons ( $\xi^a = 0$ ). It is impossible to construct the spatial isospin wave function as the product of the antisymmetric spatial function  $\psi^a$  and the symmetric isospin function  $\xi^s$  at  $T = \frac{1}{2}$ . The point is that the function  $\xi^s$  corresponds to  $T = \frac{3}{2}$ , but not to  $T = \frac{1}{2}$ . Because of this at  $S = \frac{3}{2}$  and  $T = \frac{1}{2}$  there is only one way to construct the antisymmetric spatial isospin wave function from the functions of the intermediate symmetry, namely,  $\psi' \zeta'' - \psi'' \zeta'$ . Thus, in the case  $S = \frac{3}{2}$  and  $T = \frac{1}{2}$ , the *total antisymmetric wave function of the three-nucleon system* takes the following form

$$\Psi^{\frac{3}{2} \frac{1}{2}} = \psi' \chi^s \zeta'' - \psi'' \chi^s \zeta'. \quad (2.138)$$

The analogous reasoning gives the following general form of the total antisymmetric wave function  $\Psi^{\frac{1}{2} \frac{3}{2}}$  of the three-nucleon system for the total spin  $S = \frac{1}{2}$  and isospin  $T = \frac{3}{2}$ :

$$\Psi^{\frac{1}{2}\frac{3}{2}} = \psi' \chi'' \zeta^s - \psi'' \chi' \zeta^s. \quad (2.139)$$

The structure of the total antisymmetric wave function at  $S = T = \frac{3}{2}$  is the simplest:

$$\Psi^{\frac{3}{2}\frac{3}{2}} = \psi^a \chi^s \zeta^s. \quad (2.140)$$

Wave functions (2.138) – (2.140) describe only the continuous spectrum of the three-nucleon systems, in addition, functions (2.139) and (2.140) are related to the systems of identical nucleons  $nnn$  and  $ppp$ .

If in the three-nucleon systems  $npp$  and  $ppp$  we take into account the Coulomb interaction between protons, then the total isotopic spin does not have an exact value any more. Because of this, for example, the nucleus  ${}^3He$  is described by the total wave function  $\Psi({}^3He)$ , which is a superposition of the wave functions with  $T = \frac{1}{2}$  (as for a nucleus  ${}^3H$ ) and  $T = \frac{3}{2}$ :

$$\Psi({}^3He) = \psi^a \xi^s - \psi^s \xi^a + \psi' \xi'' - \psi'' \xi' + (\varphi' \chi'' - \varphi'' \chi') \zeta^s. \quad (2.141)$$

A similar wave function describes the system proton-deuteron, as well as the system  $npp$  when three nucleons are unbound. The spatial wave functions  $\varphi'$  and  $\varphi''$ , on taking account the Coulomb interaction, are transformed at permutations as the functions  $\psi'$  and  $\psi''$ . Because of this the spatial spin wave function  $\varphi' \chi'' - \varphi'' \chi'$  is antisymmetric, and the correction to wave function (2.141), caused by the proton Coulomb interaction  $(\varphi' \chi'' - \varphi'' \chi') \zeta^s$  related to  $S = \frac{1}{2}$  and  $T = \frac{3}{2}$ , has the same structure as function (2.139). Note that the space functions  $\psi^a$ ,  $\psi^s$ ,  $\psi'$ , and  $\psi''$ , appearing in (2.141), are deformed by the Coulomb interaction and differ from the corresponding spatial functions in (2.137). It is clear the wave function of the system  $ppp$  preserves the form (2.140) when switching on the Coulomb interaction but its spatial part is deformed by the Coulomb interaction.

The spatial functions  $\psi^a$ ,  $\psi^s$ ,  $\psi'$ ,  $\psi''$ ,  $\varphi'$ , and  $\varphi''$  every so often are considered in the momentum space with the help of the Fourier transform with respect to all relative space coordinates. At the same time, the symmetry properties of the functions with respect to the nucleon permutations do not change. We do not write arguments of the spatial wave functions, which, in fact, depend on  $\mathbf{r}_{ij}$ ,  $\mathbf{x}_k$  or the corresponding relative momenta  $\mathbf{k}_{ij}$ ,  $\mathbf{p}_k$  ( $ijk=123, 231, 312$ ).

Since the explicit form of the spin-isospin wave functions, appearing in the total wave function of the three-nucleon systems, is known, the further task is to obtain the spatial parts of the total wave functions or their equations of motion. The latter can be obtained by substitution of the total

wave functions in the Schrödinger equation and eliminating the known spin and isospin wave functions.

The wave functions of bound states (2.137) and (2.141) are normalized by unity. Taking into account the normalization of the spin-isospin wave functions and by using (2.137), we obtain the following normalization condition for the spatial wave functions of  $^3H$  nucleus :

$$\int d\mathbf{r}_{23} \int d\mathbf{x}_1 (|\psi^a|^2 + |\psi^s|^2 + |\psi'|^2 + |\psi''|^2) = 1. \quad (2.142)$$

In the case of  $^3He$  nucleus, we have

$$\int d\mathbf{r}_{23} \int d\mathbf{x}_1 (|\psi^a|^2 + |\psi^s|^2 + |\psi'|^2 + |\psi''|^2 + |\varphi'|^2 + |\varphi''|^2) = 1. \quad (2.143)$$

The main contribution (more than 90%) in these six-dimensional normalization integrals gives the symmetric spatial wave function  $\psi^s$ . A few per cents contribution is given by the functions of the mixed symmetry  $\psi'$  and  $\psi''$ , the contribution of the functions  $\psi^a$ ,  $\varphi'$  and  $\varphi''$  is very small and every so often can be neglected. Thus, in a good approximation, the total wave function of three-nucleon nuclei in many cases can be written in the form

$$\Psi = \psi^s \xi^a + \psi' \xi'' - \psi'' \xi' \quad (2.144)$$

with the corresponding normalization

$$\langle \Psi | \Psi \rangle = \int d\mathbf{r}_{23} \int d\mathbf{x}_1 (|\psi^s|^2 + |\psi'|^2 + |\psi''|^2) = 1. \quad (2.145)$$

The main problem when obtaining the wave functions of three-nucleon systems is associated with the coordinate dependence (in the coordinate representation) or with the momentum dependence (in the momentum representation). There are different approximate, in particular, variational methods of solving the three-body problem when one begins with the given nucleon-nucleon potentials in the nonrelativistic Schrödinger equation and constructs more simple approximate systems of coupled differential (or integral, or integro-differential) equations. These equations can be solved numerically with the help of modern computers.

The *Faddeev equations* can be used to find the wave functions of the three-particle bound and unbound systems. They can be generalized on systems of four and more particles. Note that the Faddeev method was fruitfully used in the problems concerning to the three-particle effects in the nuclear matter.

**Integral equations for a three-nucleon system with spin-dependent nucleon-nucleon interaction.** The Faddeev equations may be extended to the case of a *three-nucleon system with spin-dependent interaction* (Sitenko and Kharchenko, 1963). One has to employ the equations (2.117) describing an arbitrary three-particle system with regard for the operator nature of the two-particle  $t$ -matrix in the nucleon spin-isospin space and spin- and isospin-dependences of the wave function. Since the nucleons are identical, the set of equations (2.117) reduces to a single equation

$$\begin{aligned} \Psi(\mathbf{p}, \mathbf{k}) = & \Phi(\mathbf{p}, \mathbf{k}) + \left( z_p - \frac{\hbar^2 k^2}{M} \right)^{-1} \int \frac{d\mathbf{p}'}{(2\pi)^3} \\ & \times \left\{ \left\langle \mathbf{k} | t_{23}(z_p) | -\frac{\mathbf{p}}{2} + \mathbf{p}' \right\rangle \Psi^{(2)} \left( \mathbf{p}', -\mathbf{p} - \frac{\mathbf{p}'}{2} \right) \right. \\ & \left. + \left\langle \mathbf{k} | t_{23}(z_p) | -\frac{\mathbf{p}}{2} - \mathbf{p}' \right\rangle \Psi^{(3)} \left( \mathbf{p}', \mathbf{p} + \frac{\mathbf{p}'}{2} \right) \right\}, \end{aligned} \quad (2.146)$$

where the functions  $\Psi^{(2)}$  and  $\Psi^{(3)}$  differ from  $\Psi \equiv \Psi^{(1)}$  by a cyclic permutation of nucleon spatial, spin, and isospin coordinates ( $z_p = Z - 3\hbar^2 p^2/4M$ ,  $M$  is the nucleon mass). The central nuclear interaction between two nucleons ( $i$  and  $j$ ) is described by the potential

$$V_{ij} = \sum_{\nu=1}^4 V^{(\nu)}(\mathbf{r}_{ij}) P_{ij}^{(\nu)}(\sigma, \tau), \quad (2.147)$$

where  $P_{ij}^{(\nu)}(\sigma, \tau)$  is the projection operator into a  $\nu$ -spin-isospin state. The values  $\nu=1, 2, 3$ , and 4 label respectively triplet-singlet  $ts$ , triplet-triplet  $tt$ , singlet-triplet  $st$ , and singlet-singlet  $ss$  spin-isospin two-nucleon states. For the central interaction (2.147), the two-nucleon  $t$ -matrix may be presented in the form

$$\langle \mathbf{k} | t_{ij}(z) | \mathbf{k}' \rangle = \sum_{\nu=1}^4 \langle \mathbf{k} | t^{(\nu)}(z) | \mathbf{k}' \rangle P_{ij}^{(\nu)}(\sigma, \tau), \quad (2.148)$$

where  $t^{(\nu)}(z)$  is the  $t$ -matrix eigenvalue in the spin-isospin state  $\nu$ . Projections of the equation (2.146) at probable three-nucleon spin-isospin states yield a set of integral equations for the spatial functions  $\Psi^s$ ,  $\Psi^a$ ,  $\Psi'$ , and  $\Psi''$ .

It is convenient to pass from the functions  $\Psi^s$ ,  $\Psi^a$ ,  $\Psi'$ , and  $\Psi''$ , which are transformed according to irreducible representations of the permutation group  $S_3$ , to the functions  $\psi_\nu(\mathbf{p}_k, \mathbf{k}_{ij})$  associated with definite spin-isospin

states  $\nu=1,2,3$ , and 4 of the nucleon pair  $ij$ . The functions  $\psi_\nu(\mathbf{p}_k, \mathbf{k}_{ij})$  are even, for  $\nu=1$  and 3, and odd, for  $\nu=2$  and 4, with respect to the permutation of particles  $i$  and  $j$  ( $\mathbf{k}_{ij} \rightarrow \mathbf{k}_{ji} = -\mathbf{k}_{ij}$ ):  $\psi_\nu(\mathbf{p}, -\mathbf{k}) = (-1)^{\nu+1} \psi_\nu(\mathbf{p}, \mathbf{k})$ . In the case  $S = 1/2$ ,  $T = 1/2$ , the spatial functions  $\Psi^s$ ,  $\Psi^a$ ,  $\Psi'$ , and  $\Psi''$  may be expressed in terms of four functions  $\psi_\nu$  with  $\nu=1,2,3$ , and 4; in the case  $S = 1/2$ ,  $T = 3/2$ , the spatial functions  $\Psi'$  and  $\Psi''$  are expressed in terms of the two functions  $\psi_2$  and  $\psi_3$ ; for  $S = 3/2$ ,  $T = 1/2$ , the spatial functions are expressed in terms of two functions  $\psi_1$  and  $\psi_2$ ; at last, for  $S = 3/2$ ,  $T = 3/2$ , the spatial function  $\Psi^a$  is expressed in terms of the single function  $\psi_2$ :

$$S = \frac{1}{2}, T = \frac{1}{2}, \left\{ \begin{array}{l} \sqrt{2}\Psi^s = T^s(\psi_1 + \psi_3), \\ \sqrt{2}\Psi^a = T^s(\psi_2 + \psi_4), \\ \sqrt{2}\Psi' = T'(\psi_1 - \psi_3) + T''(\psi_2 - \psi_4), \\ \sqrt{2}\Psi'' = T''(\psi_1 - \psi_3) - T'(\psi_2 - \psi_4); \end{array} \right. \quad (2.149)$$

$$S = \frac{1}{2}, T = \frac{3}{2}, \left\{ \begin{array}{l} \Psi' = -T'\psi_3 + T''\psi'_2 \\ \Psi'' = -T''\psi_3 - T'\psi'_2; \end{array} \right. \quad (2.150)$$

$$S = \frac{3}{2}, T = \frac{1}{2}, \left\{ \begin{array}{l} \Psi' = T'\psi_1 + T''\psi_2, \\ \Psi'' = T''\psi_1 - T'\psi_2; \end{array} \right. \quad (2.151)$$

$$S = \frac{3}{2}, T = \frac{3}{2}, \Psi^a = T^s\psi_2, \quad (2.152)$$

where the denotation

$$\psi_\nu \equiv \psi_\nu(\mathbf{p}_3, \mathbf{k}_{12}), \nu = 1, 2, 3, \text{ and } 4$$

is introduced in order to shorten the formulas. In the general case, different three-nucleon spin-isospin states associated with different  $S$  and  $T$  are described by different systems of functions  $\psi_\nu$ .

With the three-nucleon spin and isospin being arbitrary the set of integral equations for the functions  $\psi_\nu(\mathbf{p}, \mathbf{k})$  may be presented in the form

$$\begin{aligned} \psi_\nu(\mathbf{p}, \mathbf{k}) &= \varphi_\nu(\mathbf{p}, \mathbf{k}) + \left( z_p - \frac{\hbar^2 k^2}{M} \right)^{-1} \sum_{\nu'=1}^4 \int \frac{d\mathbf{p}'}{(2\pi)^3} \\ &\times \left\{ \langle \mathbf{k} | t^{(\nu)}(z_p) | -\frac{\mathbf{p}}{2} + \mathbf{p}' \rangle + (-1)^{\nu+1} \langle \mathbf{k} | t^{(\nu)}(z_p) | -\frac{\mathbf{p}}{2} - \mathbf{p}' \rangle \right\} \\ &\times C_{\nu\nu'} \psi_{\nu'} \left( \mathbf{p}', \mathbf{p} + \frac{\mathbf{p}'}{2} \right), \end{aligned} \quad (2.153)$$

The matrices  $C_{\nu\nu'}^{(ST)}$  associated with different spin-isospin states of the three-nucleon system are the following:

$$C_{\nu\nu'}^{(\frac{1}{2}\frac{1}{2})} = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} & 3 & -\sqrt{3} \\ \sqrt{3} & -1 & -\sqrt{3} & -3 \\ 3 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & -1 \end{pmatrix},$$

$$C_{\nu\nu'}^{(\frac{1}{2}\frac{3}{2})} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\sqrt{3} & 0 \\ 0 & -\sqrt{3} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.154)$$

$$C_{\nu\nu'}^{(\frac{3}{2}\frac{1}{2})} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C_{\nu\nu'}^{(\frac{3}{2}\frac{3}{2})} = -\delta_{\nu 2}\delta_{\nu' 2}. \quad (2.155)$$

In order to consider three-nucleon bound states, one has to put the free term in (2.153) equal to zero. It is to be noted that integral equation (2.120) for a system of three identic spinless particles is reproduced with (2.153) if one puts in the latter

$$C_{\nu\nu'} = \delta_{\nu 1}\delta_{\nu' 1}.$$

The equation (2.153) was derived regardless of the Coulomb interaction. Therefore, it describes only the systems consisting of either three neutrons or two neutrons and one proton. The latter system allows for the formation of a bound state (triton) as well as neutron scattering by a bound state of two other particles. In the latter case the free term in (2.153) is to be written in the form

$$\varphi_\nu(\mathbf{p}, \mathbf{k}) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}_0) \varphi(\mathbf{k}) \delta_{\nu 1}, \quad (2.156)$$

where  $\varphi(\mathbf{k})$  is the deuteron ground state wave function.

**Calculation of the three-nucleon system characteristics on the Faddeev equation basis.** The assumed two-particle character of the nuclear forces between nucleons in the above-reported method of the integral equations allows one to calculate the parameters of three-nucleon systems and describe the processes in these systems. There are many calculations of the binding energy and other characteristics of  ${}^3H$  and  ${}^3He$  nuclei within the framework of different models of the nucleon-nucleon interaction.

It is clear that the computation process complexity considerably depends on the relevant model potential. If we take into account the two-particle interaction in the  $S$ -states only and describe it by the separable potential, the problem of the binding energy determination reduces to solving of the system of one-dimensional equations. In the more general case when considering the interaction in the states with larger angular momentum, system (2.153) transforms into the system of coupled two-dimensional integral equations.

The qualitatively correct values of the binding energy of three-nucleon systems can be obtained even for the simplest potentials of two-nucleon interactions. However, the central attractive potentials reconciled with the data on the interaction of two nucleons at low energies lead to overestimates of the binding energy of triton compared with the experimental value. The correct behavior of the scattering amplitude of two nucleons on the energy shell at low energies can be ensured by describing the interaction between the nucleons by means of potentials containing two parameters, which are determined by specifying two experimental quantities – the scattering length and effective radius. To find the two-particle scattering amplitude on the energy shell in a large interval of energies, it is necessary to use more complicated potentials, characterized by a larger number of parameters. As is known, data on nucleon-nucleon scattering in the high-energy region point to the existence of a strong repulsion between the nucleons at small distances. A potential with repulsion at small distances and with subsequent short-range attraction should contain at least three parameters. Introduction of the repulsive core radius – an additional parameter characterizing the repulsion – makes it possible to describe correctly the scattering amplitude both at low and high energies. The values of the binding energy of the triton for different potentials correctly describing the phases of the two-nucleon scattering in a large energy interval are quite close to one another.

As an example we cite the results of calculations of some characteristics of  $^3H$  nuclei obtained by using one of the modifications of the *Reyd potential* with a soft repulsive core (Akaishi, 1986). The calculated binding energy of a triton equals to  $E_T=7.023$  MeV, the mean squared radius of the charge distribution in the nucleus  $\langle r^2 \rangle_c^{1/2}=1.70$  fm, the *probability of the D-state* happened to be relatively high  $P_D=0.093$ . We also obtained the probabilities of the symmetric  $S$ -state

$$P_S \equiv \int d\mathbf{r}_{23} \int d\mathbf{x}_1 |\Psi^s|^2 = 0.889$$

and state with the intermediate symmetry

$$P_{\tilde{S}} \equiv \int d\mathbf{r}_{23} \int d\mathbf{x}_1 \left( |\Psi'|^2 + |\Psi''|^2 \right) = 0.017.$$

The use of the Paris nucleon-nucleon potential gives the following numerical parameters of  ${}^3H$  nucleus (Sasakawa, 1986): the binding energy  $E_T = 7.64$  MeV, the charge distribution radius  $\langle r^2 \rangle_c^{1/2} = 1.79$  fm,

$$P_S = 0.901, P_{\tilde{S}} = 0.013, P_D = 0.085.$$

For  ${}^3He$  nucleus: the Coulomb energy  $E_c = 0.66$  MeV, the charge distribution radius  $\langle r^2 \rangle_c^{1/2} = 2.00$  fm. It is shown that the addition of the three-nucleon nuclear forces can change the calculated parameters by a few per cents.

As an illustration we give the results of the differential cross-section of the elastic neutron-deuteron scattering at relatively low energies by using the phenomenological separable potentials. The dependencies of the calculated cross-section  $\sigma(\vartheta) = \frac{\partial \sigma}{\partial \Omega}$  of the elastic neutron-deuteron scattering on the scattering angle  $\vartheta$  (in the centre-of-mass coordinate system) at two different energies of incident neutrons  $E_n = 10$  MeV and  $E_n = 22.7$  MeV and the corresponding experimental data (Adhikari and Tomio, 1995) are given in Fig. 2.4. The upper continuous curves correspond to the *Yamaguchi potential* in both the triplet and singlet states of two nucleon (the calculated binding energy of a triton in this model is  $E_T = 10.65$  MeV). The lower continuous curves correspond the Yamaguchi potential in the triplet state and the *Tabakin potential* in the singlet state (the triton binding energy is  $E_T = 7.69$  MeV).

The curves corresponding to the different versions are pretty close. The considerable departures appear in the vicinity of a minimum energy  $E_n = 22.7$  MeV. The calculated distributions are satisfactorily correspond to the experimental angular distributions. Note that the polarization phenomena are more sensitive to the interaction detail.

Basically, we can construct a system of integral equations analogous to (2.146) even for a larger number of nucleons with the pair interaction. However, if calculations of the scattering amplitude in four-nucleon systems (amplitudes of nucleon-three-nucleon system in a bound state or deuteron-deuteron scattering) can be justified, the use of these equations to calculate the four-nucleon system binding energy is not to the purpose. In fact, for nuclei with mass-numbers  $A > 4$  and even for the  ${}^4He$  nucleus the effect of saturation of the nuclear interaction becomes essential. Due to this effect the total binding energy of a nuclear system is proportional to a number of nucleons (i.e., to the mass-number  $A$ ) but not to a number of the interacting pairs.

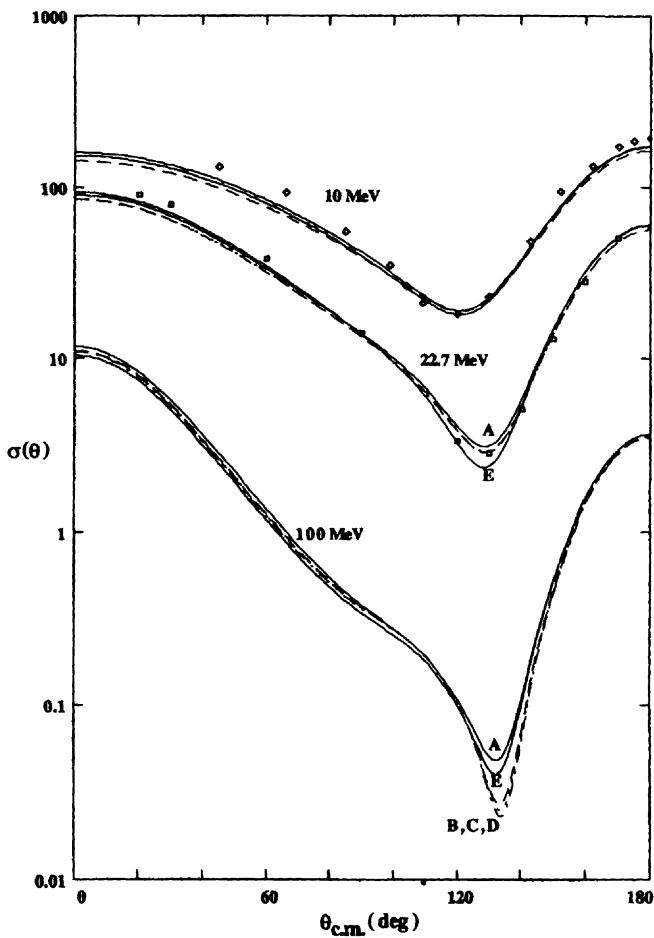


Figure 2.4. The  $n - d$  elastic differential scattering cross-section in mb/sr at different incident nucleon energies

#### 2.4. K-Harmonic Method

**Hyperspherical functions.** To find the spatial wave functions of few-particle systems from the Schrödinger equation, the method of *K*-harmonic or harmonic polynomials is frequently used. The idea of this method lies in the special representation of the spatial wave functions of few-particle systems in the form of infinite series with respect to some complete sets of the known functions of angle variables which are called *K-harmonics* or *hyperspherical functions*. The method of harmonic polynomials is especially efficient when solving the problems concerning bound states of few-particle

systems. We consider a system of  $A$  particles (nucleons) with equal masses  $M$ , with radius-vectors denoted by  $\mathbf{r}_j$  ( $j = 1, 2, \dots, A$ ). Let us denote the center-mass radius-vector as  $\mathbf{R}$ ,

$$\mathbf{R} = \frac{1}{A} \sum_j \mathbf{r}_j, \quad (2.157)$$

and introduce  $A - 1$  relative radius-vectors  $\boldsymbol{\xi}_k$ , which are conventionally called *Jacoby's vectors*:

$$\boldsymbol{\xi}_k = \sqrt{\frac{k}{k+1}} \left( \mathbf{r}_{k+1} - \frac{1}{k} \sum_{j=1}^k \mathbf{r}_j \right), \quad k = 1, 2, \dots, A-1. \quad (2.158)$$

The complete  $3A$ -dimensional Laplace operator can be written in the form

$$\sum_{j=1}^A \frac{\partial^2}{\partial \mathbf{r}_j^2} = \frac{1}{A} \frac{\partial^2}{\partial \mathbf{R}^2} + \sum_{j=1}^{A-1} \frac{\partial^2}{\partial \boldsymbol{\xi}_j^2} \equiv \frac{1}{A} \frac{\partial^2}{\partial \mathbf{R}^2} + \Delta_{3(A-1)}, \quad (2.159)$$

where  $\Delta_{3(A-1)}$  - is the Laplace operator in a space  $3(A-1)$  relative coordinates. The elementary volume in  $3A$ -dimensional space is given by

$$\Pi_{j=1}^A d\mathbf{r}_j = A^{3/2} d\mathbf{R} \Pi_{j=1}^{A-1} d\boldsymbol{\xi}_j. \quad (2.160)$$

We now introduce a vector  $\boldsymbol{\rho} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{A-1})$  in  $3(A-1)$ -dimensional space of relative coordinates, which is called the *global radius*, and its modulus  $\rho$  is determined by the expression

$$\rho^2 = \sum_{j=1}^{A-1} |\boldsymbol{\xi}_j|^2 = \frac{1}{2A} \sum_{i,j}^A (\mathbf{r}_i - \mathbf{r}_j)^2 = \sum_{j=1}^A (\mathbf{r}_j - \mathbf{R})^2. \quad (2.161)$$

A direction of this vector in  $3(A-1)$ -dimensional space, apparently, is specified by a set of  $3(A-1) - 1$  angular variables, which are denoted by one letter  $\Omega$ . They can be chosen by different ways. The global radius  $\rho$  is the unique dimensional variable in  $3(A-1)$ -dimensional space. Introducing the spherical system of coordinates, we can present the elementary volume as follows

$$d\boldsymbol{\rho} \equiv \Pi_{j=1}^{A-1} d\boldsymbol{\xi}_j = \rho^{3A-4} d\rho d\Omega, \quad (2.162)$$

where  $d\Omega$  is an element of a solid angle with the vector  $\boldsymbol{\rho}$  inside.

Note that at any choice of the angular variables  $\Omega$  the Laplace operator in  $3(A-1)$ -dimensional space can be written in the form

$$\Delta_{3(A-1)} = \frac{1}{\rho^{3A-4}} \frac{\partial}{\partial \rho} \left( \rho^{3A-4} \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \Delta_\Omega, \quad (2.163)$$

where  $\Delta_\Omega$  is the operator angular part containing derivatives in  $3A - 4$  variables in the corresponding spherical coordinate system.

We introduce now the so-called *harmonic polynomials*  $P_{Kn}$  choosing them in the form

$$P_{Kn} = \rho^K u_{Kn}(\Omega), \quad (2.164)$$

where functions  $u_{Kn}(\Omega)$  ( $K$  and  $n$  are some quantum numbers) depend on the angular variables  $\Omega$ . They satisfy the equation

$$\Delta_{3(A-1)} P_{Kn} = 0. \quad (2.165)$$

At the same time, the angular wave functions  $u_{Kn}(\Omega)$  are the eigenfunctions of the operator  $\Delta_\Omega$  in the multi-dimensional angular space:

$$\Delta_\Omega u_{Kn}(\Omega) = -K(K + 3A - 5)u_{Kn}(\Omega). \quad (2.166)$$

The angular functions  $u_{Kn}(\Omega)$  are called the  $K$ -harmonics and normalized by the condition:

$$\int d\Omega u_{Kn}^*(\Omega) u_{K'n'}(\Omega) = \delta_{KK'} \delta_{nn'}. \quad (2.167)$$

The quantum number  $K$ , which coincides with the polynomial of order  $P_{Kn}$ , describes the value of the total angular momentum in  $3(A - 1)$ -dimensional space and can take any nonnegative integers.

The rest quantum numbers are denoted by a letter  $n$  and contain the total angular momentum  $L$  of the relative motion of the system in a conventional three-dimensional space, its projection  $M$  on a selected axis, the quantum number  $\nu$  describing the symmetry of the function  $u_{Kn}(\Omega)$  with respect to permutations of particle coordinates and include additional quantum numbers, which are necessary in the case of large values of  $K$  and  $L$ . At  $A = 2$ , the number  $K$  coincides with the quantum numbers of the angular momentum  $L$ , and the functions  $u_{Kn}(\Omega)$  are the well known spherical functions  $Y_{LM}(\vartheta, \varphi)$ . Thus, the  $K$ -harmonics (hyperspherical functions), i.e., the angular functions  $u_{Kn}(\Omega)$ , at  $A > 2$  can be treated as natural generalization of the spherical functions.

The functions  $u_{Kn}$  form a complete set in  $3A - 4$ -dimensional space of angular variables. Because of this the wave function of the system relative motion can be presented in the expansion form with respect to these functions. The coefficients of the expansions depend on the global radius  $\rho$ .

**Application of the  $K$ -harmonic method in the three-particle problem.** We use now the  $K$ -harmonic method to the description of a bound state of three particles. Two relative three-dimensional vectors of three particles

$$\begin{aligned}\boldsymbol{\xi}_1 &= \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_1) \equiv \frac{1}{\sqrt{2}}\mathbf{r}_{21} \quad \text{and} \\ \boldsymbol{\xi}_2 &= \sqrt{\frac{2}{3}}\left(\mathbf{r}_3 - \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)\right) \equiv \sqrt{\frac{2}{3}}\mathbf{x}_3\end{aligned}\quad (2.168)$$

are united in one six-dimensional vector  $\boldsymbol{\rho}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ , whose modulus is given by the expression

$$\begin{aligned}\rho &= (\boldsymbol{\xi}_1^2 + \boldsymbol{\xi}_2^2)^{1/2} = \left(\frac{1}{2}\mathbf{r}_{12}^2 + \frac{2}{3}\mathbf{x}_3^2\right)^{1/2} \equiv \left(\frac{1}{2}\mathbf{r}_{23}^2 + \frac{2}{3}\mathbf{x}_1^2\right)^{1/2} \\ &\equiv \left(\frac{1}{2}\mathbf{r}_{13}^2 + \frac{2}{3}\mathbf{x}_2^2\right)^{1/2}.\end{aligned}\quad (2.169)$$

In a spherical coordinate system of the introduced six-dimensional space, the global variable has dimension of length. Rest five variables are the angles determining an orientation of the six-dimensional vector  $\boldsymbol{\rho}$ .

The kinetic energy operator of the relative motion of three particles in the spherical coordinate system takes the form

$$H_0 = -\hbar^2/2M \left\{ \frac{1}{\rho^5} \frac{\partial}{\partial \rho} \left( \rho^5 \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \Delta_{\Omega} \right\}, \quad (2.170)$$

where  $\Delta_{\Omega}$  is the angular part of the Laplace operator containing derivatives with respect to five angular variables. The *hyperspherical functions*  $u_{Kn}(\Omega)$  satisfy the equation

$$\Delta_{\Omega} u_{Kn}(\Omega) = -K(K+4)u_{Kn}(\Omega). \quad (2.171)$$

The quantum number  $K$  can take nonnegative integers, number  $n$  includes values of the total angular momentum of the relative motion of three particles  $L$  and its projection on a particular axis  $M$ , and quantum number  $\nu$  characterizes the function permutation symmetry. Out of the quantum number  $n$  we can separate an angular momentum of a pair particles  $l_{23}$  and angular momentum of one particle  $l_1$  with respect to the inertia center of two other particles. At  $L = 0$ , the quantum numbers  $K$  and  $\nu$  completely determine a set of the angular orthonormalized functions  $u_{Kn}(\Omega)$ . They present the complete set of functions of five angular variables  $\Omega$ . The wave function of relative motion of three nucleons  $\Psi \equiv \Psi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$  can be expanded with respect to these functions:

$$\Psi = \sum_{Kn} \varphi_{Kn}(\rho) u_{Kn}(\Omega). \quad (2.172)$$

The explicit form of the angular functions  $u_{Kn}(\Omega)$  specified by relations (2.171) and (2.167) can be treated as known and independent on the properties of a nucleon system. The radial functions  $\varphi_{Kn}(\rho)$  in expansion (2.172) depend on these properties and, first of all, on the nucleon-nucleon interaction.

Let us elucidate how one can introduce five angular variables  $\Omega$  and, in particular, write down in the explicit form  $d\Omega$  in formula (2.167). For this purpose we start with the expression of the volume element in six-dimensional space

$$d\mathbf{r}_{21}d\mathbf{x}_3 = 3^{3/2} d\xi_1 d\xi_2, \quad (2.173)$$

which is necessary to perform an integration in the six-dimensional normalization integral for the bound state wave function and when calculating different matrix elements of the three-nucleon system. In the spherical coordinate system we have

$$d\xi_1 d\xi_2 = \xi_1^2 d\xi_1 d\Omega \xi_2^2 d\xi_2 d\Omega_2, \quad \xi_1 \equiv |\xi_1|, \quad \xi_2 \equiv |\xi_2|, \quad (2.174)$$

where  $d\Omega_1 = \sin \vartheta_1 d\vartheta_1 d\varphi_1$  and  $d\Omega_2 = \sin \vartheta_2 d\vartheta_2 d\varphi_2$  are elements of the solid angles where the three-dimensional vectors  $\xi_1$  and  $\xi_2$  lie. Keeping in mind that  $\rho^2 = \xi_1^2 + \xi_2^2$ , we introduce the polar coordinates  $\rho$  and  $\Theta$  in the plane  $\xi_1, \xi_2$ :

$$\xi_1 = \rho \cos \Theta, \quad \xi_2 = \rho \sin \Theta. \quad (2.175)$$

Then the area element  $d\xi_1 d\xi_2$  should be changed by  $\rho d\rho d\Theta$ , and the volume element in the six-dimensional space can be presented as follows

$$d\rho = 3^{3/2} \rho^5 d\rho d\Omega, \quad d\Omega = \sin^2 \Theta \cos^2 \Theta d\Theta d\Omega_1 d\Omega_2. \quad (2.176)$$

Since  $\xi_1 \geq 0$  and  $\xi_2 \geq 0$ , the angle  $\Theta$  changes from 0 to  $\pi/2$ . At this, as usually,  $0 \leq \vartheta_1, \vartheta_2 \leq \pi$  and  $0 \leq \varphi_1, \varphi_2 \leq 2\pi$ . In this way we completely determined the element of the six-dimensional solid angle  $d\Omega$  and the integration limits in (2.167) with respect to five angular variables. Thus, accordingly (2.176) in the spherical coordinates in the six-dimensional space of the relative spatial radius-vectors for a system of three nucleons there are one dimensional variable  $\rho$  and five angular variables  $\Theta, \vartheta_1, \varphi_1, \vartheta_2, \varphi_2$ .

Table 2.1 shows the explicit form of the first  $K$ -harmonics  $u_{Kn}(\Omega)$  in the simplest case  $A = 3$  corresponding to  $K=0,1,2$  (Simonov, 1966). In the columns of Table 2.1 there are the total number  $n_K = \frac{(K+3)!(K+2)}{12K!}$  of  $K$ -harmonics of the  $K$ -th order, total number  $m_K$  ( $\sum m_K = n_K$ ) of a given

TABLE 2.1.

$K$	$n_K$	$m_K$	$L$	$l_{23}$	$l_1$	$u_{K_n}(\Omega)$
0	1	1	0	0	0	$1/\sqrt{\pi^3}$
1	6	3	1	1	0	$\frac{1}{\pi}\sqrt{8}\cos\Theta Y_{1m}(\vartheta_1, \varphi_1)$
		3	1	0	1	$\frac{1}{\pi}\sqrt{8}\sin\Theta Y_{1m}(\vartheta_2, \varphi_2)$
2	20	5	2	2	0	$\frac{8}{\sqrt{5}\pi}\cos^2\Theta Y_{2m}(\vartheta_1, \varphi_1)$
		5	2	0	2	$\frac{8}{\sqrt{5}\pi}\sin^2\Theta Y_{2m}(\vartheta_2, \varphi_2)$
		5	2	1	1	$\frac{1}{\rho^2}(\xi_1, \xi_{2k} + \xi_2, \xi_{1k} - \frac{2}{3}\vec{\xi}_1\vec{\xi}_2\delta_{ik})\frac{1}{\sqrt{\pi^3}}$ $\cdot \begin{cases} \beta\sqrt{24}, & \beta i \neq k \\ \beta\sqrt{18}, & \beta i = k \end{cases}$
		3	1	1	1	$\frac{1}{\rho^2}\sqrt{\frac{12}{\pi^3}}(\xi_1, \xi_{2k} - \xi_2, \xi_{1k}), i, k = x, y, z$
		1	0	0	0	$\frac{2}{\sqrt{\pi^3}}\cos 2\Theta$
		1	0	1	1	$\frac{4}{\sqrt{\pi^3}}\frac{\vec{\xi}_1\vec{\xi}_2}{\rho^2}$

type  $K$ -harmonics, the total angular three-dimensional relative momenta  $L$  ( $L$  runs odd values from 0 to  $K$  at  $K$  odd), momentum  $l_{23}$  of a pair and momentum  $l_1$  of the first nucleon with respect to the inertia center of a pair.

A value  $L = 0$  of the total angular momentum corresponds to  $K$ -harmonics with even  $K$  only, moreover, at  $L = 0$   $n$  in  $u_{Kn}(\Omega)$  reduces to one number  $\nu$ . (Recall that for bound states of three nucleons, i.e., for nuclei  ${}^3H$  and  ${}^3He$ , a contribution of terms with  $L = 0$  in the total wave function is dominant.) In this important particular case the functions  $u_{Kn}(\Omega)$  depend not on five variables but on two variables only:  $\lambda$  ( $0 \leq \lambda \leq 2\pi$ ) and  $\mu$  ( $0 \leq \mu \leq 1$ ) connected with the former angles  $\Theta$  and  $\varphi$  between three-dimensional vectors  $\xi_1$  and  $\xi_2$  by the relations

$$\mu = \sqrt{\cos^2 2\Theta + \sin^2 2\Theta \cos^2 \varphi}, \quad \cos \lambda = \frac{\cos 2\Theta}{\mu}. \quad (2.177)$$

Normalized accordingly (2.167) at  $n = \nu$  functions  $u_{Kn}$  can be expressed in terms of the *Jacobi polynomials*  $P_N^{ab}(z)$ :

$$u_{Kn} = \sqrt{\frac{K+2}{2\pi^3}} e^{-i\nu\lambda} \mu^{|\nu|} P_{1/2(K/2-|\nu|)}^{|\nu|, 0} (1 - 2\mu^2), \quad (2.178)$$

$$\nu = -\frac{K}{2}, -\frac{K}{2} + 2, \dots, +\frac{K}{2}.$$

After integration over three (out of five) angular variables in (2.167) the normalization condition looks much simpler ( $\Omega \rightarrow \lambda, \mu$ )

$$\begin{aligned} \int d\Omega u_{Kn}^*(\lambda, \mu) u_{K'n'}(\lambda, \mu) &= \pi^2 \int_0^1 d\mu \mu \int_0^{2\pi} d\lambda u_{Kn}^*(\lambda, \mu) u_{K'n'}(\lambda, \mu) \\ &= \delta_{KK'} \delta_{nn'}. \end{aligned} \quad (2.179)$$

To determine the radial parts  $\varphi_{Kn}(\rho)$  of the relative motion of the three-nucleon function, we substitute (2.172) in the Schrödinger equation  $\{\hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31} - E\}\Psi = 0$ . Multiplying the both parts of the obtained equation from the left by  $u_{Kn}^*(\Omega)$  and integrating over the angular variables, we obtain the infinite chain of the second order ordinary differential equations

$$\begin{aligned} \left[ \frac{d^2}{d\rho^2} + \frac{5}{\rho} \frac{d}{d\rho} - \frac{K(K+4)}{\rho^2} + \frac{2ME}{\hbar^2} \right] \varphi_{Kn}(\rho) \\ = \frac{2M}{\hbar^2} \sum_{K'n'} V_{Kn, K'n'}(\rho) \varphi_{K'n'}(\rho), \end{aligned} \quad (2.180)$$

$$V_{Kn, K'n'}(\rho) = \int d\Omega u_{Kn}^*(\Omega) (\hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31}) u_{K'n'}(\Omega). \quad (2.181)$$

For the bound system ( $E < 0$ ) consisting of a proton and two neutrons ( ${}^3H$  nucleus) we can write the asymptotic expression at  $\rho \rightarrow \infty$  for an arbitrary radial function  $\varphi_{Kn}(\rho)$ :

$$\varphi_{Kn}(\rho) \xrightarrow[\rho \rightarrow \infty]{\sim} \rho^{\frac{5}{2}} \exp\left(-\rho \sqrt{\frac{2M|E|}{\hbar^2}}\right), \quad (2.182)$$

following directly from equations (2.180), which become uncoupled at large  $\rho$ .

The infinite system of equations (2.180) is the exact one, but for all practical purposes it must be cutted, as well as expansion (2.172), at a finite value  $K$ . For bound states of three nucleons this can be justified, since, in this case, the series in  $K$ -harmonics (2.172) converges fast and even the first term with  $K = 0$  gives the dominant contribution in the ground state wave functions of nuclei  ${}^3H$  and  ${}^3He$  for the realistic nucleon-nucleon potentials. The terms with  $K \neq 0$  give only small corrections.

Note that for the model nucleon-nucleon potentials of an oscillator type

$$\hat{V}_{ij} = \gamma \mathbf{r}_{ij}, \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \quad (2.183)$$

matrix elements (2.181) have the diagonal form

$$V_{Kn, K'n'} = 3\gamma\rho^2\delta_{KK'}\delta_{nn'}. \quad (2.184)$$

At the same time, system equations (2.180) decays into independent differential equations for each radial function  $\varphi_{Kn}(\rho)$ , and  $K$  becomes a good (exact) quantum number. However, in a general case, for the different than (2.183) potentials  $\hat{V}_{ij}$ , the nondiagonal matrix elements (2.181) are not equal to zero and  $K$  is not an integral of motion any more.

When obtaining the wave functions of bound three-nucleon systems, i.e.,  ${}^3H$  and  ${}^3He$  nuclei, one can use a few first terms of expansion (2.172) with  $L = 0$  (the contribution of terms with  $L > 0$  turns out to be very small). At  $L = 0$  the set of the quantum numbers  $n$  reduces to one number  $\nu$ .

Since the nuclei  ${}^3H$  and  ${}^3He$  are characterized by the positive parity, the expansions of their spatial wave functions in the  $K$ -harmonics consist of only terms with even  $K$ . We give here the expressions of spatial wave functions  $\Psi^s$ ,  $\Psi'$  and  $\Psi''$ , keeping in the expansions terms with small  $K$  and indicating the explicit dependence on angular variables

$$\Psi^s = \frac{1}{\sqrt{\pi^3}}\varphi_{00}(\rho) + \sqrt{\frac{3}{\pi^3}}\left[1 - 2\left(\frac{\xi_1^2 - \xi_2^2}{\rho^2}\right)^2 - 8\frac{(\xi_1\xi_2)^2}{\rho^4}\right]\varphi_{40}(\rho), \quad (2.185)$$

$$\Psi' = \frac{4}{\sqrt{\pi^3}}\frac{\xi_1\xi_2}{\rho^2}\varphi_{21}(\rho), \quad \Psi'' = \frac{2}{\sqrt{\pi^3}}\frac{\xi_1^2 - \xi_2^2}{\rho^2}\varphi_{21}(\rho). \quad (2.186)$$

The factors standing before the radial functions  $\varphi_{K\nu}(\rho)$  are the  $K$ -harmonics of even order  $K=0,2,4$  (the small contributions of terms with  $K = 4$  in  $\Psi'$  and  $\Psi''$  are neglected). The radial functions  $\varphi_{K\nu}(\rho)$  can be obtained by the numerical solving of the system of differential equations (2.180) and (2.181). At the same time, we use the potentials obtained on the basis of data on the deuteron and nucleon-nucleon scattering. Note that the radial functions are interrelated by the normalization condition

$$3^{3/2}\int_0^\infty d\rho\rho^5\left[\varphi_{00}^2(\rho) + 2\varphi_{21}^2(\rho) + \varphi_{40}^2(\rho)\right] = 1. \quad (2.187)$$

Usually in the total wave functions of the nuclei  ${}^3H$  and  ${}^3He$ , the main term  $\sim \Psi^s\xi^a$  is kept. This means that the spatial function  $\Psi^s$  is presented by the term with the first  $K$ -harmonic  $K = 0$  (the next term corresponds to  $K = 4$ ). In this case for rough evaluations, the function  $\Psi^s$  can be approximated by the Gauss dependence on  $\rho$ :

$$\Psi^s = \frac{2^{3/2}\alpha^3}{3^{3/4}\sqrt{\pi^3}}\exp^{-\alpha^2\rho^2}, \quad (2.188)$$

where the structure parameter  $\alpha$ , determined from the data on the elastic electron scattering by three-nucleon nuclei, equals to  $0.37 \text{ fm}^{-1}$ .

**Interpolation model.** The  $K$ -harmonic method is convenient when obtaining the bound state wave functions of three-nucleon systems. However, it is impossible to apply the  $K$ -harmonic method directly for the obtaining of wave functions related to the continuous spectrum of the system. In fact, the direct expansion of the three-particle wave function  $\Psi(\xi_1, \xi_2)$ , describing an unbound state, with respect to  $K$ -harmonics, similar to (2.172), is useless. The corresponding expansion converges very slowly and it requires to take into account basically an entire infinite series.

Usually for the description of bound and two-cluster states of the continuous spectrum of three-particle systems the so-called interpolation model is used. The idea of this method lies in the following. First of all, we have to construct the explicit form of the asymptotics  $P(\xi_1, \xi_2)$  of the continuous spectrum wave function  $\Psi(\xi_1, \xi_2)$  at large  $\rho$ . Next, the difference  $\Psi(\xi_1, \xi_2) - \Phi(\rho)P(\xi_1, \xi_2)$  is expanded with respect to  $K$ -harmonics  $u_{Kn}(\Omega)$ . At this  $\Phi(\rho)$  is an unknown function of one variable  $\rho$ ,  $\Phi(\rho) \rightarrow 1$  at  $\rho \rightarrow \infty$ . As an example we consider the nucleon-deuteron system. The function  $P(\xi_1, \xi_2)$  is constructed as a product of the deuteron wave function  $\varphi_d(\xi_1)$  and the wave function of the relative motion of the deuteron and unbound nucleon  $\Psi(\xi_2)$ . In the first (rough) approximation, we can take the plane wave as the function  $\Psi(\xi_2)$ .

Thus, the wave function of the *nucleon-deuteron system*, from the beginning, is presented in the form

$$\Psi(\xi_1, \xi_2) = \sum_{Kn} C_{Kn}(\rho) u_{Kn}(\Omega) + \Phi(\rho) P(\xi_1, \xi_2), \quad (2.189)$$

where functions  $u_{Kn}(\Omega)$  and  $P(\xi_1, \xi_2)$  are known; functions  $C_{Kn}(\rho)$  and  $\Phi(\rho)$  do not depend on angular variables and can be found numerically from the equations which obtained when whole function (2.189) is substituted in the Schrödinger equation with given the nucleon-nucleon potentials. The part of wave function (2.189), namely  $\sum_{Kn} C_{Kn}(\rho) u_{Kn}(\Omega)$  is important and gives the main contribution in  $\Psi(\xi_1, \xi_2)$  in the domain where the nuclear forces are essential, i.e., at sufficiently small  $\rho$ . Because of this it is called the inner part of the function  $\Psi(\xi_1, \xi_2)$ . The series in  $K$ -harmonics of this part of the total function  $\Psi(\xi_1, \xi_2)$  converges fast. Usually, only the first term with  $K = 0$  is kept.

The second part of the wave function  $\Psi(\xi_1, \xi_2)$ , namely  $\Phi(\rho) P(\xi_1, \xi_2)$ , known as the outer part of total wave function (2.189), gives a dominant contribution at large  $\rho$ , where all functions  $C_{Kn}(\rho)$  tend to zero. If, at least, one nucleon of the system moves far away at large distances from the system nucleons, then, according to (2.169), the global variable  $\rho$  also

becomes large. When obtaining functions  $C_{K_n}(\rho)$  and  $\Phi(\rho)$ , the variational principle is usually used. It is clear that in the external part of function (2.189), unlike its inner part, the dominant contribution is given by a large set of  $K$ -harmonics with high  $K$ . Sometime in the external part of the wave function the first  $K$ -harmonics are omitted (which correspond to  $K$ -s that are kept in the expansion of the inner part of function (2.189)). In this case the both parts of function (2.189) are mutually orthogonal that considerably simplifies calculations.

The described approach to the obtaining the continuous spectrum function  $\Phi(\xi_1, \xi_2)$  in form (2.189) is called the interpolation model. This model can also be used when studying systems with  $A > 3$  nucleons to construct the wave functions of bound and unbound states.

**Calculation of three-nucleon system parameters.** We report now the calculation results of the main parameters of three-nucleon nuclei on the basis of the  $K$ -harmonic method. In particular, with the obtained wave functions we have calculated the binding energies of  $^3H$  and  $^3He$  nuclei, magnetic moments, mean squared radii, form factors of charge distribution and magnetic moment, Coulombic energy of the  $^3He$  nucleus.

Table 2.2 gives the calculated values of the binding energy  $E$  of  $^3H$  and  $^3He$  nuclei, calculated Coulomb energy of the  $^3He$  nucleus, calculated magnetic moments  $\mu$  (in nuclear magnetons), calculated mean squared radii of the electric charge  $\sqrt{\langle r^2 \rangle_c}$  and magnetic moment  $\sqrt{\langle r^2 \rangle_m}$  distributions, and the experimental values of these quantities.

To obtain the theoretical data of Table 2.2, we calculated preliminary the radial functions in (2.186) (see also (2.172)) for the simplest rectangular nucleon-nucleon potential. We also took account of the clusterization of  $^3H$  and  $^3He$  into a nucleon and deuteron. Because of this we found wave function (2.189) within the framework of the interpolation model. (The clusterization is taken into account by the second term of (2.189) with the radial function  $\Phi(\rho)$ .) The theoretical data of Table 2.2 obtained with these wave functions are given in the main approach, i.e., on keeping both in (2.172) and in the first term of (2.189) only one main harmonic  $K = 0$ . It is seen that even such rather simplified calculations with the help of the  $K$ -harmonic method give the nuclear parameters close to the experimental values. The clusterization effect in many cases improves the correspondence between calculations and experiment.

The  $K$ -harmonic method was also used to calculate the binding energies of nuclei with a larger nucleon number. However, as well as in the case of use the Faddeev integral equations, these calculations cannot explain the saturation effect of the nuclear interaction.

As we already noted before, the saturation of the nuclear interaction manifests itself in the fact that the total binding energy of nuclei happens to

TABLE 2.2.

	Nucleus $^3H$		Experiment
	Theory		
	without clusterization	with clusterization	
$E$ , MeV	7.875	8.959	8.482
	–	–	–
	2.793	2.715	2.979
$\langle r^2 \rangle_c^{1/2}$ , fm	1.77	1.69	$1.70 \pm 0.05$
	1.79	1.76	$1.70 \pm 0.05$
Nucleus $^3He$			
	Theory		Experiment
	without clusterization	with clusterization	
	7.088	8.240	7.718
$E$ , MeV	0.783	0.726	0.764
	-1.913	-1.829	-2.127
	1.79	1.87	$1.88 \pm 0.05$
	1.83	1.86	$1.95 \pm 0.11$

be proportional to a nucleon number  $A$ , but not a number of the interacting pairs. At the same time, nuclei have finite dimensions, and their volumes are also proportional  $A$ . In other words, the nucleon density inside nuclei is almost the same from the lightest nucleus  $^4He$  up to the heaviest ones.

This property of nuclei allows the nuclear substance to be treated as incompressible liquid whose properties do not depend on the mass-number  $A$ . Apparently, the nuclear matter properties must be determined by the character of the nuclear interaction. Therefore, the study of the nuclear substance must give additional information on the character of the nucleon-nucleon interaction. At the same time, we have to note that, in spite of the similarity of the properties of the nuclear substance and incompressible liquid, the different nuclear properties can be described by reducing the two-particle interaction to some self-consistent field and treating a nucleus as a system of quasiparticles in this self-consistent field. This approach lies in the basis of the shell nuclear model.

### Problems

**2.1.** Calculate the *Green function* for a system of three noninteracting particles .

Let us denote the particle coordinates and masses by  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ ,  $\mathbf{r}_3$  and  $m_1$ ,  $m_2$ ,  $m_3$ , respectively. We transform to the c.m.s. and introduce the relative coordinates

$$\mathbf{x} = \mathbf{r}_1 - \frac{m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_2 + m_3}, \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_3. \quad (2.190)$$

The Hamiltonian of the system may be written as

$$H_0 = -\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 - \frac{\hbar^2}{2\mu} \nabla_{\mathbf{r}}, \quad (2.191)$$

where  $m$  and  $\mu$  are the reduced masses,

$$m = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3}, \quad \mu = \frac{m_2 m_3}{m_2 + m_3}. \quad (2.192)$$

According to the general definition  $G_0^{(+)}(E) = (E - H_0 + i\mathcal{O})^{-1}$ , the Green function in the coordinate representation  $G_0(E)$  is determined by

$$\begin{aligned} \langle \mathbf{x}', \mathbf{r}' | G_0(E) | \mathbf{r}, \mathbf{x} \rangle &= \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \\ &\times \frac{\exp[i\mathbf{p}(\mathbf{x} - \mathbf{x}') + i\mathbf{q}(\mathbf{r} - \mathbf{r}')] }{E - E_p - E_q + i\mathcal{O}}, \end{aligned} \quad (2.193)$$

$$E_p = \frac{\hbar^2 p^2}{2m}, \quad E_q = \frac{\hbar^2 q^2}{2\mu}.$$

After integration in the space of the vector  $\mathbf{q}$  and over the angle variables of the vector  $\mathbf{p}$  is performed, we obtain

$$\begin{aligned} \langle \mathbf{x}', \mathbf{r}' | G_0(E) | \mathbf{r}, \mathbf{x} \rangle &= \frac{i}{8\pi^3} \frac{\mu}{\hbar^2} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \int_{-\infty}^{\infty} dp p \\ &\times \exp \left[ i \sqrt{\frac{2}{\hbar^2} (E - E_p + i\mathcal{O})} |\mathbf{r} - \mathbf{r}'| + ip |\mathbf{x} - \mathbf{x}'| \right]. \end{aligned} \quad (2.194)$$

We introduce the dimensionless quantities  $x$  and  $y$ , defined by

$$x \equiv \sqrt{\frac{2m}{\hbar^2} E} |\mathbf{x} - \mathbf{x}'|, \quad y \equiv \sqrt{\frac{2\mu}{\hbar^2} E} |\mathbf{r} - \mathbf{r}'|,$$

and the dimensionless integration variable  $\xi$ ,

$$\xi = p / \sqrt{\frac{2m}{\hbar^2} E}.$$

Then

$$\begin{aligned} \langle \mathbf{x}', \mathbf{r}' | G_0(E) | \mathbf{r}, \mathbf{x} \rangle &= \frac{1}{2\pi^3} (m\mu)^{3/2} \frac{E^2}{\hbar^6} \frac{1}{xy} \\ &\times \frac{\partial}{\partial x} \int_{-\infty}^{\infty} d\xi \exp \left( -y \sqrt{\xi^2 - 1 - i\mathcal{O}} + ix\xi \right) \end{aligned} \quad (2.195)$$

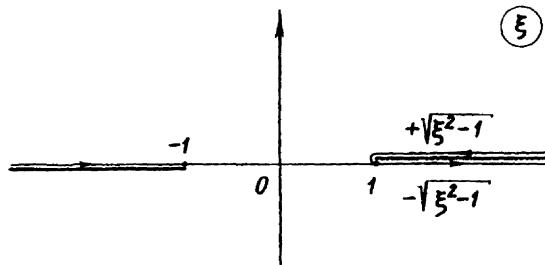


Figure 2.5. Cuts and integration contour in the complex  $\xi$  plane for (2.195).

To calculate the integral in (2.195), it is convenient to introduce a complex variable  $\xi$ . Since  $\xi = \pm(1 + i\mathcal{O})$  are the branch points, it is necessary to cut the complex  $\xi$  plane from  $-\infty$  to  $-1$  and from  $1$  to  $\infty$ . (The left cut lies somewhat lower than the real axis, the right one lies above it.) We deform the integration contour as shown in Fig. 2.5 and take the value of the root lying on the upper edge of the cut with the opposite sign. Thus we have

$$I = \int_1^\infty d\xi \exp(ix\xi) \left[ \exp\left(-y\sqrt{\xi^2 - 1}\right) - \exp\left(y\sqrt{\xi^2 - 1}\right) \right].$$

This expression easily reduces to the integral representation of the Hankel function, so that

$$I = i\pi \frac{y}{\sqrt{x^2 + y^2}} H_1^{(1)}\left(\sqrt{x^2 + y^2}\right).$$

Because

$$\frac{\partial}{\partial x} \frac{y}{\sqrt{x^2 + y^2}} H_1^{(1)}\left(\sqrt{x^2 + y^2}\right) = -\frac{xy}{x^2 + y^2} H_2^{(1)}\left(\sqrt{x^2 + y^2}\right),$$

we obtain

$$\langle \mathbf{x}', \mathbf{r}' | G_0(E) \mathbf{r}, \mathbf{x} \rangle = -\frac{i}{2\pi^2} \frac{(m\mu)^{3/2}}{\hbar^2} \frac{E}{x^2 + y^2} H_2^{(1)}\left(\sqrt{x^2 + y^2}\right), \quad (2.196)$$

where

$$x^2 + y^2 = \frac{2E}{\hbar^2} \{m(\mathbf{x} - \mathbf{x}')^2 + \mu(\mathbf{r} - \mathbf{r}')^2\}.$$

Equation (2.196) determines the Green function for a system of three noninteracting particles.

**2.2.** Write down the equations of motion of a *system of three bound particles* in the momentum representation.

Let us rewrite equation of motion  $(H_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31} - E)\Psi = 0$  in the form

$$(E - H_0)\Psi = (\hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31})\Psi. \quad (2.197)$$

For the bound state of the three-nucleon particles the factor  $(E - H_0)$  in the left-hand side of equation (2.197) is negative (because  $H_0 > 0$ ), i.e., always nonzero. For the continuous energy spectrum this factor can have both signs or be zero. Thus, in the case of the bound state of all three particles equation (2.197) can be divided by  $E - H_0$  and rewritten in the form

$$\Psi_0 = G_0(\hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31})\Psi_0, \quad G_0 = (E - H_0)^{-1} < 0. \quad (2.198)$$

For an unbound state of three particles such a representation of the motion equation is impossible in a general case.

We introduce now three auxiliary functions

$$\Psi_0^{(ij)} = G_0 \hat{V}_{ij} \Psi_0, \quad ij = 123, 231, 312. \quad (2.199)$$

If these three functions are found, then from (2.199) we can obtain the total desired wave function  $\Psi_0$ :

$$\Psi_0 = \Psi_0^{(12)} + \Psi_0^{(23)} + \Psi_0^{(31)}. \quad (2.200)$$

From definition (2.199) it follows that to find the functions  $\Psi_0^{(ij)}$ , we have to solve the following system of coupled integral equations

$$\begin{cases} \Psi_0^{(12)} = G_0 \hat{V}_{12} (\Psi_0^{(12)} + \Psi_0^{(23)} + \Psi_0^{(31)}), \\ \Psi_0^{(23)} = G_0 \hat{V}_{23} (\Psi_0^{(12)} + \Psi_0^{(23)} + \Psi_0^{(31)}), \\ \Psi_0^{(31)} = G_0 \hat{V}_{31} (\Psi_0^{(12)} + \Psi_0^{(23)} + \Psi_0^{(31)}), \end{cases} \quad (2.201)$$

In the momentum representation the results of action of the integral operators  $\hat{V}_{ij}$  on each of the functions  $\Psi_0^{(kl)}$  ( $kl=12, 23, 31$ ) can be presented in the same form as the result of action of the operators  $\hat{V}_{ij}$  on the total function  $\Psi$ . For an example we write down three functions  $\hat{V}_{12}\Psi_0^{(ij)}$ , which appear in the right-hand side of the first equation of system (2.201) by using the connections between momenta (we assume that the particle masses are the same and equal to  $M$ ):

$$\hat{V}_{12}\Psi_0^{(12)} = \int d\mathbf{k}'_{12} V_{12}(\mathbf{k}'_{12} - \mathbf{k}_{12}) \Psi_0^{(12)}(\mathbf{k}'_{12}, \mathbf{p}_3), \quad (2.202)$$

$$\hat{V}_{12}\Psi_0^{(23)} = \int d\mathbf{k}'_{12} V_{12}(\mathbf{k}'_{12} - \mathbf{k}_{12}) \Psi_0^{(23)}\left(-\frac{1}{2}\mathbf{k}'_{12} - \frac{3}{4}\mathbf{p}_3, \mathbf{k}'_{12} - \frac{1}{2}\mathbf{p}_3\right), \quad (2.203)$$

$$\hat{V}_{12}\Psi_0^{(31)} = \int d\mathbf{k}'_{12} V_{12}(\mathbf{k}'_{12} - \mathbf{k}_{12}) \Psi_0^{(31)}\left(-\frac{1}{2}\mathbf{k}'_{12} + \frac{3}{4}\mathbf{p}_3, -\mathbf{k}'_{12} - \frac{1}{2}\mathbf{p}_3\right). \quad (2.204)$$

As a result the first equation of system (2.201) can be written as follows:

$$\begin{aligned} \Psi_0^{(12)}(\mathbf{k}, \mathbf{p}) = & -\frac{1}{|E_0| + (k^2 + \frac{3}{4}p^2)/M} \int d\mathbf{k}' V_{12}(\mathbf{k}' - \mathbf{k}) \\ & \times \left\{ \Psi_0^{(12)}(\mathbf{k}', \mathbf{p}) + \Psi_0^{(23)}\left(-\frac{1}{2}\mathbf{k}' - \frac{3}{4}\mathbf{p}, \mathbf{k}' - \frac{1}{2}\mathbf{p}\right) \right. \\ & \left. + \Psi_0^{(31)}\left(-\frac{1}{2}\mathbf{k}' + \frac{3}{4}\mathbf{p}, -\mathbf{k}' - \frac{1}{2}\mathbf{p}\right) \right\}. \end{aligned} \quad (2.205)$$

Acting in the similar manner, we can write down two other equations of system (2.201).

We consider now a particular case when the two-particle potentials  $V_{ij}$  are the same for all three pairs of particles. Then we can omit indexes  $ij$  in the potentials  $V_{ij}$  and, according to (2.199), in the functions  $\Psi_0^{(ij)}$ . Since in this case in equations (2.201) the functions  $\Psi_0$  differs only by arguments, we can consider only one equation of the system. Two other equations can be reduced to it. Thus, the unique function  $\Psi_0(\mathbf{k}, \mathbf{p})$ , which determines the total wave function  $\Psi_0$

$$\Psi_0 = \Psi_0(\mathbf{k}_{12}, \mathbf{p}_3) + \Psi_0(\mathbf{k}_{23}, \mathbf{p}_1) + \Psi_0(\mathbf{k}_{31}, \mathbf{p}_2), \quad (2.206)$$

we can find from a single integral equation, which follows from (2.205)

$$\begin{aligned} \Psi(\mathbf{k}, \mathbf{p}) = & -\frac{1}{|E| + (k^2 + \frac{3}{4}p^2)/M} \int d\mathbf{k}' V(\mathbf{k}' - \mathbf{k}) \\ & \times \left\{ \Psi(\mathbf{k}', \mathbf{p}) + \Psi\left(-\frac{1}{2}\mathbf{k}' - \frac{3}{4}\mathbf{p}, \mathbf{k}' - \frac{1}{2}\mathbf{p}\right) \right. \\ & \left. + \Psi\left(-\frac{1}{2}\mathbf{k}' + \frac{3}{4}\mathbf{p}, -\mathbf{k}' - \frac{1}{2}\mathbf{p}\right) \right\}. \end{aligned} \quad (2.207)$$

Note that if the wave function  $\Psi$  is completely symmetric with respect to permutations of particles, then from (2.206) follows  $\Psi_0(-\mathbf{k}, \mathbf{p}) = \Psi_0(\mathbf{k}, \mathbf{p})$ .

We consider now a system of three bound nucleons and take into account their spins and isospins. In the case of triton  ${}^3H$ , the total wave function is expressed in terms of the known spin-isospin functions and the spatial functions  $\Psi^s$ ,  $\Psi^a$ ,  $\Psi'$  and  $\Psi''$ , we use the same denotations for these functions in the momentum space as well. Let us assume for simplicity that the nucleon-nucleon interaction is nonzero only for the  $S$ -states. In the center mass coordinate system, the spatial wave functions, on taking account of their symmetry with respect to permutations of nucleus, can be written in the form

$$\begin{aligned} \Psi^s &= u(k_{23}, p_1) + u(k_{31}, p_2) + u(k_{12}, p_3), \quad \Psi^a = 0, \\ \Psi' &= -\frac{\sqrt{3}}{2} \left[ f(k_{31}, p_2) - f(k_{12}, p_3) \right], \\ \Psi'' &= -f(k_{23}, p_1) + \frac{1}{2} \left[ f(k_{31}, p_2) + f(k_{12}, p_3) \right], \end{aligned} \quad (2.208)$$

in addition  $u(-k, p) = u(k, p)$ ,  $f(-k, p) = f(k, p)$ . (In the case of the  $S$ -interaction and zero total angular moment, the functions depend only on the modulus of vectors  $\mathbf{k}$  and  $\mathbf{p}$ .)

Substituting the total wave function of the  ${}^3H$  nucleus in the Schrödinger equation, on taking account (2.208), and introducing the denotation  $v = u + f$  and  $w = u - f$ , we obtain the coupled system of two integral equations ( $V^{tt} = V^{ss} = 0$ ):

$$\begin{aligned} v(k, p) &= -\frac{1}{|E| + (\mathbf{k}^2 + \frac{3}{4}\mathbf{p}^2)/M} \int d\mathbf{p}' \left\{ V^{ts}(\mathbf{p}' - \mathbf{k})v(p', p) \right. \\ &\quad + V^{ts} \left( \mathbf{p}' + \frac{\mathbf{p}}{2} - \mathbf{k} \right) \left[ \frac{1}{2}v \left( |\mathbf{p} + \frac{\mathbf{p}'}{2}|, p' \right) \right. \\ &\quad \left. \left. + \frac{3}{2}w \left( |\mathbf{p} + \frac{\mathbf{p}'}{2}|, p' \right) \right] \right\}, \end{aligned} \quad (2.209)$$

$$\begin{aligned} w(k, p) &= -\frac{M}{M|E| + \mathbf{k}^2 + \frac{3}{4}\mathbf{p}^2} \int d\mathbf{p}' \left\{ V^{st}(\mathbf{p}' - \mathbf{k})w(p', p) \right. \\ &\quad + V^{st} \left( \mathbf{p}' + \frac{\mathbf{p}}{2} - \mathbf{k} \right) \left[ \frac{1}{2}w \left( |\mathbf{p} + \frac{\mathbf{p}'}{2}|, p' \right) \right. \\ &\quad \left. \left. + \frac{3}{2}v \left( |\mathbf{p} + \frac{\mathbf{p}'}{2}|, p' \right) \right] \right\}. \end{aligned} \quad (2.210)$$

Adding these equations in the approximation  $V^{ts} = V^{st}$ , we obtain the same equation for the function  $u(k, p)$  as equation (2.207) of the function  $\Psi(\mathbf{k}, \mathbf{p})$  for the  $S$ -interaction at zero total orbital moment of the system. The integral equations (2.207), (2.209), (2.210) can be solved numerically by using modern computers.

### 2.3. Establish the connection between *two-particle Green's operator* $\hat{G}_{ij}$ and the *total Green operator* $\hat{G}$ for the three-particle system .

Along with the Green operators  $\hat{G}$  and  $\hat{G}_0$  for a system of three particles, we consider the two-particle Green operators

$$\hat{G}_{ij} = (z - \hat{H}_{ij})^{-1}, \quad \hat{H}_{ij} = \hat{H}_0 + \hat{V}_{ij}, \quad z = E \pm i\varepsilon, \quad (2.211)$$

which take into account the interaction only between two particles  $i$  and  $j$ . At the same time, it is assumed that the third particle does not interact with the  $i$ -th and  $j$ -th particles ( $ijl=123, 231, 312$ ). We establish now the connection between the Green operators  $\hat{G}$ ,  $\hat{G}_{ij}$  and the operators of two-particle interaction  $\hat{V}_{ij}$ . For this we employ the operator equality

$$\hat{A}^{-1} - \hat{B}^{-1} = \hat{A}^{-1}(\hat{B} - \hat{A})\hat{B}^{-1} = \hat{B}^{-1}(\hat{B} - \hat{A})\hat{A}^{-1}. \quad (2.212)$$

Setting here  $\hat{A} = z - \hat{H}$ ,  $\hat{B} = z - H_{ij}$  and noting that  $\hat{H} - \hat{H}_{ij} = \hat{V} - \hat{V}_{ij}$ , we obtain the operator relations

$$\hat{G}(z) = \hat{G}_{ij}(z) + \hat{G}_{ij}(z)(\hat{V} - \hat{V}_{ij})\hat{G}(z), \quad (2.213)$$

or

$$\hat{G}(z) = \hat{G}_{ij}(z) + \hat{G}(z)(\hat{V} - \hat{V}_{ij})\hat{G}_{ij}(z), \quad (2.214)$$

which establish the connections between the Green operators of the three particle system  $\hat{G}$  and the two-particle operators  $\hat{G}_{ij}$  (note that the two-particle operator  $\hat{G}_{ij}$  contains the interaction between the  $i$ -th and  $j$ -th particles and the third particle appears in the operator  $\hat{H}_0$ ). Since in the three-particle system there are three different pairs of interacting particles, we obtain three different relations

$$\hat{G} = \hat{G}_{12} + \hat{G}_{12}(\hat{V}_{23} + \hat{V}_{31})\hat{G}, \quad (2.215)$$

$$\hat{G} = \hat{G}_{23} + \hat{G}_{23}(\hat{V}_{31} + \hat{V}_{12})\hat{G}, \quad (2.216)$$

$$\hat{G} = \hat{G}_{31} + \hat{G}_{31}(\hat{V}_{12} + \hat{V}_{23})\hat{G}, \quad (2.217)$$

We call attention to the fact that all the Green operators in these equations are related to the same total energy of the system  $E$ .

First of all, we consider the action of the Green operators multiplied by  $\varepsilon \rightarrow 0$  ( $\varepsilon \geq 0$ ) on different wave functions. Let us act by the total Green operator, which is specified by the general formula (2.63) on an arbitrary function  $\Psi$ . We expand the function  $\Psi$  in terms of the eigenfunctions  $\Psi_\lambda$  of the system Hamiltonian  $\hat{H}$ , i.e.,

$$\Psi = \sum_{\lambda} a_{\lambda} \psi_{\lambda},$$

where  $\hat{H}\psi_{\lambda} = E_{\lambda}\psi_{\lambda}$ . Thus,

$$\hat{G}\Psi = \sum_{\lambda} a_{\lambda} (E + i\varepsilon - \hat{H})^{-1} \psi_{\lambda} \equiv \sum_{\lambda} a_{\lambda} (E + i\varepsilon - E_{\lambda})^{-1} \psi_{\lambda}. \quad (2.218)$$

If the energy  $E$  does not equal to any one of the eigenvalues of the Hamiltonian  $E_{\lambda}$ , then

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \hat{G}(E + i\varepsilon)\Psi = 0, \quad E \neq E_{\lambda}. \quad (2.219)$$

But if  $E = E_{\lambda}$ , then

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \hat{G}(E_{\lambda} + i\varepsilon)\Psi = a_{\lambda} \psi_{\lambda}. \quad (2.220)$$

We consider an unbound state of the three-particle system with the wave function  $\Psi_k = \Psi_k^{(+)}$ , which satisfies the Schrödinger equation  $(\hat{H} - E_k)\Psi_k^{(+)} = 0$  with the boundary condition for scattered particles or the *Lippmann-Schwinger equation*:

$$\Psi_k^{(+)} = \varphi_k + \hat{G}_0(E_k + i\varepsilon)\hat{V}\Psi_k^{(+)}, \quad (2.221)$$

where  $\varphi_k$  is a solution of the equation  $(\hat{H}_0 - E_k)\varphi_k = 0$ . It is easy to check that

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \hat{G}(E_k + i\varepsilon)\varphi_k = \Psi_k^{(+)}, \quad (2.222)$$

i.e., with help of the total Green operator  $\hat{G}(E_k + i\mathcal{O})$  the solution of the Lippmann–Schwinger equation  $\Psi_k^{(+)}$  with the energy  $E_k$  or the solution of the corresponding Schrödinger equation with a given boundary condition is expressed directly in terms of the incident wave  $\varphi_k$  for the same energy  $E_k$ .

The analogous relations can be obtained for the two-particle Green operators  $\hat{G}_{ij}$  and the corresponding wave functions  $\psi_k^{(ij)}$  satisfying the Schrödinger equation

$$\hat{H}_{ij}\psi_k^{(ij)} = E_k\psi_k^{(ij)}, \quad \hat{H}_{ij} = \hat{H}_0 + \hat{V}_{ij}, \quad ij = 12, 23, 31, \quad (2.223)$$

when only two particles out of three the  $i$ -th and  $j$ -th interact. At infinity the functions  $\psi_k^{(ij)}$  contain divergent waves. The wave function  $\psi_k^{(ij)}$  can describe the state of three particles when two of them ( $i$ -th and  $j$ -th) are bound, i.e., interact, and the third particle ( $l$ -th) is free (i.e., does not interact with the  $i$ -th and  $j$ -th), or the state when the mutual scattering of the  $i$ -th and  $j$ -th particles occurs, i.e., they are unbound, but interact, and the third  $l$ -th particle is free. Similarly to (2.219) and (2.220) the following relations hold true:

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \hat{G}_{ij}(E_k + i\varepsilon)\psi_{k'}^{(i'j')} = \delta_{kk'}\delta_{ij,i'j'}\psi_k^{(ij)}, \quad (2.224)$$

in addition, one of two indexes  $ij$  of the Green operator always coincides with one of the indexes  $i'j'$  of the three-particle function  $\psi_{k'}^{(i'j')}$ .

If we "switch off" the interaction of the  $l$ -th particle with  $i$ -th and  $j$ -th ones (the interaction between two latter particles is unchanged), then from (2.222) we immediately obtain the following relation

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \hat{G}_{ij}(E_k + i\varepsilon)\varphi_k = \psi_k^{(ij)}. \quad (2.225)$$

When writing the operator connections (2.215) – (2.217) we used relation (2.213). Now we employ relation (2.214). Acting by the left and right parts of this operator equality on the incident wave  $\varphi_k$  and multiplying the obtained equation by  $i\varepsilon$ , we pass to the limit  $\varepsilon \rightarrow 0$ . Utilizing the formulas (2.222) and (2.225), we obtain the following connection between the function  $\Psi_k^{(+)}$  and the functions  $\psi_k^{(ij)}$

$$\Psi_k^{(+)} = \psi_k^{(ij)} + \hat{G}(E_k + i\varepsilon)(\hat{H} - \hat{H}_{ij})\psi_k^{(ij)}. \quad (2.226)$$

Using (2.223) and the explicit form of the operator  $\hat{G}^{-1} \equiv \hat{G}^{-1}(E_k + i\varepsilon) = E_k + i\varepsilon - \hat{H}$ , we obtain from (2.226) the useful general formula

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \hat{G}(E_k + i\varepsilon)\psi_k^{(ij)} = \Psi_k^{(+)}. \quad (2.227)$$

Here the wave function  $\Psi_k^{(+)}$  of a three interacting particles system with a given boundary condition at infinity is connected with the three-particle wave function

$\psi_k^{(ij)}$  by the total Green operator  $\hat{G}$ . The asymptotics of  $\psi_k^{(ij)}$  (divergent waves at infinity) is given, but only two particles interact (the  $i$ -th and  $j$ -th).

**2.4.** Write down the *integral equation for the three-particle system* with the help of the two-particle Green operators  $\hat{G}_{ij}$ .

First of all we consider the scattering problem of a particle by two other particles in a bound state. Prescribe a number 1 to the scattered particle, and let the 2-nd and 3-d particles be bound. The corresponding asymptotic Green function is  $\Phi_1$ .

Substituting in turn the right parts of three equalities (2.215) – (2.217) in the left-hand side of general formula (2.227) and by using (2.224) and (2.227), we obtain three equations

$$\Psi_{kb}^{(+)} = \Phi_1 + \hat{G}_{23}(E_k + i0)(\hat{V}_{31} + \hat{V}_{12})\Psi_{kb}^{(+)}, \quad (2.228)$$

$$\Psi_{kb}^{(+)} = \hat{G}_{12}(E_k + i0)(\hat{V}_{23} + \hat{V}_{31})\Psi_{kb}^{(+)}, \quad (2.229)$$

$$\Psi_{kb}^{(+)} = \hat{G}_{31}(E_k + i0)(\hat{V}_{12} + \hat{V}_{23})\Psi_{kb}^{(+)}, \quad (2.230)$$

Each two-particle operator  $\hat{G}_{ij}$ , which appears in these equations, when acting as a function that stands behind it, gives the divergent waves with respect to the coordinates of the particle whose interaction is not included into the operator  $\hat{G}_{ij}$ . The waves diverge relatively coordinates of the particles whose interaction is taken into account in the operator  $\hat{G}_{ij}$ . Note that individual Green's operator  $\hat{G}_{ij}$  and the corresponding equation for this operator do not contain all the boundary conditions of the three-particle problem. In fact we have to simultaneously solve all three equations. The equations should be treated as the coupled system of consistent equations. At the same time, each of three equations, which are solved simultaneously, is responsible for the absence of the convergent waves and appearance of divergent waves at infinity with respect to corresponding relative space coordinate.

In the case of a system of three unbound particles the analogous equations for the wave function  $\Psi_{123}$  can be obtained substituting the right parts of equations (2.215) – (2.217) in (2.227) and by using (2.224) and (2.227). The obtained integral equations, unlike to equations (2.228) – (2.230), are completely symmetric with respect to coordinates of three particles

$$\Psi_{ks}^{(+)} = \Psi_{ks}^{(12)} + \hat{G}_{12}(E_k + i0)(\hat{V}_{23} + \hat{V}_{31})\Psi_{ks}^{(+)}, \quad (2.231)$$

$$\Psi_{ks}^{(+)} = \Psi_{ks}^{(23)} + \hat{G}_{23}(E_k + i0)(\hat{V}_{31} + \hat{V}_{12})\Psi_{ks}^{(+)}, \quad (2.232)$$

$$\Psi_{ks}^{(+)} = \Psi_{ks}^{(31)} + \hat{G}_{31}(E_k + i0)(\hat{V}_{12} + \hat{V}_{23})\Psi_{ks}^{(+)}. \quad (2.233)$$

To obtain the single-valued solution for the wave function  $\Psi_{ks}^{(+)}$  with the correct asymptotics at infinity, equations (2.231) – (2.233), as well as equations (2.228) – (2.230), must be solved simultaneously. Then all boundary conditions on three relative vector coordinates  $\mathbf{r}_{12}$ ,  $\mathbf{r}_{23}$ ,  $\mathbf{r}_{31}$  must be taken into account.

In conclusion we write down the integral equation of the wave function  $\psi_0$  of a bound state of the three-particle system which correspond to the energy  $E_0 < 0$ .

Unlike the above considered three-particle systems, where at least one particle is not bound with others, in the case of three bound particles the problem of the solution ambiguity is automatically removed. Since, according to (2.220), for the wave function  $\psi_0$  we have

$$\lim_{\varepsilon \rightarrow +0} i\varepsilon \hat{G}(E_0 + i\varepsilon) \psi_0 = \psi_0, \quad (2.234)$$

because  $\hat{H}\psi_0 = E_0\psi_0$  and

$$\lim_{\varepsilon \rightarrow +0} i\varepsilon G_0(E_0 + i\varepsilon) \psi_0 = \lim_{\varepsilon \rightarrow +0} i\varepsilon \hat{G}(E_0 + i\varepsilon) \psi_0 = 0, \quad (2.235)$$

due to  $\hat{H}_0\psi_0 \neq E_0\psi_0$  and  $\hat{H}_{ij}\psi_0 \neq E_0\psi_0$ , then after action by both parts of operator equalities (2.215) – (2.217) on the wave function of the bound state  $\psi_0$  multiplied by  $i\varepsilon$  (with  $\varepsilon \rightarrow +0$ ), we obtain the homogeneous with respect to the function  $\psi_0$ , three equations

$$\begin{aligned} \psi_0 &= \hat{G}_{12}(\hat{V}_{23} + \hat{V}_{31})\psi_0, \\ \psi_0 &= \hat{G}_{23}(\hat{V}_{31} + \hat{V}_{12})\psi_0, \\ \psi_0 &= \hat{G}_{31}(\hat{V}_{12} + \hat{V}_{23})\psi_0. \end{aligned} \quad (2.236)$$

To find the function  $\psi_0$ , we can now solve only one out of three equations (in addition, the boundary condition at infinity where  $\psi_0 \rightarrow 0$  is satisfied).

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