$$H = \varepsilon\left(b_1^\dagger b_1 + b_2^\dagger b_2\right) + v\left(b_1^\dagger b_2^\dagger + b_1 b_2\right)$$

2 boson operators ($n=2$)

$\downarrow$

Dynamic matrix ($2n\times 2n$ squared matrix)

$$D = \begin{pmatrix} \varepsilon & 0 & 0 & v \\ 0 & \varepsilon & v & 0 \\ 0 & -v & -\varepsilon & 0 \\ -v & 0 & 0 & -\varepsilon \end{pmatrix} \rightarrow \text{characteristic equation}$$

$$\left(w^2 - \varepsilon^2 + v^2\right)^2 = 0$$

Solutions (eigenvalues) of $D$:

$w_0 = 0 \qquad |\varepsilon| = |v| \qquad c_1$

$w_{1,2} = \pm\sqrt{\varepsilon^2 - v^2} \qquad |\varepsilon| > |v| \qquad c_2$

$w_{3,4} = \pm i\sqrt{v^2 - \varepsilon^2} \qquad |\varepsilon| < |v| \qquad c_3$

Cases $c_1, c_3$ $\rightarrow$ $D$ is not diagonalizable $\Rightarrow$ $H$ is not BV diagonalizable

Case $c_2$: pair of real eigenvalues: $(\omega, -\omega)$

$$\omega = \sqrt{\varepsilon^2 - v^2}$$

$\omega$: has two linearly independent eigenvectors

$$
w_1(\omega) = \begin{cases} \begin{bmatrix} \dfrac{v}{\sqrt{2\omega(\varepsilon-\omega)}} \\ 0 \\ 0 \\ \dfrac{\omega-\varepsilon}{\sqrt{2\omega(\varepsilon-\omega)}} \end{bmatrix}, & \varepsilon > 0 \\[2em] \begin{bmatrix} \dfrac{v}{\sqrt{2\omega(\omega-\varepsilon)}} \\ 0 \\ 0 \\ \dfrac{\omega-\varepsilon}{\sqrt{2\omega(\omega-\varepsilon)}} \end{bmatrix}, & \varepsilon < 0 \end{cases}
$$

$$
w_2(\omega) = \begin{cases} \begin{bmatrix} 0 \\ \dfrac{\omega+\varepsilon}{\sqrt{2\omega(\omega+\varepsilon)}} \\ \dfrac{-\gamma}{\sqrt{2\omega(\omega+\varepsilon)}} \\ 0 \end{bmatrix}, & \varepsilon > 0 \\[2em] \begin{bmatrix} 0 \\ \dfrac{\omega+\varepsilon}{\sqrt{-2\omega(\omega+\varepsilon)}} \\ \dfrac{\gamma}{\sqrt{-2\omega(\omega+\varepsilon)}} \\ 0 \end{bmatrix}, & \varepsilon < 0 \end{cases}
$$

With their norms just as in the previous calculation

Orthonormal vectors for $-w$ : $\begin{cases} v_1(-w) \\ v_2(-w) \end{cases}$

$\Rightarrow$ Orthonormal basis: $\{v_1(w), v_2(w), v_1(-w), v_2(-w)\}$

The normal derivative matrix $T_n$ :

$$T_n = \begin{cases} [v_1(w), v_2(w), v_1(-w), v_2(-w)], & \varepsilon > 0 \\ [v_1(-w), v_2(-w), v_1(w), v_2(w)], & \varepsilon < 0 \end{cases}$$

$$T_n^{-1} \partial T_n = \begin{cases} \begin{pmatrix} w & & & \\ & w & & 0 \\ & 0 & -w & \\ & & & -w \end{pmatrix}, & \varepsilon > 0 \\ \\ \begin{pmatrix} -w & & & \\ & -w & & 0 \\ & 0 & w & \\ & & & w \end{pmatrix}, & \varepsilon < 0 \end{cases}$$