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A Note on the Diagonalization of Quadratic Boson and Fermion Hamiltonians

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We take advantage of the symmetry present in quadratic boson and fermion hamiltonians to give a short and simple derivation of their diagonalizations. This is of particular relevance to bosons. Both procedures are critically evaluated and a striking resemblance is pointed out.

1. Introduction

The spin wave approximation [1] is a well-known method of studying the low-lying excitation spectrum of a ferromagnet. Dropping the A^2-term from the Dicke hamiltonian [2] is a convenient approximation in studying the dynamical response of a maser. In both procedures it turns out that the main problem is diagonalizing a quadratic *boson* hamiltonian. Surprisingly enough, and contrary to the expectation of most physicists, the diagonalization of such a hamiltonian is nontrivial. In addition a more fundamental, and rather intriguing question suggests itself: why is the diagonalization of a quadratic boson hamiltonian nontrivial whereas its fermion counterpart does not present any particular difficulty? Is there a deeper structure explaining this difference?

In Sect. 2 the diagonalization problem proper is analyzed, in Sect. 3 it is solved for fermions, in Sect. 4 for bosons. Through a parallel treatment the reader will notice quickly the similarities and the differences between both cases. A discussion of the mathematical structure which emerges, and its physical consequences, is to be found in Sect. 5.

2. The Diagonalization Problem

Consider the quadratic hamiltonian

$$\mathscr{H} = \sum_{i,j=1}^{n} \{\boldsymbol{\alpha}_i^+ A_{ij} \boldsymbol{\alpha}_j + \boldsymbol{\alpha}_i^+ B_{ij} \boldsymbol{\alpha}_j^+ + \boldsymbol{\alpha}_i C_{ij} \boldsymbol{\alpha}_j + \boldsymbol{\alpha}_i D_{ij} \boldsymbol{\alpha}_j^+\}. \tag{1}$$

Here I use the following convention: A^+ means hermitean conjugate, $\bar{A}$ means complex conjugate and A^t transposition; so $A^+ = \bar{A}^t$ when A is an $n \times n$ matrix. In addition $1 \leqq i,\ j \leqq n$ and $1 \leqq \mu,\ \nu \leqq 2n$. In matrix form $\mathscr{H}$ reappears as

$$\mathscr{H} = (\boldsymbol{\alpha}^+, \boldsymbol{\alpha}) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^+ \end{pmatrix} \equiv \mathbf{a}^+ \mathbb{D} \mathbf{a}; \tag{2}$$

$\mathbf{a} = \{\mathbf{a}_\mu\}$ is considered as a vector in a $2n$-dimensional space.

The operators $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^+$ are either *boson* operators satisfying the commutation relations

$$[\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j^+] = \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j^+ - \boldsymbol{\alpha}_j^+ \boldsymbol{\alpha}_i = \delta_{ij}, \qquad 1 \leqq i, j \leqq n, \tag{3}$$

the other commutators being zero, or *fermion* operators satisfying the anti-commutation relations

$$\{\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j^+\} = \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j^+ + \boldsymbol{\alpha}_j^+ \boldsymbol{\alpha}_i = \delta_{ij}, \qquad 1 \leqq i, j \leqq n, \tag{4}$$

the other anti-commutators being zero.

The problem of *diagonalizing* $\mathscr{H}$ amounts to finding a linear transformation T,

$$\mathbf{a} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^+ \end{pmatrix} = T \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\beta}^+ \end{pmatrix} = T\mathbf{b} \tag{5}$$

such that T is *canonical*, i.e. $\mathbf{b}$ is again a set of boson or fermion operators, while at the same time

$$\mathscr{H} = \mathbf{a}^+ \mathbb{D} \mathbf{a} = \mathbf{b}^+ (T^+ \mathbb{D} T) \mathbf{b} \tag{6}$$

can be written in the *diagonal* form

$$\mathscr{H} = \sum_{i=1}^{n} 2\hbar\omega_i \boldsymbol{\beta}_i^+ \boldsymbol{\beta}_i. \tag{7}$$

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One also says T diagonalizes $\mathbb{D}$, it being understood that T is canonical; I will not follow this usage however. If such a diagonalization were possible, the original problem (1) would be reduced to another one, viz. (7), whose dynamics and thermodynamics are totally trivial: one either gets n independent bosons (harmonic oscillators) or n independent fermions.
Indeed, all this can be accomplished in a variety of ways [3–8; and the references quoted therein, e.g. 9 and 10], some difficult, some direct, but most of them annoyingly cumbersome considering the simplicity of the final form (7). Here I present a self-contained derivation, nearly as simple as the result. In addition I exhibit the fundamental structure which is responsible for the difference between the fermion and the boson procedure. Thereby I am able to complement essentially the arguments of Tikochinsky [3]. The basic strategy is to exploit fully the *natural symmetry* which is present in either the boson or the fermion hamiltonian (1). In order to put the boson result in a proper perspective we consider the fermion case first.

3. Fermions

Let I be the $2n$-dimensional unit matrix, and let $\{\mathbf{u}, \mathbf{v}\}$ denote a pair of vectors from $\mathbb{C}^n$. Before proceeding we define the operator J in $\mathbb{C}^{2n}$ by

$$J\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}=\begin{pmatrix}\bar{\mathbf{v}}\\ \bar{\mathbf{u}}\end{pmatrix}. \tag{8}$$

$J^2=I$ and J is anti-unitary (evidently it is anti-linear, i.e., $J(\lambda\mathbf{x}+\mu\mathbf{y})=\bar{\lambda}J\mathbf{x}+\bar{\mu}J\mathbf{y}$). J is essential in all that follows. Notice that

$$[I, J]=0. \tag{9}$$

We first turn to be hamiltonian $\mathscr{H}$ and the natural symmetries it contains. Because of the anti-commutation relations (4) we may suppose

$$D=-A^t, \quad B=-B^t, \quad C=-C^t, \tag{10}$$

while $\mathscr{H}$ being hermitean then implies

$$A^+=A, \quad B^+=-C. \tag{11}$$

Thus we may write

$$\mathbb{D}=\begin{pmatrix}A & B\\ -\bar{B} & -\bar{A}\end{pmatrix}; \tag{12}$$

$\mathbb{D}$ is hermitean. Moreover a simple calculation shows

$$\{\mathbb{D}, J\}=0. \tag{13}$$

This is the crux of the fermion argument. As a consequence for the eigenvalue problem of $\mathbb{D}$ we notice that

$$\mathbb{D}\mathbf{x}=\omega\mathbf{x}\Rightarrow\mathbb{D}(J\mathbf{x})=-\omega(J\mathbf{x}),$$

that is to say, whenever ω is an eigenvalue of $\mathbb{D}$ with eigenvector $\mathbf{x}$, $-\omega$ is another eigenvalue of $\mathbb{D}$ with eigenvector $J\mathbf{x}$, and the eigenvalues of $\mathbb{D}$ lie symmetrically with respect to the origin.
Next we derive the necessary conditions for a matrix T to be canonical as well as diagonalizing. If T is canonical, then by (5)

$$I_{\mu\nu}=\{\mathbf{a}_\mu, \mathbf{a}_\nu^+\}=\sum_{\sigma,\tau=1}^{2n} T_{\mu\sigma}I_{\sigma\tau}T_{\tau\nu}^+ \tag{15}$$

or

$$I=TIT^+, \tag{16}$$

implying $I=TT^+$, which amounts to saying T is *unitary*. Then the inverse transformation T^{-1} exists ($\mathbf{b}=T^{-1}\mathbf{a}$) and is also unitary ($T^{-1}=T^+$). In addition we have to require that T be of the form

$$T=\begin{pmatrix}U & \bar{V}\\ V & \bar{U}\end{pmatrix}=\{\mathbf{x}_1, \dots, \mathbf{x}_n, J\mathbf{x}_1, \dots, J\mathbf{x}_n\}; \tag{17}$$

U and V are $n\times n$ matrices and the column vectors $\mathbf{x}_i=\{\mathbf{u}_i, \mathbf{v}_i\}$ are chosen from $\mathbb{C}^{2n}=\mathbb{C}^n\oplus\mathbb{C}^n$.
Up to an additive constant the final form of $\mathscr{H}$ is to be [cf. (7)]

$$\mathscr{H}=\sum_{i=1}^{n}\hbar\omega_i\{\boldsymbol{\beta}_i^+\boldsymbol{\beta}_i-\boldsymbol{\beta}_i\boldsymbol{\beta}_i^+\}\equiv\mathbf{b}^+\Omega\mathbf{b}. \tag{18}$$

When there is no fear of confusion we absorb $\hbar$ in the frequencies ω_i; so $\Omega=\mathrm{diag}(\omega_1, \dots, \omega_n, -\omega_1, \dots, -\omega_n)$. A moment's reflection on (18) certainly suffices to convince the reader that a *diagonalizing* canonical T obeys the relation

$$T^+\mathbb{D}T=\Omega. \tag{19}$$

Conversely, when we are given a $\mathbb{D}$, we take for T a unitary matrix that diagonalizes $\mathbb{D}$ and we are done. To wit, choose an orthonormal set of n eigenvectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ belonging to n eigenvalues $\{\omega_1, \dots, \omega_n\}$ of $\mathbb{D}$. It is very handy to take all the ω_i's positive [see Lieb, Schultz, Mattis [11] and van Hemmen and Vertogen [12] for the ensuing procedure]; but this is by no means necessary. The proviso is that $\mathbf{x}$ and $J\mathbf{x}$ should not appear together and that they should form an orthonormal set. Then

$$T=\{\mathbf{x}_1, \dots, \mathbf{x}_n, J\mathbf{x}_1, \dots, J\mathbf{x}_n\} \tag{20}$$

by construction diagonalizes $\mathbb{D}$, is unitary and has the required form (17). So we are done.

The importance of the operator J for linear fermion systems has been discussed extensively by van Hemmen [13] though in a slightly different context. Admittedly the procedure presently followed of diagonalizing $\mathbb{D}$ is a waste of time *as long as* no other interactions are involved. One needs correlation functions, *not* an explicit knowledge of T. Efficient procedures for obtaining the correlation functions directly, *without* diagonalizing the hamiltonian, have been given by van Hemmen & Vertogen [14] and van Hemmen [13].

4. Bosons

Let us introduce the matrix $\hat{I}$ that resembles but is not quite the unit matrix in $\mathbb{C}^{2n}$,

$$\hat{I}=\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix};\quad \hat{I}^2=I. \tag{21}$$

$\mathbb{1}$ is the unit matrix in $\mathbb{C}^n$. In terms of $\hat{I}$ we can reformulate the commutation relations (3) as

$$[\mathbf{a}_\mu,\mathbf{a}_\nu^+]=\hat{I}_{\mu\nu},\quad 1\leqq\mu,\ \nu\leqq 2n. \tag{22}$$

The change of sign in $\hat{I}$ when $\mu,\ \nu\geqq n+1$ will slightly complicate the boson argument.
The relation between J and $\hat{I}$ is

$$\{\hat{I},J\}=0, \tag{23}$$

which may be compared with (9). The interplay of J and $\hat{I}$ will be worth noticing.
Again we start by studying the symmetries present in the hamiltonian $\mathscr{H}$. Because of the commutation relations (3) we may suppose

$$D=A^t,\quad B=B^t,\quad C=C^t, \tag{24}$$

while $\mathscr{H}$ being hermitean then implies

$$A^+=A,\quad B^+=C. \tag{25}$$

Thus we may write

$$\mathbb{D}=\begin{pmatrix}A & B\\ \bar{B} & \bar{A}\end{pmatrix}; \tag{26}$$

$\mathbb{D}$ is hermitean. Moreover a simple calculation shows

$$[\mathbb{D},J]=0. \tag{27}$$

This relation will, together with (23), provide the crux of the boson argument. The counterpart of (27) is (13).
Next we derive necessary conditions for a matrix T to be canonical as well as diagonalizing. If T is canonical, then by (22) and (5)

$$\hat{I}_{\mu\nu}=[\mathbf{a}_\mu,\mathbf{a}_\nu^+]=\sum_{\sigma,\tau=1}^{2n} T_{\mu\sigma}\hat{I}_{\sigma\tau}T_{\tau\nu}^+ \tag{28}$$

or

$$\hat{I}=T\hat{I}T^+. \tag{29}$$

Since $\hat{I}^2=I$ we multiply (29) on the left by $\hat{I}$ and get

$$I=(\hat{I}T)(\hat{I}T^+). \tag{30}$$

Ergo $\hat{I}T$ and $\hat{I}T^+$ are each others inverses, and thus

$$I=(\hat{I}T^+)(\hat{I}T) \tag{31}$$

implying

$$\hat{I}=T^+\hat{I}T. \tag{32}$$

A matrix T satisfying (29) or equivalently (32) will be called *pseudo-unitary*. If T is pseudo-unitary, then the inverse transformation T^{-1} exists ($\mathbf{b}=T^{-1}\mathbf{a}$) and is also pseudo-unitary. The proof is easy.
Notice that $I=(\hat{I}T^+\hat{I})\,T$, so

$$T^{-1}=\hat{I}T^+\hat{I}. \tag{33}$$

Then take the inverse of equation (32) and you arrive at

$$\hat{I}=T^{-1}\hat{I}(T^{-1})^+ \tag{34}$$

which is nothing but (29) for T^{-1}. Two remarks are in order. Firstly, here also the existence of T^{-1} follows directly from T being canonical. Secondly, since T in addition should have the form (17) we find, using (33),

$$T=\begin{pmatrix}U & \bar{V}\\ V & \bar{U}\end{pmatrix}\Rightarrow T^{-1}=\begin{pmatrix}U^+ & -V^+\\ -\bar{V}^+ & \bar{U}^+\end{pmatrix}. \tag{35}$$

We conclude that if T is canonical then T^{-1} is also canonical and is explicitly given by (35) – a fairly straightforward procedure.
Up to an additive constant the final form of $\mathscr{H}$ is to be [cf. (7)]

$$\mathscr{H}=\sum_{i=1}^{n}\hbar\omega_i\{\beta_i^+\beta_i+\beta_i\beta_i^+\}\equiv\mathbf{b}^+\Omega\mathbf{b}. \tag{36}$$

Here $\Omega=\mathrm{diag}(\omega_1,\ldots,\omega_n,\omega_1,\ldots,\omega_n)$. Accordingly a *diagonalizing* canonical T obeys, as before in (19), the relation

$$T^+\mathbb{D}T=\Omega. \tag{37}$$

Since (36) may be interpreted as describing n independent harmonic oscillators, we arrive at the *physi-*

cal stability requirement:

$$\omega_i > 0, \quad 1 \leqq i \leqq n. \tag{38}$$

We will discard the case $\omega_i = 0$. And since $\mathbb{D} = T^{-1+}\Omega T^{-1}$ by (37), $\mathbb{D}$ also must be *strictly* positive. A second way of seeing this is to notice that $\mathbb{D}$ is conjunctive to Ω [cf. Ref. 15; T is invertible].

We want to find T and Ω. Our strategy may be described as follows. The canonical transformation T is fully determined by its columns. in particular by its first n columns $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. We, therefore, reduce (37) to an eigenvalue problem for these n vectors $\mathbf{x}_i$, $1 \leqq i \leqq n$. In so doing the use fo J will be to advantage.

Combining (37) and (29) we get

$$(\hat{I}\mathbb{D})\, T = T\hat{I}\Omega, \tag{39}$$

or

$$\hat{I}\mathbb{D}\,\mathbf{x} = \omega \mathbf{x} \tag{40}$$

where $\mathbf{x} = \mathbf{x}_\mu$, $1 \leqq \mu \leqq 2n$, and $\omega \in \{\omega_1, \ldots, \omega_n, -\omega_1, \ldots, -\omega_n\}$; notice the symmetric distribution of the ω's around zero. Whenever $\mathbf{x}_\mu$ is an eigenvector of $\hat{I}\mathbb{D}$ with eigenvalue ω, $J\mathbf{x}_\mu$ is another eigenvector of $\hat{I}\mathbb{D}$ with eigenvalue $-\omega$. This follows from (40) and (17). Once we know, say, the first n eigenvectors $(\mathbf{x}_i;\ 1 \leqq i \leqq n\}$ of $\hat{I}\mathbb{D}$, we know *all* the eigenvectors fo $\hat{I}\mathbb{D}$. The pseudo-unitary of T, i.e. (32), may be rewritten

$$(\mathbf{x}_\mu, \hat{I}\mathbf{x}_\nu) = \hat{I}_{\mu\nu} \tag{41}$$

where $(\mathbf{x}, \mathbf{y}) = \sum \bar{x}_n y_n$ is the usual inner product in $\mathbb{C}^{2n}$. The conditions (40) and (41) are necessary *and sufficient* for T to be canonical as well as diagonalizing.

If there is a canonical and diagonalizing T, stability requires that $\mathbb{D}$ be strictly positive. Conversely, when we are given a strictly positive $\mathbb{D}$, that in addition has the form (26), we have to show the existence of a canonical diagonalizing T. As in the fermion case we exhibit T via an explicit construction. For convenience we divide the proof into four steps.

(I) Consider the eigenvalue problem for the matrix $\hat{I}\mathbb{D}$. Let $\mathbf{x}$ be an eigenvector,

$$\hat{I}\mathbb{D}\,\mathbf{x} = \omega \mathbf{x}. \tag{42}$$

Equivalently,

$$\mathbb{D}^{\frac{1}{2}}\hat{I}\mathbb{D}^{\frac{1}{2}}\mathbf{y} = \omega \mathbf{y}, \quad \mathbf{y} = \mathbb{D}^{\frac{1}{2}}\mathbf{x}. \tag{43}$$

So all the eigenvalues fo $\hat{I}\mathbb{D}$ must be *real*. [If $\mathbb{D}$ itself is a real matrix, the eigenvectors $\mathbf{x}$ may be supposed to be real as well. We then may assume we are working in $\mathbb{R}^{2n}$, instead of $\mathbb{C}^{2n}$]. Moreover $\hat{I}\mathbb{D}$ must always be *diagonable*, i.e. have $2n$ independent eigenvectors (thus no nontrivial Jordan blocks), since $\mathbb{D}^{\frac{1}{2}}\hat{I}\mathbb{D}^{\frac{1}{2}}$ is diagonable.

Let us bring in J. As a consequence of (27) and (23) we then find

$$\hat{I}\mathbb{D}\,\mathbf{x} = \omega \mathbf{x} \Rightarrow \hat{I}\mathbb{D}(J\mathbf{x}) = -\omega(J\mathbf{x}). \tag{44}$$

That is to say, whenever ω is an eigenvalue of $\hat{I}\mathbb{D}$ with eigenvector $\mathbf{x}$, $-\omega$ is another eigenvalue of $\hat{I}\mathbb{D}$ with eigenvector $J\mathbf{x}$. Given a quadratic boson hamiltonian with associated positive $\mathbb{D}$, the eigenvalues of $\hat{I}\mathbb{D}$ *always* lie symmetrically with respect to the origin.

(II) By the previous argument $\hat{I}\mathbb{D}$ has n positive eigenvalues $\{\omega_i;\ 1 \leqq i \leqq n\}$, which we suppose to be simple for the moment, with eigenvectors $\{\mathbf{x}_i;\ 1 \leqq i \leqq n\}$. Form the matrix

$$T = \{\mathbf{x}_1, \ldots, \mathbf{x}_n, J\mathbf{x}_1, \ldots, J\mathbf{x}_n\}. \tag{45}$$

Plainly T has the required form (17). Moreover T contains all the $2n$ eigenvectors of $\hat{I}\mathbb{D}$; so (40) holds. But we still have to show that T is pseudo-unitary. Thereto we introduce the following definition [16] which is a natural generalization of the usual inner product notion.

Definition. We call $\langle \cdot, \cdot \rangle$ an *inner product* in $\mathbb{C}^N$ if $\langle \cdot, \cdot \rangle$ is (a) linear in the second variable, (b) symmetric: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$, and (c) nondegenerate: for each $\mathbf{x}$ we can find a $\mathbf{y}$ such that $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$. □

According to this definition (cf. also (41))

$$\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{x}, \hat{I}\mathbf{y}) \tag{46}$$

is an inner product. We assert that eigenvectors of $\hat{I}\mathbb{D}$ belonging to different eigenvalues are orthogonal with respect to the inner product (46); cf. Thouless [6] and Tikochinsky [3]. Recalling that $\mathbb{D}$ is hermitean, $\hat{I}^+ = \hat{I}$, and $\hat{I}^2 = I$, we find

$$\begin{aligned} \omega_i \langle \mathbf{x}_i, \mathbf{x}_j \rangle &= (\hat{I}\mathbb{D}\,\mathbf{x}_i, \hat{I}\mathbf{x}_j) = (\mathbf{x}_i, \mathbb{D}\,\mathbf{x}_j) \\ &= (\mathbf{x}_i, \hat{I}(\hat{I}\mathbb{D})\,\mathbf{x}_j) = \omega_j \langle \mathbf{x}_i, \mathbf{x}_j \rangle. \end{aligned} \tag{47}$$

And the assertion $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0$ follows.

(III) Next,

$$(\mathbf{x}_i, \hat{I}\mathbf{x}_i) > 0 \ (<0) \quad \text{if } \omega_i > 0 \ (\omega_i < 0). \tag{48}$$

Recalling that $\mathbb{D} > 0$ (here it is essential $\mathbb{D} > 0$!), we find

$$\omega_i (\mathbf{x}_i, \hat{I}\mathbf{x}_i) = (\hat{I}\mathbb{D}\,\mathbf{x}_i, \hat{I}\mathbf{x}_i) = (\mathbf{x}_i, \mathbb{D}\,\mathbf{x}_i) > 0. \tag{49}$$

We notice that by (49) we are free to scale the $\mathbf{x}_i$, $1 \leqq i \leqq n$, in such a way that $(\mathbf{x}_i, \hat{I}\mathbf{x}_i) = 1$. Then (41) holds, provided

$$(J\mathbf{x}_i, \hat{I}J\mathbf{x}_i) = -1, \quad 1 \leqq i \leqq n. \tag{50}$$

But this follows from (23).

(IV) Finally we eliminate the restriction that the eigenvalues of $\hat{I}\mathbb{D}$ be nondegenerate. If $\hat{I}\mathbb{D}$ were a hermitean matrix we would simply orthonormalize the eigenvectors in the eigenspace belonging to a degenerate ω, *i.e.* use the Gram-Schmidt procedure. But in the present case Gram-Schmidt works equally well [16]. Thus we are done.

5. Discussion

Whereas the procedure of diagonalizing a quadratic hamiltonian is straightforward (mathematics) in the fermion case, it remains quite remarkable that the physics and not the mere mathematics of the boson problem enables us to do the diagonalization in this case also.

The relevance of the indefinite metric (46) and $\mathbb{D}$ being strictly positive was first noticed by Thouless [6], though in a rather different context. His work was clarified in an elegant note by Chi [5]; there one can also find a discussion, though not exhaustive, of the case where $\mathbb{D}$ is not strictly positive. Both papers are concerned with the matrix problem (40) proper. An explicit but rather lengthy procedure to diagonalize a quadratic boson hamiltonian was given recently by Avery [4]. His results ar are more or less covered by Colpa [8]. The advantage of the present formulation is that its basic arguments are deduced directly from the symmetries which are to be found in the hamiltonian. These symmetries are closely related to the (anti) commutation relations (3) and (4), so to the two different kinds of statistics which apparently exist in nature. The transformation J enables us to give a unified treatment of both statistics. Let us now discuss, in some more detail, the results we have gained.

The rôle of J is indeed prominent. We begin with noticing some striking differences between the two cases: for fermions

$$[I, J]=0 \quad \text{and} \quad \{\mathbb{D}, J\}=0, \tag{51}$$

whereas for bosons

$$\{\hat{I}, J\}=0 \quad \text{and} \quad [\mathbb{D}, J]=0. \tag{52}$$

However, in the relevant eigenvalue problems we get for fermions

$$\mathbb{D}\mathbf{x}=\omega\mathbf{x} \Rightarrow \mathbb{D}(J\mathbf{x})=-\omega(J\mathbf{x}), \tag{53}$$

while for bosons

$$\hat{I}\mathbb{D}\mathbf{x}=\omega\mathbf{x} \Rightarrow \hat{I}\mathbb{D}(J\mathbf{x})=-\omega(J\mathbf{x}). \tag{54}$$

So here the rôle of J is similar. And in both cases

$$T=\left(\begin{matrix}U\\V\end{matrix}\; J\begin{pmatrix}U\\V\end{pmatrix}\right)=\{\mathbf{x}_1, \ldots, \mathbf{x}_n, J\mathbf{x}_1, \ldots, J\mathbf{x}_n\} \tag{55}$$

with the *first* n column vectors corresponding to the eigenvalue problems (53) and (54). But stability requires that the n eigenvalues in (54) be strictly positive, whereas in the fermion case we may choose them at will. The resulting canonical transformation T is *unique* up to a permutation of the first n columns in the boson case and all columns in the fermion case. Finally if $\mathbb{D}$ is a *real* matrix then T is also real; cf. the remark after (43). This will be used in an essential way in [19].

It is plain that the boson procedure we sketched in Sect. 4 works equally well when $\mathbb{D}$ is strictly negative; a $\mathbb{D}<0$ is not very useful though, because then (36) does not satisfy the stability requirement $\omega_0>0$. More precisely, since T must be canonical, it is always invertible (if it exists) and accordingly Ω has as many positive eigenvalues and as many negative eigenvalues as $\mathbb{D}$, the latter being conjunctive to Ω [15]. So the next question is a very natural one: what happens when $\mathbb{D}$ has positive *and* negative eigenvalues? Clearly, if there were a canonical diagonalizing T, then Ω would have as many negative eigenvalues as $\mathbb{D}$ and this had to be rejected on physical grounds. But in answering the above question we will see that the general result obtained in Sect. 4 is in fact optimal: if $\mathbb{D}$ is not strictly positive, it may happen that the eigenvalues of $\hat{I}\mathbb{D}$ are complex, or that we do not even have enough eigenvectors to build the matrix T; cf. (20) and (45).

Let us consider the simplest of all possible examples, the case $n=1$. When $n=1$, $A=\alpha$ is a real number, $B=\beta$ is an arbitrary complex number, and

$$\mathbb{D}=\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \alpha\end{pmatrix}. \tag{56}$$

We directly solve the associated eigenvalue problem (40)

$$\hat{I}\mathbb{D}\mathbf{x}=\omega\mathbf{x} \Rightarrow \det\begin{pmatrix}\alpha-\omega & \bar{\beta}\\ -\beta & -\alpha-\omega\end{pmatrix}=0, \tag{57}$$

and obtain

$$\omega_{\pm}=\pm\sqrt{\alpha^2-|\beta|^2}. \tag{58}$$

The ω's have the expected symmetry. They are *real* if and only if

$$\alpha^2-|\beta|^2 \geqq 0 \Leftrightarrow \alpha \geqq |\beta| \quad \text{or} \quad \alpha \leqq -|\beta|. \tag{59}$$

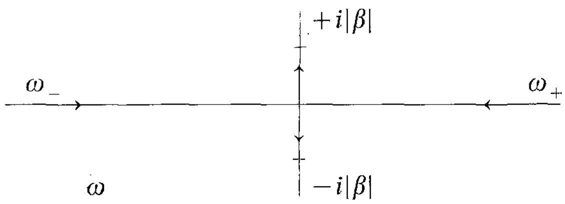
Next we turn to $\mathbb{D}$ itself and calculate its eigenvalues through

$$\det\begin{pmatrix}\alpha-\lambda & \bar{\beta}\\ \beta & \alpha-\lambda\end{pmatrix}=0, \tag{60}$$

obtaining

$$\lambda_{\pm}=\alpha\pm|\beta|. \tag{61}$$

Now $\mathbb{D}\geqq 0$ if and only if $\lambda_{\pm}\geqq 0$ (or $\mathbb{D}\leqq 0$ if and only if $\lambda_{\pm}\leqq 0$). Thus the ω's are *real* if and only if $\mathbb{D}$ is positive (or negative). When, fixing β, we let α go from $+\infty$ to 0, we get the following picture in the ω-plane,



$\alpha=0$ implies $\omega_{\pm}=\pm i|\beta|$. When α goes from 0 to $-\infty$ we only have to reverse the arrows. But purely imaginary ω's are of no use whatsoever. Moreover their mere existence already contradicts (37); so a canonical diagonal T is not to be found. The case $\omega_{\pm}=0$ is of some interest too; it appears that now $\hat{I}\mathbb{D}$ has only one eigenvector, whereas we need two [cf. (40)]. Thus we see that the diagonalization is meaningful *if and only if* $\mathbb{D}>0$, as advertised. This conclusion confirms and substantiates earlier calculations by Thouless [17] and Tikochinsky [18].
In general, if ω is an eigenvalue of $\hat{I}\mathbb{D}$ then $-\bar{\omega}$ is an eigenvalue as well,

$$\hat{I}\mathbb{D}\,\mathbf{x}=\omega\mathbf{x}\Rightarrow\hat{I}\mathbb{D}(J\mathbf{x})=-\bar{\omega}(J\mathbf{x}); \tag{62}$$

i.e., the imaginary axis is a mirror plane for the eigenvalues of $\hat{I}\mathbb{D}$. It is still an open problem however, whether $\hat{I}\mathbb{D}$ inherits much more from $\mathbb{D}$ than the symmetry of its eigenvalues (apparently a recessive character).
There exists a direct relation between the diagonalization procedure for a quadratic boson hamiltonian $\mathscr{H}$ and the dynamics $\mathscr{H}$ generates. To see this we define

$$B(f\oplus g)=a^{+}(f)+a(g) \tag{63}$$

where (notice the linearity in f *and* g)

$$a^{+}(f)=\sum_x f(x)\,\boldsymbol{\alpha}_x^{+},\qquad a(g)=\sum_x g(x)\,\boldsymbol{\alpha}_x, \tag{64}$$

and study the dynamical evolution of $B(f\oplus g)$,

$$\alpha_t[B(f\oplus g)]=e^{i\mathscr{H}t}B(f\oplus g)\,e^{-i\mathscr{H}t}. \tag{65}$$

Let $\mathscr{H}_{\text{new}}:=\frac{1}{2}\mathscr{H}_{\text{old}}$, that is, add a multiplicative factor $\frac{1}{2}$ to the right-hand side of (1). Then

$$\alpha_t[B(f\oplus g)]=B(\exp\{it\mathbb{D}\hat{I}\}\,f\oplus g). \tag{66}$$

Since $\mathbb{D}\hat{I}$ is unitarily equivalent to $\hat{I}\mathbb{D}$, we have found the relation we alluded to above: for an explicit evaluation of $\exp\{it\mathbb{D}\hat{I}\}$ in (66) one needs all the eigenvalues and eigenvectors of $\mathbb{D}\hat{I}$, or equivalently, of $\hat{I}\mathbb{D}$. Thus we have come back to (40). Equation (63) is to be compared with relation (2.1.3) of Ref. 13.
Finally: how useful is this diagonalization procedure? As remarked earlier, the diagonalization of a quadratic fermion hamiltonian is a waste of time *as long as* no other interactions are involved. For bosons, however, the situation is slightly different. The main point is that the operators $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^{+}$ are unbounded, in this way giving rise to some technical problems which have to handled properly. Clearly a diagonal hamiltonian is much more manageable, especially when interactions other than quadratic are involved and no easy method for finding the time evolution of the combined system (e.g. for a maser: the system of atoms and field) is available. Furthermore, since for *finitely* many boson modes we may nearly always discard domain problems, the only thing we have to care about in diagonalizing $\mathscr{H}$ is the collection of commutation relations (3), interpreted in a purely *algebraic* sense (as we did). We, therefore, conclude that the proposed diagonalization procedure, transforming boson operators into boson operators, is technically sound.

6. Conclusion

In this paper we have studied the diagonalization problem for quadratic boson and fermion hamiltonians of systems with finitely many degrees of freedom. Using the fact that only finitely many degrees of freedom were involved and exploiting the symmetries which are to be found in the boson and fermion hamiltonians we could give a simple, purely algebraic proof of the expected diagonability. Already within this context it appeared that in the boson case the matrix $\mathbb{D}$ which is defined by the hamiltonian (2), had to be (strictly) positive. Though on a formal level (Sect. 5) some striking similarities between both diagonalization procedures could be pointed out, one has to keep in mind that the practical work involved is rather different.

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