Our Hamiltonian:

$$H = \varepsilon b^{+}b - v\left(b^{+}b^{+} + bb\right) \qquad \varepsilon, v \in \mathbb{R}$$

Re-write H as:

$$H = \varepsilon b^{+}b + \frac{1}{2}\gamma b^{+}b^{+} + \frac{1}{2}\gamma bb, \qquad v = -\frac{1}{2}\gamma$$

$\Rightarrow$ Matrix coefficients: (1×1) matrices:

$\Rightarrow$ Dynamic matrix: (2×2) matrix:

$$D = \begin{pmatrix} \varepsilon & \gamma \\ -\gamma & -\varepsilon \end{pmatrix}, \quad \text{if } v \in \mathbb{R} \Rightarrow \gamma^{*} = \gamma$$

Eigenvalue equation:

$$\begin{bmatrix} \varepsilon & \gamma \\ -\gamma & -\varepsilon \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\omega}_{\hookrightarrow \text{ eigenvalues}} \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{\hookrightarrow\ w\text{-eigenvector}}$$

Obtain the eigenvalues of H from the characteristic equation

$$\det\left(\omega I - D\right) = 0 \qquad (A)$$

From the eigenvalue problem $\omega\varphi = D\varphi$

$$(A): \quad \omega^2 - \varepsilon^2 + \gamma^2 = 0 \qquad (B) \quad \rightarrow \text{Characteristic equation}$$

Obtain the following eigenvalues:

$$\omega: \quad \begin{cases} \text{i)} \quad \omega_{1,2} = \pm\sqrt{\varepsilon^2 - \gamma^2}, & |\varepsilon| > |\gamma| \\ \text{ii)} \quad \omega_0 = 0 & \varepsilon = \gamma \\ \text{iii)} \quad \omega_{3,4} = \pm i\sqrt{\gamma^2 - \varepsilon^2}, & |\varepsilon| < |\gamma| \end{cases}$$

Real eigenvalues $\iff |\varepsilon| > |\gamma|$

Remind that $v = -\frac{1}{2}\gamma \Rightarrow \gamma = -2v$

$|\gamma| = 2v$

$$\Rightarrow \text{if } \quad |\varepsilon| > 2v \Rightarrow \begin{cases} \omega_1 = \sqrt{\varepsilon^2 - 4v^2} \\ \omega_2 = -\sqrt{\varepsilon^2 - 4v^2} \end{cases}$$

eigenvalues (Real) of $\hat{H}$

$\Rightarrow$ By getting the eigenvalues of the 2x2 dynamical matrix associated to this system, when the condition

$$|\varepsilon| > |\gamma|$$
$$|\varepsilon| > 2v$$

is true, then the system is diagonalizable, with its real eigenvalues given by (ω_1, ω_2)