Our Hamiltonian:

$$H = \varepsilon b^{+} b - v\left(b^{+} b^{+} + b b\right) \qquad \varepsilon, v \in \mathbb{R}$$

Re-write H as:

$$H = \varepsilon b^{+} b + \frac{1}{2} \gamma b^{+} b^{+} + \frac{1}{2} \gamma b b, \qquad v = -\frac{1}{2}\gamma$$

$\Rightarrow$ Matrix coefficients: (1×1) matrices:

$\Rightarrow$ Dynamic matrix: (2×2) matrix:

$$D = \begin{pmatrix} \varepsilon & \gamma \\ -\gamma & -\varepsilon \end{pmatrix}, \quad \text{if } v \in \mathbb{R} \Rightarrow \gamma^{*} = \gamma$$

Eigenvalue equation:

$$\begin{bmatrix} \varepsilon & \gamma \\ -\gamma & -\varepsilon \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \omega \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$\hookrightarrow$ w - eigenvector

$\hookrightarrow$ eigenvalues

Obtain the eigenvalues of H from the characteristic equation

$$\det\left(\omega I - D\right) = 0 \qquad (A)$$

From the eigenvalue problem $\omega \varphi = D \varphi$

$$(A): \quad \omega^2 - \varepsilon^2 + \gamma^2 = 0 \qquad (B)$$

$\rightarrow$ Characteristic equation

Obtain the following eigenvalues:

ω : i) $\omega_{1,2} = \pm\sqrt{\varepsilon^2 - \gamma^2}$, $|\varepsilon| > |\gamma|$

ii) $\omega_0 = 0$ $\varepsilon = \gamma$

iii) $\omega_{3,4} = \pm i\sqrt{\gamma^2 - \varepsilon^2}$, $|\varepsilon| < |\gamma|$

Real eigenvalues $\Leftrightarrow$ $|\varepsilon| > |\gamma|$

Remind that $v = -\frac{1}{2}\gamma \Rightarrow \gamma = -2v$

$|\gamma| = 2v$

$\Rightarrow$ if $|\varepsilon| > 2v \Rightarrow \begin{cases} \omega_1 = \sqrt{\varepsilon^2 - 4v^2} \\ \omega_2 = -\sqrt{\varepsilon^2 - 4v^2} \end{cases}$

eigenvalues (Real) of $\hat{H}$

$\Rightarrow$ By getting the eigenvalues of the 2x2 dynamical matrix associated to this system, when the condition

$|\varepsilon| > |\gamma|$

$|\varepsilon| > 2v$

is true, then the system is diagonalizable, with its real eigenvalues given by (ω_1, ω_2)

Discussions:

1. $|\varepsilon| < |\gamma|$ $\to$ the Dynamic matrix has two imaginary eigenvalues $\Rightarrow$ H is not B.V. diagonalizable

2. $|\varepsilon| = |\gamma|$ $\to$ the Dynamic matrix is not diagonalizable
$\Downarrow$
H also not B.V. diagonalizable

$\Rightarrow$ H is BV diagonalizable $\Leftrightarrow$ $|\varepsilon| > |\gamma|$

Set $\omega = \sqrt{\varepsilon^2 - \gamma^2}$
$\downarrow$
eigenvector: $x(\omega) = \begin{cases} N^+ \begin{bmatrix} \gamma \\ \omega - \varepsilon \end{bmatrix}, & \varepsilon > 0 \\ N^- \begin{bmatrix} \gamma \\ \omega - \varepsilon \end{bmatrix}, & \varepsilon < 0 \end{cases}$

$N^+ = \dfrac{1}{\sqrt{2\omega(\varepsilon - \omega)}}$ $\quad N^- = \dfrac{1}{\sqrt{2\omega(\omega - \varepsilon)}}$

Norm: $x^*(\omega) I_- x(\omega) = \begin{cases} 1, & \varepsilon > 0 \\ -1, & \varepsilon < 0 \end{cases}$

$I_- = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, I: identity matrix of arbitrary size n

T_u: matrix containing all eigenvectors for Δ.

$T_u = \begin{cases} N^+ \begin{bmatrix} \gamma & \omega - \varepsilon \\ \omega - \varepsilon & \gamma \end{bmatrix}, & \varepsilon > 0 \end{cases}$

$$\left\{ N^{-} \begin{bmatrix} \omega-\varepsilon & \gamma \\ \gamma & \omega-\varepsilon \end{bmatrix}, \quad \varepsilon \geq 0 \right.$$

T_n is the „derivative" matrix of the „derivative" linear transformation which corresponds to the dynamical matrix D:

$$T_n^{-1} \cdot D \, T_n = \begin{cases} \begin{bmatrix} \omega & 0 \\ 0 & -\omega \end{bmatrix}, & \varepsilon > 0 \\[2ex] \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix}, & \varepsilon < 0 \end{cases}$$