# Bogoliubov transformations

$$H = E_0 + |J| Sz \sum_k \underbrace{c_k^+ c_k + d_k^+ d_k}_{\text{Independants sub-lattices}} + \gamma_k \underbrace{(c_k d_{-k} + c_{-k}^+ d_k^+)}_{\text{Coupling between sub-lattices}}$$

**diagonalization: Bogoliubov transform (see BCS theory)**

$$c_k = u_k \alpha_k + v_k \beta_{-k}^+$$

$$d_k = u_k \beta_k + v_k \alpha_{-k}^+$$

New bosonic operators with $u_k$, $v_k$ real

and: $\left[\alpha, \alpha^+\right] = \left[\beta, \beta^+\right] = 1$

$\left[\alpha, \beta\right] = 0$

**Conditions on $u_k$ $v_k$**

(1) $\left[c_k, c_k^+\right] = 1 \Leftrightarrow \left[u_k \alpha_k + v_k \beta_{-k}^+, u_k \alpha_k^+ + v_k \beta_{-k}\right] = u_k^2 \left[\alpha_k, \alpha_k^+\right] + v_k^2 \left[\beta_{-k}^+, \beta_{-k}\right] = 1$

$$\Leftrightarrow \boxed{u_k^2 - v_k^2 = 1}$$

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**<u>Conditions on $u_k$ $v_k$</u>**

(2) Eliminate cross terms like ($\alpha\beta$) to diagonalize H:

Ex: terms in

$$\left.\begin{array}{l} \alpha_k^+ \beta_{-k}^+ \Rightarrow u_k v_k + \gamma_k v_k^2 \\ \alpha_{-k}^+ \beta_k^+ \Rightarrow u_k v_k + \gamma_k u_k^2 \end{array}\right] \Rightarrow \boxed{2u_k v_k + \gamma_k (u_k^2 + v_k^2) = 0}$$

same condition for hermitic conjugates $\alpha_k \beta_{-k}$

# Bogoliubov transformations

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**diagonalization: Bogoliubov transform (see BCS)**

$$c_k = u_k \alpha_k + v_k \beta_{-k}^+ \qquad (1) \quad \boxed{u_k^2 - v_k^2 = 1}$$

$$d_k = u_k \beta_k + v_k \alpha_{-k}^+ \qquad (2) \quad \boxed{2u_k v_k + \gamma_k (u_k^2 + v_k^2) = 0}$$

Off diagonal terms are zero with these 2 conditions

**Diagonal terms in H:**

$$\left.\begin{array}{l} \alpha_k^+ \alpha_k \Rightarrow u_k^2 + \gamma_k u_k v_k \\ \alpha_k \alpha_k^+ \Rightarrow v_k^2 + \gamma_k u_k v_k \end{array}\right] \Rightarrow \alpha_k^+ \alpha_k (u_k^2 + v_k^2 + 2\gamma_k u_k v_k) + v_k^2 + \gamma_k u_k v_k$$

Idem for $\beta_k^+ \beta_k \Rightarrow (\alpha_k^+ \alpha_k + \beta_k^+ \beta_k)(u_k^2 + v_k^2 + 2\gamma_k u_k v_k) + \underbrace{2v_k^2}_{u_k^2 - 1 + v_k^2} + 2\gamma_k u_k v_k$

$$= (1 + \alpha_k^+ \alpha_k + \beta_k^+ \beta_k)(u_k^2 + v_k^2 + 2\gamma_k u_k v_k) - 1$$

# Diagonalized AF Hamiltonian

$$H = E_0 - \frac{N|J|Sz}{2} + \sum_k \omega_k (1 + \alpha_k^+ \alpha_k + \beta_k^+ \beta_k)$$

$E_0$: energy of classical Néél state

$$\omega_k = |J|Sz(u_k^2 + v_k^2 + 2\gamma_k u_k v_k) = |J|Sz\left(\cosh(2\theta_k) + \gamma_k \sinh(2\theta_k)\right)$$

$$\boxed{u_k^2 - v_k^2 = 1} \qquad \boxed{2u_k v_k + \gamma_k (u_k^2 + v_k^2) = 0}$$

**<u>Expression of magnons energy as a function of $\gamma_k$</u>**

we set: $u_k = \cosh(\theta_k)$ ➡ $\sinh(2\theta_k) + \gamma_k \cosh(2\theta_k) = 0$

$v_k = \sinh(\theta_k)$ $\qquad \sinh(2\theta_k) + \gamma_k \sqrt{1 - \sinh^2(2\theta_k)} = 0$

➡ $$\sinh(2\theta_k) = -\frac{\gamma_k}{\sqrt{1-\gamma_k^2}}$$

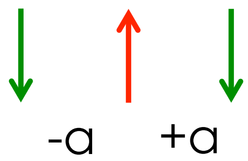
$$\cosh(2\theta_k) = \frac{1}{\sqrt{1-\gamma_k^2}}$$

➡ $$\boxed{\omega_k = |J|Sz\sqrt{1-\gamma_k^2}}$$

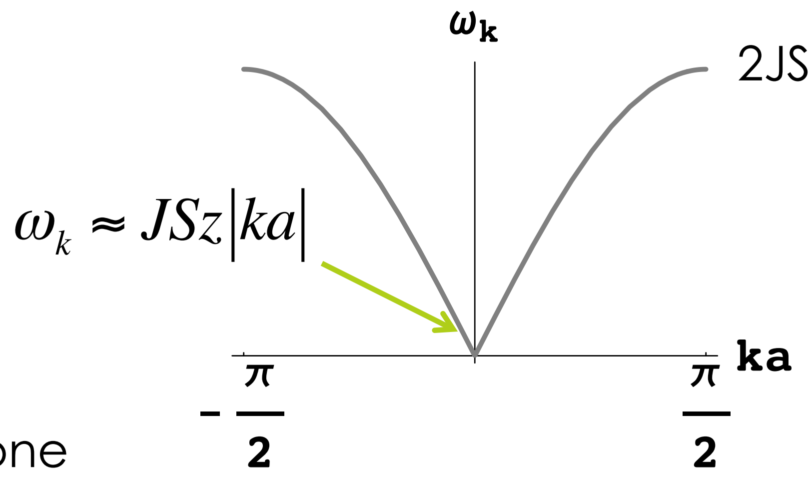
# Dispersion relation of AF magnons

AF magnons: 2 degenerate modes $\boxed{\omega_k = JSz\sqrt{1-\gamma_k^2}}$

in 1D: 2 neighbors $\gamma_k = \frac{1}{z}\sum_\delta e^{i\vec{k}.\vec{\delta}} = \frac{1}{2}\left[\cos(ka)+\cos(-ka)\right] = \cos(ka)$




$$\Rightarrow \omega_k = JSz\sqrt{1-\gamma_k^2} = JSz\left|\sin(ka)\right|$$




Periodicity $k_{BZ}=\pi/a$

Magnetic Brillouin zone ≠ cristal Brillouin zone

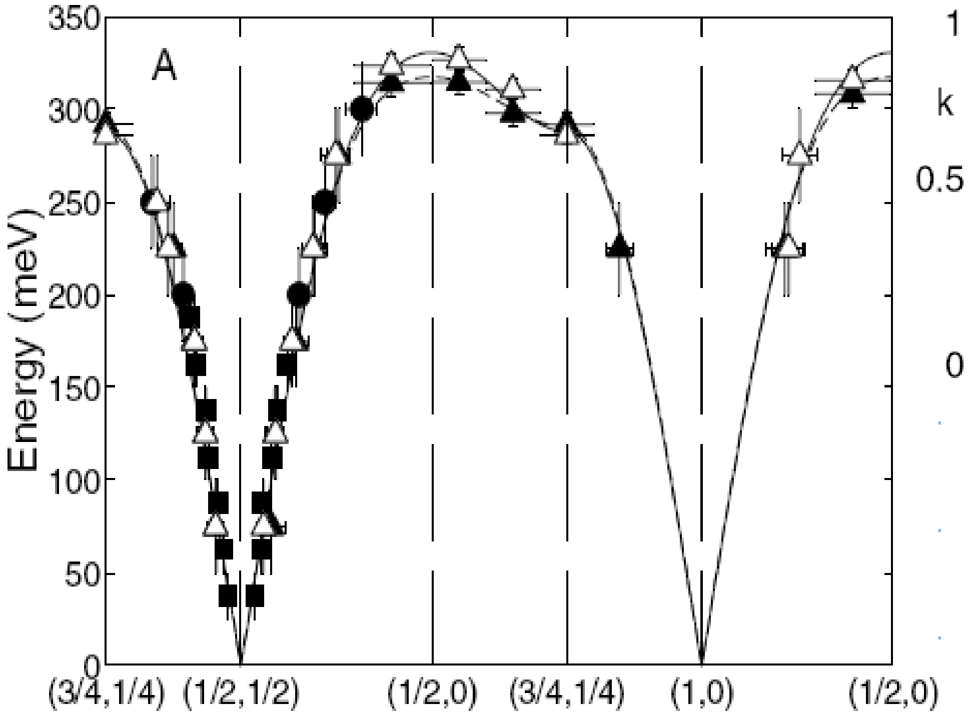
Linear dispersion at low energy / wave-vector (ferro: quadratic)

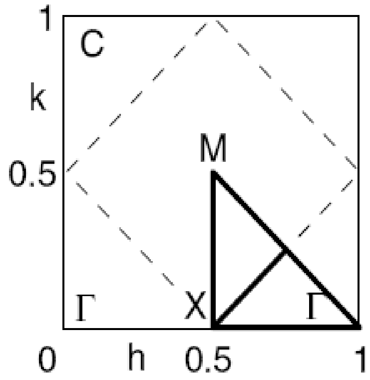
# Dispersion relation of AF magnons

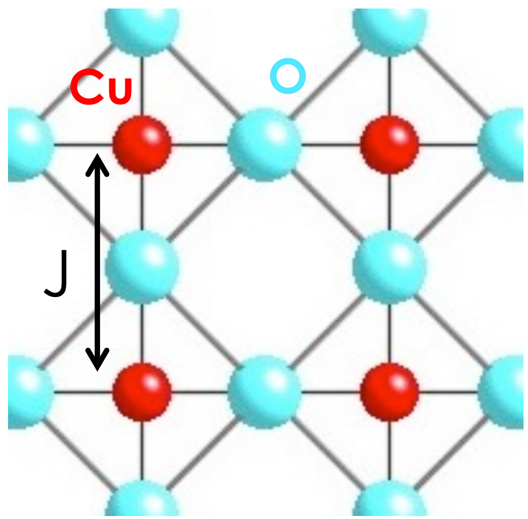
$La_{2-x}Sr_xCuO_4$: high Tc superconductor

**$La_2CuO_4$** 2D square (z=4)

$$\omega_k = 4JS\sqrt{1-\frac{1}{4}\left[\cos(k_x a)+\cos(k_y a)\right]^2}$$




First nearest neighbors only !

$CuO_2$ plane
($Cu^{2+}$: S=1/2)

# AF ground state in the harmonic approximation

$$H = E_0 - \frac{NJSz}{2} + JSz\sum_k \sqrt{1-\gamma_k^2}\,(1+\alpha_k^+\alpha_k + \beta_k^+\beta_k)$$

**Ground state (harmonic approx.)** $\langle \alpha_k^+\alpha_k \rangle = \langle \beta_k^+\beta_k \rangle = 0$ Be careful: $\langle c_k^+ c_k \rangle = \langle d_k^+ d_k \rangle \neq 0$

the gound state is not Néél state anymore !

$$E_{AF}^{harm} = \langle H \rangle = E_0 - \frac{NJSz}{2} + JSz\sum_k \sqrt{1-\gamma_k^2} = E_0 - JSz\Bigg[\underbrace{\frac{N}{2} - \sum_k \sqrt{1-\gamma_k^2}}_{>0}\Bigg] < E_0$$

$$\gamma_k = \frac{1}{z}\sum_\delta e^{i\vec{k}.\vec{\delta}} < 1 \iff \sqrt{1-\gamma_k^2} \le 1$$

**Ground state is Néél state renormalized by fluctuations**

$$E_0 = -\frac{NJS^2z}{2} \Rightarrow \boxed{E_{AF}^{harm} = -\frac{NJS^2z}{2}\Bigg[1 + \underbrace{\frac{2}{NS}\sum_k (1-\sqrt{1-\gamma_k^2})}_{}\Bigg]}$$

**Correction in 1/S**
**Recover Néél state for large S**

# AF ground state in the harmonic approximation

$$\boxed{E_{AF}^{harm} = -\frac{NJS^2 z}{2}\left[1 + \frac{2}{NS}\sum_k \left(1 - \sqrt{1-\gamma_k^2}\right)\right]} \qquad \gamma_k = \frac{1}{z}\sum_\delta e^{i\vec{k}.\vec{\delta}}$$

Can be computed numerically for different lattices

$$E_{AF}^{harm} = -\frac{NJS^2 z}{2}\left[1 + \frac{0.363}{S}\right]$$ for chain (1D)

$$E_{AF}^{harm} = -\frac{NJS^2 z}{2}\left[1 + \frac{0.158}{S}\right]$$ For square (2D)

$$E_{AF}^{harm} = -\frac{NJS^2 z}{2}\left[1 + \frac{0.097}{S}\right]$$ for cube (3D)