GEORGE WASHINGTON UNIVERSITY, DEPARTMENT OF MATHEMATICS

GRAPH THEORY, MATH 3632

Problem Set I (Revisions)

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EXERCISE II

For every k-regular graph is there a $(k+1)$-regular graph that contains the k-regular graph as a subgraph?

There is. Let $G = (V, E, \phi)$ be any k-regular graph. Let us construct $G_a = (V_a, E_a, \phi_a)$ by relabeling G such that $V(G) \cap V(G_a) = \emptyset$, $E(G) \cap E(G_a) = \emptyset$, and $\phi(G) \cap \phi(G_a) = \emptyset$

Because G_a is a relabeling of G, it is an isomorphism. This implies that there exists a bijection $f : V(G) \to V(G_a)$ which satisfies all of the properties of isomorphism.

Let us consider G' which has G and G_a as subgraphs. Now add an edge in G' from v to v' for every $v \in V(G), v' \in V(G')$ where $v' = f(v)$. (f is the isomorphism bijection from above).

The degree of each vertex $v \in V(G')$ is exactly $k+1$. V has k edges because it is in $V(G)$ or $V(G_a)$ which are both k-regular. It has an additional edge from the bijection which creates exactly one edge per $v \in V(G')$. So each vertex has degree $k+1$, so the graph G' is $(k+1)$-regular. So for every k-regular graph, there is a $(k+1)$-regular graph that contains the k-regular graph as a sub graph. $\square$

EXERCISE IV

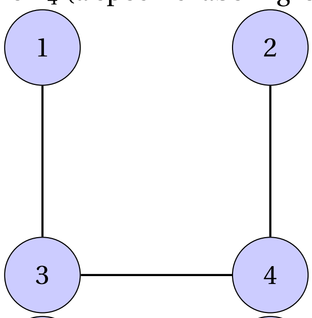
If a simple graph G is isomorphic to its complement $\overline{G}$, then G has either $4k$ or $4k+1$ vertices for some $k \in \mathbb{N}$. Find all simple graphs in on four and five vertices that are isomorphic to their complements.

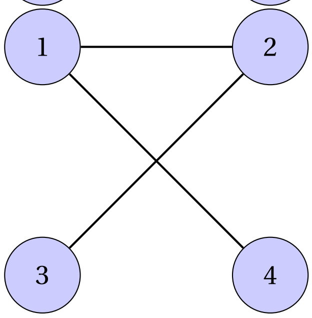
Proof that if a simple graph G is isomorphic to its complement $\overline{G}$, then G has either $4k$ or $4k+1$ vertices for some $k \in \mathbb{N}$.

Given a set of n vertices, there are $\frac{n(n+1)}{2}$ edges that a simple graph can have ($\sum_{i=1}^{i=n} i$). No edge exists in both the graph G and $\overline{G}$, while every edge exists in at least G or $\overline{G}$. Two graphs which are isomorphic will have the same number of edges. So a graph which is isomorphic to its complement will have $\frac{n(n-1)}{4}$ edges.

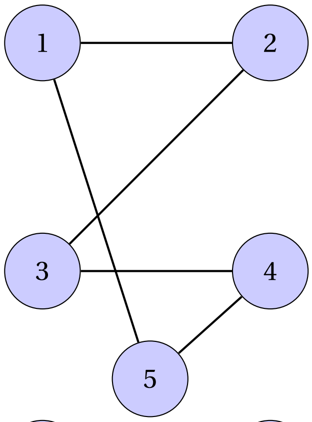
The number of edges must be an integer, so $4|n(n-1)$. This implies that $4|n$ or $4|n-1$. From this, a simple graph which is isomorphic to its complement will have $4k$ or $4k+1$ nodes for some integer k.

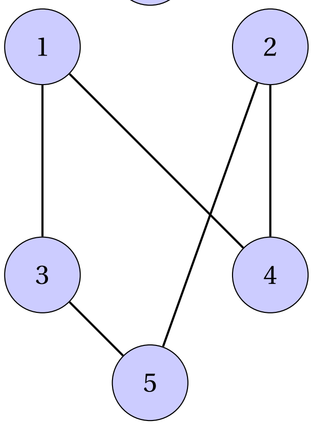
The graphs up to isomorphism on four vertices which is isomorphic to its own complement is P_4 (a specific labeling is shown along with the isomorphic complement):



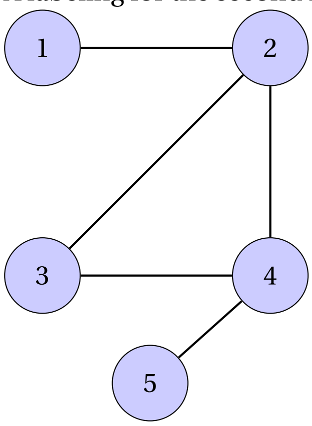


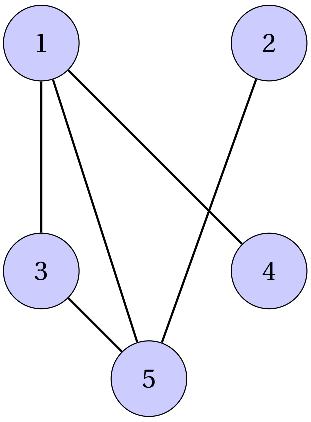
The only two graphs up to isomorphism on five vertexes which are isomorphic to their own complements are: (The first is C_n, a specific labeling is shown along with its complement)





A labeling for the second isomorphism and its isomorphic complement is:





and

EXERCISE V

For which values of n, r is there an r-regular graph on n vertices? An r-regular simple graph? On N vertices, there is an r-regular graph (including multigraphs) if $rn \equiv_2 0$.

if $rn \equiv_2 0$, then at least on of the following is true: r is even or n is even.

In the case where r is even, you can construct an r-regular multigraph on n vertices by putting $r/2$ self loops on each vertex. $2|r$ because r is even, and each self loop increases the degree of the node by 2 so each node will have a degree of 2.

In the case where n is even, you can r-regular multigraph on n vertices by splitting the graph into pairs. For each pair, drawn r edges between the vertices.

On simple graphs, it is also possible to construct an r regular graph on n vertices when $rn \equiv_2 0$, with the additional constraint that $n \geq r+1$.

To construct such a graph, relabel the vertices $\in [n]$. In the case where r is even, draw an edge Each node gets an edge from $r/2$ edges, and has $r/2$ edges drawn from it. So the graph is r-regular.

The construction is similar when n is even and r is odd. Similarly draw the edges from vertex x to each vertex $y \in [n]$ where $x < y \leq y + \lfloor r/2 \rfloor + 1$ where + is modulo-r arithmetic. Because r is odd, we must round down and connect to an additional node. Because it is odd, we will end up drawing the same edge twice. Ignore this duplicated edge which happens if $n = r+1$.

EXERCISE VIII

Suppose that the graph $G - v$ is connected for every vertex v of G. Does it follow that G is connected? (Be comprehensive.)

For a graph G where $|V(G)| = 1$, the graph is vacuously connected.
For a graph G where $|V(G)| = 2$, the graph my be connected or may not. Consider the two ismorphisms on 2 vertices. For any v, a graph of two vertices which are connected will produce $G - v$ which is always connected, and two vertices which are not connected will produce $G - v$ which is also connected.

By contradiction, I will prove that for any graph G where $|V(G)| > 2$, if $\forall v \in V(G), G - v$ is connected, the graph G is connected.

Assume that we have a graph where $|V(G)| > 2$, if $\forall v \in V(G), G - v$ is connected, but G is not connected.
$\implies \exists$distinct $x, y, z \in V(G)$ where there is no path between x and y.

The statement of the problem states that $G - z$ will be connected. If that is true, then there is a path from x to y. By contradiction, any graph G where $|V(G)| > 2$, if $\forall v \in V(G), G - v$ is connected, the graph G is connected.