1 Vector Spaces

Definition 1.1. A binary operation on a set S is a function $f : S \times S \longrightarrow S$.

Definition 1.2. A *field* is a set $\mathbb{F}$ together with two binary operations called multiplication and addition which we denote $\bullet$ and $+$ such that:

1. For each $a, b, c \in \mathbb{F}$, $a + (b + c) = (a + b) + c$
   (a) $a \cdot (b + c) = ab + ac$
   (b) $a + b = b + a$
   (c) $a \cdot b = b \cdot a$
2. There exists elements $0, 1 \in \mathbb{F}$ such that for each $f \in \mathbb{F}$
   (a) $0 + f = f$
   (b) $1 \cdot f = f$
3. For each $a \in \mathbb{F}$, there exists elements $-a, a^{-1} \in \mathbb{F}$ such that
   (a) $a + (-a) = 0$
   (b) $a \cdot a^{-1} = 1$

Example 1.3. A few examples of fields are $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{Z}/p\mathbb{Z}$ for some prime p. All of the above properties are fairly easily verified.

Definition 1.4. A *vector space* V over a field $\mathbb{F}$ is a set V with two operations, which we call vector addition and scalar multiplication. We define these functions as:

$$+ : V \times V \longrightarrow V$$
$$\bullet : \mathbb{F} \times V \longrightarrow V$$

A vector space satisfies the following properties:

1. For each $u, v, w \in V$ and $a, b \in \mathbb{F}$
   (a) $u + v = v + u$
   (b) $(u + v) + w = u + (v + w)$
   (c) $a \cdot (u + v) = a \cdot u + a \cdot v$
   (d) $(a + b) \cdot v = a \cdot v + b \cdot v$
2. There exist elements $\vec{0}, -v \in V$ and $1 \in \mathbb{F}$ such that for each $v \in V$
   (a) $1 \cdot v = v$
   (b) $v + \vec{0} = v$
   (c) $v + (-v) = \vec{0}$

Definition 1.5. We say a *vector* is an element of a vector space.

Example 1.6. The sets $\mathbb{F}^n$ and $\mathbb{F}^\infty$ for some field $\mathbb{F}$ are both vector spaces. One is finite dimensional, while the other has infinite dimension, which we will touch upon in a later section. Explicitly, we have:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_j \in \mathbb{F}\}$$
$$\mathbb{F}^\infty = \{(x_1, x_2, \ldots) \mid x_j \in \mathbb{F}\}$$

We can also have vector over finite fields such as $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Explicitly:

$$(\mathbb{Z}/p\mathbb{Z})^n = \{(a_1, \ldots, a_n) \mid a_n \in \mathbb{Z}/p\mathbb{Z}\}$$

Proposition 1.7. *The additive identity, $\vec{0} \in V$ is unique.*

Proof. Suppose $\vec{0}$ and $\hat{0}$ are both additive identities in V.

$$\vec{0} = \vec{0} + \hat{0} = \hat{0} + \vec{0} = \hat{0}$$

□

Proposition 1.8. *Additive inverses in V are unique.*

Proof. Let $v \in V$ and suppose that w and $\hat{w}$ are both additive inverses of v.

$$w = w + \vec{0} = w + (v + \hat{w}) = (w + v) + \hat{w} = \vec{0} + \vec{w} = \hat{w}$$

□

2 Subspaces and Calculus of the Subspace

Definition 2.1. A *subspace* U of V is a subset of V which is also a vector space with the same operations. We write $U \subseteq V$.

Theorem 2.2. *A subset U of V is a subset of V if and only if U satisfies:*

1. $\vec{0} \in U$
2. *U is closed under addition*
3. *U is closed under scalar multiplication*

Proof. Suppose U is a subspace of V. We know that U is a vector space, so it satisfies all three properties automatically. Suppose then that U is a subset satisfying the above properties. Property 1 assures us that we have an additive identity, while 2 and 3 assure us that vector addition and scalar multiplication are defined on U. So, if $u \in U$, then $(-1)u \in U$ by property 3. Hence, each element in U has an additive inverse. Commutativity, associativity, multiplicative inverses, and the distributive law are all satisfied as U is a subset of V. Thus, U is a subspace of V. □

Definition 2.3. Suppose $U_1, \ldots, U_n$ are subsets of V. The *sum* of $U_1, \ldots, U_n$, denoted $U_1 + \ldots + U_n$ is the set of all possible sums of elements from $U_1, \ldots, U_n$. More formally:

$$U_1 + \ldots + U_n = \{U_1 + \ldots + U_n \mid u_j \in U_j,\ j = 1, \ldots, n\}$$

Proposition 2.4. *Suppose $U_1, \ldots, U_n$ are subspaces of V. Then $U_1 + \ldots + U_n$ is the smallest subspace of V containing $U_1, \ldots, U_n$.*

Proof. Clearly $\vec{0} \in U_1 + \ldots + U_n$. We know $U_1 + \ldots + U_n$ is closed under both addition and scalar multiplication, as $U_1, \ldots, U_n$ are. By definition, $U_1 + \ldots + U_n$ is a subspace of V. Let $u_1 \in U_1$ be arbitrary. Then $u_1 + \vec{0} + \vec{0} + \ldots \in U_1 + \ldots + U_n$. Likewise, let $u_j \in U_j$ be arbitrary and note that $u_j \in U_1 + \ldots + U_n$. Furthermore, $U_j \leq U_1 + \ldots + U_n\ \forall j = 1, \ldots n$. Suppose then that W is a subspace such that $W \geq U_1 + \ldots + U_n$. We have W closed under addition by default, which implies $W \supseteq U_1 + \ldots + U_n$. $\square$

Definition 2.5. Suppose $U_1, \ldots, U_n$ are subspaces of V. If each element in $U_1 + \ldots + U_n$ can be written in only 1 way as a sum of $u_1 + \ldots + u_n$ for $u_j \in U_j$, we call $U_1 + \ldots + U_n$ a *direct sum*. We denote this as $U_1 \oplus \ldots \oplus U_n$.

Example 2.6. Consider the following direct sum of subspaces:

$$U = \{(x, 0, 0) \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$$
$$W = \{(0, y, z) \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\}$$
$$\mathbb{R}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\} = U \oplus W$$

Proposition 2.7. *Suppose $U_1, \ldots, U_n$ are subspaces of V. Then $U_1 + \ldots + U_n$ is a direct sum if and only if the only way to write $\vec{0}$ as the sum $u_1 + \ldots + u_n$ is by taking each $u_j \in U_j$ as 0.*

Proof. Suppose $U_1 + \ldots + U_n$ is a direct sum. By definition $\vec{0} \in U_1 + \ldots + U_n$ has a unique representation. Clearly $\vec{0} = \vec{0} + \ldots + \vec{0}$ for $\vec{0} \in U_j$, so $u_1 = \ldots u_n = 0$. Suppose then that the only way to write $\vec{0} = u_1 + \ldots + u_n$ is by taking $u_1 = \ldots = u_n = 0$. Let $v \in U_1 + \ldots + U_n$, and suppose v has two representations. Let's say $v = u_1 + \ldots + u_n$ and $v = v_1 \ldots v_n$ where $v_j, u_j \in U_j\ \forall j \in \{1, \ldots, n\}$. So,

$$\vec{0} = v - v = u_1 + \ldots + u_n - v_1 - \ldots - v_n = (u_1 - v_1) + \ldots + (u_n - v_n)$$

This implies $u_j = v_j = 0\ \forall j \in \{1, \ldots, n\}$. Hence, the result is true. $\square$

Proposition 2.8. *Suppose U and W are subspaces of V. $U + W$ is a direct sum if and only if $U \cap W = \{\vec{0}\}$.*

Proof. Suppose $U + W$ is a direct sum, and let $v \in U \cap W$. Then $\vec{0} = v + (-v)$ where $v \in U$ and $v \in W$. It follows then that $v = (-v) = \vec{0}$. Therefore, $U \cap W = \{\vec{0}\}$. Now suppose $U \cap W = \{\vec{0}\}$, and consider $U + W$. We will show $U + W$ is a direct sum. Suppose $u \in U$, $w \in W$, and $\vec{0} = u + w$. We have then that $u = (-w)$, and $(-w) \in W$. So, $u \in U \cap W$, as W is closed under addition. Since $U \cap W = \{0\}$, we have $u = w = \vec{0}$. $\square$