Signals And Systems

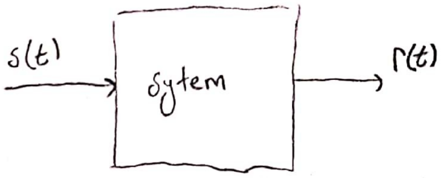
17.09.2019

Signal Processing First, by James H McClellan

Signals & Systems, by Alan V. Oppenheim

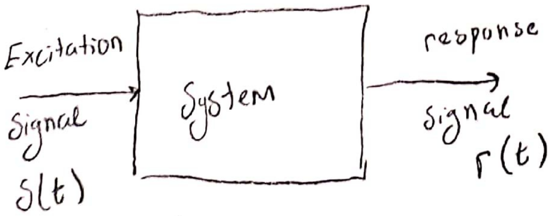
24.09.2019

Typical systems take a signal and convert it into another signal.



A system may contain many different types of signals.

- A signal a physical thing that conveys information and varies with an independent variable such as time or location.
- A system is a physical device that performs operations on signals
- A system is a
- A system is a collection of many comparents working in a cohesive manner on a purpose.
- Excitation signals are applied at system inputss and response signals are produced at system outputs.



where time is a reference, but we can use function definition to represent signals.

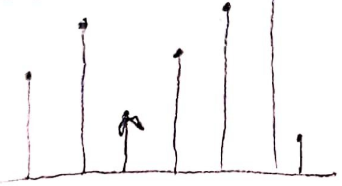
Types of Signals

- Continious time, continious value

- Continious Time, Discrete Value

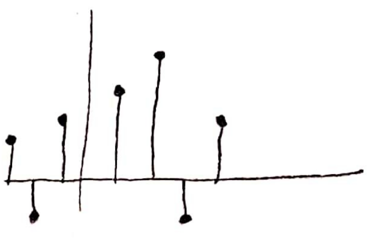
all of values countable
" " time instants uncountable

- Discrete time - continious value



all of possible values is uncountable
" " time instants " countable

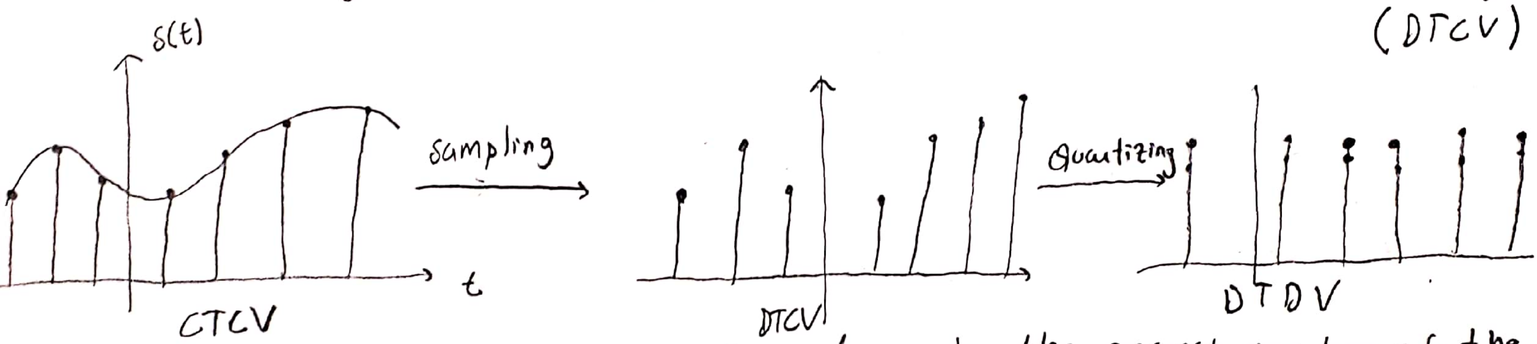
- Discrete time - Discrete value



Conversion Between Signal Types

Sampling: Aquiring values from continous time - Contínous value signal at discrete times.

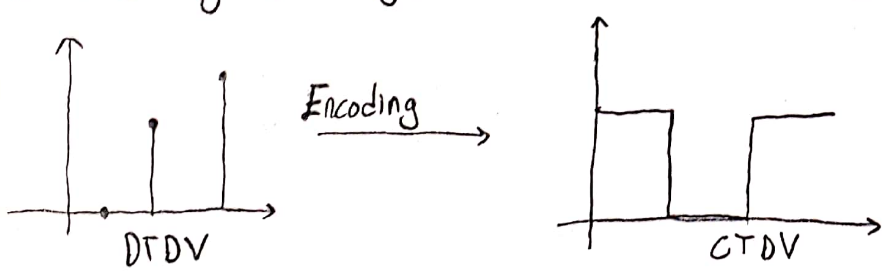
→ Turning the CTCV signal into the discrete time - continious value signals (DTCV)



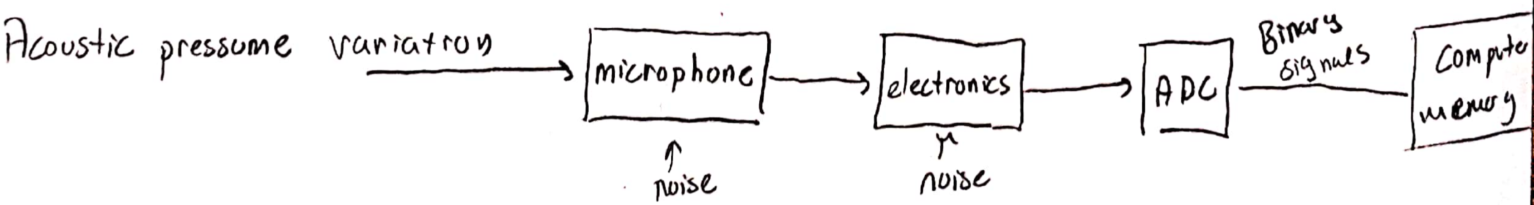
Quantizing: Approximate the signal value to the nearest number of the finite set of discrete values.

→ Turning the DTCV signal into the DTDV signal.

Encoding: Transforming the DTDV signal into rectangular pulses (CTDV) with binary (usually) discrete values. (Digital Signal)



Sound Recording System:



Mathematical Descriptions of Signals

1) Continuous Time Signal

a) Sinusoid:

frequency, time, phase

$$g(t) = A\cdot\cos(wf+\theta) = A\cdot\cos(2\pi f t+\theta) \quad , \quad w=2\pi f$$

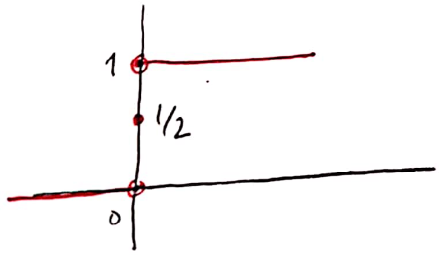
$$g(t) = A\cdot e^{(\tilde{\sigma}_0+jw_0)t} = A\cdot e^{\tilde{\sigma}t}\cdot e^{jw_0 t}$$

$$= A\cdot e^{\tilde{\sigma}t}\left(\cos(w_0 t)+j\sin(w_0 t)\right)$$

$$= A\cdot e^{\tilde{\sigma}t}\cdot\cos(w_0 t) + jAe^{\tilde{\sigma}t}\sin(w_0 t)$$

b) Heaviside Theta (step) signal

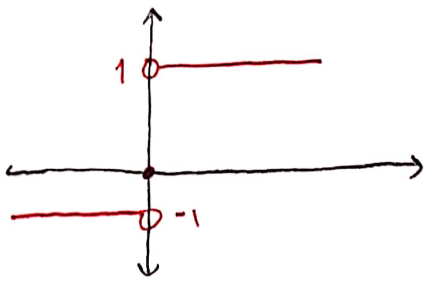
$$u(t) = \begin{cases} 1, & t>0 \\ \frac{1}{2}, & t=0 \\ 0, & t<0 \end{cases}$$



$$\delta(t) = \frac{d\,u(t)}{dt}$$

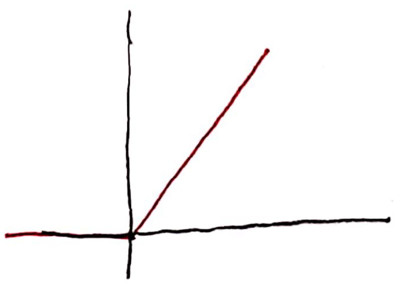
c) Signum Signal:

$$sgn(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$



d) Ramp Signal:

$$ramp(t) = \begin{cases} t, & t>0 \\ 0, & t<0 \end{cases}$$



$$u(t) = \frac{d}{dt}\cdot ramp(t)$$

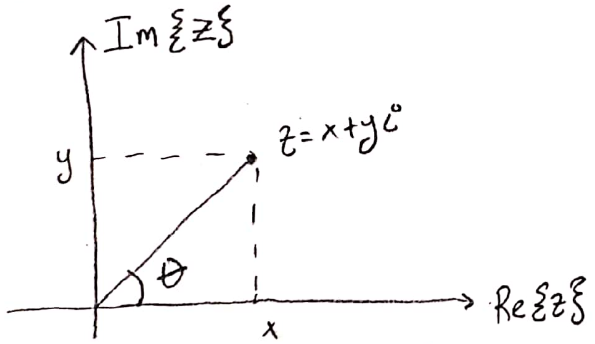
$$ramp(t) = \int_0^{t_0} u(t)\,dt$$

Complex Numbers

Complex number domain is denoted by $\mathbb{C}$

if $z \in \mathbb{C}$, then $z = x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and $i = \sqrt{-1}$

$$\underbrace{z = x + iy}_{\text{rectangular form}} \quad \underbrace{= \sqrt{x^2+y^2} \cdot \exp\left(j \arctan\left(\tfrac{y}{x}\right)\right)}_{\text{polar form}}$$



$|z| \cdot \cos\theta = x$

$|z| = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{z}\right)$

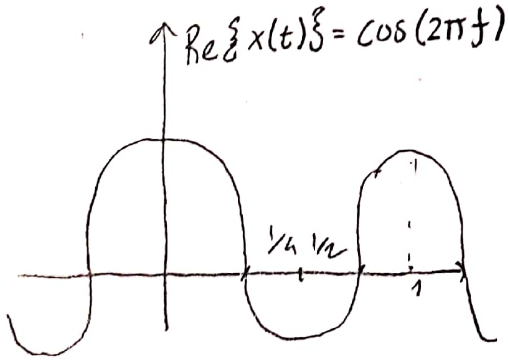
$|z| = \sqrt{x^2+y^2} = |z|^2 \cos^2\theta + |z|^2 \cdot \sin^2\theta$

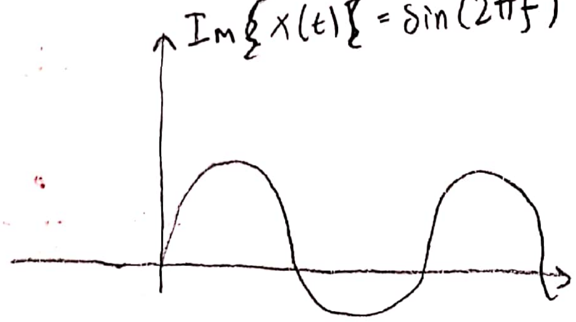
$= |z|^2 (\cos^2\theta + \sin^2\theta)$

$z = |z| \cdot \cos(\theta) + j \cdot |z| \cdot \sin(\theta)$

$= |z| \cdot (\cos(\theta) + j \cdot \sin(\theta)) = |z| \cdot e^{j\theta}$, where $j = \sqrt{-1}$

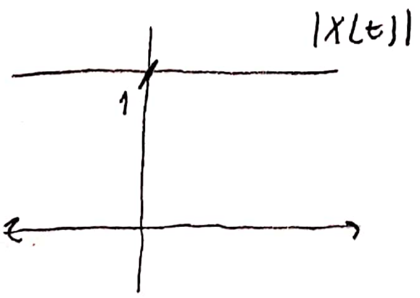
$\tilde{x}(t) = e^{i2\pi f} = \cos(2\pi f) + j \sin(2\pi f)$





$|x(t)|$ denotes magnitute of $x(t)$

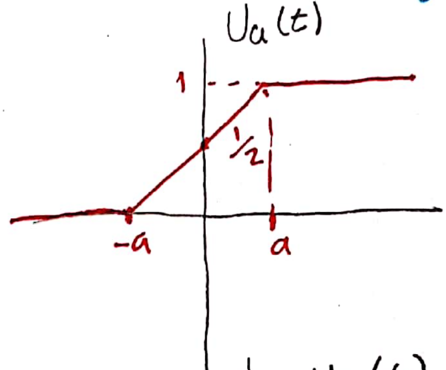
$|x(t)| = \sqrt{\cos^2(2\pi f) + \sin^2(2\pi f)} = 1$



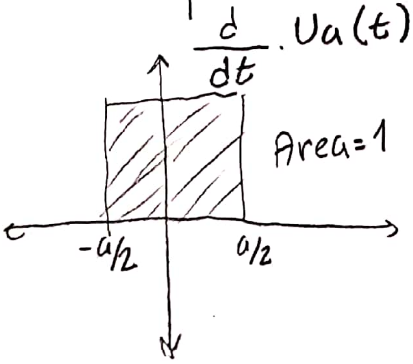
$\angle x(t)$ denotes the of $x(t)$

$\angle x(t) = \arctan\left(\dfrac{\sin(2\pi f)}{\cos(2\pi f)}\right) = \arctan(\tan(2\pi f)) = 2\pi f$

e) Unit Impulse Signal: (Dirac's Delta Signal)



$$\lim_{a \to 0} U_a(t) = U(t)$$



$$\frac{d}{dt} \lim_{a \to 0} U_a(t) = \frac{d}{dt} U(t)$$

$$\delta(t) = \frac{dU(t)}{dt}$$

- $\delta(t) = 0$ except at $t=0$

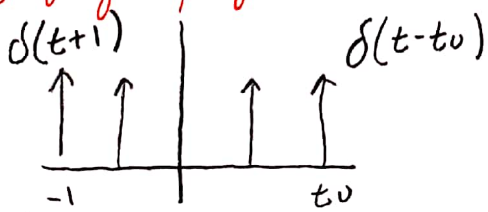
$$= \begin{cases} \pm\infty \ , & t=0 \\ 0 \ , & t \neq 0 \end{cases}$$

- $$\int_{t_1}^{t_2} \delta(t)\,dt = \begin{cases} 1 \ , & \text{if } t_1 < 0 < t_2 \\ 0 \ , & \text{otherwise} \end{cases}$$

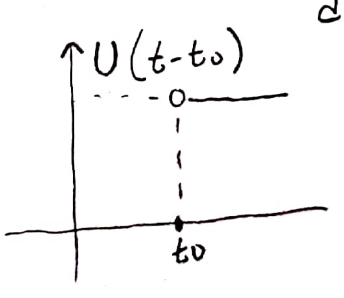
- Scaling Property

$$A \cdot \delta(t)$$

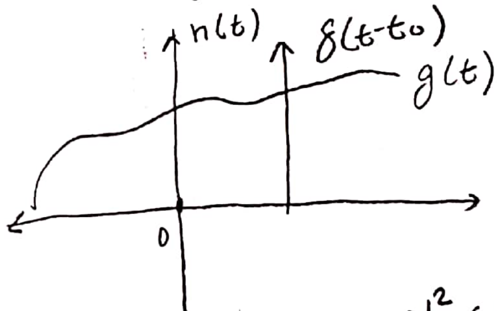
- Shifting Property



$$\delta(t-t_0) = \frac{d}{dt} U(t-t_0)$$



- $h(t) = g(t) \cdot \delta(t-t_0)$



$$h(t) = e^{-t^2} \cdot \delta(t-1) = e^{-1^2} \cdot \delta(t-1)$$

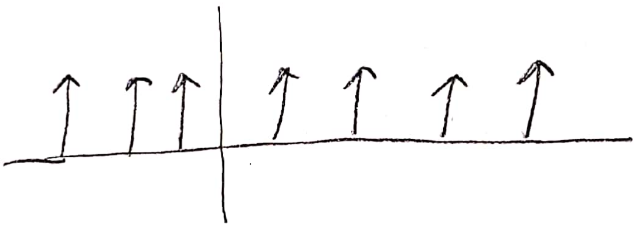
$$= \frac{1}{e} \cdot \delta(t-1) \ , \quad t \cdot \delta(t) = \underbrace{0 \cdot \delta(t)}_{=0}$$

- $$\int_{-\infty}^{+\infty} g(t) \cdot \delta(t-t_0)\,dt = \int_{-\infty}^{+\infty} g(t_0) \cdot \delta(t-t_0)\,dt$$

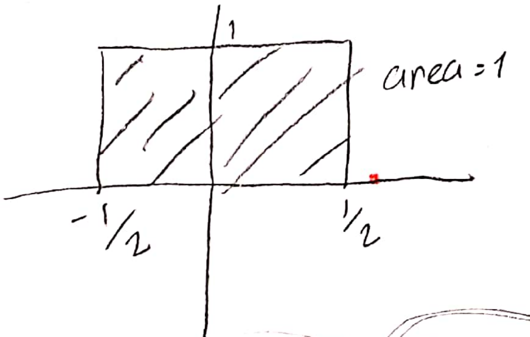
$$= g(t_0) \cdot \int_{-\infty}^{+\infty} \delta(t-t_0)\,dt = g(t_0)$$

f) The Unit Comb Signal

$$\mathrm{comb}(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$$



g) Unit Rectangular Signal

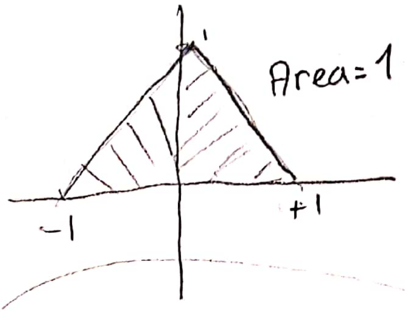


$$\mathrm{rect}(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & t > 1/2 \end{cases}$$

$$\mathrm{rect}(t) = u(t+1/2) - u(t-1/2)$$

$$\frac{d\,\mathrm{rect}(t)}{dt} = \frac{d\,u(t+1/2)}{dt} + \frac{d\,u(t-1/2)}{dt} = \delta(t+1/2) + \delta(t-1/2$$

h) Unit Triangle Signal



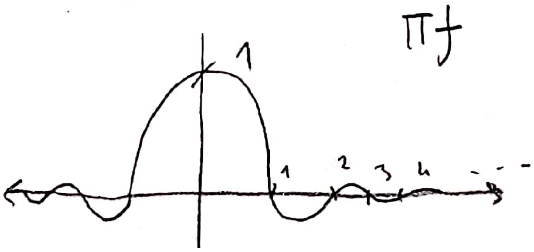
$$\mathrm{Tri}(t) = \begin{cases} 1-|t| & , \text{ if } |t| \leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

i) The Unit Sinc Signal

$$\mathrm{sinc}(t) = \frac{\sin(\pi f)}{\pi f}$$

if $t \in \mathbb{Z} \setminus \{0\}$ then $\mathrm{sinc}(t) = 0$

if $t = 0$ then $\lim_{t \to 0} \mathrm{sinc}(t) = 1$



2.13) Signal Energy and Power

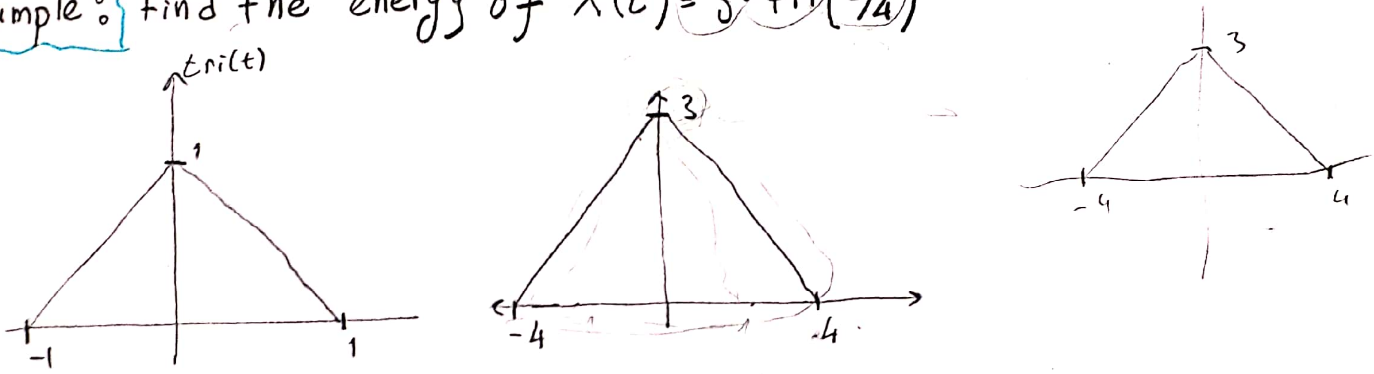
1) Signal Energy: For C.T signals, $E_s = \int_{-\infty}^{+\infty} |x(t)|^2 dt$, which is the energy of $x(t)$

$E_s \geqslant 0$

For D.T signals $E = \sum_{n=-\infty}^{+\infty} |x[n]|^2$

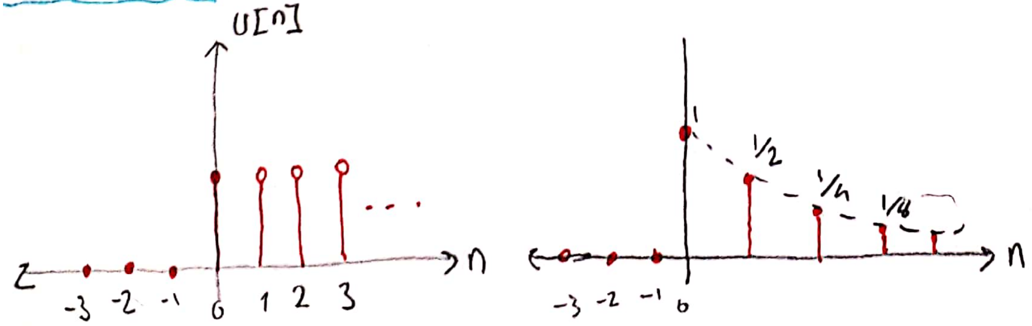
From both definitions, we have $E_s \geqslant 0$ (i.e, Energy cannot be negative)

Example: Find the energy of $x(t) = 3 \cdot \text{tri}(t/4)$



$$E_s = \int_{-\infty}^{+\infty} |x(t)|^2 = \int_{-4}^{0} \left(3\left(1+t/4\right)\right)^2 dt + \int_{0}^{4} \left(3\left(1-t/4\right)\right)^2 dt = 12+12 = 24 \text{ Joule}$$

Example: Find the energy of $x[n] = \left(\frac{1}{2}\right)^n u[n]$



$$E_s = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \sum_{n=0}^{+\infty} \left| \left(\tfrac{1}{2}\right)^n \cdot u[n] \right|^2$$

$$= \sum_{n=0}^{\infty} \left| \left(\tfrac{1}{2}\right)^n \right|^2 = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{1}{4^n} = 4/3 \text{ Joule}$$

$$\sum_{0}^{\infty} 1 \cdot \left(\tfrac{1}{4}\right)^n = \frac{1}{1-1/4} = 4/3$$

2) D.T Exponentials :

$$g[n] = e^{j2\pi f_0 n} \text{ is periodic if } g[n] = g[n+N]$$

$$= \cos(2\pi f_0 n) + j\sin(2\pi f_0 n)$$

$$e^{j2\pi f_0 n} = e^{j2\pi f_0 (n+N)} \quad \text{for all } n$$

$$\cancel{e^{j2\pi f_0 n}} = \cancel{e^{j2\pi f_0 n}} \cdot e^{j2\pi f_0 N},$$

$$e^{j2\pi f_0 N} = 1$$

$$\boxed{= e^{j2\pi K} = e^{j2\pi f_0 N} \quad \text{for all } K \in \mathbb{Z} \setminus \{0\}}$$

$$f_0 = K/N, \quad N = K/f_0$$

★ $f_0 = K/N$, both K and N are integers, so in D.T, f_0 should be a function of two integers if the signal is periodic.

C.T Signals	D.T Signals
$g_1(t) = e^{j2\pi f_1 t}$	$g_1[n] = e^{j2\pi f_1 n}$
$g_2(t) = e^{j2\pi f_2 t}$	$g_2[n] = e^{j2\pi f_2 n}$
if $f_1 \neq f_2$, then $g_1(t) \neq g_2(t)$ (i.e, distinct frequences lead to the distinct signals)	if $f_1 = f_2 + K$ where $K \in \mathbb{Z}$, then

$$g_2[n] = e^{j2\pi f_2 n} = e^{j2\pi (f_1+K)n}$$

$$= e^{j2\pi f_1 n} \cdot \underbrace{e^{j2\pi K n}}_{1}$$

$$= e^{j2\pi f_1 n}$$

$$= g_1[n]$$

$$\boxed{\text{Two different frequencies can lead to same sinizoidal signal.}}$$

2) Signal Power:

For C.T signal $X(t)$, $P_S = \lim\limits_{T \to \infty} \frac{1}{T} \cdot \int_{-T/2}^{T/2} |X(t)|^2 dt$

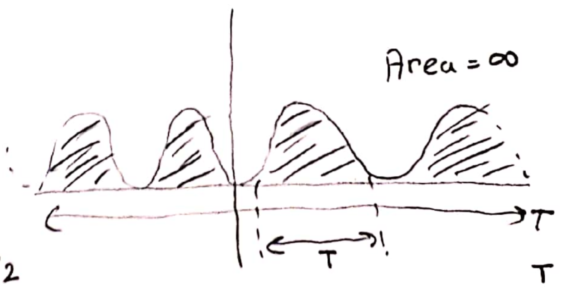
For D.T signal $X[t]$, $P_S = \lim\limits_{N \to \infty} \frac{1}{2N} \cdot \sum_{n=-N}^{N-1} |x[n]|^2$

$$P_S = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2 dt$$

$$P_S = \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} |X[n]|^2$$

Example: for $X(t) = A \cdot \cos(2\pi \underbrace{f_0}_{\text{frequency}} \overbrace{t}^{\text{time}} + \underbrace{\theta}_{\text{phase}})$ Find P_s and E_s ?

$$E_S = \int_{-\infty}^{+\infty} |X(t)|^2 dt = \int_{-\infty}^{+\infty} |A \cdot \cos(2\pi f_0 t + \theta)|^2 dt = \int_{-\infty}^{+\infty} A^2 \cdot \cos^2(2\pi f_0 t + \theta) dt = \infty$$

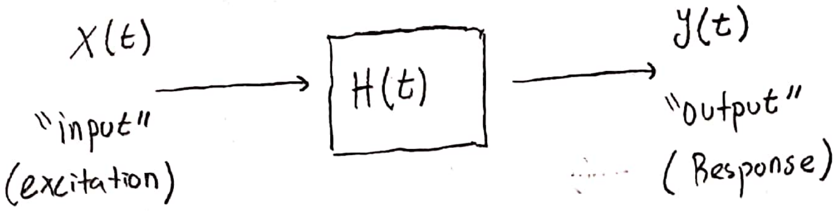


$$P_S = \lim_{T \to \infty} \frac{1}{T} \cdot \int_{-T/2}^{T/2} |X(t)|^2 dt = \lim_{T \to \infty} \frac{1}{T} \cdot \int_{0}^{T} A^2 \cdot \cos^2(2\pi f_0 t + \theta) dt$$

Solve

$$= \frac{1}{\pi} \cdot \int_{0}^{\pi} A^2 \cdot \cos^2(2\pi f_0 t + \theta) dt = \frac{A^2}{2}$$

3.1) Introduction to Description and Analysis of Systems

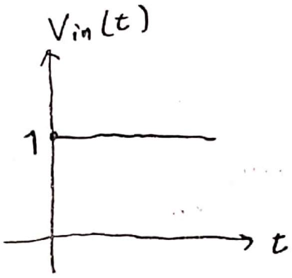


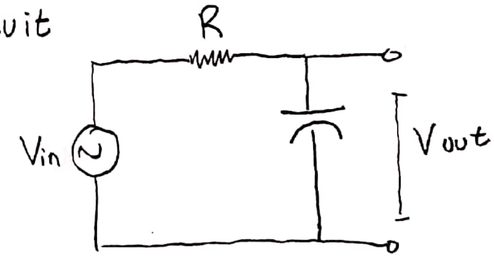
$$y(t) = x(t) \circledast H(t)$$

$\hookrightarrow$ convolution

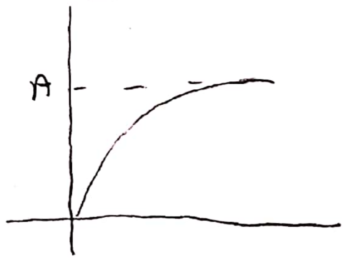
$$y(t) = x(t) \circledast H(t) = \int_{-\infty}^{t} x(t-z) \cdot h(z) \cdot dz$$

Example: 1) RC circuit

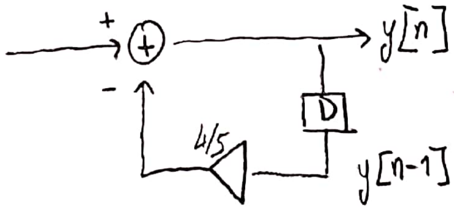




$$V_{out}(t) = A \cdot \left(1 - \exp\left(-t/RC\right)\right) u(t)$$



2) $x[n]$



$$\Rightarrow y[n] = x[n] - 4/5 \cdot y[n-1]$$

2.11) Periodic DoT Signals

$g[n]$ is a periodic function if there exists N such as $\forall m$,
$g[n] = g[n+mN]$ where $N \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{Z}$

The smallest integer N_0 such as $g[n] = g[n+N_0]$ is called fundamental periodo

Remarks

1) CoT exponentials: $g(t) = e^{j 2\pi \overset{\text{frequency}}{f_0} \overset{\text{time}}{t}} = \cos(2\pi f_0 t) + j \sin(2\pi f_0 t)$

if it is periodic, $g(t) = g(t+T)$, where T is the period

$$g(t) = g(t+T) = e^{j \cdot 2\pi f_0 (t+T)} = \underbrace{e^{j 2\pi f_0 t}}_{g(t)} \cdot \underbrace{e^{j 2\pi f_0 T}}_{\text{this must be } 1} = g(t)$$

if $g(t)$ is periodic by T, then this condition has to be supported.

$$e^{j 2\pi f_0 T} = \cos(2\pi f_0 T) + j \sin(2\pi f_0 T) = \cos(2\pi k) + j \sin(2\pi k)$$
$$= \cos(0) + j \sin(0)$$
$$= 1$$

$$\boxed{f_0 T = k}$$

2.10) Even and Odd D.T Signals

Even D.T Signals $\longrightarrow g[n] = g[-n]$

Odd D.T Signals $\longrightarrow g[n] = -g[-n]$

From here, any $g[n]$, which have odd and even parts, can be written as

$$g[n] = g_e[n] + g_o[n] \quad \text{where} \quad g_e[n] = \tfrac{1}{2}\left(g[n] + g[-n]\right)$$

$$g_o[n] = \tfrac{1}{2}\left(g[n] - g[-n]\right)$$

Remarks

1) $g_1[n]$ and $g_2[n]$ are even, Let $g_p[n] = g_1[n] \cdot g_2[n]$, then

$g_p[n]$ is even

proof: if $g_p[n] = g_p[-n]$, then $g_p[n]$ is even. Let us check it as follows

$$g_p[-n] = \underbrace{g_1[-n]}_{\text{even}} \cdot \underbrace{g_2[-n]}_{\text{even}} = g_1[n] \cdot g_2[n] = g_p[n]$$

QED

2) Let $g_1[n]$ even, $g_2[n]$ odd. Then $g_p[n]$ is odd.

proof: $g_1[n] = g_1[-n]$ and $g_2[n] = -g_2[-n]$

$g_p[-n] = g_1[-n] \cdot g_2[-n] = g_1[n] \cdot (-g_2[n]) = -g_1[n] \cdot g_2[n]$, so $g_p[n]$ is odd.

2.9) Differencing and Accumulation

$$d_F(x[n]) = \Delta x[n] = x[n+1] - x[n]$$

$$d_F(x[0]) = x[1] - x[0]$$

Unit function

$$d_F(u[n]) = u[n+1] - u[n] = \delta[n+1]$$

$$\left.\begin{aligned} n<-1 &\longrightarrow d_F(u[n])=0 \\ n=-1 &\longrightarrow d_F(u[n])=0 \\ n=0 &\longrightarrow d_F(u[n])=1 \\ n=1 &\longrightarrow d_F(u[n])=0 \\ n>1 &\longrightarrow d_F(u[n])=0 \end{aligned}\right\} = \begin{cases} 1, & n=-1 \\ 0, & \text{otherwise} \end{cases}$$

First Forward Differencing $\longrightarrow d_F(x[n]) = \Delta x[n] = x[n+1] - x[n]$

First Backward Differencing $\longrightarrow d_B(x[n]) = \Delta x[n-1] = x[n] - x[n-1]$

Accumalation

$$acc(x[n]) = \sum_{m=-\infty}^{n} x[m]$$

$\Rightarrow$ Similiar to the notion of integration for continous time signals.

Example

Let $x[n]$ be a DT signal. Find the first backward differencing of $y[n] = acc(x[n])$?

$$d_B(y[n]) = \Delta y[n-1] = y[n] - y[n-1] = \sum_{m=-\infty}^{n} x[m] - \sum_{m=-\infty}^{n-1} x[m]$$

$$= x[n]$$

$$d_B(y[n]) = y[n] - y[n-1] = acc(x[n]) - acc(x[n-1])$$

$$= \sum_{m=-\infty}^{n} x[m] - \sum_{m=-\infty}^{n-1} x[m] = x[n]$$

08.10.2019

2.8) Transformation of DT Signals

1) Time Shifting: $g[n] \longrightarrow g[n-n_0]$

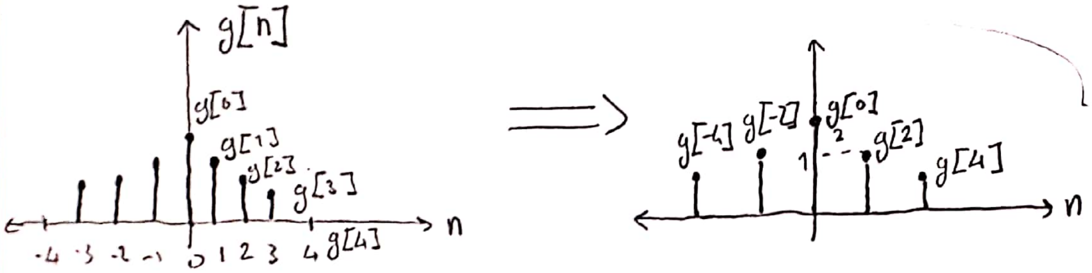
n_0 ⟶ # of shifts in samples

where the shift n_0 must be an integer.

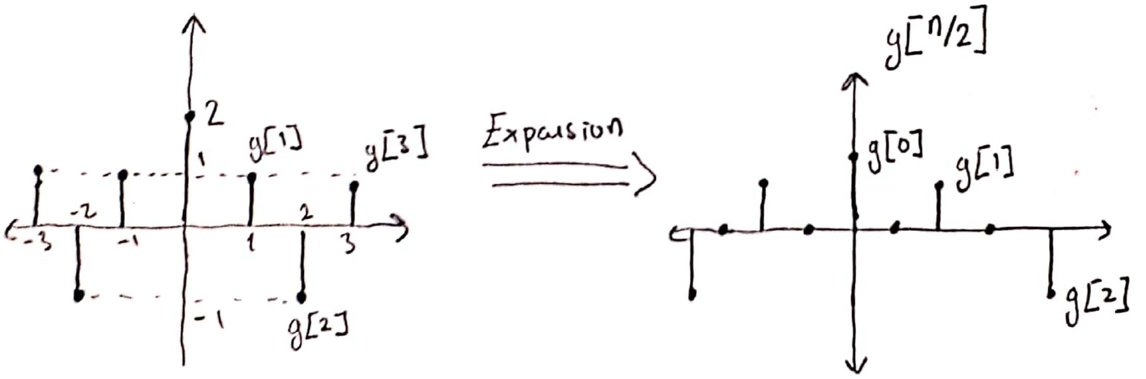
$n_0 \geq 0 \Longrightarrow$ shifts to the right

$n_0 < 0 \Longrightarrow$ shifts to the left

2) Time Scaling: Time compression if $n \longrightarrow Kn$, where $K \in \mathbb{N} \setminus \{0\}$



Time Expansion ($n \longrightarrow n/k$, where n/k is an integer)



3) Amplitude Scaling:

$g[n] \longrightarrow A\,g[n]$ where $A \in \mathbb{R}$

Properties of Systems

1) Homogenity: In a homogenous system, multiplying the excitation by constant will multiply the response output signal with the same constant

$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$, then $Ax(t) \longrightarrow \boxed{H} \xrightarrow{Ay(t)}$ so this system is homogenious.

Example: 1) $y(t) = \alpha \cdot x(t)$

$\alpha \cdot (A x(t)) = A \cdot \alpha \cdot x(t) = A y(t) \longrightarrow$ homogenious

2) $y(t) = Ax(t) + B$

$x(t) \longrightarrow K \cdot x(t)$

$A \cdot K \cdot x(t) + B \neq K \cdot y(t)$; so, not homogenious

3) $y(t) = x(t) \cdot \cos(\omega_c t)$

$x(t) \longrightarrow K \cdot x(t)$ $\quad$ $K \cdot x(t) \cdot \cos(\omega_c t) = K \cdot y(t)$; so, homogenious

$y(t) \longrightarrow K \cdot y(t)$

4) $y(t) = x^2(t)$

$x(t) \longrightarrow K \cdot x(t)$ $\quad$ $K^2 x^2(t) \neq K y(t)$; so, not homogenious

$y(t) \longrightarrow K \cdot y(t)$

2) Time Invariance: if an arbitrary excitation $x(t)$ of a system was shifted to $x(t-t_0)$, the response $y(t)$ should shift to $y(t-t_0)$ too, if the system is time-invariant.

Example: 1) $y(t) = Ax(t) \longrightarrow$ time invariant

2) $y(t) = t \cdot x(t) \longrightarrow$ time variant

$y(t) = t \cdot x(t)$

$y(t-t_0) = (t-t_0) \cdot x(t-t_0)$ ①

$t \cdot x(t-t_0) \neq (t-t_0) \cdot x(t-t_0)$ so time variant.

3) $y(t) = x^2(t) \longrightarrow$ time invariant

$x^2(t-t_0) = y(t-t_0)$

Input is shifted by t_0, which shifts the output by t_0. Thus, the system is time invariant.

? 4) $y(t) = x(2t)$

$x(2(t-t_0)) = x(2t-t_0) \neq y(t-t_0) \longrightarrow$ time variant

3) Additivity:

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow \boxed{H} \longrightarrow y_2(t)$$

if the system H is additive, then

$$x_1(t)+x_2(t) \longrightarrow \boxed{H} \longrightarrow y_1(t)+y_2(t)$$

Example:

1) $y(t)=A\,x(t) \Longrightarrow$

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t)=Ax_1(t)$$
$$x_2(t) \longrightarrow \boxed{H} \longrightarrow y_2(t)=AX_2(t)$$
$$x_1(t)+x_2(t) \longrightarrow \boxed{H} \longrightarrow y(t)=A\big(x_1(t)+x_2(t)\big)$$
$$=Ax_1(t)+Ax_2(t)$$
$$=y_1(t)+y_2(t)$$

so the system H is additive

2) $y(t)=Ax(t)+B$

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t)=Ax_1(t)+B$$
$$x_2(t) \longrightarrow \boxed{H} \longrightarrow y_2(t)=AX_2(t)+B$$
$$x_1(t)+x_2(t) \longrightarrow \boxed{H} \longrightarrow y(t)=A\big(x_1(t)+x_2(t)\big)+B$$
$$=Ax_1(t)+Ax_2(t)+B$$

$y_1(x)+y_2(x)=Ax_1+Ax_2+2B$

Not equal. Then, not additive

4) Linearity

if

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{H} \longrightarrow y_2(t)$$

Then

$$x(t) = \overbrace{\alpha}^{\text{ampitute scaling}} \cdot x_1(t) + \overbrace{\beta}^{\text{ampitute scaling}} x_2(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

* Homogenius and additive systems are linear systems.

Example:

1) $y(t) = A\,x(t) \longrightarrow$ Linear

2) $y(t) = A\,x(t) + B \longrightarrow$ Not Linear

$$A\,x_1(t) + A\,x_2(t) = y(t)$$

6) Casuality

any system for which the response occurs only during and after the time in which the excitation is applied is called Casual Sytem.

for any t_0, $y(t_0)$ depends only on $x(t)$ for $t_0 \leq t$

Example:

① $y(t) = x(t-1)$ $\quad y(1) = x(0)$

↳ this system is casual since for any t_0, $y(t)$ depends only on $x(t)$ for $t \leq t_0 - 1$

② $y(t) = x^2(t+1)$ (not casual)

③ $y(t) = \int_{-\infty}^{t} x(z)\, dz$ (casual)

④ $y(t) = \int_{t}^{\infty} x(z)\, dz$ (not casual)

⑤ $y(t) = \int_{t-1}^{t} x(z)\, dz$ (Homeworks)

⑥ $y(t) = \int_{5-t}^{5} x(z)\, dz$ (not causal)

15. 10. 2019

5) Stability

a system for which the response is bounded when the excitation is bounded is called "Bounded Input Bounded Output" (BIBO)

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

if $|x(t)| < A_x < +\infty$, then $y(t) < A_y < +\infty$ then the system is stable.

Example:

① $y(t) = \alpha \cdot x(t)$, if $|x(t)| < A_x < +\infty$

$$y(t) = \alpha \cdot x(t)$$
$$|y(t)| = |\alpha \, x(t)|$$
$$= |\alpha| \cdot |x(t)|$$
$$\leq |\alpha| \cdot A_x \leq |\alpha| \cdot \infty \leq \infty$$

so the output is bounded

② $y(t) = \dfrac{1}{(1+x(t))}$

if $x(t) = 1$ then $|y(t)| = \infty$, so it is not stable

③ $y(t) = \dfrac{1}{(1+x(t))}$ if $x(t) = -1$ then $|y(t)| = \infty$, so it is not stable.

③ $y(t) = \sum_{n=1}^{N} \exp(-n \cdot |x(t)|)$ it is stable

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r^n} \quad , \quad \sum_{n=1}^{N} r^n = \frac{1-r^{N+1}}{1-r^N} - 1$$

$$\sum_{n=1}^{N} \left(e^{-|x(t)|}\right)^n = \frac{1-e^{-(N+1)|x(t)|}}{1-e^{-|x(t)|}} - 1$$

7) Invertibility

a system is invertible if unique excitation (Input) produces unique response

$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$

$y(t) \longrightarrow \boxed{H^{-1}} \longrightarrow x(t)$

Example:

① $\underbrace{y(t)}_{\text{Output}} = \underbrace{Ax(t)}_{\text{Input}} \Longrightarrow \underbrace{x(t)}_{\text{output}} = \underbrace{1/A \cdot y(t)}_{\text{input}}$ (invertible)

② $y(t) = Ax(t) + B$, if $A, B \neq 0$ $\quad x(t) = \frac{y(t)}{A} - B/A$ (invertible)

③ $y(t) = x^2(t)$ → $x(t) = -\sqrt{y(t)}$, → $x(t) = +\sqrt{y(t)}$ } not invertible

④ $y(t) = |x(t)|$ not invertible

8) Memory

if any system response at an arbitrary time $t = t_0$ ($y(t_0)$) depends only on the excitation at $t = t_0$ the system is called memoryless sytem

Example:

① $y(t) = Ax(t)$ → memoryless

② $y(t) = Ax(t-1)$ → memory

③ $y(t) = \cos(x(t))$ → memoryless

④ $y(t) = x(t) \cdot \cos(w(t+1))$ → memory less

⑤ $y(t) = \int_{-\infty}^{t} x(z)\,dz$ → Memory

⑥ $y(t) = \frac{dx(t)}{dt}$) Homework

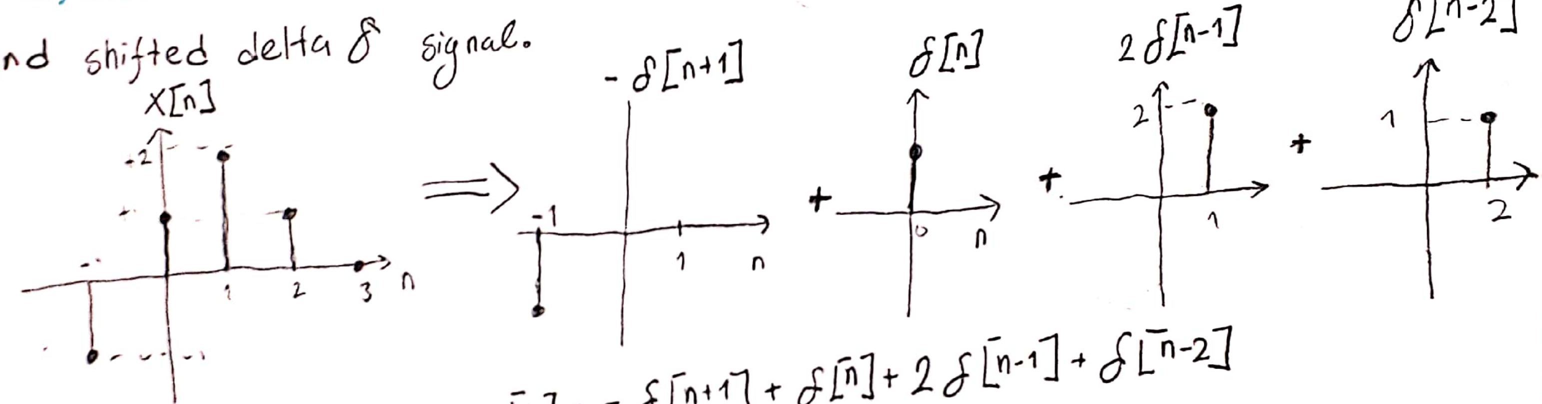
22.10.2019

$$y[n] = \sum_{k=0}^{n^2} \cos(\pi k/4) \cdot x[k]$$
→ Linear
→ not time invariant
→ non-casual
→ not stable

$$y(t) = \cos(x(t-1)) + x(t+1)$$
→ non-linear due to cos
→ time invariant
→ non-casual
→ stable

Description and Analysis of Systems

a) **Convolution Sum:** You can break any D.T signal into sum of scaled and shifted delta δ signal.



$$x[n] = -\delta[n+1] + \delta[n] + 2\delta[n-1] + \delta[n-2]$$

In general $$x[n] = \sum_{k=-\infty}^{+\infty} \delta[k] + \delta[n-k]$$

$$x[n] = x[m] \cdot \delta[n-m] = \begin{cases} x[m], & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$$

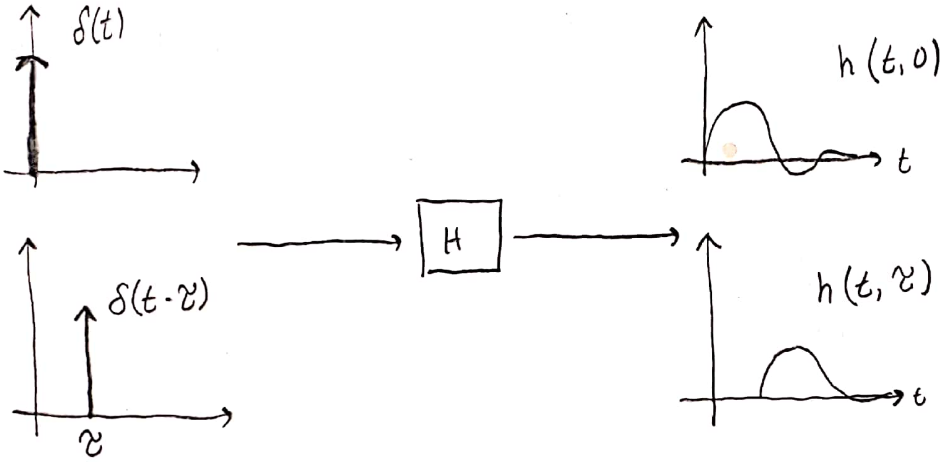
b) **Impulse Response:** The impulse respond of a system is the output of the system when we inject a δ-signal as an input.

$x(t) \longrightarrow \boxed{\text{C.T } h(t)} \longrightarrow y(t)$ $\qquad$ $x[n] \longrightarrow \boxed{\text{D.T } h[n]} \longrightarrow y[n]$

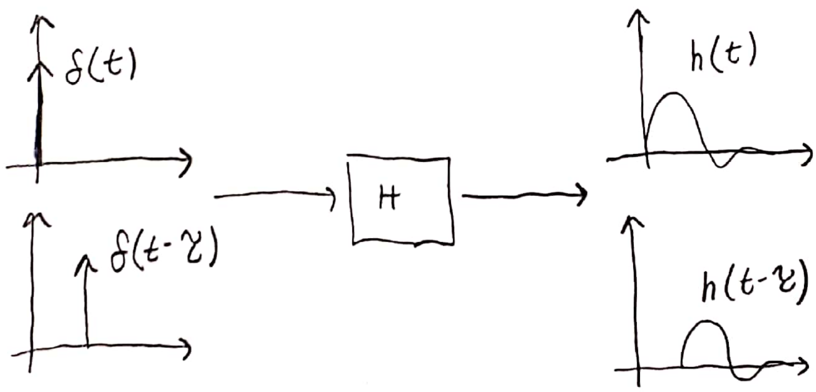
$\delta(t) \longrightarrow \boxed{\text{C.T } h(t)} \longrightarrow h(t)$ $\qquad$ $\delta[n] \longrightarrow \boxed{\text{D.T } h[n]} \longrightarrow h[n]$

Impulse Response

The impulse response of linear system $h_\tau(t)$ is the output of the sytem at time t to an impulse at time τ. This can be written as $h_\tau = H(\delta_\tau)$



if H is time invariant $\quad h_\tau(t) = h(t-\tau)$



Output of an LTI system

$x \longrightarrow \boxed{H} \longrightarrow y$ $\quad$ we can write $x(t)$ as a sample of itself

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau \quad \Bigg| \quad ① \quad y(t) = H\left(\int_{-\infty}^{+\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau\right)$$

②

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot\underbrace{H(\delta(t-\tau))}_{\to\, h_\tau(t)}\cdot d\tau \longrightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot h_\tau(t)\cdot d\tau$$

Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} e^{-st} \cdot f(t)\, dt = F(s)$$

$$\mathcal{L}^{-1}\{F(s)\} = \int_{c-j\infty}^{c+j\infty} \frac{1}{2\pi j} \cdot F(s) \cdot e^{st} ds = f(t)$$

Fourier Transform

$$\mathcal{F}_{w}\{f(t)\} = \int_{-\infty}^{+\infty} e^{jwt} \cdot f(t) dt = F(w)$$

$$\mathcal{F}_{t}^{-1}\{F(w)\} = \int_{-\infty}^{+\infty} \frac{1}{2\pi} \cdot F(w) \cdot e^{jwt} \cdot dw = f(t)$$

- $\mathcal{L}\{\delta(t-\tau)\} = \int_{-\infty}^{+\infty} e^{-st} \cdot \delta(t-\tau)\, dt = e^{-s\tau}$

- $\mathcal{L}\{\delta(t)\} = 1$

- $\mathcal{F}\{\delta(t-\tau)\} = e^{-jw\tau}$

- $\mathcal{F}\{\delta(t)\} = 1$

Convolution for Casual System & with Casual Input :

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot h(t-\tau)\,d\tau \longrightarrow \text{general LTI System}$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\cdot h(t-\tau)\cdot d\tau \longrightarrow \text{Casual LTI System}$$

$$y(t) = \int_{0}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau \longrightarrow \text{Casual Input \& General LTI System}$$

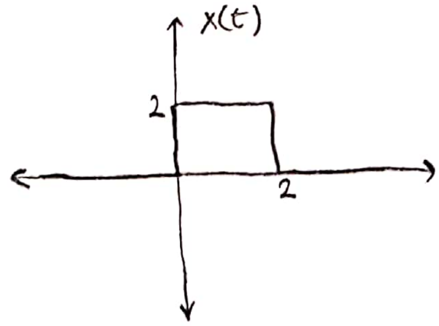
$$y(t) = \int_{0}^{t} x(\tau)\cdot h(t-\tau)\,d\tau \longrightarrow \text{Casual Input \& Casual LTI System}$$

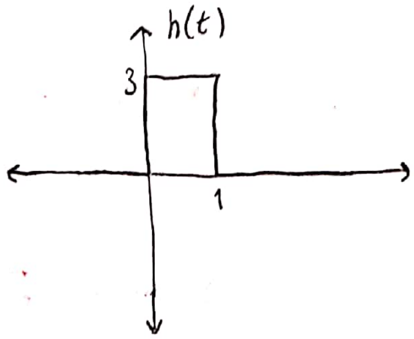
Convolution Properties

- $x(t) * h(t) = h(t) * x(t)$
- $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$
- $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$
- $\frac{d}{dt} \cdot x(t) * h(t) = \dot{x}(t) * h(t) = x(t) * \dot{h}(t)$
- Let $y(t) = x(t) * h(t)$ then,

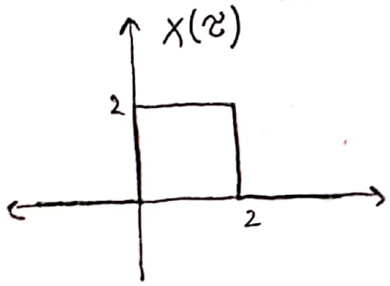
$$\int_{-\infty}^{t} y(\lambda)\,d\lambda = \left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right] * h(t) = x(t) * \left[\int_{-\infty}^{t} h(\lambda)\,d\lambda\right]$$

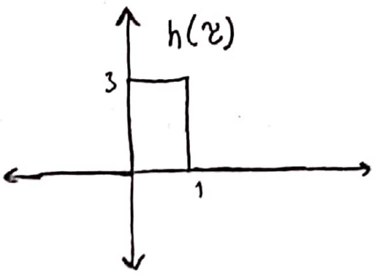
Graphically Convolve Two signal



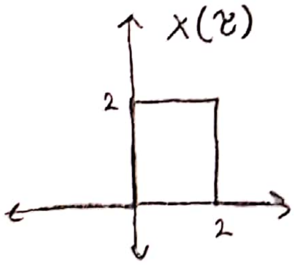


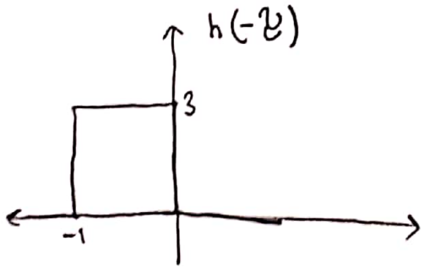
Step#1: write as function of τ





Step #2: flip $h(\tau)$ to get $h(-\tau)$





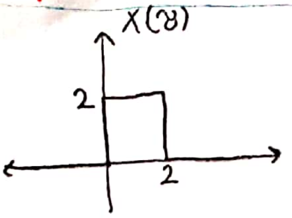
Step#3: Find edges of flipped signal

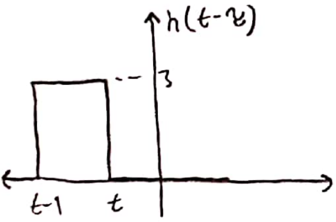
$$\boxed{\tau_{L,0} = -1 \quad \tau_{R,0} = 0}$$

Step4: Shift by t to get $h(t-\tau)$ & Its edges
↳ arbitrary

$$\boxed{\begin{array}{l} \tau_{L,t} = t-1 \\ \tau_{R,t} = t \end{array}}$$

Step #5: Find regions of τ-Overlap $- \; y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$





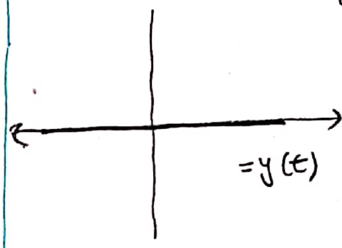
Region I

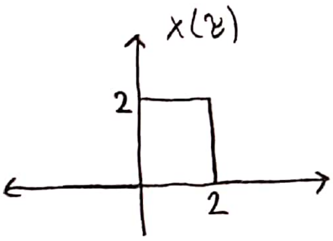
No τ-overlap

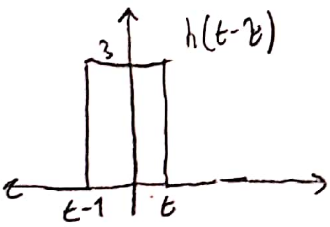
Wait $\tau_{R,0} < 0 \longrightarrow t < 0$

$y(t) = \int_{-\infty}^{+\infty} 0 \cdot d\tau = 0$

$t < 0$







Region II

Partial τ-overlap, $0 \leq t \leq 1$

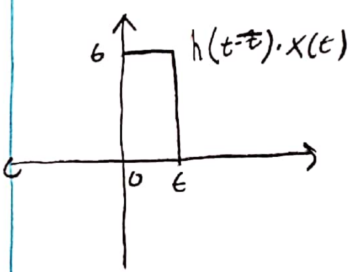
$t-1 \leq 0 \rightarrow t \leq 1$

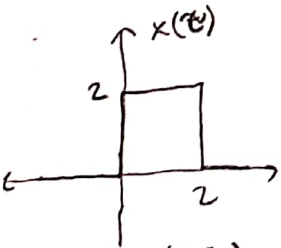
$t \geq 0$

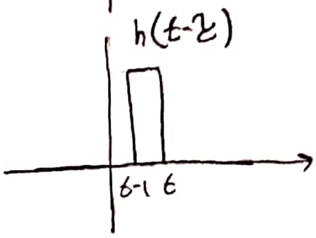
$y(t) = \int_{0}^{t} h(t-\tau) \cdot x(t) \cdot d\tau$

$= \int_{0}^{t} 6 \, d\tau = 6\tau \Big|_{0}^{t} = 6t$

$y(t) = 6t \qquad 0 \leq t < 1$







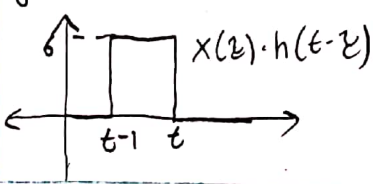
Region III

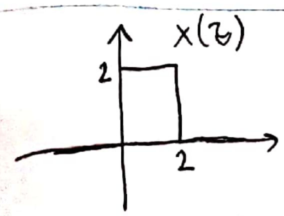
total τ-overlap $\quad 1 \leq t < 2$

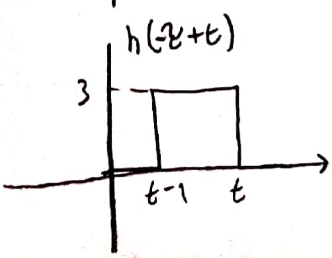
$y(t) = \int_{t-1}^{t} x(\tau) \cdot h(t-\tau) \cdot d\tau$

$= \int_{t-1}^{t} 6 \, d\tau = 6\tau \Big|_{t-1}^{t} = 6$

$y(t) = 6 \qquad 1 < t \leq 2$





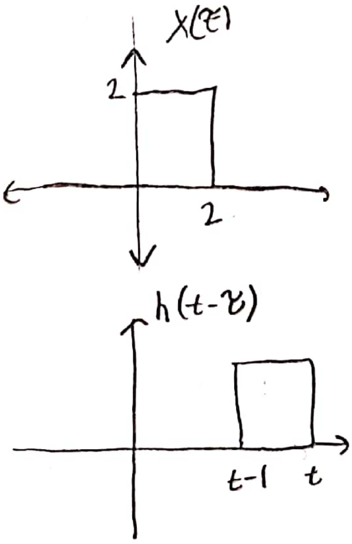


Region IV

Partial τ-overlap

$y(t) = \int_{t-1}^{2} 6 \cdot d\tau = 6\tau \Big|_{t-1}^{2} = 18 - 6t$

$2 < t \leq 3$



Region V

no τ-overlap

$$y(t) = \int_{-\infty}^{+\infty} 0 \, d\tau = 0$$

$y(t) = 0$

$t > 3$

Step #6: Assemble output signal

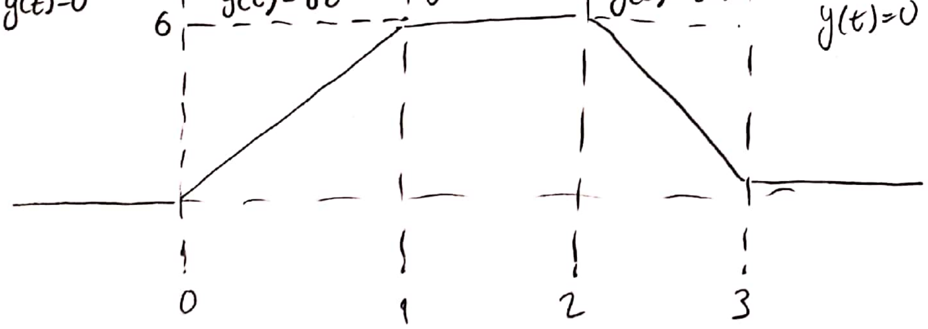
region I
$t < 0$
$y(t) = 0$

region II
$0 \leq t \leq 1$
$y(t) = 6t$

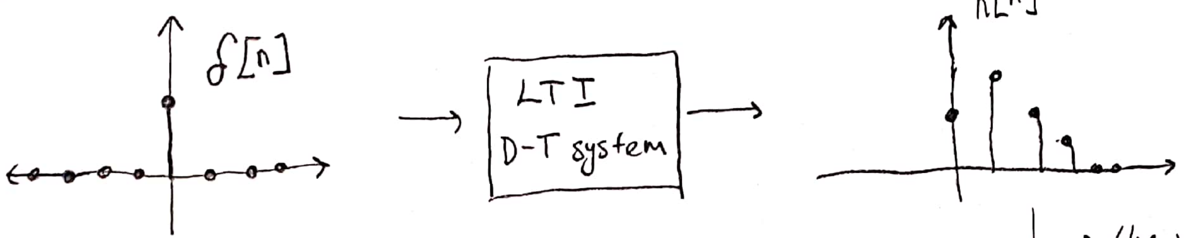
region III
$1 < t \leq 2$
$y(t) = 6$

region IV
$2 < t \leq 3$
$y(t) = -6t + 18$

region V
$t > 3$
$y(t) = 0$



Discrete Convolution



↳ the impulse response

$$x[n] \cdot h[n] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m]$$

Convolution for Casual System & with Casual Input

$$y[n] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m]$$ → General LTI system

$$y[n] = \sum_{m=-\infty}^{n} x[m] \cdot h[n-m]$$ → Casual LTI system

$$y[n] = \sum_{m=0}^{\infty} x[m] \cdot h[n-m]$$ → Casual Input & General LTI system

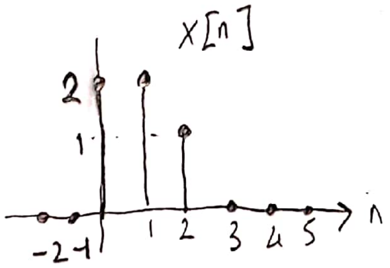
$$y[n] = \sum_{m=0}^{n} x[m] \cdot h[n-m]$$ → Casual Input & Casual LTI System

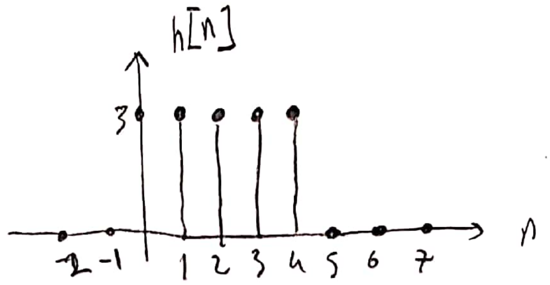
---

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] \qquad x[2n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k/2]$$

$$x[n/2] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-2k]$$

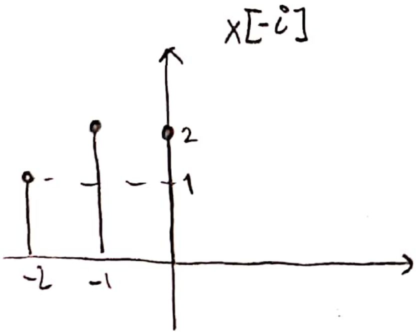
Graphical Convolution

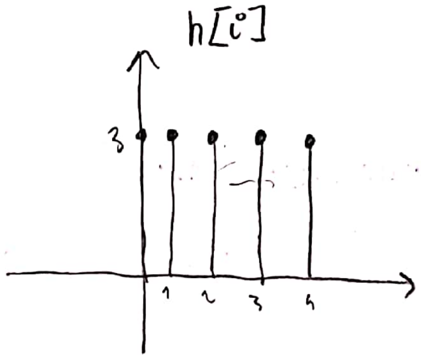




write as a function of i

flip $x[i]$ to $x[-i]$

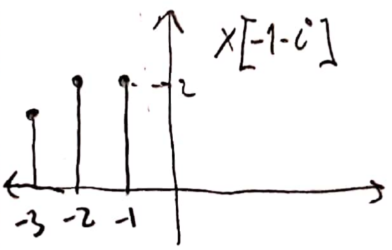


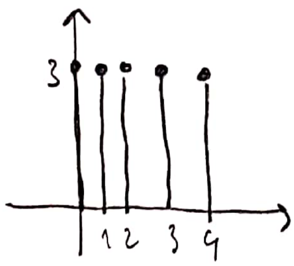


- lets look at $x[n-i]$ for $n=0$

$$y[0] = \sum_{i=0}^{0} x[0-i]\, h[i] = 6$$

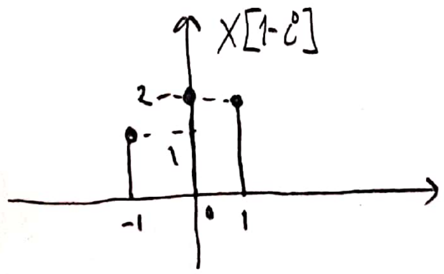
- lets look at $x[n-i]$ for $n=-1$

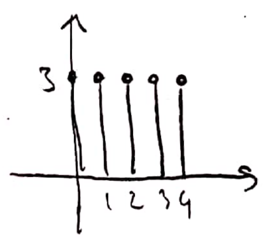




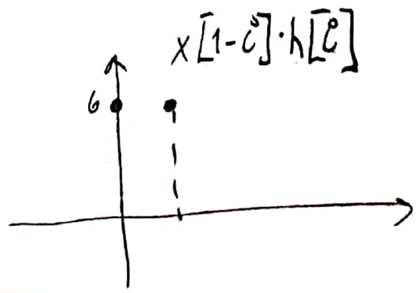
$y[n]=0 \ \forall n<0$

- lets look at $x[n-i]$ for $n=1$

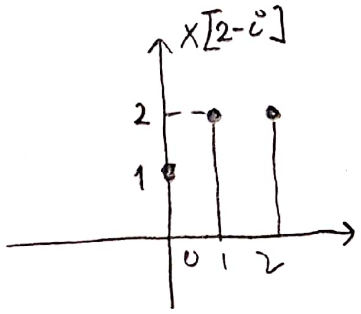


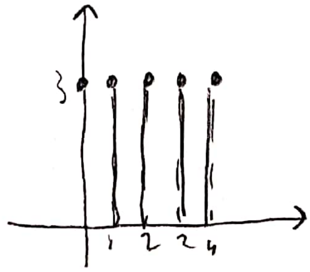


$y[1] = 6+6 = 12$



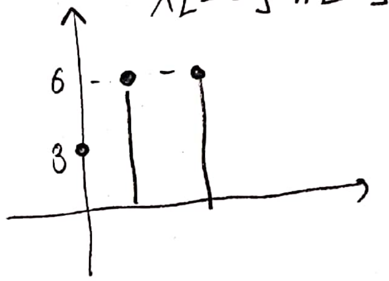
- lets look at $x[n-i]$ for $n=2$





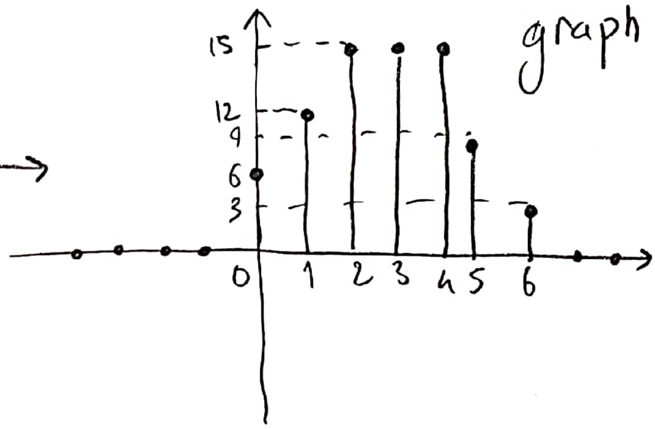
$y[2] = 3 + 6 + 6 = 15$

$x[2-i] \cdot h[i]$



- $n=0 \longrightarrow y[0] = 6$
- $n=1 \longrightarrow y[1] = 12$
- $n=2 \longrightarrow y[2] = 15$
- $n=3 \longrightarrow y[3] = 15$
- $n=4 \longrightarrow y[4] = 15$
- $n=5 \longrightarrow y[5] = 9$
- $n=6 \longrightarrow y[6] = 3$
- $n>6 \longrightarrow y[n] = 0$

graph $y[n]$



Fourier Series

$$X(t) = A_0 + \sum_{K=1}^{\infty} A_K \cdot \cos(K \cdot W_0 t + \emptyset_K) \quad , \quad -\infty < t < \infty$$

Amplitude & Phase form

Period : $T = 2\pi / W_0$

Fundamental Frequency W_0 : $W_0 = 2\pi / T$

Convert to Complex Exponential Form

$$X(t) = A_0 + A_1 \cdot \cos(1 \cdot W_0 t + \emptyset_1) + A_2 \cdot \cos(2 \cdot W_0 t + \emptyset_2) + \cdots$$

Euler Formula $\longrightarrow$ $\cos\theta = \frac{1}{2} \cdot \left[e^{J\theta} + e^{-J\theta} \right]$

$\therefore$

$$X(t) = \underbrace{A_0}_{C_0} + \underbrace{\frac{A_1}{2} e^{J\emptyset_1}}_{C_1} \cdot e^{JW_0 t} + \underbrace{\frac{A_1}{2} \cdot e^{-J\emptyset_1}}_{C_{-1}} \cdot e^{-JW_0 t} + \underbrace{\frac{A_2}{2} \cdot e^{J\emptyset_2}}_{C_2} \cdot e^{2JW_0 t} + \underbrace{\frac{A}{2} \cdot e^{-J\emptyset_2}}_{C_{-2}} \cdot e^{-2JW_0 t}$$

$\therefore$

$$X(t) = \sum_{K=-\infty}^{+\infty} a_K \cdot e^{K \cdot J \cdot W_0 \cdot t}$$

Convert to Sine-Cosine Form

$$X(t) = A_0 + A_1 \cdot \cos(W_0 t + \emptyset_1) + A_2 \cdot \cos(2 \cdot W_0 \cdot t + \emptyset_2) + \cdots$$

$\cos(\alpha + \beta) \longrightarrow \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$$X(t) = \underbrace{A_0}_{a_0} + A_1 \cdot \left[\cos(W_0 t) \cdot \cos(\emptyset_1) - \sin(W_0 t) \cdot \sin(\emptyset_1) \right] + A_2 \cdot \left[\cos(2W_0 t) \cdot \cos(\emptyset_2) - \sin(2W_0 t) \cdot \sin(\emptyset_2) \right]$$

a_1 b_1 a_2 b_2

$$X(t) = a_0 + \sum_{K=1}^{\infty} \left[a_K \cdot \cos(KW_0 t) + b_K \cdot \sin(KW_0 t) \right]$$

Relationships

1) $$X(t)= A_0 + \sum_{K=1}^{\infty} A_K \cdot \cos(K\omega_0 t + \emptyset_K)$$

2) $$X(t)= a_0 + \sum_{K=1}^{\infty} [a_K \cdot \cos(K\omega_0 t) + b_K \cdot \sin(K\omega_0 t)]$$

3) $$X(t)= \sum_{K=-\infty}^{+\infty} C_K \cdot e^{K \cdot J \cdot \omega_0 \cdot t}$$

1/2		1/3	
$A_0 = a_0$	$a_0 = A_0$	$C_0 = A_0$	$A_0 = C_0$
$A_K = \sqrt{a_K^2 + b_K^2}$	$a_K = A_K \cdot \cos(\emptyset_K)$	$C_K = \frac{1}{2} \cdot e^{J\emptyset_K} A_K$	$A_K = 2 \cdot \lvert C_K \rvert$
$\theta_K = \arctan\left(\frac{b_K}{a_K}\right)$	$b_K = -B_K \cdot \sin(\emptyset_K)$	$C_{-K} = \frac{1}{2} \cdot e^{-J\emptyset_K} A_K$	$\emptyset_K = \angle C_K$

2/3	
$C_0 = a_0$	$a_0 = C_0$
$C_K = \frac{1}{2}(a_K - J b_K)$	$a_K = 2 \operatorname{Re}\{C_K\}$
$C_{-K} = \frac{1}{2}(a_K + J b_K)$	$b_K = 2 \operatorname{Im}\{C_K\}$

Example

$X(t)= \cos(t) + 0.5 \cdot \cos(4t + \pi/3) + 0.25 \cdot \cos(8t + \pi/2)$, $\omega_0 = 1$

$A_1 = 1$, $A_4 = 0.5$, $A_8 = 0.25$

$\emptyset_1 = 0$, $\emptyset_4 = \pi/3$, $\emptyset_8 = \pi/2$

$C_1 = \frac{1}{2} \cdot A_K \cdot e^{J\emptyset_K} = \frac{1}{2} \cdot A_1 \cdot e^{J\emptyset_1} = 0.5$ $\quad C_4 = \frac{1}{2} \cdot \frac{1}{2} \cdot e^{J\pi/3} = \frac{1}{4} \cdot e^{J\pi/3}$ $\quad C_8 = \frac{1}{16} \cdot e^{J\pi/2}$

$C_{-1} = \frac{1}{2} A_K \cdot e^{-J\emptyset_K} = \frac{1}{2}$ $\quad C_{-4} = \frac{1}{4} \cdot e^{-J\pi/3}$ $\quad C_{-8} = \frac{1}{16} \cdot e^{-J\pi/2}$

$X(t) = 0.5 e^{Jt} + 0.5 e^{Jt} + 0.25 e^{J\pi/3} \cdot e^{J4t} + 0.25 e^{-J\pi/3} \cdot e^{-J4t} \ldots$

$a_1 = 1$ $\quad a_4 = 0.25$ $\quad a_8 = 0$

$b_1 = 0$ $\quad b_4 = 0.043$ $\quad b_8 = 0.25$

$a_K = A_K \cos(\theta_K)$
$b_K = -A_K \sin(\theta_K)$

Analytically Finding FS Coefficients

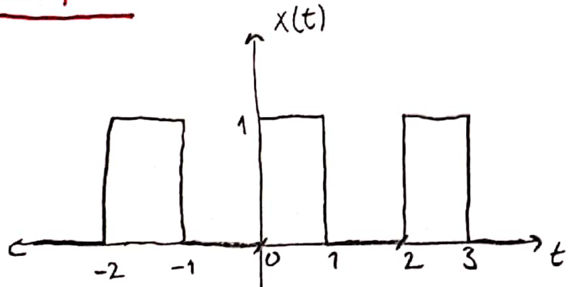
$$C_K = \frac{1}{T}\int_{t_0}^{t_0+T} X(t)\cdot e^{-jK\omega_0 t}\,dt \longrightarrow \text{Exponential Form}$$

$$a_K = \frac{2}{T}\int_{t_0}^{t_0+T} X(t)\cdot \cos(K\omega_0 t)\,dt$$

$$b_K = \frac{2}{T}\int_{t_0}^{t_0+T} X(t)\cdot \sin(K\omega_0 t)\,dt$$

$\longrightarrow$ Sine-Cosine Form

Example



$T=2 \longrightarrow \omega_0 = \frac{2\pi}{2} = \pi$ rad/sec

$t_0 = 0$

$$C_K = \frac{1}{T}\int_{t_0}^{t_0+T} X(t)\cdot e^{-j\omega_0 K t}\,dt = \frac{1}{2}\cdot\int_0^2 X(t)\cdot e^{-jK\pi t}\,dt$$

$$= \frac{1}{2}\left[\int_0^1 1\cdot e^{-jK\pi t}\,dt + \int_1^2 0\cdot e^{-jK\pi t}\,dt\right]$$

$$= \frac{1}{2}\cdot\int_0^1 e^{-jK\pi t}\,dt = \frac{1}{2}\frac{e^{-jK\pi t}}{-jK\pi}\Bigg|_0^1 = \frac{e^{-j\pi t}}{-2j\pi K} + \frac{1}{2j\pi K} = \frac{j}{2\pi K}\left[e^{-jK\pi} - 1\right]$$

$$C_K = \begin{cases} 0, & K \text{ even}, \neq 0 \\ \frac{-j}{K\pi}, & K \text{ odd} \end{cases}$$

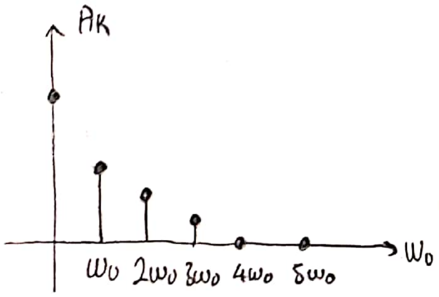
Fourier Series Spectrum

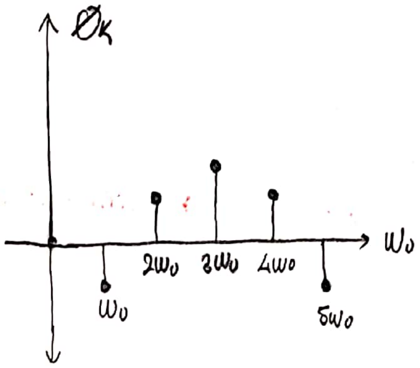
Amplitude & Phase form Spectrum

$$X(t) = A_0 + \sum_{K=1}^{\infty} A_K \cdot \cos(K\omega_0 t + \phi_K)$$

A_K = Amplitude

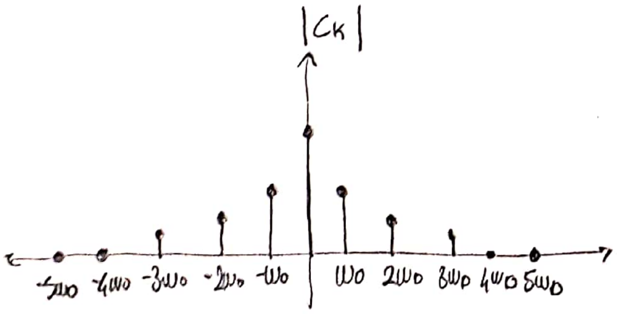
ϕ_K = Phase

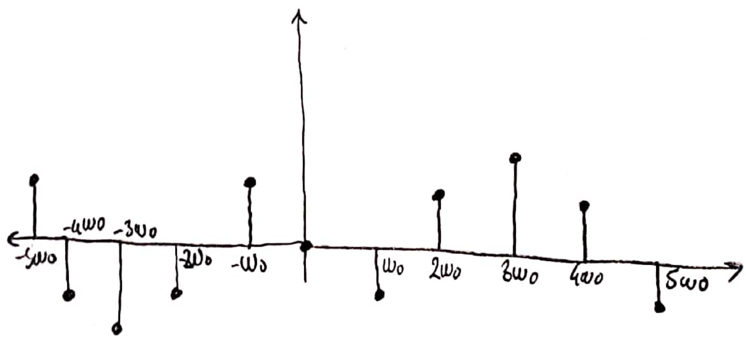




Exponential Form Spectrum

$$X(t) = \sum_{K=-\infty}^{\infty} C_K \cdot e^{j\omega_0 K \phi_K}$$



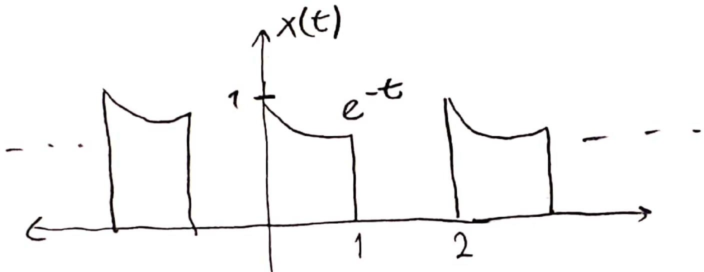


Parseval's Theorem

$$P = \frac{1}{T} \int_{t_0}^{T+t_0} x^2(t)\, dt$$ we saw that earlier

Parseval's Theorem $\longrightarrow$ $P = \sum_{K=-\infty}^{+\infty} |C_K|^2$

Example



$T = 2 \qquad \omega_0 = \frac{2\pi}{2} = \pi$ rad/sn

$$C_K = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot e^{-jk\omega_0 t}\, dt = \frac{1}{2} \int_0^2 x(t) \cdot e^{-jk\pi t}\, dt = \frac{1}{2} \left[\int_0^1 e^{-t} \cdot e^{-jk\pi t}\, dt + \cancel{\int_1^2 0\, dt} \right]$$

$$\frac{1}{2} \cdot \int_0^1 e^{-t(1+jk\pi)}\, dt = \frac{1}{2} \cdot \frac{e^{-t(1+jk\pi)}}{-(1+jk\pi)} \Bigg|_0^1 = \frac{e^{-(1+jk\pi)}}{-2(1+jk\pi)} + \frac{1}{2(1+jk\pi)}$$

Differential Equations

$$\sum_{i=0}^{n} a_i \cdot \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j f(t)}{dt^j}$$

Differential Equations like these are Linear and Time invariant

Coefficients are constant $\longrightarrow$ Time invariant

Non-linear terms $\longrightarrow$ Linear

Homogenius Differential Equation $\longrightarrow$
- There is no term that is based on function of x itself.
- all terms has the derivative of y or y itself

$\longrightarrow a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$

$\longrightarrow$ write D.E like this:

$$(D^n + a_{n-1} D^{n-1} + \dots + a_1 D^1 + a_0)\, y(t) = (b_m \cdot D^m + \dots + b_1 D^1 + b_0)\, f(t) \quad , \quad m \leq n$$

$\longrightarrow Q(D) \cdot y(t) = P(D) f(t)$

Differential Equation Examples

Find the zero input response for these tree Differential Equations.

Example (a):

$$\frac{d^2y(t)}{dt^2} + 3 \cdot \frac{dy(t)}{dt} + 2y(t) = \frac{df(t)}{dt} \quad , \; y(0) = 0 \; , \; y'(0) = -5$$

$$D^2y(t) + 3Dy(t) + 2y(t) = Df(t)$$

The zero-input form is:

$$\frac{d^2y(t)}{dt^2} + 3 \cdot \frac{dy(t)}{dt} + 2y(t) = 0$$

$$D^2y(t) + 3Dy(t) + 2y(t) = 0$$

Characteristic Equation is:

$$\lambda^2 + 3\lambda + 2 = 0 \longrightarrow (\lambda+1) \cdot (\lambda+2) = 0$$

Roots: $\lambda_1 = -1$, $\lambda_2 = -2$

Characteristic Models are:

$$e^{\lambda_1 t} = e^{-t}$$

$$e^{\lambda_2 t} = e^{-2t}$$

Zero-input solution is: $y_{zi}(t) = C_1 \cdot e^{-t} + C_2 \cdot e^{-2t}$

$$\left.\begin{array}{l} y_{zi}(0) = C_1 + C_2 = 0 \\ y_{zi}'(0) = -C_1 - 2C_2 = -5 \end{array}\right\} \quad \begin{array}{l} C_1 = -5 \\ C_2 = 5 \end{array}$$

$$y_{zi}(t) = 5e^{-2t} - 5e^{-t}$$

Example (6):

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+9y(t)=3\frac{df(t)}{dt}+5f(t)$$

$y(0)=3,\ y'(0)=-7$

$$D^2y(t)+6Dy(t)+9y(t)=3Df(t)+5f(t)$$

$\lambda^2+6\lambda+9=0 \longrightarrow (\lambda+3)^2=0$

$\lambda_1=-3\ ,\ \lambda_2=-3$

e^{-3t} and $t\cdot e^{-3t}$

$e^{-3t}-3t\cdot e^{-3t}$

$y_{zi}(t)=C_1\cdot e^{-3t}+C_2 t\cdot e^{-3t}$

$y(0)=C_1=3$

$y'(0)=-3C_1+\underset{\downarrow\atop 2}{C_2}=-7$

$y_{zi}(t)=-3e^{-3t}+2\cdot t\cdot e^{-3t}$

Example (c):

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 40y(t) = \frac{df(t)}{dt} + 2f(t)$$

$$D^2y(t) + 4Dy(t) + 40y(t) = Df(t) + 2f(t)$$

$y(0) = 2$

$y'(0) = 16.78$

$$\lambda^2 + 4\lambda + 40 = 0 \longrightarrow \lambda_1 = 2 - j6 \qquad \lambda_2 = 2 + j6$$

$$e^{\lambda_1 t} = e^{-2t} \cdot e^{j6t}$$

$$e^{\lambda_2 t} = e^{-2t} \cdot e^{-j6t}$$

$$y_{zi}(t) = C_1 \cdot e^{-2t} \cdot e^{j6t} + C_2 \cdot e^{-2t} \cdot e^{-j6t}$$

$$C_1 + C_2 = 2$$

$$y_{zi}(t) = C_1 \cdot e^{-2t} \cdot (\cos 6t + j\sin 6t) + C_2 \cdot e^{-2t} \cdot (\cos 6t - j\sin 6t)$$

$$\frac{dy_{zi}}{dt} = C_1\left[-2e^{-2t} \cdot (\cos 6t + j\sin 6t) + e^{-2t} \cdot (-6\sin 6t + 6j\cos 6t)\right]$$

$$+$$

$$C_2\left[-2e^{-2t}(\cos 6t - j\sin 6t) + (-6\sin 6t - 6j\cos 6t)\right]$$

$$\frac{dy_{zi}(0)}{dt} = C_1 \cdot [-2 + 6j] + C_2 \cdot [-2 - 6j] = 16.78$$

$$-2(C_1 + C_2) + 6j(C_1 - C_2) = 16.78$$

$$y_{zi}(t) = 4e^{-2t}\cos(6t + \pi/3)$$

controls oscillation

controls Decay

Frequency-Domain Analysis of Systems

$$x(t) = A\cos(\omega_0 t+\theta) = \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)} + \frac{A}{2}\cdot e^{-j(\omega_0 t+\theta)}$$

$$x_1(t) = \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ?$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(t-\tau)\cdot h(\tau)\,d\tau = \int_{-\infty}^{+\infty} \frac{A}{2}\cdot e^{j(\omega_0(t-\tau)+\theta)}\,h(\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)}\cdot e^{-j\omega_0\tau}\cdot h(\tau)\cdot d\tau$$

$$= \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)}\cdot \underbrace{\int_{-\infty}^{+\infty} e^{-j\omega_0\tau}\,h(\tau)\,d\tau}_{H(\omega_0)}$$

Evaluates to some complex number that depends on $h(t)$ and ω_0

The output is just this complex sinusoidal input multiplied by some complex number.

So,

$$y(t) = H(\omega_0)\cdot \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)}$$

complex valued

Because its complex we can write:

$$H(\omega_0) = |H(\omega_0)|\cdot e^{j\angle H(\omega_0)}$$, then

$$y(t) = \left(|H(\omega_0)|\cdot e^{j\angle H(\omega_0)}\right)\cdot \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)}$$

$$y(t) = \underbrace{\left(|H(\omega_0)|\cdot \frac{A}{2}\right)}_{\text{system changes the amptitude}} e^{j\left(\omega_0 t+\underbrace{\theta+\angle H(\omega_0)}_{\text{system changes the phase}}\right)}$$

if $x(t) = A\cos(\omega_0 t+\theta) = \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)} + \frac{A}{2}\cdot e^{-j(\omega_0 t+\theta)}$ then $y(t)$ will be:

$$|H(\omega)|\cdot \frac{A}{2}\left[e^{-j(\omega_0 t+\theta+\angle H(\omega_0))} + e^{j(\omega_0 t+\theta+\angle H(\omega_0))}\right]$$

LTI does changing amptitude and phase to sinusodial signal.

$$X(t) = A_1 \cdot \cos(\omega_0 t + \theta_1) + A_2 \cdot \cos(\omega_1 t + \theta_2)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = |H(\omega_0)| \cdot \frac{A_1}{2} \left[e^{-J(\omega_0 t + \theta + \angle H(\omega_0))} + e^{J(\omega_0 t + \theta + \angle H(\omega_0))} \right] + |H(\omega_1)| \frac{A_2}{2} \left[e^{-J(\omega_1 t + \theta + \angle H(\omega_1))} + e^{J(\omega_1 t + \theta \angle H(\omega_1))} \right]$$

Response to Periodic Signals

$$\text{periodic } x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(\omega) \end{matrix}} \longrightarrow y(t) = ?$$

write in FS: $\sum_{K=-\infty}^{+\infty} C_K \cdot e^{JK\omega_0 t}$

$$\sum_{K=-\infty}^{+\infty} \underbrace{C_K^x}_{\text{indicates } x(t)} \cdot e^{JK\omega_0 t} \longrightarrow \boxed{\begin{matrix} h(t) \\ H(\omega) \end{matrix}} \longrightarrow y(t) = ?$$

$$y(t) = \int_{-\infty}^{+\infty} \sum_{K=-\infty}^{+\infty} (C_K^x \cdot e^{JK(\omega_0 - \tau)t}) \cdot h(\tau)\, d\tau = \int_{-\infty}^{+\infty} \sum_{K=-\infty}^{+\infty} \left[C_K^x \cdot e^{JK\omega_0 t} \cdot e^{-JK\omega_0 \tau} \right] h(\tau)\, d\tau$$

$$y(t) = \sum_{K=-\infty}^{+\infty} H(\omega_0 K) \cdot e^{JK\omega_0 t} \cdot C_K^x$$

→ FS coefficient for $y(t)$, $C_K^y = |H(\omega_0 K)| \cdot C_K^x$

Response to Aperiodic Signal

impulse response

$$A \cdot e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega) \cdot e^{j\omega t} \cdot A \,, \quad H(\omega) = \mathcal{F}\{h(t)\}_{\omega} = \int_{-\infty}^{+\infty} e^{-j\omega t} \cdot h(t)\, dt$$

↳ frequency response

$$f(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} F(\omega) \cdot e^{j\omega t}\, d\omega \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \big(H(\omega) \cdot F(\omega)\big) \cdot e^{j\omega t}\, d\omega$$

This says FT of output = FT of input X Freq. Response

Finding $y(t)$

1 ⟶ do the convolution $x(t) * h(t)$

2 ⟶ do the:

- Compute $H(\omega)$, $F(\omega)$
- compute the product $H(\omega) F(\omega) = Y(\omega)$
- Compute the IFT $y(t) = \mathcal{F}^{-1}\{H(\omega) F(\omega)\}$

Filters

Ideal Filters:

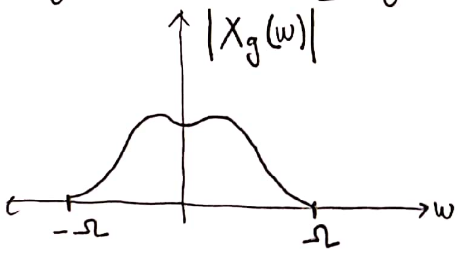
desired output

$$X(t) = x_g(t) + x_b(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x_g(t)$$

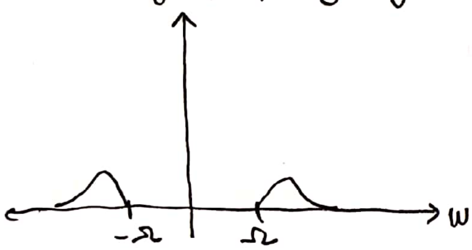
filter

Case #1:

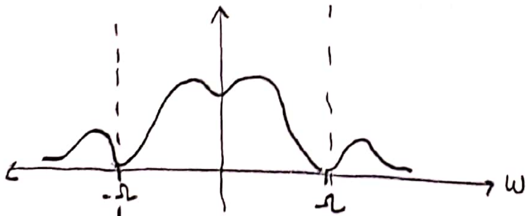
$x_g(t)$: low-frequency signal

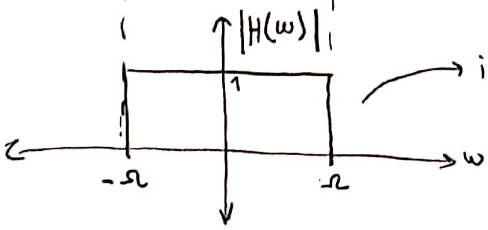


$x_b(t)$: high frequency signal



spectrum of the input signal now:





in this case we want filter like this

so $H(w) = \begin{cases} 1, & |\Omega| > w \\ 0, & \text{otherwise} \end{cases}$

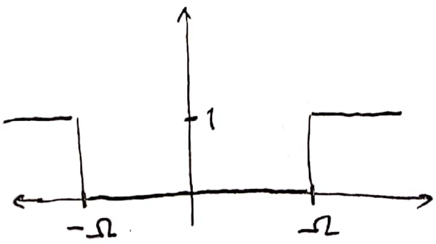
LOW-PASS filter

then $Y(w) = |X_g(w)||H(w)| + \underbrace{|X_b(w)|\cdot|H(w)|}_{0} = X_g(w)$

Case #2:

$X_g(t)$: high-frequency signal $X_b(t)$: low frequency signal

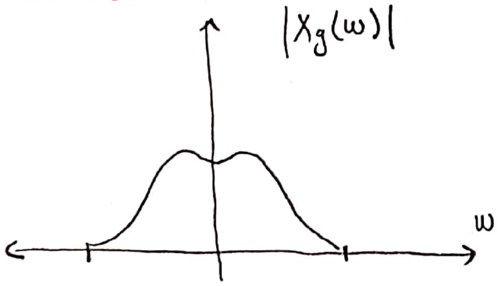
then $|H(\omega)|$ is

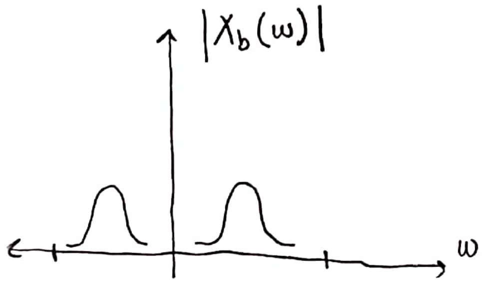


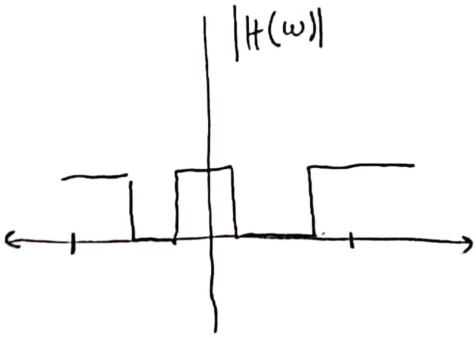
$$|H(\omega)| = \begin{cases} 0, & \omega < |\Omega| \\ 1, & \text{otherwise} \end{cases}$$

High pass filter

Case #3:

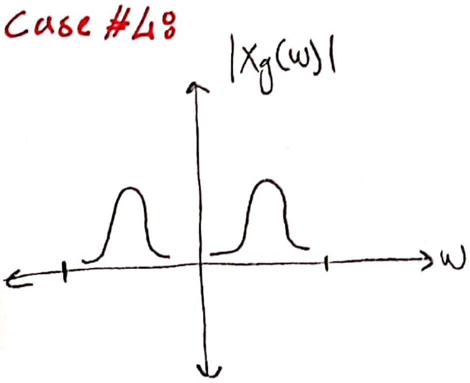


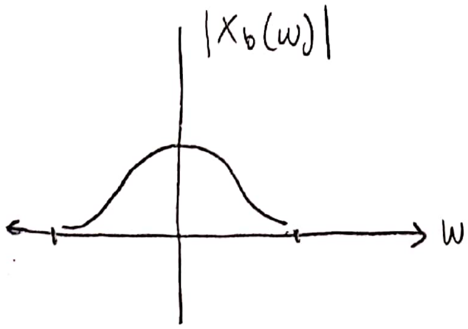


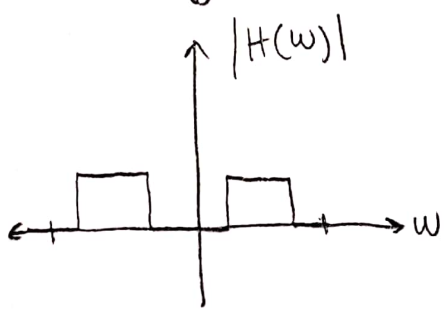


Bandstop filter

Case #4:







What about the phase of the filter's $H(\omega)$?

we could tolerate a small delay in output

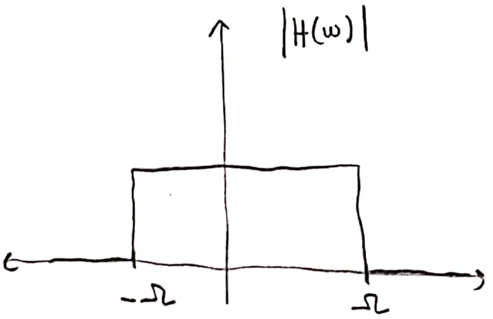
$$x_g(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(\omega) \end{matrix}} \longrightarrow y(t) = x_g(t - t_d)$$

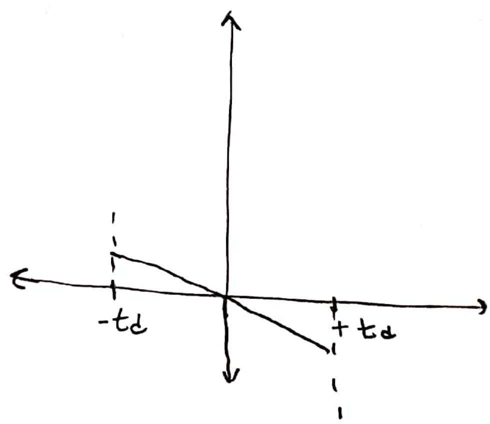
$Y(\omega) = X_g(\omega) \cdot e^{-j\omega t_d}$ is we need

then $|H(\omega)| = |e^{-j\omega t_d}| = 1$

$\angle H(\omega) = \angle e^{-j\omega t_d} = -\omega t_d$

so





Laplace Trasform

Fourier of $x(t)$ is $\mathcal{F}\{x(t)\}_\omega = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} \, dt$

when $s = \alpha + j\omega$

Laplace trasform becomes $\mathcal{L}\{x(t)\}_s = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} \, dt$

$X(s) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} \, dt \longrightarrow$ bilateral / Two-sided LT

$X(s) = \int_{0}^{+\infty} x(t) \cdot e^{-st} \, dt \longrightarrow$ unilateral / One-sided LT

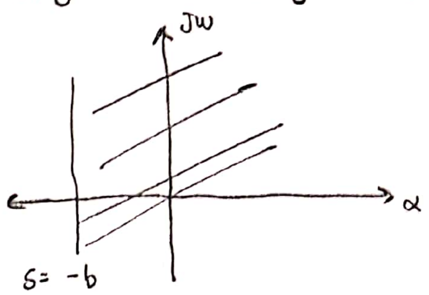
Example

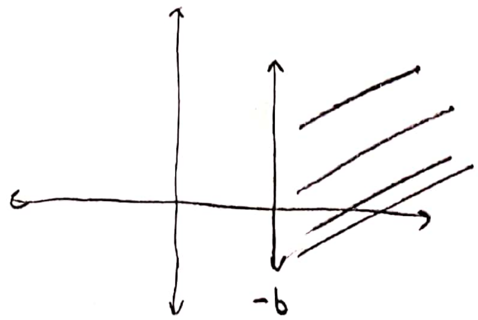
$x(t) = e^{-bt} \cdot u(t)$, $\mathcal{L}\{x(t)\}_s = \int_{-\infty}^{+\infty} e^{-bt} \cdot u(t) \cdot e^{-st} \, dt = \int_{0}^{+\infty} e^{-(b+s)t} \, dt = \frac{1}{b+s}$

$b+s > 0$, $s > -b$ two case $\longrightarrow$ $-b$ is negative

$-b$ is positive

$\hookrightarrow \alpha + j\omega + b > 0$, $\alpha + j\omega > -b$





Converge regions

Inverse Laplace

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) \cdot e^{st} \, ds$$

if $X(s) = \dfrac{b_m s^M + b_{m-1} s^{M-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$ its easy to find $x(t)$ using partial fraction expasion

Properties of Bilateral Laplace Transform

$a x(t) + b x(t) \leftrightarrow a X(s) + b X(s) \longrightarrow$ linearity

$x(t-c) \leftrightarrow e^{-cs} X(s) \longrightarrow$ time shift

$x'(t) \leftrightarrow s \cdot X(s) \longrightarrow$ Time differentation

$\int_{-\infty}^{t} x(\lambda) \, d\lambda \leftrightarrow \frac{1}{s} X(s) \longrightarrow$ Integration

$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right) \longrightarrow$ time scaling

$t^n \cdot x(t) \leftrightarrow (-1)^N \cdot \frac{d^N}{ds^N} \cdot X(s) \longrightarrow$ multiply by t^n

$e^{at} \cdot x(t) \leftrightarrow X(s-a) \longrightarrow$ multiply by exponential

$\hookrightarrow$ time shifting in s-plane

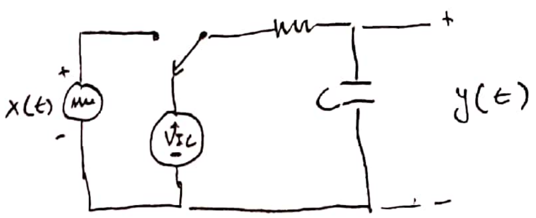
$x(t) \cdot \sin(w_0 t) \leftrightarrow \frac{J}{2} \left[X(s + Jw_0) - X(s - Jw_0) \right]$

$x(t) \cdot \cos(w_0 t) \leftrightarrow \frac{1}{2} \cdot \left[X(s + Jw_0) + X(s - Jw_0) \right]$

$\longrightarrow$ multiply by sinusoids

$x(t) * h(t) \leftrightarrow X(s) H(s) \longrightarrow$ convolution

Example



Diff eq: $\dfrac{dy(t)}{dt} + \dfrac{1}{RC} \cdot y(t) = \dfrac{1}{RC} \cdot x(t)$

$$= \frac{dy(t)}{dt} + a y(t) = b x(t)$$

$$\mathcal{L}\left\{ \frac{dy(t)}{dt} + a y(t) \right\} = \mathcal{L}\{ b x(t) \}$$

$$= \mathcal{L}\left\{ \frac{dy(t)}{dt} \right\} + \mathcal{L}\{ a y(t) \} = \mathcal{L}\{ b x(t) \}$$

$x(t) = u(t) \,/\, X(s) = 1/s$

$$= [s \cdot Y(s) - y(0^-)] + a Y(s) = b X(s)$$

$$Y(s) = \frac{y(0^-)}{s+a} + \frac{b}{s+a} \cdot X(s) = \frac{y(0^-)}{s+a} + \frac{1/RC}{s+1/RC} \cdot x(t) = \frac{y(0^-)}{s+1/RC} + \frac{1/RC}{s+1/RC} \cdot \frac{1}{s}$$

this transfers $X(s)$ to the $Y(s)$

The transfer function

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{y(0^-)}{s+1/RC} \right\} + \mathcal{L}^{-1}\left\{ \frac{1/RC}{(s+1/RC)s} \right\}$$

$$\to y(0^-) \cdot e^{-(t+1/RC)} \cdot u(t)$$

$$\to \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s+1/RC} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s+1/RC} \right\}$$

$$[u(t) - e^{-t(1/RC)} \cdot u(t)]$$

$y(t)$

$$= (1 - e^{-t(1/RC)}) \cdot u(t)$$

Partial Fractions

$$Y(s) = \frac{3s-1}{s^2+3s+2} \quad , \quad y(s) = \frac{r_1}{s+1} + \frac{r_2}{s+2} = \frac{3s-1}{(s+1)(s+2)}$$

multiply $(s+1) \longrightarrow r_1 + \frac{r_2(s+1)}{s+2} = \frac{3s-1}{s+2}$, $\boxed{s=-1} \rightarrow \boxed{r_1 = -4}$

multiply $(s+2) \longrightarrow r_2 + \frac{r_1(s+2)}{(s+1)} = \frac{3s-1}{s+1}$, $s=-2 \longrightarrow \boxed{r_2 = +7}$

$$y(s) = \frac{-4}{s+1} + \frac{7}{s+2} \quad , \quad y(t) = \mathcal{L}^{-1}\left\{\frac{-4}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{7}{s+2}\right\} = -4 \cdot e^{-t} u(t) + 7 \cdot e^{-2t} u(t)$$

Poles and Zeros

$$\underbrace{\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 \cdot y(t)} = \underbrace{b_1 \cdot \frac{dx(t)}{dt} + b_0 X(t)}$$

$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$$H(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0} \begin{array}{l} \longrightarrow B(s) \\ \longrightarrow A(s) \end{array}$$

factoring $B(s)$ and $A(s)$

$$H(s) = \frac{b_m \cdot (s-z_1)(s-z_2) \cdots (s-z_m)}{(s-p_1) \cdot (s-p_2) \cdots (s-p_n)}$$

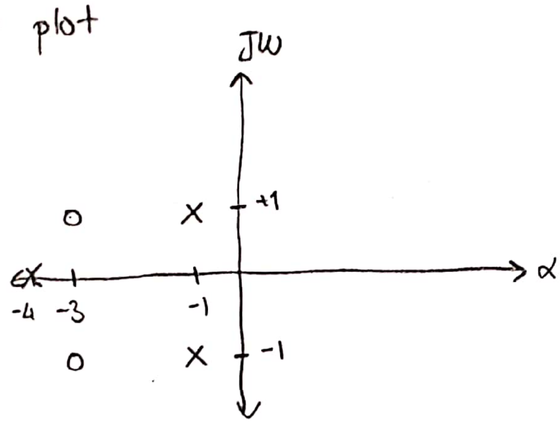
$\longrightarrow H(s)\Big|_{s=z_i} = 0 \quad i \in [1, m]$ → zeros of $H(s)$

$\longrightarrow H(s)\Big|_{s=p_i} = \infty \quad i \in [1, n]$ → poles of $H(s)$

Example

$$H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2\cdot(s+3-J)(s+3+J)}{(s+4)(s+1-J)(s+1+J)}$$

Pole-Zero plot



O → zero

X → pole

Z - Transform

$$\left.\begin{matrix} x[n] \\ h[n] \end{matrix}\right] \longrightarrow \begin{matrix} X[z] = \sum_{n} x[n] \cdot z^{-n} \\ H[z] = \sum_{n} h[n] \cdot z^{-n} \end{matrix}$$

Example

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

$$x[n] = ?$$

$$x[n] = \begin{cases} 1, & n=0 \\ -2, & n=1 \\ 0, & n=2 \\ 3, & n=3 \\ 0, & n=4 \\ -\underset{1}{\cancel{2}}, & n=5 \end{cases}$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \overset{1}{\cancel{d}}\delta[n-5]$$

Example

$$X(z) = \sum_{n} x[n] \cdot z^{-n}$$

$$= 1 + 0.8z^{-1} + 0.64 \cdot z^{-2} + 0.512z^{-3} + \cdots$$

$$= 1 + 0.8z^{-1} + (0.8 \cdot z^{-1})^2 + (0.8 \cdot z^{-1})^3 + \cdots$$

$$= \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8}$$

Example

$$X(z) = \frac{1}{z + 1.2} = \frac{z^{-1}}{1 + 1.2 \cdot z^{-1}} = z^{-1}\left(1 + (1.2 \cdot z^{-1})^2 + (1.2 \cdot z^{-2})^3 + \cdots + \right)$$

$\sim$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z = e^{j\hat{\omega}}}$$

Z-Transform of FIR Filter

$$H(z) = \sum_{K=0}^{M} b_K \cdot z^{-K} = \sum_{K=0}^{M} h[K] \cdot z^{-K}$$

$$y[n] = \sum_{K=0}^{M} b_K \cdot x[n-K] = \sum_{K=0}^{M} h[K] \cdot x[n-K]$$

Example

$y[n] = 6x[n] - 5x[n-1] + x[n-2]$ $\quad h[n] = 6\delta[n]$

$\{b_K\} = \{6, -5, 1\}$

$\hookrightarrow H(z) = 6 - 5 \cdot z^{-1} + z^{-2}$

Delay System

$x[n] \longrightarrow \delta[n-1] \longrightarrow y[n] = x[n-1]$

$H(z) = \sum \delta[n-1] \cdot z^{-n} = z^{-1}$

if $z^{-n_0} \rightarrow$ delays n_0 samples

$x[n] \longrightarrow z^{-1} \longrightarrow y[n] = x[n-1]$

Example

$y(z) = z^{-1} X(z)$ $\quad x[n] = \{3, 1, 4, 1, 5, 9, 0, 0, 0\}$

$y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$

$y(z) = 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9 \cdot z^{-6}$

$y[n] = \{0, 3, 1, 4, 1, 5, 9\}$

Convolution Property of ZT

$y[n] = x[n] * h[n]$

$Y(z) = X(z)H(z)$

Cascade Systems

$$x[n] \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$H(z) = H_1(z)H_2(z)$

$h[n] = h_1[n] * h_2[n]$

Example

$$x[n] \longrightarrow \boxed{H_1(z)} \xrightarrow{w[n]} \boxed{H_2(z)} \longrightarrow y[n]$$

$w[n] = x[n] - x[n-1]$

$h_1[n] = \delta[n] - \delta[n-1]$

$H_1(z) = 1 - z^{-1}$

$y[n] = w[n] + w[n-1]$

$h_2[n] = \delta[n] + \delta[n-1]$

$H_2(z) = 1 + z^{-1}$

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$H(z) = (1-z^{-1}) \cdot (1+z^{-1}) = 1 - z^{-2}$

$h[n] = \delta[n] - \delta[n-2]$

$y[n] = h[n] * x[n]$

$y[n] = x[n] - x[n-2]$

Convolution Example

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$x[n] = \delta[n-1] + 2\,\delta[n-2]$, $h[n] = \delta[n] - \delta[n-1]$, $y[n] = h[n] * x[n] = ?$

$X(z) = z^{-1} + 2 \cdot z^{-2}$

$H(z) = 1 - z^{-1}$

$y(z) = (z^{-1} + 2 \cdot z^{-2}) \cdot (1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3}$

$y[n] = \delta[n-1] + \delta[n-2] - 2\,\delta[n-3]$

Zeros of $H(z)$

Example

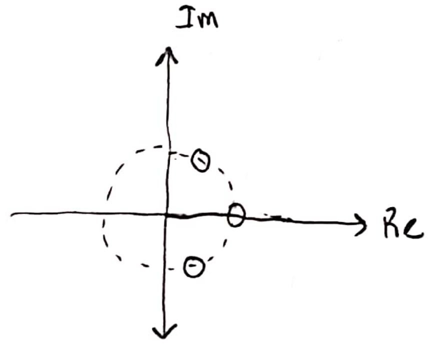
$H(z) = 1 - \frac{1}{2} \cdot z^{-1}$, find z, where $H(z) = 0$

$1 - \frac{1}{2} \cdot z^{-1} = 0$, $1 = \frac{1}{2} \cdot z^{-1} \longrightarrow 2 = z^{-1} \longrightarrow z = \frac{1}{2}$

zero at $z = \frac{1}{2}$

Example

find z, where $H(z) = 0$



$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$

$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$

Roots : 1 , $\frac{1}{2} \pm j \cdot \frac{\sqrt{3}}{2}$ $(e^{\pm j\pi/3})$

Poles of $H(z)$

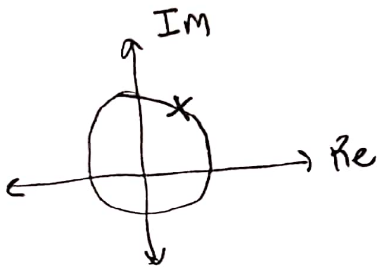
Example

find z , where $H(z) \longrightarrow \infty$

$H(z) = 1 - 2z^{-1} + 2z^{-2} - 3z^{-3}$

$H(z) = \dfrac{z^3 - 2z^2 + 2z - 3}{z^3}$, three poles at $z = 0$

$z = e^{j\omega}$



$1 - 2x + 2x^2 - x^3$

$-x^3 + 2x^2 - 2x + 1 = 0$

$+x^3 + 2x^2 - 2x = +1$

$x^3 - 2x^2 + 2x = 1$

Nulling Property of $H(z)$

When $H(z)=0$ on the unit circle

- Find inputs $x[n]$ that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - z^{-j3\hat{\omega}}$$

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n] \qquad H(e^{j\pi/3}) = ?$$

$$x[n] = e^{j(\pi/3)n} \qquad y[n] = H(e^{j(\pi/3)}) \cdot e^{j\pi/3\, n}$$

Evaulate $H(z)$ at the input frequency

$$y[n] = H(e^{j\pi/3}) \cdot e^{j\pi/3 \cdot n}$$

$$y[n] = \left(1 - 2\cdot\left(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right) + 2\cdot\left(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right) - (-1)\right)$$

$$y[n] = \left(1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1\right) \cdot e^{j(\pi/3)\cdot n} = 0$$

Infinite-Length Signal: h[n]

- Polynomial Representation

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] \cdot z^{-k}$$

- Simplify the Summation

$$H(z) = \sum_{k=0}^{+\infty} b_0 (a_1)^k \cdot u[k] \cdot z^{-k} = b_0 \sum_{n=0}^{+\infty} (a_1)^n \cdot z^{-n} \quad , \quad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

so $$H(z) = \frac{b_0}{1-a_1 z^{-1}}$$

First Order IIR Filter

$$y[n] = a_1 \cdot y[n-1] + b_0 . x[n]$$

- $$h[n] = b_0 \cdot (a_1)^n \cdot u[n]$$

$$H(z) = \frac{b_0}{1-a_1 z^{-1}}$$

$$y[n] = a_1 \cdot y[n-1] + b_0 x[n] + b_1 x[n-1]$$

- $$h[n] = b_0 \cdot a_1^{\,n} \cdot u[n] + b_1 \cdot a_1^{\,n-1} u[n-1]$$

$$H(z) = \frac{b_0}{1-a_1 \cdot z^{-1}} + \frac{b_1 z^{-1}}{1-a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}}$$

Step Response: $x[n] = u[n]$

$$y[n] = a_1 \cdot y[n-1] + b_0 x[n]$$

n	$x[n]$	$y[n]$
$n<0$	0	0
0	1	b_0
1	1	$b_0 + b_0 a_1$
2	1	$b_0 + b_0 a_1 + b_0 a_1^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
⋮	⋮	⋮

$$y[n] = b_0 (1 + a_1 + a_1^2 + \cdots + a_1^n) = b_0 \cdot \sum_{k=0}^{n} a_1^k$$

, recall: $\sum_{k=0}^{L} r^k = \dfrac{1 - r^{L+1}}{1 - r}$, $r \neq 1$

if $r = 1$, $L+1$

so $y[n] = b_0 \cdot \dfrac{1 - a_1^{n+1}}{1 - a_1}$

Example

Derive the system function $H(z)$

-Use delay property

$$y[n] = a_1 \cdot y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 \cdot z^{-1} \cdot Y(z) + b_0 \cdot X(z) + b_1 \cdot z^{-1} X(z)$$

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$Y(z)(1 - a_1 z^{-1}) = X(z)(b_0 + b_1 z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

Notes from Examples

Signals and Systems

Sampling

- $x(t) = \cos(2\pi f_0 t + \theta)$ $\quad f_0 \to$ frequency

Sampling rate f_s

$$x[n] = x(t)\Big|_{t=n\cdot T_s} = \cos(2\pi f_0 (nT_s) + \theta) = \cos(2\pi f_0 T_s n + \theta) = \cos\left(2\pi \frac{f_0}{f_s} n + \theta\right)$$

$$x[n] = \cos(\hat{\omega}_0 n + \theta) \quad , \quad \hat{\omega}_0 = 2\pi \cdot \frac{f_0}{f_s}$$

$f_0 = 440\ \text{Hz}$ $\quad f_s = 8000\ \text{Hz}$

$$\hat{\omega}_0 = 2\pi \cdot \frac{440}{8000} = 2\pi \cdot \frac{11}{200} = 2\pi \cdot \frac{N}{L} \longrightarrow x[n] = \cos(\hat{\omega}_0 n + \theta) = \cos\left(2\pi \cdot \frac{N}{L} \cdot n + \theta\right)$$

$$x[n+N_0] = \cos\left(2\pi \frac{N}{L} \cdot (n+N_0) + \theta\right) = \cos\left(2\pi \frac{N}{L} \cdot n + 2\pi \cdot \frac{N}{L} \cdot N_0 + \theta\right)$$

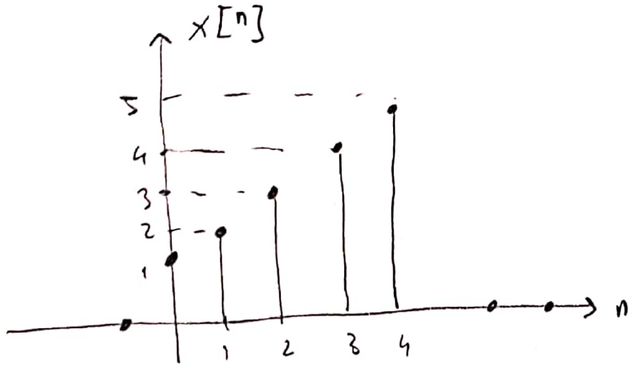
The smallest period $= N_0 = L$

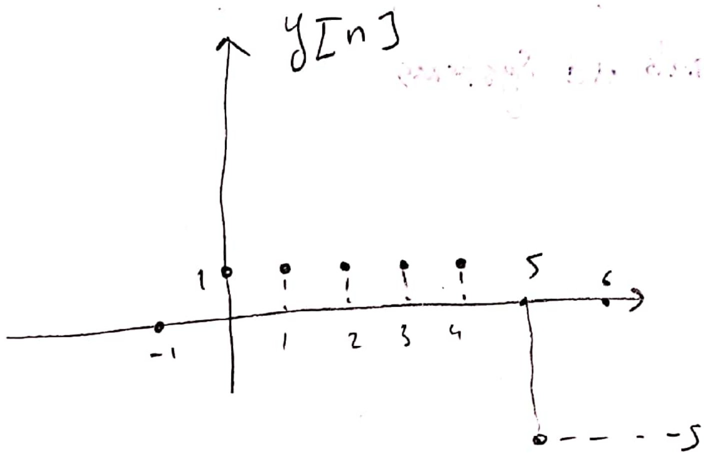
- $g(t) = \cos(2\pi f_0 t) + \cos(2\pi (3f_0) t)$

Sampling rate $f_s = 2f_0$ $\quad$ obtain formula for $g[n]$

$$g[n] = g(n \cdot T_s) = g\left(\frac{n}{f_s}\right) = \cos\left(2\pi \cdot \frac{f_0}{2f_0} \cdot n\right) + \cos\left(2\pi \cdot \frac{3f_0}{2f_0} n\right) = 2\cos(\pi n)$$

f_s = sampling rate

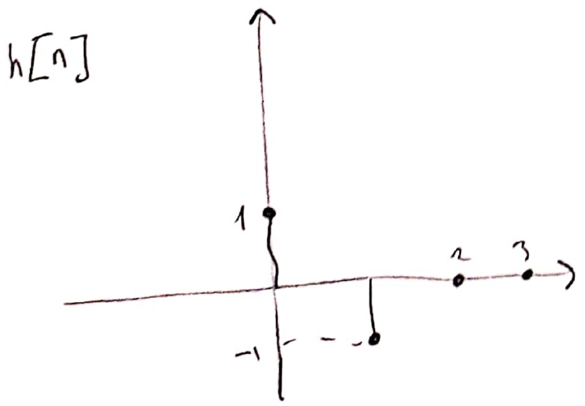




Using z-domain and z-transform, obtain input response

$$Y(z) = X(z) \cdot H(z)$$

$$H(z) = \left(\frac{X(z)}{Y(z)}\right)^{-1} = \frac{1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} - 5z^{-5}}{1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}} = 1 - z^{-1}$$



z-transform

- $x(t) = \cos(\omega_0 t)$ and $h(t) = e^{-a|t|}$ $\quad -\infty < t < \infty$

$y(t) = ?$

$$y(t) = \int_{-\infty}^{+\infty} h(\lambda) \cdot x(t-\lambda)\, d\lambda = \int_{-\infty}^{+\infty} e^{-a|\lambda|} \cdot \cos(\omega_0(t-\lambda))\, d\lambda$$

$$e^{-a|t|} = e^{at} \cdot u(-t) + e^{-at} \cdot u(t) \quad , \quad u(0) = 1/2$$

$$y(t) = \int_{-\infty}^{0} e^{a\lambda} \cdot \cos(\omega_0(t-\lambda))\, d\lambda + \int_{0}^{\infty} e^{-at} \cdot \cos(\omega_0(t-\lambda))\, d\lambda$$

$$h(t) = e^{-a|t|} = e^{at} \cdot u(-t) + e^{-at} u(t)$$

$$H(j\omega) = \frac{1}{a - j\omega_0} + \frac{1}{a + j\omega_0} = \frac{2a}{a^2 + \omega_0^2} \rightarrow y(t) = \frac{2a}{a^2 + \omega_0^2} \cdot \cos(\omega_0 t)$$

$$\mathcal{F}\{e^{at}\} = \int_{-\infty}^{0} e^{j\omega_0 t} \cdot e^{at} u(t) dt = \int_{-\infty}^{0} e^{j\omega_0 (t+a)} u(-t)\, dt = \frac{1}{j\omega_0 + a}$$

$$\mathcal{F}\{e^{-at} \cdot u(t)\} = \int_{0}^{\infty} e^{j\omega t} \cdot e^{-at} \cdot u(t) dt = \int_{-\infty}^{+\infty} e^{j\omega_0 (t-a)} u(-t)\, dt = \frac{1}{j\omega_0 - a}$$

obtain $y(t)$ using Fourier transform to impulse response

Fourier Transform Table

$$\mathfrak{F}\{x(t)\}_{w} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-jwt} \, dt$$

$$\mathfrak{F}\{x[n]\}_{\Omega} = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\Omega n}$$

$-0.5 + u(t) \longrightarrow 1/jw$

$u(t) \longrightarrow \pi \cdot \delta(w) + 1/jw$

$\delta(t) \longrightarrow 1$

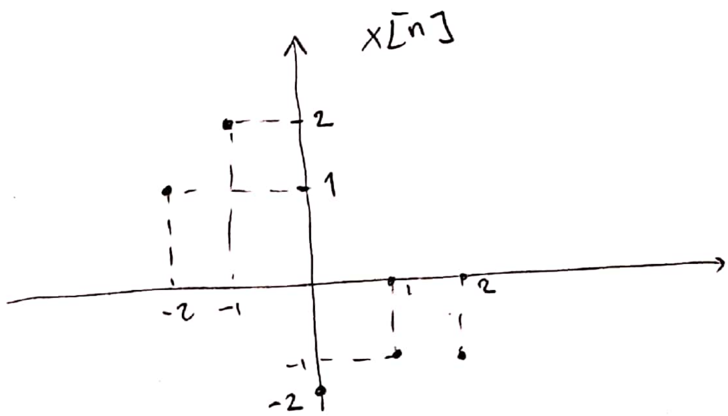
$\delta(t-c) \longrightarrow e^{-jwc}$

$e^{-at} u(t) \longrightarrow \dfrac{1}{jw+a}$

$e^{at} u(-t) \longrightarrow \dfrac{1}{jw-a}$

c

$y = 2x[n] + x[n-1] + 2x[n-3]$

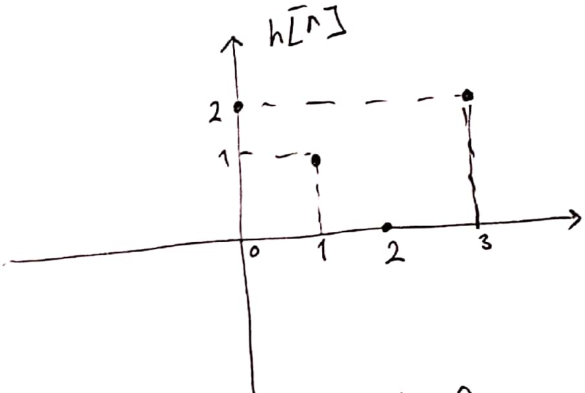


$h[n] = ?$

$y[n] = ?$

$\mathcal{F}\{x[n]\} = ?$

$\mathcal{F}\{h[n]\}$



$h[n] = 2\delta[n] + \delta[n-1] + 2\delta[n-3]$

$y[n] = x[n] * h[n]$

$$x[k] \longrightarrow \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \\ \boxed{1} & \boxed{2} & \boxed{-2} & \boxed{-1} & \boxed{-1} \end{array} \qquad h[n-k] \longrightarrow \begin{array}{cccc} n-3 & n-2 & n-1 & n \\ \boxed{2} & \boxed{0} & \boxed{1} & \boxed{2} \end{array}$$

$n < -2 \longrightarrow y[n] = 0$

$n > 5 \longrightarrow y[n] = 0$

$n = -2 \longrightarrow y[-2] = 2$

$n = -1 \longrightarrow y[-1] = 5$

$n = 0 \longrightarrow y[0] = -2$

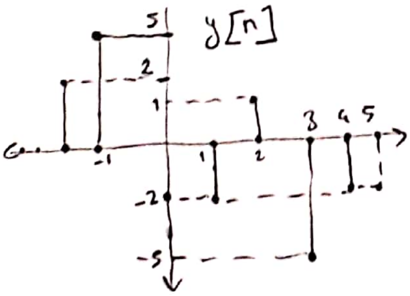
$n = 1 \longrightarrow y[1] = -2$

$n = 2 \longrightarrow y[2] = 1$

$n = 3 \longrightarrow y[3] = -5$

$n = 4 \longrightarrow y[4] = -2$

$n = 5 \longrightarrow y[5] = -2$



$$\mathcal{F}\{x[n]\}_\Omega = \sum_{n=-2}^{+2} x[n] \cdot e^{-j\Omega n} = e^{2j\Omega} + 2e^{j\Omega} - 2 - 2e^{-j\Omega} - 2e^{-2j\Omega}$$

$$\mathcal{F}\{h[n]\}_\Omega = \sum_{n=0}^{+3} h[n] \cdot e^{-j\Omega n} = 2 + e^{-j\Omega} + 2 \cdot e^{-3j\Omega}$$

$$\mathcal{F}\{y[n]\} = \mathcal{F}\{x[n]\} \cdot \mathcal{F}\{h[n]\}$$

- $x(t) = e^{-at} \cdot u(t)$ $\quad a>0 \quad$ CoT fourier transform

$\mathcal{F}\{x(t)\}_{Jw} = ?$

$$\mathcal{F}\{x(t)\}_{Jw} = \int_0^{\infty} e^{-at} \cdot u(t) \cdot e^{-Jwt}\,dt = \int_0^{\infty} e^{-at} \cdot e^{-Jwt}\,dt = \int_0^{\infty} e^{-t(Jw+a)}\,dt$$

$$= -\frac{e^{-t}}{Jw+a}\Bigg|_0^{\infty} = 0 - \left(-\frac{1}{Jw+a}\right)$$

$$= +\frac{1}{Jw+a}$$

Important Formulas

- $\mathcal{L}\{f(x)\}_s = \int_{-\infty}^{+\infty} f(x)\cdot e^{-st}\,dt$, $s = \alpha + \omega j$, Eg. $\int_{-\infty}^{+\infty} e^{-bt} u(t)\, e^{-st}\,dt = \int_{0}^{\infty} e^{-(b+s)}\,dt = \frac{1}{b+s}$

  $(b+s) > 0$

- $\mathcal{L}\{x(t-c)\}_s = e^{-cs}\cdot X(s)$
- $\mathcal{L}\{x'(t)\}_s = sX(s)$
- $\mathcal{L}\{\int x(t)\,dt\} = \frac{1}{s} X(s)$

- $\mathcal{L}\{e^{at} x(t)\} = X(s-a)$ **Example:** $\mathcal{L}^{-1}\left\{\frac{7}{s+2}\right\} = 7\cdot e^{-2t} u(t)$

- $H(\omega) = |H(\omega)|\cdot e^{j\angle H(\omega)}$
- $y(t) = \int_{-\infty}^{+\infty} x(t-\tau)\cdot h(\tau)\,d\tau$

- $\int_{-\infty}^{+\infty} \frac{A}{2}\cdot e^{j(\omega_0(t-\tau)+\theta)}\cdot h(\tau)\,d\tau = \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)} \int_{-\infty}^{+\infty} e^{-j\omega_0\tau}\cdot h(\tau)\,d\tau = \frac{A}{2}\cdot e^{j(\omega_0 t+\theta)}\cdot H(\omega_0)$

FIR = finite impulse response

- $H[z] = \sum h[n]\cdot z^{-n}$, $H[z]\big|_{z=e^{j\omega_0}} = \sum h[n]\cdot e^{-nj\omega_0}$

- $\sum_{i=0}^{\infty} a\cdot r^i = \frac{a}{1-r}$
- $H(z) = \sum h[n]\cdot z^{-n} = \sum b_k\cdot z^{-k}$, $y[n] = \sum b_k\cdot x[n-k]$

- **Delay:** $x[n] \to \delta[n-1] \to y[n] = x[n-1]$: $x[n] \to z^{-1} \to y[n] = x[n-1]$ $\Big\rangle$ $y(z) = z^{-1}\cdot X(z)$

- **Cascade:** $x[n] \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow y[n]$: $H(z) = H_1(z)H_2(z)$ | $y(z) = X(z)H_1(z)H_2(z)$

  $h[n] = h_1[n] * h_2[n]$

- **Zero/Pole Z:** $H(z) = 1-2z^{-1}+2z^{-2}-z^{-3}$, $H(z) = (1-z^{-1})\cdot(1-z^{-1}+z^{-2})$, Root: $1, \frac{1}{2} \pm j\cdot\frac{\sqrt{3}}{2}$

  $H(z) = \frac{z^3-2z^2+2z^1-1}{z^3}$, three pole at $z=0$

- **Zero/pole Laplace:** $H(s) = \frac{2s^2+12s+20}{s^3+6s^2+10s+8} = \frac{2\cdot(s+3-j)(s+3+j)}{(s+4)(s+1-j)(s+1+j)}$

  zero: $s = j-3$, $s = -j-3$

  pole: $s = -4$, $s = j-1$, $s = -j-1$

● **Infinite Length Signal:** $H(z) = \sum_{k=-\infty}^{+\infty} h[k] \cdot z^{-k}$, $\sum_{k=0}^{\infty} b_0 (a_1)^k u[k] \cdot z^{-k} = \sum_{n=0}^{\infty} b_0 (a_1)^n z^{-n}$, $H(z) = \frac{b_0}{1-a_1 z^{-1}}$

● **First Order IIR filter:** $y[n] = a_1 y[n-1] + b_0 x[n]$

$$h[n] = b_0 \cdot (a_1)^n u[n] \longrightarrow H(z) = \frac{b_0}{1 - a_1 \cdot z^{-1}}$$

$$\tilde{y}[n] = a_1 \cdot y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 \cdot a^{n-1} \cdot u[n-1]$$

$$h[n] = \frac{b_0}{1-a_1 z^{-1}} + \frac{b_1 z^{-1}}{1-a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}}$$

Also, using delay property

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 \cdot z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

● **Step Response:** $x[n] = u[n]$

$$\sum_{k=1}^{n} u[k] \cdot z^{-k} = \frac{1}{1-z^{-1}}$$

$$y[n] = a_1 y[n-1] + b_0 x[n]$$
$$= a_1 y[n-1] + b_0 u[n]$$

$$Y(z) = a_1 z^{-1} Y(z) + \frac{b_0}{1-z^{-1}} \rightarrow Y(z) = \frac{b_0}{1-z^{-1}} \cdot \frac{1}{1-a_1 z^{-1}}$$

- $A\cos(2\pi f t+\theta) \longrightarrow \omega = 2\pi f \ , \ \theta = -2\pi f t_1$

- phasor representation: $A\cos(2\pi f+\theta) = A \cdot e^{J\theta}$

- $a_k = \frac{1}{T_0} \cdot \int_0^{T_0} x(t) \cdot e^{-J\omega k t} \quad , \omega = 2\pi f = 2\pi \frac{1}{T}$

- $x(t) = x[n] \Big|_{t=n \cdot T_0}$

- $\frac{f_{max}}{f_s} \leq 0.5 \longrightarrow$ no aliasing