Personal Solutions to Problem Sheet 1 QFT by David Tong

Baset

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1.

According to Eq. (2),

$$\begin{aligned} \frac{\partial y}{\partial t} &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\Big(\frac{n\pi x}{a}\Big) \dot{q}_n \\ \frac{\partial y}{\partial x} &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \cos\Big(\frac{n\pi x}{a}\Big) \frac{n\pi}{a} q_n \end{aligned} \tag{1}$$

implying (assuming the integrals are finite and convergent, $m, n > 0$):

$$\begin{aligned} L &= \int_0^a dx \left[\frac{\sigma}{2} \frac{2}{a} \left(\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n \right)^2 - \frac{T}{2} \frac{2}{a} \left(\sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a} q_n \right)^2 \right] \\ &= \frac{1}{a} \int_0^a dx \left[\sigma \sum_{m,n} \sin\Big(\frac{n\pi}{a} x\Big) \sin\Big(\frac{m\pi}{a} x\Big) \dot{q}_n \dot{q}_m - T \sum_{m,n} \cos\Big(\frac{n\pi}{a} x\Big) \cos\Big(\frac{m\pi}{a} x\Big) (\frac{\pi}{a})^2 \, mn \, q_n q_m \right] \\ &= \frac{1}{a} \sum_{m,n} \int_0^a dx \left[\sigma \frac{\cos\big((m-n)\frac{\pi}{a} x\big) - \cos\big((m+n)\frac{\pi}{a} x\big)}{2} \dot{q}_n \dot{q}_m - T \frac{\cos\big((m-n)\frac{\pi}{a} x\big) + \cos\big((m+n)\frac{\pi}{a} x\big)}{2} \right. \\ &\quad \left. (\frac{\pi}{a})^2 \, mn \, q_n q_m \right] = \frac{1}{a} \sum_{m,n} \left[\frac{\sigma}{2} \delta_{m,n} \dot{q}_n \dot{q}_m - (\frac{\pi}{a})^2 \delta_{m,n} \, mn \, q_n q_m \frac{T}{2} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} (\frac{n\pi}{a})^2 q_n^2 \right] = L(q_n, \dot{q}_n) \end{aligned} \tag{2}$$

E-L equations imply:

$$\begin{aligned} &- \frac{\partial L}{\partial q_n} + \partial_\mu \frac{\partial L}{\partial_\mu (q_n)} = 0 \quad \equiv T(\frac{n\pi}{a})^2 q_n + \sigma \ddot{q}_n = 0 \implies \\ &\ddot{q}_n + \frac{T}{\sigma} (\frac{n\pi}{a})^2 q_n = 0 \implies \omega_n = \sqrt{\frac{T}{\sigma}} (\frac{n\pi}{a}) \end{aligned} \tag{3}$$

2.

According to the notation in the notes $\phi'(x) = \phi(y)$, implying

$$y^\nu = (\Lambda^{-1})^\nu_\mu x^\mu \implies \frac{\partial y^\nu}{\partial x^\mu} = (\Lambda^{-1})^\nu_\mu \tag{4}$$

The KG in the notes is $\partial_\mu \partial^\mu \phi(x) + m^2\phi(x) = 0$, the KG in the new coordinate system:

$$\begin{aligned} \text{KG}' :& (\Lambda^{-1})^\mu_\nu \partial_\mu (\Lambda^{-1})^\nu_\mu \partial^\mu \phi(y) + m^2\phi(y) = 0 \\ &\equiv \partial_\nu \partial^\nu \phi(y) + m^2\phi(y) = 0 \end{aligned} \tag{5}$$

3.

E-L equations:

$$\begin{cases} \text{Eq1:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \\ \text{Eq2:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \end{cases} \implies \begin{cases} \text{-} & \partial_\mu \partial^\mu \psi + m^2\psi + \lambda\psi^*\psi^2 = 0 \\ \text{-} & \partial_\mu \partial^\mu \psi^* + m^2\psi^* + \lambda\psi(\psi^*)^2 = 0 \end{cases} \tag{6}$$

The invariance part and the changed Lagrangian up to the first order:

$$\begin{aligned} \mathcal{L}' &\simeq \partial_\mu(\psi^* - i\alpha\psi^*)\partial^\mu(\psi + i\alpha\psi) - m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2 \\ &\simeq \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2 = \mathcal{L} \end{aligned} \tag{7}$$

therefore, the conserved current is:

$$\begin{aligned} j^\mu =& \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) = i\alpha\psi\partial^\mu\psi^* - i\alpha\psi^*\partial^\mu\psi \\ &\equiv i(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi) \end{aligned} \tag{8}$$

We calculate explicitly $\partial_\mu j^\mu$ and use the E-L:

$$\begin{aligned} \partial_\mu j^\mu &= i\left[\partial_\mu\psi\partial^\mu\psi^* + \psi\partial_\mu\partial^\mu\psi^* - \partial_\mu\psi^*\partial^\mu\psi - \psi^*\partial_\mu\partial^\mu\psi\right] \\ &= i\left[-\psi\left(m^2\psi^* + \lambda\psi(\psi^*)^2\right) + \psi^*(m^2\psi + \lambda\psi^*\psi^2)\right] \\ &= 0 \end{aligned} \tag{9}$$

QED.