

Personal Solutions to Problem Sheet 1 QFT by David Tong

Baset

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1.

According to Eq. (2),

$$\begin{aligned}\frac{\partial y}{\partial t} &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n \\ \frac{\partial y}{\partial x} &= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a} q_n\end{aligned}\tag{1}$$

implying (assuming the integrals are finite and convergent, $m, n > 0$):

$$\begin{aligned}L &= \int_0^a dx \left[\frac{\sigma}{2} \frac{2}{a} \left(\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n \right)^2 - \frac{T}{2} \frac{2}{a} \left(\sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a} q_n \right)^2 \right] \\ &= \frac{1}{a} \int_0^a dx \left[\sigma \sum_{m,n} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{a} x\right) \dot{q}_n \dot{q}_m - T \sum_{m,n} \cos\left(\frac{n\pi}{a} x\right) \cos\left(\frac{m\pi}{a} x\right) \left(\frac{\pi}{a}\right)^2 mn q_n q_m \right] \\ &= \frac{1}{a} \sum_{m,n} \int_0^a dx \left[\sigma \frac{\cos((m-n)\frac{\pi}{a} x) - \cos((m+n)\frac{\pi}{a} x)}{2} \dot{q}_n \dot{q}_m - T \frac{\cos((m-n)\frac{\pi}{a} x) + \cos((m+n)\frac{\pi}{a} x)}{2} \right. \\ &\quad \left. \left(\frac{\pi}{a}\right)^2 mn q_n q_m \right] = \frac{1}{a} \sum_{m,n} \left[\frac{\sigma}{2} \delta_{m,n} \dot{q}_n \dot{q}_m - \left(\frac{\pi}{a}\right)^2 \delta_{m,n} mn q_n q_m \frac{T}{2} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a}\right)^2 q_n^2 \right] = L(q_n, \dot{q}_n)\end{aligned}\tag{2}$$

E-L equations imply:

$$\begin{aligned}-\frac{\partial L}{\partial q_n} + \partial_\mu \frac{\partial L}{\partial \dot{q}_n} &= 0 \quad \equiv T \left(\frac{n\pi}{a}\right)^2 q_n + \sigma \ddot{q}_n = 0 \implies \\ \ddot{q}_n + \frac{T}{\sigma} \left(\frac{n\pi}{a}\right)^2 q_n &= 0 \implies \omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a}\right)\end{aligned}\tag{3}$$

2.

According to the notation in the notes $\phi'(x) = \phi(y)$, implying

$$y^\nu = (\Lambda^{-1})^\nu_\mu x^\mu \implies \frac{\partial y^\nu}{\partial x^\mu} = (\Lambda^{-1})^\nu_\mu \quad (4)$$

The KG in the notes is $\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0$, the KG in the new coordinate system:

$$\begin{aligned} \text{KG}' : (\Lambda^{-1})^\mu_\nu \partial_\mu (\Lambda^{-1})^\nu_\mu \partial^\mu \phi(y) + m^2 \phi(y) &= 0 \\ \equiv \partial_\nu \partial^\nu \phi(y) + m^2 \phi(y) &= 0 \end{aligned} \quad (5)$$

3.

E-L equations:

$$\begin{cases} \text{Eq1:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \\ \text{Eq2:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \end{cases} \implies \begin{cases} - & \partial_\mu \partial^\mu \psi + m^2 \psi + \lambda \psi^* \psi^2 = 0 \\ - & \partial_\mu \partial^\mu \psi^* + m^2 \psi^* + \lambda \psi (\psi^*)^2 = 0 \end{cases} \quad (6)$$

The invariance part and the changed Lagrangian up to the first order:

$$\begin{aligned} \mathcal{L}' &\simeq \partial_\mu (\psi^* - i\alpha \psi^*) \partial^\mu (\psi + i\alpha \psi) - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 \\ &\simeq \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 = \mathcal{L} \end{aligned} \quad (7)$$

therefore, the conserved current is:

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) = i\alpha \psi \partial^\mu \psi^* - i\alpha \psi^* \partial^\mu \psi \\ &\equiv i(\psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi) \end{aligned} \quad (8)$$

We calculate explicitly $\partial_\mu j^\mu$ and use the E-L:

$$\begin{aligned} \partial_\mu j^\mu &= i [\partial_\mu \psi \partial^\mu \psi^* + \psi \partial_\mu \partial^\mu \psi^* - \partial_\mu \psi^* \partial^\mu \psi - \psi^* \partial_\mu \partial^\mu \psi] \\ &= i [-\psi (m^2 \psi^* + \lambda \psi (\psi^*)^2) + \psi^* (m^2 \psi + \lambda \psi^* \psi^2)] \\ &= 0 \end{aligned} \quad (9)$$

QED.

4.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a \implies \quad (10)$$

$$\begin{aligned}
\mathcal{L}' &\simeq \frac{1}{2} \partial_\mu (\phi_a + \theta \epsilon_{abc} n_b \phi_c) \partial^\mu (\phi_a + \theta \epsilon_{abc} n_b \phi_c) - \frac{1}{2} m^2 (\phi_a \phi_a - 2\theta \epsilon_{bac} \phi_a \phi_c) \\
&\simeq \frac{1}{2} \{ \partial_\mu \phi_a \partial^\mu \phi_a + \theta n_b [\epsilon_{abc} \partial_\mu \phi_c \partial^\mu \phi_a + \epsilon_{abc} \partial^\mu \phi_c \partial_\mu \phi_a] \} - \frac{1}{2} m^2 \phi_a \phi_a \quad (11) \\
&= \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a = \mathcal{L}
\end{aligned}$$

We used the fact that ϵ_{abc} tensor is anti-symmetric, but $\partial_\mu \phi_c \partial^\mu \phi_a$ is symmetric w.r.t changing the indices a and c . The same goes for $\phi_a \phi_c$.

Computing the Noether current:

$$\begin{aligned}
j^\mu &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} X_a(\phi) \\
&= \partial^\mu \phi_a \epsilon_{abc} n_b \phi_c \implies \\
Q &= \int d^3x \epsilon_{abc} \dot{\phi}_a n_b \phi_c \quad (12) \\
&= n_b \int d^3x \epsilon_{bca} \dot{\phi}_a \phi_c \\
&= n_b \int d^3x \epsilon_{bac} \dot{\phi}_c \phi_a = n_a \int d^3x \epsilon_{abc} \dot{\phi}_b \phi_c
\end{aligned}$$

We can choose $n_a = (1, 0, 0)$, then for each choice:

$$Q_a = \int d^3x \epsilon_{abc} \dot{\phi}_b \phi_c \quad (13)$$

Direct confirmation:

$$\mathcal{L} = \frac{1}{2} \partial_t \phi_a \partial_t \phi_a - \frac{1}{2} \nabla \phi_a \nabla \phi_a - \frac{1}{2} m^2 \phi_a \phi_a \implies \quad (14)$$

E-L for each a : $\partial_t^2 \phi_a - \nabla^2 \phi_a + m^2 \phi_a = 0$

Hence:

$$\begin{aligned}
\frac{dQ_a}{dt} &= \int d^3x \epsilon_{abc} \ddot{\phi}_a \phi_c + \int d^3x \epsilon_{abc} \dot{\phi}_a \dot{\phi}_c \\
&= \int d^3x \epsilon_{abc} (\nabla^2 \phi_a - m^2 \phi_a) \phi_c = \int d^3x \epsilon_{abc} \nabla^2 \phi_a \phi_c \quad (15) \\
&= - \int d^3x \epsilon_{abc} \nabla \phi_a \cdot \nabla \phi_c = 0
\end{aligned}$$

We have used the symmetric, anti-symmetric point several times in the above derivation.