Personal solutions to GR book by Carroll chapter 1

Baset

June 2025

1.

Let the initial velocity of the particle with respect to the obesrver S' be $u'_0 \overset{S'}{\equiv} (u_0^{x'}, u_0^{y'})$. For simplicity, we assume that the collision happens at $t' = 0$. Moreover, we assume observers S and S' are aligned along the x-axis and that when $t' = 0$ then $t = 0$. Let at $t' = 0$ the line that describes the mirror be $-x' = y'$. The line that describes the particle is $\frac{x'}{u_0^{x'}} = \frac{y'}{u_0^{y'}}$. For now, $u_0^{x'}, u_0^{y'} \neq 0$. Now, we will find the angle between these two lines:

$$\tan\theta' = \left|\frac{\tan\frac{\pi}{4} - \frac{u_0^{y'}}{u_0^{x'}}}{1 + \tan\frac{\pi}{4}\frac{u_0^{y'}}{u_0^{x'}}}\right| = \left|\frac{1 - \frac{u_0^{y'}}{u_0^{x'}}}{1 + \frac{u_0^{y'}}{u_0^{x'}}}\right| \tag{1}$$

We let the $u_0^{y'} = u_0^{x'}\tan\alpha' = u_0^x \tan\alpha'$. Then, we have that:

$$\tan\theta' = \left|\frac{1 - \tan\alpha'}{1 + \tan\alpha'}\right| \tag{2}$$

But in the frame S, we have that the $u_0^y = \frac{u_0^{y'} + v}{1 + u_0^{y'} v}$. Also, the mirror's coordinates in S are:

$$y = \gamma(y' + vt') = \gamma y'|_{t'=0} \tag{3}$$

Therefore, w.r.t. to S at $t = 0$ the mirror is $y = -\frac{x}{\gamma}$. The particle's coordinates in S are $y = \frac{u_0^{y'} + v}{(1 + u_0^{y'} v)u_0^x}x$. Therefore, the angle between the particle and the mirror is:

$$\tan\theta = \left|\frac{\gamma - \frac{u_0^{y'}+v}{(1+u_0^{y'}v)u_0^x}}{1 + \gamma\frac{u_0^{y'}+v}{(1+u_0^{y'}v)u_0^x}}\right| \implies \tan\theta = \left|\frac{\gamma - \frac{\tan\alpha' + v/u_0^x}{1+\tan\alpha' u_0^x v}}{1 + \gamma\frac{\tan\alpha' + v/u_0^x}{1+\tan\alpha' u_0^x v}}\right| \tag{4}$$