Personal Solutions to Problem Sheet 1 QFT by David Tong

Baset

June 2025

1.

According to Eq. (2),

$$\begin{aligned}\frac{\partial y}{\partial t} &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi x}{a}\Big)\dot{q}_n \\ \frac{\partial y}{\partial x} &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\cos\Big(\frac{n\pi x}{a}\Big)\frac{n\pi}{a}q_n\end{aligned} \tag{1}$$

implying (assuming the integrals are finite and convergent, $m, n > 0$):

$$\begin{aligned}L = &\int_0^a dx\left[\frac{\sigma}{2}\frac{2}{a}\left(\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\dot{q}_n\right)^2 - \frac{T}{2}\frac{2}{a}\left(\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a}q_n\right)^2\right] \\ &= \frac{1}{a}\int_0^a dx\left[\sigma\sum_{m,n}\sin\Big(\frac{n\pi}{a}x\Big)\sin\Big(\frac{m\pi}{a}x\Big)\dot{q}_n\dot{q}_m - T\sum_{m,n}\cos\Big(\frac{n\pi}{a}x\Big)\cos\Big(\frac{m\pi}{a}x\Big)(\frac{\pi}{a})^2\,mn\,q_nq_m\right] \\ &= \frac{1}{a}\sum_{m,n}\int_0^a dx\left[\sigma\frac{\cos\big((m-n)\frac{\pi}{a}x\big)-\cos\big((m+n)\frac{\pi}{a}x\big)}{2}\dot{q}_n\dot{q}_m - T\frac{\cos\big((m-n)\frac{\pi}{a}x\big)+\cos\big((m+n)\frac{\pi}{a}x\big)}{2}\right. \\ &\left.(\frac{\pi}{a})^2\,mn\,q_nq_m\right] = \frac{1}{a}\sum_{m,n}\left[\frac{\sigma}{2}\delta_{m,n}\dot{q}_n\dot{q}_m - (\frac{\pi}{a})^2\delta_{m,n}\,mn\,q_nq_m\frac{T}{2}\right] \\ &= \sum_{n=1}^{\infty}\left[\frac{\sigma}{2}\dot{q}_n^2 - \frac{T}{2}(\frac{n\pi}{a})^2q_n^2\right] = L(q_n, \dot{q}_n)\end{aligned} \tag{2}$$

E-L equations imply:

$$\begin{aligned}&-\frac{\partial L}{\partial q_n} + \partial_\mu\frac{\partial L}{\partial_\mu(q_n)} = 0 \quad \equiv T(\frac{n\pi}{a})^2q_n + \sigma\ddot{q}_n = 0 \implies \\ &\ddot{q}_n + \frac{T}{\sigma}(\frac{n\pi}{a})^2q_n = 0 \implies \omega_n = \sqrt{\frac{T}{\sigma}}(\frac{n\pi}{a})\end{aligned} \tag{3}$$

2.

According to the notation in the notes $\phi'(x) = \phi(y)$, implying

$$y^\nu = (\Lambda^{-1})^\nu_\mu x^\mu \implies \frac{\partial y^\nu}{\partial x^\mu} = (\Lambda^{-1})^\nu_\mu \tag{4}$$

The KG in the notes is $\partial_\mu \partial^\mu \phi(x) + m^2\phi(x) = 0$, the KG in the new coordinate system:

$$\begin{aligned} \mathrm{KG}' : &(\Lambda^{-1})^\mu_\nu \partial_\mu (\Lambda^{-1})^\nu_\mu \partial^\mu \phi(y) + m^2\phi(y) = 0 \\ &\equiv \partial_\nu \partial^\nu \phi(y) + m^2\phi(y) = 0 \end{aligned} \tag{5}$$

3.

E-L equations:

$$\begin{cases} \text{Eq1:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \\ \text{Eq2:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \end{cases} \implies \begin{cases} \text{-} & \partial_\mu \partial^\mu \psi + m^2\psi + \lambda\psi^*\psi^2 = 0 \\ \text{-} & \partial_\mu \partial^\mu \psi^* + m^2\psi^* + \lambda\psi(\psi^*)^2 = 0 \end{cases} \tag{6}$$

The invariance part and the changed Lagrangian up to the first order:

$$\begin{aligned} \mathcal{L}' &\simeq \partial_\mu(\psi^* - i\alpha\psi^*)\partial^\mu(\psi + i\alpha\psi) - m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2 \\ &\simeq \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2 = \mathcal{L} \end{aligned} \tag{7}$$

therefore, the conserved current is:

$$\begin{aligned} j^\mu =& \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) = i\alpha\psi\partial^\mu\psi^* - i\alpha\psi^*\partial^\mu\psi \\ &\equiv i(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi) \end{aligned} \tag{8}$$

We calculate explicitly $\partial_\mu j^\mu$ and use the E-L:

$$\begin{aligned} \partial_\mu j^\mu &= i\left[\partial_\mu\psi\partial^\mu\psi^* + \psi\partial_\mu\partial^\mu\psi^* - \partial_\mu\psi^*\partial^\mu\psi - \psi^*\partial_\mu\partial^\mu\psi\right] \\ &= i\left[-\psi\left(m^2\psi^* + \lambda\psi(\psi^*)^2\right) + \psi^*(m^2\psi + \lambda\psi^*\psi^2)\right] \\ &= 0 \end{aligned} \tag{9}$$

QED.

4.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2\phi_a\phi_a \implies \tag{10}$$

$$
\begin{aligned}
\mathcal{L}' &\simeq \frac{1}{2}\partial_\mu(\phi_a + \theta\epsilon_{abc}n_b\phi_c)\partial^\mu(\phi_a + \theta\epsilon_{abc}n_b\phi_c) - \frac{1}{2}m^2(\phi_a\phi_a - 2\theta\epsilon_{bac}\phi_a\phi_c) \\
&\simeq \frac{1}{2}\{\partial_\mu\phi_a\partial^\mu\phi_a + \theta n_b\left[\epsilon_{abc}\partial_\mu\phi_c\partial^\mu\phi_a + \epsilon_{abc}\partial^\mu\phi_c\partial_\mu\phi_a\right]\} - \frac{1}{2}m^2\phi_a\phi_a \\
&= \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2\phi_a\phi_a = \mathcal{L}
\end{aligned}
\tag{11}
$$

We used the fact that ϵ_{abc} tensor is anti-symmetric, but $\partial_\mu\phi_c\partial^\mu\phi_a$ is symmetric w.r.t changing the indices a and c. The same goes for $\phi_a\phi_c$.

Computing the Noether current:

$$
\begin{aligned}
j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a(\phi) \\
&= \partial^\mu\phi_a\,\epsilon_{abc}n_b\phi_c \implies \\
Q &= \int d^3x\,\epsilon_{abc}\dot{\phi}_a n_b\phi_c \\
&= n_b\int d^3x\,\epsilon_{bca}\dot{\phi}_a\phi_c \\
&= n_b\int d^3x\,\epsilon_{bac}\dot{\phi}_c\phi_a = n_a\int d^3x\,\epsilon_{abc}\dot{\phi}_b\phi_c
\end{aligned}
\tag{12}
$$

We can choose $n_a = (1,0,0)$, then for each choice:

$$
Q_a = \int d^3x\,\epsilon_{abc}\dot{\phi}_b\phi_c \tag{13}
$$

Direct confirmation:

$$
\begin{aligned}
&\mathcal{L} = \frac{1}{2}\partial_t\phi_a\partial_t\phi_a - \frac{1}{2}\nabla\phi_a\nabla\phi_a - \frac{1}{2}m^2\phi_a\phi_a \implies \\
&\text{E-L for each } a\text{: } \partial_t^2\phi_a - \nabla^2\phi_a + m^2\phi_a = 0
\end{aligned}
\tag{14}
$$

Hence:

$$
\begin{aligned}
\frac{d\,Q_a}{dt} &= \int d^3x\,\epsilon_{abc}\ddot{\phi}_a\phi_c + \int d^3x\,\epsilon_{abc}\dot{\phi}_a\dot{\phi}_c \\
&= \int d^3x\,\epsilon_{abc}(\nabla^2\phi_a - m^2\phi_a)\phi_c = \int d^3x\,\epsilon_{abc}\nabla^2\phi_a\,\phi_c \\
&= -\int d^3x\,\epsilon_{abc}\nabla\phi_a.\nabla\phi_c = 0
\end{aligned}
\tag{15}
$$

We have used the symmetric, anti-symmetric point several times in the above derivation.

5.

- The first result is:

$$
\begin{aligned}
\eta_{\sigma\tau}x'^\sigma x'^\tau &= \eta_{\sigma\tau}\Lambda^\sigma_\mu\, x^\mu\Lambda^\tau_\nu\, x^\nu = \eta_{\mu\nu}x^\mu x^\nu \implies \\
\eta_{\mu\nu} &= \eta_{\sigma\tau}\Lambda^\sigma_\mu\Lambda^\tau_\nu
\end{aligned}
\tag{16}
$$

For any transformation to be Lorentz, it must satisfy Eq.(16):

$$\begin{aligned}\eta_{\mu\nu} &= \eta_{\sigma\tau}(\delta^\sigma_\mu + \omega^\sigma_\mu)(\delta^\tau_\nu + \omega^\tau_\nu) \\ \eta_{\mu\nu} &= \eta_{\mu\nu} + \eta_{\mu\tau}\omega^\tau_\nu + \eta_{\sigma\nu}\omega^\sigma_\mu + \eta_{\sigma\tau}\omega^\sigma_\mu\omega^\tau_\nu \\ &\simeq \eta_{\mu\nu} + (\omega_{\mu\nu} + \omega_{\nu\mu})\end{aligned} \tag{17}$$

Hence:

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \tag{18}$$

A pure rotation:

$$\begin{aligned}R(\theta) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [\delta] + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{19}$$

Anti-symmetric as one can see for $\theta \ll 1$ infinitesimally small. In cas of the boost:

$$\Lambda(v) = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$\gamma(v) \simeq 1 + \frac{1}{2}v^2 \implies$

$$\Lambda(v) \simeq \begin{bmatrix} 1 & -v & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [\delta] + \begin{bmatrix} 0 & -v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Which will be antisymmetric after we lower the index using Minkowski metric.

6.

$$\phi(x) \to \phi'(x') = \phi(x^\mu - \omega^\mu_\nu x^\nu) \simeq \phi(x) - \omega^\mu_\nu x^\nu \partial_\mu \phi \tag{22}$$

Therefore:

$$\mathcal{L}'(\phi') = \mathcal{L}(\phi - \omega^\mu_\nu x^\nu \partial_\mu \phi) \implies \delta\mathcal{L} = -\omega^\mu_\nu x^\nu \partial_\mu \mathcal{L} \tag{23}$$

Which because ω is infinitesimal and antisymmetric:

$$\delta\mathcal{L} = -\partial_\mu(\omega^\mu_\nu x^\nu \mathcal{L}) \tag{24}$$

Using Noether's theorem:

$$
\begin{aligned}
j^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\omega^\rho_\nu x^\nu \partial_\rho \phi + \omega^\mu_\nu x^\nu \mathcal{L} \\
&= -\omega^\rho_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} x^\nu \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} x^\nu\right] \\
&= -\omega^\rho_\nu \left[T^\mu_\rho x^\nu\right]
\end{aligned}
\tag{25}
$$

Accordingly, looking at Eq. (19), for arbitrary $j \neq k$, $\omega_{jk} = -\omega_{kj}$, hence:

$$
Q = -\int d^3x\, T^{\rho 0}\omega_{\rho\nu}x^\nu = -\int d^3x\,(T^{j0}\omega_{jk}x^k - T^{k0}\omega_{jk}x^j) \tag{26}
$$

We can write the above equation for three pairs, and also we have a freedome in choosing $\omega_{jk} = \pm 1$, therefore using Levi-Civita symbol:

$$
Q_i = \epsilon_{ijk}\int d^3x\,(x^j T^{0k} - x^k T^{0j}) \tag{27}
$$

Which is just the Angular momentum. The other one resulting from the Lorentz Boosts is:

$$
\begin{aligned}
&Q = \int d^3x(x^i T^{00} - x^0 T^{0i}) \implies \frac{dQ}{dt} = 0 \implies \\
&\frac{d}{dt}\int d^3x\, x^i T^{00} = \int d^3x\, T^{0i} = P^i
\end{aligned}
\tag{28}
$$

My own first impression was that this is sort of a restatement of Newton's law and sort of restatement for what momentum is, but a more precise answer in the book correctly says that this equation states that the center of energy's velocity is constant, or just the momentum we have defined which in translation invariant systems must be preserved.

7.

For the first part we only prove that the $F_{\mu\nu}$ tensor is invariant under the Guage the transformation introduced here:

$$
F'_{\mu\nu} = (\partial_\mu A'_\nu - \partial_\nu A'_\mu) = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\nu\partial_\mu \xi - \partial_\mu\partial_\nu \xi = F_{\mu\nu} \tag{29}
$$

Therefore the Lagrangian does not change under Guage transformation.

For the translation trans formation we follow the note's explanation for deriving the energy-momentum tensor, $x^\nu \to x^\nu - \epsilon^\nu$, hence: $A_\nu \to A_\nu + \epsilon^\mu \partial_\mu A_\nu$, therefore the energy-momentum tensor is:

$$
\begin{aligned}
&T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial_\nu A_\rho - \delta^\mu_\nu \mathcal{L} = -F^{\mu\rho}\partial_\nu A_\rho - \delta^\mu_\nu \mathcal{L} \implies \\
&T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
\tag{30}
$$

All the terms above except for $\partial^\nu A_\rho$ are Guage invariant. But, obviously the term mentioned isn't and therefore the obtained $T^{\mu\nu}$ can not be Guage invariant. Moreover, the excluding the Lagrangian term of the energy-momentum tensor, it can not be symmetric w.r.t. μ and ν, one can easily compute $(0,1)$ term and $(1,0)$ term to see the tensor not being symmetric. Now we investigate the proposed tensor:

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu \tag{31}$$

Using Eq.(30):

$$\begin{aligned}\Theta^{\mu\nu} &= -F^{\mu\rho}\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L} + F^{\mu\rho}\partial_\rho A^\nu \\ &= F^{\mu\rho}(\partial_\rho A^\nu - \partial^\nu A_\rho) - \eta^{\mu\nu}\mathcal{L} \\ &= F^{\mu\rho}\eta^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) - \eta^{\mu\nu}\mathcal{L} \\ &= F^{\mu\rho}F_{\rho\sigma}\eta^{\nu\sigma} - \eta^{\mu\nu}\mathcal{L}\end{aligned} \tag{32}$$

Before confirming the conserved currents we see that it is completely Guage invariant and also Symmetric.

The E-L equations according to the Lagrangian are: $\partial_\mu F^{\mu\nu} = 0$, for the currents:

$$\begin{aligned}\partial_\mu\Theta^{\mu\nu} &= \eta^{\sigma\nu}F_{\rho\sigma}\partial_\mu F^{\mu\rho} + \eta^{\sigma\nu}F^{\mu\rho}\partial_\mu F_{\rho\sigma} + \frac{1}{2}\eta^{\mu\nu}F^{\rho\sigma}\partial_\mu F_{\rho\sigma} \\ &= \{\mu \leftrightarrow \sigma\} = 0 + \eta^{\mu\nu}F^{\sigma\rho}\partial_\sigma F_{\rho\mu} + \frac{1}{2}\eta^{\mu\nu}F^{\rho\sigma}\partial_\mu F_{\rho\sigma} \\ &= \eta^{\mu\nu}F^{\sigma\rho}\left\{\partial_\sigma F_{\rho\mu} - \frac{1}{2}\partial_\mu F_{\rho\sigma}\right\} \\ &= \eta^{\mu\nu}F^{\sigma\rho}\left\{A_{\mu,\rho\sigma} - A_{\rho,\mu\sigma} - \frac{1}{2}A_{\sigma,\rho\mu} + \frac{1}{2}A_{\rho,\sigma\mu}\right\} \\ &= \frac{1}{2}\eta^{\mu\nu}F^{\sigma\rho}\left\{A_{\mu,\rho\sigma} - A_{\rho,\mu\sigma} + A_{\mu,\rho\sigma} - A_{\sigma,\rho\mu}\right\} \\ &= \frac{1}{2}\eta^{\mu\nu}F^{\sigma\rho}\left\{\partial_\sigma F_{\rho\mu} - \partial_\rho F_{\mu\sigma}\right\} = -\frac{1}{2}\eta^{\mu\nu}\left\{F^{\sigma\rho}\partial_\sigma F_{\mu\rho} + F^{\sigma\rho}\partial_\rho F_{\mu\sigma}\right\} \\ &= -\frac{1}{2}\eta^{\mu\nu}\left\{F^{\sigma\rho}\partial_\rho F_{\mu\sigma} - F^{\rho\sigma}\partial_\sigma F_{\mu\rho}\right\} = \{\rho \leftrightarrow \sigma\} = \\ &= -\frac{1}{2}\eta^{\mu\nu}\{0\} = 0\end{aligned} \tag{33}$$

Therefore this also defines four conserved currents. For the traceless proof:

$$\Theta^\mu_\mu = -F^{\mu\rho}F_{\mu\rho} - \eta^\mu_\mu\mathcal{L} = 4\mathcal{L} - 4\mathcal{L} = 0 \tag{34}$$

8.

We use the $E-L$ equation for fields:

$$\begin{aligned}0 &= -\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu C_\nu)} + \frac{\partial\mathcal{L}}{\partial C_\nu} = \partial_\mu F^{\mu\nu} + m^2 C^\nu \\ &= (\partial_\mu\partial^\mu + m^2)C^\nu - \partial^\nu\partial_\mu C^\mu\end{aligned} \tag{35}$$

We have:

$$
\begin{aligned}
m^2 C^\nu = \partial^\nu \partial_\mu C^\mu - \partial_\mu \partial^\mu C^\nu &\implies m^2 \partial_\nu C^\nu = \partial_\nu \partial^\nu \partial_\mu C^\mu - \partial_\mu \partial^\mu \partial_\nu C^\nu = 0 \\
&\implies m^2 \partial_\mu C^\mu = 0 \xrightarrow{m \neq 0} \partial_\mu C^\mu = 0
\end{aligned}
\tag{36}
$$

This, by itself helps to write one of the components based on the other (E-L, $\nu = 0$):

$$
\begin{aligned}
&\partial_t^2 C^0 + (m^2 + \partial_i \partial^i) C^0 - \partial_t^2 C^0 - \partial^i \dot{C}_i = 0 \implies \\
&(m^2 + \partial_i \partial^i) C^0 = \partial^i \dot{C}_i
\end{aligned}
\tag{37}
$$

By definition:

$$
\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{C}_i} = -F^{0i} \implies \Pi^0 = -F^{00} = 0 \tag{38}
$$

Then the Hamiltonian density:

$$
\begin{aligned}
\mathcal{H} &= \Pi^i \dot{C}_i - \mathcal{L} = \Pi^i (F_{0i} + \partial_i C_0) - \mathcal{L} = -\Pi^i \Pi_i + \Pi^i \partial_i C_0 + \frac{1}{2} \Pi^i \Pi_i + \frac{1}{2} F^{ij} F_{ij} \\
&\quad - \frac{1}{2} m^2 C_0 C^0 - \frac{1}{2} m^2 C_i C^i \implies \\
\mathcal{H} &= \frac{1}{2} (\Pi_i)^2 - \Pi_i \partial_i C_0 + \frac{1}{2} F^{ij} F_{ij} - \frac{1}{2} m^2 C_0^2 + \frac{1}{2} m^2 C_i^2
\end{aligned}
\tag{39}
$$