Personal Solutions to Problem Sheet 1 QFT by David Tong

Baset

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1.

According to Eq. (2),

$$\begin{aligned}\frac{\partial y}{\partial t} &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi x}{a}\Big)\dot{q}_n \\ \frac{\partial y}{\partial x} &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\cos\Big(\frac{n\pi x}{a}\Big)\frac{n\pi}{a}q_n\end{aligned} \tag{1}$$

implying (assuming the integrals are finite and convergent, $m, n > 0$):

$$\begin{aligned}L &= \int_0^a dx\left[\frac{\sigma}{2}\frac{2}{a}\left(\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\dot{q}_n\right)^2 - \frac{T}{2}\frac{2}{a}\left(\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a}q_n\right)^2\right] \\ &= \frac{1}{a}\int_0^a dx\left[\sigma\sum_{m,n}\sin\Big(\frac{n\pi}{a}x\Big)\sin\Big(\frac{m\pi}{a}x\Big)\dot{q}_n\dot{q}_m - T\sum_{m,n}\cos\Big(\frac{n\pi}{a}x\Big)\cos\Big(\frac{m\pi}{a}x\Big)(\frac{\pi}{a})^2\,mn\,q_nq_m\right] \\ &= \frac{1}{a}\sum_{m,n}\int_0^a dx\left[\sigma\frac{\cos\big((m-n)\frac{\pi}{a}x\big)-\cos\big((m+n)\frac{\pi}{a}x\big)}{2}\dot{q}_n\dot{q}_m - T\frac{\cos\big((m-n)\frac{\pi}{a}x\big)+\cos\big((m+n)\frac{\pi}{a}x\big)}{2}\right. \\ &\quad\left.(\frac{\pi}{a})^2\,mn\,q_nq_m\right] = \frac{1}{a}\sum_{m,n}\left[\frac{\sigma}{2}\delta_{m,n}\dot{q}_n\dot{q}_m - (\frac{\pi}{a})^2\delta_{m,n}\,mn\,q_nq_m\frac{T}{2}\right] \\ &= \sum_{n=1}^{\infty}\left[\frac{\sigma}{2}\dot{q}_n^2 - \frac{T}{2}(\frac{n\pi}{a})^2q_n^2\right] = L(q_n,\dot{q}_n)\end{aligned} \tag{2}$$

E-L equations imply:

$$\begin{aligned}&-\frac{\partial L}{\partial q_n} + \partial_\mu\frac{\partial L}{\partial_\mu(q_n)} = 0 \quad \equiv T(\frac{n\pi}{a})^2q_n + \sigma\ddot{q}_n = 0 \implies \\ &\ddot{q}_n + \frac{T}{\sigma}(\frac{n\pi}{a})^2q_n = 0 \implies \omega_n = \sqrt{\frac{T}{\sigma}}(\frac{n\pi}{a})\end{aligned} \tag{3}$$

2.

According to the notation in the notes $\phi'(x) = \phi(y)$, implying

$$y^{\nu} = (\Lambda^{-1})^{\nu}_{\mu} x^{\mu} \implies \frac{\partial y^{\nu}}{\partial x^{\mu}} = (\Lambda^{-1})^{\nu}_{\mu} \tag{4}$$

The KG in the notes is $\partial_{\mu}\partial^{\mu}\phi(x) + m^2\phi(x) = 0$, the KG in the new coordinate system:

$$\begin{aligned} \mathrm{KG}' :& (\Lambda^{-1})^{\mu}_{\nu}\partial_{\mu}(\Lambda^{-1})^{\nu}_{\mu}\partial^{\mu}\phi(y) + m^2\phi(y) = 0 \\ \equiv & \ \partial_{\nu}\partial^{\nu}\phi(y) + m^2\phi(y) = 0 \end{aligned} \tag{5}$$

3.

E-L equations:

$$\begin{cases} \text{Eq1:} \quad \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi^*)} - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0 \\ \text{Eq2:} \quad \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \end{cases} \implies \begin{cases} \text{-} \quad \partial_{\mu}\partial^{\mu}\psi + m^2\psi + \lambda\psi^*\psi^2 = 0 \\ \text{-} \quad \partial_{\mu}\partial^{\mu}\psi^* + m^2\psi^* + \lambda\psi(\psi^*)^2 = 0 \end{cases} \tag{6}$$

The invariance part: