Personal Solutions to Problem Sheet 1 QFT by David Tong

Baset

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1.

According to Eq. (2),

$$\begin{aligned}\frac{\partial y}{\partial t} &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\Big(\frac{n\pi x}{a}\Big)\dot{q}_n \\ \frac{\partial y}{\partial x} &= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\cos\Big(\frac{n\pi x}{a}\Big)\frac{n\pi}{a}q_n\end{aligned} \tag{1}$$

implying (assuming the integrals are finite and convergent, $m, n > 0$):

$$\begin{aligned}L &= \int_0^a dx\left[\frac{\sigma}{2}\frac{2}{a}\left(\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\dot{q}_n\right)^2 - \frac{T}{2}\frac{2}{a}\left(\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a}q_n\right)^2\right] \\ &= \frac{1}{a}\int_0^a dx\left[\sigma\sum_{m,n}\sin\Big(\frac{n\pi}{a}x\Big)\sin\Big(\frac{m\pi}{a}x\Big)\dot{q}_n\dot{q}_m - T\sum_{m,n}\cos\Big(\frac{n\pi}{a}x\Big)\cos\Big(\frac{m\pi}{a}x\Big)(\frac{\pi}{a})^2\, mn\, q_n q_m\right] \\ &= \frac{1}{a}\sum_{m,n}\int_0^a dx\left[\sigma\frac{\cos\big((m-n)\frac{\pi}{a}x\big) - \cos\big((m+n)\frac{\pi}{a}x\big)}{2}\dot{q}_n\dot{q}_m - T\frac{\cos\big((m-n)\frac{\pi}{a}x\big) + \cos\big((m+n)\frac{\pi}{a}x\big)}{2}\right. \\ &\quad \left.(\frac{\pi}{a})^2\, mn\, q_n q_m\right] = \frac{1}{a}\sum_{m,n}\left[\frac{\sigma}{2}\delta_{m,n}\dot{q}_n\dot{q}_m - (\frac{\pi}{a})^2\delta_{m,n}\, mn\, q_n q_m\frac{T}{2}\right] \\ &= \sum_{n=1}^{\infty}\left[\frac{\sigma}{2}\dot{q}_n^2 - \frac{T}{2}(\frac{n\pi}{a})^2 q_n^2\right] = L(q_n, \dot{q}_n)\end{aligned} \tag{2}$$

E-L equations imply:

$$\begin{aligned}&-\frac{\partial L}{\partial q_n} + \partial_\mu\frac{\partial L}{\partial_\mu(q_n)} = 0 \quad \equiv T(\frac{n\pi}{a})^2 q_n + \sigma\ddot{q}_n = 0 \implies \\ &\ddot{q}_n + \frac{T}{\sigma}(\frac{n\pi}{a})^2 q_n = 0 \implies \omega_n = \sqrt{\frac{T}{\sigma}}(\frac{n\pi}{a})\end{aligned} \tag{3}$$

2.

According to the notation in the notes $\phi'(x) = \phi(y)$, implying

$$y^\nu = (\Lambda^{-1})^\nu_\mu x^\mu \implies \frac{\partial y^\nu}{\partial x^\mu} = (\Lambda^{-1})^\nu_\mu \tag{4}$$

The KG in the notes is $\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0$, the KG in the new coordinate system:

$$\begin{aligned} \mathrm{KG}' :&(\Lambda^{-1})^\mu_\nu \partial_\mu (\Lambda^{-1})^\nu_\mu \partial^\mu \phi(y) + m^2 \phi(y) = 0 \\ &\equiv \partial_\nu \partial^\nu \phi(y) + m^2 \phi(y) = 0 \end{aligned} \tag{5}$$

3.

E-L equations:

$$\begin{cases} \text{Eq1:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \\ \text{Eq2:} & \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \end{cases} \implies \begin{cases} \text{-} & \partial_\mu \partial^\mu \psi + m^2 \psi + \lambda \psi^* \psi^2 = 0 \\ \text{-} & \partial_\mu \partial^\mu \psi^* + m^2 \psi^* + \lambda \psi (\psi^*)^2 = 0 \end{cases} \tag{6}$$

The invariance part and the changed Lagrangian up to the first order:

$$\begin{aligned} \mathcal{L}' &\simeq \partial_\mu (\psi^* - i\alpha\psi^*) \partial^\mu (\psi + i\alpha\psi) - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 \\ &\simeq \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 = \mathcal{L} \end{aligned} \tag{7}$$

therefore, the conserved current is:

$$\begin{aligned} j^\mu =& \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) = i\alpha\psi \partial^\mu \psi^* - i\alpha\psi^* \partial^\mu \psi \\ &\equiv i(\psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi) \end{aligned} \tag{8}$$

We calculate explicitly $\partial_\mu j^\mu$ and use the E-L:

$$\begin{aligned} \partial_\mu j^\mu &= i \left[\partial_\mu \psi \partial^\mu \psi^* + \psi \partial_\mu \partial^\mu \psi^* - \partial_\mu \psi^* \partial^\mu \psi - \psi^* \partial_\mu \partial^\mu \psi \right] \\ &= i \left[-\psi \left(m^2 \psi^* + \lambda \psi (\psi^*)^2 \right) + \psi^* (m^2 \psi + \lambda \psi^* \psi^2) \right] \\ &= 0 \end{aligned} \tag{9}$$

QED.

4.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a \implies \tag{10}$$

$$
\begin{aligned}
\mathcal{L}' &\simeq \frac{1}{2}\partial_\mu(\phi_a + \theta\epsilon_{abc}n_b\phi_c)\partial^\mu(\phi_a + \theta\epsilon_{abc}n_b\phi_c) - \frac{1}{2}m^2(\phi_a\phi_a - 2\theta\epsilon_{bac}\phi_a\phi_c) \\
&\simeq \frac{1}{2}\{\partial_\mu\phi_a\partial^\mu\phi_a + \theta n_b\left[\epsilon_{abc}\partial_\mu\phi_c\partial^\mu\phi_a + \epsilon_{abc}\partial^\mu\phi_c\partial_\mu\phi_a\right]\} - \frac{1}{2}m^2\phi_a\phi_a \\
&= \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2\phi_a\phi_a = \mathcal{L}
\end{aligned} \tag{11}
$$

We used the fact that ϵ_{abc} tensor is anti-symmetric, but $\partial_\mu\phi_c\partial^\mu\phi_a$ is symmetric w.r.t changing the indices a and c. The same goes for $\phi_a\phi_c$.

Computing the Noether current:

$$
\begin{aligned}
j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a(\phi) \\
&= \partial^\mu\phi_a\,\epsilon_{abc}n_b\phi_c \implies \\
Q &= \int d^3x\,\epsilon_{abc}\dot{\phi_a}n_b\phi_c \\
&= n_b\int d^3x\,\epsilon_{bca}\dot{\phi_a}\phi_c \\
&= n_b\int d^3x\,\epsilon_{bac}\dot{\phi_c}\phi_a = n_a\int d^3x\,\epsilon_{abc}\dot{\phi_b}\phi_c
\end{aligned} \tag{12}
$$

We can choose $n_a = (1,0,0)$, then for each choice:

$$
Q_a = \int d^3x\,\epsilon_{abc}\dot{\phi_b}\phi_c \tag{13}
$$

Direct confirmation:

$$
\begin{aligned}
&\mathcal{L} = \frac{1}{2}\partial_t\phi_a\partial_t\phi_a - \frac{1}{2}\nabla\phi_a\nabla\phi_a - \frac{1}{2}m^2\phi_a\phi_a \implies \\
&\text{E-L for each } a\text{: } \partial_t^2\phi_a - \nabla^2\phi_a + m^2\phi_a = 0
\end{aligned} \tag{14}
$$

Hence:

$$
\begin{aligned}
\frac{d\,Q_a}{dt} &= \int d^3x\,\epsilon_{abc}\ddot{\phi_a}\phi_c + \int d^3x\,\epsilon_{abc}\dot{\phi_a}\dot{\phi_c} \\
&= \int d^3x\,\epsilon_{abc}(\nabla^2\phi_a - m^2\phi_a)\phi_c = \int d^3x\,\epsilon_{abc}\nabla^2\phi_a\,\phi_c \\
&= -\int d^3x\,\epsilon_{abc}\nabla\phi_a.\nabla\phi_c = 0
\end{aligned} \tag{15}
$$

We have used the symmetric, anti-symmetric point several times in the above derivation.

5.

- The first result is:

$$
\begin{aligned}
&\eta_{\sigma\tau}x'^\sigma x'^\tau = \eta_{\sigma\tau}\Lambda^\sigma_\mu\,x^\mu\Lambda^\tau_\nu\,x^\nu = \eta_{\mu\nu}x^\mu x^\nu \implies \\
&\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma_\mu\Lambda^\tau_\nu
\end{aligned} \tag{16}
$$

For any transformation to be Lorentz, it must satisfy Eq.(16):

$$\begin{aligned}\eta_{\mu\nu} &= \eta_{\sigma\tau}(\delta^{\sigma}_{\mu} + \omega^{\sigma}_{\mu})(\delta^{\tau}_{\nu} + \omega^{\tau}_{\nu}) \\ \eta_{\mu\nu} &= \eta_{\mu\nu} + \eta_{\mu\tau}\omega^{\tau}_{\nu} + \eta_{\sigma\nu}\omega^{\sigma}_{\mu} + \eta_{\sigma\tau}\omega^{\sigma}_{\mu}\omega^{\tau}_{\nu} \\ &\simeq \eta_{\mu\nu} + (\omega_{\mu\nu} + \omega_{\nu\mu})\end{aligned} \tag{17}$$

Hence:

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \tag{18}$$

A pure rotation:

$$\begin{aligned}R(\theta) &= \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & -1\end{bmatrix} \\ &\simeq \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} = [\delta] + \begin{bmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned} \tag{19}$$

Anti-symmetric as one can see for $\theta \ll 1$ infinitesimally small. In cas of the boost:

$$\Lambda(v) = \begin{bmatrix}\gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \tag{20}$$

$\gamma(v) \simeq 1 + \frac{1}{2}v^2 \implies$

$$\Lambda(v) \simeq \begin{bmatrix}1 & -v & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} = [\delta] + \begin{bmatrix}0 & -v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix} \tag{21}$$

Again anti-symmetric.