Information Theory Fall 2023
Prof. Maurizio Magarini

<Abdolbaset Veisi Tabar >
<Luigi Cagliani >

First Nokia Project Report

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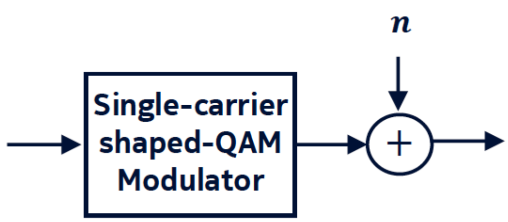
1 Introduction

This report is composed of several parts. At first, we look at the model in hand once again and simplify the optimization problem a bit using symmetry, and then we use the optimization tools of MATLAB to solve the first and then the second part of the project. The resulting graphs and figures are all included.

2 Math and model preliminaries

The model of the channel is given by the following:



What comes out of the box is a complex number chosen in the complex plane. Normally we specify that with z_m, where m is the index of the constellation index of the point in the 2-D complex plane. The notations and conventions, as suggested by the Instructor of the course, are as follows:

$$N = 16, 64QAM$$

$$z_n = x_m + ix_k, \quad n = m + \sqrt{N}k, \quad x_m = sign(m - \frac{\sqrt{N}}{2})(2^{|m-\frac{\sqrt{N}}{2}|} - 1)x_0 = \gamma_m x_0$$

$$m \in \{1, 2, ..., \sqrt{N}\}$$

Also, we know that:

$$z_{out} = z_m + N, \quad N \in \text{Gaussian Noise}$$

Both parts of this project involve finding a discrete pdf for z_m's described above in order to maximize information. As we're going to do the computation of mutual information a lot, there's a slight simplification to this due to the symmetries of 16 and 64-QAM modulations:

$$C \equiv \max_{\{p_{z_m}\}_{m=1}^{N}} I(z_m; z_{out} = Z) =$$

$$\max_{\{p_{z_m}\}_{m=1}^{N}} H(Z) - H(Z|z_m) = [\max_{\{p_{z_m}\}_{m=1}^{N}} H(Z)] - H(N)$$

Where the entropy of 2-D Gaussian is:

$$H(N) = \log_2 2\pi\sigma_N^2$$

First, we need to characterize the output of the model, Z, i.e. its pdf:

$$P\{z \leq Z < z + dz\} = f_Z(z)dz = P\{z \leq N + Z_m < z + dz\}$$

$$= \sum_{m=1}^{N} P\{z \leq N + z_m < z + dz|Z_m = z_m\}P\{Z_m = z_m\} = \sum_{m=1}^{N} P\{z - z_m \leq N < z - z_m + dz|Z_m = z_m\}p_m$$

$$= \sum_{m=1}^{N} P\{z - z_m \leq N < z - z_m + dz\}p_m = \sum_{m=1}^{N} p_m f_N(z - z_m)dz \implies$$

$$f_Z(z) = \sum_m p_m f_N(z - z_m)$$

Because of the geometry of z_m points in 16 or 64-QAM constellation, we can further simplify the probability distribution as follows:

$$p_n = P\{z_N = z_n\} = P\{z_N = x_M + ix_K == x_m + ix_k\} = P\{x_M = x_m, x_K = x_k\} = p_{m,k} = p_n$$

s.t.

$$n = m + \sqrt{n}k, \quad m, k \in \{1, 2, ..., \sqrt{N}\}$$

Obviously,

$$x_M \perp\!\!\!\perp x_K \implies p_n = p_{m,k} = p_m p_k$$

Hence:

$$f_Z(z) = \sum_{m=1,k=1}^{\sqrt{N}} p_{m,k} f_{\text{Normal Gaussian}}(z - (x_m + ix_k)) =$$

$$\sum_{m,k} p_m p_k \frac{1}{2\pi\sigma_N^2} \exp\left(-\frac{(x - x_m)^2 + (y - x_k)^2}{2\sigma_N^2}\right) =$$

$$\sum_{m}^{\sqrt{N}} p_m \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{(x - x_m)^2}{2\sigma_N^2}\right) \sum_{k}^{\sqrt{N}} p_k \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{(y - x_k)^2}{2\sigma_N^2}\right) =$$

$$f_X(x)f_X(y), \quad \text{s.t.} \quad f_X(x) = \sum_{m=1}^{\sqrt{N}} p_m \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{(x - x_m)^2}{2\sigma_N^2}\right)$$

We also exploit the fact that if $p_z = p_x p_y \implies h_z = h_x + h_y$. Therefore the entropy is:

$$h(Z) = h(X) + h(X) = 2h(X) = -2\int \sum_m p_m f_{\text{Norm.}}(x - x_m) \log_2 \left(\sum_m p_m f_{\text{Norm.}}(x - x_m)\right)dx$$

Therefore instead of evaluating a double integral and an N-variable optimization problem, we only evaluate a single integral and solve a $\sqrt{N}$-variable optimization problem. Although, we can further simplify the problem and only solve a $\frac{\sqrt{N}}{2}$ problem, but such thing isn't necessary as it doesn't simplify our problem drastically as the previous simplification did.

3 Part I of the project

For a two-variable channel due to the modulation in phase and amplitude like this the Shannon capacity is:

$$C_{Sh} = \log_2(1 + \frac{\sigma_Z^2}{\sigma_N^2}) = \log_2(1 + S), \quad S = \text{SNR}$$

We have to solve the following optimization problem:

$$\max_{\{p_i\}_{i=1}^{\sqrt{N}}} -\int_{-\infty}^{\infty} \sum_m p_m f_{\text{Norm.}}(x - x_m) \log_2(\sum_m p_m f_{\text{Norm.}}(x - x_m))dx$$

s.t.

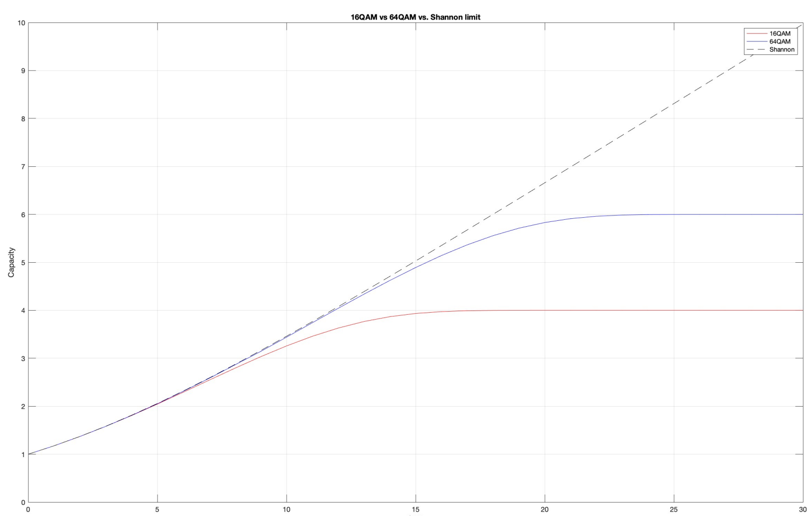
$$\sum_m p_m = 1$$

$$\sum_{m,k} p_{m,k}(x_m^2 + x_k^2) = \sigma_Z^2 = \sigma_N^2 S = S = 2\sum_m p_m x_m^2 = S$$

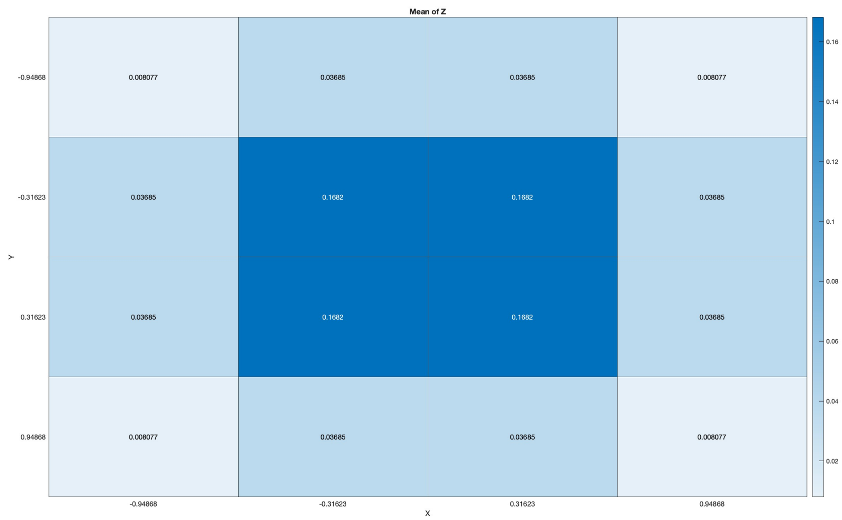
σ_N^2 is assumed to be 1 for simplicity. The last condition further simplifies as $x_m = \gamma_m x_0$:

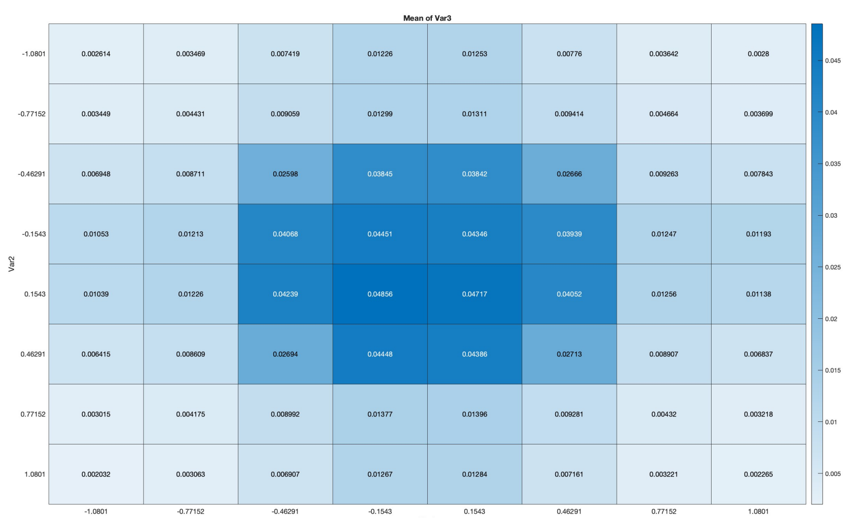
$$2(\sum_m p_m \gamma_m^2)x_0^2 = 2(\sum_m p_m \gamma_m^2) = S/(x_0^2)$$

This condition is always enforced at the beginning of the optimization by tuning x_0, therefore this condition isn't a constraint on our problem. Solving this optimization problem in MATLAB using $fmincon$ is a straightforward task. We optimize the probabilities for $N = 16, 64$ for each SNR. Then, we plot the results here vs. the Shannon limit(The picture is available in the attached file as "16QAMvs64QAMvsShannonLimit" title):



It is obvious that they match the picture provided by the project description. The heat-map of the probabilities for 16,64-QAM modulations looks like this correspondingly at a specific SNR($S = 5dB$):



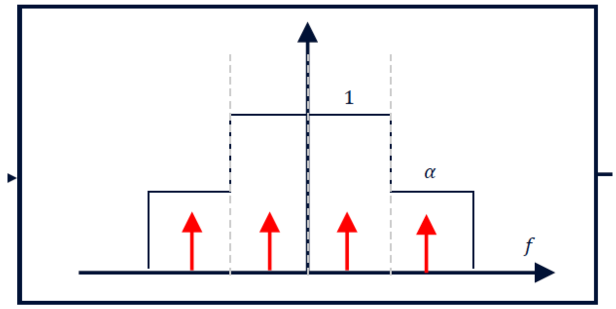


The figures are available in the attached file.

The remarks are worth noting. First, our proof of symmetry has been severely investigated. We ran a code which would optimize N probabilities for a two-variable integral. And again the result was the validity of symmetry proved. Second remark is that a further symmetry to reduce the problem to $\sqrt{N}/2$-variable problem was also mentioned and now it can be seen it's also presumable and correct.

4 Part II of the project

In this part, the model of the channel bandwidth has changed as the following:



The Shannon limit isn't the same as before for this channel. So, first, we have to compute that.

$$y_1 = x_1 + n$$

$$y_2 = \alpha x_2 + n$$

We have a limited budget in terms of power or in other words, SNR:

$$S = \text{SNR} = \beta\text{SNR} + (1-\beta)\text{SNR} =$$

$$\text{SNR}_1 + \text{SNR}_2$$

The first in the above expression is related to the first normal channel and the second term is that of the attenuated channel. The channel capacity is:

$$C = C_1 + C_2 = \log_2(1 + \beta S) + \log_2(1 + \alpha^2(1-\beta)S)$$

Hence,

$$C = \log_2(-[\alpha^2 S^2]\beta^2 + [S(1+\alpha^2 S) - \alpha^2 S]\beta + [1 + \alpha^2 S]) \implies$$
$$\phi = 2^C = -[\alpha^2 S^2]\beta^2 + [S(1+\alpha^2 S) - \alpha^2 S]\beta + [1 + \alpha^2 S]$$

We can optimize *phi* instead of C. This is obviously an optimization problem for β given the following constraint:

$$0 \leq \beta \leq 1$$

Using Lagrange's multiplier method we can easily solve this. The solution for β is:

$$\beta = 1, \quad \text{if} \quad S \leq \frac{1-\alpha^2}{\alpha^2}$$

$$\beta = \frac{1}{2} + \frac{\frac{1}{\alpha^2} - 1}{2S}, \quad \text{if} \quad S > \frac{1-\alpha^2}{\alpha^2}$$

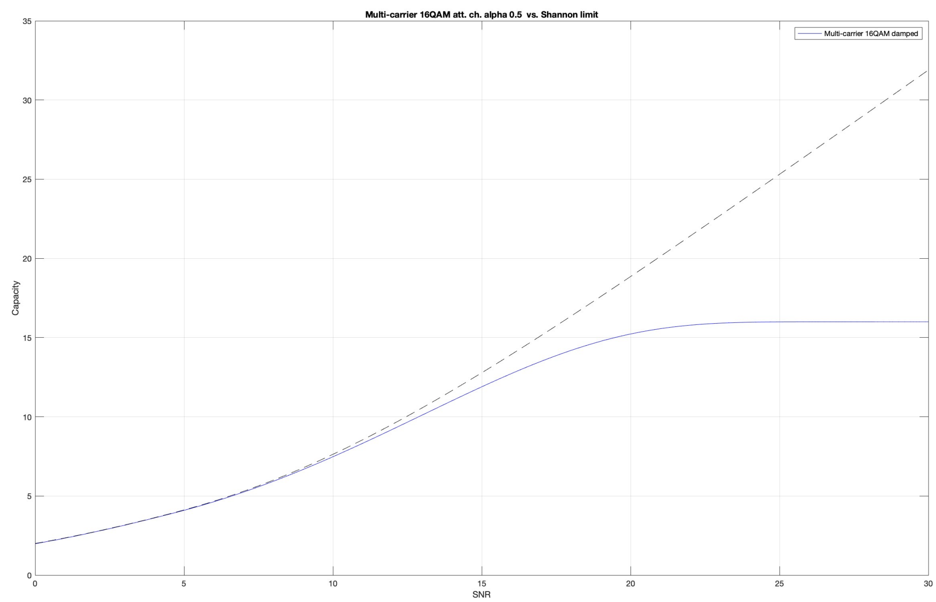
So, depending on SNR, the Shannon limit can be determined using the relations above.

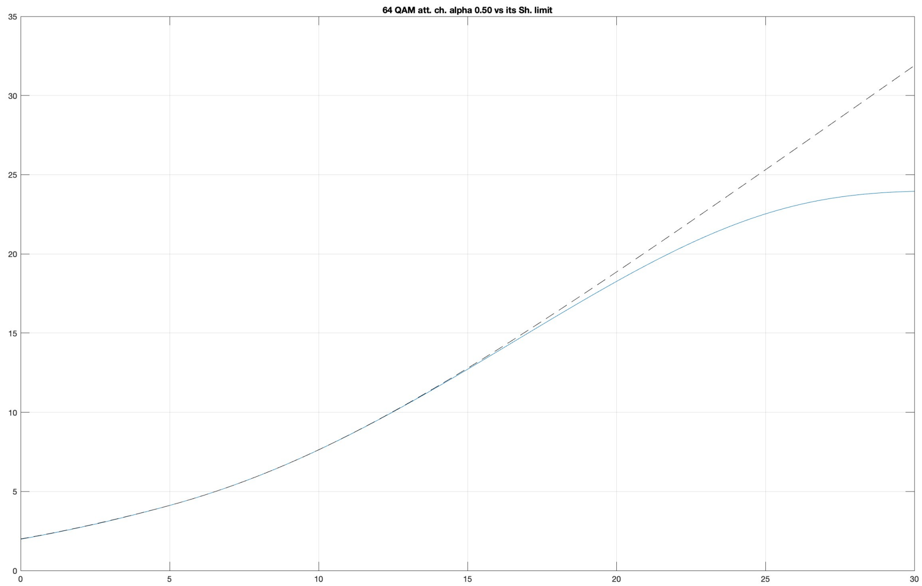
A similar algorithm has been designed to compute the optimized portion of the SNR to be devoted to the good and the rest to the attenuated channel. And, then, the probabilities for both channels are assumed to be independent and optimized for a maximum amount of capacity. This piece of code here best summarizes the thought behind the optimization:

```
fun = @(p) (my_awgn_channel_capacity( p(1), SNR_vec(i), x_64QAM, p(2:n+1) )...
        + my_awgn_channel_capacity( 1-p(1),SNR_vec(i)+10*log10(alpha^2), x_64QAM, p(n+2:end) ))
```

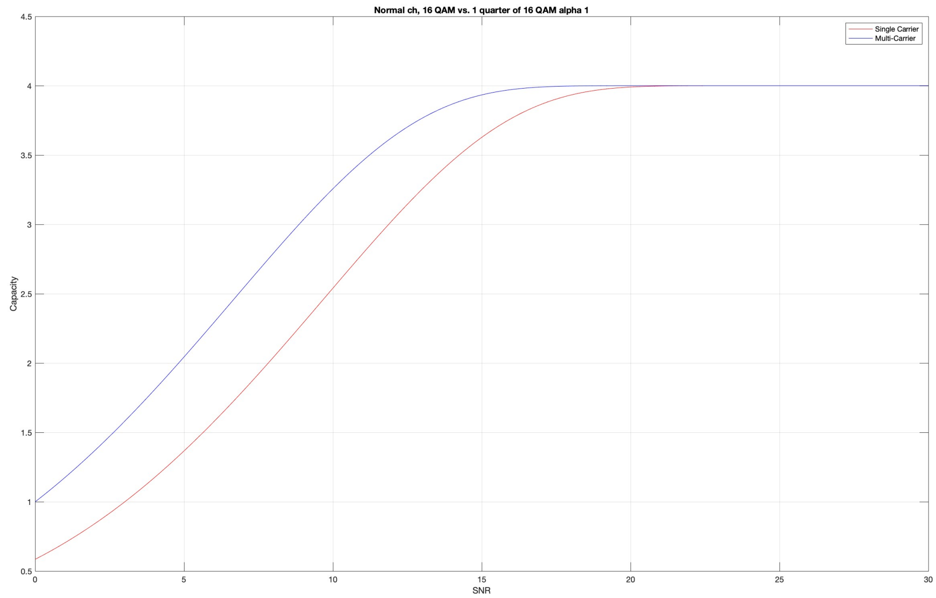
In the above piece of code, $p(1)$, is the β portion explained earlier, and the rest of the p vector are the probabilities for the good channel and the attenuated channel.

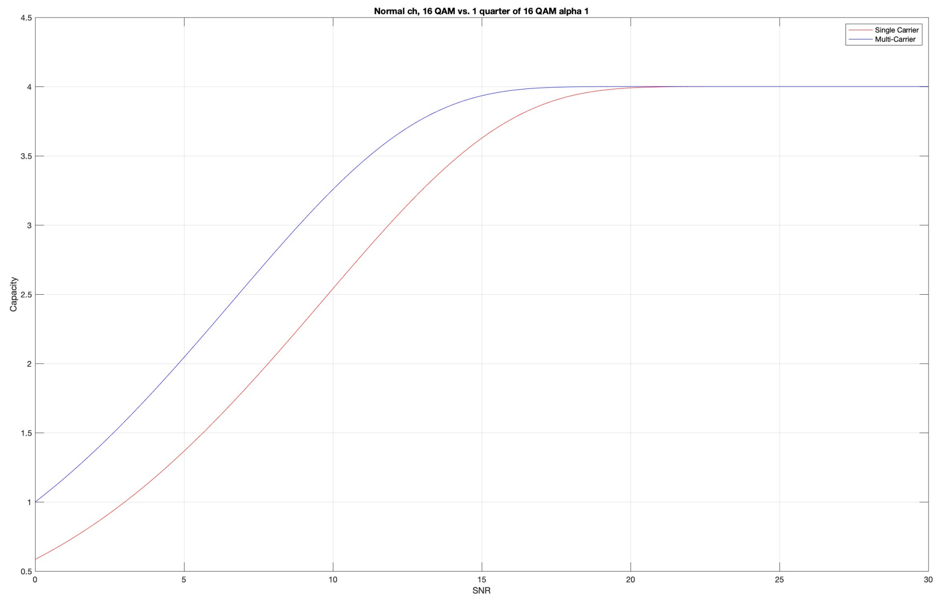
With this at hand, below you can find the graph of capacity vs. SNR for 16-QAM and 64-QAM modulations for $\alpha = 0.5$ correspondingly. Also their computed Shannon limit is depicted with dashed lines:



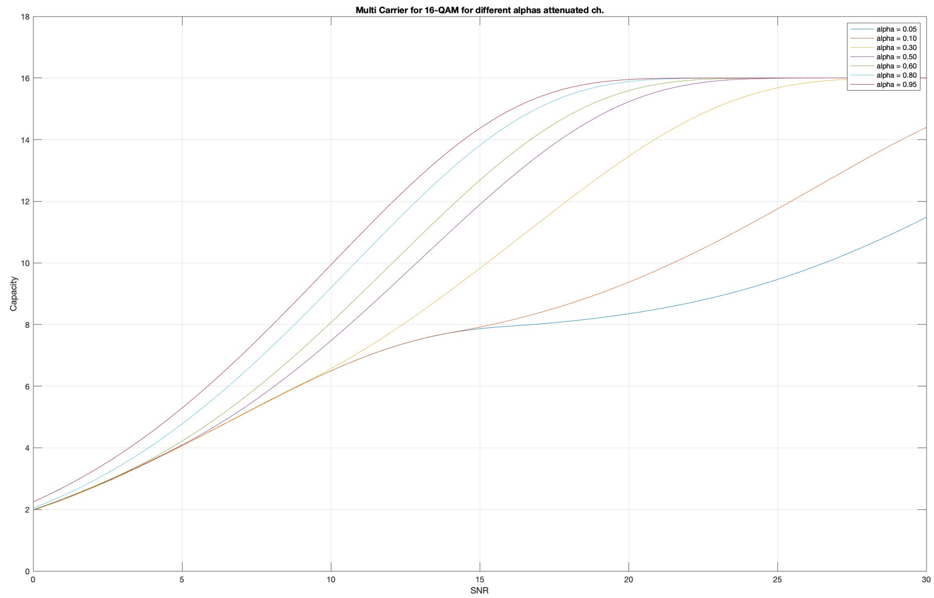


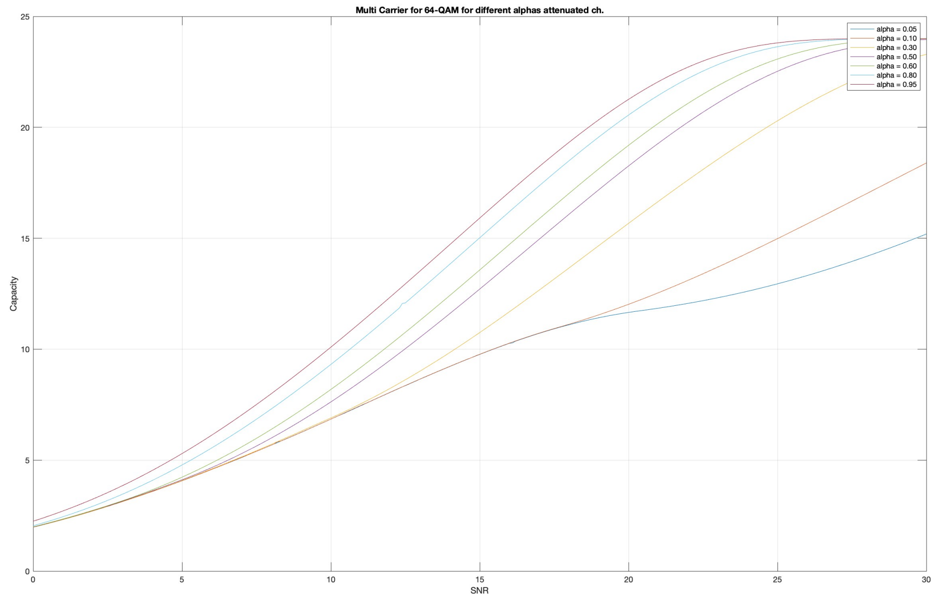
To better compare between the single-carrier and multi-carrier modes of 16-QAM and 64-QAM, I multiplied the capacity of Multi-Carrier by 4, to compare the Information rates of the two. As you see there's a subtle difference between the two. It's mostly evident in low SNRs because we subtract $\log_2(2\pi e)$ four times in the case of Multi-Carrier. To compensate for this effect, it needs a sufficiently high energy. Anyways the information rate comparisons are correspondingly depicted for 16 and 64 QAM for single and Multi-carrier modes:


Normal ch, 16 QAM vs. 1 quarter of 16 QAM alpha 1
Single Carrier
Multi-Carrier
Capacity
SNR
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
5
10
15
20
25
30



It can also be interesting to look how α can affect the Capacity vs. SNR curve. Below, you can find that as α increases, so does the capacity. This must be obvious as the less attenuated a channel is, the more information it can transfer(The first one is that of 16-QAM and the second one is for 64-QAM):


Multi Carrier for 16-QAM for different alphas attenuated ch.
alpha = 0.05
alpha = 0.10
alpha = 0.30
alpha = 0.50
alpha = 0.60
alpha = 0.80
alpha = 0.95
Capacity
SNR
0
2
4
6
8
10
12
14
16
18
5
10
15
20
25
30



5 Conclusion

For a stair-attenuated channel with single frequency pulses and 16-QAM and 64-QAM modulation, the optimized probabilities for maximized information transfer are such that sending pulses with less amplitude has to be more frequent. Also, in the case mentioned in the prev. sentence, sending in single-carrier mode is more efficient in terms of $\frac{\text{Capacity}}{\text{Bandwidth}}$ in the case where we have no attenuation.

6 Further Notes and Discussions

To make a more realistic model, and to make a better comparison between single vs multi-carrier modes a wide and realistic pulse shapes like **Raised Cosine** pulses must be considered. With this model of wide pulse in frequency, we can investigate realistically whether attenuation can be overcome better by Multi-Carrier mode or not. Also, in this case the channel noise model can be that of the Planck's distribution, $\frac{\hbar\omega}{\exp(\frac{\hbar\omega}{k_B T})-1}$. In this project, we only considered impulse shape carriers in frequency which is very unrealistic in the reality. And impulses don't care much about attenuation. What's more, I had an idea of considering an arbitrary constellation with unfixed modulations but fixed uniform probabilities. Finding the best points in the 2-D complex plane for an optimized capacity can be an interesting problem, i.e., instead of fixing the points and changing the probabilities, we can fix the probabilities and change the points.