Solutions to the Quantum Circuits and Devices by Prof. Ielmini exercise set #2

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1 Exercise 1

1.1 Question

Derive the Pauli operator for direction $\vec{n}$ described by $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4}$, calculate its corresponding eigenvectors and eigenvalues, and plot them on the Bloch sphere.

1.2 Answer

$$\vec{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \implies \sigma_{\hat{n}} = \vec{n}.\vec{\sigma} =$$

$$n_x\sigma_x + n_y\sigma_y + n_z\sigma_z = \frac{\sqrt{2}}{2}\sigma_x + \frac{\sqrt{2}}{2}\sigma_y =$$

$$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

The expectation is that:

$$|n_+\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}|1\rangle$$

$$|n_-\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - e^{i\frac{\pi}{4}}\frac{1}{\sqrt{2}}|1\rangle$$

Let's investigate if we'll meet our expectations. The eigenvalues and vectors can be found:

$$\sigma_{\hat{n}}|n\rangle = \lambda|n\rangle \equiv \det\left\{\begin{bmatrix} -\lambda & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & -\lambda \end{bmatrix}\right\} = 0 \implies$$

$$\lambda^2 - (\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \lambda^2 - 1 = 0 \implies$$

$$\lambda = \pm 1$$

We call the eigenstate associated to the ± 1 eigenvalue with $|n_\pm\rangle$. Therefore:

$$\sigma_{\hat{n}} |n_\pm\rangle = \pm |n_\pm\rangle \equiv$$

$$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \pm \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \implies$$

$$(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})n_1 = \pm n_2 \implies e^{i\frac{\pi}{4}} n_1 = \pm n_2 \implies$$

$$|n_\pm\rangle \equiv \begin{bmatrix} n_1 \\ \pm e^{i\frac{\pi}{4}} n_1 \end{bmatrix} = n_1 \begin{bmatrix} 1 \\ \pm e^{i\frac{\pi}{4}} \end{bmatrix}$$

As $\langle n_\pm | n_\pm \rangle = 1$, then $n_1 = \frac{1}{\sqrt{2}}$, hence:

$$|n_\pm\rangle = \frac{1}{\sqrt{2}} |0\rangle \pm e^{i\frac{\pi}{4}} \frac{1}{\sqrt{2}}$$

Which are equivalent to what we expected.

2 Exercise 2

2.1 Question

Let operator $\hat{O}$ be s.t. $\hat{O}|0\rangle = |-\rangle$ and $\hat{O}|1\rangle = |+\rangle$. Find an expression for $\hat{O}$.

2.2 Answer

One obvious solution that comes to mind is:

$$\hat{O} = |-\rangle \langle 0| + |+\rangle \langle 1|$$

One can try to write this in $\{|0\rangle, |1\rangle\}$ basis:

$$O_{11} = \langle 0| \hat{O} |0\rangle = \langle 0|-\rangle = \frac{1}{\sqrt{2}}, \quad O_{12} = \langle 0| \hat{O} |1\rangle = \langle 0|+\rangle = \frac{1}{\sqrt{2}}$$

$$O_{21} = \langle 1| \hat{O} |0\rangle = \langle 1|-\rangle = -\frac{1}{\sqrt{2}}, \quad O_{22} = \langle 1| \hat{O} |1\rangle = \langle 1|+\rangle = \frac{1}{\sqrt{2}}$$

Now, we test it in this basis:

$$\hat{O}|0\rangle \equiv \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \equiv |-\rangle$$

And:

$$\hat{O}|1\rangle \equiv \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \equiv |+\rangle, \quad QED.$$

3 Exercise 3

3.1 Question

Consider a state $|\psi\rangle$. Knowing that $P_0(|\psi\rangle) = 0.2$, $P_1(|\psi\rangle) = 0.8$, $P_0(H|\psi\rangle) = 0.6$, $P_1(H|\psi\rangle) = 0.4$, $P_0(HS^\dagger|\psi\rangle) = 0.7$, and $P_1(HS^\dagger|\psi\rangle) = 0.3$. Estimate the angles θ, ϕ on the Bloch sphere of the state.

3.2 Answer

Any generic state in the $\{|0\rangle, |1\rangle\}$ basis can be written as:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \quad P_0(|\psi\rangle) = |\langle 0|\psi\rangle|^2 = 0.2 = \cos^2\frac{\theta}{2}$$

$$\implies \cos\theta = 2\cos^2\frac{\theta}{2} - 1 = -0.6 \implies \theta = \pi - \arccos 0.6$$

We define $|\psi'\rangle$:

$$H|\psi\rangle = |\psi'\rangle = \cos\frac{\theta}{2}H|0\rangle + e^{i\phi}\sin\frac{\theta}{2}H|1\rangle = \cos\frac{\theta}{2}\frac{|0\rangle + |1\rangle}{\sqrt{2}} + e^{i\phi}\sin\frac{\theta}{2}\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2})|0\rangle + \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} - e^{i\phi}\sin\frac{\theta}{2})|1\rangle \implies$$

$$P_0(|\psi'\rangle) = \frac{1}{2}(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2})(\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}) =$$

$$\frac{1}{2}(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\phi) = \frac{1}{2}(1 + \sin\theta\cos\phi)$$

$$\implies 1 + \sqrt{1 - \cos^2\theta}\cos\phi = 1 + 0.8\cos\phi = 1.2 \implies \cos\phi = 0.25$$

Finally, according to the lecture notes $S = R_{\frac{\pi}{2}}$, where:

$$R_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \implies S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

Let $|\psi''\rangle = HS^\dagger|\psi\rangle$, and more over:

$$HS^\dagger|0\rangle = H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad HS^\dagger|1\rangle = -iH|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}i} \implies$$

$$|\psi''\rangle = \cos\frac{\theta}{2}\frac{|0\rangle + |1\rangle}{\sqrt{2}} + e^{i\phi}\sin\frac{\theta}{2}\frac{|0\rangle - |1\rangle}{\sqrt{2}i}$$

Which implies:

$$P_0(|\psi''\rangle) = (\frac{\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}}{\sqrt{2}})(\frac{\cos\frac{\theta}{2} + ie^{-i\phi}\sin\frac{\theta}{2}}{\sqrt{2}}) =$$

$$= \frac{1}{2}(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - i\frac{e^{i\phi} - e^{-i\phi}}{2}\sin\theta) \implies (1 + \sin\phi\sin\theta) = 2 \times 0.7$$

$$\implies 0.4 = 0.8 \times \sqrt{1 - 0.25^2}$$

Which is impossible and there's a self contradiction in the numbers given!

4 Exercise 4

4.1 Question

Consider a two-qubit system with state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Determine whether the state is a product state or entangled state.

4.2 Answer

By contradiction, let's suppose the state is a product state. The most generic product state in the basis is:

$$|\phi\rangle = (a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle) = a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + b_1 a_2 |10\rangle + a_2 b_2 |11\rangle$$

We force the inequality $|\psi\rangle = |\phi\rangle$ which implies:

$$a_1 a_2 = 0, \quad a_2 b_2 = 0, \quad a_1 b_2 \neq 0, \quad b_1 a_2 \neq 0$$

Which is impossible. Therefore the state is entangled.

5 Exercise 5

5.1 Question

Consider the two-qubit circuit in Fig. 1 where input qubits $|\psi_1\rangle$, $|\psi_2\rangle$ are prepared in the $|0\rangle$ and $|1\rangle$ state respectively. Determine the equivalent circuit operator $\hat{O}$, and the output state $|\psi_0\rangle$ of the circuit.

5.2 Answer

The equivalent circuit is:

$$\hat{O} = \hat{U}_{CNOT}(H \otimes I)(I \otimes X) = \hat{U}_{CNOT}(H \otimes X) \equiv$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

As $|\psi_0\rangle = \hat{O} |\psi_1\rangle \otimes |\psi_2\rangle$:

$$|\psi_0\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$