Solutions to the Quantum Circuits and Devices by Prof. Ielmini exercise set #1

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1 Exercise 1

1.1 Question

Calculate all possible values of the angular momentum L, of its projection on the z-axis L_z, and of the magnetic dipole momentum along z μ_z of an electron when $l = 2$, and plot them graphically.

1.2 Answer

For each value of l, we know that the total angular momentum eigenvalue is

$$L = \sqrt{l(l+1)\hbar^2)} = \sqrt{6}\hbar$$

For L_z, we know that:

$$-l \leq m \leq l \quad \text{s.t.} \quad l_z = m\hbar \implies l_z \in \{-2\hbar, -\hbar, 0, \hbar, 2\hbar\}$$

$$\mu_z = -\mu_B \frac{L_z}{\hbar} = -\frac{e\hbar}{2m_e} m = -\mu_B m \implies \mu_z \in \{2\mu_B, \mu_B, 0, -\mu_B, -2\mu_B\}$$

2 Exercise 2

2.1 Question

A qubit is localized on the Bloch sphere by angles $\theta = 50°$ and $\phi = 10°$. Draw the state on the Bloch sphere and calculate the state vector in the $\{|0\rangle, |1\rangle\}$ basis and the probability of measuring the basis states.

2.2 Answer

We know that for a generic measurement along the (θ, ϕ) axis, a general Hilbert 2D space vector can be written as:

$$|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$$

The drawing can't be much of work, so right to the state vector calculations:

$$|\psi\rangle \approx 0.906\,|0\rangle + \exp\left\{i\frac{\pi}{18}\right\}0.423\,|1\rangle$$

$\{|0\rangle, |1\rangle\}$ being the $\{|z_+\rangle, |z_-\rangle\}$ basis. Based on the Quantum Measure theory assumptions, the probability of collapsing onto any of the basis vectors is:

$$Pr(|0\rangle) = \cos^2\frac{\theta}{2} \approx 0.821, \quad Pr(|1\rangle) = \sin^2\frac{\theta}{2} \approx 0.179$$

3 Exercise 3

3.1 Question

Consider a qubit $|\psi\rangle = (\frac{1}{2} + \frac{i}{2})\,|0\rangle - (\frac{1}{2\sqrt{2}} + i\frac{\sqrt{3}}{2\sqrt{2}})\,|1\rangle$

3.1.1 a

Locate the state on the Bloch sphere by calculating the corresponding angles θ and ϕ.

3.1.2 b

Calculate the global rotation angle δ and the equivalent state $|\psi'\rangle$ with purely real α'coefficient.

3.2 Answer

3.2.1 a

$$|\psi\rangle = \frac{\sqrt{2}}{2}\exp\left\{i\frac{\pi}{4}\right\}|0\rangle - \frac{\sqrt{2}}{2}\exp\left\{i\frac{\pi}{3}\right\}|1\rangle =$$

$$\cos\frac{\pi}{4}\exp\left\{i\frac{\pi}{4}\right\} + \sin\frac{\pi}{4}\exp\left\{i\frac{4\pi}{3}\right\}|1\rangle \equiv$$

$$\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\exp\left\{i(\frac{4\pi}{3} - \frac{\pi}{4})\right\}|1\rangle =$$

$$\cos\frac{\pi}{4}|0\rangle + \sin\frac{\pi}{4}\exp\left\{i\frac{13\pi}{12}\right\}|1\rangle$$

Hence:

$$\theta = \frac{\pi}{2}, \quad \phi = \frac{13\pi}{12}$$

3.2.2 b

From the previous calculations:

$$\delta = -\frac{\pi}{4}$$

4 Exercise 4

4.1 Question

Consider two states $|\psi\rangle$, $|\psi'\rangle$ differing only by a global phase factor $e^{i\delta}$. Show that the probability of measuring a state $|s\rangle$ is identical for $|\psi\rangle$ and $|\psi'\rangle$, for any target state $|s\rangle$.

4.2 Answer

Let

$$|\psi'\rangle = e^{i\delta} |\psi\rangle$$

We have that:

$Pr\{|s\rangle\}_\psi =$ Probability of measuring states s when the current state is $\psi = |\langle\psi|s\rangle|^2$

$$Pr\{|s\rangle\}_{\psi'} = |\langle\psi'|s\rangle|^2 = \langle\psi'|s\rangle\langle s|\psi'\rangle = \langle\psi| e^{-i\delta} |s\rangle \langle s| e^{i\delta} |\psi\rangle = \langle\psi|s\rangle\langle s|\psi\rangle = |\langle\psi|s\rangle|^2$$
$$= Pr\{|s\rangle\}_\psi \implies Pr\{|s\rangle\}_{\psi'} = Pr\{|s\rangle\}_\psi \quad QED.$$

5 Exercise 5

5.1 Question

Consider the Stern-Gerlach experimental setup in Fig. 1, where the input qubit is prepared in state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - (\frac{1+\sqrt{3}}{4} - i\frac{1-\sqrt{3}}{4})|1\rangle$:

5.1.1 a

Calculate the measurement probability for states $|x_+\rangle$, $|x_-\rangle$ and $|y_+\rangle$, $|y_-\rangle$ after the corresponding experimental setups.

5.1.2 b

Suppose now to collimate both the $|x_+\rangle$, $|x_-\rangle$ output beams into the second SG setup. Calculate the measurement probability for $|y_+\rangle$, $|y_-\rangle$ after the second SG setup.

5.2 Answer

Let:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where:

$$\alpha = \frac{1}{\sqrt{2}}, \quad \beta = -(\frac{1+\sqrt{3}}{4} - i\frac{1-\sqrt{3}}{4})$$

5.2.1 a

$$Pr\{|x_+\rangle\}_\psi = |\langle x_+|\psi\rangle|^2 = |\frac{\langle 0| + \langle 1|}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)|^2 =$$

$$|\frac{\alpha+\beta}{\sqrt{2}}|^2 = \frac{1}{2}|\alpha+\beta|^2 = \frac{1}{2}[(\frac{1}{\sqrt{2}} - \frac{1+\sqrt{3}}{4})^2 + (\frac{1-\sqrt{3}}{4})^2]$$

$$= \frac{1}{2}[\frac{1}{2} + \frac{4+2\sqrt{3}}{16} - \frac{\sqrt{2}+\sqrt{6}}{4} + \frac{4-2\sqrt{3}}{16}] = \frac{1}{2} - \frac{\sqrt{2}+\sqrt{6}}{8} \approx 0.0170$$

$$\implies Pr\{|x_-\rangle\}_\psi = 1 - Pr\{|x_+\rangle\}_\psi \approx 0.983$$

Only the states of $|x_+\rangle$ are measured in the second experiment:

$$Pr\{|y_+\rangle\}_{x_+} = |\langle x_+|y_+\rangle|^2 = \frac{1}{2} = 1 - Pr\{|y_+\rangle\}_{x_-} = \frac{1}{2}$$

5.2.2 b

The density matrix after collimating both beams will be:

$$\rho = 0.017^2\,|x_+\rangle\langle x_+| + 0.893^2\,|x_-\rangle\langle x_-| = p_0\,|x_+\rangle\langle x_+| + p_1\,|x_-\rangle\langle x_-|$$

Hence:

$$Pr\{|y_+\rangle\}_\rho = \mathrm{Tr}\{\rho\,|y_+\rangle\langle y_+|\} = \mathrm{Tr}\{p_0\,\langle x_+|y_+\rangle\,|x_+\rangle\langle y_+| + p_1\,\langle x_-|y_+\rangle\,|x_-\rangle\langle y_+|\}$$

$$= p_0|\langle x_+|y_+\rangle|^2 + p_1|\langle x_-|y_+\rangle|^2 = \frac{p_0+p_1}{2} = \frac{1}{2} = 1 - Pr\{|y_-\rangle\}_\rho = \frac{1}{2}$$