Solutions to the Quantum Circuits and Devices by Prof. Ielmini exercise set #4

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1 Exercise 1

1.1 Question

Consider an electron subject to a static magnetic field $B_0 = 1T$ along $\vec{z}$ and a resonant magnetic field of amplitude $B_1 = 1mT$ along $\vec{x}$ coupled to the electron. After calculating the frequency of the resonant field, derive:

1.1.1 a

The pulse envelope amplitude to operate an X gate, when the pulse duration is $t_1 = 100ns$.

1.1.2 b

the pulse duration to operate an X gate, when the pulse envelope amplitude is $\eta = 0.5$

1.2 Answer

The Larmor and resonant frequency will be $\omega_L = |\gamma|B_0 = \frac{e}{m_e}B_0$ and $\Omega = \frac{\mu_B}{\hbar}B_1 = \frac{e}{2m_e}B_1$. Numerically:

$$\Omega = 2\pi \times 14MHz$$

1.2.1 a

For a full bit-flip operation:

$$\Omega t = \pi = 2\pi \times 14MHz \times \eta \times 10^{-7} \implies \eta = \frac{1}{2.8} \approx 0.36$$

1.2.2 b

Again:

$$\Omega t = \pi = 2\pi \times 14MHz \times \eta \times t \implies t = \frac{1}{14}10^{-6} \approx 71.43ns$$

2 Exercise 2

2.1 Question

Consider an electron subject to a static magnetic field $B_0 = 1T$ along $\vec{z}$. A resonant magnetic field with tunable phase of amplitude $B_1 = 1mT$ along $\vec{x}$ is coupled to the electron by an on/off switch with no amplitude modulation. Calculate the frequency of the resonant field and draw the pulse schedule to operate the gate $\hat{O} = \hat{H}.\hat{X}$.

2.2 Answer

We could investigate the application of O altogether and design a pulse time for it:

$$\theta(t_p) = \int_0^{t_p} \eta(t).\Omega dt$$

If it's only going to be a pulse and $\Omega = \frac{2\mu_B}{\hbar}\frac{B_1}{2} = \frac{\mu_B}{\hbar}B_1 = 2\pi \times 14MHz$. One can tune *delta* to do a precession around $\vec{x}$, $(\delta = 0)$ or around $\vec{y}$, $(\delta = \frac{\pi}{2})$. A Hadamard gate is $H = XY^{1/2}$. Proof:

$$Y^{1/2} = R_{\hat{y}}(\frac{\pi}{2}) = e^{-i\frac{\pi}{4}\sigma_y} = \cos\frac{\pi}{4}I - i\sin\frac{\pi}{4}\sigma_y = \begin{bmatrix}\cos\frac{\pi}{4} & 0\\ 0 & \cos\frac{\pi}{4}\end{bmatrix} - i\begin{bmatrix}0 & -i\sin\frac{\pi}{4}\\ i\sin\frac{\pi}{4} & 0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \implies XY^{1/2} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = H$$

Overall, the gate will be implemented as:

$$\hat{O} = XY^{1/2}X$$

- A full bit flip with $\delta = 0$, $t_1 = t_p = \frac{\pi}{\Omega} \approx 35.71ns$.
- A $\frac{\pi}{2}$ phase change with $t_2 = \frac{t_p}{2} \approx 17.86ns$, with $\delta = \frac{\pi}{2}$.
- A full bit flip again with $\delta = 0$, and $t_3 = t_p$

3 Exercise 3

3.1 Question

An electron is immersed in a static magnetic field $B_0 = 1T$ directed along $\vec{z}$ and coupled to a resonant magnetic field of amplitude $B_1 = 1mT$ along $\vec{x}$. Calculate the timing accuracy of a control system to provide a rotation angle accuracy $\Delta\theta = \frac{\pi}{100}$.

3.2 Answer

$$\Delta\theta = \eta\Omega\Delta t \implies \Delta t \le \frac{\Delta\theta}{\Omega} = \frac{\pi}{1000 \times 2\pi \times 14 \times 10^6} \approx 35.71ps$$

4 Exercise 4

4.1 Question

Consider a quantum system with natural frequency $\omega_{01} = 2\pi.5GHz$ and anharmonicity $\Delta\omega = 2\pi.200MHz$ where a π-rotation pulse along $\vec{x}$ is operated by ESR. Compare the driving amplitudes η and spectral amplitudes $\mathcal{B}$ of the driving field for the $|1\rangle \to |2\rangle$ transition for a driving pulse with:

4.1.1 a

rectangular envelope of amplitude η_1 and width $t_1 = 10ns$.

4.1.2 b

rectangular envelope of amplitude η_2 and width $t_2 = 100ns$.

4.1.3 c

Gaussian envelope of amplitude η_3 with FWHM $\Delta t = 10ns$.

4.2 Answer

The meaning of the anharmonicity is that for a transition $|0\rangle \to |1\rangle$, the freq. is ω_{01}, and for a transition $|1\rangle \to |2\rangle$, the freq. is $\omega_{01} + \Delta\omega$. The idea is to make a transition between the first two and not a third transition. Let $B_c(t) = \eta(t)B_1 \cos\omega_{01}t$. We want the power to be absent at $\omega_{01} + \Delta\omega$ freq.

4.2.1 a

$$\mathcal{F}(B_c(t)) = \mathcal{B} = \mathcal{F}(\eta(t)) * \mathcal{F}(B_1 \cos\omega_{01}t) = \mathcal{F}(\eta_1.rect(\frac{t}{t_1})) * B_1\delta(\omega - \omega_{01}) =$$

$$\eta_1 t_1.\mathrm{sinc}(\frac{\omega t_1}{2}) * B_1\delta(\omega - \omega_{01}) = \eta_1 B_1 t_1.\mathrm{sinc}(\frac{(\omega - \omega_{01})t_1}{2}) \implies$$

$$\frac{\mathcal{B}(\omega_{01} + \Delta\omega)}{\mathcal{B}(\omega_{01})} = \mathrm{sinc}(\frac{\Delta\omega.t_1}{2}) = \mathrm{sinc}(2\pi) = 0$$

The answer in the exercise class isn't correct. Different numbers should have been used to prove that Gaussian envelopes are better.

$$\Omega_1 = \eta_1\Omega \implies \theta = \pi = \int_0^{t_1} \Omega_1(t)dt = \eta_1 t_1 \Omega$$

Similarly:

$$\theta = \pi = \eta_2 t_2 \Omega$$

And finally:

$$\theta = \pi \approx \int_{-\infty}^{\infty} \eta_3 \exp\left\{-\frac{t^2}{2\sigma^2}\right\}dt = \eta_3\sqrt{2\pi\sigma^2} = \eta_3\sqrt{\frac{\pi\Delta t^2}{4\ln 2}} = \eta_3\sqrt{\frac{\pi}{4\ln 2}}\Delta t$$

Hence:

$$\frac{\eta_2}{\eta_1} = \frac{1}{10}, \quad \frac{\eta_3}{\eta_1} = 0.94, \quad \frac{\eta_2}{\eta_3} = 10.64$$

4.2.2 b

$$\frac{\mathcal{B}(\omega_{01} + \Delta\omega)}{\mathcal{B}(\omega_{01})} = \mathrm{sinc}(20\pi) = 0$$

4.2.3 c

Let the envelope be:

$$\eta(t) = \eta_3 \exp\left\{-\frac{t^2}{2\sigma^2}\right\}$$

. At $t_{FWHM} = \Delta t/2$:

$$\frac{1}{2} = \exp\left\{-\frac{\Delta t^2}{8\sigma^2}\right\} \implies 8\sigma^2 = \frac{\Delta t^2}{\ln 2} \implies \sigma = \frac{\Delta t}{2\sqrt{2\ln 2}}$$

So:

$$\mathcal{B} = \eta_3 B_1 \sqrt{2\pi\sigma^2} \exp\left\{-\frac{(\omega - \omega_{01})^2\sigma^2}{2}\right\} \implies$$

$$\frac{\mathcal{B}(\omega_{01} + \Delta\omega)}{\mathcal{B}(\omega_{01})} = \exp\left\{-\frac{(\Delta\omega)^2\sigma^2}{2}\right\} = \exp\left\{-\frac{(\Delta\omega)^2\Delta t}{8\ln 2}\right\}$$

The important message is that for a Gaussian envelope the frequency content decays far more rapidly (exponentially), with respect to a sinusoidal.