Solutions to the Quantum Circuits and Devices by Prof. Ielmini exercise set #3

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1 Exercise 1

1.1 Question

Consider an operator $\hat{O}$ with truth table given. Calculate the angles ϕ, θ, λ to implement $\hat{O}$ with two rotations around $\vec{z}$ and one rotation around $\vec{y}$.

1.2 Answer

Any rotation of a generic qubit around any $\hat{n}$ axis can by ϕ be represented as $R_{\hat{n}} = e^{-i\frac{\phi}{2}\sigma_{\hat{n}}}$. For the type of rotation mentioned:

$$U(\phi, \theta, \lambda) = R_{\hat{z}}(\phi)R_{\hat{y}}(\theta)R_{\hat{z}}(\lambda) = \hat{O} \equiv \begin{bmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{bmatrix}$$

My proposition for the operator is:

$$\hat{O} = \left|-\right\rangle\left\langle 0\right| + \left|+\right\rangle\left\langle 1\right| \equiv \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies$$

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \implies \phi + \lambda = 2\pi m, \quad m \in \mathcal{Z}$$

Let $\lambda = \pi = \phi$, and $\theta = \frac{\pi}{2} \implies u3(\frac{\pi}{2}, \pi, \pi) = \hat{O}$

2 Exercise 2

2.1 Question

Consider an electron subject to a magnetic field $B = 2T$ directed along $\vec{z}$. Determine the position of the energy levels $|0\rangle$, $|1\rangle$, the qubit frequency ω_0 and the maximum operation temperature.

2.2 Answer

The magnetic moment of an electron is $\mu = \frac{gq}{2m}S = -\frac{e}{m}S = \gamma S$. While the energy is $H = -\mu.B_0 = -\gamma B_0.S = -\gamma B_0 S_z = \frac{\hbar}{2}\omega_0\hat{\sigma}_z$. Which implies the energy gap between the eignestates is:

$$E_0 = \hbar\omega_0 = \hbar\frac{e}{m}B_0 \approx 231\mu eV \implies \omega_0 \approx 351.65G\text{rad/s}$$

$$E_0 >> k_B T \implies T \approx 2.69K/1000 = 2.69mK$$

3 Exercise 3

3.1 Question

Consider an electron subject to a magnetic field $B = 2T$ directed along $\vec{z}$. Determine the Larmor frequency ω_L and the time t_1 for which the magnetic field must be applied to operate a gate $\hat{S}$

3.2 Answer

$$\omega_L = -\gamma B = \frac{ge}{2m}B \approx 351.65 \quad G\text{rad/s}$$

Say we want to do a bit flip, meaning a full π rotation, then $t_1 = \frac{1}{2}\frac{2\pi}{\omega_0} \approx 8.93 \quad \text{ps}$

4 Exercise 4

4.1 Question

An electron is prepared in a state $|\psi_0\rangle$ with $\theta_0 = 45°$, $\phi_0 = 0$ and immersed in a static magnetic field $B = 1T$ directed along $\vec{z}$. Draw a quoted plot of the probability of measuring the $|0\rangle$ and $|+\rangle$ state along time.

4.2 Answer

Let the state at time t be as $|\psi(t)\rangle$, what we want to compute are $P_0(|\psi(t)\rangle) = |\langle 0|\psi(t)\rangle|^2$ and $P_+(|\psi(t)\rangle) = |\langle +|\psi(t)\rangle|^2$. The time evolution operator of a static Hamiltonian is given by $U(t) = e^{-i\frac{H}{\hbar}t}$, where:

$$H = -\vec{\mu}.\vec{B}, \quad \vec{B} = B_0\hat{z} \implies H = -\gamma B_0\vec{S}.\hat{z} = -\gamma B_0\frac{\hbar}{2}\sigma_z \implies$$

$$U(t) = e^{i\frac{\gamma B_0 t}{2}\sigma_z} = e^{-i\frac{\theta}{2}\sigma_z}, \quad \theta \equiv -\gamma B_0 t$$

Therefore, the Hamiltonian only leads to a precession around the direction of the applied field which is $\hat{z}$. A mere precession around $\vec{z}$, however, doesn't change the θ angle; it only changes the ϕ angle by $\phi = \phi_0 + \omega_L t = \phi_0 - \gamma B_0 t = -\gamma B_0 t$ in the Bloch sphere:

$$|\psi(t)\rangle = U(t)\,|\psi_0\rangle = \cos\frac{\theta_0}{2}\,|0\rangle + e^{i\phi(t)}\sin\frac{\theta_0}{2}\,|1\rangle = \cos\frac{\theta_0}{2}\,|0\rangle + e^{-i\gamma B_0 t}\sin\frac{\theta_0}{2}\,|1\rangle$$

The probabilities we're going to measure will be:

$$P_0(|\psi(t)\rangle) = \cos^2\frac{\theta_0}{2} = \frac{1+\cos\theta_0}{2} = \frac{2+\sqrt{2}}{4}$$

$$P_+(|\psi(t)\rangle) = \frac{1}{2}|\cos\frac{\theta_0}{2} + e^{i\omega_L t}\sin\frac{\theta_0}{2}|^2 = \frac{1}{2}(\cos\frac{\theta_0}{2} + e^{i\omega_L t}\sin\frac{\theta_0}{2})(\cos\frac{\theta_0}{2} + e^{-i\omega_L t}\sin\frac{\theta_0}{2}) = \frac{1}{2}(\cos^2\frac{\theta_0}{2} + \sin^2\frac{\theta_0}{2} + \sin\theta_0\cos\omega_L t) = \frac{1}{2}(1+\sin\theta_0\cos\omega_L t) = \frac{1}{2} + \frac{1}{2\sqrt{2}}\cos\omega_L t$$

5 Exercise 5

5.1 Question

An electron is immersed in a static magnetic field $B = 1T$ directed along $\vec{z}$. Calculate the timing accuracy of a control system to provide a rotation angle accuracy $\Delta\theta = \frac{\pi}{1000}$.

5.2 Answer

The Larmor frequency for such a magnetic field is: $\omega_L = |\gamma| B = 175.824 \quad G\text{rad/s}$. Then $\omega_L \Delta t = \Delta\theta \implies \Delta t \approx 17.85 \quad fs$.