# Solutions to the Quantum Circuits and Devices by Prof. Ielmini exercise set #5

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## 1 Exercise 1

### 1.1 Question

Consider a quantum harmonic oscillator with $C = 200fF$, $L = 50nH$. Calculate the single electron charging energy $E_C$, the inductive energy $E_L$, and the resonance frequency $\omega_0$

### 1.2 Answer

According to the lecture notes:

$$E_C = \frac{e}{2C}[eV] = \frac{1.6 \times 10^{-19}}{2 \times 200 \times 10^{-15}}[eV] = 0.4\mu eV$$

$$E_L = \frac{1}{L}(\frac{\Phi_0}{2\pi})^2 = \frac{1}{50 \times 10^{-9}e}(\frac{h}{4\pi e})^2[eV] = \frac{1}{1.6 \times 10^{-19} \times 50 \times 10^{-9}}(\frac{6.63 \times 10{-}34}{4\pi \times 1.6 \times 10^{-19}})^2[eV]$$

$$= \frac{10^{27}}{8}(0.33 \times 10^{-15})^2[eV] \approx 13.6\mu eV$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10 \quad Grad/s$$

## 2 Exercise 2

Consider two quantum harmonic oscillators with $C_1 = 700fF$, $L_1 = 15nH$, and $C_2 = 15fF$, $L_2 = 700nH$. For each oscillator, estimate the uncertainties $\Delta N$, $\Delta\phi$ on the number of Cooper pairs $N$ and superconducting phase $\phi$ for the state $|0\rangle$ and determine whether the oscillator would be most suited for a charge qubit or a phase qubit, neglecting leakage to the upper states.

## 2.1 Answer

I don't think the way the tutor solved the question was the best physical way. Let's derive something more quantum mechanical. According to the lecture notes:

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}, \quad [\Phi, Q] = i\hbar \implies \Delta\Phi\Delta Q \geq \frac{\hbar}{2}$$

We change variables to $N = \frac{Q}{2e}$ and $\phi = \frac{\Phi}{\frac{\Phi_0}{2\pi}}$ which leads to:

$$H = 4\frac{e^2}{2C}N^2 + \frac{1}{2L}(\frac{\Phi_0}{2\pi})^2\phi^2, \quad \frac{\Phi_0}{2\pi}2e[\phi, N] = i\hbar \implies [\phi, N] = 1 \implies |\Delta\phi|.|\Delta N| \geq \frac{1}{2}$$

Being at a certain state of eigen energy means $\langle 0| \Delta H |0\rangle = 0$:

$$0 = 8E_C N\Delta N + E_L\phi\Delta\phi$$

knowing $\Delta N_{max} \leftarrow \Delta\phi_{min}$:

$$8E_C N\Delta N_{max} = E_L\phi(N, E_0)\frac{1}{2\Delta N_{max}}$$

$$\implies \Delta N_{max}^2 = \frac{E_L\phi}{16E_C N} = \frac{E_L}{16E_C}\max\frac{\phi}{N} \implies$$

$$\Delta N_{max}^3 = \frac{E_L}{16E_C}[(E_0 - 4E_C\Delta N_{max}^2)\frac{2}{E_L}]^{1/2}$$

This solution is only partially completed For phase qubits $\Delta\phi < \pi$, for charge qubits $\Delta N < 1$

# 3 Exercise 3

## 3.1 Question

Consider an Al/Al2O3/Al Josephson junction ($\epsilon_{Al_2O_3} = 9$) with critical current density $J_0 = 10\text{A}/cm^2$, $W = 2\mu m$, $L = 1\mu m$, $t = 1nm$.

### 3.1.1 a

Calculate the equivalent capacitance $C_J$ and minimum equivalent inductance $L_{J0}$.

$$C_J = \frac{\epsilon\epsilon_0 W.L}{t} = \frac{9 \times 8.85 \times 10^{-12} \times 2 \times 10^{-3}}{1} = 159.3fF$$

$$I = I_0 \sin\varphi, \quad 2eV = \hbar\frac{d\varphi}{dt} = \implies 2eV = \hbar\frac{d\varphi}{dI}\frac{dI}{dt} \implies V = \frac{\Phi_0}{2\pi I_0\cos\varphi}\frac{dI}{dt}$$

$$\implies L_{J0} = \frac{\Phi_0}{2\pi J_0 W.L} = \frac{6.63 \times 10^{-34}}{4\pi \times 1.6 \times 10^{-19} \times 10^5 \times 2 \times 10^{-12}} \approx 1.65nH$$

#### 3.1.2 b

Draw a quoted plot of the equivalent inductance $L_J(\varphi)$ as a function of the junction phase $\varphi$.

### 3.2 Answer

$$L_J(\varphi) = \frac{L_{J0}}{\cos\varphi}, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

## 4 Exercise 4

### 4.1 Question

Consider an Al/Al2O3/Al Josephson junction ($\epsilon_{Al_2O_3} = 9$) with critical current density $J_0 = 10\text{A}/cm^2$, $W = 2\mu m$, $L = 1\mu m$, $t = 1nm$. Estimate the frequencies of the $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |2\rangle$ transitions.

### 4.2 Answer

We first calculate the Josephson Hamiltonian resulting from the Junction:

$$E_J = \frac{\Phi_0 I_0}{2\pi} = L_0 I_0^2 = 1.65 \times 10^{-9}(2 \times 10^{-7})^2 = 6.6 \times 10^{-23} J = 412.5\mu eV$$

If $C << C_J$:

$$E_C = \frac{e^2}{2C_J} = \frac{2.56 \times 10^{-38}}{2 \times 159.3 \times 10^{-15}} = 0.00804 \times 10^{-23} J = 0.5025\mu eV$$

The Hamiltonian is then:

$$H = -4E_C\frac{\partial^2}{\partial\phi^2} + E_J(1-\cos\phi) \approx -4E_C\frac{\partial^2}{\partial\phi^2} + E_J(\frac{\phi^2}{2} - \frac{\phi^4}{4!}) = H_0 - E_J\frac{\phi^4}{4!} = H_0 + H'$$

What we want is:

$$\hbar\omega_{12} = \langle 2| H |2\rangle - \langle 1| H |1\rangle = \langle 2| H_0 |2\rangle - \langle 1| H_0 |1\rangle + \langle 2| H' |2\rangle - \langle 1| H' |1\rangle =$$

$$\hbar\omega_{01} + \langle 2| H' |2\rangle - \langle 1| H' |1\rangle$$

$$H' = -\frac{E_J}{24}\phi^4$$

In resemblence to an H.O. $m\omega^2 \equiv E_J, \quad 4E_C \equiv \frac{\hbar^2}{2m}$

$$(\frac{2E_C}{E_J})^{\frac{1}{4}} \equiv (\frac{\hbar}{2m\omega})^{\frac{1}{2}} \implies \phi = (\frac{2E_C}{E_J})^{\frac{1}{4}}(a + a^\dagger) = \epsilon_0(a + a^\dagger) \implies$$

$$\langle n| \phi^4 |n\rangle = \langle n| \epsilon_0^4(a+a^\dagger)^4 |n\rangle = \epsilon_0^4 \langle n| (a^2+aa^\dagger+a^\dagger a+(a^\dagger)^2)(a^2+aa^\dagger+a^\dagger a+(a^\dagger)^2) |n\rangle =$$

We know we are investigating 1 and 2 states, so anything with $a^3$ or $(a^\dagger)^3$ or with more power will automatically be evaluated to 0

$\epsilon_0^4 \langle n| (a^2 a^\dagger a + a^2 (a^\dagger)2 + (a a^\dagger)^2 + a(a^\dagger)^2 a + a^\dagger a^2 a^\dagger + (a^\dagger a)^2 + a^\dagger a (a^\dagger)^2 + (a^\dagger)^2 a^2 + (a^\dagger)^2 a a^\dagger |n\rangle$

Only the diagonal elements are kept:

$$= \epsilon_0^4 \langle n| (a^2 (a^\dagger)^2 + (a a^\dagger)^2 + a(a^\dagger)^2 a + a^\dagger a^2 a^\dagger + (a^\dagger a)^2 + (a^\dagger)^2 a^2) |n\rangle$$

$$\langle 1| H' |1\rangle = -\frac{E_J}{24} \frac{2E_C}{E_J} [(n+1)(n+2) + (n+1)^2 + n(n+1) + n(n+1) + n^2 + n(n-1)] =$$

$$-\frac{E_C}{12}[6n^2 + 6n + 3] = -\frac{E_C}{4}(2n^2 + 2n + 1) = -\frac{5}{4}E_C$$

$$\langle 2| H' |2\rangle = -\frac{E_J}{24} \frac{2E_C}{E_J} [6n^2 + 6n + 3] = -\frac{E_C}{4}[2n^2 + 2n + 1] = -\frac{13}{4}E_C$$

$$\hbar\omega_{12} = \hbar\omega_{01} - 2E_C \implies \Delta\omega = -2E_C/\hbar \approx 2\pi \times 121MHz$$

$$\hbar\omega_{01} = \hbar \frac{1}{\sqrt{L_{J0} C_J}} = 40.6\mu eV$$

$$\omega_{01} = 2\pi \times 9.7GHz$$

$$\omega_{12} = 2\pi \times 9.58GHz$$