Linear Programs

Initializing The Simplex Algorithm

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Where We Stand, What We Face?

We now have an algorithm SIMPLEX (*Restricted*), which is conjectured to solve linear programs. For the time being we've shown that it terminates either by specifying that we have an unbounded linear program, or by returning a finite value *expected* to be the maximum we are looking for.

Uptill now, we've been working under the following two assumptions :

- Our linear program is ***feasible*** ; meaning that it has at least one feasible solution
- The basic solution of the initial slack form is feasible.

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During this lecture, we are going to build a function `init_simplex` taking in a linear program $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, v)$ and either returning back the fact it is ***not*** feasible or a linear program $(N, B, \underline{\boldsymbol{A}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}, v)$ in slack form having feasible basic solution.

Feasibility

Deciding On Feasibility

Let L be the linear program, given in standard form:

$$
\begin{array}{lll}
\text{maximize} & & z = v + \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & & \\
& \forall i \in \{1, \ldots, m\}, & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \\
\text{with} & \forall j \in N, & x_j \geq 0
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Out of L we build the following auxiliary linear program L_m:

$$\begin{array}{lll} \text{maximize} & & z = -x_0 \\ \text{subject to} & & \\ & \forall i \in \{1, \ldots, m\}, & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \\ \text{with} & \forall j \in N \cup \{0\}, & x_j \geq 0 \end{array}$$

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If L is feasible then there is a tuple $(t_1, \ldots, t_n)$ of non-negative real numbers satisfying all linear constraints of L. The tuple $(0, t_1, \ldots, t_n)$ does thus satisfy L_m and 0 is then an objective value of L_m, i.e. the optimal one.

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Conversely, if L_m has objective value 0 (thus optimal) it is of the form $(0, t_1, \ldots, t_n)$. Plugging this tuple in the linear constraints of L_m it implies $(t_1, \ldots, t_n)$ is a solution of L. ∎

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The difference between L and L_m is that L_m is always feasible :

If b_{min} is the smallest negative b_i for $i \in B$, the tuple $(-b_{min}, 0, \dots, 0)$ is a feasible solution of L_m.

Thus, temporarily admitting validity of the *restricted* simplex algorithm, if we're able to find a linear program equivalent to L_m which has feasible basic solution then we can decide on the feasibilty of L.

Deciding On Feasibility

Consider the slack form of L_m

$$\begin{array}{lll} \text{maximize} & & z = -x_0 \\ \text{subject to} & & \\ & \forall i \quad x_{i+m} + \sum_{j=1}^{n} a_{ij}x_j - x_0 = b_i & \\ \text{with} & \forall j \in N \cup B \cup \{0\}, \quad x_j \geq 0 & \end{array}$$

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Let's now use `pivot` with entering variable 0 and leaving one *min*. The **same** previous feasible solution of L_m is now the basic solution of the linear program we got after pivoting.

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We got an equivalent linear program to L_m having feasible basic solution !

Tableau of Auxiliary Linear Program

The tableau `T_m` of L_m is obtained out of the one for L (`T`) by adding column $(-1, 1, \ldots, 1)^T$ to tableau before column $\boldsymbol{b}^T$ and putting old coefficients of first row to 0.

	x_1	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	x_0	
	0	$\cdots$	0	0	0	$\cdots$	0	1	0
$n+1$	a_{11}	$\cdots$	a_{1n}	1	0	$\cdots$	0	-1	b_1
$n+2$	a_{21}	$\cdots$	a_{2n}	0	1	$\cdots$	0	-1	b_2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$n+m$	a_{m1}	$\cdots$	a_{mn}	0	$\cdots$	$\cdots$	1	-1	b_m

Testing Feasibility

```python
def is_feasible(N, B, T):
"""Testing feasibility of linear program.

Args:
  N, B (list[int]): lists of non-basic and
  basic variables.
  T (ndarray[float]): numpy array for
  tableau of linear program.
Output:
  (bool) True if program is feasible
  False otherwise.
"""
p, q = T.shape[0], T.shape[1]
c = T[0, :]
T[0, :] = [0]*q
new_c = np.array([1] + [-1]*(p-1), dtype=float)
T = np.insert(T, -1, new_c, axis=1)
N.append(0)
# pivoting to be basic feasible
i_min = np.argmin(T[1: ,-1])
pivot(N, B, T, i_min, 0)
V = _simplex(N, B, T)
return V[0] == 0
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Not an equivalent LP to L there! Need to understand boundary effects.

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Under the previous assumption `is_feasible` transforms L_m into an equivalent linear program P which has feasible basic solution with objective value 0.

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Fact

Putting x_0 to 0 in Q gives back an equivalent LP to L having feasible basic solution.

The InitSimplex Function

Input: $L = (\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ a linear program in standard form

Output: An equivalent program having feasible basic solution if L is feasible.

```
function InitSimplex(A, b, c)
    if IsFeasible(A, b, c) raises an exception then
        raise exception infeasible
    else
        (N, B, A′, b′, c′, v) = IsFeasible(A, b, c)
        if 0 is basic then
            Choose e such that a′_{0e} ≠ 0
            (N, B, A′, b′, c′, v) = Pivot(N, B, A′, b′, c′, v, 0, e)
        end if
        Update (c, v) to express objective value in terms non-basic indices
        Remove column corresponding to index 0 from A′
        return (N, B, A′, b′, c, v)
    end if
end function
```

The `is_feasible` function

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def is_feasible(N, B, T):
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  T (ndarray[float]): numpy array for
  tableau of linear program.
Output:
  (bool) True if program is feasible
  False otherwise.
Boundary effect:
  Transforms T into an equivalent LP having
  basic feasible solution if T is feasible.
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p, q = T.shape[0], T.shape[1]
c = T[0, :]
T[0, :] = [0]*q
new_c = np.array([1] + [-1]*(p-1), dtype=float)
T = np.insert(T, -1, new_c, axis=1)
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# pivoting to be basic feasible
i_min = np.argmin(T[1: ,-1])
pivot(N, B, T, i_min, 0)
V = _simplex(N, B, T)
if 0 in B:
  l, e = B.index(0), 0
  while T[l, e] == 0 or e == l:
    e += 1
  pivot(N, B, T, l, e)
for i in B:
  c -= c[i]*T[B.index(i), :]
T[0, :] = c
return V[0] == 0
```

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- if it is feasible we can build up an equivalent linear program which has feasible basic solution using `init_simplex`
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We are left with showing validity of *restricted* `_simplex`!

> We call `simplex` the algorithm globally going through all previous steps.

That’s it for today !