Linear Programs

What is a linear program?

Bashar Dudin

October 2, 2018

EPITA

Dummy Example I | Plane Supply

An airline is opening a new route from city A to city C transitting through city B. It can fill it's fuel tank at both A and B airports. Price of fuel at A is 3 dollars per galon and 2 gallons at B. Aim of airline is to minimize the total cost of fuel over the trip from A to C while making sure that the conditions hereby are satisfied:

Dummy Example I | Plane Supply

An airline is opening a new route from city A to city C transitting through city B. It can fill it's fuel tank at both A and B airports. Price of fuel at A is 3 dollars per galon and 2 gallons at B. Aim of airline is to minimize the total cost of fuel over the trip from A to C while making sure that the conditions hereby are satisfied:

- Safety regulations expect you to have at least $1k$ gallons to take off from A to B.

Dummy Example I | Plane Supply

An airline is opening a new route from city A to city C transitting through city B. It can fill it's fuel tank at both A and B airports. Price of fuel at A is 3 dollars per galon and 2 gallons at B. Aim of airline is to minimize the total cost of fuel over the trip from A to C while making sure that the conditions hereby are satisfied:

- Safety regulations expect you to have at least $1k$ gallons to take off from A to B.
- Safety regulations expect you to have at least $3k$ gallons on your trip from B to C. Note that you do have tank half full at B; having more fuel makes plane heavier which consumes more fuel.

Dummy Example I | Plane Supply

An airline is opening a new route from city A to city C transitting through city B. It can fill it's fuel tank at both A and B airports. Price of fuel at A is 3 dollars per galon and 2 gallons at B. Aim of airline is to minimize the total cost of fuel over the trip from A to C while making sure that the conditions hereby are satisfied:

- Safety regulations expect you to have at least $1k$ gallons to take off from A to B.
- Safety regulations expect you to have at least $3k$ gallons on your trip from B to C. Note that you do have tank half full at B; having more fuel makes plane heavier which consumes more fuel.
- Your maximum tank capacity is $6k$ gallons.

Dummy Example I | Plane Supply

An airline is opening a new route from city A to city C transitting through city B. It can fill it's fuel tank at both A and B airports. Price of fuel at A is 3 dollars per galon and 2 gallons at B. Aim of airline is to minimize the total cost of fuel over the trip from A to C while making sure that the conditions hereby are satisfied:

- Safety regulations expect you to have at least $1k$ gallons to take off from A to B.
- Safety regulations expect you to have at least $3k$ gallons on your trip from B to C. Note that you do have tank half full at B; having more fuel makes plane heavier which consumes more fuel.
- Your maximum tank capacity is $6k$ gallons.
- To keep fuel prices at their current price in both A and B airline has to buy a total higher than $5k$ gallons over the whole trip.

Dummy Example I | Plane Supply

There are a couple of steps to go through in order to tackle the previous problem. One first starts by modeling the problem. Let x_1 and x_2 be the amount of fuel in kilo-gallons respectively taken at A and B.

- Represent problem's constraints as a geometric region in 2-dimensional space.
- What is the function you're trying to minimize?
- Given an *ad hoc* cost, single out the possible values for x_1 and x_2 having that such cost.
- Does it help to figure out a strategy to figure out the minimal cost?

Dummy Example I | Plane Supply

There are a couple of steps to go through in order to tackle the previous problem. One first starts by modeling the problem. Let x_1 and x_2 be the amount of fuel in kilo-gallons respectively taken at A and B.

- Represent problem's constraints as a geometric region in 2-dimensional space.
- What is the function you're trying to minimize?
- Given an *ad hoc* cost, single out the possible values for x_1 and x_2 having that such cost.
- Does it help to figure out a strategy to figure out the minimal cost?

We've grazed here the geometric approach to solving linear programs in low dimensions. It is not central in this course, but it is the starting point for solving much harder optimisation problems.

Dummy Example II | An Election Issue

This is taken from **Introduction to Algorithms** by ***Cormen, Lieserson, Rivest and Stein*** :

> *Suppose that you are a politician trying to win an election. Your district has three different types of areas: urban, suburban, and rural. These areas have respectively: 100.000, 200.000 and 50.000 registered voters. Although not all the registered voters actually go to the polls, you decide that to govern effectively, you would like at least half the registered voters in each of the three regions to vote for you. You are honorable and would never consider supporting policies in which you do not believe. You realize, however, that certain issues may be more effective in winning votes in certain places. Your primary issues are building roads, gun control, farm subsidies, and a gasoline tax dedicated to improve public transit. According to your campaign staff's research, you can estimate how many voters you win or lose from each population segment by spending 1.000€ on advertising on each issue.*

Dummy Example II | An Election Issue

policy	urban	suburban	rural
build road	-2	5	3
gun control	8	2	-5
farm subsidies	0	0	10
gasoline tax	10	0	-2

In this table, each entry indicates the number of ***thousands*** *of either urban, suburban, or rural voters who would be won over by spending 1.000€ on advertising on support of a particular issue. Negative entries denote votes that would be lost. Your task is to figure out the minimum amount of money that you need to spend in order to win 50.000 urban votes, 100.000 suburban votes, and 25.000 rural votes.*

Modeling

The total cost of the campaign is equal to the cost of advertisement on each given issue. Let us call x_1, x_2, x_4 and x_5 the respective costs of advertisement on building roads, gun control, farm subsidies and gasoline tax. Each such cost has to be non-negative. Thus

$$x_1, x_2, x_3, x_4 \geqslant 0. \tag{1}$$

The first column gives the following constraint on each of the costs

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geqslant 50 \tag{2}$$

The LHS corresponds to the fact your aim is to have more than 50.000 voters from urban areas.

Modeling

The total cost of the campaign is equal to the cost of advertisement on each given issue. Let us call x_1, x_2, x_4 and x_5 the respective costs of advertisement on building roads, gun control, farm subsidies and gasoline tax. Each such cost has to be non-negative. Thus

$$x_1, x_2, x_3, x_4 \geqslant 0. \tag{1}$$

The first column gives the following constraint on each of the costs

$$-2x_1 + 8x_2 + 10x_4 \geqslant 50 \tag{2}$$

The LHS corresponds to the fact your aim is to have more than 50.000 voters from urban areas.

Modeling

Each one of the table's columns gives such a linear constraint on the set of costs. One can sum the issue at hand in the following way:

$$
\begin{array}{ll}
\text{minimize} & x_1 + x_2 + x_3 + x_4 \\
\text{subject to} & \\
& -2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50 \\
& 5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100 \\
& 3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25 \\
\text{with} & x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

This is now our first example of a linear program.

General Definition

Definition

A linear program is a problem of either minimizing or maximizing a linear function subject to a number of linear constraints.

A ***linear function*** in a k variables (for $k \in \mathbb{N}^*$) is a function f having the expression

$$f(x_1, x_2, \ldots, x_k) = \qquad c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$$

for k real numbers $c_1, \ldots, c_k$. A ***linear constraint*** is either an equality or a (large) inequality, only containing affine expressions.

General Definition

Definition

A linear program is a problem of either minimizing or maximizing a linear function subject to a number of linear constraints.

A ***linear function*** in a k variables (for $k \in \mathbb{N}^*$) is a function f having the expression

$$f(x_1, x_2, \ldots, x_k) = \qquad c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$$

for k real numbers c_1, …, c_k. A ***linear constraint*** is either an equality or a (large) inequality, only containing affine expressions.

The *linear function* we're trying to minimize or maximize in a linear program is called the ***objective function***♠. When the objective function is to be minimized the linear program is said to be a minimization program. It is a maximization program otherwise.

General Definition

Definition

A linear program is a problem of either minimizing or maximizing a linear function subject to a number of linear constraints.

An ***affine function*** in a k variables (for $k \in \mathbb{N}^*$) is a function f having the expression

$$f(x_1, x_2, \ldots, x_k) = c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$$

for k real numbers c_1, …, c_k. A ***linear constraint*** is either an equality or a (large) inequality, only containing affine expressions.

The *affine function* we're trying to minimize or maximize in a linear program is called the ***objective function***♠. When the objective function is to be minimized the linear program is said to be a minimization program. It is a maximization program otherwise.

General Definition

Given a linear program, our goal is to find solutions to the linear constraints such that the objective function has optimal value: a maximal value if we have a maximization linear program or a minimal one for minimization programs. A value taken by an objective function is called an ***objective value***.

General Definition

Given a linear program, our goal is to find solutions to the linear constraints such that the objective function has optimal value: a maximal value if we have a maximization linear program or a minimal one for minimization programs. A value taken by an objective function is called an ***objective value***.

A solution to the linear constraints, even if it doesn't minimize nor maximize the objective function, is called a ***feasible solution***. Most algorithms solving linear programs iteratively build feasible solutions getting closer to an ***optimal solution***.

General Definition

Given a linear program, our goal is to find solutions to the linear constraints such that the objective function has optimal value: a maximal value if we have a maximization linear program or a minimal one for minimization programs. A value taken by an objective function is called an ***objective value***.

A solution to the linear constraints, even if it doesn't minimize nor maximize the objective function, is called a ***feasible solution***. Most algorithms solving linear programs iteratively build feasible solutions getting closer to an ***optimal solution***.

Back to Elections

Give a feasible solution with as small objective value as possible.

General Definition

Given a linear program, our goal is to find solutions to the linear constraints such that the objective function has optimal value: a maximal value if we have a maximization linear program or a minimal one for minimization programs. A value taken by an objective function is called an ***objective value***.

A solution to the linear constraints, even if it doesn't minimize nor maximize the objective function, is called a ***feasible solution***. Most algorithms solving linear programs iteratively build feasible solutions getting closer to an ***optimal solution***.

Back to Elections

Give a feasible solution with as small objective value as possible.

Answer

You might have hard time guessing the right answer : $27.927927927927\ldots$

Standard Form

Here are two examples apparently different from the dummy linear program we started with:

Standard Form

Here are two examples apparently different from the dummy linear program we started with:

$$
\begin{array}{lrl}
\text{minimize} & & x_1 + x_2 + x_3 + x_4 \\
\text{subject to} & & \\
& & 2x_1 - 8x_2 - \ \ 0x_3 - 10x_4 \leq -50 \\
& & 5x_1 + 2x_2 + \ \ 0x_3 + \ \ 0x_4 \geq 100 \\
& & 3x_1 - 5x_2 + 10x_3 - \ \ 2x_4 \geq 25 \\
\text{with} & & x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Standard Form

Here are two examples apparently different from the dummy linear program we started with:

$$
\begin{array}{ll}
\text{maximize} & -x_1 - x_2 - x_3 - x_4 \\
\text{subject to} & \\
& -2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50 \\
& 5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100 \\
& 3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25 \\
\text{with} & x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Standard Form

Here are two examples apparently different from the dummy linear program we started with:

$$
\begin{array}{lr}
\text{maximize} & -x_1 - x_2 - x_3 - x_4 \\
\text{subject to} & \\
& -2x_1 + 8x_2 + \ \ 0x_3 + 10x_4 \geq 50 \\
& 5x_1 + 2x_2 + \ \ 0x_3 + \ \ 0x_4 \geq 100 \\
& 3x_1 - 5x_2 + 10x_3 - \ \ 2x_4 \geq 25 \\
\text{with} & x_1,\, x_2,\, x_3,\, x_4 \geq 0
\end{array}
$$

The first linear program is obtained out of the dummy case by multiplying the first linear inequality by -1. The second one by choosing to maximize the opposite of the objective function rather than minimizing it.

Standard Form

If our aim is to build an algorithm enabling us to solve linear programs, such an algorithm will take as input the list of coefficients of our linear program. The previous examples, giving different coefficients for the apparently same linear program, suggests that we should fix a standard input form.

Definition

A linear program is in ***standard form*** if

- it is a maximization problem
- all constraints are *less-or-equal-to* inequalities
- variables are subject to a nonnegativity constraint.

Out of the three forms our dummy program can take, none is in standard form.

Getting a Program into Standard Form

A first *sanity check* is to check that ***every*** linear program can be written in standard form. It means that, in a similar fashion as to what happens for the Gauss elimination algorithm, any linear program should be *equivalent* to a linear program which is in standard form.

The notion of equivalence of two linear programs is more involved than in the simple case of linear systems. We shall take our time explaining it.

Getting a Program into Standard Form

Definition

- Two maximization linear programs L and L' are ***equivalent*** if for each feasible solution of L having objective value v there is a feasible solution of L' having same objective value, and conversely
- A minimization linear program L is ***equivalent*** to maximization linear program L' if for each feasible solution x of L having objective value v there is a feasible solution x' of L' having objective value $-v$, and conversely.

Getting a Program into Standard Form

Definition

- Two maximization linear programs L and L' are ***equivalent*** if for each feasible solution of L having objective value v there is a feasible solution of L' having same objective value, and conversely
- A minimization linear program L is ***equivalent*** to maximization linear program L' if for each feasible solution x of L having objective value v there is a feasible solution x' of L' having objective value $-v$, and conversely.

Proposition

If L and L' are equivalent maximization linear programs then L and L' have same optimal objective values.

Getting a Program into Standard Form

Definition

- Two maximization linear programs L and L' are ***equivalent*** if for each feasible solution of L having objective value v there is a feasible solution of L' having same objective value, and conversely
- A minimization linear program L is ***equivalent*** to maximization linear program L' if for each feasible solution x of L having objective value v there is a feasible solution x' of L' having objective value $-v$, and conversely.

Proof

Try justifying with your own words the fact two equivalent maximization linear programs have same optimal solutions.

Getting a Program into Standard Form

Definition

- Two maximization linear programs L and L' are ***equivalent*** if for each feasible solution of L having objective value v there is a feasible solution of L' having same objective value, and conversely
- A minimization linear program L is ***equivalent*** to maximization linear program L' if for each feasible solution x of L having objective value v there is a feasible solution x' of L' having objective value $-v$, and conversely.

Variants

How would the statement adapt to the case of minimization linear programs? A maximization and a minimization one?

Examples of Equivalence Relations

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + x_2 + 3 \\
\text{subject to} & \\
& x_1 + x_2 + x_3 \geq 4 \\
& x_1 - x_2 - x_3 \geq 2 \\
& -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1, x_2, x_3 \geq 0
\end{array}
$$

Examples of Equivalence Relations

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 + 3 \\ \text{subject to} & \\ & x_1 + x_2 + x_3 \geq 4 \\ & x_1 - x_2 - x_3 \geq 2 \\ & -x_1 + x_2 + 2x_3 \geq 1 \\ \text{with} & x_1, x_2, x_3 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 - 3 \\ \text{subject to} & \\ & x_1 + x_2 + x_3 \geq 4 \\ & x_1 - x_2 - x_3 \geq 2 \\ & -x_1 + x_2 + 2x_3 \geq 1 \\ \text{with} & x_1, x_2, x_3 \geq 0 \end{array}$$

Examples of Equivalence Relations

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + x_2 + 3 \\
\text{subject to} & \\
 & x_1 + x_2 + x_3 \geq 4 \\
 & x_1 - x_2 - x_3 \geq 2 \\
 & -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1,\, x_2,\, x_3 \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\text{minimize} & -2x_1 - x_2 - 3 \\
\text{subject to} & \\
 & -s + x_1 + x_2 + x_3 = 4 \\
 & x_1 - x_2 - x_3 \geq 2 \\
 & -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1,\, x_2,\, x_3,\, s \geq 0
\end{array}
$$

Examples of Equivalence Relations

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + x_2 + 3 \\
\text{subject to} & \\
& x_1 + x_2 + x_3 \geq 4 \\
& x_1 - x_2 - x_3 \geq 2 \\
& -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1, x_2, x_3 \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + x_2 + 3 \\
\text{subject to} & \\
& -s + x_1 + x_2 + x_3 = 4 \\
& x_1 - x_2 - x_3 \geq 2 \\
& -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1, x_2, x_3, s \geq 0
\end{array}
$$

Examples of Equivalence Relations

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + x_2 + 3 \\
\text{subject to} & \\
& x_1 + x_2 + x_3 \geq 4 \\
& x_1 - x_2 - x_3 \geq 2 \\
& -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1, x_2, x_3 \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & t \\
\text{subject to} & \\
& -t + 2x_1 + x_2 + 3 \geq 0 \\
& x_1 + x_2 + x_3 \geq 4 \\
& x_1 - x_2 - x_3 \geq 2 \\
& -x_1 + x_2 + 2x_3 \geq 1 \\
\text{with} & x_1, x_2, x_3, t \geq 0
\end{array}
$$

Getting a Program into Standard Form

A linear program might not be in standard form for any of the four reasons hereafter :

1. The objective function might be a minimization program rather than a maximization.
2. There might be variables without nonnegativity constraints.
3. There might be *equality constraints* rather than a less-or-equal-to signs.
4. There might be *inequality constraints* that are greater-or-equal-to signs rather than less-or-equal-to signs.

Getting a Program into Standard Form

A linear program might not be in standard form for any of the four reasons hereafter :

1. The objective function might be a minimization program rather than a maximization.
2. There might be variables without nonnegativity constraints.
3. There might be *equality constraints* rather than a less-or-equal-to signs.
4. There might be *inequality constraints* that are greater-or-equal-to signs rather than less-or-equal-to signs.

How-do?

Given linear programs having one of these degeneracies at a time, how would you get equivalent programs in standard form? What if a linear program has many of these?

Getting a Program into Standard Form

A linear program might not be in standard form for any of the four reasons hereafter :

1. The objective function might be a minimization program rather than a maximization.
2. There might be variables without nonnegativity constraints.
3. There might be *equality constraints* rather than a less-or-equal-to signs.
4. There might be *inequality constraints* that are greater-or-equal-to signs rather than less-or-equal-to signs.

Dummy case

Put our dummy case into standard form[a]. What operations did you allow yourself? Why do they get you an equivalent program to the one you start with?

[a] We are abusing terminology : find an equivalent linear program which is in standard form

Slack Form

Though standard form comes naturally, to work out the ***simplex algorithm*** we choose to use another equivalent form : the slack form of a linear program.

Definition

Given a linear program in standard form L the slack form of L is the equivalent linear program obtained out of L by inserting an extra (non-negative) ***slack*** variable on the LHS of each constraint, which is not a non-negativity one. Each inequality is then replaced by an equality.

Slack Form

Though standard form comes naturally, to work out the ***simplex algorithm*** we choose to use another equivalent form : the slack form of a linear program.

Definition

Given a linear program in standard form L the slack form of L is the equivalent linear program obtained out of L by inserting an extra (non-negative) ***slack*** variable on the LHS of each constraint, which is not a non-negativity one. Each inequality is then replaced by an equality.

For instance the two inequalities

$$x_1 + x_2 \leq 20$$

$$2x_1 - x_2 \leq 2$$

Slack Form

Though standard form comes naturally, to work out the ***simplex algorithm*** we choose to use another equivalent form : the slack form of a linear program.

Definition

Given a linear program in standard form L the slack form of L is the equivalent linear program obtained out of L by inserting an extra (non-negative) ***slack*** variable on the LHS of each constraint, which is not a non-negativity one. Each inequality is then replaced by an equality.

give the equalities

$$\begin{aligned} -x_1 - x_2 + 20 &= x_3 \\ -2x_1 + x_2 + \ \ 2 &= x_4 \end{aligned}$$

Slack Form

Though standard form comes naturally, to work out the ***simplex algorithm*** we choose to use another equivalent form : the slack form of a linear program.

Definition

Given a linear program in standard form L the slack form of L is the equivalent linear program obtained out of L by inserting an extra (non-negative) ***slack*** variable on the LHS of each constraint, which is not a non-negativity one. Each inequality is then replaced by an equality.

give the equalities

$$-x_1 - x_2 + 20 = x_3$$

$$-2x_1 + x_2 + \quad 2 = x_4$$

Are they equivalent?

Slack Form

Though standard form comes naturally, to work out the ***simplex algorithm*** we choose to use another equivalent form : the slack form of a linear program.

Definition

Given a linear program in standard form L the slack form of L is the equivalent linear program obtained out of L by inserting an extra (non-negative) ***slack*** variable on the LHS of each constraint, which is not a non-negativity one. Each inequality is then replaced by an equality.

Proposition

Standard and slack forms of a linear program are equivalent.

Slack Form

Slacking

Put our dummy linear programs into slack form. Can you imagine a reason as to why adding such extra variables is natural?

That’s it for today!