

Linear Programs

The Simplex Algorithm II

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Where We Stand, What We Face

Working out a vague procedure we've managed to find supposedly optimal solutions for simple examples of linear programs (solutions were optimal, but we have no mean to know that for a fact). This heuristic is far from being satisfactory though : given a linear program L , here is what we're still missing :

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Pivoting

Pivoting | Fixing Notation

In order to answer both previous questions we'll need to properly write down involved algorithms. Consider the linear program L

$$\begin{array}{ll}\text{maximize} & z = v + \sum_{j=1}^n c_j x_j \\ \text{subject to} & \\ & \forall i \in \{1, \dots, m\}, \quad \sum_{j=1}^n a_{ij} x_j \leq b_i \\ \text{with} & \forall j \in N, \quad x_j \geq 0\end{array}$$

We write

- A for the (m, n) matrix of coefficients $(a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$
- b for the m -tuple (b_1, \dots, b_m)
- c for the n -tuple (c_1, \dots, c_n) .

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The linear program L in its standard form is determined by the data $(\mathbf{A}, \mathbf{b}, \mathbf{c}, v)$. Initial program has $v = 0$ in general.

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$$\begin{array}{ll}\text{maximize} & z = v + \sum_{j=1}^n c_j x_j \\ \text{subject to} & \\ & \forall i \in \{1, \dots, m\}, \quad x_{i+m} = b_i - \sum_{j=1}^n a_{ij} x_j \\ \text{with} & \forall j \in N \cup B, \quad x_j \geq 0.\end{array}$$

To encode the slack form we include the data N, B of non-basic and basic sets. A slack form is thus given by the data $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$. The set N is initialized at $\{1, \dots, n\}$ and B to $\{n+1, \dots, n+m\}$.

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$$\begin{array}{ll}\text{maximize} & z - \sum_{j=1}^n c_j x_j = v \\ \text{subject to} & \\ & \forall i \in \{1, \dots, m\}, \quad \sum_{j=1}^n a_{ij} x_j + x_{i+m} = b_i \\ \text{with} & \forall j \in N \cup B, \quad x_j \geq 0\end{array}$$

To encode the slack form we include the data N, B of non-basic and basic sets. A slack form is thus given by the data $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$. The set N is initialized at $\{1, \dots, n\}$ and B to $\{n+1, \dots, n+m\}$. To get closer to a standard way of writing a linear system, we slightly modify presentation of slack form.

Pivoting | Tableau of a Linear Program

Linear constraints of L in slack form have an $(m, n + m)$ matrix \underline{A} given by concatenating A and I_m along rows of A . The *tableau* T of linear program L is obtained by

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- concatenating \underline{A} and b^T along rows of A
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	x_1	\cdots	x_n	x_n	x_{n+1}	\cdots	x_{n+m}	
	$-c_1$	\cdots	$-c_n$	0	0	\cdots	0	v
$n+1$	a_{11}	\cdots	a_{1n}	1	0	\cdots	0	b_1
$n+2$	a_{21}	\cdots	a_{2n}	0	1	\cdots	0	b_2
\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
$n+m$	a_{m1}	\cdots	a_{mn}	0	\cdots	\cdots	1	b_m

The tableau T is of shape $(m + 1, n + m + 1)$.

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We are given input $(N, B, A, \mathbf{b}, \mathbf{c}, \nu)$ and two indexes $e \in N, l \in B$ respectively corresponding to entering and leaving variables to and from the set of *basic* variables B .

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- Express x_e in terms of other variables in equation l

$$T[l, :] = (1/T[l, e]) * T[l, :]$$

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- Express x_e in terms of other variables in equation l
- Replace x_e by previously obtained expression in linear constraints

$$T[l, :] = (1/T[l, e]) * T[l, :]$$

```
if i != l:  
    T[i, :] -= T[i, e] * T[l, :]
```

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- Replace x_e by corresponding expression in the value function

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- Express x_e in terms of other variables in equation l
- Replace x_e by previously obtained expression in linear constraints
- Replace x_e by corresponding expression in the value function
- Update basic and none basic sets of variables.

$$T[l, :] = (1/T[l, e]) * T[l, :]$$

```
if i != l:  
    T[i, :] -= T[i, e] * T[l, :]
```

```
N.insert(N.index(e), l).remove(e)  
B.insert(B.index(l), e).remove(l)
```

Pivoting | Full Function

```
1  def pivot(N, B, T, e, l):
2      """Pivoting in linear programs.
3
4      Pivots entering and leaving variables in linear
5      program given as tableau. Done in place.
6      """
7      T[l, :] = (1/T[l, e])*T[l, :]
8      for i in range(m+1):
9          if i != l: # ugly
10             T[i, :] -= T[i, e]*T[l, :]
11      N.insert(N.index(e), l).remove(e)
12      B.remove(B.index(l), e).remove(l)
```

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- There better be no entries e, l such that $T[l, e] == 0$. We shall ensure this is never the case when Pivot is used.
- The objective value and basic solution of obtained linear program can be read on last column to the right.

Facing Unboundedness

Testing Boundedness

Let L be the following linear program in standard form

$$\begin{array}{ll}\text{maximize} & z = v + \sum_{j=1}^n c_j x_j \\ \text{subject to} & \\ & \forall i \in B, \quad \sum_{j=1}^n a_{ij} x_j \leq b_i \\ \text{with} & \forall j \in N, \quad x_j \geq 0\end{array}$$

Fact

If there was an index j such that $c_j > 0$ and all coefficients of x_j in the linear constraints were non-positive then L is unbounded.

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Fact

Equivalently, if there was a column in the program's tableau having negative first entry and only non-positive ones afterwards then L is unbounded.

Testing Boundedness

Using the previous fact we can test, each time we call `pivot`, that the input linear program *doesn't* satisfy the previous property :

There is an index j such that $c_j > 0$ and all coefficients of x_j in the linear constraints are non-positive.

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This is a necessary healthy check, but there is yet no guarantee that this condition is fulfilled when the linear program we work with is unbounded. We will not have the needed machinery to properly answer this question until we introduce some duality.

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For the time being we'll have to accept that the naive check at hand will be enough.

The Simplex Algorithm : Second Try

The Simplex Algorithm *Restricted*

```
1  # Under construction function!
2  def _simplex(N, B, T):
3      """Restricted simplex algorithm
4
5      Runs simplex algorithm on basic feasible
6      linear program in slack form.
7
8      Args:
9          N, B (list[int]): lists of non-basic
10         and basic variables.
11         T (ndarray[float]): numpy array for
12         tableau of linear program.
13     Output:
14         (ndarray[float]) vector tail of which
15         is maximal objective value, rest is
16         optimal point.
17     """
```

```
18     m, margins = len(B), dict()
19     aug_var = [i for i in N if T[0, i] < 0]
20     while aug_var:
21         e = random.choice(augmenting_var)
22         for i in range(m):
23             if T[i, e] > 0:
24                 margins[B[i]] = T[i, -1]/T[i, e]
25         if not margins:
26             raise Exception("Unbounded LP")
27         min_margin = min(margins.values())
28         minima = [i for i in margins\
29                     if margins[i] == min_margin]
30         l = random.choice(minima)
31         pivot(N, B, T, e, l)
32         aug_var = [i for i in N if T[0, i] < 0]
33     return T[:, -1]
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The Simplex Algorithm : First Validity Checks

There are three things we need to check in order to temporarily accept the previous version of the simplex algorithm

1. after each call to `pivot` the linear program we get has feasible basic solution
2. the objective value does not decrease while looping

and a last point to *understand*:

3. what does it mean for `_simplex` not to terminate? how can we deal with it?

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Feasibility of the basic solution after each call for `pivot`

Input is an LP $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$ having feasible basic solution, i.e. $T[1:, -1]$ is non-negative. We show effect of `pivot` leaves an LP with feasible basic solution as well. Since `pivot` is called, $T[1, e] > 0$.

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Updates of \mathbf{b} are give by the relations

$$T[1, -1] = (1/T[1, e]) * T[1, -1]$$

and for each $i \in \{1, \dots, m+1\}$

$$T[i, -1] -= (T[i, e]/T[1, e]) * T[i, -1]$$

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The only possibility for such updates to give a negative result is when $T[i, e] > 0$. In that case `_simplex` chooses `l` in such a way that

$$T[1, -1]/T[1, e] \leq T[i, -1]/T[i, e]$$

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$$T[1, -1]/T[1, e] \leq T[i, -1]/T[i, e]$$

Therefore, if $-T[i, e] < 0$, mutliplying previous inequality by it and adding $T[i, -1]$ we get that update after `pivot` is non-negative.

Cycling

Each time we step into the `while` loop of the `_simplex` algorithm we either *increase or keep constant* the objective value or discover the LP is *unbounded*. A priori, `_simplex` might run on indefinitely among equivalent slack forms without ever increasing the objective value. We are going to check such behaviour can be detected.

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Proposition C

Let L be an LP $(N, B, A, \mathbf{b}, \mathbf{c}, v)$ where A is an (m, n) matrix. If `_simplex` runs more than $\binom{n+m}{m}$ iterations it does cycle, i.e. it goes indefinitely among the same finite set of slack forms with same given objective value.

Remark: This means that whenever `_simplex` runs more than $\binom{n+m}{m}$ times then one can return any current objective value and basic solution.

The proof of the above proposition is based on two facts :

- if we ever get back to a previously obtained slack form, then we are going to have the same set of options we had for equivalent slack forms once again.
- There is only a finite set of possible slack forms for the same given linear program.

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- if we ever get back to a previously obtained slack form, then we are going to have the same set of options we had for equivalent slack forms once again.
- There is only a finite set of possible slack forms for the same given linear program.

The second point is the only point we need to make clear.

Lemma *B*

Let L be an LP given by $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. A slack form of L appearing in `_simplex` is determined by the choice of a set B of basic variables.

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Proof : Such slack forms $(N, B, A, \mathbf{b}, \mathbf{c}, v)$ and $(N, B, A', \mathbf{b}', \mathbf{c}', v')$ are equivalent through the identity map.

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Proof : Such slack forms $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$ and $(N, B, \mathbf{A}', \mathbf{b}', \mathbf{c}', v')$ are equivalent through the identity map. Subtracting the linear constraints, we get the relations :

$$0 = (v - v') + \sum_{j \in N} (c_j - c'_j) x_j$$

and for each $i \in B$

$$0 = (b_i - b'_i) - \sum_{j \in N} (a_{ij} - a'_{ij}) x_j.$$

We're abusing notation here ; index of coefficients is the one corresponding to line of \mathbf{A} supporting basic variable i .

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$$0 = (\nu - \nu') + \sum_{j \in N} (c_j - c'_j) x_j$$

and for each $i \in B$

$$0 = (b_i - b'_i) - \sum_{j \in N} (a_{ij} - a'_{ij}) x_j.$$

We're abusing notation here ; index of coefficients is the one corresponding to line of \mathbf{A} supporting basic variable i .

These relations being true for any vector (x_1, \dots, x_n) it is clear that for each $i \in B$ and $j \in N$ we have $\nu = \nu'$, $b_i = b'_i$, $c_j = c'_j$, $a_{ij} = a'_{ij}$.



Proposition C

Let L be an LP $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$ where \mathbf{A} is an (m, n) matrix. If `_simplex` runs more than $\binom{n+m}{n}$ iterations it does cycle, i.e. it goes indefinitely among the same finite set of slack forms with same given objective value.

Proof: By lemma B, there are at most as many different slack forms as the number of possible choices of basic sets of variables. There are $\binom{m+n}{m}$ such choices. Thus if `SIMPLEX` runs more iterations than $\binom{m+n}{m}$ then we already obtained twice the same slack form. ■

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Remark: This solution is *not* an intelligent one. There are many different solutions in practice. We shall implement **Bland's rule**: each time we choose an index at line 21 and 30 of previous `_simplex`, we choose the smallest possible indexes.

The Simplex Algorithm *Restricted* | No Cycling Version

```
1  def _simplex(N, B, T):
2      """Restricted simplex algorithm
3
4      Runs simplex algorithm on basic feasible
5      linear program in slack form.
6
7      Args:
8          N, B (list[int]): lists of non-basic
9          and basic variables.
10         T (ndarray[float]): numpy array for
11         tableau of linear program.
12     Output:
13         (ndarray[float]) vector tail of which
14         is maximal objective value, rest is
15         optimal point.
16     """
```

```
17     m = len(B)
18     l, margin = None, float('inf')
19     aug_var = [i for i in N if T[0, i] < 0]
20     while aug_var:
21         e = min(augmenting_var)
22         for i in range(m):
23             if T[i, e] > 0:
24                 if T[i, -1]/T[i, e] < margin:
25                     margin = T[i, -1]/T[i, e]
26                     l = i
27         if not l:
28             raise Exception("Unbounded LP")
29         pivot(N, B, T, e, l)
30         aug_var = [i for i in N if T[0, i] < 0]
31     return T[:, -1]
```

That's it for today !