Linear Programs

Validity of the Simplex Algorithm

Bashar Dudin

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Where We Stand? What Is Left?

Given a linear program we are now able to check whether it is feasible or not. Under such an assumption we can find an equivalent linear program who has basic feasible solution. When that is the case we can execute the *restricted* simplex algorithm. We find ourselves in front of the upcoming two outputs

- the program is unbounded
- the algorithm terminates with a finite output.

Up so far we haven't proved that this procedure gives back an optimal solution. We only know it is feasible. Showing optimality is the subject of this set of slides.

The Dual Linear Program

Let L be the linear program (called ***primal***), given in standard form by :

$$\begin{array}{lll} \text{maximize} & & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & & \\ & \forall i \in \{1, \ldots, m\}, & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \\ \text{with} & \forall j \in \{1, \ldots, n\}, & x_j \geq 0 \end{array}$$

Out of L we build the following linear program $L^\vee$ called the ***dual*** of L :

$$\begin{array}{lll} \text{minimize} & & z = \sum_{i=1}^{m} b_i y_i \\ \text{subject to} & & \\ & \forall j \in \{1, \ldots, n\}, & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \\ \text{with} & \forall i \in \{1, \ldots, m\}, & y_i \geq 0 \end{array}$$

Duality

The main result justifying, for us, the existence and denomination of the dual linear program is the following.

Duality

A linear program L and its dual $L^\vee$ have same optimal objective values if bounded.

We have, in fact, a much more precise statement :

Proposition

If L is in a slack form having optimal basic solution then setting $y_i = -c_{n+i}$ gives an optimal solution of $L^\vee$.

Duality

We'll only be checking duality and giving a glimpse of the perception to have.

Assume we are given a feasible solution $(\bar{x}_1, \ldots, \bar{x}_n)$ of L and another one $(\bar{y}_1, \ldots, \bar{y}_m)$ of $L^\vee$. Using constraints of both linear programs we have that

$$\sum_{i=1}^{m} b_i \bar{y}_i \geq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \bar{x}_j \bar{y}_i \tag{1}$$

$$\geq \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \bar{y}_i \bar{x}_j \geq \sum_{j=1}^{n} c_j \bar{x}_j \tag{2}$$

The optimal objective value of L, if any, is bounded from above by the objective value of any feasible solution of $L^\vee$. Conversely, the optimal objective value of $L^\vee$ is bounded from below by the objective value of any feasible solution of L.

Duality

The previous reasoning stands for the following result

Lemma

If L and $L^\vee$ have respective feasible solutions $(\bar{x}_1, \ldots, \bar{x}_n)$ and $(\bar{y}_1, \ldots, \bar{y}_n)$ such that

$$\sum_{i=1}^{m} b_i \bar{y}_i = \sum_{j=1}^{n} c_j \bar{x}_j$$

then both are ***optimal*** feasible solutions of their respective linear programs.

Proof : We've just seen that the optimal objective value of L is smaller or equal to the one of $L^\vee$. The previous hypothesis just says it also has to be bigger or equal to the one of $L^\vee$, thus equality. ∎

Duality

To prove the duality theorem as well as the subsequent proposition the strategy is the following :

- run SIMPLEX on L
- if result is a solution having finite[1] objective value, check that instruction of proposition gives a feasible solution of $L^\vee$ having same objective value
- lemma implies both objective values are optimal.

This reasoning shows that if we have an optimal feasible solution for L then we have one of same objective value for $L^\vee$. To check the converse try making sense of

Bi-duality

The dual of the dual linear program is the primal one.

[1] If it is unfeasible or unbounded above lemma ensures this is also the case of $L^\vee$.

Fundamental Theorem of Linear Programming

Incidently, the previous strategy shows that SIMPLEX finds solutions of both L and $L^\vee$ that have same objective values. Thus SIMPLEX does compute an optimal solution of L.

Theorem

SIMPLEX does either check whether a linear program L is feasible, unbounded or gives back an optimal feasible solution of L having finite objective value.

This gives the following core result on linear programs :

Corollary ***(Fundamental Theorem of Linear Programming)***

Any linear program L is either infeasible, unbounded or has finite optimal objective value

Merry Christmas!