Continuous-Time Fourier Transform (CTFT)

- The Fourier series is used to represent a periodic function by a discrete sum of complex exponentials or sum of sinusoids, while the Fourier transform is then used to represent an aperiodic signal by a continuous superposition or integral of complex exponentials. Aperiodic signals don't have a Fourier series thus we use Fourier transforms to analyse them.

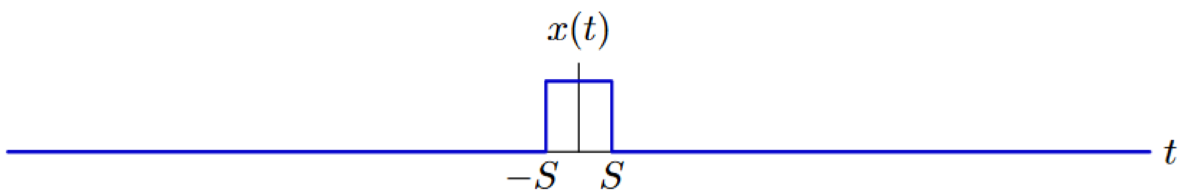
- The Fourier transform is a major cornerstone in the analysis and representation of signals and linear, time-invariant systems.

- Much of its usefulness stems directly from the properties of the Fourier transform as listed below.

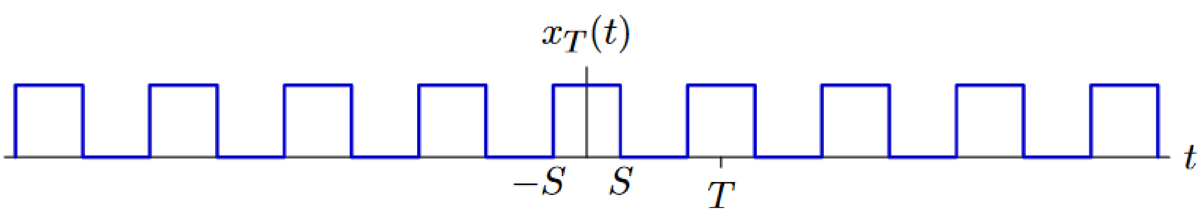
- The basic approach of Fourier transforms is to construct a periodic signal from the aperiodic one by periodically replicating it, that is, by adding it to itself shifted by integer multiples of an assumed period T.

- An aperiodic signal can be thought of as periodic signal with infinite period.

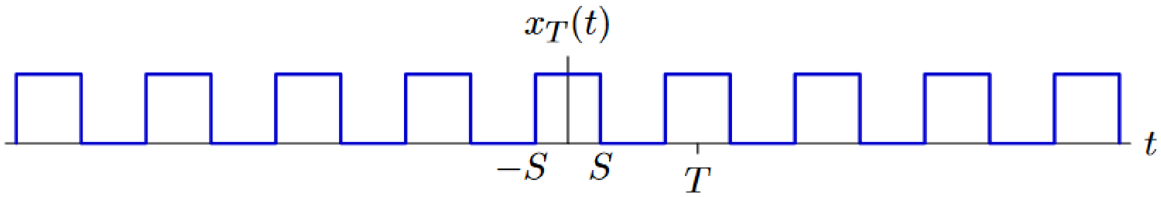
- Let $x(t)$ represent an aperiodic signal:



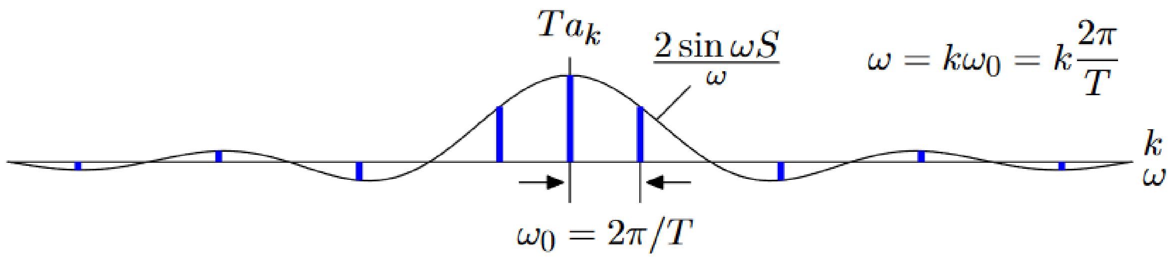
- The representation of the aperiodic signal as a periodic signal with period T, i.e. the periodic extension of $x(t)$ denoted as: $x_T(t) = \sum_{k=-\infty}^{\infty} x(t+kT)$ can be presented as:



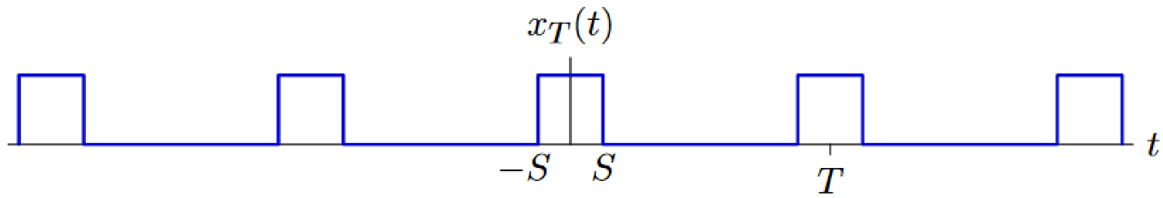
Representing $x_T(t)$ by its Fourier series, leads to:



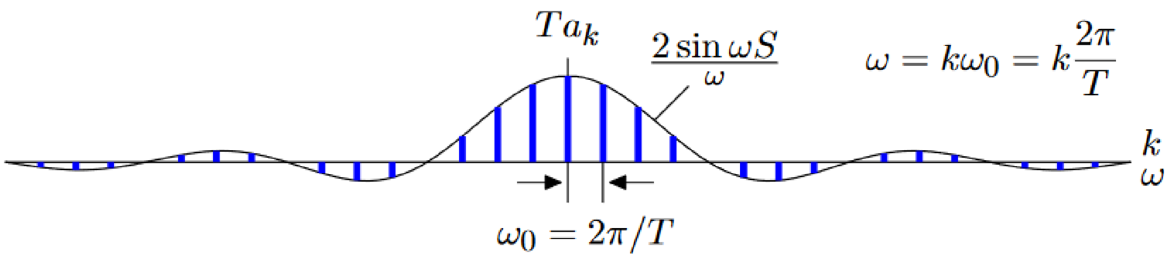
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$



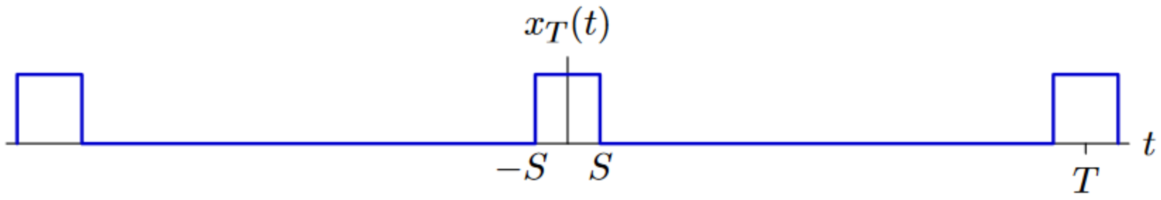
- Doubling the period doubles the number of harmonics in the given frequency interval.



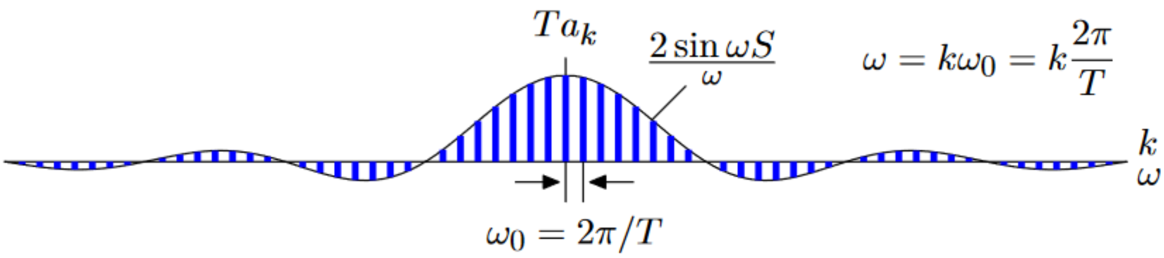
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$



- As $T \to \infty$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi k S}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$



$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

- As T increases, or equivalently, as the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases, the envelope is sampled with a closer and closer spacing.

- As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.

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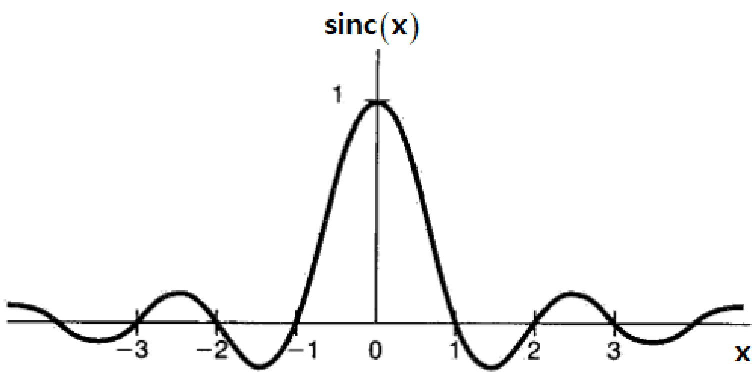
- The signal $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ is called a sinc function.



- The Fourier series of a periodic signal approaches the Fourier transform of the aperiodic signal represented by a single period as the period goes to infinity.

- This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals. Based on this idea, we can derive the Fourier transform for aperiodic signals.

- Replacing $E(\omega)$ by $X(j\omega)$ in above diagram yields the Fourier transform relations:

$$E(\omega) = X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \text{ : Fourier transform} \qquad (166)$$

- $X(j\omega)$ is known as the analysis equation.

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \text{ : inverse Fourier transform} \qquad (167)$$

- $x(t)$ is known as the synthesis equation.

- The transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the spectrum of $x(t)$.

<u>Example</u>

Find the Fourier transform of the causal complex exponential signal: $x(t) = e^{-at} u(t),\ a > 0$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = X(j\omega) = \int_{0}^{\infty} e^{(-at - j\omega t)} dt$$

$$X(j\omega) = \frac{1}{a + j\omega},\ a > 0$$

Example

Find the Fourier transform of a rectangular pulse: $x(t)=\begin{cases}1, & |t|<T_1 \\ 0, & |t|>T_1\end{cases}$.

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}dt = 2\frac{\sin(\omega T_1)}{\omega}$$

Express $X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}dt = 2\frac{\sin(\omega T_1)}{\omega}$ as a sinc function.

$$X(j\omega)=2\frac{\sin(\omega T_1)}{\omega}=2T_1\frac{\sin\left(\frac{\pi\omega T_1}{\pi}\right)}{\frac{\pi\omega T_1}{\pi}}=2T_1\,\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

Example

Find the inverse Fourier transform of this signal: $X(j\omega)=\begin{cases}1, & |\omega|<W \\ 0, & |\omega|>W\end{cases}$.

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega=\frac{\sin(Wt)}{\pi t}$$

Express $x(t)=\frac{\sin(Wt)}{\pi t}$ as a sinc function.

$$x(t)=\frac{\sin(Wt)}{\pi t}=\frac{W}{\pi}\frac{\sin\left(\frac{\pi Wt}{\pi}\right)}{\frac{\pi Wt}{\pi}}=\frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

Example

Find the Fourier transform of the following signal: $x(t) = e^{-a|t|},\ a > 0$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

The Fourier Transform of Periodic Signals

- The Fourier series representation of signal $x(t)$ which is periodic is:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad (1.68)$$

where ω_0 is the fundamental frequency and the fundamental period is $T_0 = \frac{2\pi}{\omega_0}$ and a_k are known as the complex Fourier series coefficients and are given by:

$$a_k = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt \qquad (1.69)$$

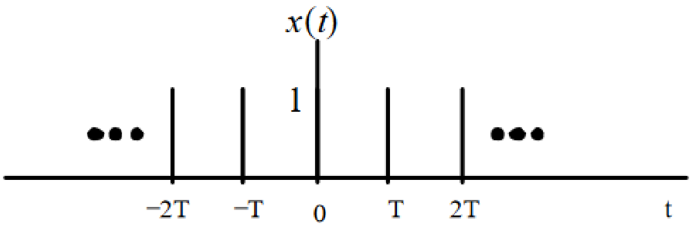
- The Fourier transform of this periodic signal $x(t)$ is:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \qquad (1.70)$$

- The inverse Fourier transform of $X(j\omega)$ is $x(t)$ which corresponds to the Fourier series representation of a periodic signal.

<u>Example</u>

Find and sketch the Fourier transform, $X(j\omega)$ for the following periodic pulse train, $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$.



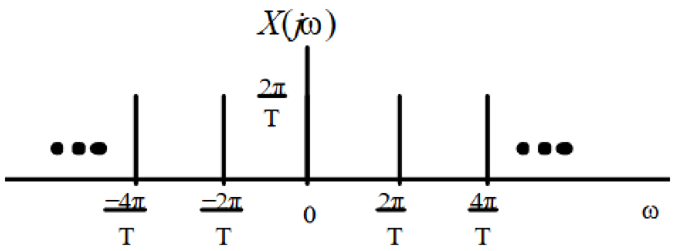
The Fourier series coefficients are given by:

$$a_k=\frac{1}{T_0}\int_{-\infty}^{\infty}x(t)e^{-jk\omega_0 t}dt$$

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T}$$

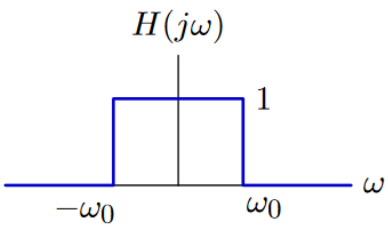
Using $X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0)$, the Fourier transform of $x(t)$ can be written as:

$$X(j\omega)=\sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta\left(\omega-\frac{2\pi k}{T}\right)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$$



Exercises

1. Using Fourier transforms, find the impulse response, $h(t)$ of the following:



2. Find the Fourier transforms of $\sin(\omega_o t)$ and $\cos(\omega_o t)$

Convergence of Fourier Transform

- If a signal $x(t)$) has finite energy, i.e. it is square integrable (meaning it is a real or complex-valued measurable function for which the integral of the square of the absolute value is finite)

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \qquad (1.71)$$

- Then it is guaranteed that $X(j\omega)$ is finite or converges.

- An alterative set of conditions that are sufficient to ensure convergence:

  1. Absolute integrability: the signal $x(t)$ should be absolutely integrable over any period, i.e. $\int_{T_0} |x(t)| dt < \infty$.
  2. In any finite interval of time $x(t)$ has a finite number of maxima and minima.
  3. In any finite interval of time, there are only a finite number of discontinuities.

Properties of Fourier Transforms (CTFT)

- The Fourier transform "inherits" properties of the Laplace transform where the s in Laplace transforms is replaced by $j\omega$ in Fourier transform.

Property Name	**Property**	
Linearity	$ax(t)+bv(t)$	$aX(\omega)+bV(\omega)$
Time Shift	$x(t-c)$	$e^{-j\omega c}X(\omega)$
Time Scaling	$x(at), \quad a\neq 0$	$\frac{1}{a}X(\omega/a), \quad a\neq 0$
Time Reversal	$x(-t)$	$X(-\omega)$ $\overline{X(\omega)}$ if $x(t)$ is real
Multiply by t^n	$t^n x(t), \quad n=1,2,3,\ldots$	$j^n \frac{d^n}{d\omega^n}X(\omega), \quad n=1,2,3,\ldots$
Multiply by Complex Exponential	$e^{j\omega_o t}x(t), \quad \omega_o$ real	$X(\omega-\omega_o), \quad \omega_o$ real
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\frac{j}{2}[X(\omega+\omega_o)-X(\omega-\omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$
Time Differentiation	$\frac{d^n}{dt^n}x(t), \quad n=1,2,3,\ldots$	$(j\omega)^n X(\omega), \quad n=1,2,3,\ldots$
Time Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Convolution in Time	$x(t)*h(t)$	$X(\omega)H(\omega)$
Multiplication in Time	$x(t)w(t)$	$\frac{1}{2\pi}X(\omega)*W(\omega)$
Parseval's Theorem (General)	$\int_{-\infty}^{\infty} x(t)\overline{v(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\overline{V(\omega)}d\omega$	
Parseval's Theorem (Energy)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$ if $x(t)$ is real $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	
Duality: If $x(t) \leftrightarrow X(\omega)$	$X(t)$	$2\pi x(-\omega)$

Example

Find the Fourier transform of $x(t)=e^{-2(t-1)}u(t-1)$.

Solution

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

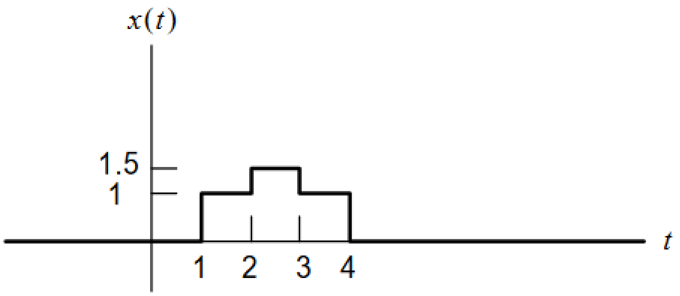
$$e^{-2t}u(t)\xleftrightarrow{F}\int_{0}^{\infty}e^{-2t}e^{-j\omega t}dt=\frac{1}{2+j\omega}$$

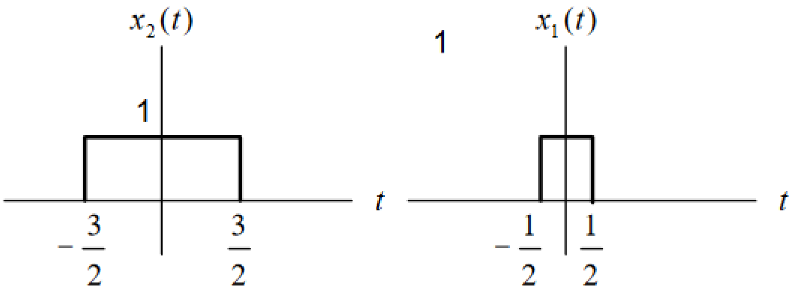
Using Time Shift: $x(t-c)\xleftrightarrow{F}e^{-j\omega c}X(j\omega)$, where in this case $c=1$ yields:

$$X(j\omega)=\frac{e^{-j\omega}}{2+j\omega}$$

Example

Signal $x(t)$ is constructed by the combination of signals $x_1(t)$ and $x_2(t)$ as shown in the figure below. Find an expression for $x(t)$ and then calculate its Fourier transform, $X(j\omega)$.





Solution

Signal $x(t)$ can be expressed as:

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

Using the Fourier transform of a rectangular pulse, the Fourier transform of $x_1(t)$ is

$$X_1(j\omega) = 2\frac{\sin(\omega T)}{\omega} = 2\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}$$

Using the Fourier transform of a rectangular pulse, the Fourier transform of $x_2(t)$ is

$$X_2(j\omega) = 2\frac{\sin(\omega T)}{\omega} = 2\frac{\sin\left(\frac{3\omega}{2}\right)}{\omega}$$

Using the linearity and time shift Fourier transform properties yields:

$$X(j\omega) = \frac{1}{2}\left[2e^{-j\omega 2.5}\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}\right] + 2e^{-j\omega 2.5}\frac{\sin\left(\frac{3\omega}{2}\right)}{\omega} = e^{-j\omega 2.5}\left[\frac{\sin\left(\frac{\omega}{2}\right) + 2\sin\left(\frac{3\omega}{2}\right)}{\omega}\right]$$

Example

Find the Fourier transform of the following periodic signal $x(t) = 1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$.

Solution

$\omega_0 t = 6\pi t$, where $\omega_0 = 2\pi f_0 = \frac{2\pi}{T} = 6\pi$ and $T = \frac{1}{3}$

The none zero Fourier series coefficients of $x(t) = 1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$ can be obtained by expressing $A\cos(\omega t + \theta) = \frac{A}{2}\left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right)$ in terms of complex exponentials, resulting in:

$$x(t)=1+\underbrace{\frac{1}{2}\left(e^{j\left(6\pi t+\frac{\pi}{8}\right)}+e^{-j\left(6\pi t+\frac{\pi}{8}\right)}\right)}_{\cos\left(6\pi t+\frac{\pi}{8}\right)}=1+\frac{1}{2}e^{j\left(6\pi t+\frac{\pi}{8}\right)}+\frac{1}{2}e^{-j\left(6\pi t+\frac{\pi}{8}\right)}$$

$$\therefore a_0=1,\ a_1=\frac{1}{2}e^{j\left(6\pi t+\frac{\pi}{8}\right)},\ a_{-1}=\frac{1}{2}e^{-j\left(6\pi t+\frac{\pi}{8}\right)}$$

Using $X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0)$, the Fourier transform of $x(t)$ can be written as:

$$X(j\omega)=2\pi a_0\delta(\omega)+2\pi a_1\delta(\omega-\omega_0)+2\pi a_{-1}\delta(\omega+\omega_0)$$

$$X(j\omega)=2\pi\delta(\omega)+2\pi\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)\delta(\omega-6\pi)+2\pi\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)\delta(\omega+6\pi)$$

$$X(j\omega)=2\pi\delta(\omega)+\pi e^{j\frac{\pi}{8}}\delta(\omega-6\pi)+\pi e^{-j\frac{\pi}{8}}\delta(\omega+6\pi)$$

Example

Find the Fourier transform of $x(3t-6)$.

Solution

$$x(3t-6)=x(3(t-2))$$

Using Time Scaling: $x(at)\overset{F}{\longleftrightarrow}\frac{1}{a}X\left(\frac{j\omega}{a}\right)$, where in this case $a=3$ yields:

$$x(3t)\overset{F}{\longleftrightarrow}\frac{1}{3}X\left(\frac{j\omega}{3}\right)$$

Then using Time Shift: $x(t-c)\overset{F}{\longleftrightarrow}e^{-j\omega c}X(j\omega)$, where in this case $c=2$ yields:

$$x(3(t-2))\overset{F}{\longleftrightarrow}\frac{1}{3}X\left(\frac{j\omega}{3}\right)e^{-2j\omega}$$

Example

Find the Fourier transform of $\frac{d^2}{dt^2}x(t-1)$.

Solution

Using Time Differentiation: $\frac{d^n}{dt^n}x(t) \xleftrightarrow{F} (j\omega)^n X(j\omega)$, where in this case $n=2$ yields:

$$\frac{d^2}{dt^2}x(t) \xleftrightarrow{F} (j\omega)^2 X(j\omega) = -\omega^2 X(j\omega)$$

Then using Time Shift: $x(t-c) \xleftrightarrow{F} e^{-j\omega c} X(j\omega)$, where in this case $c=1$ yields:

$$\frac{d^2}{dt^2}x(t-1) \xleftrightarrow{F} -\omega^2 X(j\omega)e^{-j\omega}$$

Example

Find the Fourier transform of $x(1-t)+x(-1-t)$.

Solution

$$x(1-t)+x(-1-t) = x(-t+1)+x(-t-1)$$

Using Time Reversal: $x(-t) \xleftrightarrow{F} X(-j\omega)$ yields:

Then using Time Shift: $x(t-c) \xleftrightarrow{F} e^{-j\omega c} X(j\omega)$, where in this case $c=-1$ and $c=1$ yields:

$$x(1-t) = x(-t+1) \xleftrightarrow{F} e^{j\omega} X(-j\omega)$$

$$x(-1-t) = x(-t-1) \xleftrightarrow{F} e^{-j\omega} X(-j\omega)$$

$$\therefore x(1-t)+x(-1-t) \xleftrightarrow{F} e^{j\omega} X(-j\omega) + e^{-j\omega} X(-j\omega) = 2X(-j\omega)\cos(\omega)$$

Example

Find the Fourier transform of the following periodic signal $\left[te^{-2t}\sin(4t)\right]u(t)$.

Solution

Using the Euler identity $\sin(\omega t)=\frac{1}{2j}\left(e^{j\omega t}-e^{-j\omega t}\right)$, results in:

$$\left[te^{-2t}\frac{1}{2j}\left(e^{j4t}-e^{-j4t}\right)\right]u(t)=\frac{1}{2j}te^{-2t}e^{j4t}u(t)-\frac{1}{2j}te^{-2t}e^{-j4t}u(t)$$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$X(j\omega)=\int_{0}^{\infty}\left[\frac{1}{2j}te^{-2t}e^{j4t}\right]e^{-j\omega t}dt-\int_{0}^{\infty}\left[\frac{1}{2j}te^{-2t}e^{-j4t}\right]e^{-j\omega t}dt$$

$$X(j\omega)=\int_{0}^{\infty}\frac{1}{2j}te^{(-2+j4-j\omega)t}dt-\int_{0}^{\infty}\frac{1}{2j}te^{(-2-j4-j\omega)t}dt:$$

Applying integration by parts yields:

$$X(j\omega)=\frac{1}{2j(2-j4+j\omega)^2}-\frac{1}{2j(2+j4-j\omega)^2}$$

Example

Compute the convolution of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$ using the convolution property. Then calculate $y(t)$ by finding the inverse of $Y(j\omega)=H(j\omega)X(j\omega)$.

i) $x(t)=te^{-2t}u(t),\quad h(t)=e^{-4t}u(t)$

Solution

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

Using (4) in table of continuous-time Fourier transforms, where $a=2$, yields:

$$x(t)=te^{-2t}u(t)\overset{F}{\longleftrightarrow}\frac{1}{(2+j\omega)^2}$$

Using (1) in table of continuous-time Fourier transforms, where $a=4$, yields:

$$h(t)=e^{-4t}u(t)\overset{F}{\longleftrightarrow}\frac{1}{4+j\omega}$$

Then using convolution in time property, $h(t)*x(t)\overset{F}{\longleftrightarrow}H(j\omega)X(j\omega)$ yields:

$$Y(j\omega)=H(j\omega)X(j\omega)=\left[\frac{1}{4+j\omega}\right]\left[\frac{1}{(2+j\omega)^2}\right]$$

Taking partial fractions results in:

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{1/4}{4+j\omega}-\frac{1/4}{2+j\omega}+\frac{1/2}{(2+j\omega)^2}$$

Taking the inverse Fourier transform yields:

$$y(t)=h(t)*x(t)=\frac{1}{4}e^{-4t}u(t)-\frac{1}{4}e^{-2t}u(t)+\frac{1}{2}te^{-2t}u(t)$$

ii) $x(t)=e^{-t}u(t),\quad h(t)=e^{t}u(-t)$

Solution

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Using (1) in table of continuous-time Fourier transforms, where $a = 1$, yields:

$$x(t) = e^{-t}u(t) \xleftrightarrow{F} \frac{1}{1+j\omega}$$

Using (1) in table of continuous-time Fourier transforms, where $a = -1$ yields:

$$h(t) = e^{t}u(t) \xleftrightarrow{F} \frac{1}{1-j\omega}$$

Then using convolution in time property, $h(t) * x(t) \xleftrightarrow{F} H(j\omega)X(j\omega)$ yields:

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{1}{1+j\omega}\right]\left[\frac{1}{1-j\omega}\right]$$

Taking partial fractions results in:

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega}$$

Taking the inverse Fourier transform yields:

$$y(t) = h(t) * x(t) = \frac{1}{2}e^{-|t|}$$