Analysis of Cross-Sectional Data

Machine Learning Approaches to Regression

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Analysis of Cross-Sectional Data

Best Subset Regression and Stepwise Regression

- Best Subset Regression
- Forward Stepwise Regression
- Backward Stepwise Regression
- Hybrid Stepwise Regression

Best Subset Regression

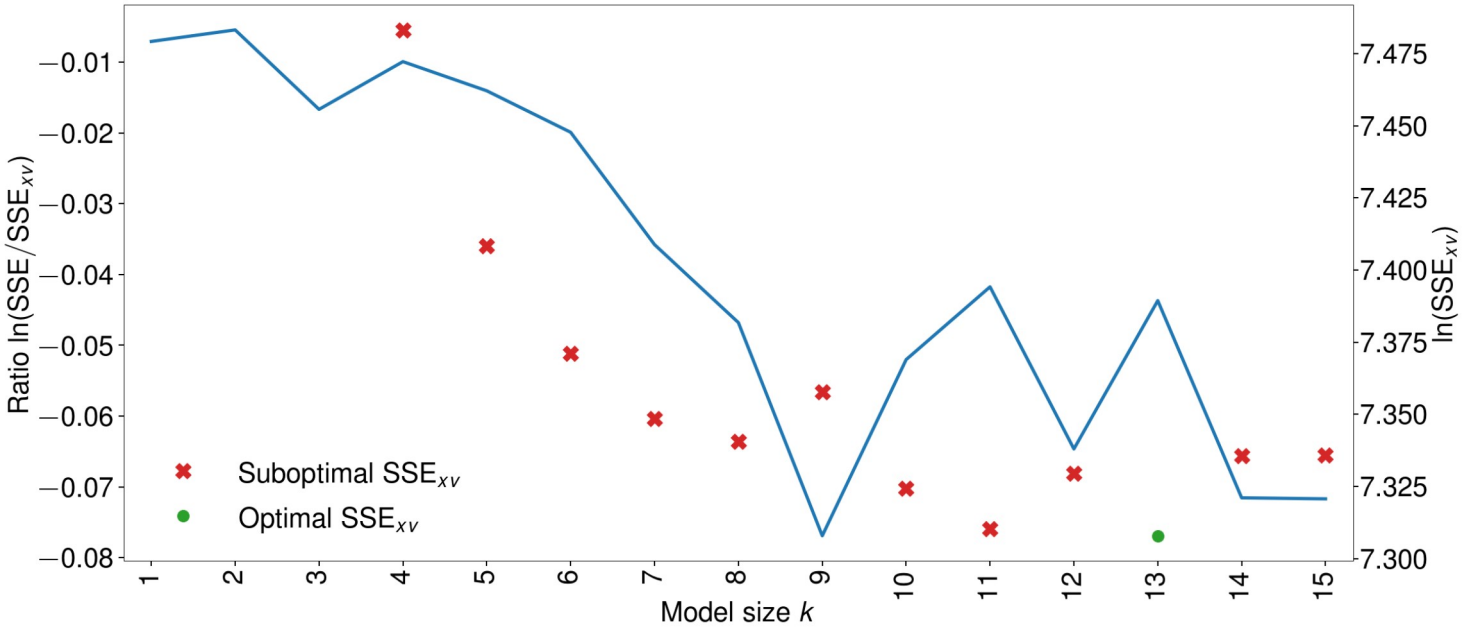
- With p candidate variables, consider all 2^p possible models
- For each $k = 1, \ldots, p$ find the best model in terms of SSE
- From set of p best models, select preferred model using cross-validation
- Example regression is portfolio tracking/replication

$$VWM_t = \sum_{i=1}^{k} \beta_i R_{i,t} + \epsilon_t$$

- $R_{i,t}$ are industry portfolio returns
- BSR example uses 15 randomly selected industry portfolios due to computational limits

Best Subset Regression

In [3]:
```
xval_plot(bsr_sse, bsr_sse_xv)
```

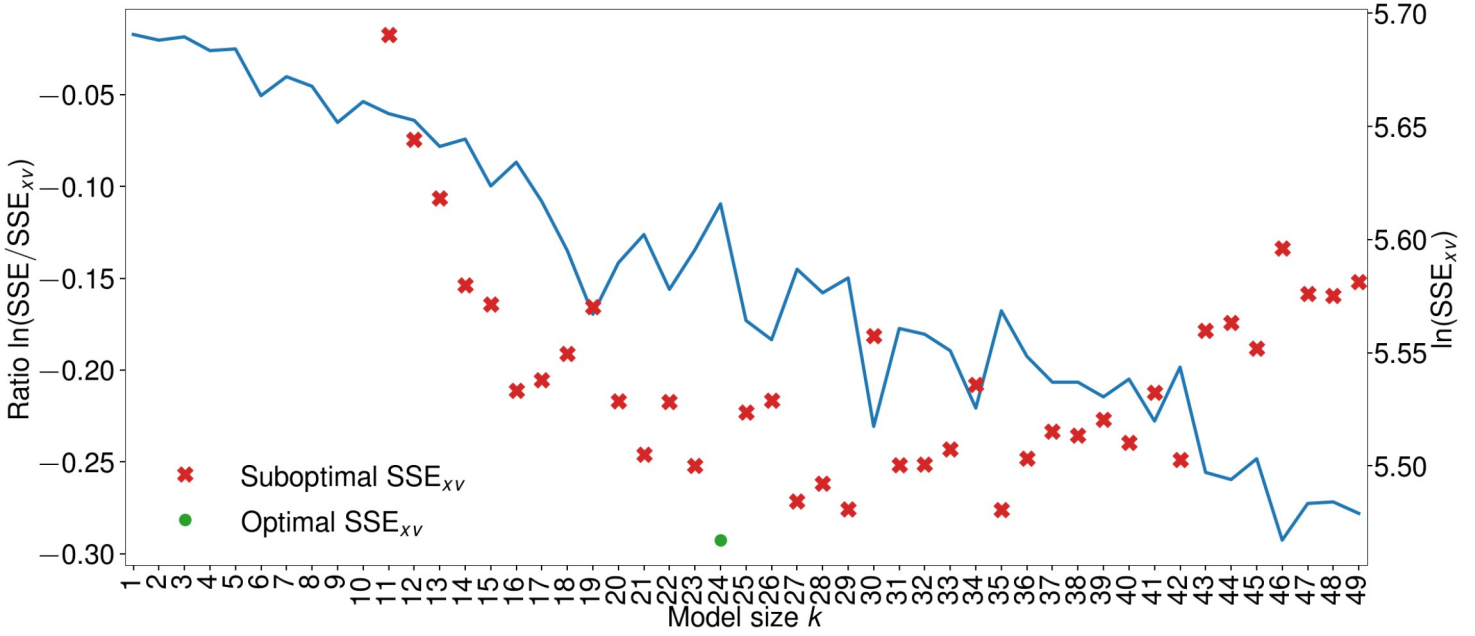


Forward Stepwise Regression

- When p is large Best Subset Regression is computationally impossible
- Forward Stepwise Regression builds a sequence of models using two principles
    1. Start from the previous model with i variables
    2. Considering all excluded variables, **add** the variable to model $i+1$ that produces the **largest reduction in SSE**
- Begins with model 0 that includes no variables (or possibly a constant)
- Produces a sequence of p (or $p+1$ if a constant) models
- Preferred model is selected from these models using cross-validated SSE

Forward Stepwise Regression

In [5]:
```
xval_plot(fs_sse, fs_sse_xv)
```

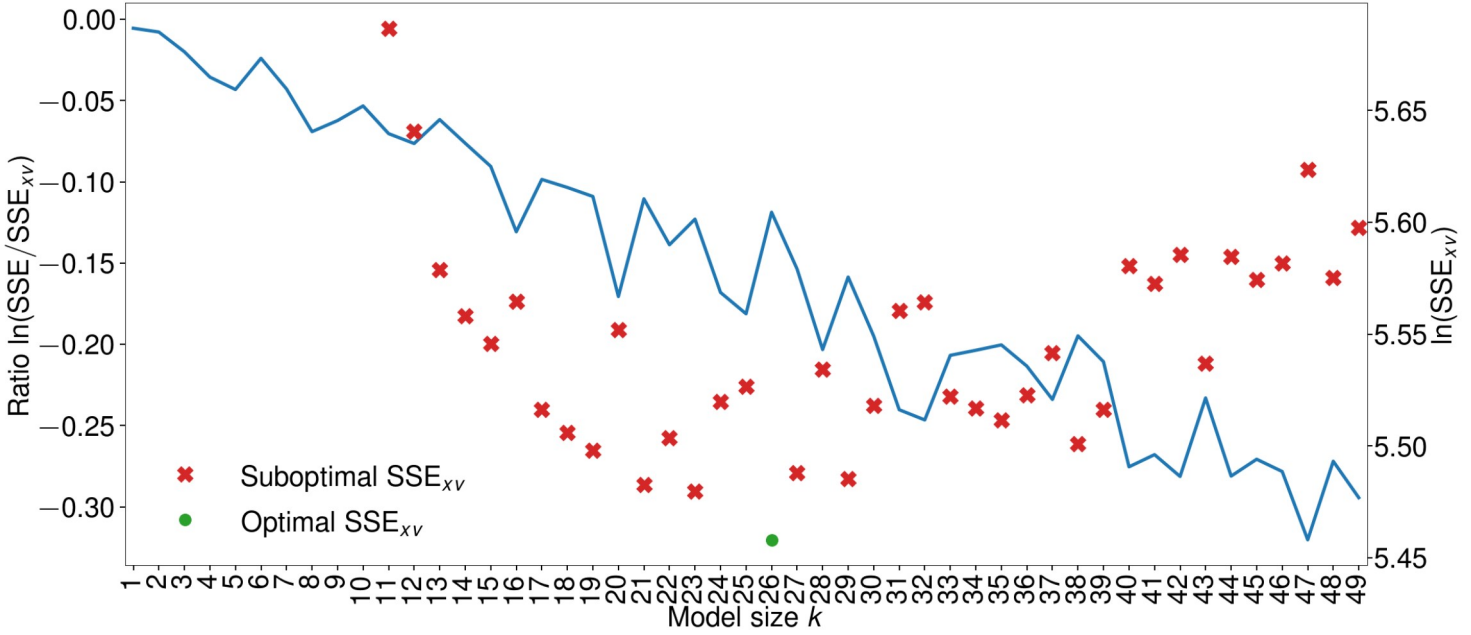


Backward Stepwise Regression

- Similar to Forward Stepwise only constructed by *removing* variables
- Backward Stepwise Regression builds a sequence of models using two principles
    1. Start from the previous model with i variables
    2. Considering all included variables in model i, **remove** the variable that **increases the SSE the least** to construct model $i-1$
- Begins with model p that includes all variables
- Like FSR, produces a sequence of p (or $p+1$ if a constant) models
- Preferred model is selected from these models using cross-validated SSE

Backward Stepwise Regression

In [7]:
```
xval_plot(bs_sse, bs_sse_xv)
```



Comparing Forward and Backward Stepwise Regression

In [8]:
```
fs_model = fs_models[pd.Series(fs_sse_xv).idxmin()]
bs_model = bs_models[pd.Series(bs_sse_xv).idxmin()]
print(f"Number of common regressors: {len(set(bs_model).intersection(fs_mode
l))}")
print(f"Only in FS: {', '.join(sorted(set(fs_model).difference(bs_model)))}")
print(f"Only in BS: {', '.join(sorted(set(bs_model).difference(fs_model)))}")
```

```
Number of common regressors: 23
Only in FS: Mach
Only in BS: Coal, Fun, Meals
```

Hybrid Approaches

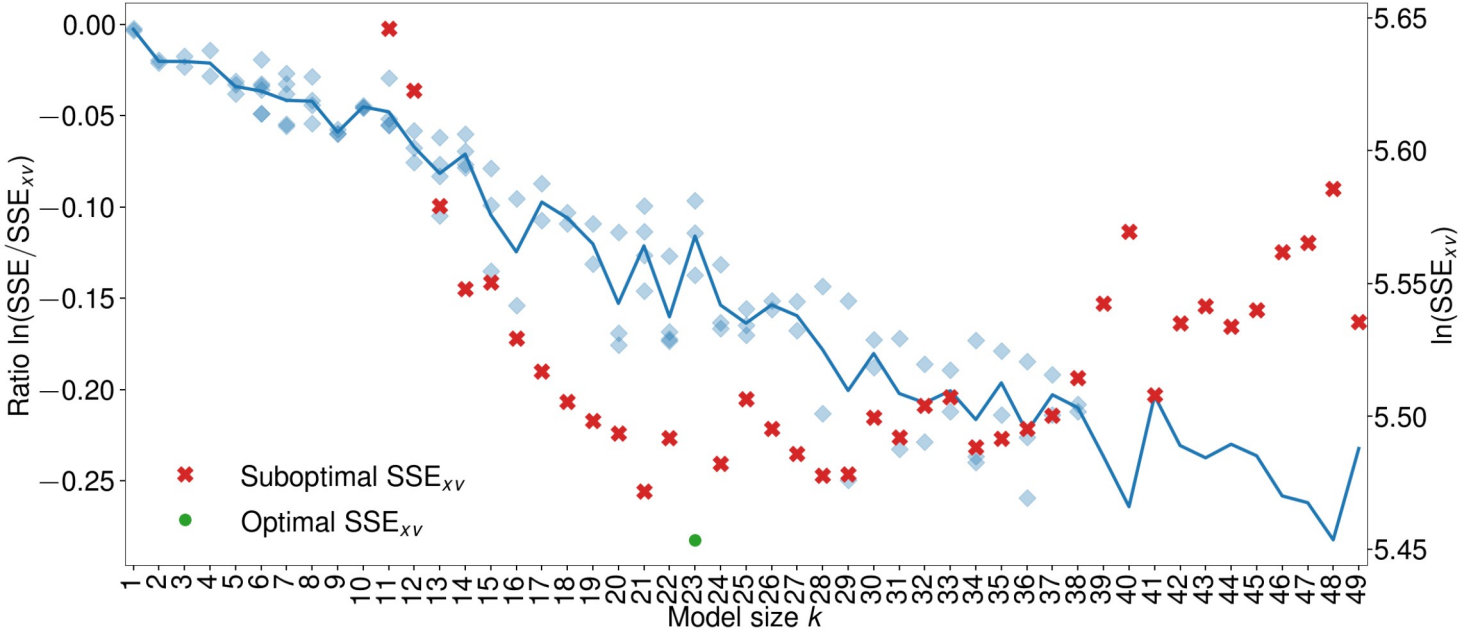
- Forward and Backward can be combined to produce a larger set of candidate models
- Forward-Backward
    - Use FSR to build select candidate models with $k = 1, \ldots, p$ variables
    - Starting with each of these p model, perform Backward Stepwise
- Can be iterated as much as one is willing to wait
- Might better approximate Best Subset Regression
- Final model selected from enlarged set of candidate models by minimizing the cross-validated SSE

In [10]:
```
print(f"The number of models selected using forward-backward is {len(fb_model
s)}")
print(f"Not in FS: {', '.join(sorted(set(fb_model).difference(fs_model)))}")
print(f"Not in BS: {', '.join(sorted(set(fb_model).difference(bs_model)))}")
```

```
The number of models selected using forward-backward is 117
Not in FS: Coal, Meals
Not in BS:
```

Hybrid Approaches

In [11]:
```
xval_plot(fb_sse, fb_sse_xv, k=fb_k)
```



Analysis of Cross-Sectional Data

Shrinkage Estimators

- Ridge Regression
- LASSO: Least Absolute Shrinkage and Selection Operator

Note: Important to standardized regressors when using shrinkage estimators

Ridge Regression

Fit a modified least squares problem

$$\underset{\boldsymbol{\beta}}{\operatorname{argmin}} \left(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)' \left(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right) \text{ subject to } \sum_{j=1}^{k} \beta_j^2 \leq \omega$$

Equivalent formulation

$$\underset{\boldsymbol{\beta}}{\operatorname{argmin}} \left(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)' \left(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right) + \lambda \sum_{j=1}^{k} \beta_j^2$$

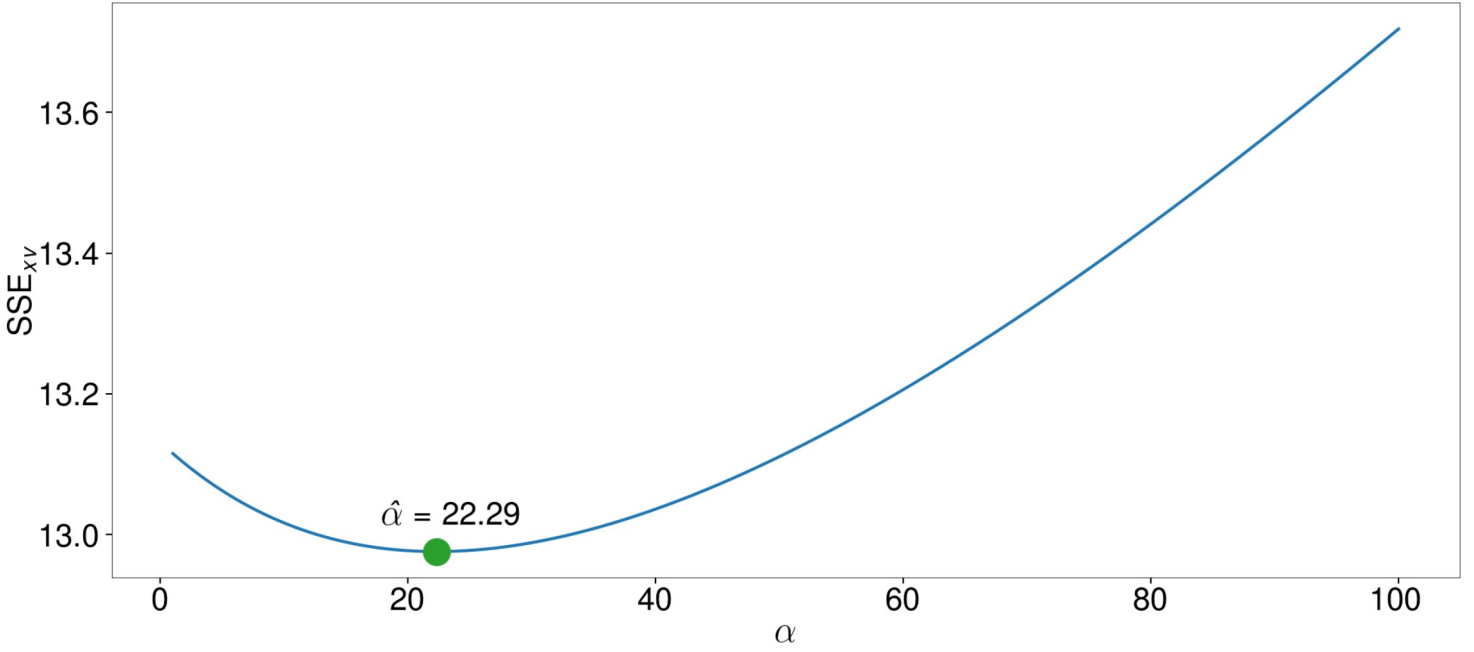
Analytical solution

$$\hat{\boldsymbol{\beta}}^{\text{Ridge}} = \left(\mathbf{X}'\mathbf{X} + \lambda \mathbf{I}_k\right)^{-1} \mathbf{X}'\mathbf{y}$$

- λ is key shrinkage (ore regularization) parameter
- Optimal λ is chosen using cross-validations across a grid of values

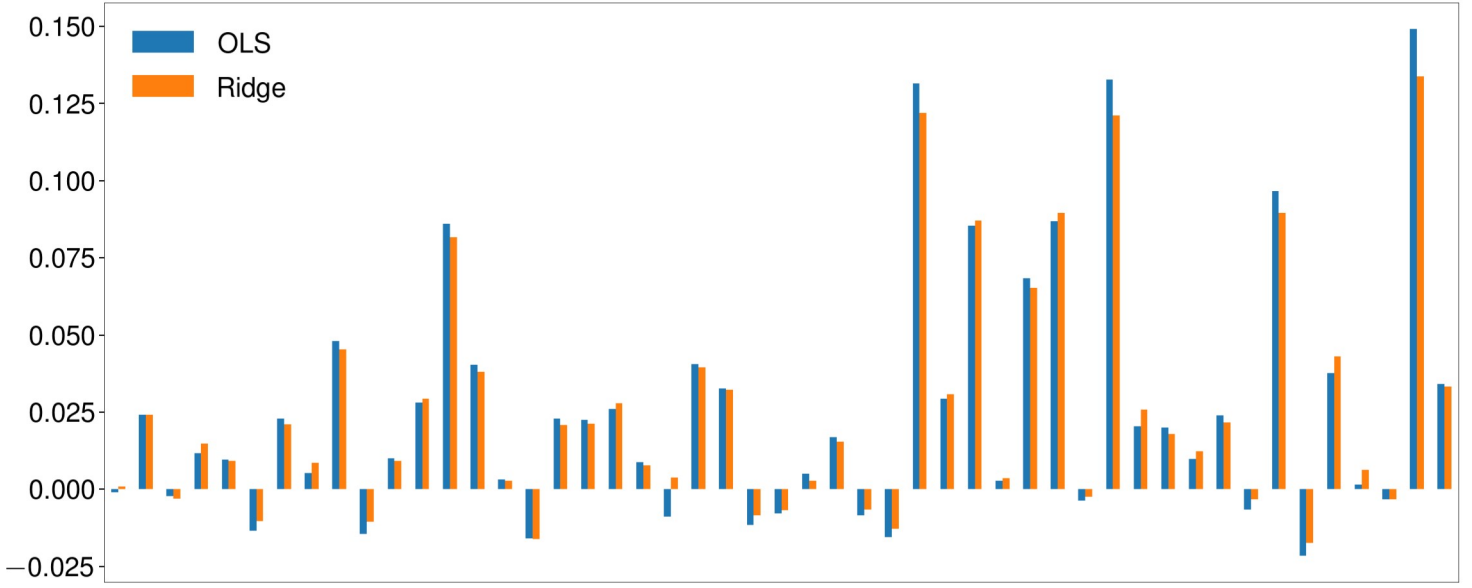
Ridge Regression α Selection

In [13]:
```
ridge_cv_plot()
```



Ridge Regression

In [14]:
```
ridge_coef_plot()
```



LASSO: Least Absolute Shrinkage and Selection Operator

Similar to Ridge but with a different penalty

$$\underset{\boldsymbol{\beta}}{\operatorname{argmin}} \ (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \ \text{subject to} \ \sum_{j=1}^{k} |\beta_j| \leq \omega$$

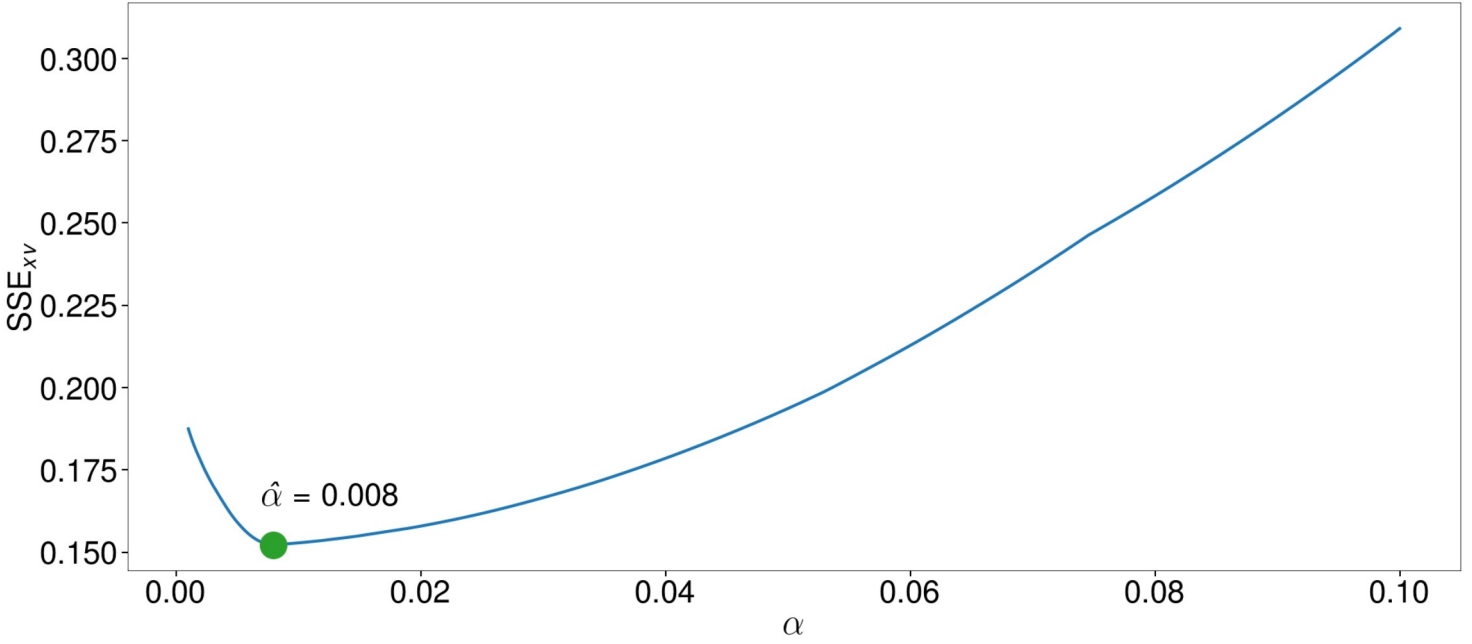
Equivalent formulation

$$\underset{\boldsymbol{\beta}}{\operatorname{argmin}} \ (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + \lambda \sum_{j=1}^{k} |\beta_j|$$

- Analytical solution but not easily presentable
- Has special property -- will usually estimate some coefficients to be **exactly** zero
- λ is key shrinkage (ore regularization) parameter
- Optimal λ is chosen using cross-validations across a grid of values

LASSO α Selection

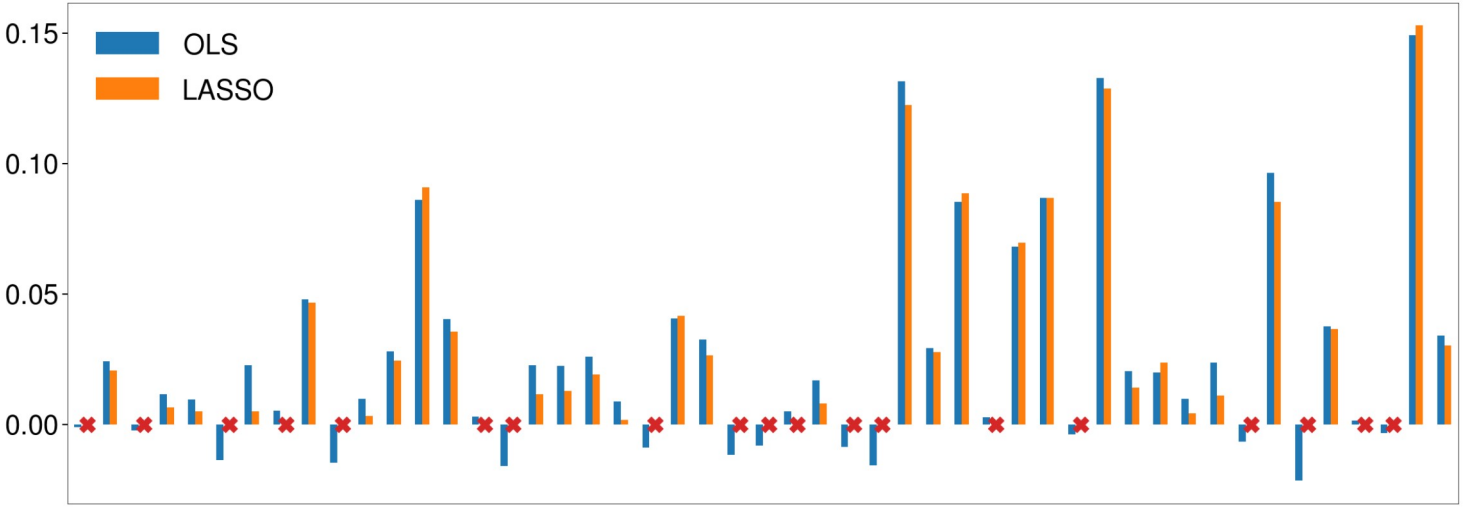
In [16]:
```
lasso_cv_plot()
```



LASSO and OLS

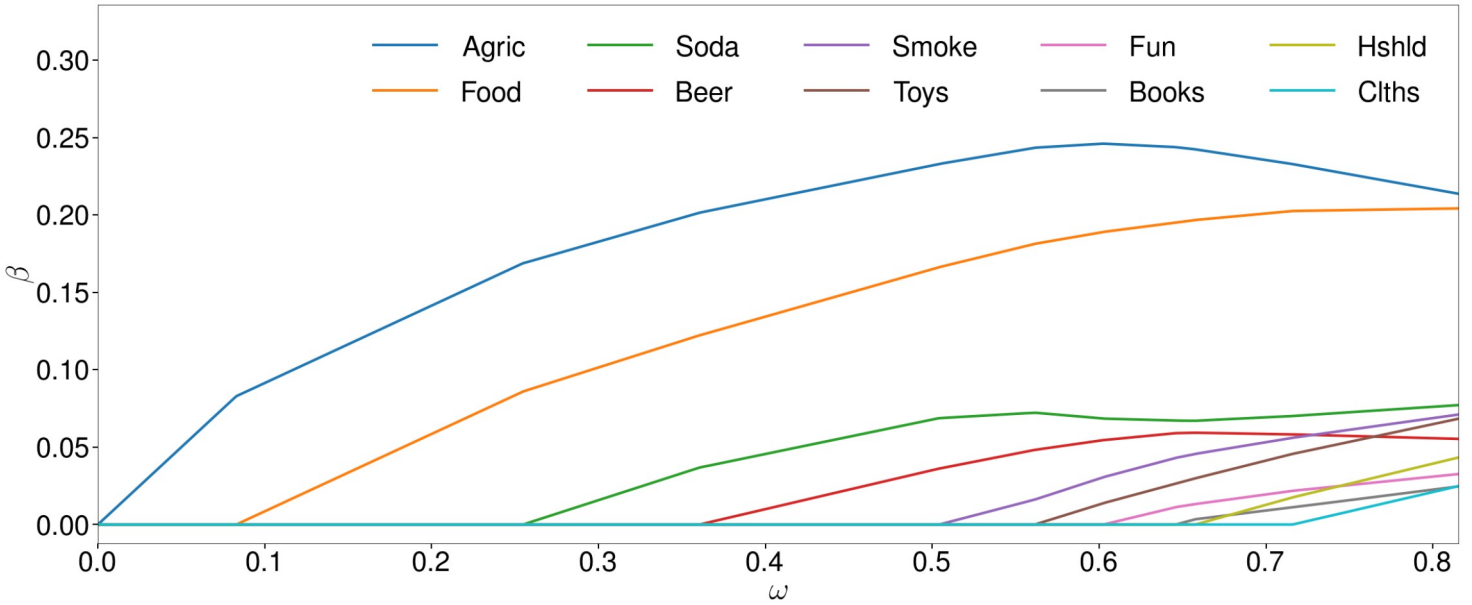
In [18]:
```
lasso_coef_plot()
```

```
Number of non-zero coefficients: 30
```



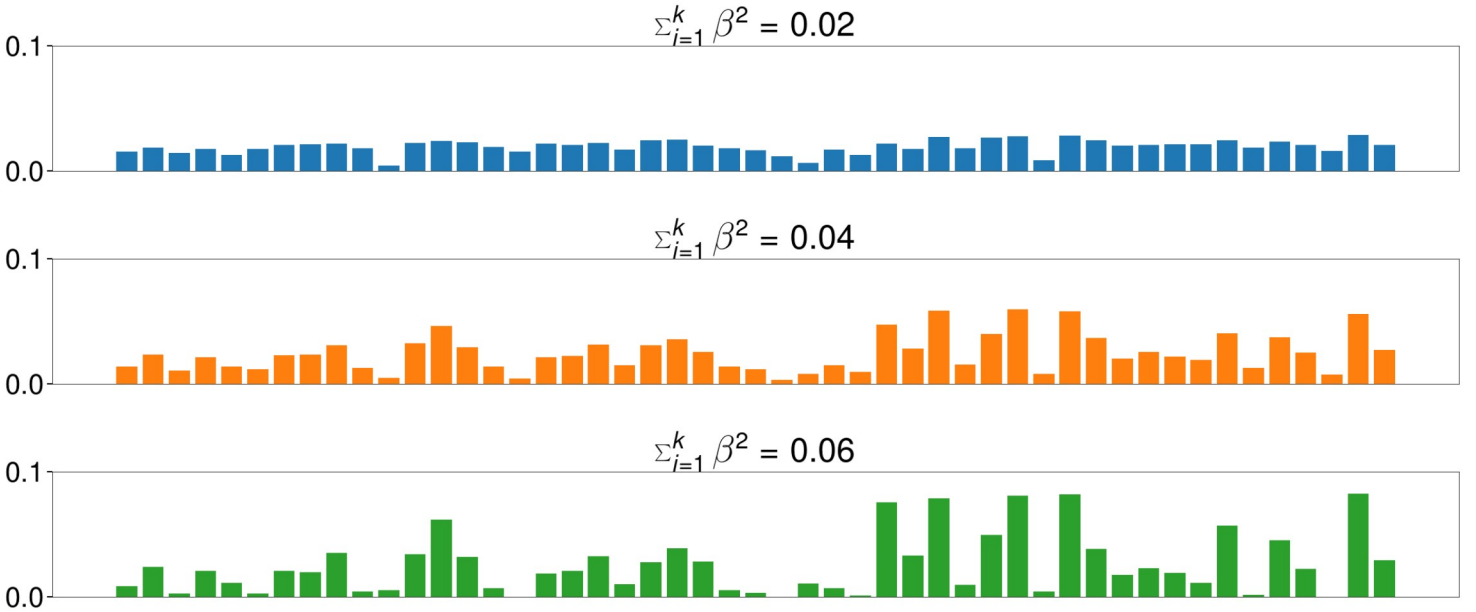
The LASSO Coefficient Path

In [20]:
```
lasso_path_plot()
```



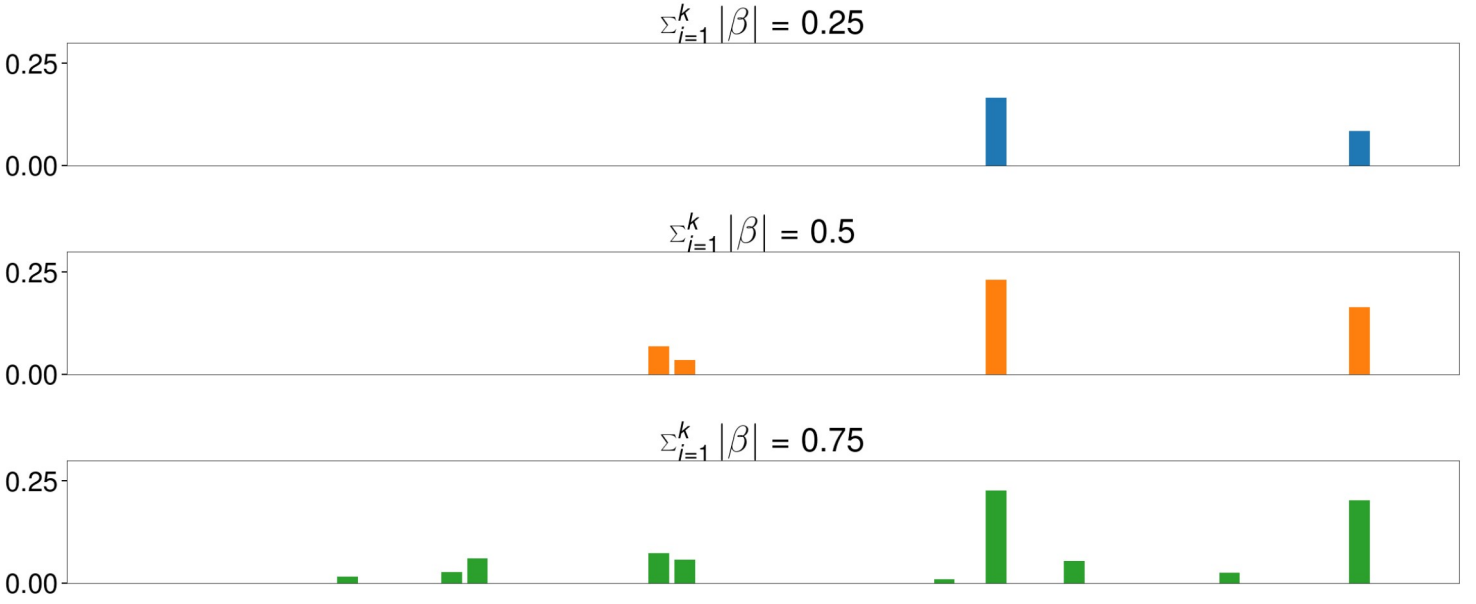
Alternative constraints in Ridge Regression

In [22]:
```
ridge_betas()
```



Alternative constraints in LASSO

In [23]:
```
lasso_betas()
```



Analysis of Cross-Sectional Data

Tree-based Estimators

- Regression Tree
- Model Representation
- Pruning
- Bagging and Random Forest
- Boosting

Regression Trees

- Regression Trees build models with **indicator** variables
- Continuous regressors are transformed into Indicator variables

$$X_{i,j} \rightarrow I_{[X_{i,j}<X_j^\star]}$$

- Model is then build using interaction of indicator variables
- Leads to a tree structure where the tree *splits* on each indicator variable

Example

- 2-level tree that splits on the market first and then on the return of the size

$$I_{[VWM<0]}\, I_{[SMB<1.2]}$$

- Parameter estimates are simply averages of Y_i values where indicators are true

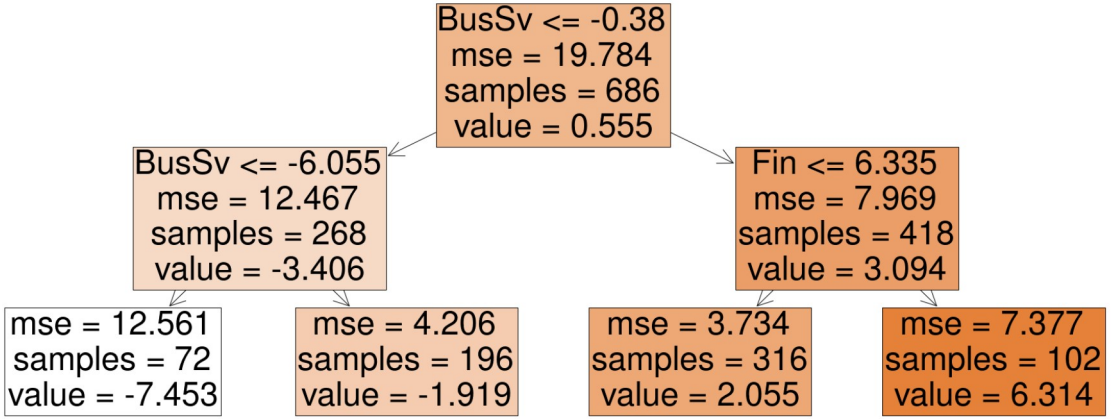
$$\frac{1}{m^\star} \sum_{i=1}^{n} Y_i \left(I_{[VWM_i<0]}\, I_{[SMB_i<1.2]} \right)$$

- $m^\star = \sum_{i=1}^{n} I_{[VWM_i<0]}\, I_{[SMB_i<1.2]}$ counts the observation in this branch

Regression Trees

Two level depiction

In [25]:
```
show_tree()
```



Regression Trees

Model Representation

- 2-level tree is regression model

$$
\begin{aligned}
VWM_i = & \quad \beta_1 I_{[X_{i,33} \leq -0.38]} I_{[X_{i,33} \leq -6.055]} \\
& + \quad \beta_2 I_{[X_{i,33} \leq -0.38]} I_{[X_{i,33} > -6.055]} \\
& + \quad \beta_3 I_{[X_{i,33} > -0.38]} I_{[X_{i,47} \leq 6.335]} \\
& + \quad \beta_4 I_{[X_{i,33} > -0.38]} I_{[X_{i,47} > 6.335]} \\
& + \quad \epsilon_i
\end{aligned}
$$

Regression Trees

Key Parameters

Optimize by choosing one of more of:

- Max depth
- Maximum number of nodes
- Minimum decrease in SSE to add a new node
- Minimum group size to allow further splitting

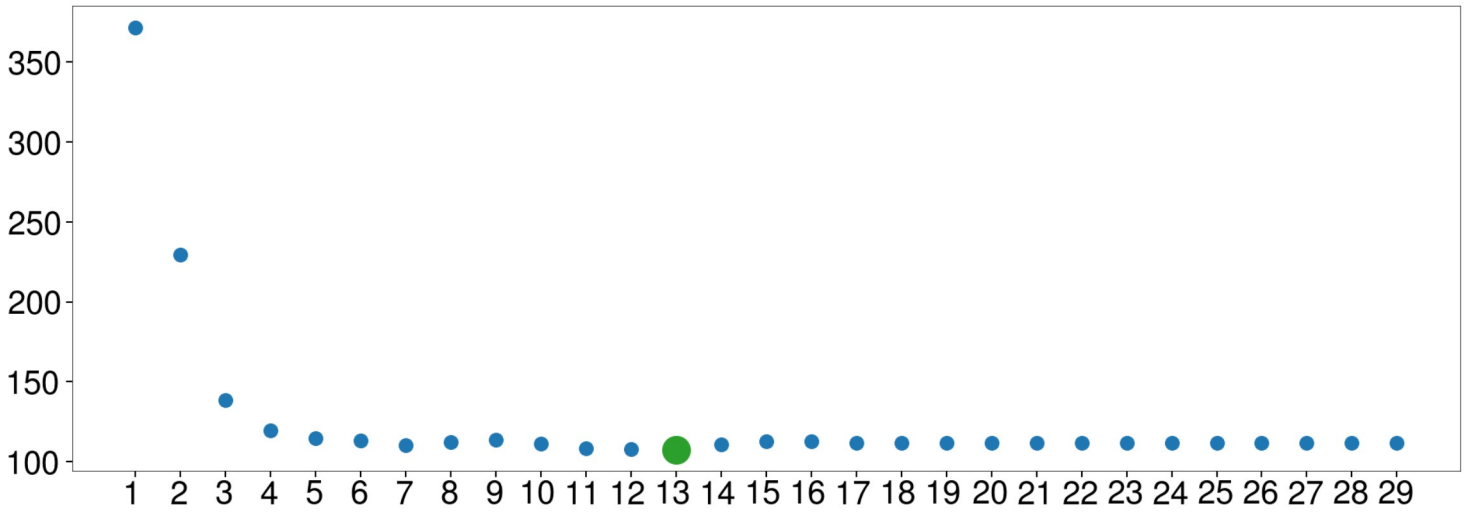
Optimize choice using cross-validation

Regression Trees

Max depth

In [27]:
```
depth_cross_val()
```

```
Number of leaves: 640
```

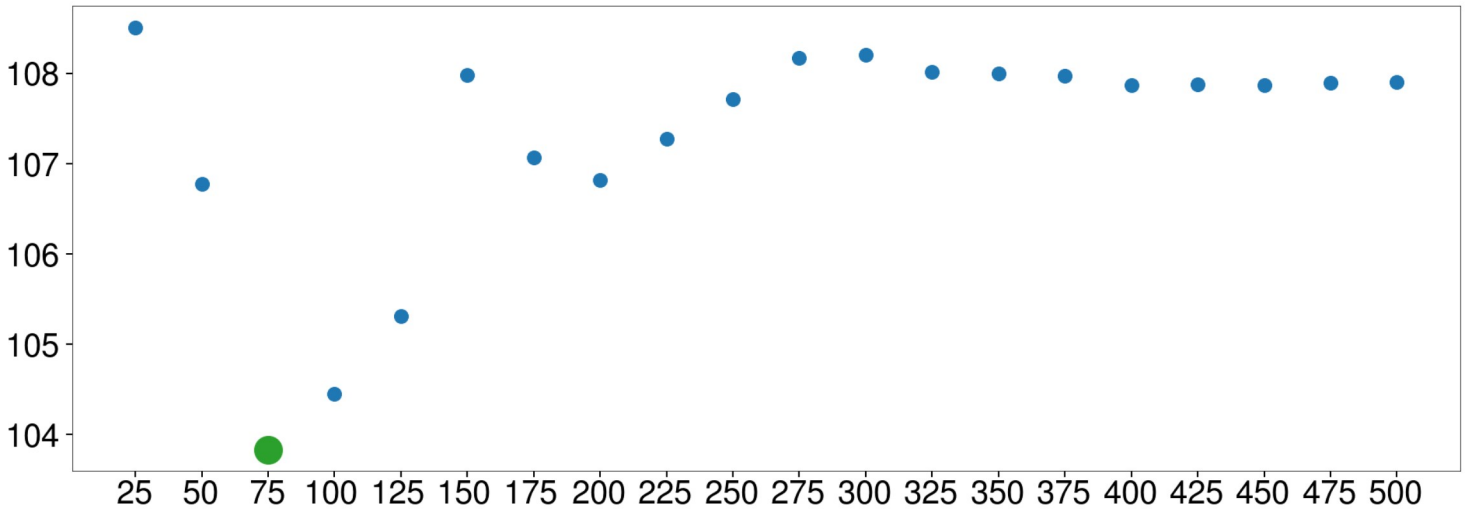


Regression Trees

Max nodes

In [28]:
```
leaves_cross_val()
```

```
Maximum Depth: 8
```



Pruning

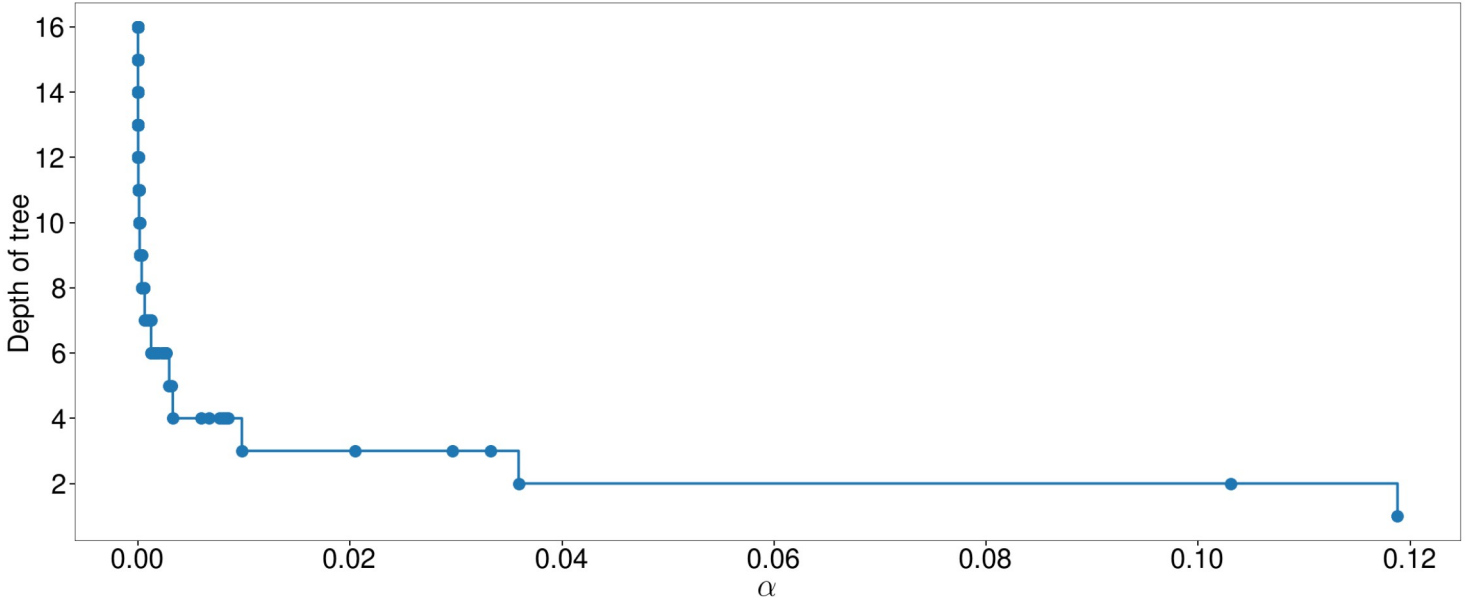
- Pruning can improve trees by removing nodes that make small improvement to the SSE
- Solve a modified objective function

$$\sum_{i=1}^{n} \left(Y_i - \hat{Y}_i \right)^2 + \alpha \left| T \right|$$

  - α is a tuning parameter
  - $\left| T \right|$ is the number of terminal nodes
- Larger values of α reduce tree size
- Select α using cross-validation by pruning a large tree for different values of α

α and tree depth

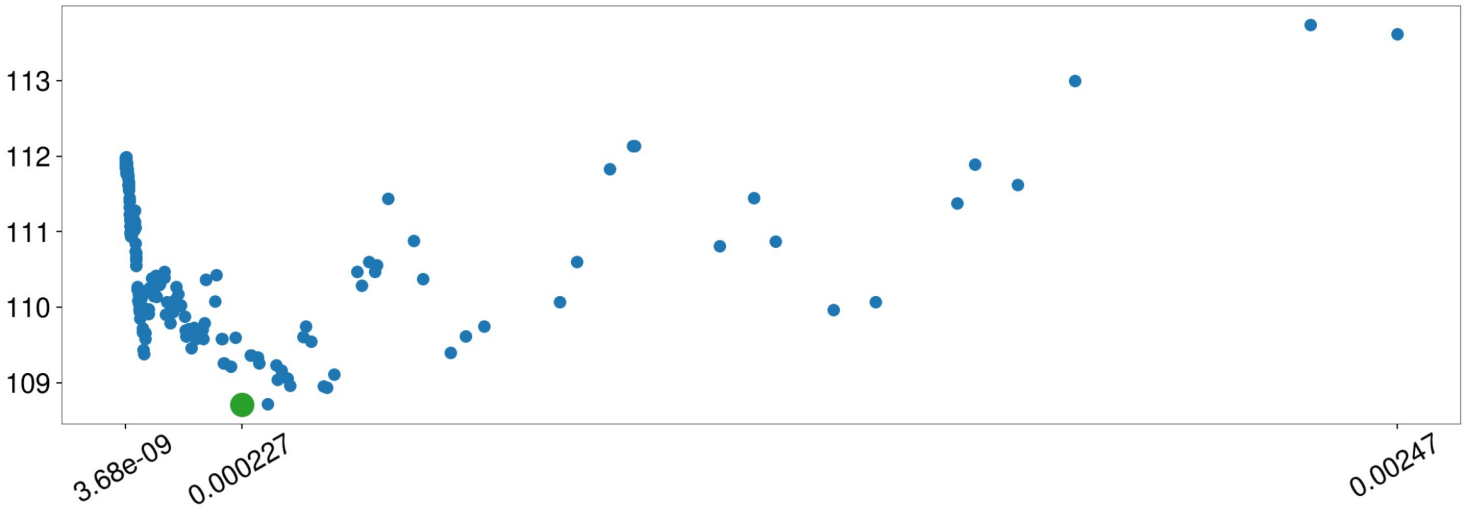
In [30]:
```
plot_tree_depth()
```



Selecting α using cross-validation

In [32]:
```
pruning_cross_val()
```

```
Number of leaves: 96
Maximum Depth: 9
```



Random Forests and Bagging

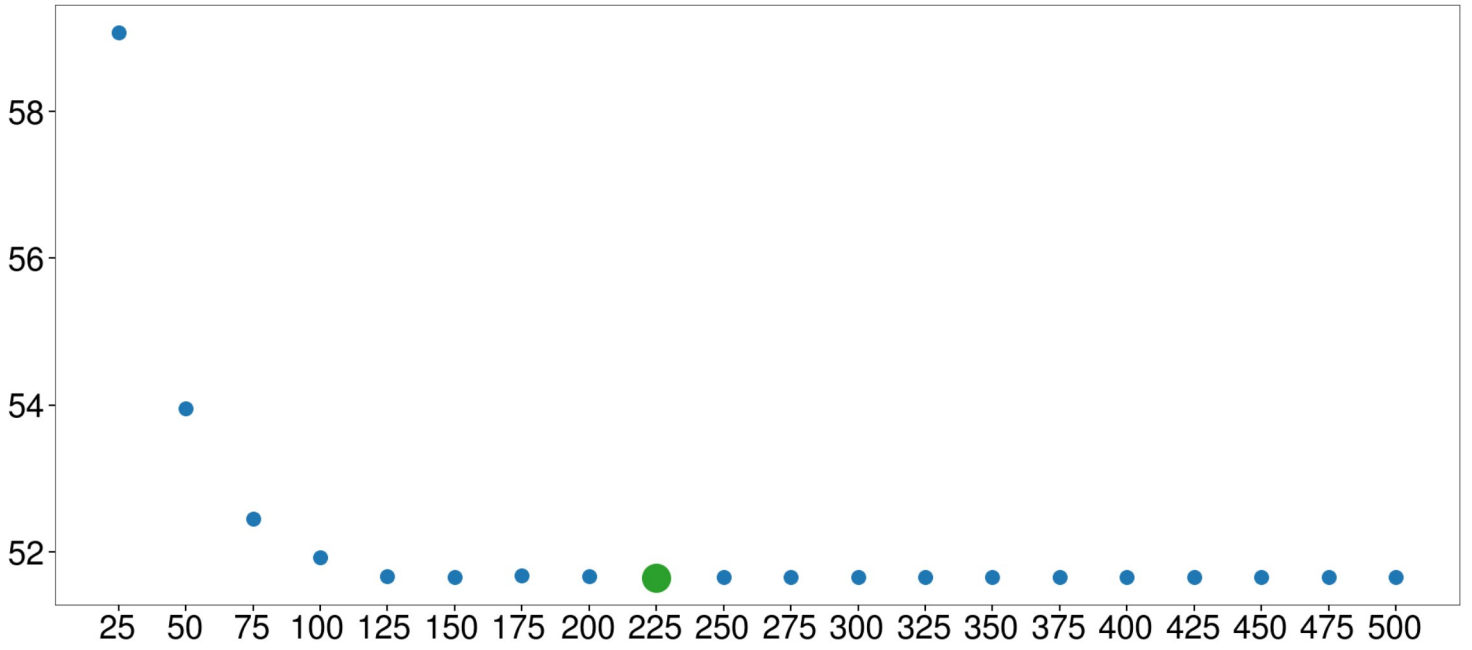
- Bagging (Bootstrap Aggregation) and Random Forests build B trees using B bootstrapped samples
- Prediction is then

$$\hat{f}^{\mathrm{RF}}(\mathbf{x}_i) = B^{-1} \sum_{b=1}^{B} \hat{f}^{(b)}(\mathbf{x}_i)$$

- Random Forest improves on Bagging since:
    - Each tree is built using only $k \approx \sqrt{p}$ of the variables
    - Produces a set of trees that are weekly correlated because most regressors are excluded from each tree
    - Used when two criteria are met
        - p is large
        - A small number of strong predictors
    - **Note**: Bagging is a special case of a Random Forest when $k = p$
- Similar tuning choices as Regression Tree plus B and k

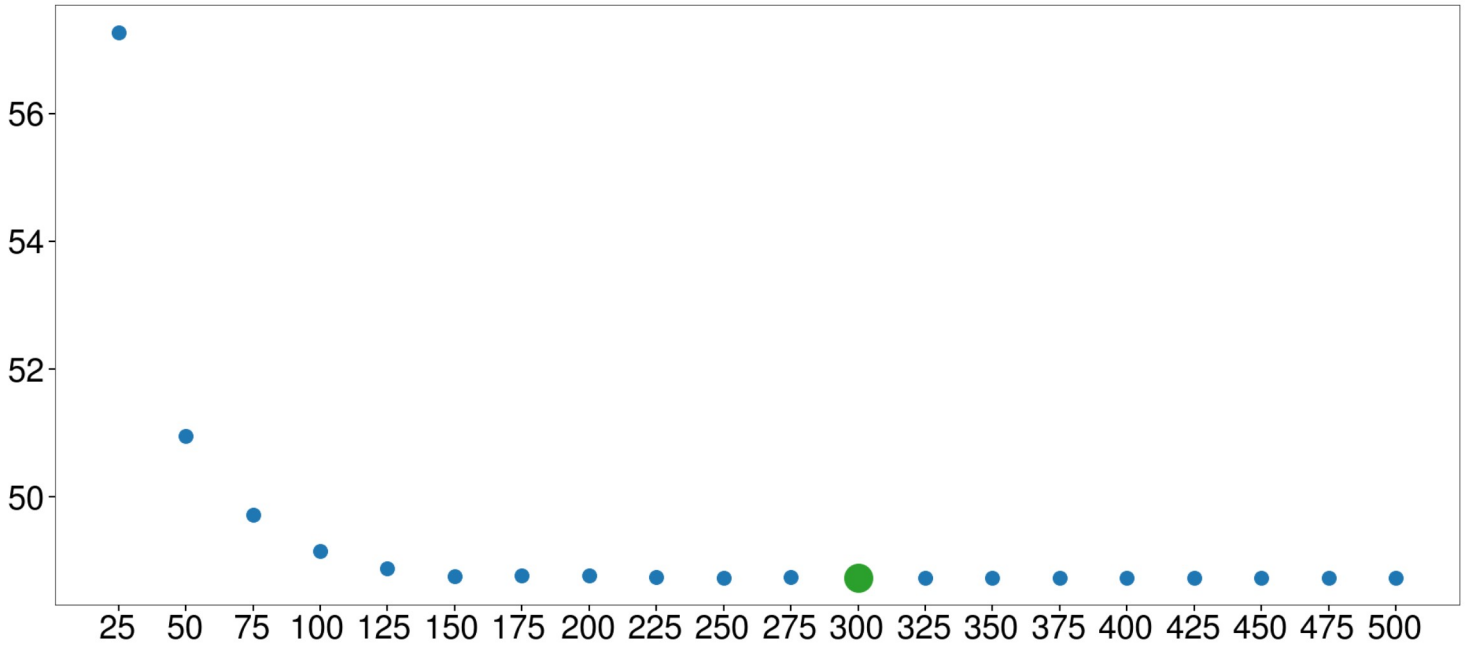
Random Forest cross-validation

In [35]:
```
rf_nodes_cross_val("sqrt")
```



Bagging cross-validation

In [36]:
```
rf_nodes_cross_val("auto")
```



Boosting

- Boosting fits a sequence of trees each with a small number d terminal nodes
- Each tree is fit to the residuals of the previous tree
    - Child trees focus on fitting observations that were hard to fit by previous trees
    - Nodes are not added for observations that have small prediction errors
    - Building a fresh tree collects all observations in to a single leaf
- Builds models with low-interaction terms
- In a nutshell: build B (large) depth d (small) trees sequentially
- One additional parameter λ (small) the slows learning

$$\hat{f}\ (\mathbf{x}) = \sum_{b=1}^{B} \lambda \hat{f}^{\,(b)}\ (\mathbf{x})$$

- $\hat{f}^{\,(b)}$ is the prediction of the residuals produced using the previous $b-1$ trees

$$\hat{\epsilon}_i^{\,(b)} = \sum_{i=1}^{b-1} Y_i - \lambda \hat{f}^{\,(i)}\ (\mathbf{x})$$

- Three tuning parameters: B, d and λ

Cross-validating Boosted Trees

In [39]:
```
boosting_cv()
```