# Financial Econometrics
# HT Week 4 Assignment Answers

February 2021

## Exercise 7.2

**Derive explicit relationships between the parameters of an APARCH(1,1,1) and**

i. **ARCH(1)**
The APARCH is
$$\sigma_t^\delta = \omega + \alpha\left(|\varepsilon_{t-1}| + \gamma\varepsilon_{t-1}\right)^\delta + \beta\sigma_{t-1}^\delta$$
so when $\gamma = \beta = 0$ and $\delta = 2$ then
$$\sigma_t^2 = \omega + \alpha\left|\varepsilon_{t-1}\right|^2$$

ii. **GARCH(1,1)**
$\gamma = 0$ and $\delta = 2$, so that
$$\sigma_t^2 = \omega + \alpha\left|\varepsilon_{t-1}\right|^2 + \beta\sigma_{t-1}^2$$

iii. **AVGARCH(1,1)**
$$\sigma_t = \omega + \alpha\left|\varepsilon_{t-1}\right| + \beta\sigma_{t-1}$$

iv. **TARCH(1,1,1)**
$$\sigma_t^\delta = \omega + \alpha\left(|\varepsilon_{t-1}| + \gamma\varepsilon_{t-1}\right)^\delta + \beta\sigma_{t-1}^\delta$$
which is the same as
$$\sigma_t = \omega + \tilde{\alpha}\left|\varepsilon_{t-1}\right| + \tilde{\gamma}|\varepsilon_{t-1}| I_{[\varepsilon_{t-1}<0]} + \beta\sigma_{t-1}$$
with $\tilde{\alpha} = \alpha + \alpha\gamma$ and $\tilde{\alpha} + \tilde{\gamma} = \alpha - \alpha\gamma$.

v. **GJR-GARCH(1,1,1)**
$$\sigma_t^2 \quad = \quad \omega + \alpha\left(|\varepsilon_{t-1}| + \gamma\varepsilon_{t-1}\right)^2 + \beta\sigma_{t-1}^2$$
which is the same as
$$\sigma_t^2 = \omega + \tilde{\alpha}\varepsilon_{t-1}^2 + \tilde{\gamma}\varepsilon_{t-1}^2 I_{[\varepsilon_{t-1}<0]} + \beta\sigma_{t-1}^2$$

so that when $\tilde{\alpha} = \alpha + 2\gamma\alpha + \gamma^2$ and $\tilde{\gamma} = -4\gamma\alpha$ . To see this, when $\varepsilon_{t-1} > 0$ then the coefficient are $\alpha\left(1+\gamma^2\right)^2$ and when negative, the coefficients are $\alpha\left(\gamma-1\right)^2$. $\tilde{\alpha}$ is then the positive coefficient and $\tilde{\gamma} = \alpha\left(\gamma-1\right)^2 - \alpha\left(1+\gamma^2\right)^2$ is the difference.

## Exercise 7.5

**Let $r_t$ follow an ARCH process**

$$\begin{aligned} r_t &= \sigma_t e_t \\ \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2 \\ e_t &\overset{\text{i.i.d.}}{\sim} N(0,1) \end{aligned}$$

1. **What are the values of the following quantities?**

   (a) $\mathrm{E}[r_{t+1}] = \mathrm{E}[e_{t+1}\sigma_{t+1}] = \mathrm{E}[\mathrm{E}_t[e_{t+1}\sigma_{t+1}]] = \mathrm{E}[\mathrm{E}_t[e_{t+1}]\sigma_{t+1}] = \mathrm{E}[0\sigma_{t+1}] = 0$

   (b) $\mathrm{E}_t[r_{t+1}] = \mathrm{E}_t[e_{t+1}\sigma_{t+1}] = \mathrm{E}_t[e_{t+1}]\sigma_{t+1} = 0\sigma_{t+1} = 0$

   (c) $\mathrm{V}[r_{t+1}] = \mathrm{E}[r_{t+1}^2] = \mathrm{E}[e_{t+1}^2\sigma_{t+1}^2] = \mathrm{E}\left[\mathrm{E}_t\left[e_{t+1}^2\sigma_{t+1}^2\right]\right] = \mathrm{E}[1\times\sigma_{t+1}^2] = \bar{\sigma}^2 = \frac{\omega}{1-\alpha}$

   (d) $\mathrm{V}_t[r_{t+1}] = \mathrm{E}_t[e_{t+1}^2\sigma_{t+1}^2] = \mathrm{E}_t[e_{t+1}^2]\sigma_{t+1}^2 = 1\times\sigma_{t+1}^2 = \sigma_{t+1}^2 = \omega + \alpha_1 r_t^2$

   (e) $\rho_1 = \mathrm{Corr}\left[r_t, r_{t-1}\right]$

$$\begin{aligned} \rho_1 &= \frac{\mathrm{E}[(e_t\sigma_t)(e_{t-1}\sigma_{t-1})]}{\mathrm{V}[r_t]} \\ &= \frac{\mathrm{E}[\mathrm{E}_{t-1}[e_t e_{t-1}\sigma_t\sigma_{t-1}]]}{\mathrm{V}[r_t]} \\ &= \frac{\mathrm{E}[\mathrm{E}_{t-1}[e_t]e_{t-1}\sigma_t\sigma_{t-1}]}{\mathrm{V}[r_t]} \\ &= \frac{\mathrm{E}[0e_{t-1}\sigma_t\sigma_{t-1}]}{\mathrm{V}[r_t]} \\ &= 0 \end{aligned}$$

2. **What is $\mathrm{E}[(r_t^2-\bar{\sigma}^2)(r_{t-1}^2-\bar{\sigma}^2)]$ where $\bar{\sigma} = \mathrm{E}[\sigma_t^2]$. Hint: Think about the AR duality.**
   The ACF of an ARCH(1) can be derived by mapping it into an AR(1) by adding $\left(r_t^2 - \sigma_t^2\right)$ to both sides (or you can add and subtract $r_t^2$ from the left side and then move the term $-r_t^2+\sigma_t^2$ to the right-hand side).

$$\begin{aligned} \sigma_t^2 &= \omega + \alpha r_{t-1}^2 \\ \sigma_t^2 + \left(r_t^2 - \sigma_t^2\right) &= \omega + \alpha r_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \\ r_t^2 &= \omega + \alpha r_{t-1}^2 + \left(r_t^2 - \sigma_t^2\right) \end{aligned}$$

From here we can apply the formula in the time-series notes to get the first autocovariance, which is

$$\alpha\frac{\mathrm{V}\left[\nu_t\right]}{1-\alpha^2}$$

1. **Describe the $h$-step ahead forecast from this model.**

$$\begin{aligned} E_t[\sigma^2_{t+1}] &= E_t[\omega + \alpha r_t^2] \\ &= \omega + \alpha r_t^2 \end{aligned}$$

$$\begin{aligned} E_t[\sigma^2_{t+2}] &= E_t[\omega + \alpha r^2_{t+1}] \\ &= \omega + \alpha E_t[r^2_{t+1}] \\ &= \omega + \alpha E_t[e^2_{t+1}\sigma^2_{t+1}] \\ &= \omega + \alpha E_t[e^2_{t+1}]\sigma^2_{t+1} \\ &= \omega + \alpha \sigma^2_{t+1} \end{aligned}$$

and substituting $\sigma^2_{t+1}$, which is known at time $t$, will produce

$$\begin{aligned} E_t[\sigma^2_{t+2}] &= \omega + \alpha\left(\omega + \alpha r_t^2\right) \\ &= \omega + \alpha\omega + \alpha^2 r_t^2 \end{aligned}$$

Finally note that $E_t[\sigma^2_{t+3}] = \omega + \alpha E_t[r^2_{t+2}]$, and so

$$E_t[\sigma^2_{t+3}] = \omega + \alpha\omega + \alpha^2\omega + \alpha^3 r_t^3$$

and the pattern emerges,

$$E_t[\sigma^2_{t+h}] = \sum_{i=0}^{h-1} \alpha^i \omega + \alpha^h r_t^2$$

The $h$-step ahead forecast is an exponentially declining function of the time $t+1$ forecast plus a constant. For large $h$, the forecast converges to $\bar{\sigma}^2$.