Financial Econometrics
HT Week 7 Assignment Answers

March 2021

Exercise 8.8

A Value-at-Risk model was fit to some return data and the series of 5% VaR violations was computed. Denote these $\widetilde{HIT}_t$. The total number of observations was $T = 50$, and the total number of violations was 4.

1. **Test the null that the model has unconditionally correct coverage using a t-test.**
   Since there were 4 out of 50 violations, $\hat{p} = 4/50 = 8\%$. This is just the mean, so that we can estimate the variance using

$$50^{-1}\sum_{i=1}^{50}(x_i-\bar{x})^2 = 50^{-1}\sum_{i=1}^{50}x_i^2 - 2x_i\bar{x}+\bar{x}^2$$
$$= 50^{-1}\sum_{i=1}^{50}x_i^2 - 50\bar{x}^2$$

   The components of this sum as $50^{-1}\sum_{i=1}^{50}x_i^2 = 50^{-1}\sum_{i=1}^{50}x_i = \hat{p}$ since $x \in \{0,1\}$and $50^{-1}(50)\bar{x}^2 = \hat{p}^2$, and so the variance estimator is $\hat{p}(1-\hat{p})$. Finally, the variance of the mean is $(n-1)^{-1}(p(1-p))$ which can be estimated using $(n-1)^{-1}\hat{p}(1-\hat{p})$ and so

$$\frac{\hat{p}-0.05}{\sqrt{(n-1)^{-1}(\hat{p}(1-\hat{p}))}} = \frac{.03}{0.0388} = 0.7741$$

   The 2-sided critical value from a normal for a 5% test is ± 1.96 and so the null cannot be rejected.

2. **Test the null that the model has unconditionally correct coverage using a LR test. The likelihood for a Bernoulli(p) random Y is**

$$f(y;p) = p^y(1-p)^{1-y}.$$

   The log likelihood **is**

$$\sum_{i=1}^{50} y_i \ln p + (1-y_i)\ln(1-p)$$

and so the MLE can be found by differentiating

$$\begin{aligned}
\frac{\partial}{\partial p} &= \sum_{i=1}^{50} \frac{y_i}{\hat{p}} - \frac{1-y_i}{1-\hat{p}} = 0 \\
0 &= \sum_{i=1}^{50} (1-\hat{p})\, y_i - (1-y_i)\, \hat{p} \\
0 &= \sum_{i=1}^{50} y_i - y_i \hat{p} - \hat{p} + y_i \hat{p} \\
n\hat{p} &= \sum_{i=1}^{50} y_i \\
\hat{p} &= \bar{y}
\end{aligned}$$

Using this I can now compute the likelihood ratio against the null model where $p = 5\%$,

$$2 \times \left\{ \sum_{i=1}^{50} y_i \ln .08 + (1-y_i) \ln .92 - \sum_{j=1}^{50} y_j \ln .05 + (1-y_j) \ln .95 \right\}$$
$$= 2 \times \{4 \ln .08 + 46 \ln .92 - 4 \ln .05 - 46 \ln .95\}$$
$$= 2 \times \{4 (\ln .08 - \ln .05) + 46 (\ln .92 - \ln .95)\} \quad = \quad 0.807$$

There is one restriction and so the distribution is a χ^2_1 and the critical value is 3.84, and so we fail to reject the null.

The following regression was estimated

$$\widetilde{HIT}_t = 0.0205 + 0.7081 \widetilde{HIT}_{t-1} + \hat{\eta}_t$$

The estimated asymptotic covariance of the parameters is

$$\hat{\sigma}^2 \hat{\Sigma}_{XX}^{-1} = \begin{bmatrix} 0.0350 & -0.0350 \\ -0.0350 & 0.5001 \end{bmatrix}, \textbf{ and } \hat{\Sigma}_{XX}^{-1} \hat{\mathbf{S}} \hat{\Sigma}_{XX}^{-1} = \begin{bmatrix} 0.0216 & -0.0216 \\ -0.0216 & 2.8466 \end{bmatrix}$$

where $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \hat{\eta}_t^2$**,** $\hat{\Sigma}_{XX} = \frac{1}{T} \mathbf{X}'\mathbf{X}$ **and** $\hat{\mathbf{S}} = \frac{1}{T} \sum_{t=1}^{T} \hat{\eta}_t^2 \mathbf{x}_t' \mathbf{x}_t$**.**

3. **Is there evidence that the model is dynamically mis-specified, ignoring the unconditional rate of violations?**

   This can be tested using a basis t-test on the lagged term. Since the data are Bernoulli, I need to use the heteroskedasticity robust standard errors. The null is tha the coefficient is 0.

$$\frac{.7081 - 0}{\sqrt{2.8466/50}} = 2.96$$

Using a 5% test, the critical value is ± 1.96 and so the null is rejected.

4. **Compute a joint test that the model is completely correctly specified. Note that**

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}.$$

We can use a Wald test of the form

$$T\left(\hat{\theta}-\theta_0\right)'\hat{\Sigma}^{-1}\left(\hat{\theta}-\theta_0\right)$$

to test 2 restrictions, $H_0 : \theta_0 = 0.05$, $\hat{\theta}_1 = 0$. Using the inverse formula above,

$$\hat{\Sigma}^{-1} = \begin{bmatrix} 46.6 & 0.353 \\ 0.353 & 0.353 \end{bmatrix}$$

so that the value of the Wald statistic is 10.1648. This test has 2 restrictions, and so the critical value is 5.99. In this case, we reject.

Note: The 5% critical values of a χ^2_ν are

ν	**CV**
1	**3.84**
2	**5.99**
3	**7.81**
47	**64.0**
48	**65.1**
49	**66.3**
50	**67.5**