Numerical Methods for Optimization and Control Theory

Daniel Kuknyo - Assignment 2

Assigned tasks: 1, 5, 10, 12

10. Implement a function for fitting a three dimensional paraboloid $z(x,y) = a \cdot x^2 + b \cdot xy + c \cdot y^2 + dx + ey + f$ to a sample of m points $(x_i, y_i, z_i) \in \mathbb{R}^3$, $(i = 1, \ldots, m)$ using linear least squares. Your method should find and return the optimal parameter values for a, b, c, d, e and f. You may use our SVD-based linear least squares solver (`linear_LSq.m`) after contstructing the coefficient matrix and the vector of expected values. Create a 3D plot of the input points and the returned paraboloid.

Generating the space

The points are sampled from a normal distribution in order to have a nicer look.

```
clear all; clc;

% Linear Space boundaries
x_min =  1;
x_max = -4;
y_min =  2;
y_max = -2;
z_min =  0;
z_max = 50;
bounds = [x_min x_max y_min y_max z_min z_max];

% Genearate the linear space
res = 1000;
X = linspace(x_min, x_max, res);
Y = linspace(y_min, y_max, res);
space = meshgrid(X,Y);
[X_m, Y_m] = meshgrid(X,Y);
```

Creating the expected values

```
% Number of points to generate
m = 1000;
f_params = [10;20;30;30;40;30]; % Parameters of the generative function

% Function for the equation of a parabola
f = @(x, y, a, b, c, d, e, f) a*x^2 + b*x*y + c*y^2 + d*x + e*y + f
```

```
f = function_handle with value:
    @(x,y,a,b,c,d,e,f)a*x^2+b*x*y+c*y^2+d*x+e*y+f
```

```
% Vector of expected values
y = generate_points(m, f, f_params, X_m, Y_m, res)
```

```
y = 3×1000
   -0.6316    0.9600   -0.5716   -0.7417   -2.0280   -3.2993   -3.3694   -2.5986 ···
    2.0000    1.9960    1.9960    1.9920    1.9880    1.9880    1.9880    1.9840
  202.0033  306.4473  167.9702  188.0764  109.8795   90.4701  124.5642   96.4404
```

Assemble the coefficient matrix

```
% Assemble J matrix
J = [];
for i = 1:m
    nx = y(1, i);
    ny = y(2, i);
    vec = [nx^2 nx*ny ny^2 nx ny 1];
    J = [J; vec];
end
```

Find the optimal parabola coefficients

The original parameters of the generator function and the estimated parameters are also visible

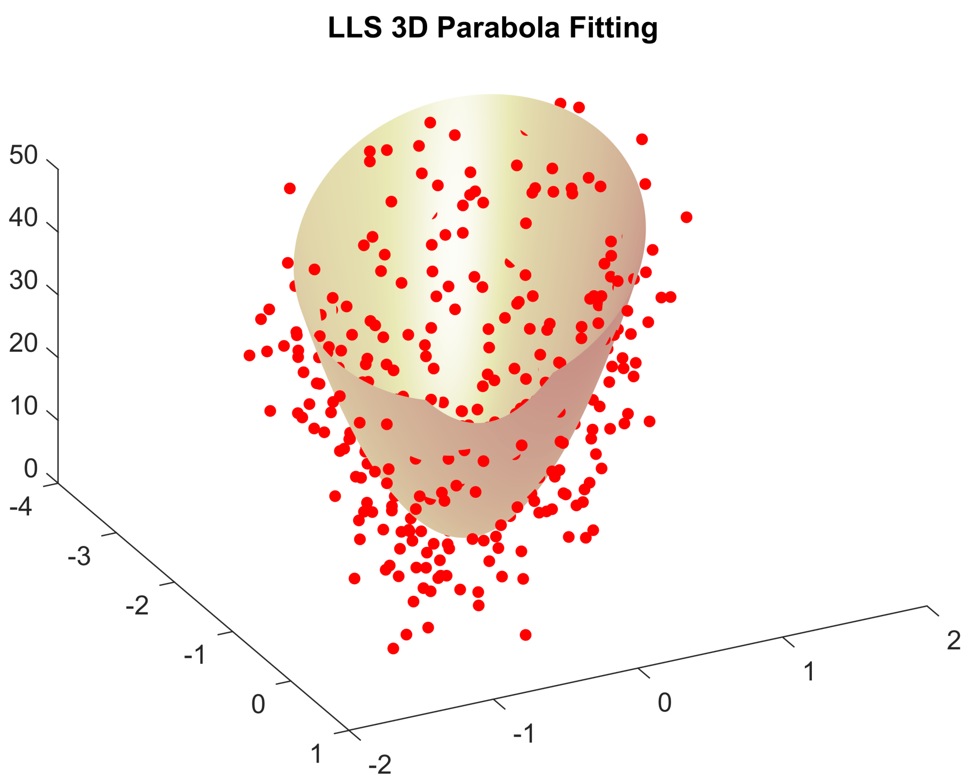
```
n_coeffs = size(J,2);
y_next = y(3,:)';

% Estimate the parameters
x = linear_svd(J, y_next, n_coeffs);

% Display the true/estimated params
[f_params'; x']
```

```
ans = 2×6
   10.0000   20.0000   30.0000   30.0000   40.0000   30.0000
   10.4012   20.0351   29.2054   31.9295   39.9882   32.8757
```

Plot the results

```
% Create the points for the parabola
Z = zeros(res,res);
for i = 1:res
    for j = 1:res
        Z(i,j) = f(X_m(i,j), Y_m(i,j), x(1), x(2), x(3), x(4), x(5), x(6));
    end
end

figure;
hold on;

% Plot the parabola
surfl(X_m,Y_m,Z);

% Plot the points
for i = 1:size(y,2)
```

```
    plot3(y(1,i), y(2,i), y(3,i), '.', ...
        'MarkerSize', 15, ...
        'MarkerEdgeColor', 'red', ...
        'MarkerFaceColor', [1 .6 .6]);
end

view([63.30 41.40])
zlim([z_min z_max]) % Remove this one to see the whole function
title("LLS 3D Parabola Fitting");
colormap("pink");
shading interp;
hold off;
```



Functions

```
function pts = generate_points(m, f, x, X_m, Y_m, res)
    % Create the Z coordinates according to f function
    Z = zeros(res,res);
    for i = 1:res
        for j = 1:res
            Z(i,j) = f(X_m(i,j), Y_m(i,j), x(1), x(2), x(3), x(4), x(5), x(6));
        end
    end

    % Add deviance
    noise_lvl = 15;
```

```
    Z = Z + noise_lvl * randn(1,res);

    % Flatten and assemble into 3*res^2 matrix
    x_coords = reshape(X_m.', 1, []);
    y_coords = reshape(Y_m.', 1, []);
    z_coords = reshape(Z.', 1, []);

    % Assemble into a single 3*m matrix
    coords = [x_coords; y_coords; z_coords];

    % Subsample the matrix to contain m points
    all_inds = 1:size(coords, 2);
    indices = sort(randsample(all_inds, min(m, size(coords,2))));
    pts = coords(:, indices);
end

function x = linear_svd(J, y, n)
    % Compute the SVD:
    [U,S,V] = svd(J);
    s = diag(S);

    % Determine the effective rank r of A using singular values
    r = 1;
    while(r < size(J,2) & s(r+1) >= max(size(J)) * eps * s(1))
        r = r + 1;
    end

    d = U' * y;
    x = V * ([d(1:r)./s(1:r); zeros(n-r,1)]);
end
```