Propositional Logic

Chapter · September 2017

1 author:

Bhowmik Subrata
Tripura University

Propositional Logic

Subrata Bhowmik
Dept. of Mathematics, Tripura University
Suryamaninagar, Tripura, India-799022
bhowmik_s@tripurauniv.in

1 Introduction:

The term 'logic' came from GREEK word 'logos', which is sometimes translated as 'sentence', 'discourse', 'reason', 'rule' and 'ratio'. The GREEK word 'logos' means thought. There are many thought processes such as 'reasoning', 'remembering', 'imagining'. Reasoning is a thought process in which inference takes place. Logic is the science of reasoning.

Briefly speaking we might define logic as the study of principles of correct reasoning. Logic is the former discipline, and it tells us how we ought to reason if we want to reason correctly. Logic is the science of how to evaluate arguments and reasoning. Critical thinking is a process of evaluation which uses logic to separate truth from falsehood, reasonable from unreasonable beliefs.

2 A Brief History of logic from ARISTOTLE to GEORGE BOOLE:

In Europe, Logic was first developed by Aristotle(384-322 BCE). Aristotelian logic became widely accepted in science and mathematics and remained in wide use in the west until the early 19th century. Aristotles system of logic was responsible for the introduction hypothetical syllogism, temporal modal logic, and the inductive logic.

In the history of logic, classical logic and symbolic logic are two important stage of development. Classical logic is also called the Aristotelian logic or Ancient logic, in contrast to symbolic logic or mathematical logic or modern logic.

According to Bassoon and O'conner [1], "modern symbolic logic is a development of the concepts and techniques which were implicit in the work of Aristotle". It is now generally agreed by logician that modern symbolic logic is a development of the concepts and techniques which were implicit in the work of Aristotle.

The first important name in the development of modern symbolic logic is that of G.W Leibnitz (1646-1716). Leibnitz put forward a two-fold plan for the reform of logic. He suggested first the establishment of a universal scientific language in which all the scientific concepts could be represented by a combination of basic ideograms. He suggested that a universal calculus of reasoning could be devised which would provide an automatic method of solution for all problems which could be expressed in the universal language. Had he carried out of his proposal, he would have provided a system of symbolic logic.

The next important name in the development of symbolic logic is that of George Boole (1815-1864). His contribution consisted in the formulation of a system of algebra which was first set out in the book The Mathematical Analysis of Logic, and in a subsequent work The Laws of Thought. Boole applied his algebra to several branches of logic including the syllogism of the classical logic. This was an important advance in that he showed that the doctrine of the Aristotelian syllogism which had hitherto been regarded as practically co-extensive with deductive logic could be shown to be a special case of a kind of logical algebra.

Other important 19th century logicians who contributed to the development of symbolic logic included Augustus de Morgan (1806-1871), W.S Jevons (1835-1882) and C.S Peirce (1839-1914).

3 Syntax and Semantics of Propositional Calculus:

Propositional logic is the most basic branch of mathematical logic. The area of logic that deals with propositions is called Propositional logic. It is also called propositional calculus(PC). In Latin, calculus means a stone used in counting. In PC, the truth or falsity of a "proposition" can be "counted" determined using "truth table".

Definition3.1: A declarative sentence ia a sentence which is true or false, but not both, is called a Proposition (or statement). Statements which are exclamatory, interrogative or imperative in nature are not propositions.

For example we consider the following sentences

(i) New Delhi is the capital of India.

(ii) How beautiful is Rose?.

(iii) 2+2=3.

(iv) What time is it?

(v) Take a cup of coffee.

In these sentences (ii), (iv) and (v) are obviously not propositions as they are not declarative in nature. Whereas (i) and (iii) are propositions.

If a proposition is true, we say that the truth value of the proposition is TRUE, denoted by T or 1. If a proposition is FALSE, the truth value is said to be false, denoted by F or 0.

Propositions which do not contain any of the logical operators or connectives (not, and, or, if-then, if-and-only-if) are called atomic propositions (or primitive/simple sentence). Many mathematical statement which can be constructed by combining one or more atomic statements using connectives are called molecular or compound propositions (or compound sentences or simply sentences).

The truth value of a compound proposition depends on those of sub propositions and the way in which they are combined using connectives.

Definition3.2: The language $\mathcal{L}$ of a propositional logic consisting of:

(1) A countable set P of simple sentences.

(2) A set $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$ of connectives. $\neg$ (not: negation) is a unary connective; the connectives $\vee$ (or: disjunction), $\wedge$ (and: conjunction), $\rightarrow$ (if-then: conditional), $\leftrightarrow$ (if-and-only-if: biconditional) are binary connectives.

We define a set W of Well Form Formulas(wffs) or simply formulas or sentences formed from P with the use of connectives with the following rule:

(i) $P \subset W$,

(ii) If $\alpha, \beta \in W$ then $\neg\alpha, \alpha \vee \beta, \alpha \wedge \beta, \alpha \rightarrow \beta, \alpha \leftrightarrow \beta \in W$,

(ii) The members of W are only defined by the above rules (i) and (ii).

Truth Value Evaluation3.3: All our propositions are either true or false, that is taking only one truth value not simultaneously both. So if a proposition is not true (denoted by T or 1) then it must be of opposite truth value that is false (denoted by F or 0), similarly if it is not false then it must be of opposite truth value that is T or 1.

From the truth value of the compound statement with connectives we need to understand the meaning of connectives and evaluate the truth value from the nature of the connectives and the component(s) of the sentence adjoint by the connectives.

(i) **Negation($\neg$):** It is an unary connective, that is it associated with only one statement. It is in the sense of "NOT" (not happening of the statement). The negation of a statement α is written as $\neg\alpha$ (NOT "α"). The truth evaluation of negation of statement is opposite to the statement, thus the negation of a statement α is true iff the statement α itself false.

Consider the example: α="City X is capital of country Y." So $\neg\alpha$="City X in NOT capital of country Y". Naturally if α is true then obviously $\neg\alpha$ is false and α is false then obviously $\neg\alpha$ is true.

The truth valuation of negation of a statement can be expressed by the following truth table:

α	$\neg\alpha$
1	0
0	1

(ii) **Disjunction($\vee$):** It is a binary connective, that is it adjoin with two statements. The nature of this connective is in the sense of "OR" (happening of at least any one of the statements). The disjunction of two statement α and β is written as $\alpha \vee \beta$ ("α OR "β"). The truth evaluation is true if we find at least any one of the adjoint statement is true, else false.

Consider the example: α="SIMRAN is dancing" and β="MITA is singing". Thus $\alpha \vee \beta$="SIMRAN is dancing" OR "MITA is singing". This compound statement is considered to be true if at least any one of the event occur or both.

See the following truth table evaluation:

α	β	$\alpha \vee \beta$
1	1	1
1	0	1
0	1	1
0	0	0

(iii) **Conjunction($\wedge$):** It is a binary connective, that is it adjoin with two statements. The nature of this connective is in the sense of "AND" (happening of both of the statements). The conjunction of two statement α and β is written as $\alpha \wedge \beta$ ("α AND "β"). The truth evaluation is true if we find both of the adjoint statement is true, else false.

Consider the example: α="SIMRAN is dancing" and β="MITA is singing". Thus $\alpha \wedge \beta$="SIMRAN is dancing" AND "MITA is singing". This compound statement is considered to be true if both of the event occur. If any one or both do not happened the conjunction should be considered false.

See the following truth table evaluation:

α	β	$\alpha \wedge \beta$
1	1	1
1	0	0
0	1	0
0	0	0

(iv) **Conditional($\rightarrow$):** It is a binary connective, that is it adjoin with two statements. The conditional statement for two statement α and β is written as $\alpha \rightarrow \beta$ (If "α" THEN "β"), where α is called antecedent and β is called consequent. Intuitively we mean $\alpha \rightarrow \beta$ as β is deducible from α.

Consider the statement α="Apply Shreedharacharya formula to quadratic equation $f(x) = 0$" and β="Solution of quadratic equation $f(x) = 0$".

(a) If α is true then we have applied Shredharacharya formula to the equation $f(x) = 0$ thus if β is true then we have a solution of the equation $f(x) = 0$ which is true, thus the statement $\alpha \rightarrow \beta$ is true.

(b) If α is true then we have applied Shredharacharya formula to the equation $f(x) = 0$ thus if β is false then we have a wrong solution of the equation $f(x) = 0$ which is not possible, thus the statement $\alpha \to \beta$ becomes false.

(c) Again if α is false then we have not applied Shredharacharya formula to solve the equation $f(x) = 0$, may be some other method applied (correct or false method) or may not applied any method to solve $f(x) = 0$, then we may get solution of $f(x) = 0$ (correct or false) or may not get any solution accordingly, that is whether β is true or false the conditional statement $\alpha \to \beta$ have to be taken true.

Consider another example: α=”KIRAN walk through the main way of University”, β=”KIRAN get the University”.

(a) If α is true then KIRAN follow the right path to the university. Thus if β is true then we must accept $\alpha \to \beta$ is true.

(b) If α is true but is β is false, then though he follow correct path, he did not get the university which we cannot accept and so we have to take the conditional statement $\alpha \to \beta$ is false.

(c) If α is false, then KIRAN did not follow the main way, he may follow another way that may lead to university or not or may did not follow any way; so accordingly he will get the university or not, that is β may be true or false, and so we have to take the conditional statement $\alpha \to \beta$ is true for any truth value of β.

The truth value evaluation of conditional statement can be demonstrated in the following truth table:

α	β	$\alpha \to \beta$
1	1	1
1	0	0
0	1	1
0	0	1

(v) **biconditional($\leftrightarrow$):** It is a binary connective, that is it adjoin with two statements. The biconditional statement for two statement α and β is written as $\alpha \leftrightarrow \beta$ (”α” IF ONLY IF ”β”). The nature of this statement is that α is happening whenever β is happening and conversely, in other words ”if α then β and if β then α”. The truth value of this statement is consider to be true if both agree that is both have the same truth value.

Consider the example: For triangles consider the statements in Euclidean plane geometry, α=”All sides are equal” and β=”All angle are equal”. We already know that if α is true then also β and conversely; in other word if α is false then β is false and conversely. Thus the statement ”α IF AND ONLY IF β” that $\alpha \leftrightarrow \beta$ is true whenever α and β are simultaneously true or false.

The truth evaluation of conditional statement can be demonstrated in the following truth table:

α	β	$\alpha \leftrightarrow \beta$
1	1	1
1	0	0
0	1	0
0	0	1

Now we define the truth evaluation of a sentence from the truth assignment of the simple sentences.

Definition3.4: A truth assignment or valuation or model of the language $\mathcal{L}$ of propositional calculus is a mapping:

$$v : P \to \{0, 1\}$$

.

Now we define the truth value of any sentence of W to be the extension of the mapping v to

$$v : W \to \{0, 1\}$$

as inductively in the following way(see the truth tables constructed in 3.3):
(i) $v(\neg\alpha) = 1$ iff $v(\alpha) = 0$.
(ii) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$.
(iii) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$.
(iv) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.
(v) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1 = v(\beta)$ or $v(\alpha) = 0 = v(\beta)$.

Let V be the set of all valuation on $\mathcal{L}$.

Definition3.5: $\alpha \in W$ is said to be a:
(i) Tautology if for any model v, $v(\alpha) = 1$. We write $\models \alpha$. If α is not a tautology we will write $\not\models \alpha$.
(ii) Contradiction if for any model v, $v(\alpha) = 0$, or in other word $\neg\alpha$ is a tautology (i.e. $\models \neg\alpha$).

Example3.6: Tautology: (i) $\alpha \vee \neg\alpha$ (law of excluded middle). (ii) $\alpha \to \alpha \vee \beta$. (iii) $\alpha \wedge \beta \to \beta$.
Contradiction: (i) $\alpha \wedge \neg\alpha$, (ii) $\neg(\alpha \to \alpha)$.

Definition3.7: Two statements α and β are said to be contradictory to each other if $\alpha \wedge \beta$ is a contradiction.

Definition3.8: Let $\alpha, \beta \in W$, we say α and β are logically equivalent if for any model v, $v(\alpha) = v(\beta)$. We write $\alpha \equiv \beta$.

Example3.9: (i) $\alpha \vee \neg\alpha \equiv \beta \vee \neg\beta$, $\alpha \wedge \neg\alpha \equiv \beta \wedge \neg\beta$,
(ii) $\alpha \vee \beta \equiv \beta \vee \alpha$, $\alpha \wedge \beta \equiv \beta \wedge \alpha$ (commutative)
(ii) $\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$, $\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma$ (associative)
(iii) $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$, $\alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$.
(iv) $\alpha \to \beta \equiv \neg\alpha \vee \beta$. (v) $\alpha \to \beta \equiv \neg\beta \to \neg\alpha$.
(vi) $\alpha \leftrightarrow \beta \equiv (\alpha \to \beta) \wedge (\beta \leftrightarrow \alpha)$. (vii) $\alpha \equiv \neg\neg\alpha$.
(viii) $\alpha \vee (\alpha \wedge \beta) \equiv \alpha$, $\alpha \wedge (\alpha \vee \beta) \equiv \alpha$ (Absorption).

Theorem3.10:
(i) $\alpha \equiv \beta$ iff $\models \alpha \leftrightarrow \beta$ iff $\models \alpha \to \beta$ and $\models \beta \to \alpha$.
(ii) If $\alpha \equiv \beta$ and $\models \alpha$ then $\models \beta$.
(iii) If $\models \alpha$ and $\models \alpha \to \beta$ then $\models \beta$.
(iv) If $\alpha \equiv \beta$ then $\neg\alpha \equiv \neg\beta$.
(v) If $\alpha_1 \equiv \alpha_2$ and $\beta_1 \equiv \beta_2$ then $\alpha_1 \vee \beta_1 \equiv \alpha_2 \vee \beta_2$ and $\alpha_1 \wedge \beta_1 \equiv \alpha_2 \wedge \beta_2$. Also $\alpha_1 \to \beta_1 \equiv \alpha_2 \to \beta_2$.

Theorem3.11: The relation $\equiv$ on W is an equivalence relation on W.
Let $< \alpha >$ be the equivalence class containing $\alpha \in W$ and $W_{/\equiv}$ be the collection of all equivalence classes.

Define the binary operations $\sqcup$; $\sqcap$ and an unary operation $\sim$ on $W_{/\equiv}$ as $<\alpha> \sqcup <\beta> = <\alpha \vee \beta>$, $<\alpha> \sqcap <\beta> = <\alpha \wedge \beta>$ and $\sim <\alpha> = <\neg\alpha>$. Then these operations are well defined and $(W_{/\equiv}, \sqcup, \sqcap, \sim, \perp, \top)$ forms a Boolean algebra, where $\perp = <\alpha \wedge \neg\alpha>$ the class of contradictions and $\top = <\alpha \vee \neg\alpha>$ the class of tautologies. (Exercise)

Exercise3.12: Consider the set of simple sentences $P = \{p, q, r\}$. Check weather the following pair of formulas are equivalent or not.

(i) $(p \to q) \to p$ and p.

(ii) $\neg p \vee q$ and $\neg q \vee p$.

(iii) $p \vee (q \leftrightarrow r)$ and $(p \vee q) \leftrightarrow (p \vee r)$.

(iv) $p \to (q \to r)$ and $(p \to q) \to (p \to r)$.

Definition3.13:(Adequate Set of Connectives) A subset C of the set of connectives $\{\neg, \vee, \wedge, \to, \leftrightarrow\}$ is said to be a adequate set of connectives if any formula is logically equivalent to a formula having only connectives from the set C.

Theorem3.14: Show that (i) $\{\neg, \vee\}$, (ii) $\{\neg, \to\}$, (iii) $\{\neg, \wedge\}$ are adequate set of connectives.

Proof: (i) and (iii) are exercise.

(ii) Let v be any model.

Claim I: $\alpha \vee \beta \equiv \neg\alpha \to \beta$.

Verification: $v(\neg\alpha \to \beta) = 1$ iff $v(\neg\alpha) = 0$ or $v(\beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$ iff $v(\alpha \vee \beta) = 1$. Thus Claim I verified.

Claim II: $\alpha \wedge \beta \equiv \neg(\alpha \to \neg\beta)$.

Verification: $v(\neg(\alpha \to \neg\beta)) = 1$ iff $v(\alpha \to \neg\beta) = 0$ iff $v(\alpha) = 1$ and $v(\neg\beta) = 0$ iff $v(\alpha) = 1$ and $v(\beta) = 1$ iff $v(\alpha \wedge \beta) = 1$. Thus Claim II verified.

Claim III: $\alpha \leftrightarrow \beta \equiv (\alpha \to \beta) \wedge (\beta \to \beta)$.

Verification: $v((\alpha \to \beta) \wedge (\beta \to \alpha)) = 1$ iff $v(\alpha \to \beta) = 1$ and $v(\beta \to \alpha) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$ and $v(\beta) = 0$ or $v(\alpha) = 1$ iff $v(\alpha) = 0 = v(\beta)$ or $v(\alpha) = 1 = v(\beta)$ iff $v(\alpha \leftrightarrow \beta) = 1$. Thus Claim III verified.

Hence by verifying Claim I, II and III we conclude that $\{\neg, \to\}$ is a adequate set of connectives. And this set is minimal as further reduction is not possible due to only one unary and only one binary connectives, further reduction will loss all the binary connectives or the unary connective.

Note3.15: Let $F \subset W$ and $v \in V$. If for any $\alpha \in F$, $v(\alpha) = 1$ we write $v \models F$ and we say v is a model of F. If $F = \{\beta\}$ we can write $v \models \beta$.

Definition3.16: Let $F \subseteq W$ and $\beta \in W$. We say F logically infer β if for any model v, if $v \models F$ then $v \models \beta$ (i.e. $v(\beta) = 1$) or in other words if for any model v if $v(\beta) = 0$ then $\exists\ \alpha \in F$ such that $v(\alpha) = 0$).

If $F = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, we write $\alpha_1, \alpha_2, ..., \alpha_n \models \beta$.

If F does not logically infer β we will write $F \not\models \beta$.

Theorem3.17: $\emptyset \models \beta$ iff $\models \beta$ (i.e. β is a tautology).

Theorem3.18:

(i) $\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$.

(ii) $\alpha_1, \alpha_2, ..., \alpha_n \models \beta$ iff $\alpha_1 \wedge \alpha_2 \wedge ...... \wedge \alpha_n \models \beta$.

(iii)For any $\alpha, \beta \in W$, $\alpha, \neg\alpha \models \beta$ (equivalently $\alpha \wedge \neg\alpha \models \beta$).

(iv) $\alpha, \alpha \to \beta \models \beta$.
(v) $\alpha \models \beta$ iff $\models \alpha \to \beta$.
(vi) $\alpha_1, \alpha_2, ..., \alpha_n \models \beta$ iff $\alpha_1, ..., \alpha_{n-1} \models \alpha_n \to \beta$ iff $\models \alpha_1 \to (\alpha_2 \to (....(\alpha_n \to \beta).....))$.

Theorem3.19: $\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$.

Theorem3.20: The relation $\preceq$ on W defined as $\alpha \preceq \beta$ iff $\alpha \models \beta$ (equivalently $\models \alpha \to \beta$) is a partial ordering on W.

Define a relation $\ll$ on $W_{/\equiv}$ defined as $< \alpha > \ll < \beta >$ iff $\alpha \preceq \beta$, then $\ll$ is an well defined and is a partial ordering on $W_{/\equiv}$. (Exercise)

Theorem3.21:
(i) $F \subseteq E \subset W$, $F \models \beta$ implies $E \models \beta$.
(ii) $F \models \alpha$ or $F \models \beta$ iff $F \models \alpha \vee \beta$.
(iii) If $F \models \alpha$ and $F \models \alpha \to \beta$ then $F \models \beta$.
(iv) $F \models \alpha$ and $F \models \alpha$ iff $F \models \alpha \wedge \beta$.
(v) If $F \models \alpha \to \beta$ and $F \models \beta \to \gamma$ then $F \models \alpha \to \gamma$ (Thus If $\models \alpha \to \beta$ and $\models \beta \to \gamma$ then $\models \alpha \to \gamma$).

4 Satisfiability:

Definition4.1: Let $F \subseteq W$. F is said to be

(i) Satisfiable if there exists a truth evaluation v, such that for any $\alpha \in F$, $v(\alpha) = 1$, that is $v \models F$. Otherwise F is said to be not satisfiable.

A wff α is said to be satisfiable if the set $\{\alpha\}$ is satisfiable, i.e. there exists a model v such that $v(\alpha) = 1$.

(ii) Finitely satisfiable if every finite subset of F is satisfiable.

Example4.2: For any $\alpha \in W$, $\{\alpha, \neg\alpha\}$ is not a satisfiable set.

Theorem4.3: (i) F is not satisfiable iff (ii) for any sentence α, $F \models \alpha$ iff (iii) there exists a sentence β such that $F \models \alpha$ and $F \models \neg\beta$ both iff (iv) there exists a sentence β such that $F \models \beta \wedge \neg\beta$.

Proof:

(i) implies (ii): Let F is not satisfiable. So we can say for any model v and for any sentence α, if $v(\alpha) = 0$, there is some $\gamma \in F$ such that $v(\gamma) = 0$, so $F \models \alpha$ for any sentence α.

(ii) implies (iii): Follows from (ii) for some fixed sentence β.

(iii) implies (iv): Immediate.

(iv) implies (i): Due to (iv) there is no model v which satisfies F.

Theorem4.4: If F is finitely satisfiable and $\beta \notin F$, then $F \cup \{\beta\}$ or $F \cup \{\neg\beta\}$ is finitely satisfiable.

Proof: Let $F \cup \{\neg\beta\}$ is not finitely satisfiable. So $\exists$ a finite set $S \subseteq F$ such that $S \cup \{\neg\beta\}$ is not satisfiable. So whenever $v \models F$, $v(\neg\beta) = 0$, i.e. $v(\beta) = 1$, hence $S \models \beta$.......(i)

Let $K \subseteq F$ be finite. Now $S \subseteq S \cup K$, so $S \cup K \models \beta$ (from (i))........(ii)

Since both S and K are finite subset of F, so $S \cup K$ is also a finite subset of F, so $\exists$ a model u such that $u \models S \cup K$. So from (ii) $u(\beta) = 1$. Thus $u \models K \cup \{\beta\}$.

Thus $F \cup \{\beta\}$ is finitely satisfiable.

Exercise4.5: Consider the set $P = \{p, q, r\}$ of simple sentences. Check wheatear the following formulas are tautology, contradiction or satisfiable:

(i) $p \to ((q \wedge \neg r) \to \neg p)$.
(ii) $(p \leftrightarrow (\neg q \vee r)) \to (\neg p \to q)$.
(iii) $(p \to (q \vee r)) \vee (p \to q)$.
(iv) $(p \wedge q) \to (p \vee r)$.
(v) $((p \to q) \to p) \to p$.

Theorem4.6:
(i) $F \models \alpha$ iff $F \cup \{\neg\alpha\}$ is not satisfiable.
(ii) $F \cup \{\alpha\}$ is satisfiable iff $F \not\models \neg\alpha$.

Proof:
(i) Let $F \models \alpha$. If possible let $F \cup \{\neg\alpha\}$ is satisfiable. So there exists $v \models F \cup \{\neg\alpha\}$, i.e. $v \models F$ and $v(\neg\alpha) = 1$, i.e. $v(\alpha) = 0$ which is not possible. Hence $F \cup \{\neg\alpha\}$ is not satisfiable.

Conversely, suppose that $F \cup \{\neg\alpha\}$ is not satisfiable. If possible let $F \not\models \alpha$, so there exists $v \in V$ sauch that $v \models F$ and $v(\alpha) = 0$, i.e. $v(\neg\alpha) = 1$, i.e. $v \models F \cup \{\neg\alpha\}$, i.e. $F \cup \{\neg\alpha\}$ is satisfiable which is not possible. Hence $F \models \alpha$.

(ii) Let $F \cup \{\alpha\}$ is satisfiable. So there exists $v \models F \cup \{\alpha\}$, i.e. $v \models F$ and $(v\alpha) = 1$, i.e. $v(\neg\alpha) = 0$ and thus $F \not\models \neg\alpha$.

Conversely, suppose that $F \not\models \neg\alpha$. So there exists a model v such that $v \models F$ but $v(\neg\alpha) = 0$, i.e. $v(\alpha) = 1$, i.e. v satisfies $F \cup \{\alpha\}$. So $F \cup \{\alpha\}$ is satisfiable.

Corr4.7:
(i) $F \models \neg\alpha$ iff $F \cup \{\alpha\}$ is not satisfiable.
(ii) $F \cup \{\neg\alpha\}$ is satisfiable iff $F \not\models \alpha$.
(iii) $\models \alpha$ iff $\neg\alpha$ is not satisfiable.
(iv) α is satisfiable iff $\not\models \neg\alpha$.
(v) $\models \neg\alpha$ iff α is not satisfiable.
(vi) $\neg\alpha$ is satisfiable iff $\not\models \alpha$.

Definition4.8: A satisfiable (finitely satisfiable) set M is said to be maximal satisfiable (resp. maximal finitely satisfiable) set if M cannot enlarge to a larger satisfiable (resp. finitely satisfiable) set, i.e. if F is satisfiable (resp. finitely satisfiable) set and $M \subseteq F$ then $F = M$.

Theorem4.9: Let M be a maximal finitely satisfiable (mfs) set then:
(i) For any $\beta \in W$, either $\beta \in M$ or $\neg\beta \in M$.
(ii) If $\alpha, \alpha \to \beta \in M$ then $\beta \in M$.
(iii) $\alpha \vee \beta \in M$ iff $\alpha \in M$ or $\beta \in M$.
(iv) $\alpha \wedge \beta \in M$ iff $\alpha \in M$ and $\beta \in M$.

Proof: (i) M is a finitely satisfiable set.

Let $\beta \notin M$, so $M \cup \{\beta\}$ or $M \cup \{\neg\beta\}$ is finitely satisfiable.

If $M \cup \{\beta\}$ is not finitely satisfiable, then $M \cup \{\neg\beta\}$ is finitely satisfiable. But $M \subseteq M \cup \{\neg\beta\}$, by maximality of M, $\neg\beta \in M$.

So $\beta \in M$ or $\neg\beta \in M$.

If $\beta, \neg\beta \in M$, then $\{\beta, \neg\beta\}$ is a finite subset of M which is not satisfiable. A contradiction arose.

Hence, either $\beta \in M$ or $\neg\beta \in M$.

(ii) Let $\alpha, \alpha \to \beta \in M$ and if possible $\beta \notin M$. By (i) $\neg\beta \in M$. Thus $\{\alpha, \alpha \to \beta, \neg\beta\}$ finite subset of M which is not satisfiable, which contradicts the finite satisfiability of M.

Hence $\beta \in M$.

(iii) Let $\alpha \vee \beta \in M$. If possible let $\alpha \notin M$ and $\beta \notin M$. By (i) $\neg\alpha \in M$ and $\neg\beta \in M$. Thus $\{\alpha \vee \beta, \neg\alpha, \neg\beta\}$ is a finite subset of M which is not satisfiable which is not satisfiable, which contradicts the finite satisfiability of M.

Hence $\alpha \in M$ or $\beta \in M$.

(iv) Exercise.

Theorem4.10: Every finitely satisfiable set of sentences contained in a maximal finitely satisfiable set of sentences.

Proof: Let F is finitely satisfiable.

Let $\mathcal{P} = \{E \subset W : F \subseteq E$ and E finitely satisfiable$\}$.

Thus $F \in \mathcal{P}$ and $(\mathcal{P}, \subseteq)$ is a poset.

Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Let $G = \cup\mathcal{C}$.

Since for any $E \in \mathcal{C}$, $F \subseteq E$, so $F \subseteq G$.

Let $\{\beta_1, ..., \beta_n\}$ be a finite subset of G, so $\exists$, $E_1, ..., E_n \in \mathcal{C}$ such that $\beta_k \in E_k$. But $\mathcal{C}$ is a chain, so all $E_k's$ subset of one E_k for some $k = 1, ..., n$ say E_j, i.e. $\{\beta_1, ..., \beta_n\} \subseteq E_j$. Since E_j is finitely satisfiable, so $\{\beta_1, ..., \beta_n\}$ is satisfiable.

Thus any finite set of G is satisfiable, i.e. G is finitely satisfiable and $F \subseteq G$, i.e. $G \in \mathcal{P}$.

But for any $E \in \mathcal{C}$, $E \subseteq G$, i.e. G is an upper bound of $\mathcal{C}$.

Hence every chain in $\mathcal{P}$ has an upper bound, so by Zorn's lemma $\mathcal{P}$ has a maximal element say M (with $F \subseteq M$) which is a maximal finitely satisfiable set.

Theorem4.11:(Compactness Theorem of Propositional Calculus): F be a set of sentences. F is satisfiable iff F is finitely satisfiable.

Proof: If F is satisfiable then F obviously finitely satisfiable.

Conversely let F is finitely satisfiable. Then F contained in a maximal finitely satisfiable set M (say).

We define $v : W \to \{0, 1\}$ as $v(\alpha) = 1$ iff $\alpha \in M$.

Now we see that for any $\alpha, \beta \in W$

(i) $v(\neg\alpha) = 1$ iff $\neg\alpha \in M$ iff $\alpha \notin M$ iff $v(\alpha) = 0$.

(iii) $v(\alpha \vee \beta) = 1$ iff $\alpha \vee \beta \in M$ iff $\alpha \in M$ or $\beta \in M$ iff $v(\alpha) = 1$ or $v(\beta) = 1$.

Thus $v/_P$ defines a valuation (which is defined since for any simple sentence p either $p \in M$ or $\neg p \in M$, i.e. either $v(p) = 1$ or $v(p) = 0$).

Thus $v \models M$, hence $v \models F$. Thus F is satisfiable.

Theorem4.12: Let M be a maximal satisfiable (ms) set then,

(i) for any $\beta \in W$, $\beta \in M$ iff $M \models \beta$.

(ii) $\models \alpha$ implies $\alpha \in M$.

(iii) If $\alpha \equiv \beta$, then $\alpha \in M$ iff $\beta \in M$, (in other word $\models \alpha \leftrightarrow \beta$ implies $\alpha \in M$ iff $\beta \in M$).

(iv) If $\alpha \to \beta \in M$, then $\alpha \in M$ implies $\beta \in M$.

(v) If $\alpha \leftrightarrow \beta \in M$, then $\alpha \in M$ iff $\beta \in M$.

Proof: (i) Let $\beta \in M$, then obviously $M \models \beta$.

Conversely let $M \models \beta$. If possible let $\beta \notin M$. Since M is satisfiable, hence finitely satisfiable and must be maximal. So $\neg\beta \in M$.

Since M is satisfiable, $\exists$ a model v such that $v \models M$, so $v(\neg\beta) = 1$, i.e. $v(\beta) = 0$.

since $M \models \beta$, $v(\beta) = 1$, which is a contradiction.

Hence $\beta \in M$.

(ii)-(v) Exercise.

Theorem4.13: $F \models \beta$ iff for a finite $E \subseteq F$, $E \models \beta$

Proof: Let for as finite $E \subseteq F$, $E \models \beta$, which readily implies $F \models \beta$ by definition.

Conversely, let $F \models \beta$, but for any finite $E \subseteq F$, $E \not\models \beta$, so $E \cup \{\neg\beta\}$ is satisfiable. Thus $F \cup \{\neg\beta\}$ is finitely satisfiable.
By compactness theorem $F \cup \{\neg\beta\}$ is satisfiable, i.e. $F \not\models \beta$, which is not possible. Thus $\exists$ a finite $E \subseteq F$ such that $E \models \beta$.

Definition4.14: $F \subset W$ we define model of F to be the set $Mod(F) = \{v : v \models F\}$. For $F = \{\alpha\}$ we will write $Mod(F) = Mod(\alpha)$.

Definition4.15: $F \subset W$ we define the theory generated by F to be the set $Th(F) = \{\beta \in W : F \models \beta\}$. For $F = \{\alpha\}$ we will write $Th(F) = Th(\alpha)$ which is the set $\{\beta \in W : \alpha \models \beta\}$.

Exercise4.16: Let $E \subseteq F \subset W$. Show that
(i) $Mod(F) \subseteq Mod(E)$.
(ii) If $F \models \beta$ iff $Mod(F) \subseteq Mod(\beta)$.
(iii) $F \subseteq Th(F)$.
(v) $Th(E) \subseteq Th(F)$.
(vi) $Th(\emptyset)$ is the set of all tautologies.
(vii) $Th(\alpha) \cap Th(\beta) = Th(\alpha \wedge \beta)$, $Mod(\alpha) \cap Mod(\beta) = Mod(\alpha \wedge \beta)$.
(viii) $Th(\alpha) \cup Th(\beta) = Th(\alpha \vee \beta)$, $Mod(\alpha) \cup Mod(\beta) = Mod(\alpha \vee \beta)$.
(ix) $\models \alpha \to \beta$ iff $Th(\beta) \subseteq Th(\alpha)$ iff $Mod(\alpha) \subseteq Mod(\beta)$.
(x) $\models \alpha \leftrightarrow \beta$ iff $Th(\alpha) = Th(\beta)$ iff $Mod(\alpha) = Mod(\beta)$.
(xi) For $\emptyset \subseteq W$, $Th(\emptyset) = \{\alpha :\models \alpha\}$ i.e. the set of all tautologies.

Definition4.17: Let $F \subset W$. F is said to be propositional theory iff it is closed under tautological consequence, that is if $F \models \beta$ then $\beta \in F$.

Theorem4.18: For any $F \subset W$,
(i) $Th(F)$ is a propositional theory.
(ii) F is a propositional theory iff $F = Th(F)$.
(iii) $Th(Th(F)) = Th(F)$
(iv) $Th(F)$ is the smallest propositional theory E such that $F \subseteq E$.
Proof:
(i) Let $Th(F) \models \beta$. If possible $\beta \notin Th(F)$. So $F \not\models \beta$, i.e. $F \cup \{\neg\beta\}$ is satisfiable. So there exists $v \models F \cup \{\neg\beta\}$, i.e. $v \models F$ and $v(\beta) = 0$. Now if $\alpha \in Th(F)$ then $F \models \alpha$, so $v(\alpha) = 1$, thus $v \models Th(F)$ and since $Th(F) \models \beta$, $v(\beta) = 1$ which is a contradiction. So $\beta \in Th(F)$. Thus $Th(F)$ is a propositional theory.

(ii) We already know that $F \subseteq Th(F)$. Now let F is a propositional theory, so $F \models \beta$ implies $\beta \in F$, i.e. $\beta \in Th(F)$ implies $\beta \in F$, i.e. $Th(F) \subseteq F$. Thus (ii) $F = Th(F)$

(iii) Follows from (i) and (ii).

(iv) We already know that $F \subseteq Th(F)$. Now if E is a propositional theory then $Th(E) = E$ and if $F \subseteq E$ then $Th(F) \subseteq Th(E) = E$, ie. $F \subseteq Th(F) \subseteq E$. Thus (iv) follows.

Definition4.19: For a truth assignment v we define the theory of v to be the set $Th(v)$ of all sentences which are true under v, i.e. $Th(v) = \{\alpha \in W : v(\alpha) = 1\}$.

Theorem4.20 For any truth assignment u and v,
(i) $Th(u) = Th(v)$ iff $u = v$.
(ii) $Th(u) \subseteq Th(v)$ iff $u = v$.
(iii) For any statement α either $\alpha \in Th(v)$ or $\neg\alpha \in Th(v)$.
(iv) $Th(v)$ is a maximal satisfiable set.

Proof. (i) Let $Th(u) = Th(v)$.
For any simple sentence p, $u(p) = 1$ iff $p \in Th(u)$ iff $p \in Th(v)$ iff $v(p) = 1$ i.e. $u = v$.
Converse follows directly.

(ii) Let $Th(u) \subseteq Th(v)$.
If possible $Th(u) \neq Th(v)$. So there exists a simple sentence α such that $\alpha \in Th(v)$ but $\alpha \notin Th(u)$. So $u(\alpha) = 0$ and $v(\alpha) = 1$. So $u(\neg\alpha) = 1$, i.e. $\neg\alpha \in Th(v)$, hence $\alpha, \neg\alpha \in Th(v)$, which is not possible. Hence $Th(u) = Th(v)$, i.e. $u = v$.
Converse follows directly.

(iii) Easy to prove.

(iv) Let $Th(v)$ is a satisfiable set, since it has a model v.
If $Th(v)$ is not maximal satisfiable set, there exists a statement $\alpha \notin Th(v)$ and $Th(v) \cup \{\alpha\}$ is satisfiable. So there exists a model u satisfying $Th(v) \cup \{\alpha\}$, thus $Th(v) \cup \{\alpha\} \subseteq Th(u)$, i.e. $Th(v) \subseteq Th(u)$. which implies $u = v$, which is not possible since $v(\alpha) = 0$ and $u(\alpha) = 1$. Thus $Th(v)$ is a maximal satisfiable set.

Theorem4.21: M is a maximal satisfiable set iff it has an unique model (truth assignment). If v that unique model then $M = Th(v)$.

Proof: Let M is maximal. Let v satisfies M, so $M \subseteq Th(v)$. but $Th(v)$ is satisfiable set, so $M = Th(v)$. Thus for any two models u and v satisfying M, $M = Th(u)$ and $M = Th(v)$, i.e. $Th(u) = Th(v)$, i.e. $u = v$. Thus M has an unique model.
The last part follows from above proof.
Converse directly follows.

Corr4.22: For any model v, v is the unique model of $Th(v)$.

Definition4.23: For any non-empty set K of truth assignments / models we define the theory of K to be the set $Th(K) = \{\alpha \in W : v(\alpha) = 1 \text{ for all } v \in K\}$.

Definition4.24: For any non-empty set K of truth assignments / models we say K is axiomatizable if for some F of sentences, $K = Mod(F)$.

Theorem4.25: (i) $Th(K) = \cap\{Th(v) : v \in K\}$.
(ii) $K \subseteq Mod(Th(K))$.
(iii) For any set of sentences F, $F \subseteq Th(Mod(F))$.
(iv) F is a propositional theory then $F = Th(Mod(F))$.
(v) $Th(F) = Th(Mod(F))$.
(vi) If K is axiomatizable then $K = Mod(Th(K))$.
Proof:

(i) Exercise.

(ii) Let $u \in K$ then $u \models Th(u)$. Since $\cap\{Th(v) : v \in K\} \subseteq Th(u)$, so $u \models \cap\{Th(v) : v \in K\}$, i.e. $u \models Mod(Th(K))$. Thus $K \subseteq Mod(Th(K))$.

(iii) $Th(Mod(F)) = \cap\{Th(v) : v \in Mod(F)\} = \cap\{Th(v) : v \models F\}$, let $\alpha \notin Th(Mod(F))$, so $\alpha \notin Th(v)$ for some $v \models F$. Since $F \subseteq Th(v)$, $\alpha \notin F$. Hence (iii) follows.

(iv) Let $\alpha \in Th(Mod(F))$, so for any $v \in Mod(F)$, $v(\alpha) = 1$, so $v \models F$ implies $v(\alpha) = 1$, i.e. $F \models \alpha$, i.e. $\alpha \in Th(F)$. Since F is a propositional theory $Th(F) = F$. Thus $\alpha \in F$. So $Th(Mod(F)) \subseteq\subseteq F$. Using (iii) (iv) follows.

(iii) Exercise.

(vi) Let K is axiomatizable. So there exists a set F of formulas such that $K = Mod(F)$. Now $F \subseteq Th(Mod(F))$, i.e. $F \subseteq Th(K)$, thus $Mod(Th(K)) \subseteq Mod(F) = K$. Then using (i) (vi) follows.

5 Topological Interpretation of Compactness Theorem:

In this section we will construct a topological space and see the interpretation of the compactness theorem from topological point of view.

Definition5.1: Let for any $\alpha \in W$, let $V_\alpha = \{v \in V : v(\alpha) = 1\} = Mod(\alpha)$. Let $\mathcal{B}_V = \{V_\alpha : \alpha \in W\}$.

Theorem5.2: (i) $V_\alpha \cup V_\beta = V_{\alpha\vee\beta}$.
(ii) $V_{\neg\alpha} = V - V_\alpha$.
(iii) $V_\alpha \cap V_\beta = V_{\alpha\wedge\beta}$.
(iv) For any tautology α, $V_\alpha = V$ and for any contradiction β, $V_\beta = \emptyset$.

Theorem5.3: $\mathcal{B}_V$ form a topology on V say $\tau(V)$ which is zero-dimensional Hausdorff space.
Proof: (i) $\mathcal{B}_V$ form a topology on V: Let $v \in V$, and $p \in P$, then either $v(p) = 0$ or 1. If $v(p) = 0$ then $v \in V_{\neg p}$ else $v \in V_p$, so $\mathcal{B}_V$ is a cover of V. Also for any $\alpha, \beta \in W$, $V_\alpha \cap V_\beta = V_{\alpha\wedge\beta} \in \mathcal{B}_V$. So $\mathcal{B}_V$ form a base for a topology on V.

(ii) $\tau(V)$ is zero dimensional: Since $V - V_\alpha = V_{\neg\alpha}$, so every member of $\mathcal{B}_V$ is clopen, so the topology is zero-dimensional.

(iii) $\tau(V)$ is Hausdorff: Let $u, v \in V$ and $u \neq v$, then there exists $p \in P$ such that $u(p) \neq v(p)$. With out loss of generality suppose $u(p) = 0$, then $v(p) = 1$. So $v \in V_{\neg p}$ and $u \in V_p$. Also $V_p \cap V_{\neg p} = V_{\neg p\wedge p} = \emptyset$.

Theorem5.4: $(\mathcal{B}_V, \cup, \cap, V - (.), \emptyset, V)$ forms a Boolean algebra.

Theorem5.5(Compactness Theorem from topological point of view): Compactness theorem holds iff the topological space $(V, \tau(V))$ is a compact space.
Proof: Let compactness theorem holds: Let $\mathcal{C}$ be any collection of of closed sets with finite intersection property. We can suppose all closed sets are proper subset of V. We define

$F = \{\neg\alpha : V_\alpha \subseteq V - C \text{ for some } C \in \mathcal{C}\}$.

Let $\{\neg\alpha_1, ..., \neg\alpha_n\}$ be a finite subset of F, So there exist $C_k \in \mathcal{C},\ k = 1, ..., n$ such that $V_{\alpha_k} \subseteq V - C_k$.

Now $S = V - \cup_{k=1}^{n} V_{\alpha_k} \supseteq \cap_{k=1}^{n} C_k \neq \emptyset$. So S is a non-empty set, and contains all $v \in S$ such that α_k are 0 for $k = 1, ..., n$, i.e. there exists a $v \in C$ such that $v(\neg\alpha_k) = 1$ for $k = 1, ..., n$.

Thus F is finitely satisfiable, so by compactness theorem F is satisfiable, so there exists a $v \in V$ such that $v(\neg\alpha) = 1$ for all $\neg\alpha \in F$.

Since each sets in $\mathcal{C}$ is proper subset of V, so each set $V - C$ for each $C \in \mathcal{C}$ contains a V_α for some α and $\neg\alpha \in F$.

Thus $v \notin V_\alpha \subseteq V - C$ for each $C \in \mathcal{C}$, i.e. $v \in C$ for each $C \in \mathcal{C}$, i.e. $\mathcal{C}$ has non-empty intersection.

Hence $(V, \tau(V))$ is a compact space.

Conversely, suppose $(V, \tau(V))$ is a compact space: Let F be an infinite subset of W which is finitely satisfiable. Let $\mathcal{C} = \{V_\alpha : \alpha \in F\}$, then $\mathcal{C}$ is a family of closed subsets. Since for any finite subset of F has a model, so $\mathcal{C}$ has finite intersection property. By compactness of $(V, \tau(V))$, $\mathcal{C}$ has non-empty intersection, i.e. there is a model satisfying all the statements in F.

6 Axiomatic Approach(Hilbert Calculi of Propositional Logic):

By the use of truth table or truth value evaluation we have decided whether a formula a tautology or not, whether two formulas are equivalent or not etc. These were some simple part of proposition logic. But dealing with more complex part of the logic cannot be handled by truth table or truth value evaluation or by any other similar effective procedure. In this part we will construct a formal axiomatic approach to deal with propositional logic.

In axiomatic approach we consider a set of tautologies as axioms and one ore more formal rule of inferring a sentence from a finite number of sentences called rule of inference.

Definition6.1: Let P be a countable set of simple sentences. We consider the primitive set of connectives $\{\neg, \to\}$ and we form the set W of wff from P with the connectives $\neg$ and $\to$ as:

(i) $P \subset W$,
(ii) If $\alpha, \beta \in W$ then $\neg\alpha, \alpha \to \beta \in W$,
(ii) The members of W are only defined by the above rules (i) and (ii).

We consider the axioms of propositional calculus (PC): for any wff α, β, γ
(PC1) $\alpha \to (\beta \to \alpha)$
(PC2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(PC3) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.

The only rule of inference is Modus Ponens (MP):
(MP) $\frac{\alpha, \alpha \to \beta}{\beta}$

We can also write the above process as: $\text{MP}(\alpha, \alpha \to \beta) = \beta$.

(Concluding/infering β from α and $\alpha \to \beta$).

We define three more connectives $\vee$, $\wedge$ and $\leftrightarrow$ defined as follows:
$\alpha \vee \beta = \neg\alpha \to \beta$
$\alpha \wedge \beta = \neg(\alpha \to \neg\beta)$

$\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$.

(NB: From truth valuation we can see that (MP) is of the form: $\frac{\alpha, \neg\alpha \vee \beta}{\beta}$).

Definition6.2: A proof is a sequence $\alpha_1, \alpha_2, ..., \alpha_n$ in W such that for each k, α_k is an axiom or α_k is a direct consequence of some of the preceding α_i and α_j $(1 \leq i, j < k)$ by MP.

Definition6.3: Let $\alpha \in W$, we say α is a theorem if there exists a proof $\alpha_1, \alpha_2, ..., \alpha_n$ such that $\alpha_n = \alpha$ called proof of α. We write $\vdash \alpha$.

Thus if α is an axiom then $\vdash \alpha$.

If α is not a theorem we will write $\nvdash \alpha$.

Theorem6.4: $\vdash \alpha \to \alpha$.

Proof:

(i) $\alpha \to (\alpha \to \alpha)$. (PC1)

(ii) $\alpha \to ((\alpha \to \alpha) \to \alpha)$. (PC1)

(ii) $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$. (PC2)

(iii) $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$. (MP (ii), (iii))

(iv) $\alpha \to \alpha$. (MP (i), (iii))

Thus $\vdash \alpha \to \alpha$.

Theorem6.5: $\vdash \alpha$ and $\vdash \alpha \to \beta$ implies $\vdash \beta$.

Proof: Let $\alpha_1, ..., \alpha_n$ be a proof of α where $\alpha_n = \alpha$ and $\beta_1, ..., \beta_m$ be a proof of $\alpha \to \beta$, then $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m, \beta$ is a proof of β where the last sentence follows from α_n and β_m be MP.

Theorem6.6: If the wff α is an axiom then $\models \alpha$.

Proof: Exercise.

Theorem6.7(Soundness Theorem of Propositional Calculus): For a sentence α, $\vdash \alpha$ implies $\models \alpha$.

Proof: Since $\vdash \alpha$, so there is a proof of α say $\alpha_1, \alpha_2, ..., \alpha_n$ such that each α_k is an axiom of PC or for $k > 2$, α_k follows from α_i, α_j for some $1 \leq i, j < k$ by MP.

Now, obviously α_1 and α_2 are axioms, so $\models \alpha_1$ and $\models \alpha_2$. Lets us prove the theorem by induction. Let $k > 2$ and for $i < k$, $\models \alpha_i$.

Now if α_k is an axiom then $\models \alpha_k$. Now if α_k is not an axiom then α_k follows from α_i and α_j by the rule MP where $1 \leq i, j < k$ and α_j is of the form $\alpha_i \to \alpha_k$.

Now $\models \alpha_i$ and $\models \alpha_j$, i.e. $\models \alpha_i \to \alpha_k$. Hence $\models \alpha_k$.

And by induction $\models \alpha_n$, i.e. $\models \alpha$.

Definition6.8: Let F be a set of sentences and $\alpha \in W$. We say α is a consequence of F if there is a sequence $\alpha_1, \alpha_2, ..., \alpha_n$ in W called proof of α from F such that $\alpha_n = \alpha$ and for each k, α_k is an axiom or member of F or α_k is a direct consequence of some of the preceding α_i and α_j $(1 \leq i, j < k)$ by MP. We write $F \vdash \alpha$.

If $F = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, we write $\alpha_1, \alpha_2, ..., \alpha_n \vdash \alpha$.

If α is not a consequence of F we write $F \nvdash \alpha$.

Theorem6.9: $\alpha \in F$ implies $F \vdash \alpha$.

Theorem6.10: $\emptyset \vdash \alpha$ iff $\vdash \alpha$.

Theorem6.11: $\alpha \vdash \alpha$.

Theorem6.12: If $F \subseteq E \subset W$ then $F \vdash \alpha$ implies $E \vdash \alpha$.

Theorem6.13: If $\vdash \alpha$ then for any set F of sentences $F \vdash \alpha$.
Theorem6.14: If $F \vdash \alpha$ and $F \vdash \alpha \to \beta$ then $F \vdash \beta$. (Exercise)

Theorem6.15:(Soundness Theorem[General Form]): If $F \vdash \alpha$ then $F \models \alpha$.
Proof: Exercise.

Theorem6.16(Deduction Theorem): If $F, \beta \vdash \alpha$ then $F \vdash \beta \to \alpha$.
Proof: Let $\alpha_1, ..., \alpha_n$ be the proof of α from $F \cup \{\beta\}$, where $\alpha_n = \alpha$. Now α_1 is an axiom or $\alpha_1 \in F$ or $\alpha_1 = \beta$.
(i) If α_1 is an axiom then $\vdash \alpha_1$. Again $\vdash \alpha_1 \to (\beta \to \alpha_1$ (PC1), hence $\vdash \beta \to \alpha_1$.
(ii) If $\alpha \in F$ then $F \vdash \alpha_1$, again $\vdash \alpha_1 \to (\beta \to \alpha_1)$ (PC1), i.e. $F \vdash \alpha_1 \to (\beta \to \alpha_1)$, thus $F \vdash \beta \to \alpha_1$.
(iii) If $\alpha_1 = \beta$, since $\vdash \beta \to \beta$, so $\vdash \beta \to \alpha_1$.
Thus in general $F \vdash \beta \to \alpha_1$.

Let $k > 1$ and for any $i < k$, $F \vdash \beta \to \alpha_i$.
If α_k is an axiom or in F or β we can prove as above that $F \vdash \beta \to \alpha_i$.
Let α_k follows from α_i, α_j for $1 \leq i, j < k$ by MP where $\alpha_j = \alpha_i \to \alpha_k$.
Now
$F \vdash \beta \to \alpha_i$ ....(iv)
$F \vdash \beta \to (\alpha_i \to \alpha_k)$ ......(v)
Also, $F \vdash (\beta \to (\alpha_i \to \alpha_k)) \to ((\beta \to \alpha_i) \to (\beta \to \alpha_k))$ (PC2).....(vi)
From (v) and (vi) using MP we have $\vdash (\beta \to \alpha_i) \to (\beta \to \alpha_k)$ ....(vii)
Again from (iv) and (vii), $F \vdash \beta \to \alpha_k$.

Thus by induction $F \vdash \beta \to \alpha_i$ for all $1 \leq i \leq n$, thus $F \vdash \beta \to \alpha_n$, i.e. $F \vdash \beta \to \alpha$.

Note6.17: The converse of **Deduction Theorem** is also true, i.e. $F \vdash \beta \to \alpha$ implies $F, \beta \vdash \alpha$.
Proof: If $F \vdash \beta \to \alpha$, so $F, \beta \vdash \beta \to \alpha$ ...(i) and also $F, \beta \vdash \beta$ ...(ii), thus by MP((i),(ii)) $F, \beta \vdash \alpha$.

Corr6.18: (i) $\alpha \vdash \beta$ iff $\vdash \alpha \to \beta$.
(ii) $\alpha_1, \alpha_2, ..., \alpha_n \vdash \beta$ iff $\vdash \alpha_1 \to (\alpha_2 \to (....(\alpha_n \to \beta).....))$

Theorem6.19:(MP) $\alpha, \alpha \to \beta \vdash \beta$.

Theorem6.20: If $\alpha \vdash \beta$ and $\beta \vdash \gamma$ then $\alpha \vdash \gamma$.
Proof: Since $\alpha \vdash \beta$, let $\alpha, \beta_1, ..., \beta_n = \beta$ be a proof of β from α where α_k are axiom or follows from α_i, α_j for $1 \leq i, j < k$ by MP.
Again, since $\beta \vdash \gamma$, let $\beta, \gamma_1, ..., \gamma_n = \gamma$ be a proof of γ from β where γ_k are axiom or follows from α_i, α_j for $1 \leq i, j < k$ by MP.
Now, $\alpha, \beta_1, ..., \beta_n, \gamma_1, ..., \gamma_n = \gamma$ be a proof of γ from α, thus $\alpha \vdash \gamma$.

Corr6.21: (Syllogism): If $\vdash \alpha \to \beta$ and $\vdash \beta \to \gamma$ then $\vdash \alpha \to \gamma$. Exercise.
Theorem6.22: (Syllogism): $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$.
Proof:
(i) $\alpha \to \beta$ (Assumption 1)
(ii) $\beta \to \gamma$ (Assumption 2)
(iii) $(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$ (PC1)
(iv) $\alpha \to (\beta \to \gamma)$ MP(ii, iii)

(v) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ (PC2)
(vi) $(\alpha \to \beta) \to (\alpha \to \gamma)$ MP(iv, v)
(vii) $\alpha \to \gamma$ MP(i, vi).
Thus $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$.

Theorem6.23:(Syllogism(General Form)): If $F \vdash \alpha \to \beta$ and $F \vdash \beta \to \gamma$ then $F \vdash \alpha \to \gamma$.

Theorem6.24: $\vdash \neg\alpha \to (\alpha \to \beta)$.
Proof:
(i) $\neg\alpha$ (Assumption 1)
(ii) α (Assumption 2)
(iii) $\alpha \to (\neg\beta \to \alpha)$ (PC1)
(iv) $\neg\alpha \to (\neg\beta \to \neg\alpha)$ (PC1)
(v) $\neg\beta \to \alpha$ (MP(ii, iii))
(vi) $\neg\beta \to \neg\alpha$ (MP(i, iv))
(vii)$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$ (PC3)
(viii) $\alpha \to \beta$ (MP(vi, vii))
(ix) β (MP(ii, viii).
Thus $\neg\alpha, \alpha \vdash \beta$. Using deduction theorem $\neg\alpha \vdash \alpha \to \beta$ and again using deduction theorem $\vdash \neg\alpha \to (\alpha \to \beta)$.

Theorem6.25: $\vdash \neg\neg\alpha \to \alpha$.
Proof:
(i) $\neg\neg\alpha$ (We assume)
(ii) $\neg\neg\alpha \to (\neg\alpha \to \neg\neg\neg\alpha)$ (Theorem)
(iii) $\neg\alpha \to \neg\neg\neg\alpha$ (MP(i, ii))
(iv) $(\neg\alpha \to \neg\neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)$ (PCA3)
(v) $\neg\neg\alpha \to \alpha$ (MP(iii, iv))
(vi) α (MP(i, v)).
Thus $\neg\neg\alpha \vdash \alpha$, hence by deduction theorem $\vdash \neg\neg\alpha \to \alpha$.

Theorem6.26: $\vdash \alpha \to \neg\neg\alpha$.
Proof:
(i) $\neg\neg\neg\alpha \to \neg\alpha$ (Theorem)
(ii) $(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$ (PC3)
(iii) $\alpha \to \neg\neg\alpha$ (MP(i, ii)).
Hence $\vdash \alpha \to \neg\neg\alpha$.

Theorem6.27: $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.
Proof:
(i) $\alpha \to \beta$ (Assumption)
(ii) $\neg\neg\alpha \to \alpha$ (Theorem)
(iii) $\neg\neg\alpha \to \beta$ (Syllogism(ii, i))
(iv) $\beta \to \neg\neg\beta$ (Theorem)
(v) $\neg\neg\alpha \to \neg\neg\beta$ (Syllogism(iii, iv))
(vi) $(\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$ (PC3)
(vii) $\neg\beta \to \neg\alpha$ MP(v, vi)
Thus $\alpha \to \beta \vdash \neg\beta \to \neg\alpha$ and using deduction theorem $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

Theorem6.28:(MT: Modus Tollens) If $\vdash \alpha \to \beta$ and $\vdash \neg\beta$ then $\vdash \neg\alpha$.
Proof: We have $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$. Since $\vdash \alpha \to \beta$, so $\vdash \neg\beta \to \neg\alpha$.

Again since, $\vdash \neg\beta$, thus $\vdash \neg\alpha$.

Theorem6.29:(MP: Modus Ponens(General Form)) If $F \vdash \alpha \to \beta$ and $F \vdash \alpha$ then $F \vdash \beta$.

Theorem6.30:(MT: Modus Tollens(General Form)) If $F \vdash \alpha \to \beta$ and $F \vdash \neg\beta$ then $F \vdash \neg\alpha$.

Theorem6.31: $\alpha \vdash \beta$ iff $\neg\beta \vdash \neg\alpha$. (Exercise)

Theorem6.32: $\vdash (\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$.
Proof:
(i) $\alpha \to \neg\beta$ (Assumption)
(ii) $\vdash (\alpha \to \neg\beta) \to (\neg\neg\beta \to \neg\alpha)$. (Theorem)
(iii) $\neg\neg\beta \to \neg\alpha$ (MP(i, ii))
(iv) $\beta \to \neg\neg\beta$ (Theorem)
(v) $\beta \to \neg\alpha$ (Syllogism(iv, iii)).
Thus $\alpha \to \neg\beta \vdash \beta \to \neg\alpha$ and the remaining follows by deduction theorem.

Theorem6.33: $\vdash (\neg\alpha \to \beta) \to (\neg\beta \to \alpha)$. (Exercise).
Theorem6.34: $\vdash (\neg\alpha \to \alpha) \to \alpha$.
Proof:
(i) $\neg\alpha \to (\neg(\neg\alpha \to \alpha) \to \neg\alpha)$ (PCA1),
(ii) $(\neg\alpha \to (\neg(\neg\alpha \to \alpha) \to \neg\alpha)) \to ((\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to (\neg\alpha \to \neg\alpha))$ (PCA2),
(iii) $(\neg\alpha \to \neg(\neg\alpha \to \alpha)) \to (\neg\alpha \to \neg\alpha)$ MP(ii, iii)
(iv) $((\neg\alpha \to \alpha) \to \alpha) \to (\neg\alpha \to \neg(\neg\alpha \to \alpha))$ (Theorem)
(v) $((\neg\alpha \to \alpha) \to \alpha) \to (\neg\alpha \to \neg\alpha)$ Syllogism(iv, iii)
(vi) $(\neg\alpha \to \neg\alpha) \to (\alpha \to \alpha)$ (PC3)
(vii) $((\neg\alpha \to \alpha) \to \alpha) \to (\alpha \to \alpha)$ Syllogism(v, vi)
(viii) $(\neg\alpha \to \alpha) \to \alpha)$ Syllogism in (vii)

Theorem6.35: $\vdash (\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$.
Proof:
(i) $\alpha \to \beta$ (Assumption 1)
(ii) $\neg\alpha \to \beta$ (Assumption 2)
(iii) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ (Theorem)
(iv) $\neg\beta \to \neg\alpha$ (MP(i, iii))
(v) $\neg\beta \to \beta$ (Syllogism(iv, ii))
(vi)$(\neg\beta \to \beta) \to \beta$ (Theorem)
(vii) β (MP(v, vi)).
Thus $\alpha \to \beta, \neg\alpha \to \beta \vdash \beta$ and using deduction theorem we have $\vdash (\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$.

Theorem6.36: $\vdash \alpha \to (\neg\beta \to \neg(\alpha \to \beta))$.
Proof:
(i) α (Assumption 1)
(ii) $\alpha \to \beta$ (Assumption 2)
(iii) β (MP(i, ii).
Thus $\alpha, \alpha \to \beta \vdash \beta$, using deduction theorem
$\alpha \vdash (\alpha \to \beta) \to \beta$.....(iv)
Since $\vdash ((\alpha \to \beta) \to \beta) \to (\neg\beta \to \neg(\alpha \to \beta))$.....(v) (Theorem)

From Syllogism(iv, v) we have $\alpha \vdash \neg\beta \to \neg(\alpha \to \beta)$ and using deduction theorem we have the required result.

Exercise6.37:
(i) $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$.
(ii) For any two formulas α, β, show that $\alpha, \neg\alpha \vdash \beta$.
(iii) If $F \vdash \alpha$ and $\alpha \vdash \beta$ then $F \vdash \beta$.
(iv) If $F \vdash \alpha_1$,......, $F \vdash \alpha_n$ and $\alpha_1, \ldots\ldots, \alpha_n \vdash \beta$ then $F \vdash \beta$.

Lemma6.37: Let α be a sentence in which the only occurrence of simple sentences are $p_1, p_2, \ldots., p_k$. Let v be a truth assignment. We define $p_i^{/}$ to be p_i if $v(p_i) = 1$, otherwise $p_i^{/}$ is $\neg p_i$ and $\alpha^{/}$ to be α if $v(\alpha) = 1$, otherwise $\alpha^{/}$ is $\neg\alpha$. Then
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \alpha^{/}$
Proof: Let the assume the only connectives occurring in α are $\neg$ and $\to$.
We will prove the claim by induction on the number n of all the total number of occurrence of $\neg$ and $\to$ in α.
If $n = 0$ then, α is a simple sentence p_i for some $i = 1, 2, \ldots., k$. Thus $\alpha^{/}$ is $p_i^{/}$ and the claim follows immediately.

Let the claim be true for less than n occurrence of $\neg$ and $\to$.
Case 1: α is of the form $\neg\beta$.
By induction hypothesis, $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \beta^{/}$.......(i)

Subcase 1.1: Let $v(\beta) = 1$, so $v(\alpha) = 0$. Thus $\beta^{/} = \beta$ and $\alpha^{/} = \neg\alpha = \neg\neg\beta$.
Now $\vdash \beta \to \neg\neg\beta$, so from (i) $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \neg\neg\beta^{/}$ by (MP). Thus $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \alpha^{/}$.

Subcase 1.2: Let $v(\beta) = 0$, so $v(\alpha) = 1$. Thus $\beta^{/} = \neg\beta$ and $\alpha^{/} = \alpha = \neg\beta$.
From (i) $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \neg\neg\beta^{/}$, i.e. $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \alpha^{/}$.

Case 2: α is of the form $\beta \to \gamma$.
By induction hypothesis, $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \beta^{/}$.......(ii)
and $p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \gamma^{/}$........(iii)

Subcase 2.1: $v(\gamma) = 1$, so $v(\alpha) = 1$. Thus $\gamma^{/} = \gamma$, $\alpha^{/} = \alpha$. From (iii)
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \gamma$........(iii)
Since $\vdash \gamma \to (\beta \to \gamma)$ ......(iv) (PC1)
From (iii) and (iv) by MP we have:
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \beta \to \gamma$, i.e. the required claim.

Subcase 2.2: $v(\beta) = 0$, then so $v(\alpha) = 1$, $\beta^{/} = \neg\beta$ and $\alpha^{/} = \alpha = \beta \to \gamma$. So from (iii)
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \neg\beta$........(v)
Since $\vdash \neg\beta \to (\beta \to \gamma)$ ......(vi)
From (v) and (vi) by MP we have:
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \beta \to \gamma$, i.e. the required claim.

Subcase 2.3: $v(\beta) = 1$ and $v(\gamma) = 0$, then so $v(\alpha) = 0$, $\beta^{/} = \beta$, $\gamma^{/} = \neg\gamma$ and $\alpha = \neg\alpha = \neg(\beta \to \gamma)$. So from (iii) and (iv)
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \beta$........(vii)
$p_1^{/}, p_2^{/}, \ldots., p_k^{/} \vdash \neg\gamma$........(viii)

Since $\vdash \beta \to (\neg\gamma \to \neg(\beta \to \gamma))$.......(ix)
From (vii) and (ix) by MP we have:
$p_1^/, p_2^/, ..., p_k^/ \vdash \neg\gamma \to \neg(\beta \to \gamma)$.....(xi)
Again, by MP(viii, ix) we have:
$p_1^/, p_2^/, ..., p_k^/ \vdash \neg(\beta \to \gamma)$ and the claim follows.

Theorem6.38(Completeness Theorem of Propositional Calculus): For a sentence α, $\models \alpha$ implies $\vdash \alpha$.

Proof: Let $\models \alpha$. Let $p_1, p_2, ..., p_k$ be the distinct simple sentences occurring α. Let $v \in V$.

We define We define $p_i^/$ to be p_i if $v(p_i) = 1$, otherwise $p_i^/$ to be $\neg p_i$ and $\alpha^/$ to be α if $v(\alpha) = 1$, otherwise $\alpha^/$ to be $\neg\alpha$, but $\alpha^/ = \alpha$ (being tautology). Then we have
$p_1^/, p_2^/, ..., p_k^/ \vdash \alpha$.
Let us consider a case when $v(p_k) = 1$, then
$p_1^/, p_2^/, ..., p_{k-1}^/, p_k \vdash \alpha$........(i)
Again we consider a case when $v(p_k) = 0$, then
$p_1^/, p_2^/, ..., p_{k-1}^/, \neg p_k \vdash \alpha$........(ii)
Using deduction theorem we have from (i) and (ii) respectively:
$p_1^/, p_2^/, ..., p_{k-1}^/ \vdash p_k \to \alpha$........(iii)
$p_1^/, p_2^/, ..., p_{k-1}^/ \vdash \neg p_k \to \alpha$........(iv)
We have $\vdash (p_k \to \alpha) \to ((\neg p_k \to \alpha) \to \alpha)$........(v)
Using MP(iii, v) we have,
$p_1^/, p_2^/, ..., p_{k-1}^/ \vdash (\neg p_k \to \alpha) \to \alpha$........(vi)
Again using MP(iv, vi) we have,
$p_1^/, p_2^/, ..., p_{k-1}^/ \vdash \alpha$.

Thus the assumption $p_k^/$ is eliminated. Repeating the above process for more $(k-1)$-times we have $\vdash \alpha$.

Theorem6.39(Completeness Theorem of Propositional Calculus)(General Form): Let $F \subseteq W$ and for a sentence α if $F \models \alpha$ then $F \vdash \alpha$.

Proof: Let $F \models \alpha$, then as a consequence of compactness theorem of PC there exists a finite set $\{\alpha_1, ..., \alpha_n\} \subseteq F$ such that $\alpha_1, ..., \alpha_n \models \alpha$, then $\models \alpha_1 \to (\alpha_1 \to (......(\alpha_n \to \alpha).....))$.

Now using completeness theorem we have $\vdash \alpha_1 \to (\alpha_1 \to (......(\alpha_n \to \alpha).....))$ and using converse of deduction theorem we have $\alpha_1, ..., \alpha_n \vdash \alpha$ and so $F \vdash \alpha$.

Theorem6.40:
(i) $\vdash \alpha \vee \neg\alpha, \nvdash \alpha \wedge \neg\alpha$.
(ii) If $F \vdash \alpha$ and $\alpha \vdash \beta$ iff $F \vdash \alpha \wedge \beta$. So $\vdash \alpha$ and $\vdash \beta$ iff $\vdash \alpha \wedge \beta$.
(iii) $\vdash \alpha \to (\alpha \vee \beta)$
(iv) $\vdash (\alpha \wedge \beta) \to \alpha$.
(v) If $\vdash \alpha \leftrightarrow \beta$ then $\vdash \alpha$ iff $\vdash \beta$.

7 Consistency:

Definition7.1: A set F of formulas is said to be

(i) consistent if there does not exists any formula β such that $F \vdash \beta$ and $F \vdash \neg\beta$ holds simultaneously, equivalently there does not exists any formula β such that $F \vdash \beta \wedge \neg\beta$. Other wise F is said to be inconsistent.

(ii) F is said to be maximal consistent if F is consistent and F cannot to enlarge to a bigger set as consistent set, i.e. if E is consistent and $F \subseteq E$ then $E = F$.

Exercise7.2: If $\{\alpha.\neg\alpha\} \subseteq F$ then F is inconsistent.

Theorem7.3: (i) If $\vdash \alpha$ then $\{\alpha\}$ is consistent.
(ii) For every simple sentence $p \in P$, $\{p\}$ is consistent.
(iii) $E \subseteq F \subset W$. If F is consistent then E also consistent.
Proof: (i) If $\{\alpha\}$ is inconsistent, then there exists β such that $\alpha \vdash \beta$ and $\alpha \vdash \neg\beta$. By soundness theorem
$\alpha \models \beta$ and $\alpha \models \neg\beta$.......(*)
But $\vdash \alpha$, so $\models \alpha$, so (*) is not possible. Hence the proof.

(ii) If $\{p\}$ is inconsistent, then there exists β such that $p \vdash \beta$ and $p \vdash \neg\beta$. By soundness theorem
$p \models \beta$ and $p \models \neg\beta$.......(**)
We take a model v such that $v(p) = 1$, then $v(\beta) = 1 = v(\neg\beta)$, which is not possible. Hence (**) is not possible. Thus the proof follows.

(iii) Exercise.

Exercise 7.4: Prove 7.3(i) and (ii) without use of soundness theorem.

Definition7.5: For any $F \subseteq W$ the theorem of F is the set $TH(F) = \{\beta : F \vdash \beta\}$.
If $F = \{\alpha\}$, we write this set as $TH(\alpha)$.

Theorem7.6:
(i) $TH(\emptyset) = \{\alpha :\vdash \alpha\}$.
(ii) $E \subseteq F \subseteq W$ implies $TH(E) \subseteq TH(F)$.
(iii) If $\alpha \vdash \beta$ or equivalently $\vdash \alpha \to \beta$, then $TH(\beta) \subseteq TH(\alpha)$.
(iv) $\vdash \alpha \leftrightarrow \beta$ iff $TH(\alpha) = TH(\beta)$.
(vi) $F \vdash \beta$ then $\beta \in TH(F)$.
(vii) If $F \vdash \beta$ then $TH(\beta) \subseteq TH(F)$.
Proof: Exercise.

Theorem7.7:
(i) If $\{\alpha\}$ is consistent then $TH(\alpha)$ is a constituent set.
(ii) $TH(\alpha) = Th(\alpha)$.
(iii) $TH(\alpha) \cap TH(\beta) = TH(\alpha \wedge \beta)$.
(iv) $TH(\alpha) \cup TH(\beta) = TH(\alpha \vee \beta)$.
(vi) $\vdash \alpha \leftrightarrow \beta$ iff $TH(\alpha) = TH(\beta)$.
(vii) $\vdash \alpha \to \beta$ iff $TH(\beta) \subseteq TH(\alpha)$.
Proof: (i) If $TH(\alpha)$ is inconsistent, then there exists β such that $TH(\alpha) \vdash \beta$ and $TH(\alpha) \vdash \neg\beta$. Now $\exists$ a finite subset $\{\alpha_1, ..., \alpha_n\} \subseteq TH(\alpha)$ such that $\alpha_1, ..., \alpha_n \vdash \beta$ and $\alpha_1, ..., \alpha_n \vdash \neg\beta$.
But $\alpha \vdash \alpha_k$ for any $k = 1, ..., n$, so $\alpha \vdash \beta$ and $\alpha \vdash \neg\beta$, which is not possible. Hence the proof follows.

(ii) - (vii) Exercise.

Theorem7.8: If F is a consistent set iff there is an α such that $F \nvdash \alpha$.

Proof: Let F be consistent. So there is no formula β such that $F \vdash \beta$ and $F \vdash \neg\beta$ holds simultaneously and so there exists some α (β or $\neg\beta$) such that $F \nvdash \alpha$.

Conversely, let there is an α such that $F \nvdash \alpha$. If possible let F is inconsistent, so there exists β such that $F \vdash \beta$ and $F \vdash \neg\beta$. Since $\beta, \neg\beta \vdash \alpha$, so $F \vdash \alpha$, which is not possible, hence F is a consistent.

Theorem7.9: Every consistent set of formula(s) is contained in a maximal consistent set of formulas.

Proof: Let F be a consistent set of formula(s).

Let $\mathcal{P} = \{E \subset W : F \subseteq E$ and E is consistent$\}$.

Thus $F \in \mathcal{P}$ and $(\mathcal{P}, \subseteq)$ is a poset.

Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Let $G = \cup\mathcal{C}$.

Since for any $E \in \mathcal{C}$, $F \subseteq E$, so $F \subseteq G$.

If G is not consistent then there exists a formula β such that $G \vdash \beta$ and $G \vdash \neg\beta$. So we can find a finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1, \alpha_2, ..., \alpha_n \vdash \beta$ and $\alpha_1, \alpha_2, ..., \alpha_n \vdash \neg\beta$.

Now $\exists$, $E_1, ..., E_n \in \mathcal{C}$ such that $\alpha_k \in E_k$. But $\mathcal{C}$ is a chain, so all $E_k's$ subset of one E_k for some $k = 1, ..., n$ say E_j, i.e. $\{\alpha_1, ..., \alpha_n\} \subseteq E_j$. And we conclude that E_j is inconsistent, which is impossible. Hence G is consistent and so $G \in \mathcal{P}$.

So every chain in $\mathcal{P}$ has an upper bound. So by Zorn's lemma $\mathcal{P}$ has a maximal element say M (with $F \subseteq M$) which is a maximal consistent set containing F.

Theorem7.10: F is consistent iff F is finitely consistent, i.e. every finite subset of F is consistent.

Proof: If F is consistent then obviously F is finitely consistent.

Conversely suppose that F is finitely consistent. If possible let F is inconsistent so there exists β such that both $F \vdash \beta$ and $F \vdash \neg\beta$ holds simultaneously. Hence there exists a finite subset $\{\alpha_1, ..., \alpha_n\}$ of F such that $\{\alpha_1, ..., \alpha_n\} \vdash \beta$ and $\{\alpha_1, ..., \alpha_n\} \vdash \neg\beta$. So $\{\alpha_1, ..., \alpha_n\}$ is a finite subset of F which is inconsistent, which is a contradiction. Hence F is consistent.

Theorem7.11:

(i) $F \vdash \alpha$ iff $F \cup \{\neg\alpha\}$ is inconsistent.

(ii) $F \cup \{\alpha\}$ is consistent iff $F \nvdash \neg\alpha$.

Proof: (i) Let $F \vdash \alpha$. So $F \cup \{\neg\alpha\} \vdash \alpha$, and also $F \cup \{\neg\alpha\} \vdash \neg\alpha$. Hence $F \cup \{\neg\alpha\}$ is inconsistent.

Conversely, suppose that $F \cup \{\neg\alpha\}$ is inconsistent. So there is a formula β such that $F \cup \{\neg\alpha\} \vdash \beta$ and $F \cup \{\neg\alpha\} \vdash \neg\beta$. By deduction Theorem $F \vdash \neg\alpha \to \beta$ and $F \vdash \neg\alpha \to \neg\beta$, so $F \vdash \beta \to \alpha$ and then using Syllogism $F \vdash \neg\alpha \to \alpha$. Since $\vdash (\neg\alpha \to \alpha) \to \alpha$, by MP we have $F \vdash \alpha$.

(ii) Let $F \cup \{\alpha\}$ is consistent. Let if possible $F \vdash \neg\alpha$. So $F \cup \{\alpha\} \vdash \neg\alpha$, and also $F \cup \{\alpha\} \vdash \alpha$. which shows that $F \cup \{\alpha\}$ is inconsistent, which is not possible, so $F \nvdash \neg\alpha$.

Conversely suppose that $F \nvdash \neg\alpha$. Let if possible $F \cup \{\alpha\}$ is inconsistent. So there is a formula β such that $F \cup \{\alpha\} \vdash \beta$ and $F \cup \{\alpha\} \vdash \neg\beta$. As similar above we can show $F \vdash \neg\alpha$, which is not possible, so $F \cup \{\alpha\}$ is consistent.

Corollary7.12:

(i) $F \vdash \neg\alpha$ iff $F \cup \{\alpha\}$ is inconsistent.

(ii) $F \cup \{\neg\alpha\}$ is consistent iff $F \nvdash \alpha$.

(iii) $\vdash \alpha$ iff $\{\neg\alpha\}$ is inconsistent.

(iv) $\{\alpha\}$ is consistent iff $\nvdash \neg\alpha$.

(v) $\vdash \neg\alpha$ iff $\{\alpha\}$ is inconsistent.

(vi) $\{\neg\alpha\}$ is consistent iff $\nvdash \alpha$.

Exercise7.13: $F \cup \{\alpha\}$ is consistent iff $F \cup \{\neg\neg\alpha\}$ is consistent.

Theorem7.14: Let M is maximal set then
(i) $\alpha \in M$ iff $M \vdash \alpha$.
(ii) for $\alpha \in W$, $\alpha \in M$ iff $\neg\alpha \notin M$ (i.e. for any α either $\alpha \in M$ or $\neg\alpha \in M$).
(iii) $\alpha \vee \beta \in M$ iff $\alpha \in M$ or $\beta \in M$.
(iv) $\alpha \wedge \beta \in M$ iff $\alpha \in M$ and $\beta \in M$.
(v) if $\alpha, \alpha \to \beta \in M$ then $\beta \in M$.
(vi) If $\alpha \leftrightarrow \beta \in M$ then $\alpha \in M$ iff $\beta \in M$.
(vii) $\vdash \alpha$ implies $\alpha \in M$ (so $\vdash \neg\alpha$ implies $\alpha \notin M$).
Proof: (i) If $\alpha \in M$, then $M \vdash \alpha$.
Conversely, suppose $M \vdash \alpha$. If $\alpha \notin M$, then by maximality of M, $M \cup \{\alpha\}$ is inconsistent. So $M \vdash \neg\alpha$, which shows that M is inconsistent. So $\alpha \in M$.

(ii) Let $\alpha, \neg\alpha \in M$, then $M \vdash \alpha$ and $M \vdash \neg\alpha$, i.e. M is inconsistent which is not possible, so both α and $\neg\alpha$ cannot be both of them to be members of M.
Again let neither α and $\neg\alpha$ is in M, by maximality of M, $M \cup \{\alpha\}$ and $M \cup \{\neg\alpha\}$ both inconsistent. So $M \vdash \neg\alpha$ and $M \vdash \alpha$. So $\alpha, \neg\alpha \in M$ which is a contradiction, so $\alpha \in M$ or $\neg\alpha \in M$ but not both.

(iii)-(vii) Exercise.

Definition7.15: Let $\mathcal{M}$ be the collection of all maximal consistent subsets of W. It is not so hard to see that this is a non-empty set.
We define $[.] : W \to \wp(\mathcal{M})$ (where $\wp(\mathcal{M})$ is power set of $\mathcal{M}$) defined as: for each sentence $\alpha \in W$ we define

$$[\alpha] = \{M \in \mathcal{M} : \alpha \in M\}$$

.

Theorem7.16:
(i) $[\alpha] = \{M \in \mathcal{M} : M \vdash \alpha\}$.
(ii) $\vdash \alpha \to \beta$ iff $[\alpha] \subseteq [\beta]$.
(iii) $\vdash \alpha \leftrightarrow \beta$ iff $[\alpha] = [\beta]$.
(iv) $[\neg\alpha] = \mathcal{M} - [\alpha]$.
(v) For any theorem α, $[\alpha] = \mathcal{M}$.
(vi) For any theorem α, $[\neg\alpha] = \emptyset$.
(viii) $[\alpha \vee \beta] = [\alpha] \cup [\beta]$.
(vii) $[\alpha \wedge \beta] = [\alpha] \cap [\beta]$.
Proof: Exercise.

Corollary7.17: The collection $\mathcal{B}(\mathcal{M}) = \{[\alpha] : \alpha \in W\}$ forms a base for a topology on $\mathcal{M}$. We denote this topology by $\tau(\mathcal{M})$.

Theorem:
(i) $(\mathcal{M}, \tau(\mathcal{M}))$ is a zero-dimensional space, i.e. the topology $\tau(\mathcal{M})$ has a base consisting of clopen sets.
(ii) $(\mathcal{M}, \tau(\mathcal{M}))$ is a Hausdorff space.

(iii) $(\mathcal{M}, \tau(\mathcal{M}))$ is a second countable space.

(v) $\mathcal{B}(\mathcal{M})$ is a base for closed sets also.

Proof: (i) Since $[\alpha] \cap [\neg\alpha] = [\alpha \wedge \neg\alpha] = \emptyset$, so if $N \in \mathcal{M} - [\alpha]$, $N \in [\neg\alpha] = \mathcal{M} - [\alpha]$, so $[\alpha]$ is a closed set. So $\mathcal{B}(\mathcal{M})$ consisting of clopen sets. Hence (i) follows.

(ii) Let $M, N \in \mathcal{M}$ and $M \neq N$, so there exists $\alpha \in W$ such that α is only one of them, not in both, say $\alpha \in M$ and $\alpha \notin N$ (other case similar). So $\neg\alpha \in N$. Thus $M \in [\alpha]$ and $N \in [\neg\alpha]$. But we have $[\alpha] \cap [\neg\alpha] = \emptyset$. Hence (ii) follows.

(iii) The space is second countable since P is countable and also the set of connectives, so the set W is countable. So $\mathcal{B}(\mathcal{M})$ is a second countable space.

(v) Every open set U is of the form $\cup_{i\in\Lambda}[\alpha_i]$ where Λ is an indexing set. So every closed set is of the form $\mathcal{M} - \cup_{i\in\Lambda}[\alpha_i] = \cap_{i\in\Lambda}(\mathcal{M} - [\alpha_i]) = \cap_{i\in\Lambda}([\neg\alpha_i]) = \cap_{i\in\Lambda}([\beta_i])$.

Theorem7.18: $(\mathcal{B}(\mathcal{M}), \cup, \cap, \mathcal{M} - [.], \emptyset, \mathcal{M})$ forms a Boolean algebra.

8 Topological Semantics:

We have studied the semantics of propositional calculus as two two valued mappings. In this section we will study the topological semantics of propositional calculus. Here we consider the primitive connectives to be $\{\neg, \vee\}$ and other connectives are defined as defined before with $\neg$ and $\vee$.

For any topological space $(X, \tau(X))$ we denote $\tau_{clopen}(X)$ be the collection of all clopen sets in $(X, \tau(X))$.

Definition8.1: By a topological model $(X, \tau(X), v)$ of the language $\mathcal{L}$ of propositional logic a non-empty set X with a topology $\tau(X)$ on X and a mapping $v : P \to \tau_{clopen}(X)$.

Now we extend the mapping v to $v : W \to \tau_{clopen}(X)$ inductively as:

(i) $v(\neg\alpha) = X - v(\alpha)$.

(ii) $v(\alpha \vee \beta) = v(\alpha) \cup v(\beta)$.

Let $\mathcal{T}$ be the collection of all topological models of $\mathcal{L}$.

Exercise8.2: $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$.

We will say a formula α is valid for the model $(X, \tau(X), v)$, if $v(\alpha) = X$. α is said to be topologically valid if for any model in $\mathcal{T}$, α is valid. We write $\Vdash \alpha$.

Let $F \subseteq W$ and $\beta \in W$, we will write $F \Vdash \alpha$, if for any $(X, \tau(X), v) \in \mathcal{T}$, $v(\alpha) = X$ for any $\alpha \in F$ then $v(\beta) = X$.

If $F = \{\alpha_1, ..., \alpha_n\}$ we will write $\alpha_1, ..., \alpha_n \Vdash \beta$.

Exercise8.3: $\emptyset \Vdash \beta$ iff $\Vdash \beta$.

Exercise8.4: (i)For any topological model $(X, \tau(X), v)$ $v(\alpha) \subseteq v(\beta)$ iff $\Vdash \alpha \to \beta$.

(ii) $\Vdash \alpha \leftrightarrow \beta$ iff for any topological model $(X, \tau(X), v)$ $v(\alpha) = v(\beta)$.

Theorem8.5:(Soundness Theorem for Topological Semantics): If $\vdash \alpha$ then $\Vdash \alpha$.

Proof: Exercise.

Theorem8.6:(Completeness Theorem for Topological Semantics): If $\Vdash \alpha$ then $\vdash \alpha$.

Proof: We consider the topological space $(\mathcal{M}, \tau(\mathcal{M}))$. Now we define the mapping:

$$v : P \to \tau_{clopen}(\mathcal{M})$$

defined as $v(p) = [p]$. So $(\mathcal{M}, \tau(\mathcal{M}), v)$ is topological model for $\mathcal{L}$.

We claim that for any $\alpha \in W$, $v(\alpha) = [\alpha]$. We prove it by induction on the connectives:

(i) Let $\alpha = \neg\beta$ and $v(\beta) = [\beta]$. So $v(\alpha) = v(\neg\beta) = \mathcal{M} - v(\beta) = \mathcal{M} - [\beta] = [\neg\beta] = [\alpha]$.

(ii) $\alpha = \beta \vee \gamma$ and $v(\beta) = [\beta]$, $v(\gamma) = [\gamma]$. Now $v(\alpha) = v(\beta \vee \gamma) = v(\beta) \cup v(\gamma) = [\beta] \cup [\gamma] = [\beta \vee \gamma] = [\alpha]$.

Thus the claim is true.

Let $\Vdash \alpha$, so $v(\alpha) = \mathcal{M}$. So $\alpha \in M$ for every maximal consistent set M. If α in not a theorem then $\neg\alpha$ must be consistent (since if $\neg\alpha$ is inconsistent, then $\vdash \alpha$ which is not possible). Now $\{\neg\alpha\}$ can be extended to a maximal consistent set say M so that $\neg\alpha \in M$, but $\alpha \in M$, which is not possible. So $\vdash \alpha$.

9 Algebraic Semantics:

We have studied the semantics of propositional calculus as truth valued semantics and topological semantics.

It is easy to see that for any topological space $(X, \tau(X))$, $(\tau_{clopen}(X), \cup, \cap, X - (.), \emptyset, X)$ forms a Boolean algebra.

In this section we will study the algebraic semantics of propositional calculus. Here we consider the primitive connectives to be $\{\neg, \vee\}$ and other connectives are defined as defined before with $\neg$ and $\vee$.

Definition9.1: By an algebraic model we mean a Boolean algebra $(B, +, \cdot, ^{/}, 0, 1)$ or simply B along with a valuation $v : P \to B$. We denote the model by (B, v).

Now we extend the mapping v to $v : W \to B$ inductively as:

(i) $v(\neg\alpha) = v(\alpha)^{/}$.

(ii) $v(\alpha \vee \beta) = v(\alpha) + v(\beta)$.

Let $\mathfrak{B}$ be the collection of all Boolean models of $\mathcal{L}$.

Exercise9.2: $v(\alpha \wedge \beta) = v(\alpha) \cdot v(\beta)$.

We will say a formula α is valid for the model (B, v), if $v(\alpha) = 1$. α is said to be algebraically valid if for any model in $\mathfrak{B}$ α is valid. We write $\Vvdash \alpha$.

Let $F \subseteq W$ and $\beta \in W$, we will write $F \Vvdash \alpha$, if for any $(B, v) \in \mathfrak{B}$, $v(\alpha) = 1$ for any $\alpha \in F$ implies $v(\beta) = 1$.

If $F = \{\alpha_1, ..., \alpha_n\}$ we will write $\alpha_1, ..., \alpha_n \Vvdash \beta$.

Exercise9.3: $\emptyset \Vvdash \beta$ iff $\Vvdash \beta$.

Exercise9.4: (i) For any algebraic model (B, v) $v(\alpha) \leq v(\beta)$ iff $\Vvdash \alpha \to \beta$. Here $\leq$ is the partial order relation induced from the Boolean algebra B defined as $a \leq b$ iff $a + b = b$ equivalently iff $a \cdot b = a$.

(ii) $\Vvdash \alpha \leftrightarrow \beta$ iff for any algebraic model (B, v) $v(\alpha) = v(\beta)$.

Theorem9.5:(Soundness Theorem for Algebraic Semantics): If $\vdash \alpha$ then $\Vdash \alpha$.
Proof: Exercise.

Theorem9.6:(Completeness Theorem for Algebraic Semantics): If $\Vdash \alpha$ then $\vdash \alpha$.
Proof: We consider Boolean algebra $(\mathcal{B}(\mathcal{M}), \cup, \cap, \mathcal{M}-[.], \emptyset, \mathcal{M})$. Now we define the mapping:

$$v : P \to \mathcal{B}(\mathcal{M})$$

defined as $v(p) = [p]$. So $((\mathcal{B}(\mathcal{M}), \cup, \cap, \mathcal{M} - [.], \emptyset, \mathcal{M}), v)$ is algebraic model for $\mathcal{L}$.

It is easy already verified that for any $\alpha \in W$, $v(\alpha) = [\alpha]$. The remaining proof is same as that of topological semantics.

References

[1] A. H. Bassoon; D. J. O'connor, *Introduction to Symbolic Logic*, Oxford university press (1998).

[2] P. Hinman, *Fundamentals of Mathematical Logic*, A. K. Peters (2005).

[3] A. Margaris, *First Order Mathematical Logic*, Dover (1990).

[4] E. Mendelson, *Introduction to Mathematical Logic, 6e*, CRC Press (2015).

[5] H. Rasiowa, *An algebraic approach of non-classical logic*, Elsevier Science (1974).

[6] H. Rasiowa, R. Sikorski, *The Mathematics of Metamathematics*, PWN-Polish Scientific Publishers (1970).

[7] S. M. Sriastava, *A Course on Mathematical Logic, 2e*, Springer (2013).

[8] R. R. Stoll, *Set Theory and Logic*, Dover, (1979).