Statistics and inferencing

Activity 05

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⇒ Joint PDF:-

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{12}(x+y)e^{-y} & 0 \le x \le 4,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) find the optimal MSE estimator of X, $\hat{X}_{MSE}$ as follows:

$$\hat{X}_{MSE} = E[X|Y]$$

① Marginal PDF of Y

to find the conditional Pdf $f_{X|Y}(x|y)$ we first need to find the marginal Pdf of Y, → $f_Y(y)$

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\, dx$$

Substituting $f_{X,Y}(x|y) = \frac{1}{12}(x+y)e^{-y}$

$$f_Y(y) = e^{-y} \int_0^y \frac{1}{12}(x+y)\, dx$$

$$f_Y(y) = \frac{1}{12}\left[\frac{x^2}{2} + yx\right]_0^y \cdot e^{-y}$$

$$f_Y(y) = \frac{1}{12}\left[\frac{y^2}{2} + y(y)\right] \cdot e^{-y}$$

$$f_Y(y) = \frac{1}{12}[8+4y] \cdot e^{-y}$$

$$f_Y(y) = \frac{1}{12}(8+4y) \cdot e^{-y}$$

$$f_Y(y) = \frac{e^{-y}(8+4y)}{12}$$

(2) conditional pdf $f_{X|Y}(x|y)$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Substituting the values;

$$f_{X|Y}(x|y) = \frac{\frac{1}{12}(x+y)\cancel{e^{-y}}}{\frac{(8+4y)\cancel{e^{-y}}}{12}}$$

$$f_{X|Y}(x|y) = \frac{x+y}{8+4y} \quad (0 \leq x \leq 4)$$

(3) conditional expectation $E[X|Y=y]$

$$E[X|Y=y] = \int_0^4 x f_{X|Y}(x|y)\,dx$$

$$E[X|Y=y] = \int_0^4 x\left(\frac{x+y}{8+4y}\right)dx$$

$$E[X|Y=y] = \frac{1}{8+4y}\int_0^4 x(x+y)\,dx$$

$$E[x|y=y] = \frac{1}{8+4y}\left(\frac{64}{3} + 8y\right)$$

$$E[x|y=y] = \frac{64+24y}{3(8+4y)}$$

optimal estimator =

$$\hat{X}_{MSE} = \frac{64+24y}{3(8+4y)}$$

b) $$\hat{X}_{lMse} = \mu_x + \frac{\sigma_{xy}}{\sigma_y^2}(Y-\mu_y)$$

→ (1) finding $\mu_x = E[x]$

$$\mu_x = \int_0^\infty \int_0^4 x\, f_{x,y}(x,y)\, dx\, dy$$

$$\mu_x = \int_0^\infty \int_0^4 x \frac{1}{12}(x+y)e^{-y}\, dx\, dy$$

$$\Downarrow$$

Solving from matlab;

$$\boxed{\mu_x = 2}$$

↳ 2 finding $\mu_y \doteq E[y]$:

$$\mu_y = \int_0^\infty y f_y(y)\, dy$$

$$\mu_y = \int_0^\infty \frac{y(8+4y)e^{-y}}{12}\, dy$$

$\Downarrow$

evaluated directly through matlab

$$\boxed{\mu_y = 1}$$

↳ 3 finding $\sigma_{xy} \rightarrow$ covariance

$$\sigma_{xy} = E[xy] - \mu_x \mu_y$$

$$E[xy] = \int_0^\infty \int_0^y xy\, f_{x,y}(x,y)\, dx\, dy$$

$$E[xy] = \int_0^\infty \int_0^y xy \frac{1}{12}(x+y)e^{-y}$$

$$E[xy] = \frac{1}{12} \int_0^{\infty} ye^{-y}\, dy \int_0^{4} x(x+y)\, dx$$

$\Rightarrow$ inner integral

$$\int_0^{4} x^2 + xy \, dx = \frac{64}{3} + 8y$$

$$E[xy] = \frac{1}{12} \int_0^{\infty} ye^{-y} \left(\frac{64}{3} + 8y\right) dy$$

$$\Downarrow$$

Solving using matlab

$$E[xy] = \frac{28}{9}$$

$$\sigma_{xy} = \frac{28}{9} - \mu_x \mu_y$$

$$\sigma_{xy} = \frac{28}{9} - (2)(1)$$

$$\sigma_{xy} = \frac{28}{9} - 2 = \frac{10}{9}$$

(4) finding σ_y^2 $\rightarrow$ the variance of y

$$\sigma_y^2 = E[y^2] - \mu_y^2$$

$$E[y^2] = \int_0^\infty y^2 f_y(y)\, dy$$

$$E[y^2] = \int_0^\infty y^2 \frac{(8+4y)e^{-y}}{12}\, dy$$

$$E[y^2] = \frac{1}{12}\left(8\int_0^\infty y^2 e^{-y}\, dy + 4\int_0^\infty y^3 e^{-y}\, dy\right)$$

$$\Downarrow$$

Solved directly using matlab

$$E[y^2] = \frac{1}{12}(16+24) = \frac{10}{3}$$

$$\sigma_y^2 = E[y^2] - \mu_y^2 = \frac{10}{3} - 1 = \frac{7}{3}$$

$$\hat{X}_{lmse} = \mu_x + \frac{\sigma_{xy}}{\sigma^2_y}(y - \mu_y)$$

Substituting the known values

$$\hat{X}_{lmse} = 2 + \frac{10/9}{7/3}(y-1)$$

$$\hat{X}_{lmse} = 2 + \frac{10}{21}(y-1)$$