20-oct-24

Statistics and inferencing
Homework 1

Name: Basil Khowaja BK08432

Problem 1)

$$P(x=0) = \frac{2}{3}\theta$$

$$P(x=1) = 1/3\,\theta$$

$$P(x=2) = \frac{2}{3}(1-\theta)$$

$$P(x=3) = \frac{1}{3}(1-\theta)$$

$\Rightarrow$ finding $E[x]$:

$$E[x] = 0\cdot P(x=0) + 1\cdot P(x=1) + 2\cdot P(x=2) + 3\cdot P(x=3)$$

$$E[x] = 0\cdot\frac{2}{3}\theta + 1\cdot\frac{1}{3}\theta + 2\cdot\frac{2}{3}(1-\theta) + 3\cdot\frac{1}{3}(1-\theta)$$

$$E[x] = \frac{1}{3}\theta + \frac{7}{3}(1-\theta)$$

$$E[x] = \frac{1}{3}\theta + \frac{7}{3} - \frac{7}{3}\theta$$

$$E[x] = \frac{7}{3} - \frac{6}{3}\theta = \boxed{\frac{7}{3} - 2\theta}$$

Sample mean $\bar{X}$:

$$\bar{X} = \frac{3+0+2+1+3+2+1+0+2+1}{10} = \frac{15}{10}$$

$$\boxed{\bar{X} = 1.5}$$

Equating the $\bar{x} = E[x]$

$$\frac{7}{3} - 2\theta = 1.5$$

$$7 - 6\theta = 4.5$$

$$\theta = \frac{2.5}{6} = \frac{5}{12}$$

moments estimate $\theta = \frac{5}{12}$

b) find the mean and variance of MoM estimator

$\rightarrow$ expected value of $\bar{X}$

$\downarrow$

$$E[\bar{x}] = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$E[\bar{x}] = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = E[x] = \frac{7}{3} - 2\theta$$

①

$$\bar{x} = \frac{7}{3} - 2\theta$$

taking expected value of b/s

$$E[\bar{x}] = \frac{7}{3} - 2E(\hat{\theta})$$

Substituting ① here

$$\cancel{\frac{7}{3}} - 2\theta = \cancel{\frac{7}{3}} - 2E(\hat{\theta})$$

$$-2\theta = -2E(\hat{\theta})$$

mean $\Rightarrow$ $E(\hat{\theta}) = \theta$

Now variance:-

$$\theta = \frac{7}{6} - \frac{1}{2}\bar{x}$$

using the Property $y = a + bx$

$$\text{var}(y) = b^2 \text{var}(x)$$

here; $Y = \hat{\theta}$, $a = \frac{7}{6}$ and $b = \frac{-1}{2}$

$$Var(\hat{\theta}) = \left(\frac{-1}{2}\right)^2 var(\bar{X})$$

$$var(\hat{\theta}) = \frac{1}{4} var(\bar{x})$$

$$var(\bar{x}) = \frac{1}{n} var(x)$$

$$var(x) = E(x^2) - (E(x))^2$$

$$E(x) = \frac{7}{3} - 2\theta$$

$$E(x^2) = 0 + 1 \cdot \frac{1}{3}\theta + 4 \cdot \frac{2}{3}(1-\theta) + 9 \cdot \frac{1}{3}(1-\theta)$$

$$E(x^2) = \frac{17}{3} - \frac{16}{3}\theta$$

$$var(x) = \left(\frac{17}{3} - \frac{16}{3}\theta\right) - \left(\frac{7}{3} - 2\theta\right)^2$$

$$var(x) = \frac{17}{3} - \frac{16}{3}\theta - \frac{49}{9} + \frac{28}{3}\theta - 4\theta^2$$

$$var(x) = \frac{2}{9} + 4\theta(1-\theta)$$

$$var(\hat{\theta}) = \frac{1}{4} \cdot \frac{1}{n}\left(\frac{2}{9} + 4\theta(1-\theta)\right)$$

$$\Rightarrow \quad var(\hat{\theta}) = \frac{1}{4n}\left(\frac{2}{9} + 4\theta(1-\theta)\right)$$

c) find an approximate Standard error of your estimate;

Substituting $n=10$ and $\theta = 5/12$

$$var(\hat{\theta}) = \frac{1}{40}\left(\frac{2}{9} + \frac{35}{36}\right)$$

$$\boxed{var(\hat{\theta}) = \frac{43}{1440}}$$

$$SE(\hat{\theta}) = \sqrt{\frac{43}{1440}}$$

$$\boxed{SE(\hat{\theta}) = \sqrt{0.02986} \approx 0.1728}$$

d)

$$3, 0, 2, 1, 3, 2, 1, 0, 2, 1$$

$n_0 =$ no of 0's

$n_1 =$ no of 1's

$n_2 =$ no of 2's

$n_3 =$ no of 3's

$\Longrightarrow$ from the data

$n_0 = 2$

$n_1 = 3$

$n_2 = 3$

$n_3 = 2$

$$L(\theta) = \left(\frac{2}{3}\theta\right)^{n_0} \left(\frac{1}{3}\theta\right)^{n_1} \left(\frac{2}{3}(1-\theta)\right)^{n_2} \left(\frac{1}{3}(1-\theta)\right)^{n_3}$$

$$L(\theta) = \left(\frac{2}{3}\theta\right)^{2} \left(\frac{1}{3}\theta\right)^{3} \left(\frac{2}{3}(1-\theta)\right)^{3} \left(\frac{1}{3}(1-\theta)\right)^{2}$$

$$L(\theta) = \frac{2^5}{3^{10}} \theta^5 (1-\theta)^5$$

$$L(\theta) = \frac{32}{59049} \theta^5 (1-\theta)^5$$

$$\log L(\theta) = \log\left(\frac{32}{59049} \theta^5 (1-\theta)^5\right)$$

$$\log L(\theta) = 5\log(\theta) + 5\log(1-\theta)$$

differentiating;

$$\frac{d}{d\theta} \log L(\theta) = 5\frac{1}{\theta} - 5\frac{1}{1-\theta}$$

Setting derivative equal to zero

$$\frac{5}{\theta} - \frac{5}{1-\theta} = 0$$

$$\frac{1}{\theta} = \frac{1}{1-\theta}$$

$$\boxed{\theta = 1/2}$$

$$\boxed{\hat{\theta}_{MLE} = 1/2} \Rightarrow \text{Ans}$$

e) fisher information $I(\theta) = -E\left[\frac{d^2}{d\theta^2} \log L(\theta)\right]$

$$I(\hat{\theta}) = -\frac{d^2}{d\theta^2} \log L(\theta) \Big|_{\theta = \hat{\theta}_{MLE}}$$

$$\log L(\theta) = 5 \log(\theta) + 5 \log(1-\theta)$$

$$\frac{d}{d\theta} \log L(\theta) = 5 \cdot \frac{1}{\theta} - 5 \cdot \frac{1}{1-\theta}$$

$$\frac{d^2}{d\theta^2} \log L(\theta) = \frac{-5}{\theta^2} - 5 \cdot \frac{1}{(1-\theta)^2}$$

$$I(\hat{\theta}) = -\left[5 \cdot \frac{1}{(1/2)^2} - 5 \cdot \frac{1}{(1-1/2)^2}\right]$$

$$I(\hat{\theta}) = 20 + 20 = 40$$

$$SE(\hat{\theta}) = \sqrt{\frac{1}{I(\hat{\theta})}} = \sqrt{\frac{1}{40}}$$

$$\boxed{SE(\hat{\theta}) \approx 0.1581}$$

approximate Standard error of max likelihood

f) $P(\theta) = 1$ for $\theta \in [0,1]$

the Posterior distribution is Proportional to the Product of the likelihood and the Prior.

$$P(\theta \mid data) \propto L(\theta) \cdot P(\theta)$$

$$P(\theta \mid data) \propto \theta^5 (1-\theta)^5$$

$$P(\theta \mid data) = C \cdot \theta^5 (1-\theta)^5$$

finding constant C;

$$\int_\theta^1 P(\theta \mid data)\, d\theta = 1$$

$$\int_0^1 C \cdot \theta^5 (1-\theta)^5 \, d\theta = 1$$

using beta distribution;

$$\int_0^1 \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\alpha = 6$ and $\beta = 6$;

$$= \frac{\Gamma(6)\Gamma(6)}{\Gamma(12)}$$

using the fact that $\Gamma(n) = (n-1)!$

$$\Gamma(6) = 5! = 120$$

$$\Gamma(12) = 11! = 39916800$$

$$\int_0^1 \theta^5 (1-\theta)^5 d\theta = \frac{14400}{39916800} = \frac{1}{2772}$$

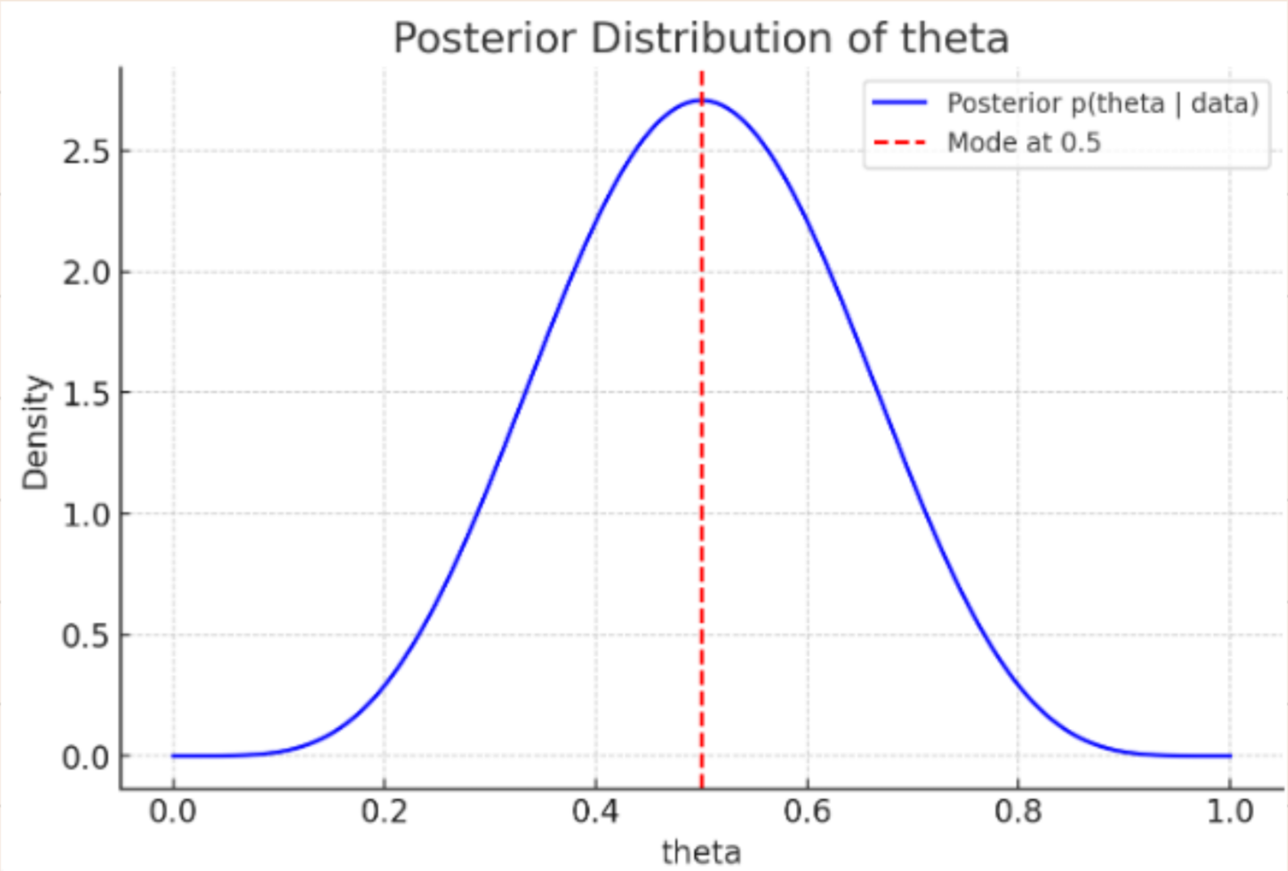
$$\boxed{c = 2772}$$

$$P(\theta | data) = 2772 \cdot \theta^5 (1-\theta)^5$$

$$\text{Mode} = \frac{\alpha - 1}{\alpha + \beta - 2}$$

$$\text{Mode} = \frac{5}{10} = \frac{1}{2}$$

Plotted using matlab;



red dashed line

Shows the mode of

the Posterior $\theta = 0.5$

Problem 2)

a)

$$P(x=1) = \theta$$

$$P(x=2) = 1-\theta$$

$$E[x] = 1 \cdot P(x=1) + 2 \cdot P(x=2)$$

$$E[x] = 1 \cdot \theta + 2(1-\theta)$$

$$E[x] = 2-\theta$$

$$\bar{x} = \frac{x_1+x_2+x_3}{3} = \frac{1+2+2}{3} = \frac{5}{3}$$

$$\bar{x} = E[x]$$

$$\frac{5}{3} = 2-\theta$$

$$\theta = \frac{6}{3} - \frac{5}{3} = \frac{1}{3}$$

$$\boxed{\hat{\theta}_{MOM} = \frac{1}{3}}$$

b)

$$L(\theta) = P(x=x_1) \cdot P(x=x_2) \cdot P(x=x_3)$$

$$L(\theta) = P(x=1) \cdot P(x=2) \cdot P(x=2)$$

$$L(\theta) = \theta \cdot (1-\theta) \cdot (1-\theta)$$

$$L(\theta) = \theta(1-\theta)^2 \quad \text{Ans}$$

c) maximum likelihood estimate of θ?

$$\log l(\theta) = \log\left[\theta(1-\theta)^2\right]$$

$$\log l(\theta) = \log(\theta) + 2\log(1-\theta)$$

$$\frac{d}{d\theta}\log l(\theta) = \frac{d}{d\theta}\left[\log(\theta) + 2\log(1-\theta)\right]$$

$$\frac{d}{d\theta}\log l(\theta) = \frac{1}{\theta} - 2\cdot\frac{1}{1-\theta}$$

$$\frac{d}{d\theta}\log l(\theta) = \frac{1}{\theta} - \frac{2}{1-\theta}$$

$$\frac{1}{\theta} - \frac{2}{1-\theta} = 0$$

$$\frac{1}{\theta} = \frac{2}{1-\theta}$$

$$1-\theta = 2\theta$$

$$\theta = \frac{1}{3}$$

$$\boxed{\hat{\theta}_{MLE} = \frac{1}{3}}$$

d) $$P(\theta \mid data) = \frac{L(\theta) \cdot P(\theta)}{\int_0^1 L(\theta) \cdot P(\theta)\, d\theta}$$

$$P(\theta \mid data) = \frac{\theta(1-\theta)^2}{\int_0^1 \theta(1-\theta)^2\, d\theta}$$

$$P(\theta \mid data) \propto \theta(1-\theta)^2$$

finding normalizing constant

$$\int_0^1 \theta(1-\theta)^2\, d\theta$$

$$\int_0^1 \theta(1-2\theta+\theta^2)\, d\theta$$

$$\int_0^1 \theta\, d\theta = \frac{1}{2}, \quad \int_0^1 \theta^2\, d\theta = \frac{1}{3}$$

$$\int_0^1 \theta^3\, d\theta = \frac{1}{4}$$

$$\frac{1}{2} - 2 \cdot \frac{1}{3} + \frac{1}{4} = \frac{1}{12}$$

$$P(\theta \mid data) = \frac{\theta(1-\theta)^2}{1/12}$$

$$\boxed{P(\theta \mid data) = 12 \cdot \theta (1-\theta)^2}$$

Problem 3)

a) $$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$L(p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\log L(p) = \log \left[\binom{n}{k} p^k (1-p)^{n-k} \right]$$

$$\log L(p) = \log \binom{n}{k} + \log \left[p^k (1-p)^{n-k} \right]$$

$$\log L(p) = \log \binom{n}{k} + k \log(p) + (n-k) \log(1-p)$$

$$\log L(p) = k \log(p) + (n-k) \log(1-p)$$

$$\frac{d}{dp} \log L(p) = \frac{k}{p} - \frac{n-k}{1-p}$$

$$\frac{k}{p} - \frac{n-k}{1-p} = 0$$

$$k(1-p) = (n-k)p$$

$$k - kp = np - kp$$

$$k = np$$

$$\boxed{p = \frac{k}{n}}$$

$$\boxed{\hat{p} = \frac{X}{n}} \quad \text{Ans//}$$

b) $$I(p) = -E\left[\frac{d^2}{dp^2} \log L(p)\right]$$

$$\log L(p) = k \log(p) + (n-k)\log(1-p)$$

$$\frac{d}{dp} \log L(p) = \frac{k}{p} - \frac{n-k}{1-p}$$

$$\frac{d^2}{dp^2} \log L(p) = \frac{-k}{p^2} - \frac{n-k}{(1-p)^2}$$

for $X \sim Bin(n, p)$

$$E(k) = np$$

$$I(p) = -E\left[\frac{-k}{p^2} - \frac{n-k}{(1-p)^2}\right]$$

$$I(p) = E\left[\frac{k}{p^2}\right] + E\left[\frac{n-k}{(1-p)^2}\right]$$

$$I(p) = \frac{np}{p^2} + \frac{n(1-p)}{(1-p)^2}$$

$$I(p) = \frac{n}{p(1-p)}$$

Cramer - Rao lower Bound;

$$\operatorname{var}(\hat{p}) \geq \frac{1}{I(p)}$$

$$\operatorname{var}(\hat{p}) \geq \frac{1}{\frac{n}{p(1-p)}} = \frac{p(1-p)}{n}$$

$$\operatorname{var}(\hat{p}) \geq \frac{p(1-p)}{n}$$

$$\operatorname{var}(x) = np(1-p)$$

$$\operatorname{var}(\hat{p}) = \frac{\operatorname{var}(x)}{n^2} = \frac{np(1-p)}{n^2}$$

$$\operatorname{var}(\hat{p}) = \frac{p(1-p)}{n}$$

variance of MLE $\hat{p}$: $\operatorname{var}(\hat{p}) = \frac{p(1-p)}{n}$

Cramer Rao lower bound

$$\text{var}(\hat{p}) \geq \frac{p(1-p)}{n}$$

conclusion:- $\hat{p} = \frac{X}{n}$ attains the

Cramer rao lower bound

$$\text{var}(\hat{p}) = \frac{p(1-p)}{n}$$

c) $$L(p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$L(p) = \binom{10}{5} p^5 (1-p)^5$$

$$\binom{10}{5} = 252$$

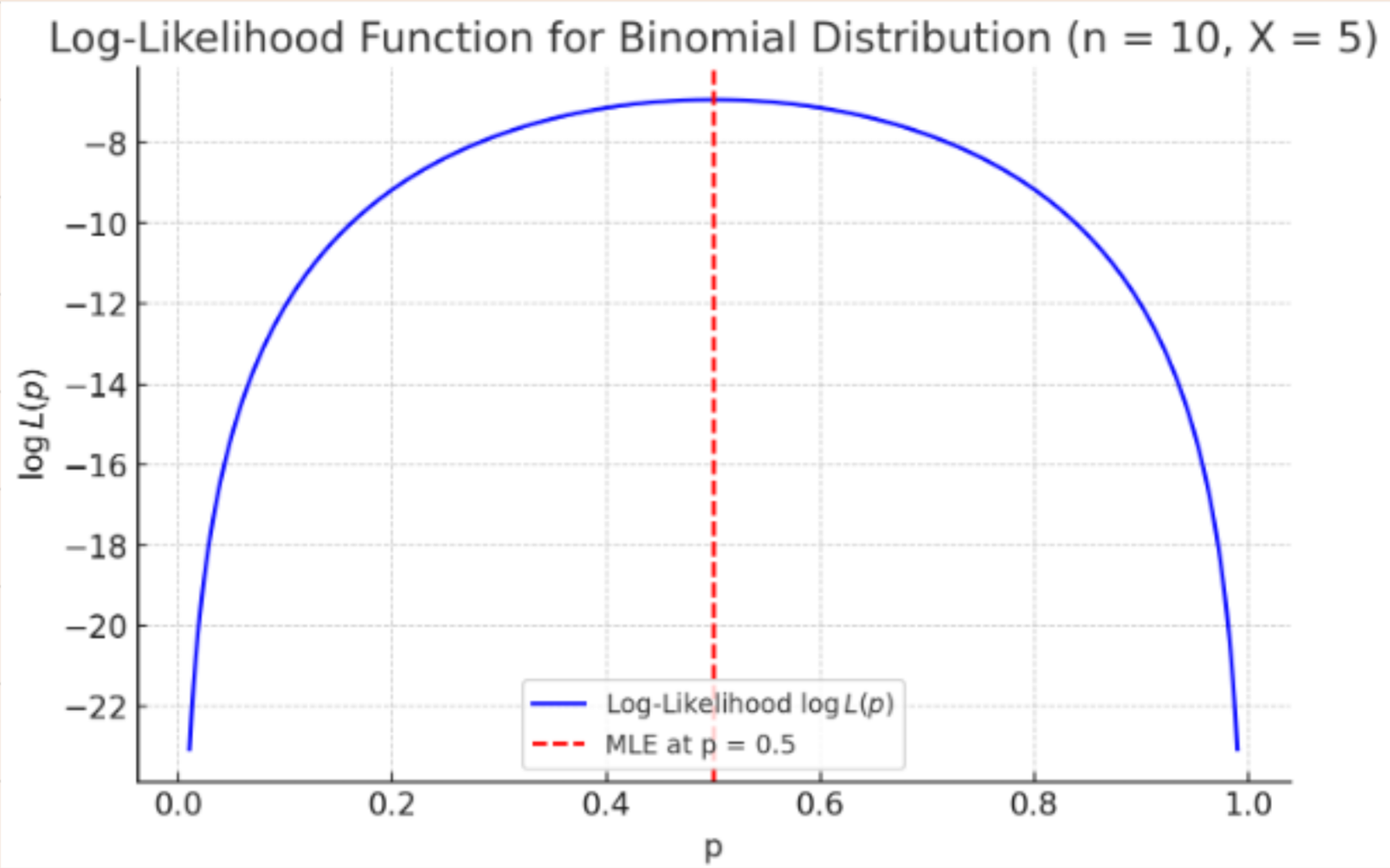
$$L(p) = 252 \cdot p^5 (1-p)^5$$

log-likelihood:-

$$\log L(p) = \log\left[252 \cdot p^5 (1-p)^5\right]$$

$$\log L(p) = \log(252) + 5\log(p) + 5\log(1-p)$$

$$\log L(p) = 5\log(p) + 5\log(1-p)$$



the red dashed line shows MLE of p, which occurs at $\boxed{p = 0.5}$

Problem 4)

a) $P(X=k) = p(1-p)^{k-1}, \quad k = 1, 2, 3$

$$E(X) = 1/p$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\bar{X} = E(X)$$

$$\bar{X} = \frac{1}{p}$$

$$p = \frac{1}{\bar{X}}$$

$$\boxed{\hat{p}_{mom} = \frac{1}{\bar{X}}}$$ Ans//

b) $\log L(p) = \log\left(p^n (1-p)^{s-n}\right)$

$$\log L(p) = n\log(p) + (s-n)\log(1-p)$$

$$\frac{d}{dp}\log L(p) = \frac{n}{p} - \frac{s-n}{1-p}$$

$$\frac{n}{p} = \frac{s-n}{1-p}$$

$$n(1-p) = p(s-n)$$

$$n - np = ps - np$$

$$p = \frac{n}{s} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\boxed{\hat{p}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i}}$$ Ans

c)

$$\frac{d}{dp} \log L(p) = \frac{n}{p} - \frac{S-n}{1-p}$$

$$\frac{d^2}{dp^2} \log L(p) = \frac{-n}{p^2} - \frac{S-n}{(1-p)^2}$$

$$I(p) = -E\left[\frac{d^2}{dp^2} \log L(p)\right]$$

$$= E\left[\frac{n}{p^2} + \frac{S-n}{(1-p)^2}\right]$$

using $E(S-n) = E(S) - n = \frac{n}{p} - n$

$$I(p) = \frac{n}{p^2} + \frac{n\left(\frac{1}{p}-1\right)}{(1-p)^2}$$

$$I(p) = \frac{n}{p^2}$$

$$\text{var}\left(\hat{p}_{MLE}\right) \approx \frac{1}{I(p)} = \frac{1}{n/p^2} = \frac{p^2}{n}$$

$$\boxed{\text{var}\left(\hat{p}_{MLE}\right) \approx \frac{p^2}{n}}$$ Ans

d) Prior distribution $P(p)$:

$$P(p) = 1 \quad 0 \leq p \leq 1$$

$$L(p) = p^n (1-p)^{s-n}$$

$$P(p \mid \text{data}) = \frac{L(p) \cdot P(p)}{\int_0^1 L(p) \cdot P(p) \, dp}$$

$$P(p \mid \text{data}) \propto L(p) = p^n (1-p)^{s-n}$$

$$p(p \mid \text{data}) \propto p^n (1-p)^{s-n}$$

$$\text{Beta}(\alpha, \beta) \propto p^{\alpha-1} (1-p)^{\beta-1}$$

comparing with $p^n (1-p)^{s-n}$

$$\alpha = n+1 \quad , \quad \beta = s-n+1$$

$$P(p \mid \text{data}) \sim \text{Beta}(n+1, s-n+1)$$

$$\text{Posterior mean} = \frac{\alpha}{\alpha+\beta}$$

$$// = \frac{n+1}{(n+1)+(s-n+1)} = \frac{n+1}{s+2}$$

Problem 5)

a) the theoritical mean of laplace distribution

$$E(x) = 0$$

but;

$$E(|x|) = \sigma$$

$$\bar{X}_{abs} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

$$\bar{X}_{abs} = \sigma$$

$$\hat{\sigma}_{MoM} = \bar{X}_{abs} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

b) $$L(\sigma) = \prod_{i=1}^{n} f(x_i \mid \sigma)$$

$$L(\sigma) = \frac{1}{(2\sigma)^n} \exp\left(- \frac{\sum_{i=1}^{n} |x_i|}{\sigma} \right)$$

$$\frac{d}{d\sigma} \log L(\sigma) = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} |x_i|}{\sigma^2}$$

$$-n\sigma + \sum_{i=1}^{n} |x_i| = 0$$

$$\hat{\sigma}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

c) $$I(\sigma) = -E\left[\frac{d^2}{d\sigma^2} \log L(\sigma)\right]$$

$$\frac{d}{d\sigma} \log L(\sigma) = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n} |x_i|}{\sigma^2}$$

$$\frac{d^2}{d\sigma^2} \log L(\sigma) = \frac{n}{\sigma^2} - \frac{2\sum_{i=1}^{n} |x_i|}{\sigma^3}$$

$$I(\sigma) = \frac{n}{\sigma^2}$$

$$\boxed{\operatorname{var}(\hat{\sigma}_{mle}) = \frac{1}{I(\sigma)} = \frac{\sigma^2}{n}}$$

d) $$L(\sigma) = \frac{1}{(2\sigma)^n} \exp\left(-\frac{\sum_{i=1}^{n} |x_i|}{\sigma}\right)$$

$$L(\sigma) = g(T(x), \sigma)\, h(x)$$

$$T(x) = \sum_{i=1}^{n} |x_i| \rightarrow \text{Ans}$$