Lecture 08
Approximation Theory

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Evaluation of Mean of $Y = g(X)$ Using Approximation Method

Linearization is carried out through a Taylor series expansion of g about μ_X, where $\mu_X = E[X]$. To the second order, we obtain

$$Y = g(X) \approx g(\mu_X) + (X - \mu_X)g'(\mu_X) + \frac{1}{2}(X - \mu_X)^2 g''(\mu_X)$$

We have expressed Y as approximately equal to a polynomial function of X.

This may be understood as if X is changing about its mean position μ_X by an amount ϵ, where ϵ has the same distribution as that of X except that it is zero-mean. This interpreation leads to an equivalent manipulation as given by (to the 2nd order):

$$\begin{aligned} Y = g(X) = g(\mu_X + \epsilon) &\approx g(\mu_X) + \epsilon g'(\mu_X) + \tfrac{1}{2!}\epsilon^2 g''(\mu_X) \\ &= g(\mu_X) + (X - \mu_X)g'(\mu_X) + \frac{1}{2}(X - \mu_X)^2 g''(\mu_X) \end{aligned}$$

Now, we take mean of both sides to obtain

$$E[Y] =: \mu_Y \approx g(\mu_X) + E[X - \mu_X]g'(\mu_X) + \frac{1}{2}E\left[(X - \mu_X)^2\right]g''(\mu_X)$$

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Note that $E[X - \mu_X] = E[X] - \mu_X = \mu_X - \mu_X = 0$, thus the middle term vanishes, and in the last term, we consider $\sigma_X^2 := E\left[(X - \mu_X)^2\right]$, this gives

$$E(Y) \approx g(\mu_X) + \frac{1}{2}\sigma_X^2 g''(\mu_X)$$

Evaluation of Variance of $Y = g(X)$ Using Approximation Method

For variance, the linearization by Taylor series expansion is sufficient to the first order,

$$\begin{aligned} Y = g(X) &\approx g(\mu_X) + (X - \mu_X)g'(\mu_X) \\ \sigma_Y^2 := \mathrm{var}(Y) &= \mathrm{var}(g(X)) \\ &\approx \mathrm{var}\left(g(\mu_X) + (X - \mu_X)g'(\mu_X)\right) \end{aligned}$$

Recalling that if $U = a + bV$, then $E(U) = a + bE(V)$ and $\mathrm{Var}(U) = b^2\,\mathrm{Var}(V)$, we find

$$\begin{aligned} \sigma_Y^2 &\approx \mathrm{var}\left(g(\mu_X) + (X - \mu_X)\right) \\ &= \left[g'(\mu_X)\right]^2 \mathrm{var}\left(X - \mu_X\right) \\ &= \left[g'(\mu_X)\right]^2 \mathrm{var}\left(X\right) =: \left[g'(\mu_X)\right]^2 \sigma_X^2 \end{aligned}$$

Evaluation of Mean and Variance of an Estimator $\hat{\theta} = h(T(X))$ Using Approximation Method, where $T(X)$ is some statistics, and X is the available data set

Using Taylor's series based expansion, we may obtain

$$\widehat{\theta} = h(T(X)) = h(\mu_T) + (T(X) - \mu_T)h'(\mu_T) + \tfrac{1}{2}(T(X) - \mu_T)^2 h''(\mu_T)$$

where $\mu_T = E\big[T(X)\big]$; next, we take mean of both sides to obtain

$$E[\widehat{\theta}] =: \mu_{\hat{\theta}} \approx h(\mu_T) + \tfrac{1}{2}E\Big[(T - \mu_T)^2\Big]h''(\mu_T) = h(\mu_T) + \tfrac{1}{2}\mathbf{var}\,\big(T(X)\big)h''(\mu_T)$$

Evaluation of Mean and Variance of an Estimator $\hat{\theta} = h(T(X))$ Using Approximation Method, where $T(X)$ is some statistics, and X is the available data set

As discussed earlier, we may easily find

$$\sigma^2_{\widehat{\theta}} = \left[g'(\mu_T)\right]^2 \sigma^2_T.$$

The estimation error is defined as $\widetilde{\theta} = \widehat{\theta} - \theta$, where θ is the true value, constant in nature, to be estimated, therefore the variance of $\widetilde{\theta}$ is same as that of $\widehat{\theta}$, this gives

$$\sigma^2_{\widetilde{\theta}} = \left[g'(\mu_T)\right]^2 \sigma^2_T.$$

Example

Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with the density function

$$f(x \mid \theta) = \frac{\theta}{(1+x)^{\theta+1}}, \quad 0 < \theta < \infty \text{ and } 0 \leq x < \infty$$

Find the method of moments (MoM) estimate of θ.
Find the mean and variance of MoM estimate.

Find the maximum likelihood (ML) estimate of θ.
Find the mean and variance of ML estimate.

(i) <u>ML Estimate</u>:

Owing to i.i.d. property we have

$$f(x|\theta) = \prod_{i=1}^{n} f(X_i|\theta) = \frac{\theta^n}{\prod_{i=1}^{n}(1+x_i)^{\theta+1}}$$

$$\ell(\theta) = \log f(x|\theta) = n\log(\theta) - (\theta+1)\sum_{i=1}^{n}\ln(1+X_i)$$

Taking derivative

$$\ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n}\ln(1+X_i) \Rightarrow \theta = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\ln(1+X_i)}$$

The ML estimator of θ is thus

$$\boxed{\hat{\theta}_{ML} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\ln(1+X_i)}}$$

② $\text{Var}(\hat{\theta}) = \dfrac{1}{n\, I(\theta_0)}$ <u>Variance of $\widehat{\theta}_{ML}$</u>

where $I(\theta_0) := \dfrac{-1}{n} E[\ell''(\theta_0)]$

So, in simple manner, we may write that

$$\text{var}(\hat{\theta}) = \text{Var}(\tilde{\theta}) = \frac{-1}{E[\ell''(\theta_0)]}$$

From previous slide, we obtain

$$\ell'(\theta_0) = \frac{n}{\theta_0} - \sum_{i=1}^{n} \ln(1+x_i)$$

$$\ell''(\theta_0) = \frac{-n}{\theta_0^2} \implies \boxed{\text{Var}(\tilde{\theta}) = \frac{\theta_0^2}{n}}$$

③

Mean of $\hat{\theta}_{ML}$

Let $T(x) := \frac{1}{n} \sum_{i=1}^{n} \ln(1+X_i)$

$$\Rightarrow \hat{\theta}_{ML} = h(T(x)) = \frac{1}{T(x)}$$

Let us find mean and variance of $T(x)$

$$\mu_T = E[T(x)] = E\left[\frac{1}{n} \sum_{i=1}^{n} \ln(1+X_i)\right] = E[\ln(1+X)]$$

$$E[\ln(1+X)] = \int_0^\infty \ln(1+x) \frac{\theta}{(1+x)^{\theta+1}} dx$$

$$= -\frac{(\theta \ln(1+x) + 1)}{\theta \cdot (1+x)^\theta} \Bigg|_0^\infty$$

$$= \frac{1}{\theta} \quad \text{(applying lower limit only)}$$

Using L'Hospital's rule, You may prove that the function vanishes when upper limit is applied

(4) $\mu_T = \frac{1}{\theta}$

$$\text{var}(T(X)) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} \ln(1+X_i)\right)$$

$$= \frac{1}{n^2}\,\text{var}\left(\sum_{i=1}^{n} \ln(1+X_i)\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \text{var}(\ln(1+X_i)) \qquad \text{i.i.d. property}$$

$$= \frac{1}{n}\,\text{var}(\ln(1+X))$$

$$\text{var}(\ln(1+X)) = E[\ln^2(1+X)] - \underbrace{\left(E[\ln(1+X)]\right)^2}_{\text{already calculated}}$$

$$E[\ln^2(1+X)] = \int_0^\infty \frac{\ln^2(1+x)\,\theta}{(1+x)^{\theta+1}}\,dx = \frac{2}{\theta^2} \quad (\text{using Mathematica})$$

⑤

$$\mathrm{Var}(\ln(1+x)) = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$\Rightarrow \mathrm{var}(T(x)) = \frac{1}{n\theta^2}$$

$$E[\hat{\theta}_{ML}] = h(\mu_T) + \frac{1}{2}\mathrm{var}(T(x))\, h''(\mu_T)$$

$$h(T) = \frac{1}{T} \Rightarrow h'(T) = -\frac{1}{T^2} \Rightarrow h''(T) = \frac{2}{T^3}$$

$$\Rightarrow h''(\mu_T) = \frac{2}{\mu_T{}^3} = \frac{2}{(1/\theta)^3} = 2\theta^3.$$

$$\Rightarrow E[\hat{\theta}_{ML}] = \theta + \frac{1}{2}\cdot\frac{1}{n\theta^2}\cdot 2\theta^3 = \underset{\text{true value}}{\theta} + \underbrace{\frac{\theta}{n}}_{\text{bias}}.$$

⑥ MoM Estimate

$$E[X] = \int_0^\infty \frac{x\,\theta}{(1+x)^{\theta+1}}\,dx = \frac{1}{\theta-1} \quad \text{(you solve it by yourself)}$$

$$\Rightarrow \theta = \frac{1}{E[X]} + 1 \quad \Rightarrow \quad \hat{\theta}_{MOM} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i} + 1$$

Let $T(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\Rightarrow \hat{\theta}_{MOM} = h(T(X)) = \frac{1}{T(X)} + 1$$

We need to find mean and variance of $T(X)$.

This is shown on next slides.

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$$E[T(x)] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = E[X] = \frac{1}{\theta-1} \cdot \left[\begin{array}{l}\text{As obtained}\\ \text{earlier}\end{array}\right]$$

$$\text{var}(T(x)) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\,\text{var}\left(\sum_i X_i\right) = \frac{\text{var}(x)}{n}$$

$$\text{var}(X) = E[X^2] - [E(X)]^2$$

$$E[X^2] = \int_0^\infty \frac{x^2\theta}{(1+x)^{\theta+1}}\,dx = \frac{2}{(\theta-2)(\theta-1)} \quad \left[\begin{array}{l}\text{Using online}\\ \text{Mathematica}\end{array}\right]$$

$$\Rightarrow \text{var}(x) = \frac{2}{(\theta-2)(\theta-1)} - \frac{1}{(\theta-1)^2} = \frac{\theta}{(\theta-2)(\theta-1)^2}$$

(valid for $\theta > 2$ only)

$$\Rightarrow \text{var}(T(x)) = \frac{\theta}{n(\theta-2)(\theta-1)^2}.$$

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$$\mathrm{var}(\hat{\theta}_{MOM}) = \mathrm{var}\left(\frac{1}{T(x)} + 1\right) =: \mathrm{var}\left(h(T(x))\right)$$

$$\approx \sigma^2_{T(x)} \left[h'(\mu_T)\right]^2$$

Since, $h(T) = \frac{1}{T} + 1 \Rightarrow h'(T) = -\frac{1}{T^2}$

$$\Rightarrow h'(\mu_T) = -\frac{1}{\mu_T^2} = -(\theta-1)^2$$

$$\sigma^2_{T(x)} = \mathrm{var}(T(x)) = \frac{\theta}{n(\theta-2)(\theta-1)^2}$$ (as computed earlier)

$$\Rightarrow \mathrm{var}(\hat{\theta}_{mom}) = \frac{\theta}{n(\theta-2)(\theta-1)^2} \cdot (\theta-1)^4 = \frac{\theta(\theta-1)^2}{n(\theta-2)}.$$

9 Now we find the mean of $\hat{\theta}_{MOM}$.

$$E[\hat{\theta}_{MOM}] \approx h(\mu_T) + \frac{1}{2}\sigma^2_{T(x)} h''(\mu_T)$$

$$h''(T) = \frac{2}{T^3} \Rightarrow h''(\mu_T) = \frac{2}{\mu_T{}^3} = 2(\theta-1)^3 \quad \left(\mu_T = \frac{1}{\theta-1}\right)$$

$$\Rightarrow E[\hat{\theta}_{MOM}] \approx \left(\frac{1}{1/(\theta-1)} + 1\right) + \frac{1}{\cancel{2}} \cdot \frac{\theta}{n(\theta-2)(\cancel{\theta-1})^{\cancel{2}}} \cdot \cancel{2}(\theta-1)^{\cancel{3}}$$

$$E[\hat{\theta}_{MOM}] = \underbrace{\theta}_{\text{true value}} + \underbrace{\frac{(\theta-1)\theta}{n(\theta-2)}}_{\text{bias in the MoM estimate.}}$$