

Problem 6: Suppose that X_1, X_2, \dots, X_n are i.i.d. random variables on the interval $[0, 1]$ with the density function

$$f(x | \alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} [x(1-x)]^{\alpha-1}$$

where $\alpha > 0$ is a parameter to be estimated from the sample. It can be shown that

$$E(X) = \frac{1}{2}, \quad \text{Var}(X) = \frac{1}{4(2\alpha + 1)}$$

- How does the shape of the density depend on α ?
- How can the method of moments be used to estimate α ?
- What equation does the MLE of α satisfy?
- What is the asymptotic variance of the MLE?
- Find a sufficient statistic for α .

Problem 7: The Pareto distribution has been used in economics as a model for a density function with a slowly decaying tail:

$$f(x | x_0, \theta) = \theta x_0^\theta x^{-\theta-1}, \quad x \geq x_0, \quad \theta > 1$$

Assume that $x_0 > 0$ is given and that X_1, X_2, \dots, X_n is an i.i.d. sample.

- Find the method of moments estimate of θ .
- Find the MLE of θ .
- Find the asymptotic variance of the MLE.
- Find a sufficient statistic for θ .

Problem 8: Let X_1, \dots, X_n be an i.i.d. sample from a Rayleigh distribution with parameter $\theta > 0$:

$$f(x | \theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad x \geq 0$$

- Find the method of moments estimate of θ .
- Find the MLE of θ .
- Find the asymptotic variance of the MLE.

Problem 9: Let X_1, \dots, X_n be i.i.d. random variables with the density function

$$f(x | \theta) = (\theta + 1)x^\theta, \quad 0 \leq x \leq 1$$

- Find the method of moments estimate of θ .
- Find the MLE of θ .
- Find the asymptotic variance of the MLE.
- Find a sufficient statistic for θ .

Problem 10: If gene frequencies are in equilibrium, the genotypes AA , Aa , and aa occur with probabilities $(1 - \theta)^2$, $2\theta(1 - \theta)$, and θ^2 , respectively. Plato et al. (1964) published the following data on haptoglobin type in a sample of 190 people:

| Haptoglobin Type | | |
|------------------|-------|-------|
| Hp1-1 | Hp1-2 | Hp2-2 |
| 10 | 68 | 112 |

- Find the MLE of θ .
- Find the asymptotic variance of the MLE.
- Find an approximate 99% confidence interval for θ .
- Use the bootstrap to find the approximate standard deviation of the MLE and compare it to the result of part (b).
- Use the bootstrap to find an approximate 99% confidence interval and compare to part (c).