Statistics and inferencing

Activity - 07

Name: Basil Khowaja BK08432

⟹ Problem 01)

a) using the fact that the pdf of the sum of two independent random variables is the convolution of the individual PDF's show that

$$f_Y(y) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_2 y}\left(e^{(\lambda_2-\lambda_1)y} - 1\right)$$

↳ which is the two parameter hypo-exponential distribution.

Sol://

we are given that $Y = X + V$

where $X \sim \exp(\lambda_1)$ and $V \sim \exp(\lambda_2)$

with $\lambda_1 \neq \lambda_2$, the pdf's of X and V are:-

$$f_X(x) = \lambda_1 e^{-\lambda_1 x}, \quad x \geq 0$$

$$f_V(v) = \lambda_2 e^{-\lambda_2 v}, \quad v \geq 0$$

$\Rightarrow$ we need to show that the PDF of $Y = X + V$, $f_Y(y)$ is the convolution

$$f_Y(y) = \int_0^y f_X(x) f_V(y-x)\,dx$$

Substituting the PDF's of X and V:

$$\Rightarrow \quad f_Y(y) = \int_0^y \lambda_1 e^{-\lambda_1 x}\, \lambda_2 e^{-\lambda_2 (y-x)}\,dx$$

Simplifying:

$$\Rightarrow \quad f_Y(y) = \lambda_1 \lambda_2 e^{-\lambda_2 y} \underbrace{\int_0^y e^{(\lambda_2-\lambda_1)x}\,dx}$$

$$\int_0^y e^{(\lambda_2-\lambda_1)x}\,dx = \frac{e^{(\lambda_2-\lambda_1)y}-1}{\lambda_2-\lambda_1}$$

$$\Rightarrow \quad f_Y(y) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 y}\left(e^{(\lambda_2-\lambda_1)y}-1\right)$$

$\Rightarrow$ Simplifying further;

$$f_Y(y) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_2 y}\left(e^{(\lambda_2-\lambda_1)y} - 1\right)$$

b) establish that

$$f_{X,Y}(x,y) = \lambda_1 \lambda_2 e^{(\lambda_2-\lambda_1)x - \lambda_2 y}$$

for $x \geq 0$ and $y \geq 0$

Sol:// Since X and V are independent the joint distribution can be written as:-

$$f_{X,Y}(x,y) = f_X(x) f_V(y-x)$$

Since; $f_x(x) = \lambda_1 e^{-\lambda_1 x}$, $f_V(y-x) = \lambda_2 e^{-\lambda_2(y-x)}$

thus the joint pdf is:-

$$f_{X,Y}(x,y) = \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2(y-x)}$$

Simplifying the expression;

$$f_{X,Y}(x,y) = \lambda_1 \lambda_2 e^{-\lambda_2 y} e^{(\lambda_2-\lambda_1)x}$$

c) from Part (a) the marginal pdf $f_Y(y)$

$$f_Y(y) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 y}\left(e^{(\lambda_2-\lambda_1)y}-1\right)$$

from Part (b) the joint PDF

$$f_{X,Y}(x,y) = \lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)x-\lambda_2 y}$$

$$\hat{X}_{MSE} = \frac{\int_0^y x\,\lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)x-\lambda_2 y}\,dx}{f_Y(y)}$$

Substituting $f_Y(y)$ as found from Part (a)

$$\hat{X}_{MSE} = \frac{\int_0^y x\,\lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)x-\lambda_2 y}\,dx}{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 y}\left(e^{(\lambda_2-\lambda_1)y}-1\right)}$$

$$\hat{X}_{MSE} = \frac{(\lambda_2-\lambda_1)\int_0^y x e^{(\lambda_2-\lambda_1)x}\,dx}{e^{(\lambda_2-\lambda_1)y}-1}$$

using integration by Parts

$$dv = e^{(\lambda_2-\lambda_1)x}\,dx$$

$$v = \frac{e^{(\lambda_2-\lambda_1)x}}{\lambda_2-\lambda_1}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^y x e^{(\lambda_2-\lambda_1)x}\,dx = \underbrace{\left[\frac{x e^{(\lambda_2-\lambda_1)x}}{\lambda_2-\lambda_1}\right]_0^y}_{= \frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1}} - \underbrace{\int_0^y \frac{e^{(\lambda_2-\lambda_1)x}}{\lambda_2-\lambda_1}\,dx}_{\frac{1}{(\lambda_2-\lambda_1)^2}\left(e^{(\lambda_2-\lambda_1)y}-1\right)}$$

$$\int_0^y x e^{(\lambda_2-\lambda_1)x}\,dx = \frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \frac{e^{(\lambda_2-\lambda_1)y}-1}{(\lambda_2-\lambda_1)^2}$$

$$\hat{X}_{MSE} = \frac{(\lambda_2 - \lambda_1)\left(\frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \frac{e^{(\lambda_2-\lambda_1)y}-1}{(\lambda_2-\lambda_1)^2}\right)}{e^{(\lambda_2-\lambda_1)y} - 1}$$

$\Rightarrow$ Simplifying the expression;

$$\hat{X}_{MSE} = \frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \frac{e^{(\lambda_2-\lambda_1)y}-1}{(\lambda_2-\lambda_1)^2} \cdot \frac{(\lambda_2-\lambda_1)}{e^{(\lambda_2-\lambda_1)y}-1}$$

$$\hat{X}_{MSE} = \frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \frac{1}{\lambda_2-\lambda_1}$$

$$\hat{X}_{MSE} = \frac{1}{\lambda_2-\lambda_1} - \frac{y e^{(\lambda_1-\lambda_2)y}}{e^{(\lambda_2-\lambda_1)y}-1}$$

$\Rightarrow$ this is the required mean-squared error estimate of X, given $Y=y$.