HABIB UNIVERSITY

Midterm-I Exam (Part B)
CE 362: Statistics and Inference
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Timing: 4:00 - 6:00 PM
Dated: Oct 30, 2022
Duration: 120 min

Q1 [20 points]: Consider identically and independently distributed (iid) samples X_i for $i = 1, 2, \ldots, n$ from the Rayleigh probability density function (pdf)

$$f(x_i \mid \sigma) = \frac{x_i}{\sigma^2} \exp\left(-\frac{1}{2}\frac{x_i^2}{\sigma^2}\right).$$

Derive the Neyman-Pearson test

$$L(\boldsymbol{X}) = \frac{f(\boldsymbol{X}; \mathcal{H}_1)}{f(\boldsymbol{X}; \mathcal{H}_0)} > \gamma$$

for the hypothesis testing problem

$$\begin{aligned} \mathcal{H}_0 &: \sigma^2 = \sigma_0^2 \\ \mathcal{H}_1 &: \sigma^2 = \sigma_1^2 > \sigma_0^2. \end{aligned}$$

Q2 [30 points]: Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ where $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$ are unknown parameters. Let

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

be the sample mean which aims to estimate the unknown μ.
Note that $\overline{X}$ is a random variable.

(a) Prove that the pdf of $\overline{X}$ is given by

$$f(x) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp\left(-\frac{n}{2\sigma^2}(x-\mu)^2\right) \qquad \text{for } x \in (-\infty, \infty)$$

Note: You need to evaluate mean and variance of $\overline{X}$.

(b) Evaluate the Cramer-Rao lower bound (crlb), i.e., $\operatorname{var}(\widehat{\mu} - \mu) \approx \dfrac{1}{nI(\mu)}$, where $nI(\mu) = -E[\ell''(\mu)]$, and $\ell(\mu) = \sum_{i=1}^{n} \log f(x_i \mid \mu)$ is the log likelihood function.

(c) Based on crlb, argue if $\overline{X} = \widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the optimal estimator of μ.

Q3 [20 points]: Suppose that $X_1, \ldots, X_n$ are iid from the Weibull pdf

$$f(x \mid \alpha) = \alpha^{-1} \beta\, x^{\beta-1} \exp\left(-x^{\beta}/\alpha\right), \quad x > 0$$

where $\alpha (> 0)$ is the unknown parameter, but $\beta (> 0)$ is assumed known.

(a) Using Neyman factorization theorem, obtain the sufficient statistic for α.

(b) Using maximum likelihood method, obtain an estimator of α.

Q4 [20 points]: Suppose that $X_1, \ldots, X_n$ are iid from the Beta pdf

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1} (1-x)^{\beta-1},$$
$$0 < x < 1, \alpha > 0, \beta > 0$$

where α is the unknown parameter, but β is assumed known.

(a) Using Neyman factorization theorem, obtain the sufficient statistic for α.

(b) Using maximum likelihood method, obtain an estimator of α.

Q5 [30 points]: This problem is concerned with the estimation of the variance of a normal distribution with unknown mean from a sample $X_1, \ldots, X_n$ of i.i.d. normal random variables. During problem sessions, we have discussed the evaluation of mean-squared errors (mse) of the following two estimators of variance:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2, \quad \text{and} \quad \widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

Now consider another estimator of variance as given by

$$P = \rho \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

where $\rho > 0$. We can instantly notice that P can be written in terms of S^2 or $\widehat{\sigma^2}$ as

$$P = \rho \cdot (n-1) \cdot S^2 = \rho \cdot n \cdot \widehat{\sigma^2}$$

Neatly proving all steps, prove that the value of ρ which minimizes the mse of P is

$$\rho = \frac{1}{n+1}.$$

Q1

$$\mathcal{H}_0 : \sigma^2 = \sigma_0^2$$

$$\mathcal{H}_1 : \sigma^2 = \sigma_1^2$$

$$p(x_i|\mathcal{H}_0) = \frac{x_i}{\sigma_0^2} \exp\left(-\frac{1}{2}\frac{x_i^2}{\sigma_0^2}\right)$$

$$p(\bar{x}|\mathcal{H}_0) = \frac{1}{(\sigma_0^2)^n}\left[\prod_{i=1}^{n} x_i\right] \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i}^{n} x_i^2\right)$$

$$p(x_i|\mathcal{H}_1) = \frac{x_i}{\sigma_1^2} \exp\left(-\frac{1}{2}\frac{x_i^2}{\sigma_1^2}\right)$$

$$p(\bar{x}|\mathcal{H}_1) = \frac{1}{(\sigma_1^2)^n}\left[\prod_{i=1}^{n} x_i\right] \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n} x_i^2\right)$$

$$\mathcal{L}(x) = \frac{p(\bar{x}|\mathcal{H}_1)}{p(\bar{x}|\mathcal{H}_0)} > \gamma$$

$$= \frac{(\sigma_0^2)^n \exp\left(-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n} x_i^2\right)}{(\sigma_1^2)^n \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} x_i^2\right)}$$

$$\left(\frac{\sigma_0}{\sigma_1}\right)^{2n} \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{i=1}^{n} x_i^2\right) > \gamma$$

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2 > \ln\left(\frac{\sigma_1^{2n}}{\sigma_0^{2n}}\gamma\right)$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 > \frac{\ln\left(\frac{\sigma_1^{2n}}{\sigma_0^{2n}}\gamma\right)}{\frac{N}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)} = \gamma'$$

Q2. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$E\bar{X} = \frac{1}{n}\sum_{i=1}^{n} EX_i$$

since $X_i \sim N(\mu, \sigma^2)$, $EX_i = \mu$.

$$E\bar{X} = \frac{1}{n}\cdot n \cdot \mu = \mu \text{ (unbiased)}$$

$$\text{var}(\bar{X}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{var}\left(\sum_i X_i\right)$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n}\text{var}(X_i) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq n}}^{n}\text{cov}(X_i, X_j)\right]$$

$$\text{cov}(X_i, X_j) = E(X_i - \bar{X}_i)(X_j - \bar{X}_j) = 0$$

because X_i & X_j are independent.

$$\text{var}(\bar{X}) = \frac{1}{n^2}\cdot n \cdot \text{var}(X_i) = \frac{\text{var}(X_i)}{n}$$

$$\text{var}(\bar{X}) = \sigma^2/n$$

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

which immediately gives.

$$f_{\bar{X}}(x) = \frac{1}{\sqrt{\frac{2\pi\sigma^2}{n}}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2/n}\right).$$

(b) $$\text{lik}(x) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2/n}\right)$$

$$\begin{aligned} l(x) &= \log \text{lik}(x) \\ &= \frac{1}{2}\log\left(\frac{n}{2\pi\sigma^2}\right) - \frac{n(x-\mu)^2}{2\sigma^2} \end{aligned}$$

$$l'(x) = \frac{\partial l(x)}{\partial \mu} = \frac{+2n(x-\mu)}{2\sigma^2}$$

$$l''(x) = \frac{\partial^2 l(x)}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\text{var}(\mu-\hat{\mu}) = \frac{1}{-E\, l''(\mu)} = \frac{\sigma^2}{n} = \text{C.R.L.B.}$$

CRLB stands for Cramer - Rao lower bound

Since $\text{var}(\bar{X}) = \text{var}(\mu-\hat{\mu}) = \frac{\sigma^2}{n}$

Therefore $\bar{X} = \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the optimal estimator of μ.

Q3. $f(x|\alpha) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right)$

(a) Since β is known, only α is an only unknown parameter.

$$f(x|\alpha) = \underbrace{\beta x^{\beta-1}}_{=: h(x)} \underbrace{\frac{1}{\alpha} \exp\left(-\frac{x^\beta}{\alpha}\right)}_{g(x;\alpha)}$$

if $X = \{X_1, X_2, \ldots\ldots, X_n\}$, then

$$f(x|\alpha) = \underbrace{\beta^n \prod_{i=1}^{n} x_i^{\beta-1}}_{=: h(x)} \underbrace{\frac{1}{\alpha^n} \exp\left(-\frac{1}{\alpha} \sum_{i=1}^{n} x_i^\beta\right)}_{=: g(x;\alpha)}$$

The sufficient statistic for α is

$$T(X) = \sum_{i=1}^{n} X_i^\beta \quad \left(\begin{array}{c}\text{assuming} \\ \beta \text{ is known}\end{array}\right)$$

Q3 (b)

$$lik(\alpha) = \frac{1}{\alpha^n} \exp\left(-\frac{1}{\alpha} \Sigma_{i=1}^{n} X_i^{\beta}\right)$$

$$l(\alpha) = \log lik(\alpha)$$
$$= -n \log(\alpha) - \frac{1}{\alpha} \Sigma_{i=1}^{n} X_i^{\beta}$$

$$l'(\alpha) = \frac{\partial l(\alpha)}{\partial \alpha} = -\frac{n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} X_i^{\beta}$$

Substituting $l'(\alpha) = 0$, we get

$$\hat{\alpha} = \frac{1}{n} \Sigma_{i=1}^{n} X_i^{\beta}$$

The m.l. estimator of α is thus found in closed-form.

Q4. (a) β is known; α is unknown.

$$f(x|\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Applying Neyman factorisation, we get

$$f(x|\alpha) = h(x)\, g(x|\alpha)$$

where

$$h(x) = \frac{(1-x)^{\beta-1}}{\Gamma(\beta)}$$

and $g(x|\alpha) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha-1}$

for multivariate $X = \{X_1, X_2, \ldots, X_n\}$

$$g(x|\alpha) = \frac{\Gamma(\alpha+\beta)^n}{\Gamma(\alpha)^n} \prod_{i=1}^{n} x_i^{\alpha-1}.$$

Q5. $P = \rho(n-1)S^2$

$$EP = E[\rho(n-1)S^2]$$
$$= \rho(n-1)\,E[S^2]$$
$$= \rho(n-1)\sigma^2$$

For the proof of $ES^2 = \sigma^2$, refer to problem session II ppt slides.

Note $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\Rightarrow \quad S^2 = \frac{P}{\rho(n-1)} \Rightarrow \frac{P}{\rho\sigma^2} \sim \chi^2_{n-1}$$

$$var(P) = (\rho\sigma^2)^2\, var(\chi^2_{n-1})$$
$$= \rho^2\sigma^4\, 2(n-1)$$
$$= 2\rho^2\sigma^4(n-1)$$

$$MSE(P) = var(P) + Bias^2(P)$$

where $var(P) = 2\rho^2\sigma^4(n-1)$

$Bias(P) = EP - \sigma^2 = \rho(n-1)\sigma^2 - \sigma^2$

$$MSE(P) = 2\rho^2\sigma^4(n-1) + [\rho(n-1)-1]^2\sigma^4$$

$\frac{\partial MSE(P)}{\partial \rho} = 0$ to find optimal value of ρ

$$(2\rho)(2\sigma^4(n-1)) + 2\sigma^4(\rho(n-1)-1)(n-1)$$

$$\rho(4\sigma^4(n-1) + 2\sigma^4(n-1)^2) = 2\sigma^4(n-1)$$

$$\rho = \frac{2(n-1)}{4(n-1) + 2(n-1)^2}$$

$$= \frac{2(n-1)}{2n^2 - 4n + 2 + 4n - 4}$$

$$= \frac{2(n-1)}{2(n^2-1)} = \frac{n-1}{n^2-1} = \frac{1}{n+1}$$ [proved].