Lab Manual: Introduction to Z-Transform and Pre-Multiplication Technique

1 Introduction

In many signal processing applications, certain sequences may not be absolutely summable, making it difficult to compute their Fourier transform. To address this issue, we modify the sequence by pre-multiplying it with an exponential sequence, ensuring absolute summability. This method allows us to analyze and process a wider class of signals using the Z-transform.

This lab explores the pre-multiplication of a discrete-time signal $h[n]$ by an exponential term r^n, where r is a positive real constant that we control. This transformation ensures that the modified sequence can be analyzed using the discrete-time Fourier transform (DTFT) and subsequently the Z-transform.

2 Pre-Multiplication and Summability

Consider the sequence:

$$h[n] = 2^n u[n] \tag{1}$$

where $u[n]$ is the unit step function. This sequence grows exponentially and is not absolutely summable, meaning its Fourier transform does not exist. However, if we multiply it by an exponential factor r^n, we obtain:

$$h[n]r^n = (2r)^n u[n] \tag{2}$$

By choosing an appropriate r, we can ensure that the resulting sequence is absolutely summable. To analyze this, consider the sum:

$$S = \sum_{n=0}^{\infty} (2r)^n \tag{3}$$

which is a geometric series with a common ratio $2r$. The series converges if:

$$|2r| < 1 \Rightarrow r < \frac{1}{2} \tag{4}$$

This means that for $r < \frac{1}{2}$, the sequence becomes absolutely summable.

2.1 Effect of Different Values of r

- When $r = 1$, the sequence remains unchanged and still diverges.
- When $r = 2$, the sequence grows even faster, worsening divergence.
- When $r = 4$, the sequence transforms into a decreasing power-law sequence, ensuring absolute summability.
- For any $r < \frac{1}{2}$, the sum converges, making the sequence absolutely summable.

3 Mathematical Formalization

To formalize this, we define the DTFT of $h[n]r^n u[n]$ as:

$$H(r, \omega) = \sum_{n=-\infty}^{\infty} (h[n]r^n)e^{-j\omega n} \tag{5}$$

which depends on two parameters: r and ω. This transformation expresses $h[n]$ in terms of basis functions $re^{j\omega}$ instead of $e^{j\omega}$.

3.1 Defining a Complex Variable

$$z = re^{j\omega} \tag{6}$$

We rewrite the above equation as the bilateral Z-transform:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{7}$$

which is a fundamental tool for analyzing discrete-time systems.

For example, consider:

$$H(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} \tag{8}$$

Using the sum of an infinite geometric series:

$$H(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for} \quad |z| > |\alpha| \tag{9}$$

This result shows that the Z-transform is valid in the region where $|z| > |\alpha|$, which defines the Region of Convergence (ROC).

4 Poles and Zeros in Z-Transform

In the Z-transform representation of a system, poles and zeros play a crucial role in defining system behavior.

- **Zeros:** The values of z for which $H(z) = 0$.
- **Poles:** The values of z for which $H(z)$ tends to infinity.

For the example function:

$$H(z) = \frac{1}{1 - \alpha z^{-1}} \tag{10}$$

- The zero occurs at $H(z) = 0$, which does not exist in this case.
- The pole occurs when the denominator is zero:

$$1 - \alpha z^{-1} = 0 \Rightarrow z = \alpha \tag{11}$$

The locations of poles and zeros determine system stability and frequency response characteristics. The Region of Convergence (ROC) is crucial in identifying whether the system is stable or causal:

1. **Causal Systems:** The ROC extends outward from the outermost pole.
2. **Anti-Causal Systems:** The ROC extends inward from the innermost pole.
3. **Two-Sided Systems:** The ROC lies between two poles.

Graphically, poles and zeros are plotted on the Z-plane, which provides insights into system behavior, filtering properties, and response characteristics.

5 Understanding Partial Fraction Expansion Using MATLAB's `residuez`

To analyze a system in the Z-domain, one common approach is partial fraction expansion. This method helps break down a transfer function into simpler components, making it easier to analyze the system's behavior.

5.1 Steps to Perform Partial Fraction Expansion

1. **Define the System's Coefficients**
2. **Compute the Partial Fraction Expansion** using MATLAB's `residuez` function.
3. **Represent the System in Partial Fractions**.
4. **Visualizing Poles and Zeros** using MATLAB functions.

6 Task 1

1. Implement the pre-multiplication technique on $h[n] = 2^n u[n]$ and plot its summability for different values of r.
2. Compute and visualize the Z-transform of the modified sequence using MATLAB.
3. Identify and plot the poles and zeros of a given transfer function using MATLAB.
4. Use `residuez` to perform partial fraction expansion on a given system.
5. Analyze the system stability and ROC based on the computed poles and zeros.

7 Understanding Partial Fraction Expansion with `residuez`

Before performing partial fraction expansion, recall that the **Z-transform** of a rational function can be decomposed into simpler terms using MATLAB's `residuez` function:

$$X(z) = \sum \frac{r_i}{1 - p_i z^{-1}}$$

where:

- r_i are the **residues** (numerator coefficients of partial fractions),
- p_i are the **poles** (denominator roots of the system),
- The function may also include **direct terms** if applicable.

Task:

1. Express the given system as a rational function in z^{-1}.
2. Use `residuez` to compute its partial fraction expansion.
3. Identify and plot the **poles and zeros**.

Task 2: Compute the Z-Transform using `residuez`

Given the sequence:

$$x[n] = 2^n u[n]$$

Instructions:

- Derive the corresponding **Z-transform** in rational form.
- Compute the **partial fraction expansion** using `residuez`.
- **Plot poles and zeros** to visualize the system behavior.

Task 3

Evaluate the partial fraction expansion and analyze the poles and zeros for the following rational functions:

1.
$$X(z) = \frac{1 + 0.5z^{-1}}{1 - 1.5z^{-1} + 0.7z^{-2}}$$

2.
$$X(z) = \frac{2 + 3z^{-1}}{1 - 0.8z^{-1} - 0.2z^{-2}}$$

3.
$$X(z) = \frac{4 - z^{-1} + 0.5z^{-2}}{1 + 0.3z^{-1} - 0.6z^{-2}}$$

For each case:

- Compute the partial fraction expansion.
- Determine the **poles and zeros**.

8 Right-Hand and Left-Hand Sequences

8.1 Introduction

In discrete-time signal processing, sequences are classified as **right-sided (causal)** or **left-sided (anti-causal)** based on their definition.

- **Right-Sided (Causal) Sequences:** Defined for $n \geq n_0$. Their **Region of Convergence (ROC)** is the exterior of a circle bounded by a pole.
- **Left-Sided (Anti-Causal) Sequences:** Defined for $n \leq n_0$. Their **ROC** is the interior of a circle. While they may seem unrealizable, they are useful for inverse filtering.

Example of a left-sided impulse response:

$$h[n] = \alpha^n u[-n-1]$$

This sequence is mathematically unusual but has practical filtering applications.

9 Instructions

9.1 Task 4: Pole-Zero Analysis

1. Given the transfer function:

$$H(z) = \frac{z}{z - \alpha}$$

   Identify **poles** and **zeros** for different values of α.

2. Consider the following values for α:

$$\alpha = 0.5, 2, -0.5, \frac{1}{2}, -2$$

3. **Tasks to complete:**
   - **Pole-Zero Plots:**
     - Plot the pole-zero diagram for each α.
     - Clearly label poles ('X') and zeros ('O').
     - Plot the transfer function.

4. **Complete the following table:**

α	Monotonic	Alternating	Convergent	Constant	Divergent
$\alpha = 0.5$					
$\alpha = 1$					
$\alpha = 2$					
$\alpha = -0.5$					
$\alpha = -1$					
$\alpha = -2$					

9.2 Task 5: Sequence Properties and Behavior

1. **Repeat the pole-zero analysis** for the given transfer function:

$$H(z) = \frac{z}{z - \alpha}$$

   Identify poles and zeros for each α.

2. **Plot the pole-zero diagrams.**

3. **Complete the following table analyzing sequence behavior:**

10 Analysis Questions

1. How does the pole position affect system stability?
2. What determines whether a sequence is monotonic or alternating?
3. How does the pole location impact system convergence?

α	Monotonic	Alternating	Convergent	Constant	Divergent
$\alpha = 0.5$					
$\alpha = 1$					
$\alpha = 2$					
$\alpha = -0.5$					
$\alpha = -1$					
$\alpha = -2$					

11 Importance of Stability, Convergence, Divergence, Causality, and Non-Causality in Z-Transform

11.1 Stability

- A stable system ensures bounded output for a bounded input (BIBO stability).
- Stability is crucial in control systems, signal processing, and communication systems to prevent unbounded growth in response.
- Stability is determined by the Region of Convergence (ROC) of the Z-transform and the location of poles.
- If all poles lie inside the unit circle in the z-plane, the system is stable.

11.2 Convergence and Divergence

- The Z-transform must converge to ensure a meaningful frequency-domain representation.
- Convergence depends on the ROC; if the ROC exists and includes the unit circle, the system is stable.
- Divergent systems yield unbounded outputs and are not practically useful in most applications.

11.3 Causality and Non-Causality

- A causal system depends only on present and past inputs, making it realizable in real-time applications.
- A system is causal if its ROC extends outward from the outermost pole.
- Non-causal systems may be useful in theoretical analysis and pre-processing applications but are not physically realizable in real-time.
- Causality is important in digital filters, communication systems, and time-domain processing.

12 Transform - Theoretical Concepts and Exercises

12.0.1 Zeros and Poles

- The values of z in the finite z-plane for which $H(z) = 0$ are called the zeros of $H(z)$.
- The values of z in the finite z-plane for which $H(z) \to \infty$ are called the poles of $H(z)$.

12.0.2 Region of Convergence (ROC) and Stability

- If the ROC includes the unit circle, then the system is stable.
- If the ROC of $H(z)$ includes the unit circle, then $H(\omega)$ exists.
- The ROC of the Z-transform of a right-sided sequence extends from the outermost pole to infinity.
- If a right-sided system is stable, all the poles must be inside the unit circle.

12.0.3 Properties of ROC

- The ROC cannot include a pole.
- The ROC can include a zero.
- A right-sided system is causal if the number of poles is greater than or equal to the number of zeros.

13 Task 6

13.1 MATLAB Implementation

For each of the given transfer functions, perform the following tasks:

- Find the zeros and poles.
- Determine the impulse response.
- Check if the system is stable, unstable, causal, or non-causal.
- Explain whether the system is converging or diverging.

13.2 Given Transfer Functions

1. $H(z) = \frac{5-26z^{-1}+39.5z^{-2}-23.5z^{-3}+8z^{-4}-2z^{-5}}{1-5z^{-1}+8.5z^{-2}-6z^{-3}+2z^{-4}}$
2. $H(z) = \frac{3z^2-z-0.75}{z^2}, \quad |z| > 0.5$
3. $H(z) = \frac{3-2z^{-1}+0.5z^{-2}}{1-1.5z^{-1}+z^{-2}-0.25z^{-3}}$
   - Case 1: $|z| > 0.5$
   - Case 2: $|z| > 2$
4. $H(z) = \frac{1}{1-1.25z^{-1}+0.5z^{-2}-0.0625z^{-3}}$
   - Given in factored form: $H(z) = \frac{1}{(1-0.25z^{-1})(1-0.5z^{-1})^2}$
5. $H(z) = \frac{1+0.4z^{-1}-2.2z^{-2}+0.8z^{-3}}{1-1.3z^{-1}+0.4z^{-2}}$
   - Case 1: $|z| > 0.8$
   - Case 2: $|z| > 0.2$

14 LCCDE-Based Analysis

14.1 Overview of LCCDE Systems

Linear Constant-Coefficient Difference Equations (LCCDE) describe discrete-time linear systems where the relationship between input and output is expressed using past and present values. These equations are fundamental in signal processing, control systems, and digital filtering. The general form of an LCCDE is:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

where $y[n]$ is the output, $x[n]$ is the input, and a_k, b_m are the system coefficients. The impulse response $h[n]$ characterizes system behavior and stability.

15 Task 7

15.1 Problem 1

A stable system is characterized by the Linear Constant-Coefficient Difference Equation (LCCDE):

$$y[n] + \frac{1}{4}y[n-1] = x[n] + \frac{1}{2}x[n-1]$$

(a) Determine the impulse response $h[n]$.

(b) Use MATLAB to verify the result from part (a) by computing the first five values of the impulse response.

15.2 Problem 2

Consider the given LCCDE:

$$y[n] + \frac{1}{4}y[n-2] = x[n] + \frac{1}{2}x[n-2]$$

(a) Find the impulse response $h[n]$.

(b) Determine the input $x[n]$ that results in the output:

$$y[n] = \left(\frac{1}{2}\right)^n u[n]$$