. Digital Signal Processing | Instructor: Dr. Shafayat Abrar
. Lab 08 Analog filter Design: II | Demonstrator: Ahmad Bilal

1 Objectives

1. Explore methods for designing analog lowpass filters using the Chebyshev polynomials.
2. Explore methods for designing analog bandpass filters using lowpass filters.

2 Chebyshev Lowpass Filter

The squared-magnitude function for a Chebyshev lowpass filter is given by

$$|H(\omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\omega/\omega_1)} \tag{1}$$

Parameter ε is a positive constant, N is the order of the filter, and ω_1 is the passband edge frequency in rad/s. The parameter ε may be termed a *ripple factor* as it depends on the allowable ripple amplitude $-R_p$ (dB).[1] The term $C_N(\nu)$ represents the **Chebyshev polynomial of order** N. Chebyshev filters of this type are sometimes called Chebyshev type-I filters.

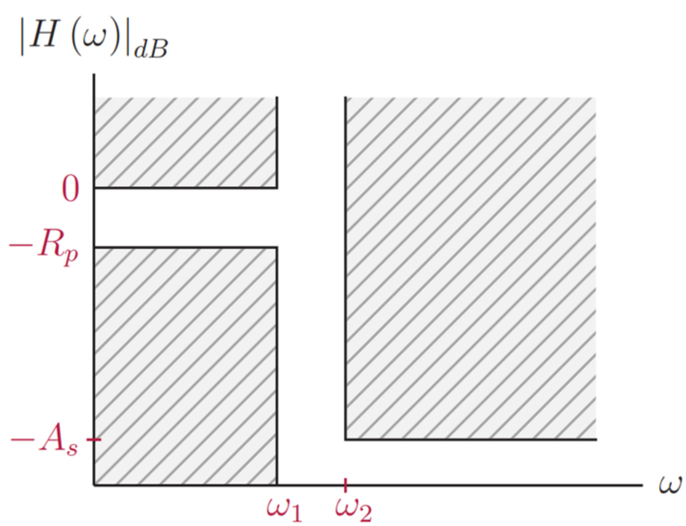

Figure 1: Decibel tolerance specifications for analog lowpass filter.

[1]The value of ε satisfies the range $0 < \varepsilon < 1$. The larger the value of ε the smaller the value of R_p (dB).

Given dB tolerances R_p and A_s and critical frequencies ω_1 and ω_2:

$$\omega_0 = \frac{\omega_2}{\omega_1}, \quad \textbf{and} \quad F = \sqrt{\frac{10^{A_s/10} - 1}{10^{R_p/10} - 1}}$$

$$N = \frac{\cosh^{-1}(F)}{\cosh^{-1}(\omega_0)} \quad \Longrightarrow \quad \textbf{Round up to next integer}$$

Compute ε from one of the following:

$$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -R_p \quad \textbf{or} \quad 10\log_{10}\left(\frac{1}{1+\varepsilon^2 C_N^2(\omega_0)}\right) = -A_s$$

Example 01: Suppose, $R_p = 2$ dB. We compute ε as follows:

$$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -2.$$

Divide both sides by 10:

$$\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.2.$$

$$\frac{1}{1+\varepsilon^2} = 10^{-0.2} \approx 0.63096.$$

$$1+\varepsilon^2 = \frac{1}{0.63096} \approx 1.5849.$$

$$\varepsilon^2 = 1.5849 - 1 = 0.5849.$$

$$\varepsilon = \sqrt{0.5849} \approx 0.7648.$$

Example 02: Find N and ε given $\omega_1 = 50$, $\omega_2 = 60$, $R_p = 3$ dB, and $A_s = 30$ dB.

Compute ω_0

$$\omega_0 = \frac{\omega_2}{\omega_1} = \frac{60}{50} = \boxed{1.2}.$$

Use the passband ripple formula:

$$\varepsilon = \sqrt{10^{R_p/10} - 1}.$$

Substitute $R_p = 3$ dB:

$$\varepsilon = \sqrt{10^{0.3} - 1} \approx \sqrt{1.99526 - 1} \approx \sqrt{0.99526} \approx \boxed{0.9976}.$$

To compute F, use the formula, and substitute $A_s = 30\,\text{dB}$ and $R_p = 3\,\text{dB}$:

$$F = \sqrt{\frac{10^{A_s/10} - 1}{10^{R_p/10} - 1}} = \sqrt{\frac{10^3 - 1}{10^{0.3} - 1}} = \sqrt{\frac{999}{0.99526}} \approx \sqrt{1003.85} \approx \boxed{31.69}.$$

To compute N, use the formula:

$$N = \frac{\cosh^{-1}(F)}{\cosh^{-1}(\omega_0)}.$$

1. Compute $\cosh^{-1}(F)$: For $F \approx 31.69$, use $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ in calculator:

$$\cosh^{-1}(31.69) \approx \ln(31.69 + \sqrt{31.69^2 - 1}) \approx \ln(63.36) \approx 4.147.$$

2. Compute $\cosh^{-1}(\omega_0)$: For $\omega_0 = 1.2$:

$$\cosh^{-1}(1.2) \approx \ln(1.2 + \sqrt{1.2^2 - 1}) \approx \ln(1.8633) \approx 0.622.$$

3. Calculate N:

$$N = \frac{4.147}{0.622} \approx 6.667.$$

Since filter order N must be an integer, **round up**: we get $\boxed{N = 7}$

Once the values of the parameters ε, N and ω_1 are specified, the design procedure proceeds with determining the poles of the filters.

2.1 Chebyshev Polynomials

The Chebyshev polynomial of order N is defined as

$$C_N(\nu) = \cos\left(N \cos^{-1}(\nu)\right)$$

A better approach, to understanding the definition, would be to split it into two equations as

$$\nu = \cos(\theta)$$

and

$$C_N(\nu) = \cos(N\theta)$$

The Chebyshev polynomial of a specified order N can be obtained by
1. Using trigonometric identities to express $\cos(N\theta)$ as a function of $\cos(\theta)$.
2. Replacing each $\cos(\theta)$ term with ν.
The first two polynomials are easy to obtain from this definition:

$$C_0(\nu) = 1$$
$$C_1(\nu) = \nu$$

To obtain $C_2(\nu)$ let us write $\cos(2\theta)$ in terms of $\cos(\theta)$ using a trigonometric identity:

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

Thus, the second-order Chebyshev polynomial is

$$C_2(\nu) = 2\nu^2 - 1$$

Higher-order Chebyshev polynomials may be obtained by continuing in this fashion. As the order increases, however, the procedure outlined above becomes increasingly tedious. Fortunately, it is possible to derive a recursive formula to facilitate the derivation of higher-order Chebyshev polynomials.

Let us write $\cos((N+1)\theta)$ and $\cos((N-1)\theta)$ using trigonometric identities:

$$\cos((N+1)\theta) = \cos(\theta)\cos(N\theta) - \sin(\theta)\sin(N\theta)$$
$$\cos((N-1)\theta) = \cos(\theta)\cos(N\theta) + \sin(\theta)\sin(N\theta)$$

Adding the two equations results in

$$\cos((N+1)\theta) + \cos((N-1)\theta) = 2\cos(\theta)\cos(N\theta)$$

which can be rearranged to produce the recursive formula we seek:

$$\cos((N+1)\theta) = 2\cos(\theta)\cos(N\theta) - \cos((N-1)\theta)$$

Thus, the recursive formula for obtaining Chebyshev polynomials is

$$C_{N+1}(\nu) = 2\nu C_N(\nu) - C_{N-1}(\nu)$$

The recursive formula in Eqn. (10.70) allows any order Chebyshev polynomial to be found provided that the polynomials for the previous two orders are known. With the knowledge of $C_0(\nu) = 1$ and $C_1(\nu) = \nu$, the polynomial $C_2(\nu)$ can be found as

$$C_2(\nu) = 2\nu C_1(\nu) - C_0(\nu) = 2\nu^2 - 1$$

Similarly $C_3(\nu)$ is found as

$$C_3(\nu) = 2\nu C_2(\nu) - C_1(\nu) = 2\nu\left(2\nu^2 - 1\right) - \nu = 4\nu^3 - 3\nu$$

2.2 Poles for the Chebyshev Lowpass Filter

In constructing the system function for a Chebyshev lowpass filter we will use an approach similar to that taken with a Butterworth lowpass filter in Section 10.4.1. First, we will determine the poles of the product $H(s)H(-s)$. Recall that the product $H(s)H(-s)$ is obtained by starting with the squared magnitude function $|H(\omega)|^2$ and replacing $j\omega$ with s :

$$H(s)H(-s) = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{s}{j\omega_1}\right)} \tag{2}$$

The poles p_k of $H(s)H(-s)$ are the solutions of the equation

$$1+\varepsilon^2 C_N^2\left(\frac{p_k}{j\omega_1}\right) = 0 \tag{3}$$

for $k = 0, \ldots, 2N-1$.

Let us define

$$\nu_k = \frac{s}{j\omega_1}$$

so that

$$1+\varepsilon^2 C_N^2\left(\nu_k\right) = 0$$

Using the definition of the Chebyshev polynomial, we obtain

$$1+\varepsilon^2 \cos^2\left(N\theta_k\right) = 0$$

where $\nu_k = \cos\left(\theta_k\right)$. Isolating $\cos\left(N\theta_k\right)$, it yields

$$\cos\left(N\theta_k\right) = \pm\frac{j}{\varepsilon}$$

Let $\theta_k = \alpha_k + j\beta_k$ with α_k and β_k both as real parameters, this gives

$$\cos\left(N\alpha_k + jN\beta_k\right) = \pm\frac{j}{\varepsilon}$$

which, using the appropriate trigonometric identity, can be written as

$$\cos\left(N\alpha_k\right)\cos\left(jN\beta_k\right) - \sin\left(N\alpha_k\right)\sin\left(jN\beta_k\right) = \pm\frac{j}{\varepsilon}$$

Recognizing that $\cos\left(jN\beta_k\right) = \cosh\left(N\beta_k\right)$ and $\sin\left(jN\beta_k\right) = j\sinh\left(N\beta_k\right)$, we obtain

$$\cos\left(N\alpha_k\right)\cosh\left(N\beta_k\right) - j\sin\left(N\alpha_k\right)\sinh\left(N\beta_k\right) = \pm\frac{j}{\varepsilon}$$

Equating real and imaginary parts of both sides of the above equation yields

$$\cos\left(N\alpha_k\right)\cosh\left(N\beta_k\right) = 0$$

and

$$\sin\left(N\alpha_k\right)\sinh\left(N\beta_k\right) = \pm\frac{1}{\varepsilon}$$

Since $\cosh\left(N\beta_k\right)$ cannot be equal to zero, the cosine term must be set equal to zero, leading to

$$\cos\left(N\alpha_k\right) = 0 \quad \Longrightarrow \quad \alpha_k = \frac{(2k+1)\pi}{2N}, \quad k = 0, \ldots, 2N-1 \tag{4}$$

To solve for β_k, we can show that

$$\sin\left(N\alpha_k\right) = \pm 1 \quad \Longrightarrow \quad \beta_k = \frac{\sinh^{-1}(1/\varepsilon)}{N} \quad \forall k \tag{5}$$

Using the values of α_k and β_k found, the poles of $H(s)H(-s)$ are

$$p_k = j\omega_1 \left[\cos\left(\alpha_k\right)\cosh\left(\beta_k\right) - j\sin\left(\alpha_k\right)\sinh\left(\beta_k\right)\right]$$

or

$$p_k = \omega_1 \sin\left(\alpha_k\right)\sinh\left(\beta_k\right) + j\omega_1 \cos\left(\alpha_k\right)\cosh\left(\beta_k\right) \tag{6}$$

for $k = 0, \ldots, 2N-1$. The poles in the left half s-plane are associated with $H(s)$ to obtain a causal and stable *Chebyshev* lowpass filter.

Task 1: Write a generic MATLAB script to compute and plot the magnitude and phase spectra of a lowpass Chebyshev filter. Your code

1. **evaluates N and ε for the given values of ω_1, ω_2, R_p dB, and A_s dB.**
2. **identifies the necessary poles of the transfer function in the left half of the s-plane and determines them accordingly (for the computed N and given ω_1).**
3. **displays the magnitude and phase spectra in separate subplots.**

Task 2: Write a simpler MATLAB script to compute and plot the magnitude and phase spectra of a lowpass Chebyshev filter. Your code

1. **evaluates ε for the given value of R_p dB.**
2. **identifies the necessary poles of the transfer function in the left half of the s-plane and determines them accordingly for the given N and ω_1.**
3. **displays the magnitude and phase spectra in separate subplots.**

Task 3: Compare the magnitude and phase spectrum of the lowpass Chebyshev filters for N considering $N = 3, 4,$ and 5. Repeat the exercise for $N = 6, 7,$ and 8.

1. **Use `hold on` in the magnitude and phase spectrum subplots to enable comparison.**
2. **Use different line styles to distinguish plots for various values of N and include a `legend` indicating the corresponding N values.**
3. **Finally, record your observations on the flatness or ripples of the magnitude spectrum in the pass and stop bands (the effects of ε), the fall-off beyond the edge frequencies, and the frequency range of the linear phase spectrum as the filter order N increases.**

2.3 Chebyshev Bandpass Filters

A Chebyshev bandpass filter may easily be obtained from a Chebyshev lowpass filter. All we need to do is use the modulation property of the Fourier transform. Consider the spectra of LowPass and BandPass filters as shown below:

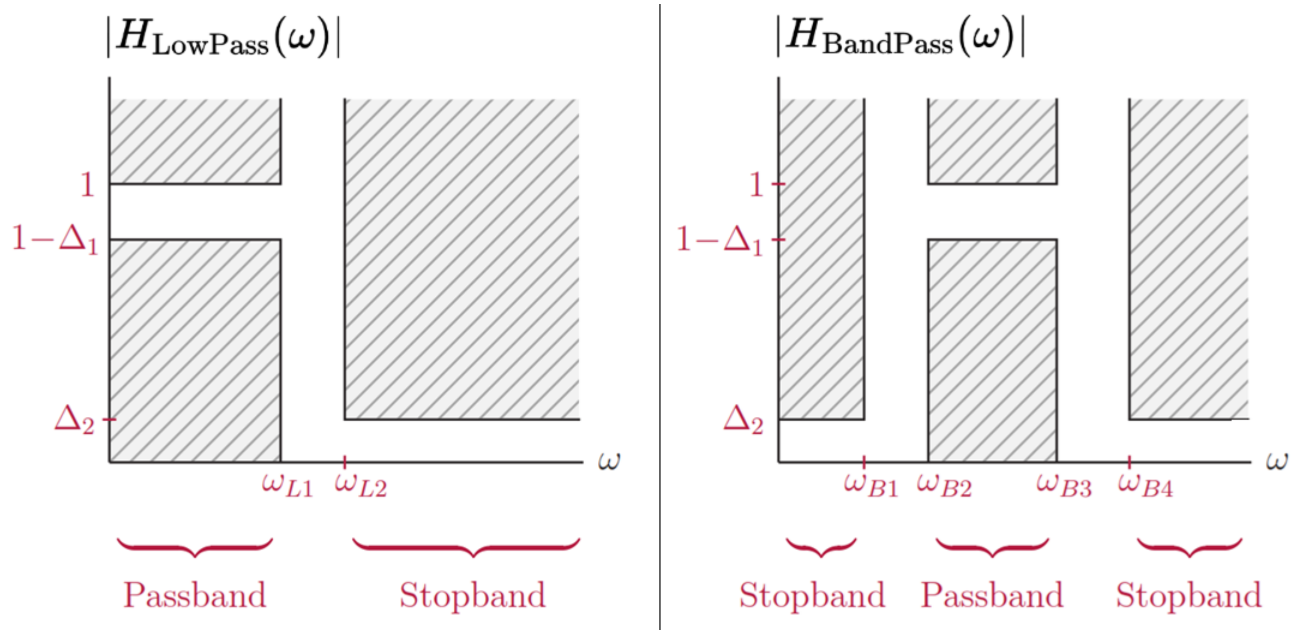

Figure 2: The frequency spectra of the LowPass and BandPass filters.

Let ω_{mid} be the middle frequency of the passband and computed as $\omega_{\text{mid}} = (\omega_{\text{B2}} + \omega_{\text{B3}})/2$; see Fig. 2. The impulse response $h_{\text{BandPass}}(t)$ of BandPass filter may be obtained from the impulse response $h_{\text{LowPass}}(t)$ of LowPass filter as follows:

$$h_{\text{BandPass}}(t) = 2h_{\text{LowdPass}}(t)\cos(\omega_{\text{mid}}t)$$

So, if the LowPass filter bandwidth is W, then the bandwidth of BandPass filter is $2W$.

Task 4: Design an experiment using an Nth-order Chebyshev bandpass filter to eliminate both low- and high-frequency components from input signals, and retain only the middle frequency.

1. **Perform this experiment for $N = 3, 5,$ and 7.**
2. **Note: The time axis should be sufficiently long to ensure that the impulse response of the filter fully settles to zero (i.e., $h(t)$ reaches its steady-state value).**
3. **Show the plots for each experiment using a 2×2 subplots. The first subplot displays the original signal containing three tones. The second subplot shows the bandpass filter impulse response $h(t)$. The third subplot presents the amplitude spectrum $H(f)$, and the fourth subplot depicts the filtered signal, which is supposed to extract the middle tone.**