. Digital Signal Processing — Instructor: Dr. Shafayat Abrar

. Lab 06 Analog filter Design — Demonstrator: Ahmad Bilal

1 Objectives

1. Develop the concept of analog filters.
2. Conditions for distortionless transmission
3. Learn ideal filter behavior.
4. Learn design specifications for realizable filters.
5. Explore methods for designing realizable analog filters using the Butterworth formula.

2 Introduction

filtering modifies the strength or significance of frequency components in a signal. Sometimes, it removes certain frequencies; other times, it amplifies a specific range. This process is called **filtering**, and the system enabling it is a **filter**.

In the prerequisite course on Signals and Systems, we analyzed continuous-time linear time-invariant (CT-LTI) systems using transform-domain techniques. The input-output relationship in the transform domain is given by:

$$Y(f) = H(f)X(f), \quad \text{where } H(f) = |H(f)|e^{i\measuredangle H(f)}$$

in terms of Fourier transforms or

$$Y(s) = H(s)X(s)$$

using Laplace transforms. The system function $H(f)$ or $H(s)$ shapes the input signal's spectrum to produce the output.

Any CT-LTI system can be viewed as a **filter**. For example, a communication channel that transmits an information-bearing signal can be modeled as a CT-LTI system, as shown in fig. 1.

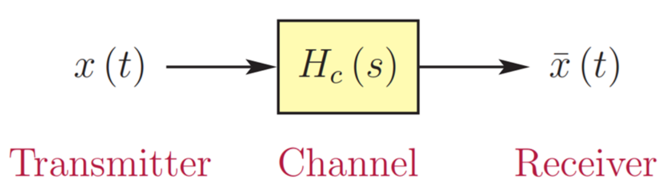

Figure 1: CT-LTI model for a communication channel.

Ideally, the output signal $\bar{x}(t)$ should match the transmitted signal $x(t)$, but in practice, $\bar{x}(t)$ is a distorted version of $x(t)$. Thus, the channel acts as an **unintended filter**. An **equalizer** can be cascaded with the channel to counteract this distortion, as shown in fig. 2. The equalizer's role is to minimize distortion, ensuring the output $y(t)$ better represents the original signal $x(t)$.

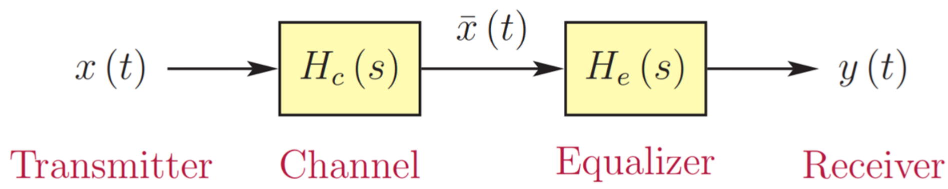

Figure 2: Cascade connection of the channel with an equalizer.

For the output $y(t)$ to match the input $x(t)$, the channel and equalizer must together *must* form an *identity* system:

$$H_c(s)H_e(s) = 1$$

This means the equalizer's transfer function must be equal to the reciprocal of the channel's transfer function:

$$H_e(s) = \frac{1}{H_c(s)}$$

Here, $H_c(s)$ represents a filter (channel) that distorts the signal, while $H_e(s)$ is designed to counteract or reduce this distortion.

3 Distortionless Transmission

A system modifies the input signal $x(t)$ as it produces the output $y(t)$, changing its shape. This modification, whether intentional or not, is called **distortion**. A CT-LTI system with system function $H(s)$ relates the input $x(t)$ and output $y(t)$ through convolution:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

In the frequency domain, this relationship is:

$$Y(f) = H(f)X(f) \tag{1}$$

The system introduces linear distortion, which occurs in two forms: amplitude and phase distortion. From Eq. (1), the output's magnitude spectrum is:

$$|Y(f)| = |H(f)|\,|X(f)| \tag{2}$$

Since the input $x(t)$ consists of sinusoidal components at various frequencies, Eq. (2) shows how the system scales each component differently. This uneven scaling leads to **amplitude distortion**.

The phase spectrum of the output is given by:

$$\measuredangle Y(f) = \measuredangle H(f) + \measuredangle X(f) \tag{3}$$

Here, the system shifts the phase of each sinusoidal component, introducing time delays. If these delays vary across frequencies, the signal's alignment is disrupted, causing **phase distortion**.

To achieve a distortionless output, the system must produce $y(t) = x(t)$, requiring

$$Y(f) = X(f) \tag{4}$$

which implies

$$H(f) = 1, \quad \text{for all } f \tag{5}$$

and

$$h(t) = \delta(t) \tag{6}$$

This means the system's impulse response must be an ideal impulse, which is physically unrealizable since no system can respond instantaneously without delay.

A realizable alternative allows for a scaled and delayed output:

$$y(t) = Kx(t - t_d) \tag{7}$$

where K is a scale factor and t_d is a time delay. Taking the Fourier transform gives:

$$Y(f) = Ke^{-j2\pi f t_d} X(f) \tag{8}$$

Thus, the required system transfer function is: $H(f) = Ke^{-j2\pi f t_d}$, with magnitude and phase characteristics:

$$|H(f)| = K \tag{9}$$

$$\measuredangle H(f) = -2\pi f t_d \tag{10}$$

A distortionless system maintains a constant magnitude and a phase that varies linearly with frequency, as shown in fig. 3.

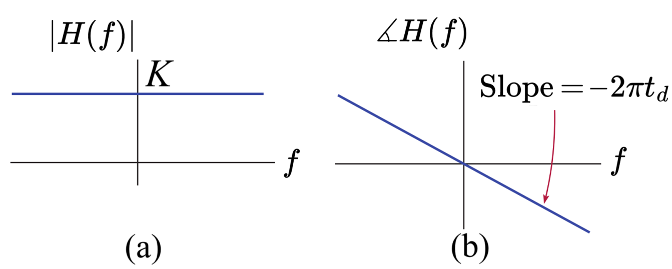

Figure 3: Frequency-domain characteristics for a distortionless system: (a) magnitude, (b) phase.

4 Ideal filters

One type of the ideal filter is an ideal lowpass filter that is characterized by the system function

$$H_{LP}(f) = \Pi\left(\frac{f}{2f_0}\right) e^{-j2\pi f t_d} = \begin{cases} e^{-j2\pi f t_d}, & |f| \leq f_0 \\ 0, & |f| > f_0 \end{cases}$$

Any signal that contains only the frequencies in the range $|f| \leq f_0$ is transmitted through this system without any distortion of its magnitude or phase characteristics. The magnitude and the phase of the system function $H_{LP}(f)$ of an ideal lowpass filter are graphed in Fig. ??.

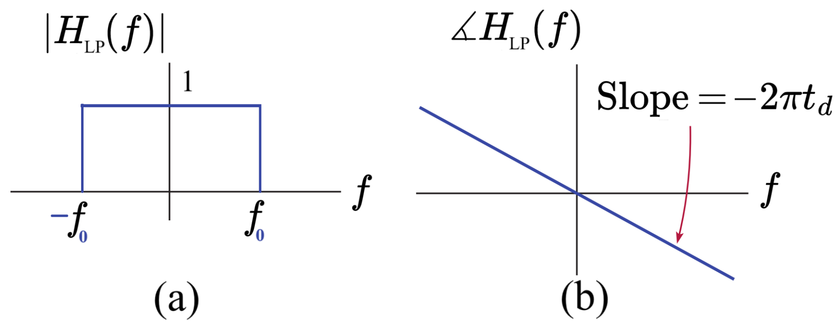

Figure 4: The frequency spectrum of the ideal lowpass filter: (a) magnitude, (b) phase.

The system function has a constant magnitude for $-f_0 \leq f \leq f_0$ and a linear phase with slope $-\omega t_d$. Signals within $|f| \leq f_0$ pass through undistorted. Here, f_0 is the cutoff frequency of an ideal lowpass filter, and t_d represents the time delay. The ideal lowpass filter is not practically realizable. Its impulse response, obtained via the inverse Fourier transform, is

$$h_{LP}(t) = 2f_0 \,\mathrm{sinc}\,(2f_0(t - t_d))$$

as shown in the figure below.

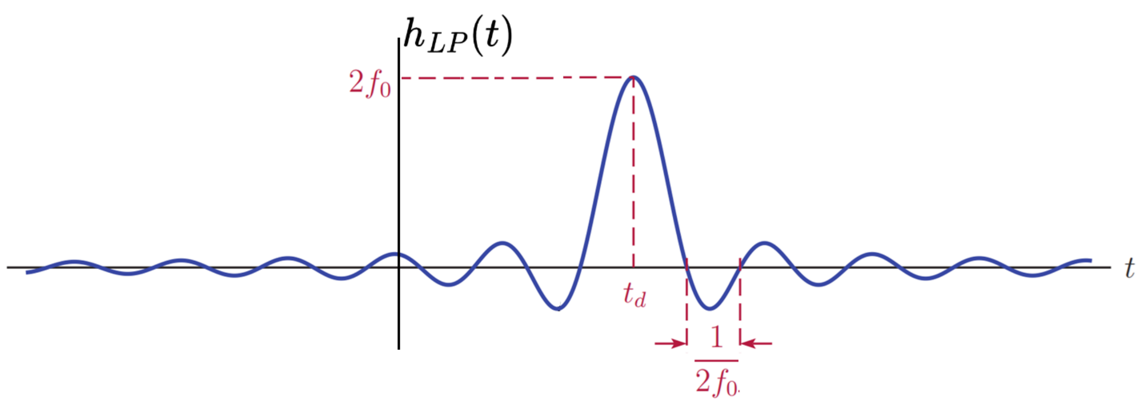

Figure 5: Impulse response of the ideal lowpass filter.

Key observations about the impulse response of the ideal lowpass filter:

1. **It follows a sinc function with a peak amplitude of $2f_0$ at $t = t_d$.**
2. **Zero crossings are evenly spaced, occurring every $1/(2f_0)$.**
3. **The response exists for all t, including $t < 0$, making the filter non-causal and physically unrealizable.**
4. **Adding a time delay cannot make the response causal.**

5 Design of Analog Filters

This section focuses on designing analog filters that approximate ideal behavior while remaining realizable. Since ideal filters are non-causal and impractical, real-world designs must relax frequency response constraints.

Analog filter specifications are typically given as tolerance limits for the magnitude response $|H(f)|$. Fig. A illustrates specifications for four filter types: lowpass, highpass, bandpass, and band-reject.

For a lowpass filter (Fig. A(a)), the passband $0 < f < f_1$ requires $|H(f)|$ to stay within $1 - \Delta_1 \leq |H(f)| \leq 1$. In the stopband $f > f_2$, $|H(f)|$ must be below Δ_2. The range $f_1 < f < f_2$ is the transition band, where no constraints apply. The parameters Δ_1 and Δ_2 define passband and stopband tolerances.

Highpass filter specifications (Fig. A(b)) mirror those of lowpass filters. Bandpass and band-reject filters (Fig. A(c), (d)) have two transition bands. Specifications may also be expressed in decibels (Fig. B). The impulse response $h(t)$ must be real-valued.

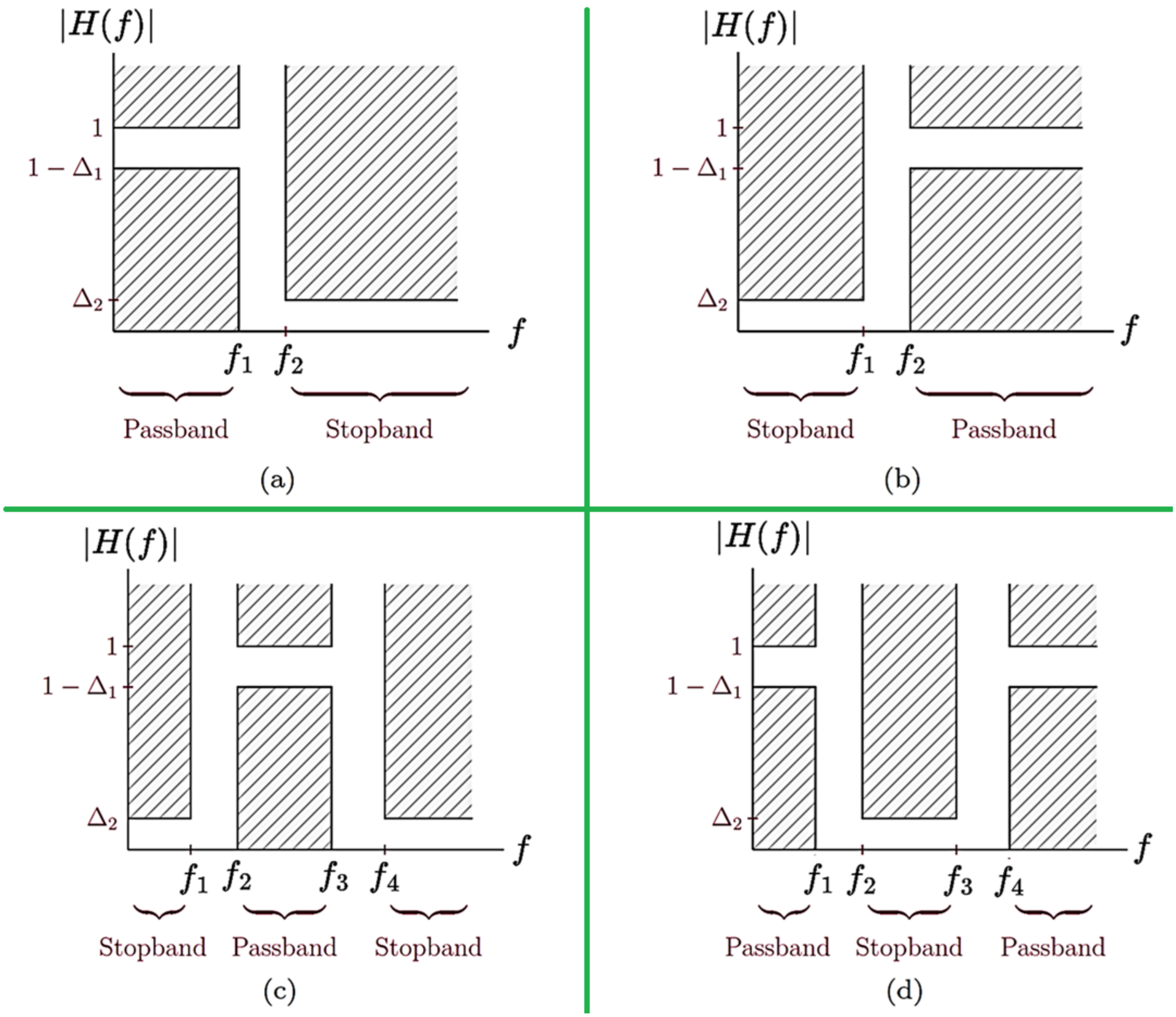

Figure 6: Specification diagrams for frequency selective filters: (a) lowpass, (b) highpass, (c) bandpass, (d) band-reject.

The maximum value of the magnitude response is set as 0 dB (which is equal to unity on the normal scale). The maximum allowed dB passband ripple R_p and the minimum required dB stopband attenuation A_s are related to the parameters Δ_1 and Δ_2 as

$$R_p = 20\log_{10}\left(\frac{1}{1-\Delta_1}\right)$$

and

$$A_s = 20\log_{10}\left(\frac{1}{\Delta_2}\right)$$

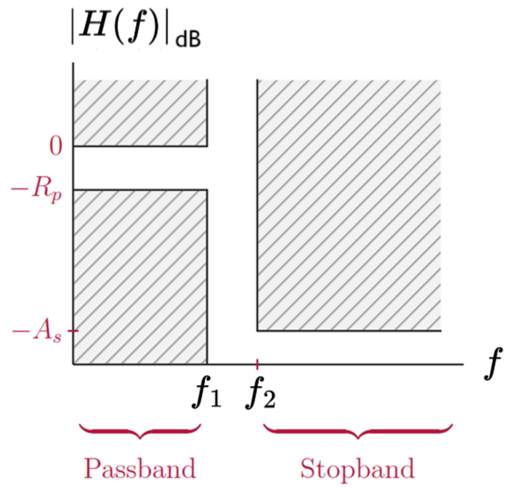

Figure 7: Decibel tolerance specifications for analog lowpass filter.

Given the filter specifications in Fig. 6, the design problem is:

Analog Filter Design Problem: Find a system function $H(s)$ whose magnitude response stays within the allowed range of the specification diagram.

Filter design typically starts with a lowpass prototype, regardless of the final filter type. Highpass, bandpass, or band-reject filters are then derived using frequency transformations. This method leverages well-established lowpass filter design techniques.

6 Butterworth Lowpass Filters

Butterworth lowpass filters are characterized by the squared-magnitude function

$$|H(f)|^2 = \frac{1}{1 + (f/f_c)^{2N}} \tag{11}$$

where the parameters N and f_c are the filter order and the cutoff frequency respectively. The magnitude characteristic $|H(f)|$ for a Butterworth filter is said to be maximally flat. Magnitude spectra for Butterworth filters of various orders are shown in Fig. 8.

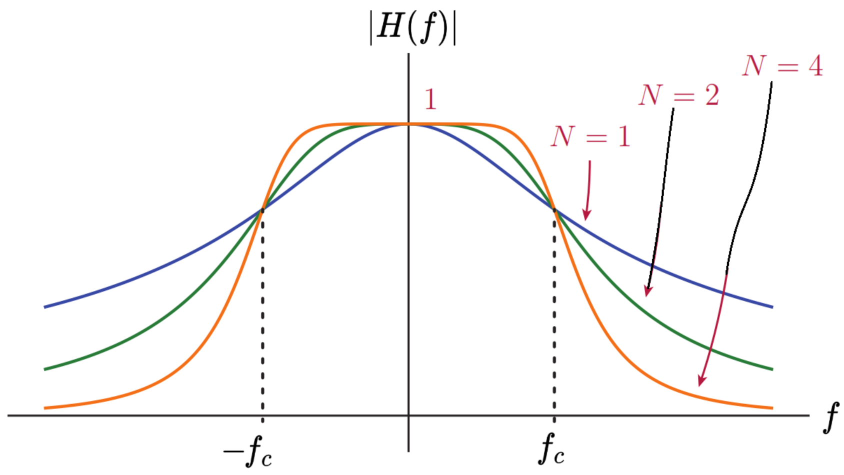

Figure 8: Magnitude spectra for Butterworth lowpass filters or order $N = 1, N = 2$, and $N = 4$.

6.1 Laplace Transform Based Butterworth LPF Design

The squared-magnitude function is given by

$$|H(f)|^2 = H(f)H^*(f)$$

Since the filter's impulse response $h(t)$ is real, its system function satisfies conjugate symmetry:

$$H^*(f) = H(-f)$$

Thus,

$$|H(f)|^2 = H(f)H(-f)$$

For a system function $H(s)$ with real coefficients,

$$|H(f)|^2 = H(s)H(-s)|_{s=j2\pi f}$$

Substituting $s = j2\pi f$, we obtain

$$H(s)H(-s) = \frac{1}{1 + (-s^2/\omega_c^2)^N}, \quad \omega_c = 2\pi f_c$$

If $H(s)$ has a pole at $p_1 = \sigma_1 + j\omega_1$ in the left half-plane, then $H(-s)$ has a corresponding pole at $\bar{p}_1 = -\sigma_1 - j\omega_1$. The poles of $H(s)$ and $H(-s)$ are symmetric about the origin. Extracting $H(s)$ from $H(s)H(-s)$ requires separating these poles, with different methods for even and odd N.

Case I: N is odd

The poles of the product $H(s)H(-s)$ are the values of s that satisfy the characteristic equation

$$s^{2N} - \omega_c^{2N} = 0$$

which can be written in the alternative form

$$s^{2N} = \omega_c^{2N} e^{j2\pi k}$$

where we have used the fact that $e^{j2\pi k} = 1$ for all integer k. Poles of $H(s)H(-s)$ are

$$p_k = \omega_c e^{j\pi k/N}, \quad k = 0, \dots, 2N-1$$

Here are a few key observations:

1. The poles of $H(s)H(-s)$ lie on a circle with a radius of ω_c.
2. These poles are evenly distributed around the circle, with an angular spacing of π/N radians between consecutive poles.
3. Since $H(s)$ and $H(-s)$ have real coefficients, all complex poles occur in conjugate pairs.

Fig. 9 depicts the situation for $\omega_c = 2$ rad/s, and $N = 5$.

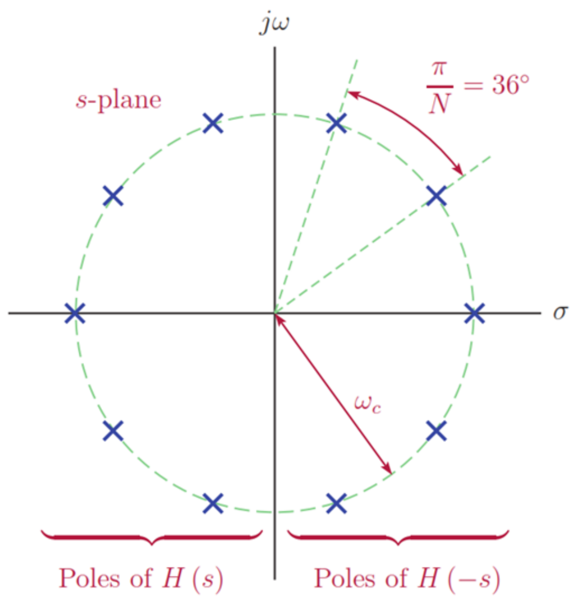

Figure 9: Poles of the product $H(s)H(-s)$ for Butterworth lowpass filter of order $N = 5$ with $3-\text{dB}$ cutoff frequency $\omega_c = 2\text{rad/s}$.

To ensure the filter is causal and stable, we select the poles in the left half of the s-plane for $H(s)$, while the right-half poles belong to $H(-s)$. The Butterworth lowpass filter is given by:

$$H(s) = \frac{\omega_c^N}{\prod_k (s - p_k)} \quad \text{(all poles in left-half of the } s\text{-plane are multiplied)}$$

for values of k satisfying

$$\frac{\pi}{2} < \frac{k\pi}{N} < \frac{3\pi}{2}. \quad \text{(this range corresponds to left-half of the } s\text{-plane)}$$

Case I: N is even
The poles of the product $H(s)H(-s)$ are the solutions of the equation

$$s^{2N} + \omega_c^{2N} = 0$$

which can be written as

$$s^{2N} = \omega_c^{2N} e^{j\pi(2k+1)}$$

In this case, the general solution for the poles is

$$p_k = \omega_c e^{j\pi(2k+1)/2N}, \quad \text{for} \quad k = 0, \ldots, 2N-1$$

For even filter orders, the poles of the magnitude-squared system function remain evenly spaced on a circle. Unlike the odd-order case, there are no poles on the real axis. The system function $H(s)$ is constructed similarly to the odd-order case.

Poles of $H(s)H(-s)$ for the Butterworth lowpass filter:

$$p_k = \begin{cases} \omega_c e^{jk\pi/N}, & k = 0, \ldots, 2N-1 \quad \text{if } N \text{ is odd} \\ \omega_c e^{j(2k+1)\pi/2N}, & k = 0, \ldots, 2N-1 \quad \text{if } N \text{ is even} \end{cases}$$

6.2 Example: Butterworth LPF Design

Determine the system function $H(s)$ of a third-order Butterworth lowpass filter with a 3 - dB cutoff frequency of $\omega_c = 20\pi \text{rad/s}$.

Solution:

Using the Butterworth approximation formula in Eqn. (11), the squared-magnitude function of the desired filter can be written as

$$|H(f)|^2 = \frac{1}{1 + (2\pi f/20\pi)^6}$$

Substituting $(2\pi f)^2 \to -s^2$ in Eqn. (11), we obtain

$$H(s)H(-s) = \frac{1}{1 + (-s^2/400\pi^2)^3}$$

or equivalently

$$H(s)H(-s) = \frac{(20\pi)^6}{(20\pi)^6 - s^6}$$

The poles of the function $H(s)H(-s)$ are those values of s that satisfy

$$s^6 = (20\pi)^6$$

Solutions of which are

$$p_k = (20\pi)e^{j2\pi k/6}, \quad k = 0, \ldots, 5$$

as shown in Fig. 10.

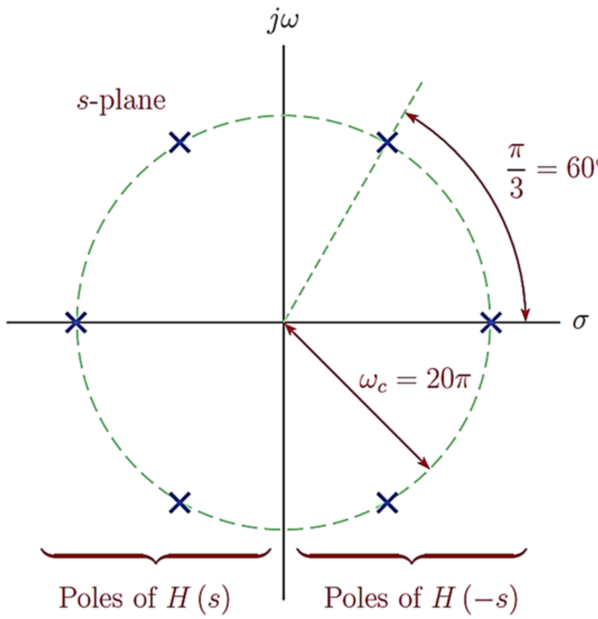

Figure 10: Poles of $H(s)H(-s)$ for the third-order Butterworth lowpass filter of the Example.

We focus on the left-half-plane poles, corresponding to $H(s)$. From Fig. 10, these poles are:

$$\begin{aligned}
p_2 &= (20\pi)e^{j2\pi/3} \\
p_3 &= (20\pi)e^{j\pi} = -20\pi \\
p_4 &= (20\pi)e^{j4\pi/3} = (20\pi)e^{-j2\pi/3}
\end{aligned}$$

The transfer function is constructed using these poles:

$$H(s) = \frac{(20\pi)^3}{(s + 20\pi)(s^2 + 20\pi s + 400\pi^2)} \tag{12}$$

Substituting $s = j\omega$, where $\omega = 2\pi f$, gives:

$$H(f) = \frac{(20\pi)^3}{(j\omega + 20\pi)(-\omega^2 + j20\pi\omega + 400\pi^2)}$$

The magnitude is:

$$|H(f)| = \frac{(20\pi)^3}{\sqrt{\omega^2 + (20\pi)^2}\sqrt{(-\omega^2 + 400\pi^2)^2 + (20\pi\omega)^2}}$$

and the phase of the system function is

$$\measuredangle H(f) = -\tan^{-1}\left(\frac{\omega}{20\pi}\right) - \tan^{-1}\left(\frac{20\pi\omega}{-\omega^2 + 400\pi^2}\right)$$

The magnitude and phase responses of the designed filter are shown in Fig. 11.

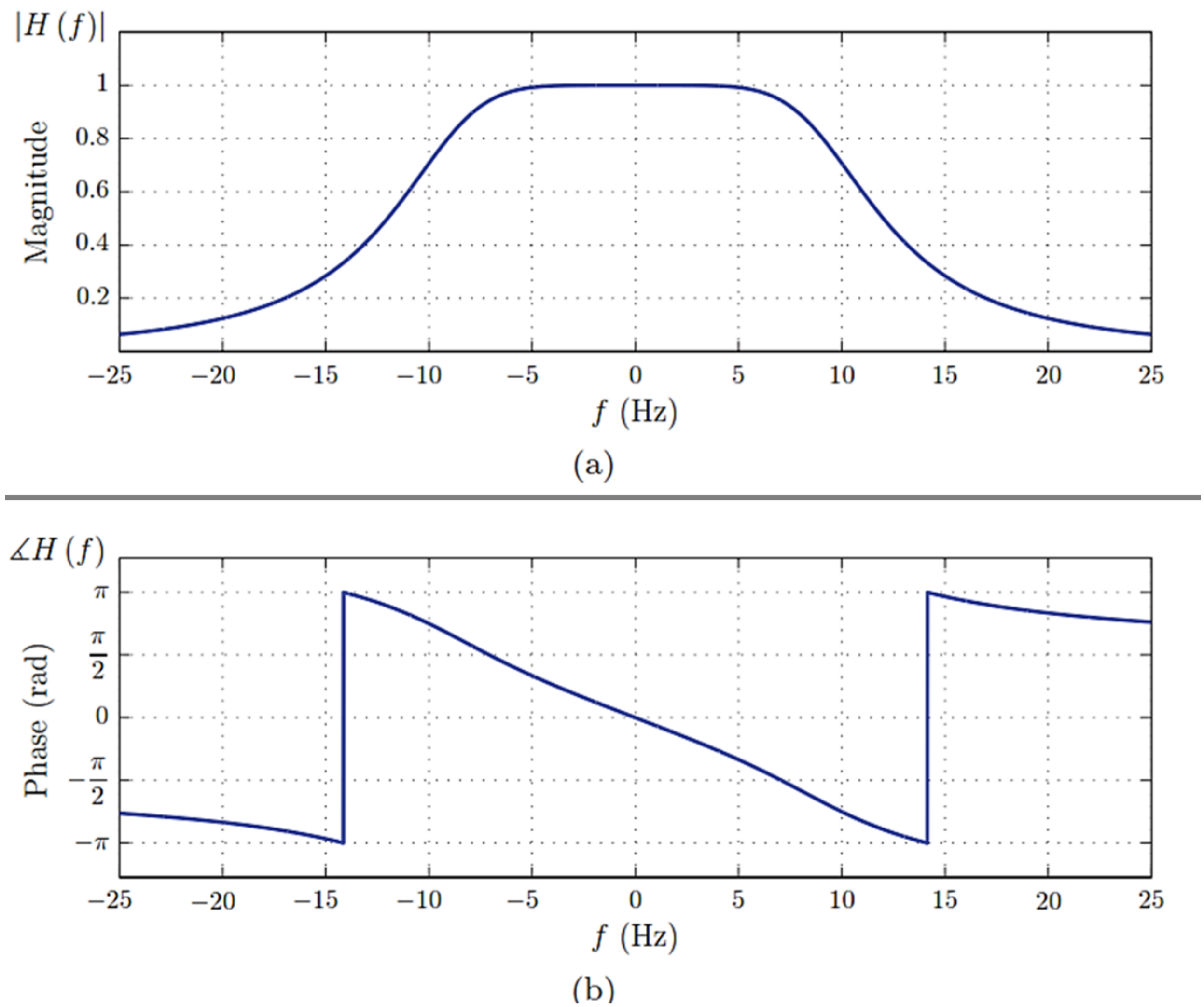

Figure 11: The frequency spectrum of the third-order Butterworth lowpass filter designed in the Example: (a) magnitude, (b) phase. Note the design specification is $f_c = 10$ Hertz.

The spectrum is *flat* (unity) and the phase is *linear* for $-10 \le f \le 10$ Hz. To check causality, we find the inverse Laplace transform of $H(s)$ in Eq. (12). The denominator consists of two factors: $(s + 20\pi)$ and $(s^2 + 20\pi s + 400\pi^2)$. Let's express $H(s)$ as:

$$H(s) = \frac{A}{s + 20\pi} + \frac{Bs + C}{s^2 + 20\pi s + 400\pi^2}$$

Multiplying both sides by the denominator:

$$(20\pi)^3 = A(s^2 + 20\pi s + 400\pi^2) + (Bs + C)(s + 20\pi)$$

Expanding,

$$As^2+20\pi As+400\pi^2 A+Bs^2+20\pi Bs+Cs+20\pi C = (A+B)s^2+(20\pi A+20\pi B+C)s+400\pi^2 A+20\pi C$$

Comparing coefficients,

- For s^2 terms: $A + B = 0 \Rightarrow B = -A$
- For s terms: $20\pi A + 20\pi B + C = 0$
- For constant terms: $400\pi^2 A + 20\pi C = (20\pi)^3 = 8000\pi^3$

Substituting $B = -A$ into the second equation:

$$20\pi A - 20\pi A + C = 0 \Rightarrow C = 0$$

Substituting $C = 0$ into the third equation:

$$400\pi^2 A = 8000\pi^3$$

Solving for A, we get $A = \frac{8000\pi^3}{400\pi^2} = 20\pi$. Since $B = -A$, we get: $B = -20\pi$; also $C = 0$. Thus, the decomposition is:

$$H(s) = \frac{20\pi}{s + 20\pi} - \frac{20\pi s}{s^2 + 20\pi s + 400\pi^2}$$

Using known Laplace inverse formulas:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t) = e^{-at}, \;\; t \geq 0$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2 + b^2}\right\} = e^{-at}\cos(bt)u(t) = e^{-at}\cos(bt), \;\; t \geq 0$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s+a)^2 + b^2}\right\} = e^{-at}\sin(bt)u(t) = e^{-at}\sin(bt), \;\; t \geq 0$$

We rewrite the second term:

$$\frac{20\pi s}{s^2 + 20\pi s + 400\pi^2} = \frac{20\pi[(s + 10\pi) - 10\pi]}{(s + 10\pi)^2 + (10\pi\sqrt{3})^2}$$

Splitting:

$$\frac{20\pi(s + 10\pi)}{(s + 10\pi)^2 + (10\pi\sqrt{3})^2} - \frac{20\pi(10\pi)}{(s + 10\pi)^2 + (10\pi\sqrt{3})^2}$$

The inverse Laplace transform of $F(s)$ is:

$$f(t) = 20\pi e^{-20\pi t} - \left(20\pi e^{-10\pi t}\cos(10\pi\sqrt{3}t) + \frac{20\pi}{\sqrt{3}}e^{-10\pi t}\sin(10\pi\sqrt{3}t)\right).$$

Simplify the expression:

$$f(t) = 20\pi e^{-20\pi t} - 20\pi e^{-10\pi t}\left(\cos(10\pi\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(10\pi\sqrt{3}t)\right).$$

which is *causal.*