EE/CE 453/352: Digital Signal Processing

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Homework 2 SOLUTION

Question 01 [8 pts]: Determine the DTFT and Z-transform for each of the following sequences and compare their plots:

a) $x_1[n] = \alpha^n u[n-1]$

b) $x_2[n] = \alpha^n u[-n-1]$

Solution:

a) Assuming $|\alpha| < 1$, the DTFT is given by:

$$X_{1d}(e^{j\omega}) = \sum_{n=1}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n - 1 = \frac{1}{1-\alpha e^{-j\omega}} - 1 = \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}$$

$$X_{1z}(z) = \sum_{n=1}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n - 1 = \frac{1}{1-\alpha z^{-1}} - 1 = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} = \frac{\alpha}{z-\alpha}$$

And ROC = |z|>|a|

b) Assuming $|\alpha| < 1$, the DTFT is given by:

$$X_{2d}(e^{j\omega}) = \sum_{n=-\infty}^{-1} \alpha^n e^{-j\omega n} = \sum_{m=1}^{\infty} \alpha^{-m} e^{j\omega m} = \sum_{m=0}^{\infty} \left(\alpha^{-1} e^{j\omega}\right)^m - 1 = \frac{1}{1-\alpha^{-1} e^{j\omega}} - 1$$

$$X_2(e^{j\omega}) = \frac{e^{j\omega}}{\alpha - e^{j\omega}}$$

$$X_{2z}(z) = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = \sum_{m=1}^{\infty} \alpha^{-m} z^m = \sum_{m=0}^{\infty} (\alpha^{-1} z)^m - 1 = \frac{1}{1-\alpha^{-1} z} - 1 = \frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$
$$= \frac{z}{\alpha - z}$$

And ROC = |z|<|a|

Question 02 [14 pts]: Consider the following sequences:

1. $x_1[n] = (0.3)^n \mu[n+1]$
2. $x_2[n] = (0.7)^n \mu[n-1]$
3. $x_3[n] = (0.4)^n \mu[n-5]$
4. $x_4[n] = (-0.4)^n \mu[-n-2]$

a) Determine the ROCs of the z-transform of each of the above sequences.

b) From the ROCs indicated in part (a), determine the ROCs of the following sequences:

i. $y_1[n] = x_1[n] + x_3[n]$
ii. $y_2[n] = x_1[n] + x_4[n]$
iii. $y_3[n] = x_2[n] + x_4[n]$

Solution:

a) The ROC of $Z\{x_1[n]\}$ is $|z| > 0.3$, the ROC of $Z\{x_2[n]\}$ is $|z| > 0.7$, the ROC of $Z\{x_3[n]\}$ is $|z| > 0.4$ and the ROC of $Z\{x_4[n]\}$ is $|z| < 0.4$.

b) The ROC of $Z\{y_1[n]\}$ is $|z| > 0.7$, the ROC of $Z\{y_2[n]\}$ is $|z| > 0.7$ and the ROC of $Z\{y_3[n]\}$ does not converge.

Question 03 [08 pts]: The transfer function of a causal LTI discrete-time system is given by:

$$H(z) = \frac{1 - 3.6z^{-1} + 0.33z^{-2}}{1 + 0.8z^{-1} - 0.13z^{-2}}$$

Determine the output $y[n]$ of the above system for all values of n, for the following input:

$$x[n] = 2.4(0.1)^n\mu[n] + 0.3(-0.3)^n\mu[n]$$

Solution:

We can evaluate $Y(z) = X(z)H(z)$:

$$X(z) = \frac{2.4}{1 - 0.1z^{-1}} + \frac{0.3}{1 + 0.3z^{-1}} = \frac{2.4 + 0.51z^{-1}}{(1 - 0.1z^{-1})(1 + 0.3z^{-1})}$$

$$Y(z) = \left[\frac{2.4 + 0.51z^{-1}}{(1 - 0.1z^{-1})(1 + 0.3z^{-1})}\right]\left[\frac{1 - 3.6z^{-1} + 0.33z^{-2}}{1 + 0.8z^{-1} - 0.13z^{-2}}\right]$$

$$\frac{Y(z)}{z} = \frac{(2.4z^{-1} - 8.13z^{-2} - 1.05z^{-3} + 0.17z^{-4})}{(1 - 0.1z^{-1})(1 + 0.3z^{-1})(1 + 0.94z^{-1})(1 - 0.14z^{-1})}$$

A partial fraction expansion of $Y(z)$ in z^{-1} yields:

$$Y(z) = \frac{0.84}{1 - 0.1z^{-1}} + \frac{4.95}{1 + 0.3z^{-1}} - \frac{8.83}{1 + 0.94z^{-1}} - \frac{6.83}{1 - 0.14z^{-1}}, \qquad |z| > 0.94$$

$$y[n] = \mu(n)[0.84(0.1)^n + 4.95(-0.3)^n - 8.83(-0.94)^n - 6.83(0.14)^n]$$

Question 04 [5 pts]: Determine the frequency response $H(e^{j\omega})$ of the following transfer function, and show that the magnitude response $|H(e^{j\omega})|$ assumes its maximum value of $2/(1-\alpha)$ at $\omega = \omega_C$:

$$H(z) = \frac{1 - z^{-2}}{1 - (1 + \alpha)\cos(\omega_C)\, z^{-1} + \alpha z^{-2}}$$

Solution:

Replacing $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{1 - e^{-j2\omega}}{1 - (1+\alpha)\cos(\omega_C)\, e^{-j\omega} + \alpha e^{-j2\omega}}$$

If $\omega = \omega_C$:

$$\left|H(e^{j\omega_C})\right|^2 = \frac{2(1-\cos 2\omega_C)}{1 + (\cos\omega_C)^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos 2\omega_C - 2\cos\omega_C\,(1+\alpha)^2\cos\omega_C}$$

$$\left|H(e^{j\omega_C})\right|^2 = \frac{4\sin^2\omega_C}{(1-\alpha)^2\sin^2\omega_C} = \frac{4}{(1-\alpha)^2} \quad \rightarrow \quad H(e^{j\omega_C}) = \frac{2}{1-\alpha}$$

Question 05 [8 pts]: Consider the two finite-length sequences $\{2,4,-3\}, 0 \le n \le 2$ and $h[n] = \{-8,4,12,-3\}, 0 \le n \le 3$.

a) Determine the linear convolution: $y_L[n] = g[n] * h[n]$.
b) Determine the circular convolution (after zero padding): $y_C[n] = g[n] ④ h[n]$.
c) Determine $y_C[n]$ using DFT-based approach (non-matrix).
d) Determine circular convolution (after zero padding): $y_C[n] = g[n] ⑥ h[n]$ and compare your result with part (a).

Solution:

a) $y_L[0] = g[0]h[0] = -16$
$y_L[1] = g[0]h[1] + g[1]h[0] = -24$
$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] = 64$
$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] = 30$
$y_L[4] = g[1]h[3] + g[2]h[2] = -48$
$y_L[5] = g[2]h[3] = -9$

$$y_L = \{-16, -24, 64, 30, -48, -9\}, \qquad 0 \le n \le 5$$

b) After zero padding, $g[n] = \{2,4,-3,0\}$
$y_C[0] = g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1] = -64$
$y_C[1] = g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2] = -15$
$y_C[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3] = 64$
$y_C[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] = 30$

$$y_C = \{-64, -15, 64, 30\}, \qquad 0 \le n \le 3$$

c) Using $Y_C(k) = G(k)H(k)$

$$G(k)=\sum_{n=0}^{3} g(n)e^{-j2\pi k\frac{n}{N}}$$

$G(0)=2+4-3+0$

$\boldsymbol{G(0)=3}$

$G(1)=2+4e^{-j2\pi\frac{1}{4}}-3e^{-j2\pi\frac{2}{4}}+0$

$\boldsymbol{G(1)=5-j4}$

$G(2)=2+4e^{-j2\pi\frac{2}{4}}-3e^{-j2\pi\frac{4}{4}}+0$

$\boldsymbol{G(2)=-5}$

$G(3)=2+4e^{-j2\pi\frac{3}{4}}-3e^{-j2\pi\frac{6}{4}}+0$

$\boldsymbol{G(3)=5+j4}$

$$\boldsymbol{G(k)=\{4,-1+j,-5,-1-j\}}$$

$$H(k)=\sum_{n=0}^{3} h(n)e^{-j2\pi k\frac{n}{N}}$$

$H(0)=-8+4+12-3$

$\boldsymbol{H(0)=5}$

$H(1)=-8+4e^{-j2\pi\frac{1}{4}}+12e^{-j2\pi\frac{2}{4}}-3e^{-j2\pi\frac{3}{4}}$

$\boldsymbol{H(1)=-20-j7}$

$H(2)=-8+4e^{-j2\pi\frac{2}{4}}+12e^{-j2\pi\frac{4}{4}}-3e^{-j2\pi\frac{6}{4}}$

$\boldsymbol{H(2)=3}$

$H(3)=-8+4e^{-j2\pi\frac{3}{4}}+12e^{-j2\pi\frac{6}{4}}-3e^{-j2\pi\frac{9}{4}}$

$\boldsymbol{H(3)=-20+j7}$

$$\boldsymbol{H(k)=\{5,-20-j7,3,-20+j7\}}$$

$$Y(k)=G(k)H(k)=\{15,-128+j45,-15,-128-j45\}$$

Performing IDFT:

$$y(n)=\frac{1}{N}\sum_{n=0}^{3} Y(k)e^{j2\pi k\frac{n}{N}}$$

$$y(0)=\frac{1}{4}(15-128+j45-15-128-j45)=-64$$

$$y(1)=\frac{1}{4}\left(15+(-128+j45)e^{j2\pi\frac{1}{4}}-15e^{j2\pi\frac{2}{4}}+(-128-j45)e^{j2\pi\frac{3}{4}}\right)=-15$$

$$y(2)=\frac{1}{4}\left(15+(-128+j45)e^{j2\pi\frac{2}{4}}-15e^{j2\pi\frac{4}{4}}+(-128-j45)e^{j2\pi\frac{6}{4}}\right)=64$$

$$y(3)=\frac{1}{4}\left(15+(-128+j45)e^{j2\pi\frac{3}{4}}-15e^{j2\pi\frac{6}{4}}+(-128-j45)e^{j2\pi\frac{9}{4}}\right)=30$$

$$y(n)=\{-64,-15,64,30\}$$

Which is the same as the circular convolution result in part (b).

d) After zero padding, $g[n]=\{2,4,-3,0,0,0\}$ and $h[n]=\{-8,4,12,-3,0,0\}$.
$y_C[0]=g[0]h[0]+g[1]h[5]+g[2]h[4]+g[3]h[3]+g[4]h[2]+g[5]h[1]=-16$
$y_C[1]=g[0]h[1]+g[1]h[0]+g[2]h[5]+g[3]h[4]+g[4]h[3]+g[5]h[2]=-24$
$y_C[2]=g[0]h[2]+g[1]h[1]+g[2]h[0]+g[3]h[5]+g[4]h[4]+g[5]h[3]=64$
$y_C[3]=g[0]h[3]+g[1]h[2]+g[2]h[1]+g[3]h[0]+g[4]h[5]+g[5]h[4]=30$
$y_C[4]=g[0]h[4]+g[1]h[3]+g[2]h[2]+g[3]h[1]+g[4]h[0]+g[5]h[5]=-48$

$$y_C[5] = g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0] = -9$$

$$y_C = \{-16, -24, 64, 30, -48, -9\}, \qquad 0 \le n \le 5$$

Which is the same as the linear convolution result in part (a).

Question 06 [6 pts]: Compute the 4-point FFT of the following sequence using the Decimation in Time algorithm:

$$x[n] = \sin\left(\frac{n\pi}{2}\right), \qquad n = 0,1,2,3 \ldots$$

Solution:

$$x[n] = \{0,1,0,-1\}, \qquad x_e[n] = [0,0], \qquad x_o[n] = [1,-1]$$

$$G(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} G(0) \\ G(1) \end{bmatrix}$$

$$H(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} H(0) \\ H(1) \end{bmatrix}$$

$X(0) = G(0) + W_4^0 H(0) = 0 + 1(0) = 0$

$X(1) = G(1) + W_4^1 H(1) = 0 - j(2) = -j2$

$X(2) = G(0) - W_4^0 H(0) = 0 - 1(0) = 0$

$X(3) = G(1) - W_4^1 H(1) = 0 - (-j)(2) = j2$

$$X(k) = \{0, -j2, 0, j2\}$$

Question 07 [4 pts]: Find the IDFT of the following function with $N = 4$:

$$X(k) = \{\text{last 4 digits of your HU ID}\}$$

Solution:

Let $X(k) = \{1,0,1,0\}$.

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$