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. **Lab 03: Sampling and Reconstruction: FFT-Based Analysis of Filtered Signals**

List of Reading Material

1. Revisit/read the concept of sampling for converting a continuous-time signal to a discrete-time signal.

2. Revisit/read how sampling affects the frequency-domain characteristics of the signal, and what precautions must be taken to ensure that the signal obtained through sampling accurately represents the original.

Lab Objectives

1. Sample the analog signals and reconstruct it back from its sampled version. Understand various interpolation methods used and the spectral relationships involved in reconstruction.

2. Analyze the sampled and reconstructed signals using MATLAB DSP-toolbox function FFT.

1 Sampling of the Analog Signals: Reading Material

The term **sampling** refers to periodically measuring the amplitude of a continuous-time signal and constructing a discrete-time signal with the measurements. If certain conditions are satisfied, a continuous-time signal can be completely represented by measurements (samples) taken from it at uniform intervals. This allows us to store and manipulate continuous-time signals on a digital computer.

A general relationship between the continuous-time signal $x_a(t)$ and its discrete-time copy $x[n]$ is

$$x[n] = x_a(t)\Big|_{t=nT_S}$$

Where T_S is the sampling interval, that is the time interval between consecutive samples. It is also referred to as the sampling period. The reciprocal of the sampling interval is called the sampling rate or the sampling frequency:

$$f_S = \frac{1}{T_S}$$

The relationship between a continuous-time signal and its discrete-time version is illustrated in fig. 1. In signal processing, sampling a continuous signal involves discretizing its values at specific inter-

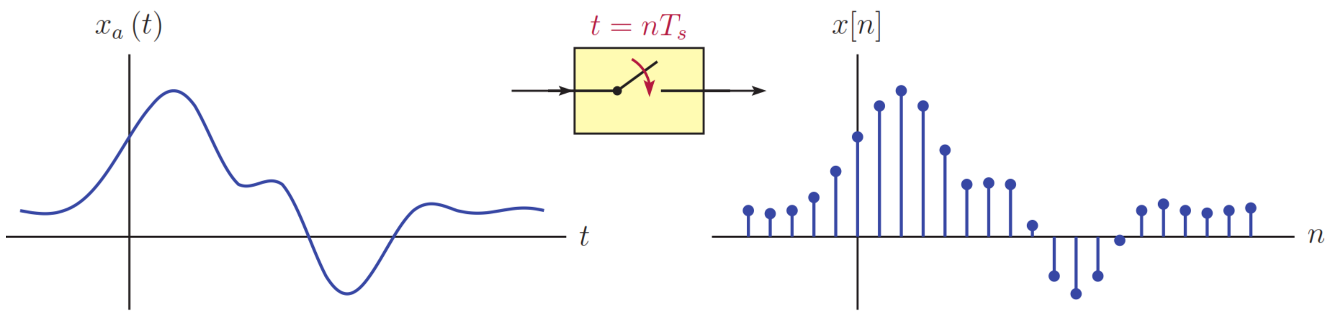

Figure 1: Caption

vals. This process is mathematically represented by convolving the continuous signal with a series of impulses known as Dirac delta functions, positioned at each sampling instance.

Consider a periodic impulse train $p(t)$ with period T_S :

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_S)$$

Multiplication of any signal $x(t)$ with this impulse train $p(t)$ would result in amplitude information for $x(t)$ being retained only at integer multiples of the period T_S. Let the signal $x\ s(t)$ be defined as the product of the original signal and the impulse train, i.e.,

$$x_S(t) = x_a(t)p(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_S) = \sum_{n=-\infty}^{\infty} x_a(nT_S)\delta(t - nT_S)$$

We will refer to the signal $x_S(t)$ as the impulse-sampled version of $x(t)$. fig. 2 illustrates this.

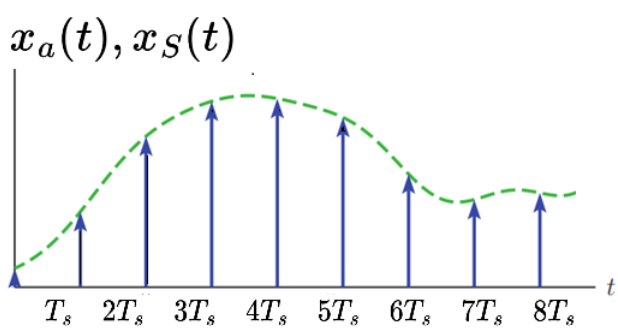

Figure 2: Caption

It is important to understand that the impulse-sampled signal $x_S(t)$ is still a continuoustime signal. What is the value of sampled signal $x_S(t)$ between any two successive samples? Answer: Zero.

With the aid of the Fourier transform, this relationship can also be written using frequencies in Hertz as follows

$$X_S(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} X_a\left(f - k f_S\right)$$

where $X_S(f)$ and $X_a(f)$ are Fourier transforms of signals $x_S(t)$ and $x_a(t)$, respectively. This is a very significant result. The **spectrum** of the **impulse-sampled** signal is obtained by adding frequency-shifted versions of the spectrum of the original signal and then scaling the sum by $1/T_S$. The terms of the summation are shifted by all integer multiples of the sampling rate f_S. fig. 3 illustrates this.

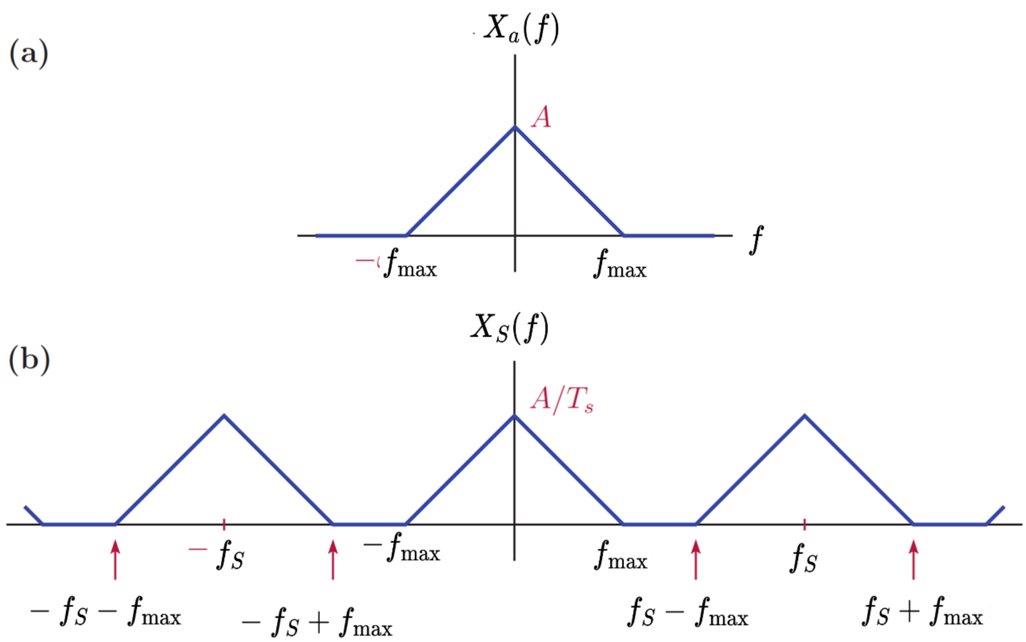

Figure 3: Caption

1.1 Nyquist sampling criterion

Let $x_a(t)$ be a signal the spectrum $X_a(f)$ of which is band-limited to $f_{\max}$, meaning it exhibits no frequency content for $|f| > f_{\max}$. We sample $x_a(t)$ using impulses to obtain the signal $x_S(t)$. If we want to be able to recover the signal $x_a(t)$ back from its impulse-sampled version $x_S(t)$, then the spectrum $X_a(f)$ must also be recoverable from the spectrum $X_S(f)$. This in turn requires that no overlaps occur between periodic repetitions of spectral segments.

To keep the left edge of the spectral segment centered at $f = f_S$ from interfering with the right edge of the spectral segment centered at $f = 0$, we need

$$f_S - f_{\max} \geq f_{\max}$$
$$f_S \geq 2 f_{\max}$$

For the impulse-sampled signal to form an accurate representation of the original signal, the sampling rate must be at least twice the highest frequency in the spectrum of the original signal. **This is known as the Nyquist sampling criterion**.

Activity 1: Sampling and sampled signals in time and frequency domains

Consider an analog sinusoidal *sampling* signal $x(t)$,

$$x(t) = A\cos(2\pi f_0 t)$$

where A is positive amplitude and f_0 is frequency ($T_0 = 1/f_0$ is the time period). The Fourier transform of which is

$$X(f) = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$$

Based on the theory, the Fourier transform of $X_S(f)$ is

$$X_S(f) = \frac{A}{2T_S}\sum_{n=0}^{\infty}\delta(f - f_0 \pm n f_S) + \frac{A}{2T_S}\sum_{n=0}^{\infty}\delta(f + f_0 \pm n f_S).$$

Your task is to *sample* the *sampling* signal $x(t)$ to obtain a *sampled* signal $x_S(t)$ such that the values of $x_S(t)$ is specifically *zero* between any two consecutive samples, and at $t = nT_S$, $x_S(t) = x(t)$, where $n = 0, 1, 2, \ldots$. Plot and compare sampling and sampled signals in both time and frequency domains.

It is important to note that the zero-filling in the sampled signal is essential for reconstruction, as it allows the interpolation of values between consecutive samples. However, this zero-filling makes the Fourier analysis more complex.

Hints for Activity 1:

1. first select some appropriate values for A and f_0. As an example, consider $A = 1$ and $f_0 = 3$ Hertz. You are required to try some other values.

2. Secondly, select an appropriate sampling frequency, f_S. Let us say $f_s = K f_0$, where $K \geq 2$ (the

Nyquist criterion). Recommended value of K is $K \geq 5$. So, the sampling time is

$$T_S = \frac{1}{f_S}, \text{ where } f_S = Kf_0$$

3. Since we are required to define the values of the sampled signal $x_S(t)$ between consecutive samples, we define a time stepsize that must be ***smaller*** than T_S. Let T_Δ be the time stepsize, and it is obtained as

$$T_\Delta = \frac{T_S}{M}$$

where M be an integer. It is recommended to select M such that $M \geq 5$.

4. Next, we determine the number of cycles of the sampling sinusoid for the simulation. For illustration purposes, 3 or 4 cycles suffice for periodic signals. However, frequency-domain analysis benefits from a larger number of cycles. A practical approach is to use many cycles for frequency analysis while restricting the time-domain plots to a few cycles using the "**xlim**" command for aesthetic reason and clarity of depiction. Let C denote the number of cycles. The recommended value for C is 100 or more.

5. Based on the discussion above, we can define the time variable as follows:

```
% T0 is the time period of x(t)
% C is the number of cycles of x(t)
% TS is the sampling period
% M is an integer used to compute time Stepsize
Tmax = T0*C; % The maximum value of time
TDelta = TS/M; % The Stepsize of time
t = linspace(0,Tmax,round(Tmax/TDelta)); % The time axis
% Once we have t, one can compute x(t)
```

6. The next challenge is how to obtain a *zero-filled* $x_S(t)$ from $x(t)$ such that

$$x_S(t) = \begin{cases} x(t), & \text{for } t = nT_S \\ 0, & \text{for } nT_S < t < (n+1)T_S \end{cases}$$

Convince yourself that this may easily be done in MATLAB as follows:

```
data_points = length(xt); % Number of data points in x(t)
xSt = zeros(1, data_points); % The Sampled signal is initialized with zeros
xSt(1:M:end) = xt(1:M:end); % The Sampled signal as specified above
```

Observe that each sample in $x_S(t)$ is surrounded by $M - 1$ zeros on both sides.

7. Obtain the time-domain plots of signals $x(t)$ and $x_S(t)$ in MATLAB. Refer to fig. 4 for an example.

8. Let us evaluate frequency-domain plots.

   (a) Since, both $x(t)$ and $x_S(t)$ are periodic in nature, we must divide their `fft` values by the observation time $T_{\max}$.

   (b) The `fft` values of data `xt` are multiplied with time stepsize T_Δ as well because it is a continuous-time signal (Recall what you learned in Lab 02). Therefore

$$X(f) = \mathcal{F}\{x(t)\} = \frac{T_\Delta}{T_{\max}}\texttt{fft(xt)}, \text{ where } 0 \le f \le Mf_S$$

   To obtain a two-sided plot, we may use `fftshift` as follows:

$$X(f) = \mathcal{F}\{x(t)\} = \frac{T_\Delta}{T_{\max}}\texttt{fftshift(fft(xt))}, \text{ where } -Mf_S/2 \le f \le Mf_S/2$$

   (c) The spectrum $X_S(f)$ obtained as follows:

$$X_S(f) = \mathcal{F}\{x_S(t)\} = \frac{1}{T_{\max}}\texttt{fftshift(fft(xSt))}, \text{ where } -Mf_S/2 \le f \le Mf_S/2$$

   This is shown in fig. 5. The evaluation of $X_S(f)$ does not require multiplication by T_Δ, as it represents a discrete-time (sampled) signal.

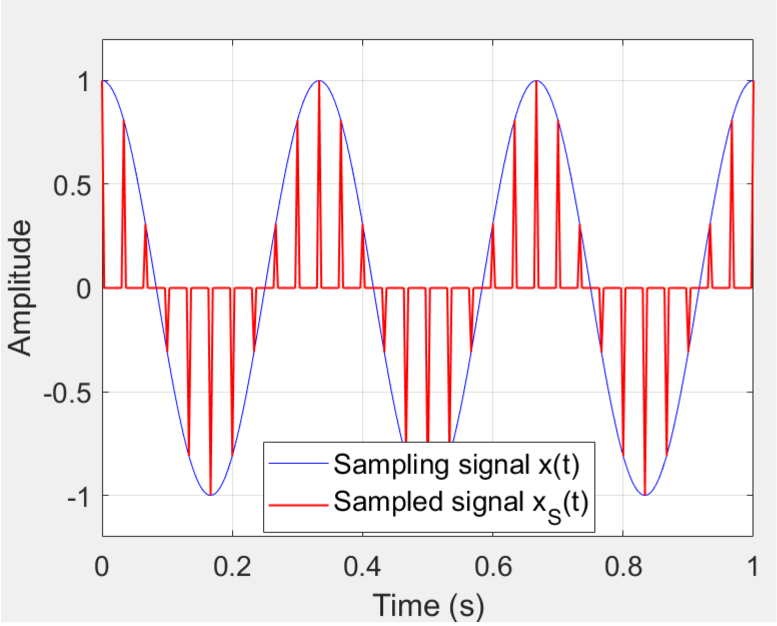

Figure 4: The **plot** command is used for both traces. In this plot, we consider $f_0 = 3$, $M = 10$, $K = 10$. You may also use **stem** command to plot $x_S(t)$ with small or 'none' marker size.

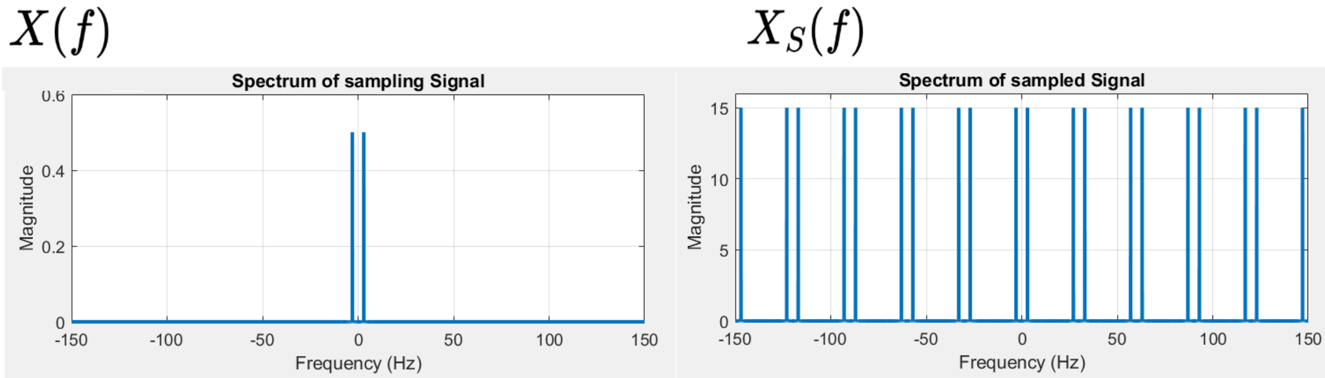

Figure 5: For $f_0 = 3$ and $K = 10$, we have $f_S = 30$. With $A = 1$, the sidelobes in $X(f)$ are $A/2 = 1/2 = 0.5$. Similarly, for $T_S = 1/30$, the sidelobes in $X_S(f)$ are $A/(2T_S) = 30/2 = 15$. Therefore, the spectrum of $X(f)$ includes two tones at $f = \pm 3\,\text{Hz}$ with an amplitude of 0.5, as expected. Likewise, the spectrum of $X_S(f)$ contains tones at $\pm 3 \pm 30n\,\text{Hz}$ for $n = 0, 1, 2, 3, 4,$ and 5, where each tone has an amplitude of 0.5. All of these observations are in complete agreement with the following analytical result: $X_S(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} X\left(f - kf_S\right)$.

2 Reconstruction of the Sampled Signal: Reading Material

The purpose of sampling an analog signal is often to store, process, or transmit it digitally and later convert it back to its analog form. This raises an important question: How can the original analog signal be reconstructed from its sampled version? Specifically, given the discrete-time signal $x[n]$ or the impulse-sampled signal $x_S(t)$, how can we recover a signal identical to, or at least closely resembling, $x_a(t)$? Essentially, we need a way to **fill the gaps** between the impulses in $x_S(t)$ in a meaningful way. In technical terms, this involves computing the signal amplitudes between sampling instants using interpolation.

2.1 Zero-Order Hold Interpolation

Consider obtaining a signal similar to $x_a(t)$ using a simple interpolation method. One approach is to use the impulse-sampled signal $x_S(t)$ and hold the amplitude of each sample constant for the duration T_S immediately following it. The resulting signal is a **staircase** approximation of the original. **This approach, known as zero-order hold interpolation, uses horizontal lines (polynomials of order zero) between sampling instants**. This is illustrated in fig. 7.

Zero-order hold interpolation can be achieved by processing the impulse sampled signal $x_S(t)$ through a zero-order hold reconstruction filter, a linear system the impulse response of which is a rectangle with unit amplitude and a duration of T_S.

$$h_{\text{zoh}}(t) = \Pi\left(\frac{t - T_S/2}{T_S}\right)$$

This is illustrated in fig. **??**. **The linear system that performs the interpolation is called a reconstruction filter.** The ZOH reconstruction filter output is $x_{\text{zoh}} = x_S(t) * h_{\text{zoh}}(t)$. In the Signals and

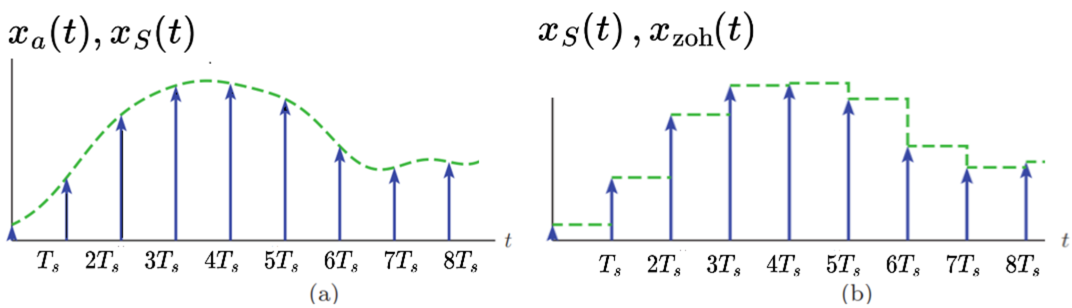

Figure 6: Caption

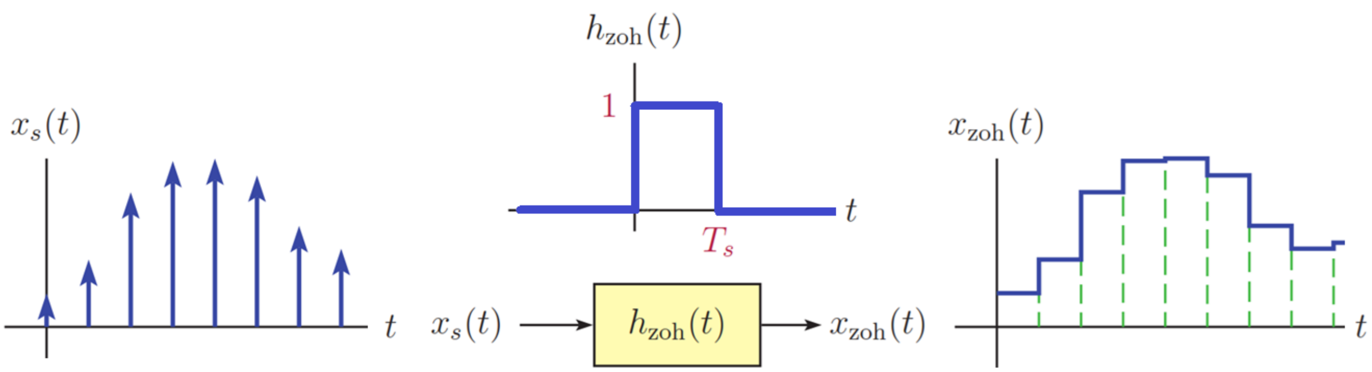

Figure 7: Zero-order hold interpolation using an interpolation filter.

Systems course, you learned that the Fourier transform of x_{zoh} is obtained as

$$X_{\text{zoh}}(\omega) = (f) = \text{sinc}\,(T_S f)\, e^{-j\pi T_S f} \sum_{k=-\infty}^{\infty} X_a\,(f - k f_S) \tag{1}$$

In this Lab, you perform convolution using the MATLAB `filter` command.

Activity 2: Consider the zero-filled sampled signal (as you obtained in Activity 01) and pass it through the Zero-Order Hold (ZOH) reconstruction filter, and plot and examine the response in both time and frequency domains.

1. Determine the impulse response of the ZOH filter, $h_{zoh}(t)$, and visualize it using the **`plot`** or **`stem`** command.
2. Pass the zero-filled sampled signal through the ZOH filter using the **`filter`** command.
3. Plot both the sampled signal and the reconstructed signal in the time domain on the same graph to observe the effect of ZOH interpolation. If done correctly, your result should resemble fig. 8.
4. Compute the frequency spectrum of the reconstructed signal (e.g., using the Fourier Transform) and plot it to observe how the interpolation affects the frequency components. If done correctly,

your result should resemble fig. 9.

5. Analyze the frequency response of the reconstructed signal to evaluate ZOH performance. Verify that the simulated sideband harmonics align with the theoretical predictions in Equation (1).

Note: Do it yourself, or ask the lab demonstrator for some hints if you are lost.

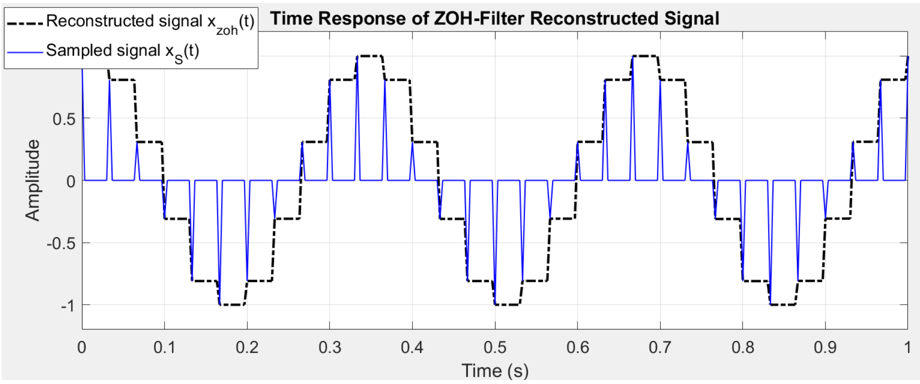

Figure 8: Time response of ZOH reconstruction filter for a sampled sinusoid signal.

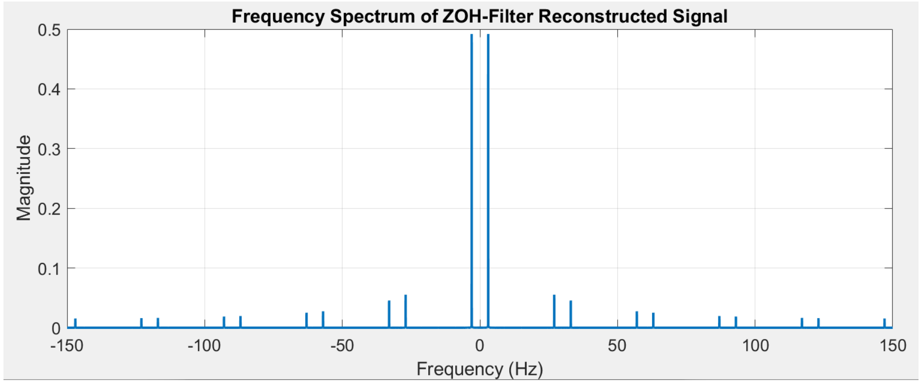

Figure 9: Frequency response of ZOH reconstruction filter for a sampled sinusoid signal.

2.2 First-Order Hold Interpolation

As an alternative to zero-order hold, the gaps between the sampling instants can be filled by **linear interpolation**, that is, by connecting the tips of the samples with straight lines as shown in fig. 10. This is also known as **first-order hold interpolation** since the straight line segments used in the interpolation correspond to first-order polynomials.

first-order hold interpolation can also be implemented using a **causal** first-order hold reconstruction

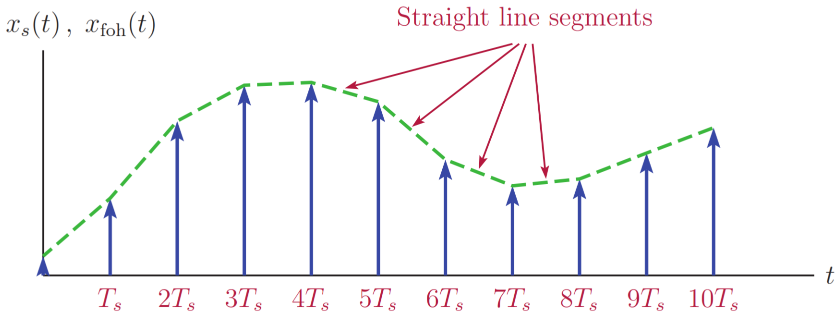

Figure 10: Reconstruction using first-order hold interpolation.

filter. The impulse response of such a filter is a **triangle** in the form

$$h_{\text{foh}}(t) = \begin{cases} 0, & t < 0 \\ t/T_s, & 0 < t < T_s \\ 2 - t/T_s, & T_s < t < 2T_s \\ 0, & t \geq 2T_s \end{cases}$$

In the Signals and Systems course, you learned that the Fourier transform of x_{foh} is obtained as

$$X_{\text{foh}}(f) = \text{sinc}^2\left(T_S f\right) e^{-j2\pi T_S f} \sum_{k=-\infty}^{\infty} X_a\left(f - k f_S\right) \tag{2}$$

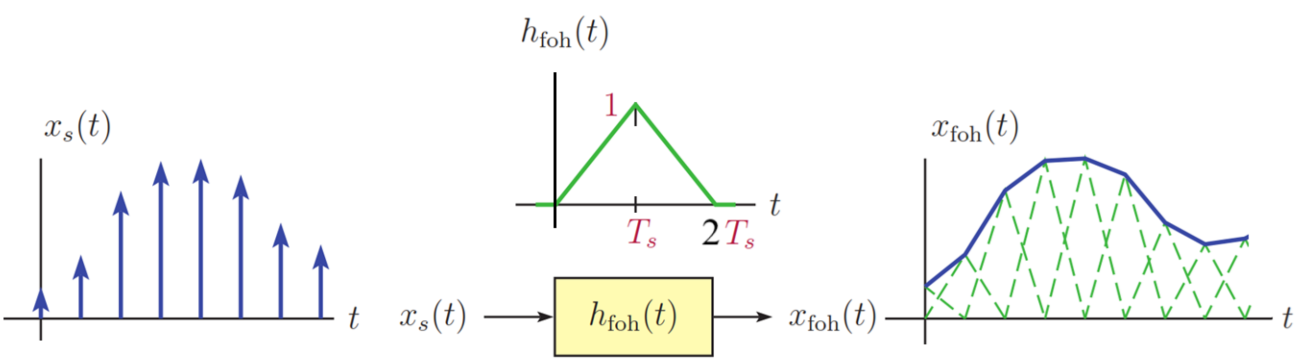

Figure 11: First-order hold interpolation using an interpolation filter.

Activity 3: Consider the zero-filled sampled signal (as you obtained in Activity 01) and pass it through the First-Order Hold (FOH) reconstruction filter, and plot and examine the response in both time and frequency domains.

1. Determine the impulse response of the FOH filter, $h_{foh}(t)$, and visualize it using the **`plot`** or **`stem`** command.

2. Pass the zero-filled sampled signal through the FOH filter using the **`filter`** command.

3. Plot both the sampled signal and the reconstructed signal in the time domain on the same graph to observe the effect of FOH interpolation. If done correctly, your result should resemble Fig. 12.

4. Compute the frequency spectrum of the reconstructed signal (e.g., using the Fourier Transform) and plot it to observe how the interpolation affects the frequency components. If done correctly, your result should resemble Fig. 13.

5. Analyze the frequency response of the reconstructed signal to evaluate FOH performance. Verify that the simulated sideband harmonics align with the theoretical predictions in Equation (2).

6. How does FOH compare to ZOH in terms of performance? Consider a metric to quantify the improvement of FOH over ZOH. For instance, what if you compute the sideband harmonics? (This question is optional.)

Note: Do it yourself, or ask the lab demonstrator for some hints if you are lost.

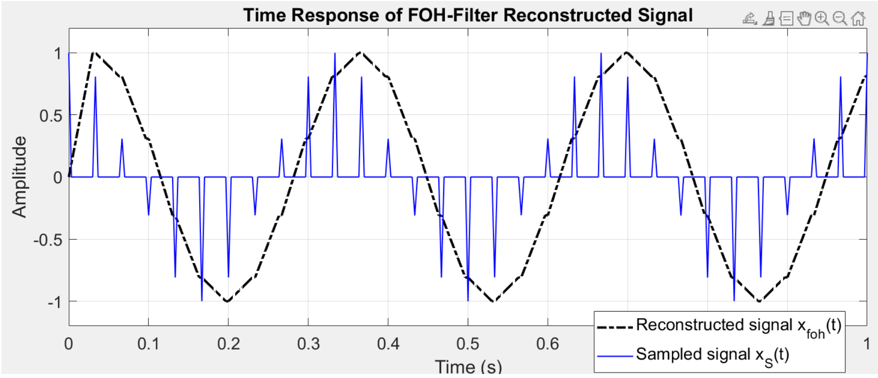

Figure 12: Time response of FOH reconstruction filter for a sampled sinusoid signal.

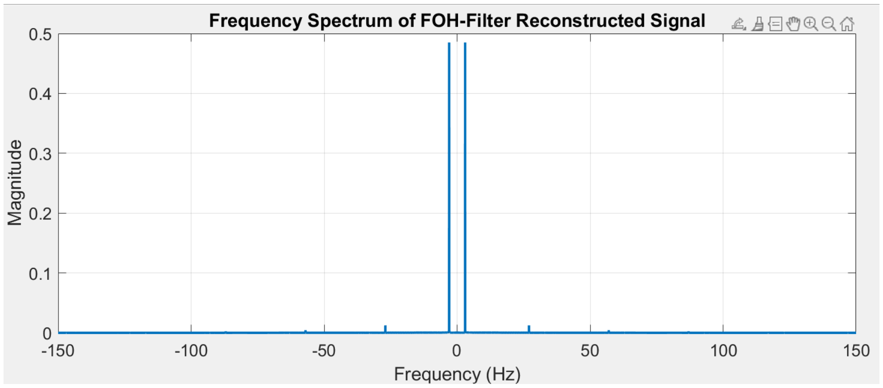

Figure 13: Frequency response of FOH reconstruction filter for a sampled sinusoid signal.

2.3 Raised-Cosine Interpolation

The raised-cosine filter is a filter frequently used for pulse-shaping in digital modulation due to its ability to minimize intersymbol interference (ISI). The raised-cosine filter is an implementation of a low-pass Nyquist filter, and in this lab, we use it as a reconstruction filter. Its frequency-domain description is a piecewise-defined function, given by: (see Fig. 14)

$$H_{\mathrm{rc}}(f) = \begin{cases} 1, & |f| \leq \dfrac{1-\beta}{2T_S} \\ \dfrac{1}{2}\left(1 + \cos\left(\dfrac{\pi T_S}{\beta}\left(|f| - \dfrac{1-\beta}{2T_S}\right)\right)\right), & \dfrac{1-\beta}{2T_S} < |f| \leq \dfrac{1+\beta}{2T_S} \\ 0, & \text{otherwise} \end{cases}$$

for $0 \leqslant \beta \leqslant 1$ and characterized by two values; β, the **roll-off factor**, and T_S, the reciprocal of the sampling frequency.

The impulse response of such a filter is given by: (see Fig. 15)

$$h_{\mathrm{rc}}(t) = \begin{cases} \dfrac{\pi}{4T_S}\,\mathrm{sinc}\left(\dfrac{1}{2\beta}\right), & t = \pm\dfrac{T_S}{2\beta} \\ \dfrac{1}{T_S}\,\mathrm{sinc}\left(\dfrac{t}{T_S}\right)\dfrac{\cos\left(\dfrac{\pi\beta t}{T_S}\right)}{1 - \left(\dfrac{2\beta t}{T_S}\right)^2}, & \text{otherwise} \end{cases}.$$

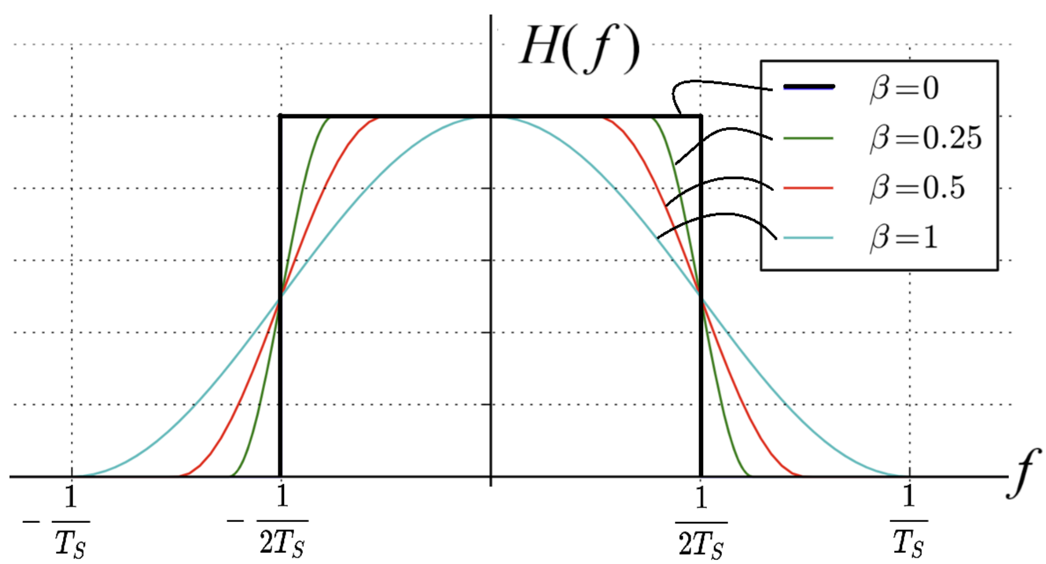

Figure 14: Frequency response of raised-cosine filter with various roll-off factors.

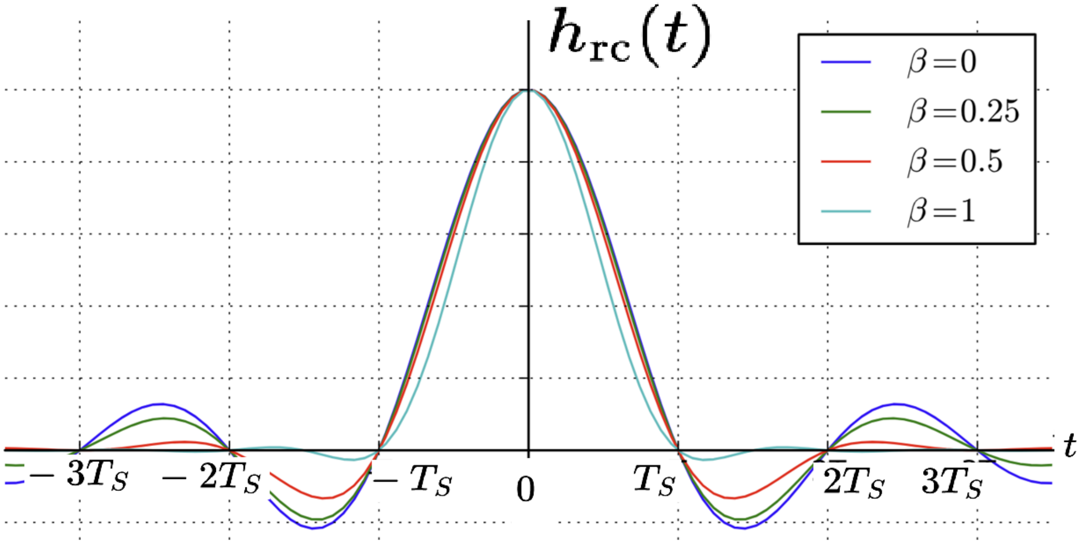

Figure 15: Impulse response of raised-cosine filter with various roll-off factors.

Activity 4: Consider the zero-filled sampled signal (as you obtained in Activity 01) and pass it through the Raised-Cosine reconstruction filter, and plot and examine the response in both time and frequency domains.

1. Determine the impulse response of the Raised-Cosine filter, $h_{rc}(t)$, and visualize it using the **`plot`** command for $\beta = 0.25$.
2. Pass the zero-filled sampled signal through the RC filter using the **`filter`** command.
3. Plot both the sampled signal and the reconstructed signal in the time domain on the same graph to observe the effect of RC interpolation.
4. Compute the frequency spectrum of the reconstructed signal (e.g., using the Fourier Transform) and plot it to observe how the interpolation affects the frequency components.

Note: Do it yourself, or ask the lab demonstrator for some hints if you are lost.