Mechanics of a Gyrocompass

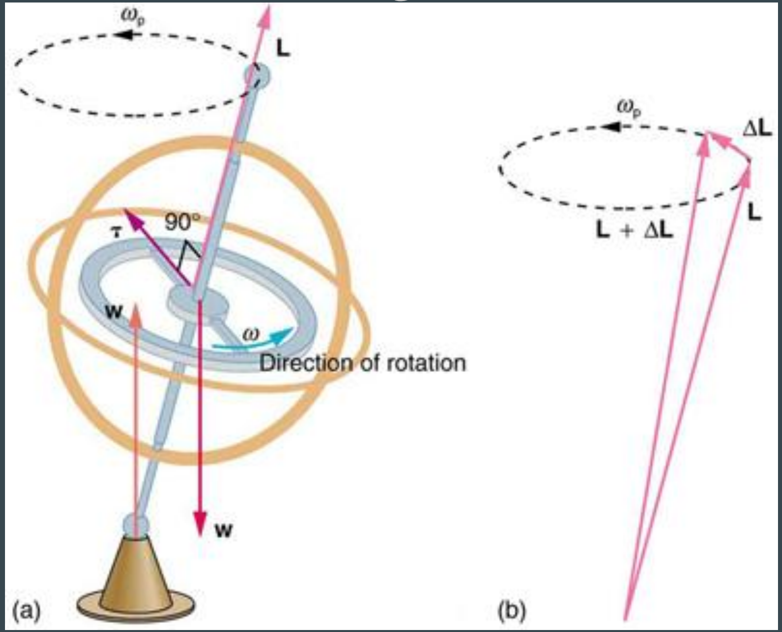
Working Properties

- Newton's First Law

"Every body continues in its state of rest or uniform motion in a straight line, unless compelled by forces to change that state."

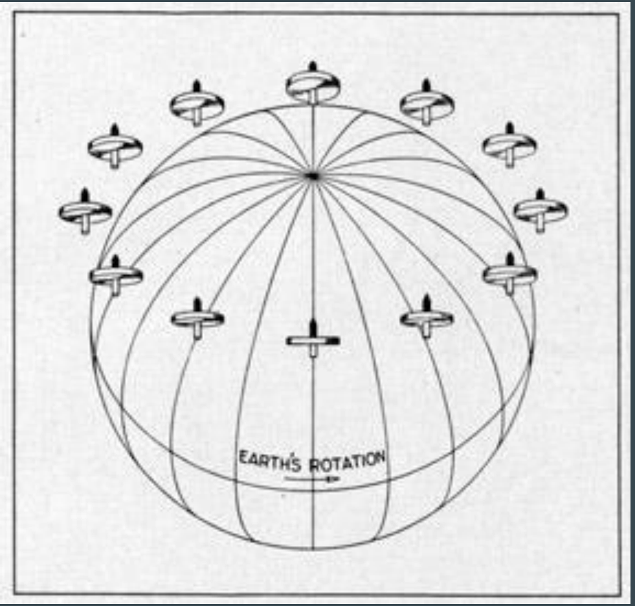
- Precession

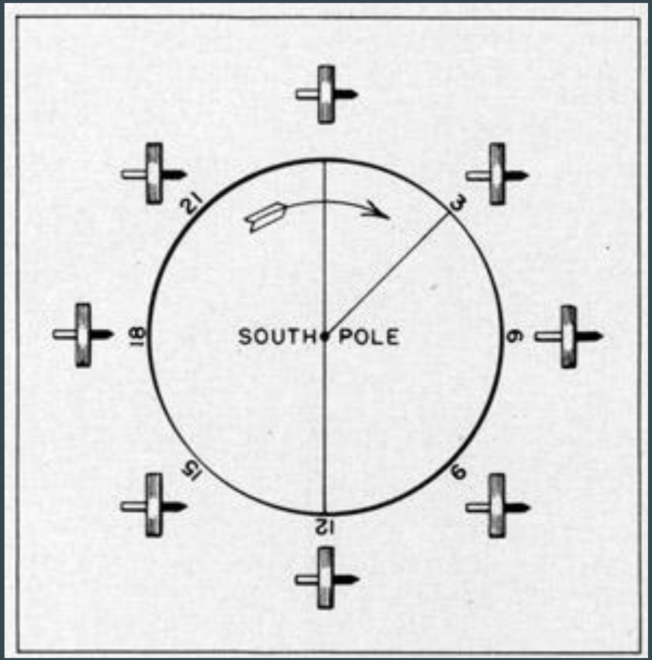
$\boldsymbol{\tau} = \Delta \mathbf{L} / \Delta t$

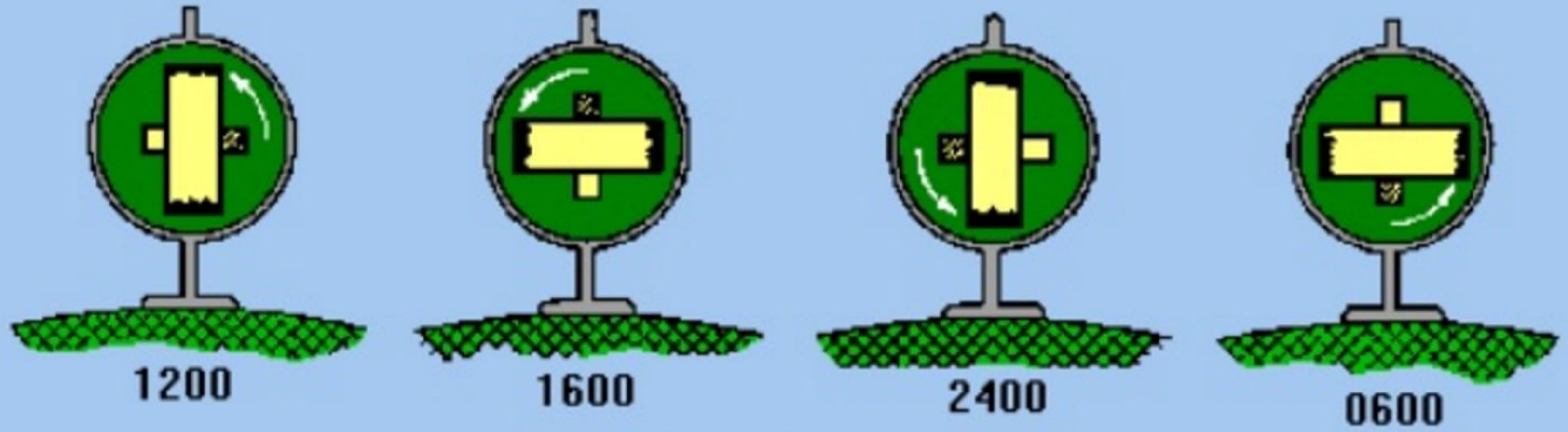


On the Surface

- At the equator...
- At the poles...
- In between...

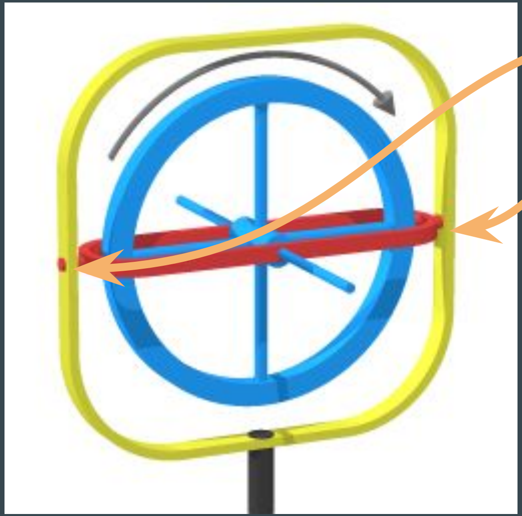






C. Gyro on equator viewed from earth.

Constraining the Gyroscope



Tighten these connections

- Only allows for movement in horizontal

Constructing the Lagrangian

- Construct a rotating frame from an inertial frame.
- Inertia tensor will not be changing with time.
- Diagonalize inertia tensor.
- Find angular velocities and Lagrangian.

Constructing a Rotating Frame

- Start at the centre of the Earth
- By applying a series of time-dependent and independent rotation matrices to the inertial frame we arrive at a coordinate system fixed along the axis of symmetry of the gyrocompass

$$\begin{pmatrix} cos(\Omega t) & sin(\Omega t) & 0 \\ -sin(\Omega t) & cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \rightarrow \begin{pmatrix} cos(\Phi) & sin(\Phi) & 0 \\ -sin(\Phi) & cos(\Phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} cos(\delta) & 0 & -sin(\delta) \\ 0 & 1 & 0 \\ sin(\delta) & 0 & cos(\delta) \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x_4 \\ y_4 \\ z_4 - R \end{pmatrix} \rightarrow \begin{pmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\psi) & sin(\psi) \\ 0 & -sin(\psi) & cos(\psi) \end{pmatrix}\begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix}$$

Angular Velocity

- Three terms
  - ψ - describes rotation of the gyrocompass rotor
  - α - precession of the gyrocompass in the horizontal plane
  - Ω - rotation of the Earth

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\psi) & sin(\psi) \\ 0 & -sin(\psi) & cos(\psi) \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\psi} \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\psi) & sin(\psi) \\ 0 & -sin(\psi) & cos(\psi) \end{pmatrix} \begin{pmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{\alpha} sin(\psi) \\ \dot{\alpha} cos(\psi) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\psi) & sin(\psi) \\ 0 & -sin(\psi) & cos(\psi) \end{pmatrix} \begin{pmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\delta) & 0 & -sin(\delta) \\ 0 & 1 & 0 \\ sin(\delta) & 0 & cos(\delta) \end{pmatrix}$$

$$\begin{pmatrix} cos(\Phi) & sin(\Phi) & 0 \\ -sin(\Phi) & cos(\Phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\Omega t) & sin(\Omega t) & 0 \\ -sin(\Omega t) & cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\Omega} \end{pmatrix} = \begin{pmatrix} -\Omega \sin(\delta)\cos(\alpha) \\ \Omega(\sin(\delta)\sin(\alpha)\cos(\psi) + \cos(\delta)\sin(\psi)) \\ \Omega(-\sin(\delta)\sin(\alpha)\sin(\psi) + \cos(\delta)\cos(\psi)) \end{pmatrix}$$

Result

$$\begin{pmatrix} -\Omega\sin(\delta)\cos(\alpha) \\ \Omega(\sin(\delta)\sin(\alpha)\cos(\psi)+\cos(\delta)\sin(\psi)) \\ \Omega(-\sin(\delta)\sin(\alpha)\sin(\psi)+\cos(\delta)\cos(\psi)) \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\alpha} sin(\psi) \\ \dot{\alpha} cos(\psi) \end{pmatrix} + \begin{pmatrix} \dot{\psi} \\ 0 \\ 0 \end{pmatrix}$$

The Lagrangian

- No potential energy
- $I_2 = I_3$

$$\mathcal{L} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 (\omega_2^2 + \omega_3^2) + \frac{1}{2} M V^2$$

$$\mathcal{L} = \frac{1}{2} I_1 (\dot{\psi} - \Omega \sin(\delta) \cos(\alpha))^2 + \frac{1}{2} I_2 (\dot{\alpha}^2 + \Omega^2 \sin(\alpha)^2 \sin(\delta)^2)$$
$$+ \frac{1}{2} I_2 \Omega^2 \cos(\delta)^2 + I_2 \dot{\alpha} \Omega \cos(\delta)$$

Motion of α Coordinate

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_1}{\partial \dot{\alpha}}\right) = \frac{\partial \mathcal{L}_1}{\partial \alpha}$$

- Solved numerically in our simulation

$$I_2\ddot{\alpha} = I_1\Omega(\dot{\psi} - \Omega\sin(\delta)\cos(\alpha))\sin(\delta)\sin(\alpha) + \frac{1}{2}I_2\Omega^2\sin(\delta)^2\sin(2\alpha)$$

Case 1

- $\sin(\delta) = 0$

$$L_x = I_1 \dot{\psi} = \text{constant}$$
$$I_2 \ddot{\alpha} = 0 \,.$$

Case 2

$$\mathcal{L} = \frac{1}{2}I_1(\dot\psi - \Omega\sin(\delta)\cos(\alpha))^2 + \frac{1}{2}I_2(\dot\alpha^2 + \Omega^2\sin(\alpha)^2\sin(\delta)^2)$$
$$+\frac{1}{2}I_2\Omega^2\cos(\delta)^2 + I_2\dot\alpha\Omega\cos(\delta)$$

- $\sin(\delta) \neq 0$

$$I_2\ddot\alpha = I_1\Omega(\dot\psi - \Omega\sin(\delta)\cos(\alpha))\sin(\delta)\sin(\alpha) + \frac{1}{2}I_2\Omega^2\sin(\delta)^2\sin(2\alpha)$$

- For system exhibiting small oscillations about the North-South line,

$$L_x \approx I_1(\dot\psi - \Omega\sin\delta)\,,$$
$$I_2\ddot\alpha \approx (L_x\Omega\sin\delta + I_2\,\Omega^2\sin^2\delta)\,\alpha$$

$L_x < 0\,,\qquad |\dot\psi| \gg \Omega$

$$L_x \approx -I_1|\dot\psi| \approx \text{constant}\,,$$
$$I_2\ddot\alpha \approx -I_1|\dot\psi|\Omega\sin\delta\,\alpha\,.$$

Solution $\alpha \approx A\sin(\tilde{\omega}t + B)$

- Angular velocity of the axis of symmetry about the North-South line,

$$\tilde{\omega} = \sqrt{\frac{I_1 \sin\delta}{I_2}}\sqrt{|\dot{\psi}|\Omega}$$

- Period of oscillations,

$$T = \frac{2\pi}{\sqrt{|\dot{\psi}|\Omega}}\sqrt{\frac{I_2}{I_1 \sin\delta}}$$