Solution Note: Let T 3 be the three dimensional unit size torus and let μ be the Φ 4 3 measure on the space of distributions D′ (T 3)

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February 14, 2026

Abstract

We investigate quasi-invariance of the three-dimensional dynamical Phi-four measure on the unit torus under deterministic shifts by a fixed nonzero smooth function. Specifically, for the shift map sending a distribution to that distribution plus a smooth test function, we ask whether the original measure and its pushforward assign zero mass to the same measurable sets. The manuscript develops criteria and techniques to compare these measures on the distribution space and to determine when such translations preserve, or fail to preserve, equivalence of null sets.

Contents

1 Introduction

Let $\mathbb{T}^3$ denote the three-dimensional unit torus and let μ be a renormalized Φ^4_3 measure on the space of distributions $\mathcal{D}'(\mathbb{T}^3)$. A basic structural question about such non-Gaussian field theories concerns their behavior under deterministic translations. Given a fixed smooth function $\psi \in C^\infty(\mathbb{T}^3)$, $\psi \not\equiv 0$, consider the shift map

$$T_\psi : \mathcal{D}'(\mathbb{T}^3) \to \mathcal{D}'(\mathbb{T}^3), \qquad T_\psi(u) = u + \psi.$$

The problem addressed here is whether μ is quasi-invariant under T_ψ in the strong sense of *equivalence of measures*: do μ and $(T_\psi)_*\mu$ have the same null sets? For the underlying Gaussian free field, this question is governed by the Cameron–Martin theorem and reduces to whether ψ lies in the Cameron–Martin space associated with the covariance. For the interacting Φ^4_3 measure, the shift interacts nontrivially with the renormalized quartic potential, and the issue becomes an ultraviolet stability question for the resulting change-of-measure density.

This manuscript does not resolve the equivalence question for the limiting Φ^4_3 measure. Instead, it provides a precise reduction from the continuum problem to a cutoff-to-limit passage. The starting point is a standard cutoff approximation μ_N (or μ_ε) built from a finite-dimensional Gaussian reference law and a renormalized Wick-ordered interaction. At each fixed cutoff, translation by a smooth ψ is an admissible map on the cutoff state space, and $(T_\psi)_*\mu_N$ is mutually absolutely continuous with respect to μ_N with an explicit Radon–Nikodym derivative. This derivative decomposes into a Gaussian Cameron–Martin contribution and an interaction term expressible, via Wick translation identities, as an exponential of a finite sum of integrals of Wick monomials paired against smooth functions derived from ψ.

The main contribution of the note is to isolate what remains to be proved in order to upgrade cutoff-level absolute continuity to equivalence at the continuum level. Concretely, the analysis identifies uniform integrability of the cutoff Radon–Nikodym derivatives as the key missing ingredient, along with identification of a strictly positive limiting density. The note also clarifies several scope conditions that are logically prior to the analytic difficulty. First, addition by a smooth function is continuous on the negative-regularity spaces in which μ is expected to concentrate, so μ and its translate live naturally on the same measurable state space. Second, the exact form of the Gaussian covariance matters for edge cases involving the zero mode: under a massive covariance, smooth shifts are Cameron–Martin admissible, whereas under a mean-zero convention one must restrict to shifts with zero average or enlarge the state space.

Organization. Section 2 states the current verdict and formulates a conditional equivalence theorem that reduces quasi-invariance of μ to uniform integrability and convergence of the cutoff Radon–Nikodym densities. Section 3 develops the cutoff framework, derives the explicit cutoff change-of-measure formula, and explains how Wick translation identities organize the interaction contribution.

2 Answer

Answer: UNKNOWN. The equivalence of the renormalized Φ^4_3 measure μ on $\mathcal{D}'(\mathbb{T}^3)$ and its pushforward $(T_\psi)_*\mu$ under a nontrivial smooth deterministic shift $\psi \in C^\infty(\mathbb{T}^3)$ is not settled in this manuscript. What can be given is a precise cutoff-to-limit reduction: at fixed ultraviolet cutoff the shift has an explicit Radon–Nikodym derivative, and equivalence in the limit follows from uniform integrability and identification of the limiting density. The present obstruction is that the required uniform integrability and nondegenerate limiting density are not proved here.

Notation. Throughout, $(T_\psi)_*\mu$ denotes the pushforward measure defined by $(T_\psi)_*\mu(A) = \mu(T_\psi^{-1}A) = \mu(A-\psi)$ for Borel sets $A \subset \mathcal{D}'(\mathbb{T}^3)$. This coincides with the notation $T_\psi^*\mu$ employed in the problem statement; we adopt $(\cdot)_*$ to emphasize that T_ψ acts on the underlying random distribution.

Cutoff measures and the Wick translation identity. Let $N \geq 1$ and let u_N denote the Fourier truncation of the field to frequencies $|k| \leq N$, viewed as a smooth random function on $\mathbb{T}^3$. Let ν_N be the corresponding centered Gaussian free field law on this finite-dimensional space with covariance given by the truncated Green operator. The standard renormalized Φ^4_3 cutoff measure can be written abstractly as

$$\mu_N(du) = Z_N^{-1} \exp\bigl(-V_N(u)\bigr)\, \nu_N(du), \tag{1}$$

where V_N is the renormalized interaction functional, typically of the schematic form

$$V_N(u) = \int_{\mathbb{T}^3} \Bigl(\tfrac{\lambda}{4} \, : u_N(x)^4 :_N + \tfrac{\beta_N}{2} \, : u_N(x)^2 :_N + \gamma_N \Bigr)\, dx, \tag{2}$$

with constants β_N, γ_N chosen to renormalize divergences and $: \cdot :_N$ denoting Wick ordering with respect to ν_N.

Fix a deterministic $\psi \in C^\infty(\mathbb{T}^3)$ and consider the shift map $T_\psi(u) = u + \psi$. At finite N, T_ψ preserves the underlying finite-dimensional affine space and $(T_\psi)_*\mu_N$ is absolutely continuous with respect to μ_N with a Radon–Nikodym derivative R_N that can be expressed by separating the Gaussian and interaction contributions. Writing

$$R_N(u) = \frac{d(T_\psi)_*\mu_N}{d\mu_N}(u) = \exp\bigl(\Delta_N^{\mathrm{G}}(u;\psi) + \Delta_N^{\mathrm{int}}(u;\psi)\bigr), \tag{3}$$

one obtains formally the Gaussian part from the Cameron–Martin formula for ν_N,

$$\Delta_N^{\mathrm{G}}(u;\psi) = \langle u_N, \psi\rangle_{\mathcal{H}_N} - \tfrac{1}{2}\|\psi\|^2_{\mathcal{H}_N}, \tag{4}$$

where $\mathcal{H}_N$ is the Cameron–Martin space of ν_N and the pairing is the corresponding finite-dimensional inner product. The interaction part is the renormalized difference

$$\Delta_N^{\mathrm{int}}(u;\psi) = -V_N(u-\psi) + V_N(u). \tag{5}$$

Because V_N is expressed in terms of Wick polynomials, Δ_N^{int} can be expanded into a finite sum of integrals of Wick monomials in u_N with coefficients determined by ψ and the renormalization constants. At the purely algebraic level, the identities $: (u_N+\psi)^m :_N = \sum_{j=0}^m \binom{m}{j} \psi^{m-j} : u_N^j :_N$ yield an explicit expression for $\Delta_N^{\mathrm{int}}(u;\psi)$ as a polynomial functional of $\{\int : u_N^j : (x)\, f(x)\, dx\}$ for $j \le 4$ and smooth weights f built from ψ.

Conditional verdict. Let μ be the Φ^4_3 measure on $\mathcal{D}'(\mathbb{T}^3)$ constructed as the weak limit of approximating measures μ_N as $N \to \infty$. For each N, extend the cutoff density R_N to the common state space $\mathcal{D}'(\mathbb{T}^3)$ by setting $R_N = 0$ on the complement of the N-th frequency support; thus each R_N is a measurable function on $(\mathcal{D}'(\mathbb{T}^3), \mu_N)$.

Theorem 2.1 (Conditional equivalence)**.** *Assume that $\mu_N \Rightarrow \mu$ weakly and that there exists $p > 1$ such that $\sup_N \|R_N\|_{L^p(\mu_N)} < \infty$. Assume in addition that there exists a measurable function R on $\mathcal{D}'(\mathbb{T}^3)$ with $R > 0$ μ-almost surely, $\mathbb{E}_\mu[R] = 1$, and such that for every bounded continuous function F on $\mathcal{D}'(\mathbb{T}^3)$,*

$$\int F(u) R_N(u)\, d\mu_N(u) \longrightarrow \int F(u) R(u)\, d\mu(u). \tag{6}$$

Then μ and $(T_\psi)_*\mu$ are equivalent and

$$\frac{d(T_\psi)_*\mu}{d\mu} = R. \tag{7}$$

The proof is standard: the L^p bound gives uniform integrability, which allows one to pass the identity $\int F\, d(T_\psi)_*\mu_N = \int F\, R_N\, d\mu_N$ to the limit for bounded continuous F and then extend by monotone class arguments. The strict positivity of R yields mutual absolute continuity rather than mere absolute continuity.

Status and the most-blocking gap. At finite cutoff, absolute continuity holds and the formula for R_N above is rigorous in the finite-dimensional setting. The unresolved step is the passage $N \to \infty$ at the level of the densities. The key requirement is a uniform integrability estimate for $\{R_N\}$ under μ_N, equivalently a nontrivial $L^p(\mu_N)$ bound with $p > 1$, together with identification of the limiting density and its positivity. In the present manuscript we do not prove such a bound, and we also do not exclude the possibility that R_N concentrates in a manner that produces a degenerate limit in which R vanishes on a set of positive μ-measure. Consequently, Theorem 2.1 should be read as a reduction: establishing uniform integrability and a strictly positive L^1 limit of the cutoff Radon–Nikodym derivatives would yield equivalence, whereas failure of these properties would prevent upgrading cutoff-level absolute continuity to equivalence of the limiting measures.

3 Solution

We work on the three-dimensional unit torus $\mathbb{T}^3$. The objective is to decide whether the Φ^4_3 measure μ is equivalent, in the sense of having the same null sets, to its pushforward under a deterministic smooth shift $T_\psi(u) = u + \psi$, where $\psi \in C^\infty(\mathbb{T}^3)$ is not identically zero. The approach adopted here is to write an explicit cutoff Radon–Nikodym formula and to isolate the single ultraviolet step needed to pass from cutoff quasi-invariance to a statement about the continuum measure. All continuum statements that depend on a particular construction of μ or on limiting distributional products are stated as assumptions.

3.1 Construction of a cutoff Φ^4_3 measure

Fix a real-valued orthonormal Fourier basis $\{e_k\}_{k\in\mathbb{Z}^3}$ of $L^2(\mathbb{T}^3)$, with $e_0 \equiv 1$. For $N \in \mathbb{N}$, let P_N denote the spectral projector onto modes $|k| \le N$. Write $E_N := P_N L^2(\mathbb{T}^3)$, a finite-dimensional real vector space. The cutoff field variable is $\varphi_N \in E_N$, viewed as a smooth function. Let ν_N be a centered non-degenerate Gaussian measure on E_N with covariance C_N that is diagonal in the Fourier basis. The precise covariance is not fixed here because the arguments below use only that ν_N has a smooth strictly positive density with respect to Lebesgue measure on E_N, and that polynomial moments exist. The cutoff interacting measure is defined as

$$\mu_N(d\varphi) := Z_N^{-1} \exp\bigl(-V_N(\varphi)\bigr)\, \nu_N(d\varphi), \tag{8}$$

where V_N is a renormalized interaction functional. A standard template is

$$V_N(\varphi) := \int_{\mathbb{T}^3} \Bigl(\tfrac{\lambda}{4} :\varphi(x)^4\!: \_N + \tfrac{m}{2} :\varphi(x)^2\!: \_N\Bigr)\, dx, \tag{9}$$

with parameters $\lambda > 0$, $m \in \mathbb{R}$, and Wick ordering $:\cdot:\_N$ taken with respect to ν_N. The present note does not fix whether additional counterterms are needed (for example, a linear counterterm) because this depends on the normalization convention for the base Gaussian and on the chosen construction of the Φ^4_3 measure. The translation identities used below are algebraic and remain valid for any polynomial Wick ordering at fixed N.

Assumption (continuum limit and renormalized objects). There exists a probability measure μ on a distribution space $\mathcal{D}'(\mathbb{T}^3)$ and a spectral cutoff approximation scheme $(\mu_N)_{N\ge1}$ with Wick renormalization such that $\mu_N \Rightarrow \mu$ as $N \to \infty$, after embedding $E_N \subset \mathcal{D}'(\mathbb{T}^3)$ in the natural way. Moreover, there exist limiting renormalized Wick objects compatible with this approximation, in particular a distribution $:\varphi^3:$ under μ such that the pairing $\langle :\varphi^3:, \psi\rangle$ is well-defined as a real-valued random variable for each fixed $\psi \in C^\infty(\mathbb{T}^3)$. No further regularity or integrability of these random variables is asserted here.

3.2 Renormalization of translated interactions

Let $\psi \in C^\infty(\mathbb{T}^3)$ be fixed and nonzero. For each N, define $\psi_N := P_N\psi \in E_N$. Since ψ is smooth, $\psi_N \to \psi$ rapidly in all Sobolev norms. The translation map on cutoff fields is $T_{\psi_N}(\varphi) = \varphi + \psi_N$. Define the shifted cutoff measure by $T_{\psi_N,*}\mu_N$. Because E_N is finite dimensional and both μ_N and $T_{\psi_N,*}\mu_N$ have smooth strictly positive densities with respect to Lebesgue measure, the two measures are equivalent for each fixed N.

Lemma (Wick translation in finite dimension). Let X be a centered real Gaussian vector in $\mathbb{R}^d$ with covariance Σ. For any deterministic vector $h \in \mathbb{R}^d$ and any integer $n \ge 0$, the Wick power satisfies the algebraic identity

$$:(X+h)^n: \;:= \sum_{j=0}^{n} \binom{n}{j} h^{n-j} :X^j:, \tag{10}$$

where $:\cdot:$ is Wick ordering with respect to X.

Proof. In finite dimension, Wick polynomials are characterized by their exponential generating function

$$\exp\bigl(\langle t, X\rangle - \tfrac{1}{2}\langle t, \Sigma t\rangle\bigr) = \sum_{n\ge0} \frac{1}{n!} :\langle t, X\rangle^n: \,. \tag{11}$$

Multiplying by $\exp(\langle t,h\rangle)$ gives the generating function of $:\langle t, X+h\rangle^n:$ and yields (10) by comparing coefficients. In the field setting, (10) applies pointwise for the scalar Gaussian variable $\varphi_N(x)$ under ν_N, with $h=\psi_N(x)$. For the quartic interaction (9), expanding $:(\varphi+\psi)^4:\_N$ yields

$$:(\varphi+\psi)^4:\_N =:\varphi^4:\_N + 4\psi\ :\varphi^3:\_N + 6\psi^2\ :\varphi^2:\_N + 4\psi^3\varphi + \psi^4, \tag{12}$$

and similarly

$$:(\varphi+\psi)^2:\_N =:\varphi^2:\_N + 2\psi\varphi + \psi^2. \tag{13}$$

These identities are purely algebraic at fixed N. Applying (12)–(13) with ψ replaced by $-\psi_N$ yields the corresponding expansions for $(\varphi-\psi_N)$ used below.

3.3 Radon–Nikodym derivative at cutoff

For each N, define the cutoff Radon–Nikodym derivative

$$R_N(\varphi) := \frac{d\,T_{\psi_N,*}\mu_N}{d\mu_N}(\varphi). \tag{14}$$

A direct computation from (8) gives

$$R_N(\varphi) = \exp\bigl(-V_N(\varphi-\psi_N) + V_N(\varphi)\bigr)\,\frac{d\,T_{\psi_N,*}\nu_N}{d\nu_N}(\varphi). \tag{15}$$

The Gaussian factor is explicit at finite dimension. Writing $\langle\cdot,\cdot\rangle$ for the L^2 inner product on E_N and C_N for the covariance operator of ν_N, one has

$$\frac{d\,T_{\psi_N,*}\nu_N}{d\nu_N}(\varphi) = \exp\Bigl(\langle C_N^{-1}\psi_N,\varphi\rangle - \tfrac{1}{2}\langle C_N^{-1}\psi_N,\psi_N\rangle\Bigr). \tag{16}$$

Assumption (invertibility on the translated subspace). We assume that C_N is strictly positive on E_N, for example via a massive covariance, or else that the translation preserves the subspace where C_N is invertible (for example, by restricting to mean-zero fields and requiring $\int_{\mathbb{T}^3}\psi\,dx=0$). Under this assumption, C_N^{-1} exists on E_N and (16) is valid. Substituting (12) and (13) into (15) yields an explicit exponential of an integral of Wick polynomials. In the template case (9), one obtains

$$\log R_N(\varphi) = \Bigl\langle C_N^{-1}\psi_N,\varphi\Bigr\rangle - \tfrac{1}{2}\langle C_N^{-1}\psi_N,\psi_N\rangle + \frac{\lambda}{4}\int_{\mathbb{T}^3}\Bigl(4\psi_N\ :\varphi^3:\_N - 6\psi_N^2\ :\varphi^2:\_N + 4\psi_N^3\varphi - \psi_N^4\Bigr)\,dx \tag{17}$$

$$+\frac{m}{2}\int_{\mathbb{T}^3}\Bigl(2\psi_N\varphi - \psi_N^2\Bigr)\,dx. \tag{18}$$

All Wick products above are at cutoff level N and hence are classical polynomials of the coordinates of $\varphi\in E_N$.

Proposition (cutoff equivalence). For each fixed $N\in\mathbb{N}$ and $\psi\in C^\infty(\mathbb{T}^3)$, the measures μ_N and $T_{\psi_N,*}\mu_N$ on E_N are equivalent, with Radon–Nikodym derivative R_N given by (15). Moreover, $\int R_N\,d\mu_N = 1$.

Proof. Both measures have strictly positive smooth densities with respect to Lebesgue measure on E_N. The formula (15) follows by the change of variables $\varphi\mapsto\varphi+\psi_N$ in (8), together with (16) for the Gaussian factor. The identity $\int R_N\,d\mu_N=1$ is immediate from the definition of a Radon–Nikodym derivative.

3.4 A cutoff-to-limit criterion for absolute continuity

At cutoff level, the shift always produces an equivalent measure. The continuum problem is to decide whether this equivalence persists as $N\to\infty$. A useful reduction is to phrase the desired conclusion in terms of the convergence of the weighted measures $R_N\mu_N$. Let $\mathcal{X}$ be a Polish space in which the limiting field is realized, for instance a negative-regularity Sobolev or Besov space continuously embedded in $\mathcal{D}'(\mathbb{T}^3)$. We regard μ_N as measures on $\mathcal{X}$ by embedding $E_N\subset\mathcal{X}$.

Proposition (sufficient condition to pass the Radon–Nikodym derivative to the limit). Assume 3.1 and 3.3. Suppose there exists a measurable function $R : \mathcal{X} \to [0, \infty)$ with $R \in L^1(\mu)$ such that for every bounded continuous functional $F : \mathcal{X} \to \mathbb{R}$ one has

$$\lim_{N\to\infty} \int_{\mathcal{X}} F(\varphi)\, R_N(\varphi)\, \mu_N(d\varphi) = \int_{\mathcal{X}} F(\varphi)\, R(\varphi)\, \mu(d\varphi). \tag{19}$$

Then the measures $T_{\psi_N,*}\mu_N = R_N\mu_N$ converge weakly on $\mathcal{X}$ to a probability measure $\widetilde{\mu}$ satisfying $\widetilde{\mu} \ll \mu$ with Radon–Nikodym derivative $d\widetilde{\mu}/d\mu = R$. In particular, the weak limit $\widetilde{\mu}$ of $T_{\psi_N,*}\mu_N$ is absolutely continuous with respect to μ. If the same conclusion holds with ψ replaced by $-\psi$ and with density $\widetilde{R} \in L^1(\widetilde{\mu})$ such that $\mu \ll \widetilde{\mu}$, then μ and $\widetilde{\mu}$ are equivalent.

Proof. For each N, $T_{\psi_N,*}\mu_N = R_N\mu_N$ by definition. The left-hand side of (19) is therefore the integral of F against $T_{\psi_N,*}\mu_N$. Hence (19) states that $T_{\psi_N,*}\mu_N \Rightarrow \widetilde{\mu}$ and identifies the limit as $\widetilde{\mu}(d\varphi) = R(\varphi)\, \mu(d\varphi)$. Since $F \equiv 1$ is bounded and continuous, (19) yields $\int R\, d\mu = \lim_N \int R_N\, d\mu_N = 1$, so $\widetilde{\mu}$ is a probability measure. The final statement follows by applying the same reasoning to the inverse translation.

Remark (where the nontrivial input enters). Proposition 3.4 reduces the continuum absolute continuity question to the joint control of (μ_N) and (R_N). The weak convergence $\mu_N \Rightarrow \mu$ in Assumption 3.1 does not, by itself, control the weighted measures $R_N\mu_N$. A typical sufficient condition is uniform integrability of (R_N) under (μ_N), for example $\sup_N \|R_N\|_{L^p(\mu_N)} < \infty$ for some $p > 1$, together with enough convergence information to identify subsequential limits of $R_N\mu_N$ as absolutely continuous with respect to μ. Establishing such bounds (or proving a precise degeneration mechanism) requires model-specific ultraviolet analysis that is not provided in this chapter.

3.5 The ultraviolet decision point: uniform integrability versus degeneration

The structure of (17) shows that the ultraviolet difficulty is concentrated in the cross terms generated by translation, especially

$$\int_{\mathbb{T}^3} \psi_N(x)\ :\varphi(x)^3:\_N\, dx \qquad \text{and} \qquad \int_{\mathbb{T}^3} \psi_N(x)^2\ :\varphi(x)^2:\_N\, dx. \tag{20}$$

Although these are classical random variables at fixed cutoff, their size under μ_N can depend strongly on N. The continuum question becomes an integrability problem for the exponential of these quantities.

Assumption (distributional pairing is meaningful). Under 3.1, the limiting distribution $:\varphi^3:$ can be paired with a smooth ψ to produce a real random variable. This ensures that the formal continuum analogue of (17) is at least syntactically meaningful. No exponential moment bounds are assumed.

Open problem (uniform integrability of cutoff densities). Fix nonzero $\psi \in C^\infty(\mathbb{T}^3)$. Find verifiable conditions, within a specific construction of the Φ^4_3 measure and its cutoff approximations, under which there exists $p > 1$ such that

$$\sup_{N\geq 1} \|R_N\|_{L^p(\mu_N)} < \infty, \tag{21}$$

and such that a convergence statement of the form (19) holds with some limiting density $R \in L^1(\mu)$. By Proposition 3.4, such a result yields a subsequential weak limit $\widetilde{\mu}$ of $T_{\psi_N,*}\mu_N$ that satisfies $\widetilde{\mu} \ll \mu$.

Alternative open problem (detect degeneration). Fix nonzero $\psi \in C^\infty(\mathbb{T}^3)$. Provide a proof that the family (R_N) degenerates as $N \to \infty$, in a way that rules out uniform integrability, for example by showing that $\log R_N$ diverges in μ_N-probability or that the relative entropy

$$\mathrm{Ent}(T_{\psi_N,*}\mu_N \mid \mu_N) = \int \log R_N(\varphi)\, T_{\psi_N,*}\mu_N(d\varphi) \tag{22}$$

diverges to $+\infty$ as $N \to \infty$. A verified degeneration statement, combined with a stability principle that connects cutoff-level singular behavior of $R_N\mu_N$ to the limiting measure μ, would point toward singularity between μ and any weak subsequential limit of $(T_{\psi_N,*}\mu_N)$. This chapter does not establish such a result.

3.6 A conditional criterion implying failure of uniform integrability

This subsection records a simple mechanism that forces divergence of $\|R_N\|_{L^p(\mu_N)}$ for every fixed $p > 1$, provided the leading translation functional has asymptotically Gaussian tails with diverging variance. The criterion is purely probabilistic; verifying its hypotheses is a separate ultraviolet task.

Definition (dominant cubic translation functional). Fix $\psi \in C^\infty(\mathbb{T}^3)$ nonzero and set

$$X_N(\varphi) := \int_{\mathbb{T}^3} \psi_N(x) \ :\varphi(x)^3:\_N \, dx. \tag{23}$$

In the template formula (17), the term λX_N appears inside $\log R_N$ with coefficient λ.

Lemma (Gaussian exponential growth). Let G_N be a centered real Gaussian random variable with variance σ_N^2. Then for every $p > 1$,

$$\mathbb{E} \exp\bigl(p\lambda G_N\bigr) = \exp\bigl(\tfrac{1}{2}p^2\lambda^2\sigma_N^2\bigr), \tag{24}$$

and in particular $\sigma_N^2 \to \infty$ implies $\mathbb{E}\exp(p\lambda G_N) \to \infty$.

Proposition (conditional divergence of $\|R_N\|_{L^p(\mu_N)}$). Fix $p > 1$ and nonzero $\psi \in C^\infty(\mathbb{T}^3)$. Assume that there exist real constants a_N and random variables Y_N such that

$$\log R_N = \lambda X_N + Y_N + a_N, \tag{25}$$

and such that the following two conditions hold.

(i) Asymptotically Gaussian lower bound for the cubic term. There exists a sequence $\sigma_N \to \infty$ such that X_N is centered under μ_N and for every fixed $t \in \mathbb{R}$,

$$\liminf_{N\to\infty} \log \mathbb{E}_{\mu_N} \exp\bigl(tX_N\bigr) \geq \tfrac{1}{2}t^2\sigma_N^2. \tag{26}$$

(ii) Uniform exponential integrability of the remainder. There exists $\delta > 0$ such that

$$\sup_{N\geq 1} \mathbb{E}_{\mu_N} \exp\bigl(\delta|Y_N|\bigr) < \infty, \tag{27}$$

and $\sup_N |a_N| < \infty$. Then for every $p > 1$,

$$\|R_N\|_{L^p(\mu_N)} \to \infty \qquad \text{as } N \to \infty, \tag{28}$$

so the family (R_N) is not uniformly integrable under (μ_N).

Proof. Fix $p > 1$. Set $q > 1$ and $q' = \frac{q}{q-1}$, and define $r := p/q \in (0, p)$. From (**??**) we have

$$R_N^p = \exp(pa_N)\, \exp(p\lambda X_N + pY_N).$$

To lower bound the last expectation, write

$$\exp(r\lambda X_N) = \exp(r\lambda X_N + rY_N)\, \exp(-rY_N)$$

and apply Hölder with exponents (q, q') to obtain

$$\mathbb{E}_{\mu_N} \exp(r\lambda X_N) \leq \bigl(\mathbb{E}_{\mu_N} \exp(p\lambda X_N + pY_N)\bigr)^{1/q} \bigl(\mathbb{E}_{\mu_N} \exp(-q'rY_N)\bigr)^{1/q'}. \tag{29}$$

Rearranging (29) gives

$$\mathbb{E}_{\mu_N} \exp(p\lambda X_N + pY_N) \geq \frac{\bigl(\mathbb{E}_{\mu_N} \exp(r\lambda X_N)\bigr)^q}{\bigl(\mathbb{E}_{\mu_N} \exp(-q'rY_N)\bigr)^{q/q'}}. \tag{30}$$

Choose $q > 1$ large enough that $q'r = p/(q-1) \le \delta$. Then by (27) we have

$$\sup_{N\ge1} \mathbb{E}_{\mu_N} \exp(-q'rY_N) \le \sup_{N\ge1} \mathbb{E}_{\mu_N} \exp(\delta|Y_N|) < \infty.$$

Moreover, (26) applied with $t = r\lambda$ and $\sigma_N \to \infty$ yields

$$\liminf_{N\to\infty} \log \mathbb{E}_{\mu_N} \exp(r\lambda X_N) \;\ge\; \tfrac{1}{2}r^2\lambda^2\sigma_N^2 \;=\; +\infty.$$

Combining these bounds with (30) and $\sup_N |a_N| < \infty$ shows that $\mathbb{E}_{\mu_N} R_N^p \to \infty$, hence (**??**).

Remark (status of the criterion). Proposition 3.6 does not assert that (26) and (27) hold for the Φ^4_3 cutoff measures in (8); verifying or refuting these hypotheses is the ultraviolet problem. The criterion is recorded to make explicit that controlling the translated measure amounts to controlling cumulant growth of the cubic pairing (23) together with uniform exponential integrability of the remainder terms generated in (17).

3.7 Conclusion

At cutoff level, μ_N is always equivalent to its smooth shift, and the Radon–Nikodym derivative is explicit in (15), with a concrete Wick expansion in (17). The continuum equivalence problem is reduced to a single ultraviolet decision point: one must control the weighted measures $R_N\mu_N$ relative to $\mu_N \Rightarrow \mu$, for instance through a uniform integrability estimate such as (21) and a convergence statement such as (19), or else demonstrate a precise form of degeneration of R_N as $N \to \infty$ that obstructs such a limit passage. Proposition 3.6 spells out one concrete degeneration mechanism: diverging cumulants of the cubic translation functional force divergence of $\|R_N\|_{L^p(\mu_N)}$ provided the remainder terms are uniformly exponentially integrable. The present chapter isolates the exact random functionals generated by translation and makes explicit the additional ultraviolet input required to conclude absolute continuity or singularity for the limiting Φ^4_3 measure.

4 Conclusion and Outlook

This note examined the quasi-invariance problem for the renormalized Φ^4_3 measure μ on $\mathcal{D}'(\mathbb{T}^3)$ under deterministic smooth shifts $T_\psi(u) = u + \psi$. At the level of ultraviolet cutoffs, the picture is clear: for each cutoff measure μ_N (or μ_ε), the translated measure $(T_\psi)_*\mu_N$ is mutually absolutely continuous with respect to μ_N, and the Radon–Nikodym derivative can be written explicitly as the product of a Cameron–Martin factor for the Gaussian reference law and an interaction factor obtained by comparing the renormalized potentials and expanding Wick powers under translation.

For the limiting continuum measure μ, the equivalence question remains open in this manuscript. The main outcome is a reduction principle: if the cutoff Radon–Nikodym derivatives admit a cutoff-uniform L^p bound for some $p > 1$ (hence uniform integrability) and converge, in the sense of testing against bounded continuous observables, to a strictly positive $L^1(\mu)$ density, then μ and $(T_\psi)_*\mu$ are equivalent and the limit provides the Radon–Nikodym derivative. Conversely, without such cutoff-uniform control one cannot rule out degeneration of the densities in the limit, which would obstruct equivalence.

The most concrete remaining gap is therefore analytic rather than algebraic: one needs cutoff-uniform integrability estimates under the interacting measures for the exponential functionals generated by the shift, dominated by pairings such as $\langle :\phi^3:_\varepsilon, \psi\rangle$ together with the lower-order terms arising from the same expansion. Establishing such bounds requires arguments that genuinely control interacting expectations uniformly in the cutoff, not merely Gaussian comparisons. A credible next step is to develop a cutoff-uniform exponential-moment (or logarithmic Sobolev/transport-type) estimate tailored to these Wick-polynomial observables under μ_ε, and then to prove convergence and almost-sure positivity of the limiting density. In parallel, a careful treatment of the massless/mean-zero convention would be needed to formulate the problem for shifts with nonzero spatial average, since in that setting the state space itself may not be preserved by T_ψ.

Acknowledgments

Portions of this manuscript were prepared with the assistance of *Basis* (*Research that compiles.*).