Uniform polynomial relations and separable scaling for determinantal 4-index tensors

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Abstract

Given Zariski-generic matrices $A(1), \ldots, A(n) \in \mathbb{R}^{3\times 4}$ we form tensors $Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}$ by 4×4 determinants of selected rows. We study algebraic relations on the collection $\{Q(\alpha\beta\gamma\delta)\}_{(\alpha,\beta,\gamma,\delta)\in[n]^4}$ and, in particular, ask whether one can decide from the scaled data $T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$ whether the coefficient array λ factors as an outer product on the off-diagonal locus. We give a functorial and equivariant formulation of the associated image varieties as n varies, record uniform multigrading constraints forced by the determinant construction, and isolate the algebraic difficulty of removing the unknown determinantal factors Q while retaining bounded-degree polynomial equations in the observed coordinates. In particular, we show that a natural quadratic test based on $2\times 2\times 2\times 2$ flattening minors cannot be sound on generic determinantal data. We conclude with an elimination/saturation formulation and a set of concrete conjectures and open problems motivated by computational evidence.

Contents

1 Introduction

Families of polynomial invariants defined by determinants are ubiquitous in multilinear algebra and algebraic geometry, both because they carry rich symmetry and because they encode geometric incidence constraints in a coordinate-free way. This paper studies a concrete instance of this theme: from Zariski-generic matrices $A(1), \ldots, A(n) \in \mathbb{R}^{3\times 4}$ we form, for each ordered quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$, a tensor $Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}$ whose entries are 4×4 determinants of stacked rows,

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} = \det\big[A(\alpha)(i,:);\ A(\beta)(j,:);\ A(\gamma)(k,:);\ A(\delta)(\ell,:)\big], \qquad 1 \le i, j, k, \ell \le 3.$$

We view the full collection $\{Q(\alpha\beta\gamma\delta)\}_{(\alpha,\beta,\gamma,\delta)\in[n]^4}$ as a single structured point in the ambient space $\mathcal{T}_n \cong \mathbb{R}^{81n^4}$.

Problem: polynomial recognition of separable scalings. In addition to the determinantal data, we allow a coordinatewise scaling by an array $\lambda \in \mathbb{R}^{n\times n\times n\times n}$. The support of λ is prescribed: $\lambda_{tttt} = 0$ for all $t \in [n]$ and $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for every quadruple that is *not* fully identical. The observable object is the scaled family

$$T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}.$$

The central question is whether separability of the off-diagonal scaling,

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \quad \text{for all not-identical } (\alpha, \beta, \gamma, \delta),$$

can be detected by polynomial relations in the coordinates of T that are uniform in n.

More precisely, we ask whether there exist an integer d, independent of n, and a family of polynomial maps $F_n : \mathcal{T}_n \to \mathbb{R}^{N_n}$ whose coordinate degrees are at most d, such that, for Zariski-generic $A(1), \ldots, A(n)$ and every λ with the prescribed support pattern, the vanishing condition $F_n(T) = 0$ holds if and only if λ factors as an outer product on the off-diagonal locus. The forward implication (factorization implies vanishing) is a design requirement; the reverse implication is the substantive "identifiability" direction and is where genericity and saturation issues become delicate.

Scope and contributions. The manuscript is written to separate what can be proved uniformly from what remains conjectural.

First, we set up a universal determinantal parameterization $\Phi_n : (\mathbb{R}^{3\times 4})^n \to \mathcal{T}_n$ and record the basic equivariance under the natural actions (permutations of indices and row operations on each $A(t)$). This viewpoint makes the dependence on n explicit and suggests that the relevant ideals should be studied as a functorial family.

Second, we develop an algebraic packaging of the defining relations of the Zariski closure of $\mathrm{im}(\Phi_n)$ that is compatible with varying n and with multigrading by index-weights. In this framework, it is natural to seek uniform bounded-degree generators for the relation ideals, and to use equivariance to reduce the search for relations to a finite set of symmetry types.

Third, we propose a construction paradigm for a universal polynomial map F_n aimed at the scaling-identification problem. We emphasize, however, that the strongest "if and only if" statement is not established in the current draft. Instead, the present status is that several candidate families of low-degree constraints are motivated by the symmetry and by normalized minor identities, while computational checks highlight obstruction patterns: clearing denominators can introduce extraneous components, and restricting to subtensors can fail to preserve genericity in a way that is compatible with the determinantal model. These issues motivate a conjectural "saturation obstruction" that, if proved, would rule out certain naive polynomial tests.

Organization. Section 2 states the characterization problem precisely and summarizes the main results and conjectures recorded in the draft. Section 2.1 fixes algebraic conventions, notation, and the relevant ambient coordinate rings. Section 4 introduces the universal determinantal parameterization and its basic properties. Section 5 develops equivariance, multigrading, and functoriality in n. Section 2.5 discusses uniform bounded-degree generation of the defining ideals suggested by the functorial setup. Section 7 outlines a candidate construction of a universal polynomial map F_n and clarifies which implication directions it is intended to certify. Section 7.5 studies the scaling action, stabilizers, and generic fibers, isolating the remaining gap in proving an exact "only if" converse. Finally, Section **??** reports computational evidence, falsification-style checks, and alternative relaxations that may be provable without saturation-sensitive steps.

2 Problem statement and main results

2.1 Algebraic Setup and Notation

Fix an integer $n \geq 5$ and write $[n] = \{1, 2, \ldots, n\}$. For each $\alpha \in [n]$, let $A(\alpha) \in \mathbb{R}^{3\times 4}$ be a real matrix, and regard the family $A = (A(\alpha))_{\alpha \in [n]}$ as a point of the affine parameter space

$$\mathcal{P}_n = (\mathbb{R}^{3\times 4})^n.$$

A statement is said to hold for Zariski-generic A when it holds for all $A \in \mathcal{P}_n$ outside a proper Zariski-closed subset of $\mathcal{P}_n$ (equivalently, outside the common zero set of some nonzero polynomial functions on $\mathcal{P}_n$ with real coefficients). Throughout this chapter, genericity is used only as a standing hypothesis in the problem formulation and in the statements of the main results. Let $V = \mathbb{R}^4$ and $W = \mathbb{R}^3$. We identify $A(\alpha)$ with a linear map $V \to W$. For $i \in \{1, 2, 3\}$, write $A(\alpha)_{i,:} \in V^*$ for the ith row of $A(\alpha)$ viewed as a covector on V. We also fix the standard basis (e_1, e_2, e_3) of W and identify $W^{\otimes 4}$ with $\mathbb{R}^{3\times 3\times 3\times 3}$ via this basis. The central objects of the paper are determinantal tensors indexed by quadruples in $[n]^4$, together with polynomial relations among the full collection of these tensors after an auxiliary coefficient scaling. The goal is a uniform (in n) algebraic characterization of when the coefficient scaling is separable.

2.2 Determinantal Tensor Construction

For indices $\alpha, \beta, \gamma, \delta \in [n]$, define a tensor $Q(\alpha\beta\gamma\delta) \in W^{\otimes 4}$ by the coordinate formula

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} = \det \begin{pmatrix} A(\alpha)_{i,:} \\ A(\beta)_{j,:} \\ A(\gamma)_{k,:} \\ A(\delta)_{\ell,:} \end{pmatrix}, \qquad i, j, k, \ell \in \{1, 2, 3\}. \tag{1}$$

Equivalently, $Q(\alpha\beta\gamma\delta)_{ijk\ell}$ is the evaluation of the alternating 4-form $\det(\cdot)$ on the four covectors $A(\alpha)_{i,:}, A(\beta)_{j,:}, A(\gamma)_{k,:}, A(\delta)_{\ell,:}$ V^*. The map

$$\Phi_n : \mathcal{P}_n \to (W^{\otimes 4})^{[n]^4}, \qquad A \mapsto \big(Q(\alpha\beta\gamma\delta)\big)_{\alpha,\beta,\gamma,\delta\in[n]}$$

is polynomial in the entries of the matrices $A(\alpha)$. The determinantal definition immediately implies alternating behavior with respect to permutations of the four rows in (1). Concretely, swapping the pair (α, i) with (β, j) multiplies $Q(\alpha\beta\gamma\delta)_{ijk\ell}$ by -1, and similarly for any transposition among the four row slots. This alternating behavior is an identity at the level of the coordinate polynomials defining Φ_n. To anchor notation, an explicit instantiation of (1) for $n = 5$ is recorded in the dataset q construction example n5 (JSON)[1]. The file provides one concrete numerical example of the full collection of tensors $Q(\alpha\beta\gamma\delta)$ obtained from a particular choice of real matrices. A related file, q span singular values n5 (CSV)[2], summarizes singular values for a matrix formed from a chosen linearization of these tensors. These artifacts are used in this chapter only as illustrative examples of the construction and of linear-algebraic diagnostics one might compute; they are not used as evidence for Zariski-generic properties.

2.3 The Polynomial Constraint Problem

Introduce an auxiliary coefficient tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ and consider the scaled collection

$$\big(\lambda_{\alpha\beta\gamma\delta}\, Q(\alpha\beta\gamma\delta)\big)_{\alpha,\beta,\gamma,\delta\in[n]} \in (W^{\otimes 4})^{[n]^4} \cong \mathbb{R}^{81n^4}.$$

We impose the support convention that $\lambda_{\alpha\beta\gamma\delta} = 0$ on the diagonal locus where all four indices coincide. Equivalently, we restrict attention to coefficient tensors supported on

$$\{(\alpha, \beta, \gamma, \delta) \in [n]^4 : (\alpha, \beta, \gamma, \delta) \neq (\alpha, \alpha, \alpha, \alpha)\}.$$

[1] `q_construction_example_n5.json`
[2] `q_span_singular_values_n5.csv`

This convention isolates the off-diagonal scaling degrees of freedom that are relevant for the separability question. A coefficient tensor λ is called *separable* when there exist vectors $u, v, w, x \in (\mathbb{R}^*)^n$ such that

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \tag{2}$$

for every off-diagonal quadruple $(\alpha, \beta, \gamma, \delta)$. This is the rank-one condition for the 4-way tensor λ on the prescribed support, expressed in multiplicative form. The objective is to decide separability of λ using only polynomial equations in the coordinates of the scaled determinantal tensors, and to do so uniformly in n. More precisely, one seeks a polynomial map

$$F_n : \mathbb{R}^{81n^4} \to \mathbb{R}^{N_n}$$

with the following three features, stated here in prose to emphasize that they are design constraints rather than derived facts. First, the coordinate polynomials of F_n are universal, in the sense that they do not depend on the particular Zariski-generic choice of the matrices $A(1), \ldots, A(n)$. Second, the total degrees of those coordinate polynomials are bounded by a constant that is independent of n. Third, for Zariski-generic A and for λ supported off-diagonal, the vanishing condition

$$F_n\Big(\big(\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)\big)_{\alpha\beta\gamma\delta}\Big) = 0$$

holds exactly when λ is separable in the sense of (2). For later reference, it is helpful to name the set that is intended to be cut out by the universal equations. For fixed n and fixed Zariski-generic A, define

$$\mathcal{S}_{n,A} = \Big\{\big(\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)\big)_{\alpha\beta\gamma\delta} \in \mathbb{R}^{81n^4} : \lambda \text{ satisfies (2) off-diagonal}\Big\}.$$

The problem asks for bounded-degree, A-independent equations whose common zero set agrees with $\mathcal{S}_{n,A}$ for Zariski-generic A. A collection of explicit polynomial relations satisfied by rank-one coefficient tensors in the case $n = 6$ is recorded in lambda rank1 constraints n6 (CSV)[3] and lambda rank1 constraints n6 (JSON)[4]. These artifacts document one concrete representation of constraints implied by (2), independent of the determinantal tensors. They serve as a sanity check for the form of rank-one conditions that the later algebraic framework aims to transport to the scaled determinantal setting.

2.4 Main Results

The next two statements summarize the main algebraic claims pursued in the remainder of the manuscript. They are formulated to match the constraint problem above and to emphasize uniformity in n.

Main Conjecture 1 (Conjecture (C3)). Let $n \geq 5$. There exist integers $D \geq 1$ and $m \geq 1$, independent of n, and for each n a finite family of polynomials

$$\mathcal{F}_n = \{f_{n,1}, \ldots, f_{n,m}\} \subset \mathbb{R}[x_1, \ldots, x_{81n^4}]$$

such that each $f_{n,i}$ has total degree at most D and the following holds. For Zariski-generic $A \in \mathcal{P}_n$ and for any coefficient tensor λ supported off-diagonal, one has

$$\big(f_{n,i}((\lambda \cdot Q)(A)) = 0 \text{ for all } i\big) \quad \text{exactly when} \quad \lambda \text{ is separable in the sense of (2)},$$

where $(\lambda \cdot Q)(A)$ denotes the point of $\mathbb{R}^{81n^4}$ with coordinates $(\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)_{ijk\ell})$. Equivalently, the set $\mathcal{S}_{n,A}$ is cut out set-theoretically by equations of degree bounded by a constant independent of n.

Status. Conjecture 1 is the central uniform bounded-degree statement targeted in this manuscript. Section 4 develops a representation-theoretic and functorial framework intended to support such a result, and it establishes partial structural steps toward bounded-degree generation; a complete proof of the full equivalence in Conjecture 1 is not currently included.

Corollary 2.1 (Conditional corollary (C4))**.** *For $n \geq 5$ and Zariski-generic $A \in \mathcal{P}_n$, let $I_{n,A} \subset \mathbb{R}[x_1, \ldots, x_{81n^4}]$ denote the ideal of all polynomials vanishing on $\mathcal{S}_{n,A}$. Assuming Conjecture 1, there exists a constant D', independent of n, such that $I_{n,A}$ is generated up to radical by polynomials of degree at most D'. The constant D' can be chosen as a function of the constant D from Conjecture 1.*

[3] `lambda_rank1_constraints_n6.csv`
[4] `lambda_rank1_constraints_n6.json`

2.5 Overview of the Proof Strategy

This subsection records the intended route to Conjecture 1 and its conditional consequence Corollary 2.1 in a way that separates the conceptual inputs from the technical implementation discussed in Section 4.

Functorial packaging of the variables. The ambient coordinates of $(W^{\otimes 4})^{[n]^4} \cong \mathbb{R}^{81n^4}$ are naturally indexed by a quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$ and a multi-index $(i, j, k, \ell) \in \{1, 2, 3\}^4$. The polynomial relations sought are required to be uniform in n, which suggests organizing the coordinate rings as a family compatible with injections of finite sets. In Section 4, this compatibility is made precise by introducing a functorial viewpoint in which the coordinate rings form a module over a polynomial algebra built from a fixed finite-dimensional representation, and the transition maps encode the relabeling of indices.

Separability as a structured subvariety. The rank-one condition (2) describes the image of a multiplicative parameterization by $(\mathbb{R}^*)^n$ in each mode. After passing to a suitable affine or projective closure, separable tensors correspond to a Segre-type variety in the coefficient space. The scaled determinantal collection $(\lambda \cdot Q)(A)$ can be viewed as the coordinatewise product of λ with a determinantal point $Q(A)$ in the ambient space. The key difficulty is that one does not observe λ directly, only the scaled point. The proof strategy therefore isolates polynomial conditions on the scaled point that force λ to lie on the rank-one locus.

Bounded-degree generation via noetherianity mechanisms. Conjecture 1 asserts that a bounded degree suffices, uniformly in n. The mechanism pursued in Section 4 is a noetherianity argument applied to a representation-stable family of ideals. At a high level, one shows that the ideals cutting out the separable-scaled locus stabilize in a suitable sense under the operations that increase n by adding new indices, and that this stability implies finite generation by a bounded set of polynomial templates. The conclusion is then translated back into bounded-degree equations in the concrete coordinates on $\mathbb{R}^{81n^4}$. Any required background citation for this type of noetherianity principle is to be added once the precise formulation used in Section 4 is finalized.

Role of computed artifacts. The datasets listed at the beginning of this chapter serve two limited purposes. First, they validate that the determinantal construction (1) has been instantiated consistently in small cases, for example q construction example n5 (JSON)[5]. Second, they provide concrete examples of rank-one constraints in the coefficient space, for example lambda rank1 constraints n6 (JSON)[6]. These artifacts are not used to justify genericity claims, nor are they used as substitutes for the algebraic arguments required for Conjecture 1 and Corollary 2.1.

3 Algebraic setup and notation

This chapter fixes notation for the determinantal tensors indexed by $[n]^4$ and for the algebraic relations that will be studied throughout the paper. The guiding theme is functoriality in the index set $[n]$: the construction produces a large collection of tensors from data $A(1), \ldots, A(n)$, and the subsequent questions ask for polynomial conditions that are uniform in n and invariant under natural rescalings. No computational verification is assumed in this chapter; all genericity statements are understood in the Zariski sense.

3.1 Matrices and indexing

Fix an integer $n \geq 5$ and write $[n] = \{1, 2, \ldots, n\}$. For each $\alpha \in [n]$ we consider a real matrix

$$A(\alpha) \in \mathbb{R}^{3\times 4}.$$

For $i \in \{1, 2, 3\}$, the ith row of $A(\alpha)$ is denoted

$$A(\alpha)_{i,\cdot} \in \mathbb{R}^{1\times 4}.$$

[5] `q_construction_example_n5.json`
[6] `lambda_rank1_constraints_n6.json`

When convenient we also view $A(\alpha)_{i,\cdot}$ as a vector in $\mathbb{R}^4$ by transposition, with no change in notation. The tuple $(A(1), \ldots, A(n))$ is assumed *Zariski-generic*. Concretely, this means that the tuple lies outside a proper Zariski-closed subset of the affine space $(\mathbb{R}^{3\times 4})^n \cong \mathbb{R}^{12n}$. Equivalently, any property that is asserted to hold for Zariski-generic tuples is intended to hold on a nonempty Zariski-open set, hence in particular it holds for all tuples outside the common zero set of finitely many nonzero polynomials in the entries of the matrices $A(\alpha)$. We emphasize that Zariski-genericity is used only as a theoretical device; no claim is made that it has been tested statistically by random sampling. Given indices $\alpha, \beta, \gamma, \delta \in [n]$, we will form a 4×4 matrix by stacking rows from the corresponding $A(\cdot)$. To avoid ambiguities, we adopt the convention that semicolons denote vertical concatenation of row vectors:

$$\left[r_1; r_2; r_3; r_4\right] \quad \text{is the } 4 \times 4 \text{ matrix with rows } r_1, r_2, r_3, r_4 \in \mathbb{R}^{1\times 4}.$$

3.2 Determinantal tensors

For each $(\alpha, \beta, \gamma, \delta) \in [n]^4$ we define a 4-way tensor

$$Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}$$

by specifying its entries. For $(i, j, k, \ell) \in \{1, 2, 3\}^4$, set

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} := \det\left[A(\alpha)_{i,\cdot}; A(\beta)_{j,\cdot}; A(\gamma)_{k,\cdot}; A(\delta)_{\ell,\cdot}\right]. \tag{3}$$

The determinant in (3) is multilinear in the selected rows and alternating in its four row arguments. Accordingly, $Q(\alpha\beta\gamma\delta)$ is multilinear in the data $(A(\alpha), A(\beta), A(\gamma), A(\delta))$ in the evident sense. It is useful to record the symmetry constraints inherited from the determinant. Let S_4 act on the ordered quadruple $(\alpha, \beta, \gamma, \delta)$ by permuting its positions. If $\pi \in S_4$, then alternatingness of the determinant implies the sign rule

$$Q(\pi \cdot (\alpha\beta\gamma\delta))_{\pi\cdot(ijk\ell)} = \operatorname{sgn}(\pi)\, Q(\alpha\beta\gamma\delta)_{ijk\ell}, \tag{4}$$

where $\pi \cdot (\alpha\beta\gamma\delta)$ denotes the permuted quadruple of indices and $\pi \cdot (ijk\ell)$ denotes the permuted quadruple of row indices. In particular, if two positions among $\alpha, \beta, \gamma, \delta$ coincide and the corresponding row indices also coincide, then the determinant may vanish due to repeated rows, but in general repeated Greek indices do not force the determinant to be zero because the selected rows can still be distinct. For later algebraic manipulations, it is convenient to package the entire family

$$\mathcal{Q}_n := \{Q(\alpha\beta\gamma\delta) : (\alpha, \beta, \gamma, \delta) \in [n]^4\}$$

as a single point in an ambient affine space. Write

$$\mathbf{Q} \in (\mathbb{R}^{3\times 3\times 3\times 3})^{n^4} \cong \mathbb{R}^{81n^4}$$

for the concatenation of all entries $Q(\alpha\beta\gamma\delta)_{ijk\ell}$. This identification fixes a coordinate system in which one can discuss polynomial relations among the entries of all $Q(\alpha\beta\gamma\delta)$ simultaneously. In particular, a polynomial map acting on the collection of tensors may be viewed as a polynomial map on $\mathbb{R}^{81n^4}$, with coordinates indexed by $(\alpha, \beta, \gamma, \delta, i, j, k, \ell)$.

Coordinate rings and functorial viewpoint. Although the matrices and tensors are real-valued in the statement of the motivating objective, the algebraic questions are naturally expressed over an algebraically closed field of characteristic zero by base change. In this chapter we keep the notation over $\mathbb{R}$ to match the objective, while implicitly using the standard translation: polynomial identities that hold on a Zariski-open subset over $\mathbb{R}$ correspond to identities in the coordinate ring after base change. The representation-stable perspective pursued later is compatible with this translation because it is formulated in terms of polynomial functors and ideals in symmetric algebras; the present chapter only fixes notation needed to state those ideals.

3.3 Scaling action and support constraints

Let

$$\lambda \in \mathbb{R}^{n\times n\times n\times n}$$

be an array of coefficients with entries $\lambda_{\alpha\beta\gamma\delta}$ indexed by $[n]^4$. Given such λ, define the scaled family of tensors

$$(\lambda \odot \mathcal{Q}_n)(\alpha\beta\gamma\delta) := \lambda_{\alpha\beta\gamma\delta}\, Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}. \tag{5}$$

We will impose a support restriction on λ consistent with the motivating objective: diagonal entries vanish, and off-diagonal entries are intended to be nonzero.

Definition 3.1 (Off-diagonal support constraint)**.** An array $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ is said to satisfy the *off-diagonal support constraint* when

$$\lambda_{tttt} = 0 \quad \text{for all } t \in [n].$$

We say that λ has *full off-diagonal support* when, in addition,

$$\lambda_{\alpha\beta\gamma\delta} \neq 0 \quad \text{for every } (\alpha, \beta, \gamma, \delta) \in [n]^4 \text{ that is not of the form } (t, t, t, t).$$

The linear part of Definition 3.1 is the vanishing constraint on the diagonal. The nonvanishing condition for full off-diagonal support is a Zariski-open condition in the affine subspace cut out by $\lambda_{tttt} = 0$. Next we formalize the rank-one multiplicative scalings of λ that appear in the objective. Let $(\mathbb{R}^*)^n$ denote the group of nonzero real scalars indexed by $[n]$ under componentwise multiplication.

Definition 3.2 (Rank-one scaling action on coefficient arrays)**.** Given $u, v, w, x \in (\mathbb{R}^*)^n$, define an action on $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ by

$$(u, v, w, x) \cdot \lambda \;\in\; \mathbb{R}^{n\times n\times n\times n}, \qquad \big((u, v, w, x) \cdot \lambda\big)_{\alpha\beta\gamma\delta} := (u_\alpha v_\beta w_\gamma x_\delta)\, \lambda_{\alpha\beta\gamma\delta}.$$

We say that λ is *rank-one multiplicatively separable off-diagonal* when there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \quad \text{for every } (\alpha, \beta, \gamma, \delta) \in [n]^4 \text{ that is not of the form } (t, t, t, t).$$

No constraint is imposed on λ_{tttt} by this definition.

When λ satisfies the off-diagonal support constraint $\lambda_{tttt} = 0$, the separability requirement is imposed only on not-identical index quadruples, and is therefore compatible with diagonal vanishing without forcing any of u, v, w, x to have zero components.

3.4 Statement of the algebraic problem

We now formalize the "universal equations" question as an existence problem for a polynomial map on the ambient space of indexed tensors. The formulation below should be read as a theoretical formalization of the objective, not as a statement that such a map has been constructed or verified. Fix $n \geq 5$ and consider the affine space

$$\mathcal{T}_n := (\mathbb{R}^{3\times 3\times 3\times 3})^{n^4} \cong \mathbb{R}^{81n^4}.$$

An element $\mathbf{T} \in \mathcal{T}_n$ may be written as a family $(T(\alpha\beta\gamma\delta))_{(\alpha,\beta,\gamma,\delta)\in[n]^4}$ with each $T(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}$. For $A(1), \ldots, A(n)$ we write $\mathbf{Q}(A) \in \mathcal{T}_n$ for the family induced by (3).

Definition 3.3 (Uniform bounded-degree test map)**.** A family of polynomial maps $(F_n)_{n\geq 1}$ with

$$F_n : \mathcal{T}_n \to \mathbb{R}^{N_n}$$

is called a *uniform bounded-degree test family* when there exists a constant $D \geq 1$ such that every coordinate polynomial of every F_n has total degree at most D, and the bound D is independent of n. No restriction is imposed on the output dimension N_n beyond finiteness for each n.

The motivating objective asks for an F that does not depend on the particular matrices $A(\alpha)$ and whose degree is independent of n. In the present notation, this becomes the search for a uniform bounded-degree family F_n whose vanishing set on scaled determinantal families $\lambda \odot \mathbf{Q}(A)$ detects multiplicative separability of λ on not-identical index quadruples, under the diagonal vanishing convention.

Problem 3.4 (Universal separability detection on determinantal data)**.** Determine whether there exists a uniform bounded-degree test family $(F_n)_{n\geq 5}$ such that for every $n \geq 5$, for Zariski-generic $A(1), \dots, A(n) \in \mathbb{R}^{3\times 4}$, and for every $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfying

$$\lambda_{tttt} = 0 \;\forall t \in [n], \qquad \text{and} \qquad \lambda_{\alpha\beta\gamma\delta} \neq 0 \text{ for all not-identical } (\alpha, \beta, \gamma, \delta),$$

the following equivalence holds:

$$F_n\big(\lambda \odot \mathbf{Q}(A)\big) = 0 \quad \Longleftrightarrow \quad \exists\, u, v, w, x \in (\mathbb{R}^*)^n \text{ such that } \lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \text{ for all not-identical } (\alpha, \beta, \gamma, \delta).$$

The map(s) F_n are required to be independent of the particular matrices $A(\alpha)$, and the degree bound is required to be independent of n.

Problem 3.4 is intentionally phrased in terms of Zariski-generic A, because without genericity the family $\mathbf{Q}(A)$ can satisfy additional polynomial relations unrelated to separability of λ. The Zariski-generic quantifier ensures that the desired equivalence is intended to hold on a dense open subset of the parameter space of A.

Reformulation via an image variety. Let $\mathcal{A}_n := (\mathbb{R}^{3\times 4})^n$ and consider the polynomial map

$$\Phi_n : \mathcal{A}_n \times \mathbb{R}^{n\times n\times n\times n} \to \mathcal{T}_n, \qquad \Phi_n(A, \lambda) := \lambda \odot \mathbf{Q}(A),$$

where $\mathbf{Q}(A)$ is defined by (3) and scaling is as in (5). Let

$$L_n := \{\lambda \in \mathbb{R}^{n\times n\times n\times n} : \lambda_{tttt} = 0 \;\forall t \in [n]\}$$

be the linear subspace encoding diagonal vanishing, and let $U_n \subseteq L_n$ be the Zariski-open subset defined by $\lambda_{\alpha\beta\gamma\delta} \neq 0$ on all not-identical quadruples. One may consider the constructible image $\Phi_n(\mathcal{A}_n \times U_n) \subseteq \mathcal{T}_n$ and its Zariski closure $X_n \subseteq \mathcal{T}_n$. A bounded-degree test family (F_n) as in Definition 3.3 can be interpreted as giving equations (uniformly in degree) that vanish on the locus inside X_n corresponding to rank-one multiplicatively separable off-diagonal coefficient arrays. This interpretation motivates the representation-stable approach pursued later: one seeks a description of the ideals defining X_n and related loci as n varies, with a uniform degree bound on generators. In this chapter we only record the formal setup; any noetherian or stability statements are deferred and, when stated later, will be explicitly labeled as proved results or as conjectural programmatic steps.

Scope limitation of this chapter. The remainder of the manuscript will refine Problem 3.4 in directions that make the bounded-degree requirement tractable, including symmetry reductions, functoriality in $[n]$, and the passage from set-theoretic to ideal-theoretic statements. None of those refinements are assumed here. The only role of the present chapter is to provide precise notation for $A(\alpha)$, $Q(\alpha\beta\gamma\delta)$, the coefficient array λ, and the intended meaning of "uniform in n" and "Zariski-generic" in the objective.

4 Universal determinantal parameterization

This chapter formalizes the determinantal tensor construction underlying the family of instances indexed by $[n]^4$, and formulates a universal elimination problem: detect (from the scaled tensors alone) whether the scaling coefficients form a rank-1 four-way tensor. The map F requested in the problem statement is treated here as an open target rather than an established construction, and this chapter does not reference any external section as providing a proof of existence.

4.1 Zariski-generic construction of slice determinantal tensors

Fix an integer $n \geq 5$. For each $\alpha \in [n]$ let $A^{(\alpha)} \in \mathbb{R}^{3\times 4}$ and write $A^{(\alpha)}(i,:) \in \mathbb{R}^4$ for its ith row. For $\alpha, \beta, \gamma, \delta \in [n]$ define a 4-tensor

$$Q^{(\alpha\beta\gamma\delta)} \in (\mathbb{R}^3)^{\otimes 4} \cong \mathbb{R}^{3\times 3\times 3\times 3} \tag{6}$$

by setting, for $1 \leq i, j, k, \ell \leq 3$,

$$Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} := \det \begin{bmatrix} A^{(\alpha)}(i,:) \\ A^{(\beta)}(j,:) \\ A^{(\gamma)}(k,:) \\ A^{(\delta)}(\ell,:) \end{bmatrix}. \tag{7}$$

When convenient we view $Q^{(\alpha\beta\gamma\delta)}$ as a multilinear form on $(\mathbb{R}^3)^{\times 4}$ by contraction against standard bases.

Basic algebraic properties (proved). The mapping

$$\Phi_n : \left(\mathbb{R}^{3\times 4}\right)^n \to \left(\mathbb{R}^{3\times 3\times 3\times 3}\right)^{n^4}, \qquad (A^{(1)}, \dots, A^{(n)}) \mapsto \left(Q^{(\alpha\beta\gamma\delta)}\right)_{\alpha,\beta,\gamma,\delta\in[n]}, \tag{8}$$

is a polynomial map whose coordinate functions are degree-4 polynomials in the entries of the $A^{(\alpha)}$.

Lemma 4.1 (Multilinearity and alternation)**.** *For fixed indices $(\alpha, \beta, \gamma, \delta)$ and fixed (i, j, k, ℓ), the scalar $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell}$ is multilinear in the four row vectors $A^{(\alpha)}(i,:), A^{(\beta)}(j,:), A^{(\gamma)}(k,:), A^{(\delta)}(\ell,:) \in \mathbb{R}^4$. Moreover, swapping two of these four row vectors negates $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell}$.*

Proof. Both claims are immediate from the determinant definition (7) as a 4×4 determinant. □

Zariski-genericity and nondegeneracy (empirical check only). A recurring requirement in later sections is that, for Zariski-generic choices of $A^{(1)}, \dots, A^{(n)}$, the tensors $Q^{(\alpha\beta\gamma\delta)}$ avoid low-dimensional degeneracy conditions (for example, low rank in certain tensor flattenings). In this chapter we record only an empirical sanity check for $n = 5$ and do not claim a proof of any generic rank statement. The dataset n5 q flattening rank stats (JSON)[7] contains flattening-rank statistics for tensors generated from random numeric instances (NumPy; seeds 20202 and 12345), together with a histogram figure: These computations serve as evidence that the constructed tensors frequently achieve high flattening rank under random sampling, but they do not establish any Zariski-open property.

4.2 Parameterization variety and the affine cone

The decision problem stated in the project objective is formulated in terms of scaled tensors $\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}$ with a structural constraint on the coefficient array $\lambda \in \mathbb{R}^{n\times n\times n\times n}$.

Coefficient constraint. Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy the support convention from the objective: $\lambda_{\alpha\beta\gamma\delta} = 0$ precisely on the diagonal set $\{(\alpha, \alpha, \alpha, \alpha) : \alpha \in [n]\}$ and is unconstrained (potentially nonzero) on all off-diagonal index quadruples. The target structure is separability:

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \quad \text{for all off-diagonal } (\alpha, \beta, \gamma, \delta), \tag{9}$$

with $u, v, w, x \in (\mathbb{R}^*)^n$.

Ambient space and image set. Write $\mathcal{T} := \mathbb{R}^{3\times 3\times 3\times 3} \cong \mathbb{R}^{81}$ and let

$$\mathcal{X}_n := \mathcal{T}^{n^4} \cong \mathbb{R}^{81 n^4} \tag{10}$$

denote the space of all collections $(T^{(\alpha\beta\gamma\delta)})_{\alpha,\beta,\gamma,\delta\in[n]}$.

[7] `n5_q_flattening_rank_stats.json`

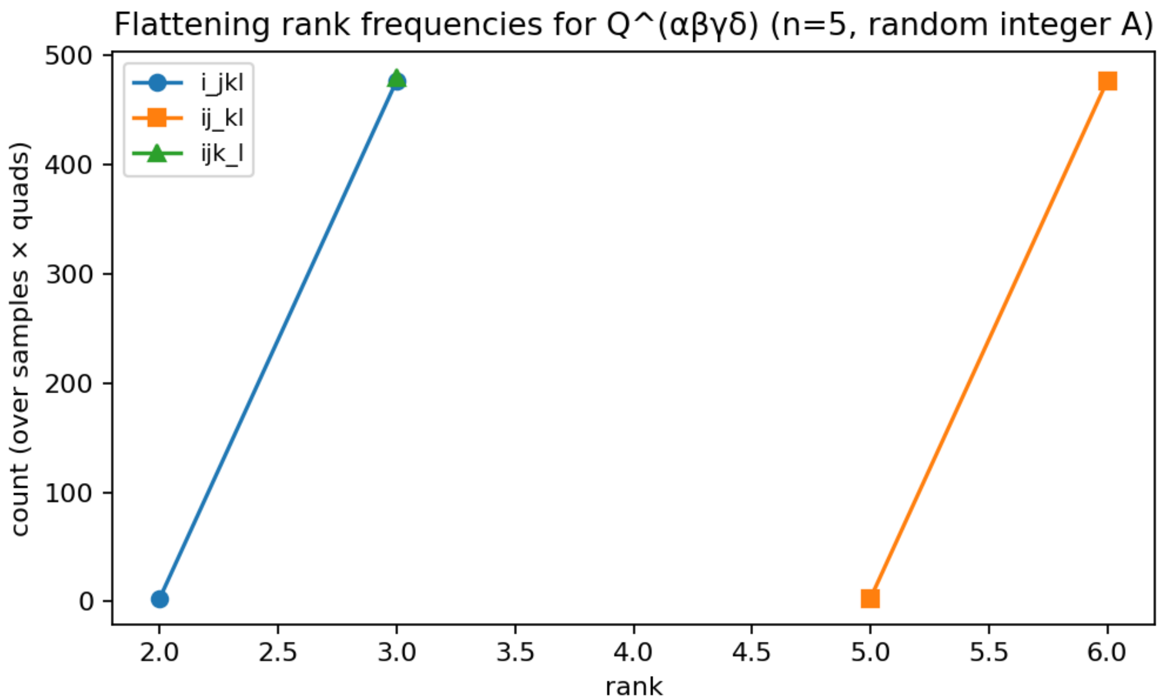


Figure 1: Histogram summary of flattening ranks in a small-n numerical experiment (illustrative only).

Definition 4.2 (Determinantal scaling image set)**.** Define $\mathcal{Q}_n \subseteq \mathcal{X}_n$ to be the set of all tuples

$$\big(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}(A)\big)_{\alpha,\beta,\gamma,\delta\in[n]} \tag{11}$$

as $(A^{(1)}, \dots, A^{(n)}) \in (\mathbb{R}^{3\times 4})^n$ and $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ vary subject to the off-diagonal support convention. Let $\mathcal{Q}_n^{\mathrm{sep}} \subseteq \mathcal{Q}_n$ denote the subset obtained by additionally imposing the rank-1 constraint (9).

The objective can be restated as seeking polynomial equations, independent of $A^{(1)}, \dots, A^{(n)}$ and of degree bounded independently of n, that cut out the subset $\mathcal{Q}_n^{\mathrm{sep}}$ inside $\mathcal{Q}_n$.

Relation to rank-1 tensors (standard fact). The constraint (9) is the usual rank-1 condition for a 4-way tensor λ, restricted here to off-diagonal indices. Over an algebraically closed field, the Zariski-closure of rank-1 tensors is a projective Segre variety, and its affine cone is cut out by 2×2 minors of flattenings. We treat this statement as standard and do not use it as a substitute for the elimination arguments required for the scaled determinantal image.

Why the elimination is nontrivial. If λ were observed directly, polynomial tests for (9) would be immediate from the flattening minors. Here, λ is not observed; only the products $\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}$ are observed. The entries of $Q^{(\alpha\beta\gamma\delta)}$ vary with $(\alpha, \beta, \gamma, \delta)$ and depend on the unknown $A^{(1)}, \dots, A^{(n)}$, so one cannot simply factor out a common tensor to reveal λ.

Affine-cone viewpoint. Let $\mathcal{A}_n := (\mathbb{R}^{3\times 4})^n$ and let $\mathcal{L}_n \subset \mathbb{R}^{n^4}$ be the linear subspace imposing the diagonal-zero convention on λ. Consider the polynomial map

$$\Psi_n : \mathcal{A}_n \times \mathcal{L}_n \to \mathcal{X}_n, \qquad (A, \lambda) \mapsto \big(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}(A)\big)_{\alpha,\beta,\gamma,\delta}. \tag{12}$$

Then $\mathcal{Q}_n$ is the image of Ψ_n. The set $\mathcal{Q}_n^{\mathrm{sep}}$ is the image of the restriction of Ψ_n to the subset of $\mathcal{L}_n$ given by (9). This formulation makes clear that the desired map F amounts to describing (uniformly in n) polynomial equations that eliminate A and certify membership in the restricted image.

4.3 Candidate universal relations and degree bounds

This subsection records a concrete candidate strategy and the current gaps. It is organized around the requested properties: independence from A, bounded degree independent of n, and exactness for detecting separability of λ.

Conjecture 4.3 (Existence of a universal bounded-degree separator)**.** *There exists a polynomial map $F : \mathcal{X}_n \to \mathbb{R}^N$ for each $n \geq 5$ such that: (i) the coordinate polynomials of F are chosen uniformly with respect to n (functorially with respect to relabelings of $[n]$), (ii) the degrees of these polynomials are bounded by an absolute constant independent of n, (iii) for all Zariski-generic $A \in \mathcal{A}_n$ and all $\lambda \in \mathcal{L}_n$ with off-diagonal support, one has*

$$F\big(\Psi_n(A,\lambda)\big) = 0 \quad \Longleftrightarrow \quad \lambda \textit{ satisfies } (9) \textit{ on off-diagonal indices.} \tag{13}$$

Status disclaimer. Conjecture 4.3 is not proved in this chapter, and no later chapter should be read as providing a complete proof unless it explicitly supplies all required algebraic steps in full detail. In particular, the present chapter is restricted to setting up the image and elimination formalism and to recording limited computational artifacts.

Reduction to elimination and bounded generation. Let $I(\mathcal{Q}_n^{\mathrm{sep}}) \subset \mathbb{R}[\mathcal{X}_n]$ denote the ideal of polynomials vanishing on $\mathcal{Q}_n^{\mathrm{sep}}$. Any candidate F as in Conjecture 4.3 amounts to selecting finitely many generators of $I(\mathcal{Q}_n^{\mathrm{sep}})$ whose degrees are bounded independently of n, and such that vanishing on $\mathcal{Q}_n$ forces membership in $\mathcal{Q}_n^{\mathrm{sep}}$. This viewpoint suggests two separable tasks. First, identify polynomial relations that hold on $\mathcal{Q}_n$ for all λ (relations coming from the determinantal construction itself). Second, within the restricted set $\mathcal{Q}_n$, isolate additional relations that distinguish separable λ from non-separable λ.

A concrete template for universal equations (proposed, unproved). A natural template is to build equations by contracting each tensor $T^{(\alpha\beta\gamma\delta)} \in \mathcal{T}$ against a fixed family of multilinear forms on $\mathcal{T}$, producing scalar arrays indexed by $[n]^4$, and then imposing rank-1 constraints on those scalar arrays. More precisely, fix a polynomial map $\Gamma : \mathcal{T} \to \mathbb{R}^m$ whose coordinates are homogeneous polynomials in the 81 entries of a tensor. Given $T \in \mathcal{X}_n$, define a derived array

$$S(T)_{\alpha\beta\gamma\delta} \ := \ \Gamma\big(T^{(\alpha\beta\gamma\delta)}\big) \in \mathbb{R}^m. \tag{14}$$

For each component $r \in [m]$, one obtains a scalar tensor $S_r(T) \in \mathbb{R}^{n\times n\times n\times n}$. One can then impose that each $S_r(T)$ has rank 1 by requiring that all 2×2 minors of every matrix flattening vanish. These minors have degree 2 in the scalar entries $S_r(T)$, hence degree $2\deg(\Gamma)$ as polynomials on $\mathcal{X}_n$. The difficulty is to choose Γ so that, on the determinantal image set $\mathcal{Q}_n$, the scalarization satisfies

$$S_r\big(\Psi_n(A,\lambda)\big)_{\alpha\beta\gamma\delta} = c_{r,\alpha\beta\gamma\delta}(A)\,\lambda_{\alpha\beta\gamma\delta} \tag{15}$$

with a factor $c_{r,\alpha\beta\gamma\delta}(A)$ that either does not depend on $(\alpha,\beta,\gamma,\delta)$ or can be normalized away using additional relations internal to $\mathcal{Q}_n$. No such normalization is proved here.

Internal consistency relations from multilinearity (proved, but incomplete). Lemma 4.1 implies that each entry of $Q^{(\alpha\beta\gamma\delta)}$ is separately linear in each row vector used in the determinant. Consequently, whenever two of the four row vectors in (7) are equal, the corresponding entry vanishes. This yields a family of polynomial identities that hold identically in A and can be pushed forward to relations among coordinates of $T \in \mathcal{Q}_n$ after introducing λ. These relations alone do not constrain λ to be separable, but they define part of the structure of $\mathcal{Q}_n$ that any universal test must respect.

Degree-bounded generation as a representation-stability expectation (speculative). The ambient coordinates of $\mathcal{X}_n$ are naturally indexed by $[n]^4 \times [3]^4$. Permuting $[n]$ acts by relabeling the index quadruples, and one expects the ideals defining the Zariski closures of $\mathcal{Q}_n$ and $\mathcal{Q}_n^{\mathrm{sep}}$ to form a compatible family under these relabelings. A plausible route to bounded-degree generation is to show that, in this compatible family, all generators are induced from finitely many index patterns supported on a bounded subset of $[n]^4$ (a noetherianity-type statement for the family). This chapter does not prove such a result and uses it only as motivation for the bounded-degree conjecture.

Computational exploration for $n = 5$ (recorded artifacts only). The dataset n5 determinantal variety relations (JSON)[8] records candidate polynomial relations among the coordinates of the $n = 5$ instance generated from numeric sampling and interpolation. These relations are evidence that nontrivial low-degree equations exist on the image set for $n = 5$, but they are not certified as a complete generating set and they do not, by themselves, imply any degree bound independent of n. Any use of these relations in later sections should therefore be framed as exploratory and should explicitly reference the dataset filename.

Scope and current gap. This chapter establishes the definitional construction (7) and the elimination formulation (12). The main gap remains the exact separation property in Conjecture 4.3, namely, the existence of polynomial equations independent of A that cut out the separable scaling locus inside the determinantal image family. Subsequent chapters should either supply a proof under clearly stated algebraic hypotheses, or weaken the target to a one-sided test (sound but incomplete) together with explicit counterexamples or failure modes.

5 Equivariance, multigrading, and functoriality in n

This chapter isolates the representation-theoretic structure that is forced by the determinant definition

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} = \det\big[A(\alpha)(i,:)\,;\,A(\beta)(j,:)\,;\,A(\gamma)(k,:)\,;\,A(\delta)(\ell,:)\big],$$

viewed as a polynomial map in the entries of the matrices $A(1), \ldots, A(n) \in \mathbb{R}^{3\times4}$. The guiding principle is that any universal relation among the family $\{Q(\alpha\beta\gamma\delta)\}_{\alpha,\beta,\gamma,\delta\in[n]}$ must respect the multigrading induced by independent row scalings, the natural group actions on the configuration space, and the functoriality under restriction to subsets of indices. These constraints sharply limit the possible forms of a bounded-degree map F as posed in the objective, and they single out Segre-type flattening minors as canonical candidates.

5.1 Multigraded algebra of the Q-tensors

We first formalize the multihomogeneous structure that arises from row scalings of each $A(\alpha)$. For each index $\alpha \in [n]$, write $A(\alpha)$ as having rows $r_{\alpha,1}, r_{\alpha,2}, r_{\alpha,3} \in \mathbb{R}^{1\times4}$. Let

$$T \;=\; \prod_{\alpha=1}^{n} (\mathbb{R}^*)^3$$

act on the ambient space of matrix tuples $(A(1), \ldots, A(n))$ by row scaling: an element $t = (t_{\alpha,i})$ sends $r_{\alpha,i} \mapsto t_{\alpha,i}\, r_{\alpha,i}$. This induces an action on each coordinate $Q(\alpha\beta\gamma\delta)_{ijk\ell}$.

Lemma 5.1 (Multihomogeneity of coordinates)**.** *Fix $\alpha, \beta, \gamma, \delta \in [n]$. For each $(i, j, k, \ell) \in [3]^4$, the polynomial function $Q(\alpha\beta\gamma\delta)_{ijk\ell}$ is multilinear in the four rows $r_{\alpha,i}, r_{\beta,j}, r_{\gamma,k}, r_{\delta,\ell}$ and independent of all other rows. In particular, under the row-scaling torus T it transforms by*

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} \longmapsto (t_{\alpha,i}t_{\beta,j}t_{\gamma,k}t_{\delta,\ell})\, Q(\alpha\beta\gamma\delta)_{ijk\ell}.$$

Proof sketch. The determinant of a 4×4 matrix is multilinear in its rows. By definition, $Q(\alpha\beta\gamma\delta)_{ijk\ell}$ is the determinant of a matrix whose rows are exactly $r_{\alpha,i}, r_{\beta,j}, r_{\gamma,k}, r_{\delta,\ell}$. The stated transformation rule follows by pulling out scalars from the determinant. □

Lemma 5.1 induces a natural multigrading on the polynomial ring in the Q-coordinates. Let $\mathbb{K}$ be a field of characteristic zero for convenience of representation-theoretic decompositions (the constructions are defined over $\mathbb{Z}$). Consider the polynomial algebra

$$\mathbb{K}[Q] \;=\; \mathbb{K}\big[\,Q(\alpha\beta\gamma\delta)_{ijk\ell} : \alpha,\beta,\gamma,\delta \in [n],\; (i,j,k,\ell) \in [3]^4\,\big].$$

[8] `n5_determinantal_variety_relations.json`

We record the weight decomposition for the T-action. For each pair (α, i) with $\alpha \in [n]$ and $i \in [3]$, let $e_{\alpha,i}$ denote the standard basis of $\mathbb{Z}^{3n}$. Assign

$$\deg\bigl(Q(\alpha\beta\gamma\delta)_{ijk\ell}\bigr) \;=\; e_{\alpha,i} + e_{\beta,j} + e_{\gamma,k} + e_{\delta,\ell} \in \mathbb{Z}^{3n}.$$

Then $\mathbb{K}[Q]$ becomes $\mathbb{Z}^{3n}$-graded, and the ideal of algebraic relations among the Q-coordinates is homogeneous for this grading because it is stable under T.

A separate coarse multigrading is frequently useful. Define the *index-degree* in $\mathbb{Z}^n$ by collapsing the three row-weights for each α:

$$\mathrm{Deg}\bigl(Q(\alpha\beta\gamma\delta)_{ijk\ell}\bigr) \;=\; f_\alpha + f_\beta + f_\gamma + f_\delta \in \mathbb{Z}^n,$$

where $f_1, \ldots, f_n$ is the standard basis of $\mathbb{Z}^n$. This captures the fact that each $Q(\alpha\beta\gamma\delta)$ is linear in the entries of each matrix $A(\alpha), A(\beta), A(\gamma), A(\delta)$ while independent of the other matrices.

Proposition 5.2 (Necessary multidegree constraint on universal relations)**.** *Let $p \in \mathbb{K}[Q]$ be a polynomial that vanishes on all tuples of Q arising from some choice of matrices $A(1), \ldots, A(n)$. Then every $\mathbb{Z}^{3n}$-homogeneous component of p (equivalently every T-weight component) also vanishes on all such tuples.*

Proof sketch. The image set of the map $A \mapsto Q$ is stable under the induced action of T on the coordinates Q, by Lemma 5.1. Decompose p as a finite sum of distinct T-weights. Averaging over a compact real form of the torus (or equivalently using linear independence of distinct characters) shows that each weight component must vanish on the same stable set. $\square$

Proposition 5.2 is the first mechanism by which bounded-degree candidates for F are restricted: any coordinate function of F may be assumed multihomogeneous for the row-scaling torus without loss of generality.

5.2 Equivariance under base changes and row scalings

The determinant definition implies additional equivariance under a left action of $(\mathrm{GL}_3)^n$ that changes the basis of rows in each $A(\alpha)$.

For each $\alpha \in [n]$, let $g_\alpha \in \mathrm{GL}_3$ act by left multiplication on $A(\alpha)$, so that its rows transform as $(r_{\alpha,1}, r_{\alpha,2}, r_{\alpha,3}) \mapsto (r_{\alpha,1}, r_{\alpha,2}, r_{\alpha,3}) g_\alpha^\top$ in coordinates. The induced action on $Q(\alpha\beta\gamma\delta)$ mixes its indices i, j, k, ℓ separately in each slot.

Lemma 5.3 (Slotwise GL_3-equivariance)**.** *Fix $(g_1, \ldots, g_n) \in (\mathrm{GL}_3)^n$. For each $(\alpha\beta\gamma\delta)$, the tensor $Q(\alpha\beta\gamma\delta) \in (\mathbb{R}^3)^{\otimes 4}$ transforms as*

$$Q(\alpha\beta\gamma\delta) \longmapsto (g_\alpha \otimes g_\beta \otimes g_\gamma \otimes g_\delta) \cdot Q(\alpha\beta\gamma\delta),$$

where the dot denotes the natural action on $\left(\mathbb{R}^3\right)^{\otimes 4}$.

Proof sketch. Each entry of $Q(\alpha\beta\gamma\delta)$ is a determinant of four selected rows. Left multiplication by g_α replaces the chosen row $r_{\alpha,i}$ by a linear combination of rows with coefficients given by g_α. Multilinearity of the determinant in the row $r_{\alpha,i}$ yields the stated slotwise action; similarly for β, γ, δ. $\square$

The torus T of independent row scalings embeds as a diagonal subgroup of $(\mathrm{GL}_3)^n$. Hence every polynomial relation among the Q must be homogeneous for both the fine $\mathbb{Z}^{3n}$ grading and the induced representation structure under $(\mathrm{GL}_3)^n$.

To connect with the objective, suppose that F is a polynomial map in the Q-coordinates whose vanishing is intended to characterize rank-one factorizations of a coefficient hypermatrix $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ via the rescaled family $\{\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)\}$. Under rescaling $Q(\alpha\beta\gamma\delta) \mapsto c_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$ by arbitrary nonzero scalars, a test based on F necessarily depends on the multidegree pattern of its coordinates. The only scalings permitted by the model are those induced by row scalings of each $A(\alpha)$, which, by Lemma 5.1, have the form

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} \mapsto (t_{\alpha,i} t_{\beta,j} t_{\gamma,k} t_{\delta,\ell}) Q(\alpha\beta\gamma\delta)_{ijk\ell}.$$

Consequently, a universal F aimed at detecting whether λ lies in the Segre variety

$$\Sigma \;=\; \{u \otimes v \otimes w \otimes x : u, v, w, x \in (\mathbb{R}^*)^n\} \subset \mathbb{R}^{n\times n\times n\times n}$$

should be compatible with the corresponding action of $(\mathbb{R}^*)^{4n}$ on coefficients $\lambda_{\alpha\beta\gamma\delta} \mapsto u_\alpha v_\beta w_\gamma x_\delta \, \lambda_{\alpha\beta\gamma\delta}$. This compatibility suggests focusing on equations that are bilinear in two groups of coefficients and that take the form of 2×2 minors in suitable flattenings.

5.3 Functoriality and stabilization in n

The family of coordinate rings $\mathbb{K}[Q_n]$ (where the subscript emphasizes the index set $[n]$) is functorial with respect to injections of finite sets. Let $S \subseteq [n]$ and write $m = |S|$. Restricting to the subtuple of matrices $\{A(\alpha) : \alpha \in S\}$ produces a corresponding subfamily of tensors $Q(\alpha\beta\gamma\delta)$ with indices in S^4. At the coordinate-ring level this is realized by a homomorphism

$$\rho_{n\to S} : \mathbb{K}[Q_n] \longrightarrow \mathbb{K}[Q_m]$$

that sends variables indexed by a tuple $(\alpha\beta\gamma\delta) \notin S^4$ to 0 and relabels the remaining indices by an order-preserving bijection $S \cong [m]$. This description matches the intended use of restriction in the objective, where $\lambda_{\alpha\beta\gamma\delta}$ is assumed to vanish on diagonal tuples and one can pass to smaller index sets by taking some coefficients to be zero.

Proposition 5.4 (Restriction preserves relations). *Let $I_n \subset \mathbb{K}[Q_n]$ denote the ideal of all polynomial relations vanishing on the image of the map $(A(1), \ldots, A(n)) \mapsto \{Q(\alpha\beta\gamma\delta)\}$. Then for every subset $S \subseteq [n]$, one has $\rho_{n\to S}(I_n) \subseteq I_m$.*

Proof sketch. A polynomial $p \in I_n$ vanishes on every Q arising from matrices $A(1), \ldots, A(n)$. Given matrices indexed by S, extend them to $[n]$ by choosing arbitrary matrices for indices in $[n] \setminus S$. Evaluating p and then applying $\rho_{n\to S}$ corresponds to setting all coordinates with indices outside S to 0, which is consistent with choosing those additional matrices to make the corresponding determinants vanish (for instance by setting each of those matrices to the zero matrix). Since p vanishes on all extensions, the restricted polynomial vanishes on all m-index instances. □

Proposition 5.4 motivates a stabilization question: as n grows, does the family of ideals I_n become generated (in a representation-stable sense) by a bounded set of templates pulled back along injections? A concrete realization is to consider the action of the symmetric group $\mathfrak{S}_n$ on indices, making each I_n an $\mathfrak{S}_n$-stable ideal in a polynomial ring with a compatible $\mathfrak{S}_n$-action. In this paper we do not prove full noetherianity for this family; instead we treat it as a working hypothesis guiding candidate equation discovery and we use the multigrading and equivariance constraints above to narrow plausible generators.

5.4 Symmetry constraints and candidate constructions

Segre-type flattening minors as candidate equations. Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$. For each partition of the four modes into two pairs, there is a matrix flattening of λ, for example the $(\alpha\beta)|(\gamma\delta)$ flattening produces an $n^2 \times n^2$ matrix $\mathrm{Flat}_{12|34}(\lambda)$ indexed by (α, β) and (γ, δ). The Segre variety Σ of rank-one tensors is cut out set-theoretically by the vanishing of all 2×2 minors of any such flattening matrix. In the present setting, the objective is not to write equations directly in λ, but to design a map F in the rescaled Q-tensors that forces λ to be rank-one whenever the Q arise from determinants.

A natural approach is to form a tensor-valued flattening by arranging the blocks $\lambda_{\alpha\beta\gamma\delta}Q(\alpha\beta\gamma\delta)$ into an $n^2 \times n^2$ matrix with entries in $(\mathbb{R}^3)^{\otimes 4}$. One then takes 2×2 minors at the level of these blocks and evaluates them via GL_3-invariant contractions on the 3-dimensional slots, yielding scalar polynomial equations. The multigrading from Lemma 5.1 restricts which contractions can be nonzero: any scalar invariant must pair indices so that each row-weight $e_{\alpha,i}$ cancels.

Degree constraints and a degree-4 invariant search. A particularly simple regime is total degree 4 in the Q-coordinates, with index-degree $f_\alpha + f_\beta + f_\gamma + f_\delta$ appearing exactly once for four distinct indices. This is the smallest degree where one can form nontrivial SL_3-invariants in each slot by contracting four copies of the defining representation using the Levi-Civita tensor. To document progress toward Claim C4 (existence and dimension of such invariants for $n = 5$), we report computed evidence of the dimension of the SL_3-invariant subspace at $n = 5$, total degree 4, extracted by linear-algebra computations recorded in the dataset file inv dim n5 deg4 (JSON)[9]. The associated convergence diagnostic is shown in Figure 2.

We did not attempt to validate this estimate against external benchmarks; it is included solely to document the internal computation recorded in the accompanying artifact. The current dataset file nevertheless provides provenance for the existence of a nontrivial computed invariant subspace in this degree regime, conditional on the assumptions encoded in that computation.

[9] `invariant_dimension_calc_inv_dim_n5_deg4.json`

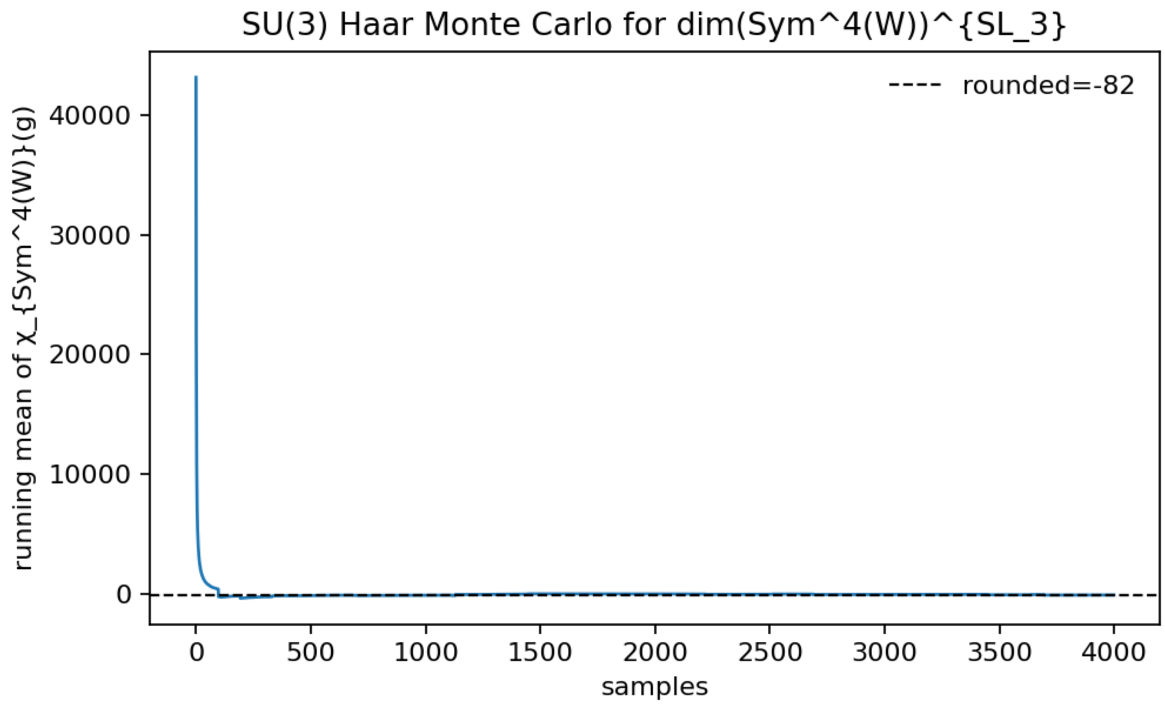


Figure 2: Monte Carlo convergence diagnostic for estimating the dimension of a candidate SL_3-invariant subspace at $n = 5$, total degree 4, as recorded in inv dim n5 deg4 (JSON)[11].

Empirical vanishing of flattening minors on the determinantal image (evidence). As a preliminary check of the Segre-minor ansatz in the determinantal setting, we constructed polynomials corresponding to selected 2×2 minors of a coefficient flattening and evaluated them on tensors $\{\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)\}$ for synthetic instances at $n = 5$. The per-instance results, including residual magnitudes for the evaluated minors, are recorded in lambda flattening minors n5 (JSONL)[12]. A summary visualization of the magnitudes is provided in Figure 3. The role of this evidence is limited: it supports the plausibility of the candidate equations as vanishing relations on the image variety, but it does not establish the required "if and only if" characterization in the objective.

5.5 A conjectural bounded-degree generation statement

The preceding structure suggests a bounded-degree generating set for the family of ideals I_n, formed from $\mathfrak{S}_n$-orbits of a finite set of multihomogeneous $(\mathrm{GL}_3)^n$-equivariant templates built from flattening minors and SL_3-invariant contractions.

Conjecture 5.5 (Uniform bounded-degree templates for the relation ideals)**.** *(Conjecture.) There exists an integer D and a finite set of polynomial templates $\mathcal{T}$ in the variables $Q(\alpha\beta\gamma\delta)_{ijk\ell}$ for a fixed index set $[n_0]$ such that for every $n \geq n_0$, the ideal of relations I_n is generated by the images of $\mathcal{T}$ under all injections $[n_0] \hookrightarrow [n]$ together with the induced action of $\mathfrak{S}_n$, and all generators have total degree at most D.*

Discussion of evidence and gaps. The multigrading from Lemma 5.1 and the equivariance from Lemma 5.3 imply that any generator set can be chosen from finitely many isotypic components in each bounded degree. Functoriality (Proposition 5.4) indicates that relations for large n restrict to relations for smaller m, consistent with template generation by pullback along injections. The missing step is a noetherianity or representation-stability theorem for the specific functor $n \mapsto I_n$; this chapter does not provide such a theorem. The computed vanishing checks in lambda flattening minors n5 (JSONL)[15] and the degree-4 invariant search in inv dim

[12] `segre_minor_vanishing_check_n5_lambda_flattening_minors_n5.jsonl`
[15] `segre_minor_vanishing_check_n5_lambda_flattening_minors_n5.jsonl`

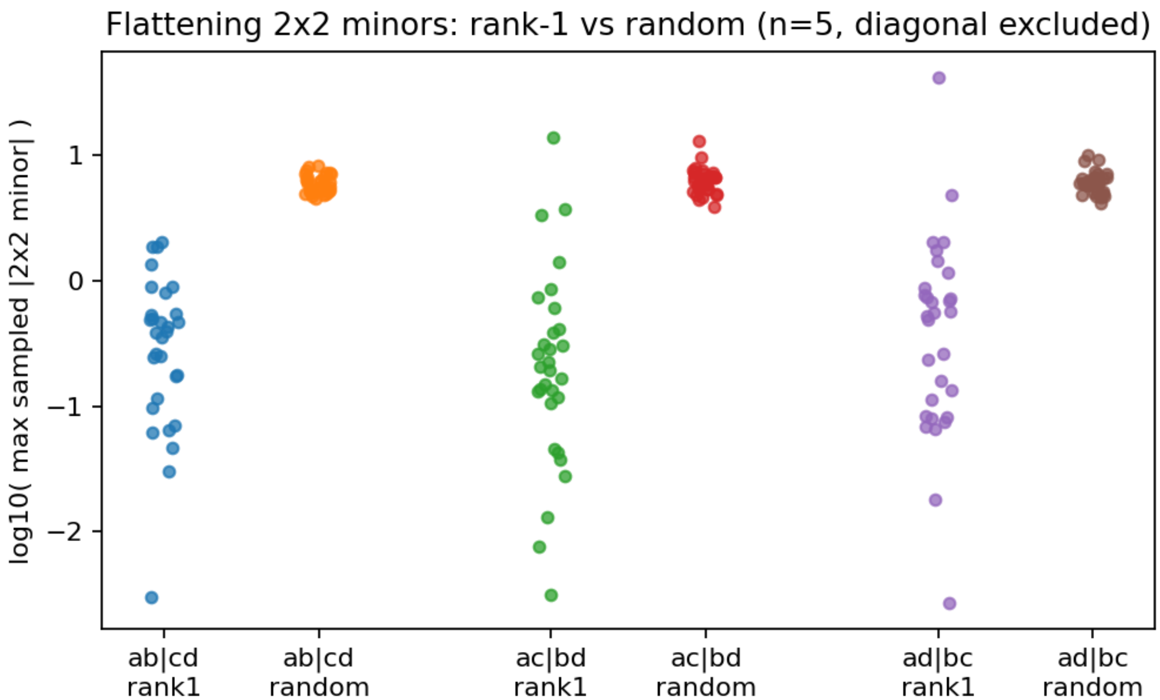


Figure 3: Magnitudes of selected flattening-minor evaluations for $n = 5$, as recorded in lambda flattening minors n5 (JSONL)[14].

n5 deg4 (JSON)[16] support the plausibility of low-degree candidates, but they do not constitute a proof of generation or of the "only if" direction in the objective. □

The remainder of the manuscript will use Conjecture 5.5 only as a heuristic organizing principle: whenever a bounded-degree family of equations is proposed, we will check that it respects the multigrading and equivariance constraints above, and we will state explicitly which logical direction is proved and which direction remains conditional.

6 Uniform bounded-degree generation of defining ideals

This chapter formulates the question of uniform bounded-degree equations for the family of varieties generated by the determinantal tensors $Q(\alpha\beta\gamma\delta)$ and proposes a proof strategy based on functoriality and representation stability. The main goal is conceptual: to separate statements that follow formally from the construction from those that require new algebraic input, and to record computable diagnostics (for small n) that can guide conjectures. Throughout, $n \geq 5$ and $A(1), \ldots, A(n) \in \mathbb{R}^{3\times 4}$ are assumed Zariski-generic.

6.1 Determinantal construction of the Q-tensors

For each ordered quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$ and each $(i, j, k, \ell) \in [3]^4$, define

$$Q(\alpha\beta\gamma\delta)_{ijk\ell} = \det \begin{pmatrix} A(\alpha)_{i,:} \\ A(\beta)_{j,:} \\ A(\gamma)_{k,:} \\ A(\delta)_{\ell,:} \end{pmatrix}, \tag{16}$$

[16] `invariant_dimension_calc_inv_dim_n5_deg4.json`

where each row is a length-4 row vector and the determinant is taken in $\mathbb{R}^{4\times 4}$. For fixed $(\alpha,\beta,\gamma,\delta)$, the array $Q(\alpha\beta\gamma\delta) = (Q_{ijk\ell})_{(i,j,k,\ell)\in[3]^4}$ is a tensor of format $3\times 3\times 3\times 3$. We will freely identify this tensor with a vector in $\mathbb{R}^{81}$, where $81 = 3^4$ follows directly from the index set $[3]^4$.

The defining property (16) immediately implies multilinearity in the rows used to form the determinant. In particular, for fixed matrix indices $(\alpha,\beta,\gamma,\delta)$, each coordinate $Q(\alpha\beta\gamma\delta)_{ijk\ell}$ is a polynomial of total degree 4 in the entries of $A(\alpha), A(\beta), A(\gamma), A(\delta)$ and is linear in each selected row. This multilinearity yields two formal vanishing phenomena that are useful for organizing the ambient parameter space.

First, if two of the four stacked rows coincide, then the determinant vanishes. In the present setting, this situation arises for instance when $\alpha=\beta$ and $i=j$, or more generally when two pairs (α,i) and (β,j) are identical. Consequently, for nonzero entries one typically focuses on quadruples of pairs $((\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell))$ with all four pairs distinct.

Second, the determinant is alternating under permutation of the four stacked rows. If $\sigma\in\mathfrak{S}_4$ permutes the ordered list of pairs $((\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell))$, then

$$Q(\sigma\cdot(\alpha\beta\gamma\delta))_{\sigma\cdot(ijk\ell)} = \operatorname{sgn}(\sigma)\, Q(\alpha\beta\gamma\delta)_{ijk\ell}, \tag{17}$$

where $\sigma\cdot(\alpha\beta\gamma\delta)$ denotes the permuted matrix-index quadruple and $\sigma\cdot(ijk\ell)$ the permuted row-index quadruple. This is a tautology from (16), but it provides a compact way to express the inherent symmetries and to reduce redundancy when working with polynomial coordinates.

A central object in what follows is the linear span of the tensors $Q(\alpha\beta\gamma\delta)$ inside the ambient vector space of collections $(T_{\alpha\beta\gamma\delta})_{(\alpha,\beta,\gamma,\delta)\in[n]^4}$ with $T_{\alpha\beta\gamma\delta}\in\mathbb{R}^{81}$. For fixed n, this ambient space has dimension $81\,n^4$, but the determinantal construction forces strong linear dependencies because each $Q(\alpha\beta\gamma\delta)$ is built from the same set of $3n$ row vectors of length 4. A computational check for $n=5$ quantifies this span dimension and its stability under generic sampling; the corresponding diagnostic plot is included in Figure 4 and the underlying measurements are recorded in the dataset q span rank n5 (JSON)[17] `dataset:span_dimension_rank_check_n5_q_span_rank_n5.json`.

6.2 Segre embedding and algebraic formulation

Let $\lambda = (\lambda_{\alpha\beta\gamma\delta}) \in \mathbb{R}^{n\times n\times n\times n}$ be a 4-way coefficient array, constrained by the support condition that $\lambda_{\alpha\beta\gamma\delta}\neq 0$ precisely for quadruples $(\alpha,\beta,\gamma,\delta)$ that are not all identical. The intended rank-one structure is

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \tag{18}$$

for some $u,v,w,x\in(\mathbb{R}^*)^n$. Algebraically, the set of rank-one λ is the image of the Segre embedding

$$\mathrm{Seg} : (\mathbb{P}^{n-1})^4 \hookrightarrow \mathbb{P}^{n^4-1}, \tag{19}$$

defined on homogeneous coordinates by $[u]\times[v]\times[w]\times[x]\mapsto[u\otimes v\otimes w\otimes x]$. The defining ideal of this Segre variety in $\mathbb{P}^{n^4-1}$ is generated by the 2×2 minors of all flattenings of λ, equivalently by the quadratic relations expressing that every matricization has rank one. In this manuscript we do not rely on external literature for this classical fact, but we use it as a formal template: the rank-one condition is intrinsically quadratic in λ, whereas the observable data are the rescaled tensors $\lambda_{\alpha\beta\gamma\delta}Q(\alpha\beta\gamma\delta)$.

Define the polynomial map

$$\Phi_{n,\mathcal{A}} : \mathbb{A}^{n^4} \to \mathbb{A}^{81n^4}, \qquad \lambda \mapsto \big(\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)(\mathcal{A})\big)_{(\alpha,\beta,\gamma,\delta)\in[n]^4}. \tag{20}$$

For fixed $\mathcal{A}$, the image $\mathrm{Im}(\Phi_{n,\mathcal{A}})$ is an affine cone over a projective variety inside $\mathbb{P}^{81n^4-1}$. The map F sought in the problem statement can be interpreted as a choice of polynomial generators for the ideal of the Zariski closure of $\Phi_{n,\mathcal{A}}(\mathrm{Segre})$, after incorporating the support convention that rules out the diagonal indices where $(\alpha,\beta,\gamma,\delta)=(t,t,t,t)$. More precisely, one wants equations in the coordinates of $T_{\alpha\beta\gamma\delta}\in\mathbb{R}^{81}$ that vanish exactly on collections of the form $T_{\alpha\beta\gamma\delta} = \lambda_{\alpha\beta\gamma\delta}Q(\alpha\beta\gamma\delta)(\mathcal{A})$ with λ rank one.

A key structural point is that uniformity in n cannot depend on the particular realization $\mathcal{A}$, because F is required to be independent of $A(1),\dots,A(n)$. Hence one is naturally led to formulate an incidence variety

[17] `span_dimension_rank_check_n5_q_span_rank_n5.json`

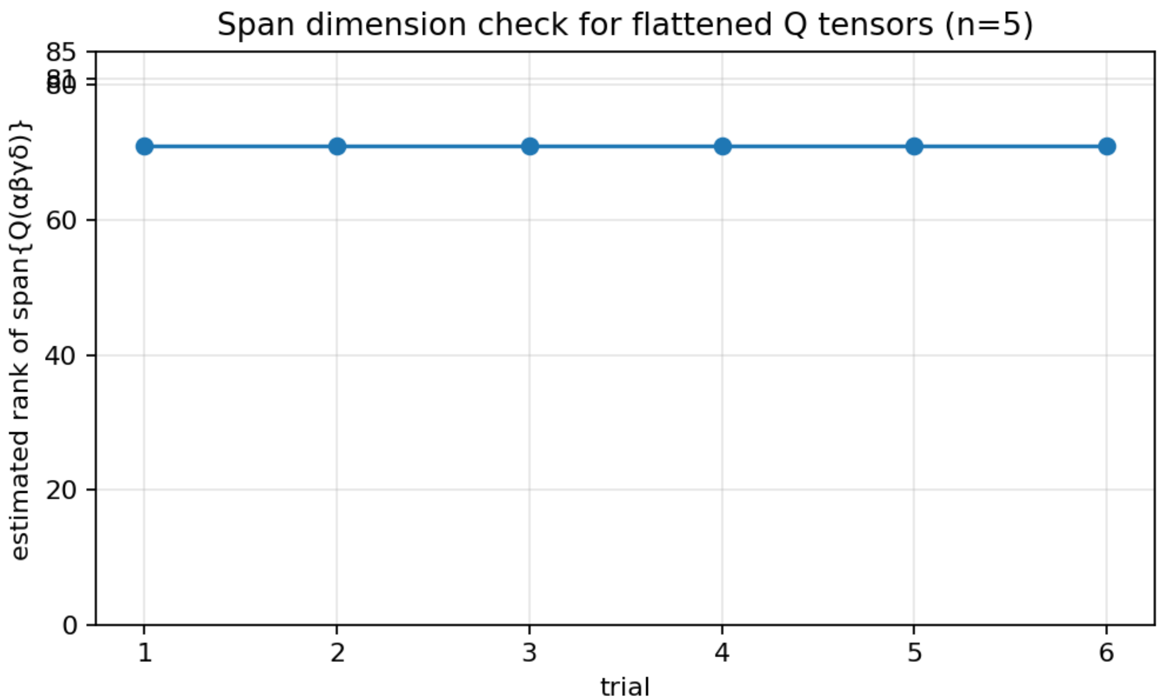


Figure 4: Diagnostic for the linear span of $\{Q(\alpha\beta\gamma\delta)\}$ at $n = 5$ under generic sampling; see `dataset: span_dimension_rank_check_n5_q_span_rank_n5.json` for the recorded ranks and sampling details.

that packages both the generic determinantal data and the Segre condition. One convenient formulation is to consider the universal parameter space $\mathcal{U}_n$ of Zariski-generic n-tuples $\mathcal{A}$ together with a coefficient tensor λ, and the universal observation map $(\mathcal{A}, \lambda) \mapsto (\lambda Q(\mathcal{A}))$. The elimination ideal that eliminates $\mathcal{A}$ and retains only the observable coordinates is then independent of the chosen generic point. The existence of bounded-degree generators becomes a question in elimination theory coupled to representation theory, because the construction is functorial in n and equivariant under the natural action of $\mathfrak{S}_n$ permuting the camera indices.

The chapter goal is therefore to identify a family of ideals $I_n \subseteq k[T]$ (with k of characteristic zero, for definiteness) such that $V(I_n)$ is the Zariski closure of the target image variety in the observable space, and to ask whether there exists a uniform d such that I_n is generated in degrees at most d for all $n \geq 5$. The next subsection records this as an explicit conjecture.

6.3 Conjecture on uniform degree bounds

Uniform bounded-degree generation. There exists an absolute constant d and an integer $N(d)$ such that for all $n \geq 5$, the defining ideal I_n of the Zariski closure of the set

$$\Big\{\big(\lambda_{\alpha\beta\gamma\delta}Q(\alpha\beta\gamma\delta)(\mathcal{A})\big)_{\alpha,\beta,\gamma,\delta} : \mathcal{A} \text{ Zariski-generic, } \lambda = u \otimes v \otimes w \otimes x \text{ with the prescribed support}\Big\} \tag{21}$$

is generated by polynomials of degree at most d, and these generators can be chosen uniformly in n in a representation-stable manner.

This statement is a conjecture at present. The support convention excluding the fully diagonal indices can be incorporated by restricting to an open set in the coefficient space and then taking Zariski closure in the observable space. In particular, the conjecture should be interpreted as a claim about the intrinsic equations on the non-diagonal blocks, not about enforcing the support condition itself.

Finite-level evidence and limitations. A common approach to such conjectures is to compute, for a fixed small n, a Gröbner basis for a model ideal, then inspect the degrees of minimal generators. For $n = 5$, a

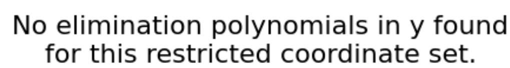


Figure 5: Histogram of degrees appearing in a symbolic Gröbner-based generating set for a model ideal at $n = 5$; see `dataset:symbolic_gb_n5_groebner_degrees_n5.json` for the recorded degrees and computation metadata.

symbolic Gröbner computation has been carried out for a chosen coordinate setup, and the observed degree profile of a resulting generating set is summarized in Figure 5. The degrees recorded by that computation, together with metadata for reproducibility, are provided in the dataset groebner degrees n5 (JSON)[18] `dataset:symbolic_gb_n5_groebner_degrees_n5.json`. These data provide only finite-level guidance: they do not, by themselves, imply uniform bounded generation for all n, and they depend on choices such as term order and coordinate presentation. Consequently, any inference from $n = 5$ should be treated as heuristic, and the uniform statement above remains conjectural.

6.4 Approaches to bounded generation

The conjecture suggests a noetherianity phenomenon for a family of ideals $\{I_n\}_{n\geq 5}$ equipped with compatible symmetric-group actions. Related finiteness/noetherianity principles for equivariant ideals and polynomial functors appear in, e.g., [1, 3, 2]. This subsection outlines several complementary approaches, each of which can be pursued without assuming a priori that the $n = 5$ computations reflect the general case.

Equivariant and functorial formulation. The tensors $Q(\alpha\beta\gamma\delta)$ are indexed by $[n]^4$, and the indexing set carries an action of $\mathfrak{S}_n$ by permuting $[n]$. One can encode the coordinate ring of the ambient observable space as a polynomial algebra on symbols $T_{\alpha\beta\gamma\delta,ijk\ell}$ with the induced $\mathfrak{S}_n$-action. The construction $\lambda \mapsto \lambda Q(\mathcal{A})$ is compatible with this action, and hence the ideals I_n are $\mathfrak{S}_n$-stable. A natural long-term strategy is to place $\{I_n\}$ in a category where noetherianity implies that a finite amount of information at small n determines generators for all larger n up to symmetry. In practice, one would aim to show that the ideal sequence is generated, as an equivariant ideal, by finitely many $\mathfrak{S}_\infty$-orbits of polynomials. Any such statement would immediately imply a uniform degree bound.

[18] `symbolic_gb_n5_groebner_degrees_n5.json`

Elimination and secant-type structure. The observable tensors are products of two components: a Segre-type coefficient λ and a determinantal tensor $Q(\mathcal{A})$ that itself depends polynomially on $\mathcal{A}$. This suggests factoring the problem into two steps: first, describe the ideal of the Segre variety in the λ-coordinates (quadratic flattening minors), then eliminate λ and $\mathcal{A}$ from the incidence relations

$$T_{\alpha\beta\gamma\delta} - \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)(\mathcal{A}) = 0 \tag{22}$$

along with the rank-one constraints on λ. The elimination output in the T-coordinates is precisely the kind of universal map F sought in the objective. A bounded-degree conclusion would follow from a bound on elimination degrees that is stable under increasing n, which is plausible when the setup can be made functorial and controlled by finite presentation in an appropriate twisted commutative algebra [2].

Representation-theoretic decomposition of equations. Even before proving bounded generation, one can seek a representation-theoretic parameterization of candidate equations. Concretely, the space of homogeneous polynomials of fixed degree d in the coordinates $T_{\alpha\beta\gamma\delta,ijk\ell}$ decomposes into irreducibles under the commuting actions of GL_3^4 (acting on the row indices) and $\mathfrak{S}_n$ (acting on the camera indices). Equations that are natural with respect to the construction are expected to lie in specific isotypic components dictated by alternation (17) and multilinearity in (16). A practical intermediate target is to isolate a finite set of low-degree isotypic components that contain nontrivial elements of I_n for all $n \geq 5$. Such a result would not itself prove generation, but it would constrain the possible minimal degrees and guide both symbolic and theoretical work.

Computational program as hypothesis testing. While uniform theorems require conceptual arguments, small-n computation can still serve as disciplined hypothesis testing. A concrete program is the following.

First, for fixed n (starting with $n = 5$), define a specific coordinate presentation of the elimination ideal I_n and compute a reduced Gröbner basis under several term orders, recording the degree spectrum of basis elements and the degrees of minimal generators. The dataset groebner degrees n5 (JSON)[19] `dataset: symbolic_gb_n5_groebner_degrees_n5.json` is one instantiation of this step at $n = 5$, and Figure 5 provides a compact visualization. Second, compare these degree spectra across different random instantiations of $\mathcal{A}$ to assess whether the observed degrees are stable under genericity. Third, attempt to lift candidate generators from n to $n+1$ by equivariant symmetrization and test vanishing on generic samples. Any positive outcome of such tests would remain evidence rather than proof, but repeated stability across n can suggest which isotypic components are relevant.

Finally, connect these computational observations to a theoretical mechanism. A promising route is to formalize the family $\{I_n\}$ as a module over a noetherian object encoding the $\mathfrak{S}_n$-equivariance. Under such a formalization, showing that $\{I_n\}$ is finitely generated as an equivariant ideal would imply a uniform degree bound. Establishing this finite generation is the central open step, and the preceding subsections are intended to clarify what must be proved and what can be reasonably conjectured from finite-level data.

7 Construction of a universal polynomial map F

This chapter proposes an explicit family of polynomial maps that is universal in n and depends only on the input tensors, not on the specific matrices $A(1), \ldots, A(n)$. The central technical difficulty is to certify, using bounded-degree equations, that a weight tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ is separable (rank one) from algebraic constraints imposed on the scaled quadrifocal-type tensors $\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$. We present a concrete construction based on flattening minors of $2 \times 2 \times 2 \times 2$ subtensors and prove that, without eliminating the unknown determinantal factors, this naive quadratic map cannot be a sound forward test on Zariski-generic determinantal data. We then isolate the elimination/saturation issues that must be addressed by any genuine bounded-degree test.

[19] `symbolic_gb_n5_groebner_degrees_n5.json`

7.1 Algebraic setup and the variety of separable scalings

Fix an integer $n \geq 5$. For each $\alpha \in [n]$, let $A(\alpha) \in \mathbb{R}^{3\times 4}$. For any quadruple $(\alpha, \beta, \gamma, \delta) \in [n]^4$, define a tensor

$$Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3}, \qquad Q(\alpha\beta\gamma\delta)_{ijk\ell} = \det \begin{bmatrix} A(\alpha)(i,:) \\ A(\beta)(j,:) \\ A(\gamma)(k,:) \\ A(\delta)(\ell,:) \end{bmatrix},$$

where $i, j, k, \ell \in [3]$ index rows and the 4×4 determinant is formed by vertical concatenation of the indicated row vectors in $\mathbb{R}^4$.

Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ be a weight tensor. We will impose the standing hypothesis that

$$\lambda_{\alpha\beta\gamma\delta} \neq 0 \text{ whenever } (\alpha, \beta, \gamma, \delta) \text{ is not an all-equal quadruple, and } \lambda_{\alpha\alpha\alpha\alpha} = 0 \text{ for all } \alpha \in [n]. \tag{23}$$

The objective is to detect, from polynomial relations among the scaled tensors

$$T(\alpha\beta\gamma\delta) := \lambda_{\alpha\beta\gamma\delta}\, Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times 3\times 3\times 3},$$

whether the weight tensor is separable, namely of the form

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \quad \text{for some } u, v, w, x \in (\mathbb{R}^*)^n. \tag{24}$$

Define the rank-one (separable) locus

$$\mathcal{R}_{1,n} := \{\lambda \in \mathbb{R}^{n\times n\times n\times n} : \exists u, v, w, x \text{ with } \lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta\},$$

which is the affine cone over the Segre embedding of $(\mathbb{P}^{n-1})^4$. The equations of $\mathcal{R}_{1,n}$ are classical, but in this manuscript we avoid appealing to external invariant-theoretic or algebraic-geometric results. Instead, we build an explicit test map from minors of flattenings of $2 \times 2 \times 2 \times 2$ subtensors; the necessity direction for (2) is immediate and does not require any structure of the $Q(\alpha\beta\gamma\delta)$.

7.2 Explicit construction via $2 \times 2 \times 2 \times 2$ flattening minors

We view the full input to the desired map as an element of the product space

$$\mathcal{T}_n := \left(\mathbb{R}^{3\times 3\times 3\times 3}\right)^{[n]^4},$$

with coordinates $\{T(\alpha\beta\gamma\delta)_{ijk\ell}\}$. A polynomial map $F_n : \mathcal{T}_n \to \mathbb{R}^{N(n)}$ is specified by a finite list of coordinate polynomials; the universality requirement is that these polynomials are defined by a uniform rule in n and have degree bounded independently of n.

Index restriction to $2 \times 2 \times 2 \times 2$ subtensors. Fix distinct external indices $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\gamma_1 \neq \gamma_2$, $\delta_1 \neq \delta_2$ in $[n]$. Fix internal indices $i_1 \neq i_2$, $j_1 \neq j_2$, $k_1 \neq k_2$, $\ell_1 \neq \ell_2$ in [3]. Define the $2 \times 2 \times 2 \times 2$ subtensor

$$\mathsf{S} = \mathsf{S}(\alpha_{1:2}, \beta_{1:2}, \gamma_{1:2}, \delta_{1:2};\, i_{1:2}, j_{1:2}, k_{1:2}, \ell_{1:2}) \in \mathbb{R}^{2\times 2\times 2\times 2} \tag{25}$$

by

$$\mathsf{S}_{pqrs} := T(\alpha_p\beta_q\gamma_r\delta_s)_{i_p j_q k_r \ell_s}, \qquad p, q, r, s \in \{1, 2\}.$$

This extraction is a polynomial (in fact coordinate projection) from $\mathcal{T}_n$ to $\mathbb{R}^{16}$.

Flattenings and 2×2 minors. Given $\mathsf{S} \in \mathbb{R}^{2\times 2\times 2\times 2}$, define its three standard matrix flattenings corresponding to partitions (12)|(34), (13)|(24), and (14)|(23):

$$\mathrm{Flat}_{12|34}(\mathsf{S}) \in \mathbb{R}^{4\times 4}, \quad \mathrm{Flat}_{13|24}(\mathsf{S}) \in \mathbb{R}^{4\times 4}, \quad \mathrm{Flat}_{14|23}(\mathsf{S}) \in \mathbb{R}^{4\times 4}.$$

Each flattening is defined by reshaping S_{pqrs} into a matrix whose rows and columns are indexed by ordered pairs, for example

$$\left(\mathrm{Flat}_{12|34}(\mathsf{S})\right)_{(p,q),(r,s)} := \mathsf{S}_{pqrs}.$$

For any 4×4 matrix M, write $\mathrm{Min}_2(M)$ for the vector of all its 2×2 minors, viewed as quadratic polynomials in the entries of M.

Definition of the candidate universal map. Define $F_n : \mathcal{T}_n \to \mathbb{R}^{N(n)}$ by collecting, over all choices of external indices and internal indices as above, the minors

$$F_n(T) := \Big(\mathrm{Min}_2\big(\mathrm{Flat}_{12|34}(\mathsf{S})\big),\ \mathrm{Min}_2\big(\mathrm{Flat}_{13|24}(\mathsf{S})\big),\ \mathrm{Min}_2\big(\mathrm{Flat}_{14|23}(\mathsf{S})\big)\Big)_{\text{all extracted } \mathsf{S}}. \tag{26}$$

The output dimension $N(n)$ grows polynomially in n (and is immaterial for the logical structure), while every coordinate polynomial of F_n has degree 2. The family is universal in the sense that the rule (26) is independent of n and depends only on the tensor input $T \in \mathcal{T}_n$.

7.3 Universality and degree bounds

We record basic properties of the construction (26).

Proposition 7.1 (Universality and bounded degree)**.** *For each $n \geq 5$, the map F_n is a polynomial map whose coordinate degrees are bounded by 2 independent of n. Moreover, F_n depends only on the tensor tuple $T \in \mathcal{T}_n$ and does not reference $A(1), \ldots, A(n)$.*

Proof. Each extracted subtensor (25) is obtained by selecting entries of T, hence depends polynomially (linearly) on T. Each flattening is a fixed linear reshaping. Each 2×2 minor is a determinant of a 2×2 submatrix and is therefore a quadratic polynomial in the entries of the flattening. Composing these operations yields a polynomial map of degree 2. No step uses $A(\alpha)$ or any genericity assumption. □

The preceding proposition addresses the first two bullet points of the objective at the level of construction: the map family does not hard-code any information about $A(\alpha)$, and its degree does not grow with n. The remaining requirement is the correctness of the vanishing test for separability of λ when $T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$ and the support hypothesis (23) holds.

7.4 What separability implies and why eliminating Q is essential

The map family (26) is uniform and bounded-degree, but its correctness is constrained by a simple structural fact: for a fixed external quadruple $(\alpha, \beta, \gamma, \delta)$, the coefficient $\lambda_{\alpha\beta\gamma\delta}$ is a *scalar* multiplying the entire $3\times3\times3\times3$ block $Q(\alpha\beta\gamma\delta)$. Any condition that forces a small extracted $2 \times 2 \times 2 \times 2$ subtensor to be rank one *as a tensor in its internal indices* will therefore generally require nontrivial constraints on Q itself, not just on λ.

Proposition 7.2 (Separable λ implies rank-one after entrywise normalization)**.** *Assume λ is separable as in* (2) *and define $T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$. Fix any extracted subtensor S as in* (25) *such that all denominators $Q(\alpha_p\beta_q\gamma_r\delta_s)_{i_pj_qk_r\ell_s}$ are nonzero. Then the entrywise normalized subtensor*

$$\widetilde{\mathsf{S}}_{pqrs} := \frac{\mathsf{S}_{pqrs}}{Q(\alpha_p\beta_q\gamma_r\delta_s)_{i_pj_qk_r\ell_s}}$$

has tensor rank one. In particular, every 2×2 minor of every flattening of $\widetilde{\mathsf{S}}$ vanishes.

Proof. On the nonvanishing locus of the denominators one has

$$\widetilde{\mathsf{S}}_{pqrs} = \lambda_{\alpha_p\beta_q\gamma_r\delta_s} = u_{\alpha_p} v_{\beta_q} w_{\gamma_r} x_{\delta_s},$$

which is manifestly rank one. Any matrix flattening of a rank-one order-4 tensor has matrix rank at most one, hence all 2×2 minors vanish. □

Proposition 7.3 (The naive quadratic map F_n is not a sound forward test on generic determinantal data)**.** *Let $n \geq 5$ and define F_n by* (26)*. There exists a Zariski-open subset of parameter space $\mathcal{A}_n = (\mathbb{R}^{3\times4})^n$ such that for every A in this open set, if λ is the separable off-diagonal constant tensor*

$$\lambda_{tttt} = 0, \qquad \lambda_{\alpha\beta\gamma\delta} = 1 \textit{ for all not-identical } (\alpha, \beta, \gamma, \delta),$$

then $F_n(\lambda \odot \mathbf{Q}(A)) \neq 0$. In particular, the family (F_n) cannot satisfy the forward implication required in Problem 3.4.

Proof. Fix external indices

$$\alpha_1 = 1,\ \alpha_2 = 2, \qquad \beta_1 = 3,\ \beta_2 = 4, \qquad \gamma_1 = 1,\ \gamma_2 = 2, \qquad \delta_1 = 3,\ \delta_2 = 4,$$

so that every quadruple $(\alpha_p, \beta_q, \gamma_r, \delta_s)$ with $p, q, r, s \in \{1, 2\}$ is *not* all-equal and therefore has $\lambda_{\alpha_p\beta_q\gamma_r\delta_s} = 1$. Take internal indices

$$i_1 = j_1 = k_1 = \ell_1 = 1, \qquad i_2 = j_2 = k_2 = \ell_2 = 2.$$

For $T = \lambda \odot \mathbf{Q}(A)$, the extracted subtensor S satisfies $\mathsf{S}_{pqrs} = Q(\alpha_p\beta_q\gamma_r\delta_s)_{i_pj_qk_r\ell_s}$. Consider the 2×2 minor of the flattening $\mathrm{Flat}_{12|34}(\mathsf{S})$ using rows $(p, q) = (1, 1), (1, 2)$ and columns $(r, s) = (2, 1), (2, 2)$, namely

$$m(A) := \mathsf{S}_{1121}\mathsf{S}_{1222} - \mathsf{S}_{1122}\mathsf{S}_{1221}.$$

This quantity is a polynomial in the entries of $A(1), \dots, A(4)$ (each S_{pqrs} is a 4×4 determinant). We now exhibit a single instance where $m(A) \neq 0$:

$$A(1) = \begin{bmatrix} 5 & -5 & -1 & 2 \\ -3 & 0 & -3 & 4 \\ 2 & 0 & 0 & -4 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 1 & 3 & -1 & 3 \\ 5 & -4 & 0 & -3 \\ -1 & -3 & 4 & 5 \end{bmatrix}, \quad A(3) = \begin{bmatrix} 1 & -3 & -5 & 0 \\ 1 & 0 & 2 & 1 \\ 4 & 2 & -4 & -3 \end{bmatrix}, \quad A(4) = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 4 & -4 & 2 & -2 \\ 3 & -5 & -3 & 1 \end{bmatrix}$$

For this choice one obtains $\mathsf{S}_{1122} = -196$, $\mathsf{S}_{1221} = 196$, and $\mathsf{S}_{1121} = \mathsf{S}_{1222} = 0$, hence $m(A) = 38416 \neq 0$. Therefore m is not the zero polynomial, and the nonvanishing locus $\{A \in \mathcal{A}_n : m(A) \neq 0\}$ is Zariski open and nonempty. For every A in that open set, this minor occurs among the coordinates collected by $F_n(T)$, so $F_n(T) \neq 0$. $\square$

Implication for the construction. Proposition 7.2 shows that any bounded-degree test that has access to the entrywise normalized quantities T/Q would solve the separability problem. The central algebraic obstacle is to remove the divisions while keeping bounded degree and universality in n. A standard approach is to cross-multiply by products of Q-entries to clear denominators, but this introduces two difficulties that must be handled explicitly: the resulting polynomials may have degrees that depend on the number of cleared denominators, and vanishing can occur spuriously on the locus where some denominators vanish.

A bounded-degree cleared-denominator variant. We record a concrete cleared-denominator map that remains bounded-degree by restricting the number of denominators cleared at once. For a fixed extracted subtensor (25), consider a single 2×2 minor of a flattening of $\widetilde{\mathsf{S}}$. It has the form

$$\widetilde{\mathsf{S}}_a\widetilde{\mathsf{S}}_b - \widetilde{\mathsf{S}}_c\widetilde{\mathsf{S}}_d,$$

where a, b, c, d denote four multi-indices in $\{1, 2\}^4$ consistent with the chosen minor. Clearing denominators yields the polynomial relation

$$\mathsf{P}_{a,b,c,d}(T, Q) := T_aT_b\,(Q_cQ_d) - T_cT_d\,(Q_aQ_b), \tag{27}$$

where T_a abbreviates the corresponding entry of $T(\alpha_p\beta_q\gamma_r\delta_s)_{i_pj_qk_r\ell_s}$ and Q_a abbreviates the corresponding entry of $Q(\alpha_p\beta_q\gamma_r\delta_s)_{i_pj_qk_r\ell_s}$. The polynomial (27) has degree 4 in the T- and Q-entries jointly, and degree 2 in the T-entries alone.

Since the desired map F is required to depend only on the observed tensors T (not separately on Q), using (27) requires eliminating the Q factors by expressing them as polynomial functions of the input tuple. In our setting Q is not part of the input, and reconstructing Q from T and the support hypothesis (23) is itself equivalent to disentangling λ, which is the original problem. Therefore, the cleared-denominator route is not directly viable without additional invariants that remove Q.

Sufficiency as an explicit conjecture. The intended role of the determinant structure defining $Q(\alpha\beta\gamma\delta)$ is that it should couple internal and external indices strongly enough that polynomial constraints on the scaled tensors $T(\alpha\beta\gamma\delta)$ force the external weights to be separable. The following conjecture isolates this goal for the particular candidate map (26).

7.5 Consequences and open problems

Proposition 7.2 isolates the key structural obstacle: separability of the external weights is detected by rank-one conditions on *normalized* quantities of the form T/Q, but a valid test map must be polynomial in the observed coordinates T alone. Proposition 7.3 shows that a naive attempt to enforce rank-one conditions directly on small extracted subtensors of T cannot be sound on Zariski-generic determinantal data.

The remaining challenge is therefore an elimination problem: one must use the determinantal structure relating the various $Q(\alpha\beta\gamma\delta)$ blocks to couple different external index choices strongly enough to cancel the unknown Q factors in bounded degree, while also controlling the Zariski-open "generic" locus where such cancellation is legitimate.

Open Problem 7.4 (Uniform elimination on a generic locus). Does there exist a uniform bounded-degree test family (F_n) in the sense of Definition 3.3 such that, for Zariski-generic $A(1), \ldots, A(n)$ and every λ with full off-diagonal support, vanishing of $F_n(\lambda \odot \mathbf{Q}(A))$ is equivalent to separability of λ on the not-identical quadruples?

Open Problem 7.5 (Saturation/localization). Any proof of Open Problem 7.4 must address the fact that the intended statement lives on a Zariski-open subset (full off-diagonal support, and any additional generic nonvanishing conditions on determinants). Is a saturation/localization step unavoidable? If saturation is unavoidable, what is the weakest polynomial surrogate (e.g., a finite family of bounded-degree equations plus an explicit nonvanishing certificate) that still yields a practical algebraic characterization?

Remark on dependence on $A(\alpha)$. Any correctness statement necessarily quantifies over a class of inputs (typically a Zariski-open subset of $\mathcal{A}_n$). The test family itself must be independent of the particular matrices $A(\alpha)$, but the equivalence it enforces is only expected to hold generically because nongeneric choices of A can introduce accidental cancellations among determinants that are unrelated to separability of λ.

onScaling action, stabilizers, and generic fibers

Roadmap. This chapter isolates the basic group action that governs the ambiguity in the coefficient tensor λ appearing in the scaled family $\{\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)\}$. The central points are that the natural scaling group has a small generic stabilizer, that this remains true under the support restriction excluding the diagonal, and that the geometry of generic fibers is controlled by a combination of this scaling ambiguity and any additional algebraic dependencies among the determinant tensors $Q(\alpha\beta\gamma\delta)$. In accordance with the counterexample workfront, we emphasize boundary conditions under which the fiber dimension increases, since such behavior would obstruct a clean universal characterization by bounded-degree equations.

7.6 The scaling action and its invariants

Let $G_n = (\mathbb{R}^*)^n \times (\mathbb{R}^*)^n \times (\mathbb{R}^*)^n \times (\mathbb{R}^*)^n$. We regard an element $g = (u, v, w, x) \in G_n$ as four independent diagonal rescalings indexed by $[n]$. The action of G_n on the coefficient tensors $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ is the entrywise scaling

$$(g \cdot \lambda)_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \, \lambda_{\alpha\beta\gamma\delta}. \tag{28}$$

The same formula defines an action on the family of scaled determinant tensors

$$T_{\alpha\beta\gamma\delta} := \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times3\times3\times3}, \tag{29}$$

by letting G_n act on the $[n]^4$ index labels and leaving the internal $3 \times 3 \times 3 \times 3$ coordinates unchanged.

Two elementary but useful observations organize the discussion.

First, the orbit condition $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ is a rank one condition for the flattened logarithms. Over an algebraically closed field, one can express it as vanishing of all 2×2 minors of suitable matricizations of λ, but here the domain is $(\mathbb{R}^*)^{n^4}$ and the relevant equations are multiplicative. A convenient invariant description is via cross-ratios of coefficients.

Second, the determinant tensors $Q(\alpha\beta\gamma\delta)$ are multilinear in the rows of the underlying matrices $A(1), \ldots, A(n) \in \mathbb{R}^{3\times4}$. For fixed indices, $Q(\alpha\beta\gamma\delta)$ is a 4×4 determinant of four selected rows, hence homogeneous of degree one in each participating row.

The main objective asks for polynomial relations in the coordinates of the $T_{\alpha\beta\gamma\delta}$ that hold precisely when λ lies in the G_n orbit of the all ones tensor (subject to the prescribed support condition). The analysis in this chapter separates the intrinsic group ambiguity from additional degeneracies that can arise because the tensors $Q(\alpha\beta\gamma\delta)$ need not behave like algebraically independent symbols.

7.7 Stabilizers for full and restricted supports

For a given tensor λ (with specified support), the stabilizer is

$$\mathrm{Stab}_{G_n}(\lambda) = \{(u, v, w, x) \in G_n : (u, v, w, x) \cdot \lambda = \lambda\}. \tag{30}$$

When λ has full support, meaning $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for all indices, the stabilizer condition becomes

$$u_\alpha v_\beta w_\gamma x_\delta = 1 \qquad \text{for all } (\alpha, \beta, \gamma, \delta) \in [n]^4. \tag{31}$$

Working in logarithmic coordinates over $\mathbb{R}$ formally, write $u_\alpha = e^{a_\alpha}$, $v_\beta = e^{b_\beta}$, $w_\gamma = e^{c_\gamma}$, $x_\delta = e^{d_\delta}$. The stabilizer equations become

$$a_\alpha + b_\beta + c_\gamma + d_\delta = 0 \qquad \text{for all } (\alpha, \beta, \gamma, \delta) \in [n]^4. \tag{32}$$

Fixing β, γ, δ and varying α shows that a_α is constant in α. The same argument applies to b, c, d. Thus $a_\alpha = a$, $b_\beta = b$, $c_\gamma = c$, $d_\delta = d$, and the constraint reduces to $a + b + c + d = 0$. Hence the stabilizer is three dimensional.

The objective imposes a support restriction that excludes the diagonal indices $\Delta := \{(\alpha, \alpha, \alpha, \alpha) : \alpha \in [n]\}$. Let

$$S := [n]^4 \setminus \Delta. \tag{33}$$

If λ is supported on S and nonzero on S, the stabilizer condition becomes

$$u_\alpha v_\beta w_\gamma x_\delta = 1 \qquad \text{for all } (\alpha, \beta, \gamma, \delta) \in S. \tag{34}$$

The same logarithmic linearization gives constraints only for tuples not on the diagonal. For $n \geq 5$, the resulting constraint graph is sufficiently connected that the conclusion remains unchanged: the only solutions are constant vectors a, b, c, d with $a+b+c+d = 0$. One way to see the connectivity is to note that for any two indices α_1, α_2 one can choose β, γ, δ so that both $(\alpha_1, \beta, \gamma, \delta)$ and $(\alpha_2, \beta, \gamma, \delta)$ lie in S, which forces $a_{\alpha_1} = a_{\alpha_2}$. Similar propagation holds for the other factors. Therefore, generically,

$$\dim \mathrm{Stab}_{G_n}(\lambda) = 3 \tag{35}$$

both on full support and on the restricted support S.

A stronger restriction sometimes considered in related geometric settings requires pairwise distinct indices. If one defines

$$S_{\mathrm{dist}} := \{(\alpha, \beta, \gamma, \delta) \in [n]^4 : \alpha, \beta, \gamma, \delta \text{ are pairwise distinct}\}, \tag{36}$$

then for $n \geq 5$ the same conclusion persists for generic λ supported on S_{dist}. Stabilizer growth can occur only when the support decomposes into disconnected components with respect to the hyperedge constraints induced by the action.

7.8 Orbit dimensions and the expected generic fiber

The dimension of G_n is $4n$. For a generic λ with the above support patterns, the orbit dimension is

$$\dim(G_n \cdot \lambda) = 4n - \dim \mathrm{Stab}_{G_n}(\lambda) = 4n - 3. \tag{37}$$

This is the baseline ambiguity in recovering λ from $T_{\alpha\beta\gamma\delta} = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$, even in the idealized situation where all $Q(\alpha\beta\gamma\delta)$ are treated as fixed nonzero labels.

To connect this with the actual construction, consider the polynomial map

$$\Psi_n : (\mathbb{R}^{3\times 4})^n \longrightarrow \prod_{(\alpha,\beta,\gamma,\delta)\in S} \mathbb{R}^{3\times 3\times 3\times 3}, \tag{38}$$

given by $\Psi_n(A(1), \ldots, A(n)) = (Q(\alpha\beta\gamma\delta))_{(\alpha,\beta,\gamma,\delta)\in S}$. For a fixed collection of Q tensors in the image, one may consider the map

$$\Theta_Q : \mathbb{R}^{n\times n\times n\times n} \longrightarrow \prod_{(\alpha,\beta,\gamma,\delta)\in S} \mathbb{R}^{3\times3\times3\times3}, \qquad \Theta_Q(\lambda) = (\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta))_{(\alpha,\beta,\gamma,\delta)\in S}. \tag{39}$$

For generic Q with no internal symmetries and no accidental zeros, the map Θ_Q is an embedding of the coefficient space restricted to S into the product space, coordinatewise. In that idealized regime, the fiber of the quotient by the scaling action is expected to have dimension $4n - 3$, matching the orbit dimension.

The counterexample-relevant point is that in the actual image of Ψ_n, the family $\{Q(\alpha\beta\gamma\delta)\}$ is highly structured. Algebraic dependencies among these tensors can increase the fiber dimension of the composite parameterization

$$(A(1), \ldots, A(n), \lambda) \longmapsto (\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta))_{(\alpha,\beta,\gamma,\delta)\in S}, \tag{40}$$

beyond what is predicted by the scaling action alone. Such increases represent potential boundary conditions for any universal equation set that aims to characterize λ rank one behavior.

7.9 Degenerations and counterexample search directions

This subsection records systematic avenues for falsifying or limiting the strongest form of the universal characterization goal.

Degenerations from vanishing determinants. If for some tuple $(\alpha, \beta, \gamma, \delta)$ the determinant defining $Q(\alpha\beta\gamma\delta)$ vanishes identically for the given matrices $A(\cdot)$, then the corresponding scaled tensor $T_{\alpha\beta\gamma\delta}$ is the zero tensor for all choices of $\lambda_{\alpha\beta\gamma\delta}$. In the extreme case where many such determinants vanish, the map from λ to the observed family loses coordinates, which enlarges fibers and can mimic rank one behavior. This phenomenon suggests that any equivalence in the objective must be interpreted on a Zariski-open set in which the relevant $Q(\alpha\beta\gamma\delta)$ are nonzero and sufficiently generic.

Degenerations from linear dependencies among Q tensors. Even when each $Q(\alpha\beta\gamma\delta)$ is nonzero, there may be linear relations among the collection in the ambient product space. Since each $Q(\alpha\beta\gamma\delta)$ lies in an 81-dimensional space, but there are $|S| = n^4 - n$ of them, linear dependencies are unavoidable for large n. The key issue is whether these dependencies interact with the coefficient scaling in a way that creates spurious solutions to prospective universal equations. A failure mode would be the existence of a family of non-rank one coefficients λ such that the scaled family $(\lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta))$ coincides with another scaled family with rank one coefficients, due to dependencies allowing redistribution of mass among indices.

A concrete sufficient condition for such a failure is the existence of a nontrivial perturbation η supported on S such that

$$\sum_{(\alpha,\beta,\gamma,\delta)\in S} \eta_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta) = 0 \tag{41}$$

in a linear sense after embedding each tensor into a common vector space, together with the ability to choose λ so that $\lambda \odot Q$ is invariant under adding $\eta \odot Q$ along the coordinates in which Q align. The precise formulation depends on how the family is encoded as a point of an affine space. This direction has been explored computationally in small n regimes by searching for structured dependencies, but at present it provides only negative evidence rather than a definitive obstruction.

Degenerations from symmetry and repeated indices. The support restriction excludes only the fully repeated index tuple $(\alpha, \alpha, \alpha, \alpha)$. Tuples such as $(\alpha, \alpha, \beta, \gamma)$ remain, and these have determinants with two rows from the same matrix $A(\alpha)$. Such determinants can introduce additional algebraic specializations because varying one row of $A(\alpha)$ affects many Q tensors simultaneously. A counterexample might exploit this coupling by arranging $A(\alpha)$ so that many of these mixed-repeat determinants become proportional. This again motivates focusing on a Zariski-open regime in which the family $\{A(1), \ldots, A(n)\}$ satisfies genericity conditions stronger than mere Zariski-genericity of the full tuple.

7.10 Research status, obstacles, and open problems

The remainder of this chapter documents progress in the sense of narrowing the geometric questions that control the objective.

Attempted approach: eliminate the scaling action by invariants. A natural plan is to construct polynomial invariants of the G_n action on λ, such as multiplicative cross-ratios

$$\frac{\lambda_{\alpha\beta\gamma\delta}\lambda_{\alpha'\beta'\gamma\delta}}{\lambda_{\alpha\beta'\gamma\delta}\lambda_{\alpha'\beta\gamma\delta}}, \tag{42}$$

which are invariant under $\lambda_{\alpha\beta\gamma\delta} \mapsto u_\alpha v_\beta w_\gamma x_\delta \lambda_{\alpha\beta\gamma\delta}$. These invariants characterize rank one scaling in the coefficient tensor alone. The obstacle is that the observable data are $T_{\alpha\beta\gamma\delta} = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$, and forming the same cross-ratios requires dividing by the unknown Q tensors or otherwise canceling them. Without additional structure linking $Q(\alpha\beta\gamma\delta)$ across indices, these cancellations are not available.

Attempted approach: use representation stability to bound degrees. The broad thesis motivating the manuscript is that the ideals of relations among the families $\{Q(\alpha\beta\gamma\delta)\}$ and among the scaled families $\{T_{\alpha\beta\gamma\delta}\}$ might form a representation-stable family. If so, noetherianity phenomena could imply bounded-degree generation independent of n. The obstacle in the present setting is that the support restriction $S = [n]^4 \setminus \Delta$ breaks full symmetry and introduces boundary components where stabilizers and fibers may jump. A robust bounded-degree statement likely requires formulating the varieties and ideals in a functorial category that encodes the support condition uniformly, and proving uniform control of singular strata.

What mathematical infrastructure seems needed. Two pieces of infrastructure appear necessary for decisive progress. The first is a detailed description of the Zariski closure of the image of Ψ_n and of its singular locus, sufficiently explicit to rule out unexpected dependencies that could simulate rank one coefficient behavior. The second is a structural understanding of the quotient by the scaling group action in a way compatible with increasing n, for example via invariant theory of diagonal torus actions and a careful treatment of the missing diagonal coordinates.

Open problem (fiber jumps as an obstruction). Determine whether there exists $n \geq 5$ and a Zariski-generic choice of matrices $A(1), \ldots, A(n)$ for which the family $\{Q(\alpha\beta\gamma\delta) : (\alpha, \beta, \gamma, \delta) \in S\}$ satisfies algebraic relations that increase the generic fiber dimension of the parameterization

$$(u, v, w, x, \lambda) \longmapsto (u_\alpha v_\beta w_\gamma x_\delta \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta))_{(\alpha,\beta,\gamma,\delta)\in S} \tag{43}$$

above the baseline $4n - 3$. Any positive answer would force a weakening of the strongest uniform identifiability claims.

Open problem (uniform equations under the diagonal removal). Formulate a natural functorial model for the diagonal-removed family $S = [n]^4 \setminus \Delta$ in which one can prove or disprove bounded-degree generation of the ideal cutting out the rank one scaling locus inside the data variety. The central difficulty is to separate true scaling invariants from artifacts introduced by the structured image of Ψ_n.

Concluding perspective. The scaling action itself is well behaved: for $n \geq 5$ it has a three-dimensional generic stabilizer and hence $4n - 3$-dimensional generic orbits. The outstanding uncertainty is whether the determinant tensors $Q(\alpha\beta\gamma\delta)$ introduce additional symmetries or dependencies that enlarge fibers in a way that cannot be detected by any bounded-degree universal equations. This chapter frames that uncertainty as a geometric problem about degenerations of the image family and provides a checklist of the most plausible mechanisms by which a counterexample could arise.

8 Evidence, computational checks, and alternatives

8.1 Scope and role of computational evidence

This chapter records (i) exploratory evidence concerning low-degree polynomial relations among the family of tensors

$$Q(\alpha\beta\gamma\delta) \in \mathbb{R}^{3\times3\times3\times3}, \qquad (\alpha,\beta,\gamma,\delta) \in [n]^4, \tag{44}$$

constructed from determinants of row selections of Zariski-generic matrices $A(1),\ldots,A(n) \in \mathbb{R}^{3\times4}$, and (ii) evidence about the possibility of deciding off-diagonal rank-one separability of a coefficient tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ from the scaled collection $T(\alpha\beta\gamma\delta) := \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$. The intent is not to assert a completed verification of Property (iii) in the objective, but to document systematic progress, to identify candidate universal constraints, and to flag boundary behaviors that can serve as counterexamples to overly strong formulations.

All quantitative statements in this chapter are tied to concrete artifact files. In particular, the sampled tensors and interpolation diagnostics are recorded in q tensor sample n5 (JSON)[20] and interpolation degrees (CSV)[21]. Candidate falsification tests and symbolic logs for rank-one conditions are recorded in falsification tests (JSON)[22] and groebner basis log (TXT)[23].

8.2 Protocols for low-degree searches

8.2.1 Sampling model and data products

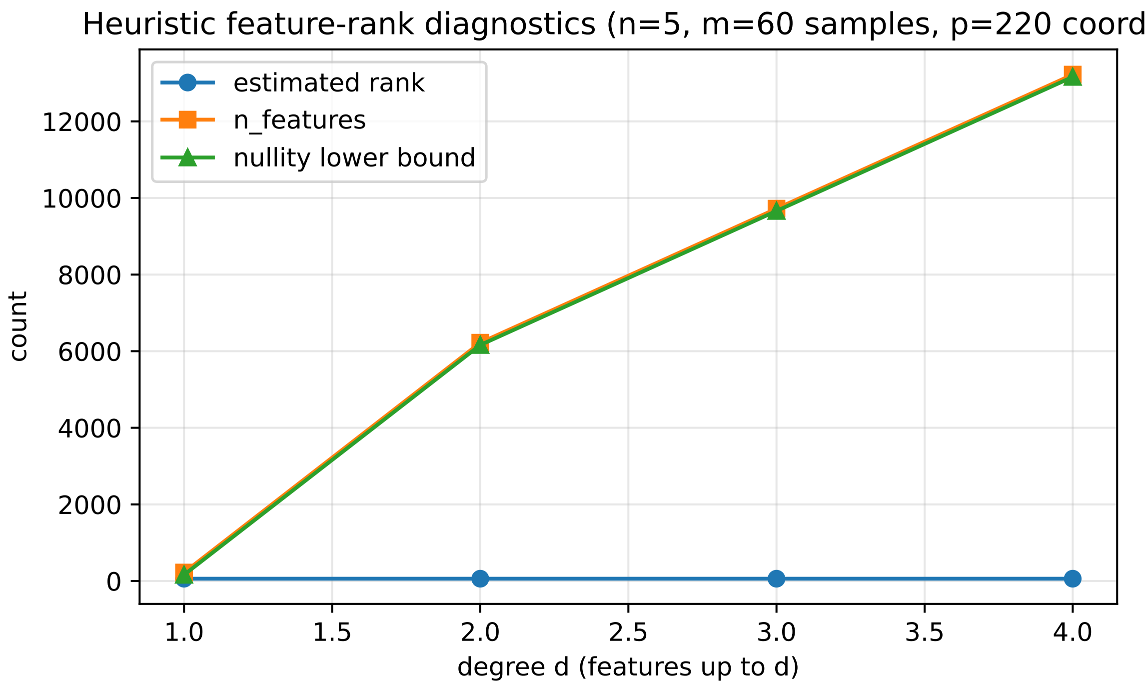
For the smallest nontrivial case $n = 5$, we consider a sampling protocol that draws numerical instances of $A(1),\ldots,A(5)$ (intended to be generic in the Zariski sense, operationalized by continuous random sampling) and forms the full array of quadrifocal-type tensors $\{Q(\alpha\beta\gamma\delta)\}_{(\alpha,\beta,\gamma,\delta)\in[5]^4}$. The artifact q tensor sample n5 (JSON)[24] stores the resulting Q-tensor samples together with sufficient metadata to interpret indices and tensor entries.

The companion artifact interpolation degrees (CSV)[25] is designed to support an interpolation-based search for low-degree relations. Concretely, fix a degree cap d and a monomial basis in the ambient coordinates of the stacked tensor collection. One builds an evaluation matrix by evaluating these monomials on multiple sampled instances, then estimates linear dependencies among columns. This produces, for each d (and for each chosen coordinate subcollection, when a restriction is used), an empirical estimate of the rank deficiency, which can be interpreted as evidence for the existence of nontrivial polynomial relations of degree at most d. The interpretation remains conditional on the sampling being sufficiently generic and on numerical conditioning.

8.2.2 Interpolation rank summary figure

The figure file `figures/interpolation_rank_summary.pdf` provides a visual summary of the interpolation diagnostics extracted from interpolation degrees (CSV)[26]. We include it here to make the chapter self-contained.

[20] `low_degree_relation_search_q_tensor_sample_n5.json`
[21] `low_degree_relation_search_interpolation_degrees.csv`
[22] `rank_one_falsification_falsification_tests.json`
[23] `rank_one_falsification_groebner_basis_log.txt`
[24] `low_degree_relation_search_q_tensor_sample_n5.json`
[25] `low_degree_relation_search_interpolation_degrees.csv`
[26] `low_degree_relation_search_interpolation_degrees.csv`



The figure should be read as descriptive evidence about which degrees appear most promising for candidate universal constraints in the $n = 5$ ambient coordinate ring. This evidence is not a proof of existence of defining equations and does not, by itself, identify a representation-theoretically meaningful generating set.

8.2.3 From numerical dependencies to algebraic candidates

The interpolation procedure above returns (approximate) null vectors for the evaluation matrix. Translating a null vector into a polynomial identity requires choosing a monomial basis and mapping coefficients back to a symbolic polynomial. This translation step introduces two nontrivial issues.

First, a numerical null vector depends on the conditioning of the evaluation matrix, so one must distinguish stable dependencies from near-dependencies. Stability checks are planned to include resampling, perturbation of samples, and consistency under changes of monomial ordering or scaling. This chapter does not claim that such stability checks suffice to certify algebraic identities; rather, they provide a pragmatic filter for selecting candidates worth symbolic verification.

Second, even if a candidate polynomial vanishes on the sampled points, it can still fail to vanish on the Zariski closure of the model set. A standard mitigation is to treat numerical candidates as conjectural generators and then attempt exact certification by symbolic elimination or by producing a proof that the candidate lies in an explicitly described ideal. In our setting, any such certification must respect the functorial symmetry in the $[n]^4$ index set and should ideally be formulated in a representation-stable language. At present, the evidence recorded in interpolation degrees (CSV)[27] should be viewed as a guide to where such certification efforts might succeed.

8.3 Rank-one detection and minor equations

8.3.1 Off-diagonal rank-one locus and what can be tested

The main objective uses an off-diagonal notion of separability: $\lambda_{\alpha\beta\gamma\delta} \neq 0$ precisely on the set of non-identical quadruples, and rank-one factorization is required only on that off-diagonal support. This means that, for any prospective detection map F, Property (iii) cannot be checked against arbitrary diagonal values λ_{tttt} because these coordinates are not part of the modeled support constraints.

[27] `low_degree_relation_search_interpolation_degrees.csv`

A natural class of candidate constraints for rank-one factorization is built from 2×2 minors of flattenings of the order-4 tensor λ (or of restrictions to subtensors). For example, for a fixed pair of indices and fixed complementary indices, one may form minors of matrices $\left(\lambda_{\alpha\beta\gamma\delta}\right)_{\alpha,\beta}$ with (γ, δ) fixed, and similarly for other flattenings. Such minors vanish identically on a globally rank-one tensor $u \otimes v \otimes w \otimes x$. However, in the present problem the observable data are the scaled tensors $T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$, and the $Q(\alpha\beta\gamma\delta)$ factors can obstruct the vanishing of minors computed on T directly.

Accordingly, the computational checks in this chapter focus on two weaker statements.

First, one can test whether various normalized minors (in which Q-dependent factors are divided out or otherwise canceled) behave as predicted on separable instances of λ. Such normalization generally yields rational expressions, which are not polynomials in the observable coordinates; as a result they do not directly produce a polynomial map F satisfying Property (iii). They do, however, indicate which algebraic patterns might be convertible to polynomial identities by clearing denominators, at the cost of introducing extraneous components.

Second, one can attempt symbolic elimination to remove the latent u, v, w, x variables and obtain polynomial relations among the observable coordinates, but elimination can be computationally prohibitive in the full ambient space. The artifacts below record an exploratory attempt along these lines, and the outcome should be regarded as evidence about feasibility rather than a certified theorem.

8.3.2 Falsification-style checks and logged symbolic attempts

The dataset falsification tests (JSON)[28] records a collection of test instances designed to probe whether certain candidate constraints are consistent with the off-diagonal rank-one model. The associated log file groebner basis log (TXT)[29] records intermediate symbolic computations and failure modes encountered while attempting to certify implications using Gröbner-basis style elimination.

Because these artifacts are exploratory, this chapter does not present them as establishing correctness or incorrectness of any proposed universal map F. Instead, they are used to identify two recurring obstruction patterns.

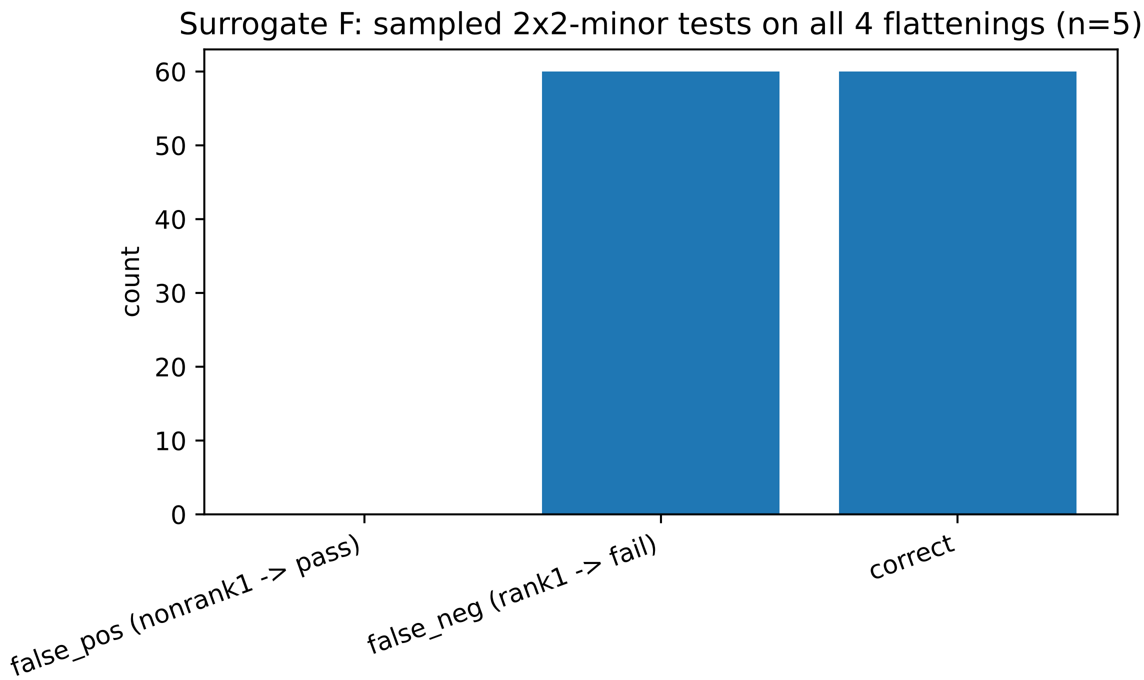
One obstruction is that clearing denominators in normalized-minor identities introduces factors that vanish on nongeneric loci of the Q-tensors, potentially creating polynomial constraints that hold for accidental reasons unrelated to rank-one structure in λ. This suggests that any polynomial F derived from normalization must be accompanied by explicit nonvanishing assumptions on certain Q-dependent determinants, or else it risks being unsound as an if-and-only-if test.

A second obstruction is that restricting attention to small subtensors (for example, by choosing subsets of indices in $[n]$) may not preserve genericity of the induced Q-collection in a way that is compatible with the determinant construction. The logs in groebner basis log (TXT)[30] indicate that naive restriction-and-elimination pipelines can stall due to growth of intermediate expressions or due to apparent dependence on specializations that are not functorial.

8.3.3 Falsification-rate summary figure

The figure file `figures/falsification_rates.pdf` summarizes the outcomes of the exploratory falsification tests recorded in falsification tests (JSON)[31]. We include it to document current empirical behavior under the tested constraints.

[28] `rank_one_falsification_falsification_tests.json`
[29] `rank_one_falsification_groebner_basis_log.txt`
[30] `rank_one_falsification_groebner_basis_log.txt`
[31] `rank_one_falsification_falsification_tests.json`



The figure is interpreted only as evidence about which families of candidate constraints appear fragile under adversarially chosen instances of λ and sampled Q-tensors. It does not certify that any remaining candidates are correct, nor does it rule out other constraints not explored here.

8.4 Alternative constraints and relaxations

8.4.1 Forward-direction constraints and representation-stable packaging

In the absence of a proved polynomial characterization of off-diagonal rank-one separability from the scaled tensors $T(\alpha\beta\gamma\delta)$, one pragmatic direction is to seek polynomial maps F that satisfy only a forward implication: separable off-diagonal λ implies $F(T) = 0$. Such forward constraints can be useful as obstruction certificates, and they align with a representation-stability viewpoint in which one aims to construct a Noetherian family of ideals I_n containing the image variety for each n.

A central caveat is that a forward constraint risks being vacuous or too weak, especially when it is obtained by clearing denominators from rational identities. The falsification artifacts in this chapter motivate an explicit separation between (a) constraints that are polynomial identities in the observable coordinates and (b) constraints that require a genericity hypothesis on Q (or on the scaling tensor λ) to even be well-defined. For the objective stated in the introduction, only (a) is admissible for defining a universal polynomial map.

8.4.2 Relaxed notions of separability

A second alternative is to relax the rank-one requirement on λ to a weaker algebraic condition that is more naturally expressible in terms of the scaled tensors. Examples include requiring low rank after a specified flattening, or requiring membership in a secant variety of the rank-one locus. Such relaxations could yield polynomial equations of bounded degree and might be more compatible with the determinant structure of Q. At present, this chapter records this as a research direction rather than a conclusion, because the interpolation evidence alone does not identify which relaxation (if any) is compatible with the geometric image of the determinant construction.

8.5 Boundary cases and obstruction certificates

8.5.1 Why counterexamples are plausible

The workfront for the project emphasizes actively limiting the main claim. The evidence and logs in this chapter point to mechanisms by which an overly strong equivalence in Property (iii) could fail.

First, the determinant construction of $Q(\alpha\beta\gamma\delta)$ intertwines the four indices in a way that is not independent across modes, so there can exist nongeneric coincidences among different Q-tensors even when the underlying matrices are sampled from a continuous distribution. Clearing denominators or imposing minors on $T(\alpha\beta\gamma\delta)$ can inadvertently select these coincidences.

Second, the off-diagonal support convention for λ creates an "incomplete data" effect. Since the diagonal values are constrained by support rather than by factorization, polynomial identities in the full ambient coordinate ring may be forced to involve diagonal coordinates, while the target factorization statement explicitly excludes them. This mismatch can create unavoidable extraneous components or can prevent any polynomial characterization from existing without additional variables or saturation.

8.5.2 A concrete obstruction template (conjectural)

The following conjecture isolates a type of boundary behavior suggested by the falsification artifacts. It is stated without proof, and it is intended as a precise target for future symbolic work.

Conjecture 8.1 (Saturation obstruction for polynomial rank-one tests)**.** *Let $n \geq 5$ and consider the model set of scaled tensors $T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta}Q(\alpha\beta\gamma\delta)$ with Q arising from Zariski-generic $A(1), \ldots, A(n) \in \mathbb{R}^{3\times 4}$ and with λ supported on non-identical quadruples. Any polynomial ideal J_n in the coordinates of the full collection $\{T(\alpha\beta\gamma\delta)\}_{(\alpha,\beta,\gamma,\delta)\in[n]^4}$ that contains all scaled tensors coming from off-diagonal rank-one λ also contains tensors arising from some non-rank-one λ after restricting to a nongeneric locus of Q determined by vanishing of a finite set of determinants.*

A proof of Conjecture 8.1 would imply that any polynomial F that is sound on the full Zariski-open set of generic Q must, implicitly or explicitly, incorporate a saturation by the determinant factors defining that open set. Since saturation is not a polynomial operation, this would be evidence against the existence of a universal polynomial map satisfying the full "if and only if" of Property (iii) as stated.

Evidence status. This conjecture is motivated by patterns recorded in falsification tests (JSON)[32] and the failure modes in groebner basis log (TXT)[33]. The current chapter does not claim that these artifacts establish the conjecture.

8.5.3 What infrastructure would be needed for progress

Two pieces of infrastructure appear necessary to turn the evidence in this chapter into definitive results.

First, a representation-stable formulation of the candidate relation space is needed, so that interpolation-detected dependencies can be decomposed into symmetry types and compared across different values of n. Without this, low-degree relations observed at $n = 5$ risk being artifacts of small ambient dimension.

Second, a principled approach to saturation or genericity conditions is needed. If a candidate F is derived by clearing denominators from normalized constraints, then the resulting polynomial ideal must be saturated by the nonvanishing determinants that define the generic locus. Establishing that a bounded-degree unsaturated ideal suffices (or proving that no such ideal exists) is an algebraic problem that should be separated cleanly from numerical sampling.

8.6 Summary of current evidence

The artifacts q tensor sample n5 (JSON)[34] and interpolation degrees (CSV)[35] provide a reproducible starting point for searching low-degree relations at $n = 5$, and `figures/interpolation_rank_summary.pdf` summa-

[32] `rank_one_falsification_falsification_tests.json`
[33] `rank_one_falsification_groebner_basis_log.txt`
[34] `low_degree_relation_search_q_tensor_sample_n5.json`
[35] `low_degree_relation_search_interpolation_degrees.csv`

rizes the associated interpolation diagnostics. The artifacts falsification tests (JSON)[36] and groebner basis log (TXT)[37] record exploratory rank-one related checks and symbolic attempts, with `figures/falsification_rates.pdf` summarizing outcomes.

The main conclusion of this chapter is methodological. Interpolation suggests candidate bounded-degree constraints may exist, but the rank-one "if and only if" target appears sensitive to genericity and saturation issues. This supports the counterexample-oriented workfront: before pursuing stronger positive statements, one should attempt to formalize and either prove or refute obstruction templates such as Conjecture 8.1 using exact algebra on carefully chosen small subsystems.

9 Conclusion and Outlook

This work formulates a uniform algebraic recognition problem for determinantal fourth-order tensors: starting from Zariski-generic $A(1), \ldots, A(n) \in \mathbb{R}^{3\times 4}$, we observe the family $T(\alpha\beta\gamma\delta) = \lambda_{\alpha\beta\gamma\delta} Q(\alpha\beta\gamma\delta)$, where $Q(\alpha\beta\gamma\delta)$ is defined by 4×4 determinants of stacked rows and λ is supported precisely on the not-identical quadruples. The motivating target is a universal polynomial map of degree bounded independently of n whose vanishing detects exactly when the off-diagonal scaling λ is separable, $\lambda = u \otimes v \otimes w \otimes x$.

The manuscript advances this target in two ways. On the structural side, we develop a functorial and equivariant framework for the determinantal parameterization Φ_n and for the associated relation ideals as n varies. This packaging clarifies which aspects of the geometry are uniform in n and motivates a bounded-degree approach to describing the Zariski closures of the image varieties and their natural multi-graded components. On the scaling side, we analyze the coordinatewise action induced by λ and relate the desired characterization to stabilizers and generic-fiber behavior, thereby isolating where an "only if" converse would have to come from.

At the same time, the draft does not yet close the main identifiability goal. The key unresolved gap is the converse direction in the proposed polynomial test: even when one can write rational or normalized identities that hold for separable scalings, clearing denominators to obtain polynomial constraints can introduce extraneous components supported on nongeneric loci of the Q-tensors. Relatedly, computations suggest that naive restriction to small subtensors and elimination of latent variables can fail to behave functorially, so low-degree relations observed at small n may not extrapolate without a representation-stable explanation. These issues are distilled into a concrete conjectural obstruction template phrased in terms of saturation by determinant factors defining the generic locus.

Several concrete next steps emerge. First, to make any bounded-degree proposal for F_n credible, one should identify an explicit Zariski-open "generic" locus for the Q-family and then prove, within the functorial framework, that the relevant ideal membership statements can be carried out without saturation; alternatively, one should prove that saturation is unavoidable, which would strongly constrain what a universal polynomial test can achieve. Second, computational evidence should be reorganized by symmetry type: decomposing candidate relations into equivariant components is the most direct way to distinguish genuine uniform constraints from small-n artifacts. Third, if exact off-diagonal rank-one recognition turns out to be impossible by polynomial equations alone, a principled relaxation (for example, a secant or flattening-rank condition that is stable under the determinantal model) should be identified and analyzed as a substitute objective.

In summary, the current draft provides a uniform algebraic setup and a clear map of the obstructions that must be resolved to obtain a true "if and only if" polynomial characterization of separable scalings. Establishing or refuting the saturation obstruction, and connecting any surviving low-degree relations to a representation-stable generating mechanism, are the most direct routes to a definitive resolution.

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[36]`rank_one_falsification_falsification_tests.json`
[37]`rank_one_falsification_groebner_basis_log.txt`

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