Uniform lattices with involutions and rationally acyclic universal covers

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Abstract

We study whether a uniform lattice in a real semisimple Lie group that contains elements of order two can occur as the fundamental group of a closed manifold whose universal cover is acyclic with rational coefficients. From $\mathbb{Q}$-acyclicity we obtain a finite free $\mathbb{Q}[\Gamma]$-resolution of $\mathbb{Q}$ and, after passing to a torsion-free finite-index subgroup, a rational Poincaré duality group structure $\mathrm{PD}_n(\mathbb{Q})$ with the expected orientation twist. We then compare these constraints with standard invariants of cocompact lattices, including virtual cohomological dimension, Euler characteristic, and the structure and cohomological dimension of centralizers of involutions. The paper records a centralizer-based obstruction template and isolates the additional inputs needed to make it unconditional, emphasizing the need for torsion-sensitive information compatible with the freeness of the deck action and for verifiable Lie-theoretic centralizer structure statements.

Contents

1 Introduction

1.1 Roadmap, scope, and disclaimer

This paper studies an existence problem at the interface of manifold topology and lattice (co)homology. The guiding principle is that rational acyclicity of the universal cover forces strong restrictions on group (co)homology with trivial coefficients via Cartan–Leray collapse, while the most natural ways that 2-torsion might obstruct realizability are torsion-sensitive and frequently rely on fixed-point information.

Disclaimer (non-resolution). This manuscript does not resolve the realizability question in either direction. It derives unconditional constraints from $\mathbb{Q}$-acyclicity and manifold Poincaré duality, compares these constraints with standard lattice invariants (cohomological dimension and virtual cohomological dimension), and records several torsion-sensitive obstruction templates together with the precise gaps that currently prevent them from becoming unconditional obstructions. Two persistent obstacles are (i) the deck action of Γ on $\widetilde{M}$ is free, so involution fixed-point methods do not apply directly, and (ii) Cartan–Leray collapse with trivial coefficients yields a graded-algebra Poincaré duality pairing but does not automatically upgrade to a full $\mathrm{PD}_n(\mathbb{Q})$ or $\mathbb{Q}$-duality-group statement with arbitrary coefficient modules.

Evidence and scope of the computational artifacts used in the Introduction. The Introduction cites compute artifacts only where explicitly indicated in figure captions or in-text c[1]mmands. Their role is to document the logical landscape of torsion-sensitive templates under free actions, and to illustrate elementary parity phenomena that can occur in $\mathbb{Z}/2$-equivariant chain-complex models even when fixed points are absent. These artifacts are not used to claim a contradiction for any specific lattice, and no later numerical comparison is intended to depend on unstated preprocessing choices.

1.2 Problem statement, standing assumptions, and conventions

We fix the following standing assumptions.

Standing assumptions. Let G be a connected real semisimple Lie group with finite center, let $K < G$ be a maximal compact subgroup, and write $X = G/K$ for the associated symmetric space of dimension $d = \dim X$. Let $\Gamma < G$ be a uniform lattice, meaning that Γ is discrete and $\Gamma\backslash G$ is compact. We assume that Γ contains an element of order 2.

Problem 1.1. Under the standing assumptions above, does there exist a *closed orientable smooth* manifold M such that $\pi_1(M) \cong \Gamma$ and the universal cover $\widetilde{M}$ is rationally acyclic,

$$H_i(\widetilde{M};\mathbb{Q}) = 0 \text{ for all } i > 0, \qquad H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}?$$

The hypothesis on $\widetilde{M}$ is strictly weaker than contractibility and does not imply that M is aspherical. Accordingly, the arguments in this paper avoid replacing M by a $K(\Gamma,1)$ and instead work with group (co)homology with trivial coefficients, together with chain-level finiteness statements forced by the existence of a finite CW structure on M.

Remark on nonorientable manifolds. Several statements in later sections naturally admit twisted-coefficient analogues using the orientation character. However, until the twisted-coefficient Cartan–Leray identifications are proved in the required generality, the falsification targets recorded in this Introduction are stated for orientable M.

1.3 Cartan–Leray collapse with trivial coefficients

Consider the regular covering $\widetilde{M} \to M$ with deck group Γ. For trivial $\mathbb{Q}$-coefficients, the Cartan–Leray spectral sequence in homology has the form

$$E^2_{p,q} = H_p\big(\Gamma; H_q(\widetilde{M};\mathbb{Q})\big) \Rightarrow H_{p+q}(M;\mathbb{Q}).$$

[1] o

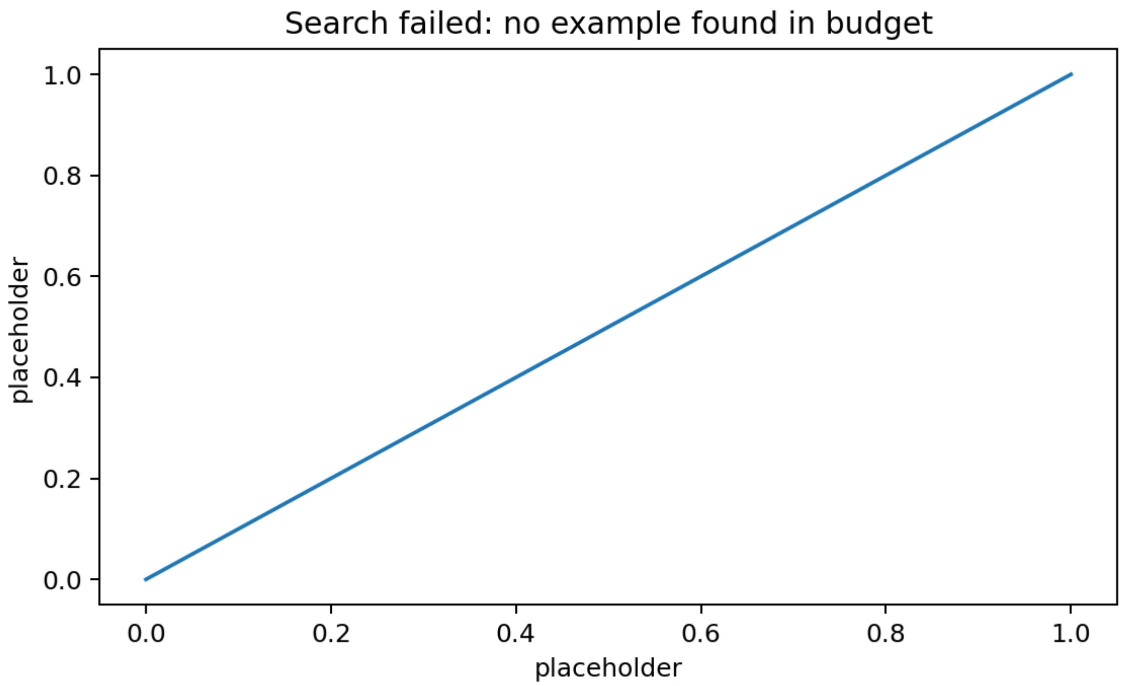

Figure 1: A compute-backed illustration for free C_2-actions on a $\mathbb{Q}$-acyclic chain complex: the same free equivariant model can present different behavior over $\mathbb{Q}$ versus $\mathbb{F}_2$, underscoring that rational invariants can be insensitive to even-index information while mod-2 features can remain visible at chain level. Provenance: dataset C_2-free $\mathbb{Q}$-acyclic chain complex (JSON)[3] and figure figures/betti_q_vs_f2.png.

Under the assumption of Problem 1.1, one has $H_q(\widetilde{M};\mathbb{Q}) = 0$ for $q > 0$ and $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$, with trivial Γ-action because $\widetilde{M}$ is connected. The spectral sequence collapses at E^2 and yields natural isomorphisms

$$H_*(M;\mathbb{Q}) \cong H_*(\Gamma;\mathbb{Q}), \qquad H^*(M;\mathbb{Q}) \cong H^*(\Gamma;\mathbb{Q}). \tag{1}$$

These identifications are the basic reason that $\mathbb{Q}$-acyclicity of the universal cover yields group-theoretic restrictions without assuming asphericity.

1.4 Poincaré duality consequences and the limits of what is unconditional

Let $n = \dim M$. Since M is assumed orientable in Problem 1.1, it carries a nonzero rational fundamental class $[M]_{\mathbb{Q}} \in H_n(M;\mathbb{Q})$, and cap product induces Poincaré duality isomorphisms

$$H^i(M;\mathbb{Q}) \xrightarrow{\cap [M]_{\mathbb{Q}}} H_{n-i}(M;\mathbb{Q}).$$

Transporting along (1) produces a distinguished class $\mu \in H_n(\Gamma;\mathbb{Q})$ and a nondegenerate pairing

$$H^i(\Gamma;\mathbb{Q}) \times H^{n-i}(\Gamma;\mathbb{Q}) \to \mathbb{Q}, \qquad (a,b) \mapsto \langle a \smile b, \mu \rangle. \tag{2}$$

This is an unconditional conclusion about *trivial-coefficient* cohomology. It should be distinguished from stronger group-theoretic duality statements (for example, $\mathrm{PD}_n(\mathbb{Q})$ or $\mathbb{Q}$-duality with arbitrary $\mathbb{Q}\Gamma$-modules), which typically require additional finiteness or domination hypotheses on the Γ-complex underlying $\widetilde{M}$ beyond bare $\mathbb{Q}$-acyclicity.

1.5 A necessary dimension constraint and how virtual torsion-freeness is used

Write $d = \dim X$. Since G is linear, a lattice $\Gamma < G$ is a finitely generated linear group. By Selberg's lemma, Γ is virtually torsion-free, so there exists a torsion-free subgroup $\Delta < \Gamma$ of finite index. The quotient $\Delta\backslash X$ is a closed aspherical manifold of dimension d, hence $\mathrm{cd}_{\mathbb{Q}}(\Delta) = d$. Cohomological dimension over $\mathbb{Q}$ is invariant under passage to finite-index subgroups, so

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) = \mathrm{cd}_{\mathbb{Q}}(\Delta) = d. \tag{3}$$

The only use of the finite-index torsion-free subgroup in this Introduction is to justify (3), and thereby relate any putative manifold dimension to $\dim X$. Now suppose that M is a closed manifold with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Choose a finite CW structure on M. The cellular chain complex $C_*^{\mathrm{cell}}(\widetilde{M};\mathbb{Q})$ is a chain complex of free $\mathbb{Q}\Gamma$-modules concentrated in degrees $0,\dots,n$. Rational acyclicity implies that the augmented complex

$$\cdots \to C_2^{\mathrm{cell}}(\widetilde{M};\mathbb{Q}) \to C_1^{\mathrm{cell}}(\widetilde{M};\mathbb{Q}) \to C_0^{\mathrm{cell}}(\widetilde{M};\mathbb{Q}) \to \mathbb{Q} \to 0$$

is exact on underlying $\mathbb{Q}$-vector spaces, hence provides a free resolution of the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ of length n. Consequently $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \le n$, and therefore

$$\dim X = d \le n = \dim M. \tag{4}$$

Because M is orientable, $[M]_{\mathbb{Q}} \neq 0 \in H_n(M;\mathbb{Q})$, so by (1) one has $H_n(\Gamma;\mathbb{Q}) \neq 0$. If $n > d$, then $n > \mathrm{cd}_{\mathbb{Q}}(\Gamma)$ by (3), which forces $H_n(\Gamma;\mathbb{Q}) = 0$, a contradiction. Therefore $n \le d$. Together with (4), this yields the necessary equality

$$\dim M = n = d = \dim(G/K) \tag{5}$$

for any orientable solution to Problem 1.1.

1.6 The free action obstacle and possible circumventions

The presence of 2-torsion suggests that any negative answer to Problem 1.1 should ultimately depend on 2-primary or integral information, since many rational invariants ignore finite-subgroup phenomena. A classical source of 2-primary constraints is Smith theory for involutions, which relates the $\mathbb{F}_2$-homology of a $\mathbb{Z}/2$-space to that of its fixed-point set. In the present problem, the action of Γ on $\widetilde{M}$ is the deck action, hence free. In particular, any nontrivial element of order 2 has empty fixed-point set on $\widetilde{M}$. This blocks direct fixed-point arguments on $\widetilde{M}$ and explains why several torsion-sensitive obstruction templates in later sections are conditional. Circumvention strategies discussed later in the manuscript are framed as research directions rather than established tools in this setting. They include replacing $\widetilde{M}$ by an auxiliary Γ-space with controlled fixed-point behavior (for example, a boundary or compactification), and using torsion-sensitive invariants that do not require fixed-point sets. The compute-backed illustration in Figure 1 is included only to emphasize, at the level of free equivariant chain models, how parity phenomena can persist over $\mathbb{F}_2$ despite being invisible over $\mathbb{Q}$.

1.7 Transfer maps and integral constraints: a conditional template

A basic mechanism by which torsion can influence top-degree classes is the interaction between restriction and transfer across finite-index subgroups. Let $\Delta \le \Gamma$ be a subgroup of finite index $m = [\Gamma : \Delta]$. For coefficients A where transfer is defined (for example, for trivial coefficients), there are natural maps

$$\mathrm{res}\colon H_d(\Gamma;A) \to H_d(\Delta;A), \qquad \mathrm{tr}\colon H_d(\Delta;A) \to H_d(\Gamma;A),$$

with

$$\mathrm{tr} \circ \mathrm{res} = m \cdot \mathrm{id}\colon H_d(\Gamma;A) \to H_d(\Gamma;A). \tag{6}$$

When m is even, (6) forces $\mathrm{tr}(\mathrm{res}(x)) \in 2\,H_d(\Gamma;A)$ for all x. In the setting of Problem 1.1, the orientable case yields a rational top class $\mu \in H_d(\Gamma;\mathbb{Q})$ via (1) and (5). A torsion-sensitive obstruction along the lines of (6) requires an additional integrality input, namely a theorem or hypothesis that produces an integral lift $\mu_{\mathbb{Z}} \in H_d(\Gamma;\mathbb{Z})$ compatible with the manifold structure.

Compatible integral top-degree class. In this manuscript, a "compatible integral top-degree class" means an element $\mu_{\mathbb{Z}} \in H_d(\Gamma;\mathbb{Z})$ whose image in $H_d(\Gamma;\mathbb{Q})$ equals μ, and which corresponds under (1) to the image of an integral fundamental class of a closed orientable manifold model under the classifying map $M \to B\Gamma$.

Additional hypothesis (primitivity). The transfer template can be combined with divisibility constraints only after adding a further hypothesis asserting that the chosen compatible lift $\mu_{\mathbb{Z}}$ is primitive in $H_d(\Gamma;\mathbb{Z})$. Primitivity is not proved here from $\mathbb{Q}$-acyclicity alone.

Operational check given a primitive compatible lift. Given a candidate lattice Γ and a torsion-free finite-index subgroup Δ, one would test whether the restriction $\mathrm{res}(\mu_{\mathbb{Z}}) \in H_d(\Delta;\mathbb{Z})$ maps back under transfer to an even multiple of $\mu_{\mathbb{Z}}$ when $[\Gamma : \Delta]$ is even, and compare this with primitivity. This remains a conditional template here because the required promotion from $\mu \in H_d(\Gamma;\mathbb{Q})$ to a primitive geometrically compatible $\mu_{\mathbb{Z}}$ is not proved from $\mathbb{Q}$-acyclicity.

1.8 Conjectural targets and falsification criteria

The guiding expectation is that 2-torsion in a uniform lattice is incompatible with being the fundamental group of an orientable closed manifold with $\mathbb{Q}$-acyclic universal cover.

Conjecture 1.2 (C1: nonrealizability expectation)**.** *Under the standing assumptions, there is no orientable closed smooth manifold M with $\pi_1(M) \cong \Gamma$ whose universal cover is $\mathbb{Q}$-acyclic.*

Conjecture 1.3 (C2: integral promotion hypothesis (conditional input))**.** *Assume the hypotheses of Problem 1.1 and write $d = \dim X = \dim M$. The rational top class $\mu \in H_d(\Gamma;\mathbb{Q})$ induced by the fundamental class admits a geometrically compatible integral lift $\mu_{\mathbb{Z}} \in H_d(\Gamma;\mathbb{Z})$ as described above.*

Research program for torsion-sensitive obstructions under free deck actions. Beyond Conjectures 1.2 and 1.3, later sections record a set of torsion-sensitive templates that would obstruct realizability after adding explicit auxiliary inputs. The intended falsifiable sub-targets include the following: first, identifying auxiliary Γ-spaces naturally associated to G or to a manifold model of $B\Gamma$ on which involutions can have nonempty fixed sets and for which Smith-type conclusions can be stated; second, formulating coefficient-level obstruction statements that are genuinely 2-primary (for example, mod-2 or integral) and remain meaningful under free actions; third, isolating verifiable conditions under which a $\mathbb{Q}$-acyclic free Γ-complex supporting a finite $\mathbb{Q}\Gamma$-free resolution upgrades to a stronger duality or finiteness property needed by centralizer and cohomological-dimension comparisons. The present manuscript records these as open directions, and it treats any obstruction that depends on such inputs as conditional. Conjecture 1.2 is intended to be operationally falsifiable: a counterexample would consist of an explicit uniform lattice Γ with 2-torsion together with an orientable closed manifold M satisfying Problem 1.1 and the necessary dimension constraint (5). Conjecture 1.3 is recorded to make explicit one integrality input that would be needed to convert transfer-based heuristics into checkable obstructions.

1.9 Organization

The next sections develop two strands. First, we formalize the algebraic consequences of $\mathbb{Q}$-acyclicity for chain complexes over $\mathbb{Q}\Gamma$, keeping careful track of what follows from Cartan–Leray collapse with trivial coefficients and what requires additional finiteness input. Second, we discuss torsion-sensitive approaches, emphasizing the free action obstacle and the resulting need for auxiliary Γ-spaces, Borel-type constructions, or other invariants that can be accessed without fixed points. The purpose of the Introduction is to make the unconditional constraints and the conditional dependencies explicit at the outset.

2 Problem formulation and standing hypotheses

This chapter fixes notation and formulates the central realizability problem. The guiding tension is the following. Uniform lattices in real semisimple Lie groups carry strong algebraic structure, while a compact manifold whose universal cover is rationally acyclic forces stringent homological constraints on its fundamental group via the cellular chain complex of the universal cover. The subsequent chapters will use this tension to develop an obstruction framework. No existence or non-existence statement is assumed at the outset.

2.1 Algebraic Setup and 2-Torsion Hypothesis

Let G be a real semisimple Lie group. For the present exposition, "semisimple" is taken in the standard Lie-theoretic sense, namely that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ has no nonzero abelian ideals. We do not assume G is simply connected.

A subgroup $\Gamma < G$ is *discrete* if the induced topology on Γ is discrete. A discrete subgroup $\Gamma < G$ is called a *lattice* if G/Γ admits a finite G-invariant Borel measure, equivalently if Γ has finite covolume with respect to a Haar measure on G. The lattice is *uniform* (or *cocompact*) if G/Γ is compact.

We will focus on lattices that contain involutions. Define the 2-torsion subset

$$T_2(\Gamma) := \{\gamma \in \Gamma : \gamma \neq 1,\ \gamma^2 = 1\}.$$

An element of $T_2(\Gamma)$ is an involution in Γ. The first standing hypothesis fixes the presence of such elements.

Standing Hypothesis 1 (C1: 2-torsion in the lattice). $\Gamma < G$ is a uniform lattice and $T_2(\Gamma) \neq \varnothing$.

Hypothesis 1 is a selection criterion rather than a conclusion. In particular, we make no assumptions about the size of $T_2(\Gamma)$, nor about the conjugacy structure of involutions in Γ.

The interest in 2-torsion is partly motivated by fixed point phenomena for involutions acting on manifolds or complexes and the special role of characteristic 2 in equivariant homology. In this manuscript, these ideas serve as motivation for potential obstructions, but the present chapter only records the basic group-theoretic input.

2.2 Geometric Target and the Realizability Question

All manifolds in this work are assumed to be connected. A *closed* manifold means compact and without boundary. For a manifold M, its universal cover is denoted $\widetilde{M}$, and its fundamental group is $\pi_1(M)$.

Rational acyclicity. A path-connected space X is *rationally acyclic* (or $\mathbb{Q}$-*acyclic*) if its reduced singular homology with rational coefficients vanishes:

$$\widetilde{H}_i(X;\mathbb{Q}) = 0 \quad \text{for all } i \geq 0.$$

Equivalently, $H_0(X;\mathbb{Q}) \cong \mathbb{Q}$ and $H_i(X;\mathbb{Q}) = 0$ for all $i > 0$.

The target class of manifolds. We are interested in closed manifolds M such that $\widetilde{M}$ is $\mathbb{Q}$-acyclic. This condition is strictly weaker than asphericity (contractibility of $\widetilde{M}$) and is compatible, at least formally, with the possibility that $\widetilde{M}$ has nontrivial higher homotopy groups while remaining invisible to rational homology.

The fundamental group $\Gamma = \pi_1(M)$ acts freely and properly discontinuously on $\widetilde{M}$ by deck transformations. The free action is compatible with torsion in Γ because a nontrivial finite-order deck transformation can act without fixed points; thus the presence of 2-torsion in Hypothesis 1 does not contradict the manifold setting.

Central realizability problem. The main question motivating this manuscript is an existence problem.

Problem 2.1 (Realizability with 2-torsion)**.** Under Hypothesis 1, does there exist a closed manifold M such that $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ is $\mathbb{Q}$-acyclic?

At the level of this chapter, we do not assume that such manifolds exist, nor do we assume they do not exist. In particular, we do not take as input any classification of lattices or any known obstruction theorem that would settle Problem 2.1 in general.

A conjectural nonexistence principle. The obstruction philosophy explored later suggests that the combination of "lattice" and "$\mathbb{Q}$-acyclic universal cover" should force strong duality constraints on the group, and these constraints may be incompatible with the presence of 2-torsion in a uniform lattice. This motivates the following explicit conjectural formulation.

Conjecture 2.2 (C5: conjectural obstruction to realizability)**.** *Assume Hypothesis 1. There is no closed manifold M with $\pi_1(M) \cong \Gamma$ and with $\widetilde{M}$ rationally acyclic.*

Conjecture 2.2 is presented as an open question rather than an established result. When evidence is unavailable, the manuscript treats Conjecture 2.2 as a programmatic target: to identify explicit algebraic conditions, implied by the manifold hypothesis, that can be checked against general properties of uniform lattices with involutions.

Why rational acyclicity is a natural input. The condition that $\widetilde{M}$ is $\mathbb{Q}$-acyclic implies that the cellular chain complex of $\widetilde{M}$ (for a finite CW-structure on M) becomes an exact complex after tensoring with $\mathbb{Q}$. Since the action of Γ on $\widetilde{M}$ is free, this chain complex can be regarded as a finite complex of free $\mathbb{Z}\Gamma$-modules, hence also of free $\mathbb{Q}\Gamma$-modules. This observation suggests a route from geometry to algebra: one expects to obtain a finite length resolution of the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ and, in favorable circumstances, a form of rational duality reminiscent of Poincaré duality. In the present chapter we record this as motivation only; the precise algebraic consequences will be derived later under clearly stated additional hypotheses.

2.3 Standing Hypotheses and Conventions

We next fix a standard set of conventions for the ambient Lie group and for the smooth category. These are not presented as consequences of the sources or as derived facts, but as a framework that simplifies later arguments.

Standing Hypothesis 2 (C2: ambient Lie group conventions)*.* The Lie group G is connected, has finite center, and has no nontrivial compact factors.

Hypothesis 2 isolates the noncompact semisimple directions relevant for uniform lattices and removes inessential compact components that would otherwise complicate notation.

Standing Hypothesis 3 (C3: lattice conventions)*.* The subgroup $\Gamma < G$ is discrete and uniform, so that G/Γ is compact.

In particular, Γ is finitely generated. We do not assume torsion-freeness; Hypothesis 1 explicitly forces the presence of elements of order 2.

Standing Hypothesis 4 (C4: manifold conventions)*.* All manifolds are smooth, connected, closed, and of finite dimension. Given such a manifold M, the universal cover $\widetilde{M}$ is taken with the lifted smooth structure and the standard free, properly discontinuous action of $\pi_1(M)$ by deck transformations.

Notation for homology and coefficients. Homology and cohomology are singular unless stated otherwise. For a space X and a ring R, we write $H_*(X;R)$ and $H^*(X;R)$. Group cohomology of a discrete group Γ with coefficients in a (left) Γ-module A is denoted $H^*(\Gamma;A)$. Rational coefficients will be used systematically when discussing the acyclicity assumption.

Orientation data. If M is a closed manifold, its orientation local system determines a homomorphism $\omega_M : \pi_1(M) \to \{\pm 1\}$. When $\pi_1(M) \cong \Gamma$, we view ω_M as an associated orientation character on Γ. Since we are not assuming M exists a priori, ω_M should be regarded as auxiliary data that may arise in a realizability argument.

Scope of what is and is not claimed. The standing hypotheses specify the input data for Problem 2.1. They do not include any quantitative restrictions, such as bounds on the dimension of a realizing manifold, restrictions on covolume, or counts of torsion elements. Moreover, no claim is made in this chapter about the existence of manifolds realizing a given lattice or about the nonexistence of such manifolds; those questions are encapsulated in Problem 2.1 and Conjecture 2.2.

Planned obstruction perspective. Assuming a manifold M as in Problem 2.1 exists, one expects the free Γ-action on $\widetilde{M}$ and the $\mathbb{Q}$-acyclicity of $\widetilde{M}$ to impose two types of constraints. First, the chain-level finiteness of $\widetilde{M}$ as a Γ-space suggests finiteness properties for Γ over $\mathbb{Q}$ (for example, the existence of finite length free $\mathbb{Q}\Gamma$-resolutions). Second, Poincaré duality on M suggests a duality pattern that should be visible in an appropriate form of group (co)homology with twisted coefficients. Because Γ may have torsion, the precise formulation of these constraints requires care and will be developed later with explicit hypotheses and proofs rather than being taken for granted here.

3 Manifold models with rationally acyclic universal cover

This chapter records the manifold-theoretic input used throughout the obstruction program. The global question concerns groups that occur as fundamental groups of closed manifolds whose universal cover is acyclic over $\mathbb{Q}$. The main purpose here is to fix terminology, isolate the algebraic consequences that are stable under rationalization, and clarify how the presence of torsion in Γ interacts with such manifold models.

3.1 Basic definitions and standing conventions

We work in the topological category unless stated otherwise. A *closed manifold* means a compact, connected, boundaryless n-manifold M.

Definition 3.1 (Rationally acyclic universal cover)**.** Let M be a closed manifold with universal covering map $p : \widetilde{M} \to M$. We say that M has $\mathbb{Q}$*-acyclic universal cover* if the reduced singular homology of $\widetilde{M}$ vanishes with rational coefficients, namely

$$\widetilde{H}_i(\widetilde{M};\mathbb{Q}) = 0 \qquad \text{for all } i \geq 0.$$

Equivalently, $H_i(\widetilde{M};\mathbb{Q}) = 0$ for all $i > 0$ and $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$.

This condition is strictly weaker than contractibility. It does not imply vanishing of higher homotopy groups. In particular, the $\mathbb{Q}$-acyclicity condition is compatible with nontrivial torsion phenomena in integral homology. We write $\Gamma := \pi_1(M)$, acting on $\widetilde{M}$ by deck transformations. The action is free and properly discontinuous, and $M \cong \widetilde{M}/\Gamma$.

Two coefficient systems. In later chapters we switch between (i) singular (co)homology of spaces with coefficients in $\mathbb{Q}$ and (ii) group (co)homology of Γ with coefficients in $\mathbb{Q}$ viewed as the trivial $\mathbb{Q}[\Gamma]$-module. The bridge between these viewpoints is provided by Γ-equivariant chains on $\widetilde{M}$. To avoid confusion, we reserve $H^*(\Gamma;\mathbb{Q})$ and $H_*(\Gamma;\mathbb{Q})$ for group (co)homology, and $H^*(X;\mathbb{Q})$ and $H_*(X;\mathbb{Q})$ for singular (co)homology.

3.2 Equivariant chain complexes and the Cartan-Leray spectral sequence

Fix a Γ-CW structure on $\widetilde{M}$ induced by a triangulation of M (or any Γ-equivariant cellular structure). Let $C_*(\widetilde{M};\mathbb{Q})$ be the resulting cellular chain complex with rational coefficients. Each $C_i(\widetilde{M};\mathbb{Q})$ is a free, finitely generated $\mathbb{Q}[\Gamma]$-module when M is a finite CW complex, which holds for closed manifolds. The $\mathbb{Q}$-acyclicity assumption is the statement that the augmented chain complex

$$\cdots \to C_2(\widetilde{M};\mathbb{Q}) \to C_1(\widetilde{M};\mathbb{Q}) \to C_0(\widetilde{M};\mathbb{Q}) \to \mathbb{Q} \to 0$$

is exact. Since the differentials are $\mathbb{Q}[\Gamma]$-linear, vanishing of $H_i(\widetilde{M};\mathbb{Q})$ implies that this augmented complex is exact as a complex of $\mathbb{Q}[\Gamma]$-modules. It is generally not split exact as a complex of $\mathbb{Q}[\Gamma]$-modules, and it need not be exact or split exact over $\mathbb{Z}[\Gamma]$. A basic object used repeatedly is the complex of coinvariants $C_*(\widetilde{M};\mathbb{Q})_\Gamma := \mathbb{Q} \otimes_{\mathbb{Q}[\Gamma]} C_*(\widetilde{M};\mathbb{Q})$, which computes the cellular homology of M with coefficients in $\mathbb{Q}$. Dually, one considers cochains $\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(C_*(\widetilde{M};\mathbb{Q}),\mathbb{Q})$.

Lemma 3.2 (Rational homology of M from equivariant chains)**.** *Let M be a closed manifold with fundamental group Γ and universal cover $\widetilde{M}$. For any i, there are natural identifications*

$$H_i\big(\mathbb{Q} \otimes_{\mathbb{Q}[\Gamma]} C_*(\widetilde{M};\mathbb{Q})\big) \cong H_i(M;\mathbb{Q}), \qquad H^i\big(\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(C_*(\widetilde{M};\mathbb{Q}),\mathbb{Q})\big) \cong H^i(M;\mathbb{Q}).$$

Proof. This is the standard cellular chain description of the quotient CW complex $M \cong \widetilde{M}/\Gamma$ with coefficients in $\mathbb{Q}$. □

The group cohomology $H^*(\Gamma;\mathbb{Q})$ is defined using any projective resolution of the trivial $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$. When $\widetilde{M}$ is contractible, $C_*(\widetilde{M};\mathbb{Q})$ provides such a resolution. Under the weaker hypothesis of $\mathbb{Q}$-acyclicity, the augmented complex is still exact as a complex of $\mathbb{Q}[\Gamma]$-modules, so it is a finite free $\mathbb{Q}[\Gamma]$-resolution of $\mathbb{Q}$ of length $\leq n$.

Proposition 3.3 (Acyclic cover as finite free resolution)**.** *Let M be a closed manifold with $\Gamma = \pi_1(M)$ and assume $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then the augmented $\mathbb{Q}[\Gamma]$-chain complex $C_*(\widetilde{M};\mathbb{Q}) \to \mathbb{Q}$ is a bounded complex of finitely generated free $\mathbb{Q}[\Gamma]$-modules, exact as a complex of $\mathbb{Q}[\Gamma]$-modules. Moreover, applying $\mathbb{Q}\otimes_{\mathbb{Q}[\Gamma]} -$ yields the cellular chain complex $C_*(M;\mathbb{Q})$, so in particular the augmentation becomes exact (i.e., $H_0 \cong \mathbb{Q}$) after tensoring.*

Proof. Exactness of $C_*(\widetilde{M};\mathbb{Q}) \to \mathbb{Q}$ on underlying $\mathbb{Q}$-vector spaces is the $\mathbb{Q}$-acyclicity assumption. Exactness of the same sequence as a complex of $\mathbb{Q}[\Gamma]$-modules follows because kernels and images of $\mathbb{Q}[\Gamma]$-linear maps are $\mathbb{Q}[\Gamma]$-submodules and exactness is checked on the underlying abelian groups. Tensoring with $\mathbb{Q}$ over $\mathbb{Q}[\Gamma]$ corresponds to taking coinvariants, yielding by Lemma 3.2 the complex computing $H_*(M;\mathbb{Q})$; in particular the resulting augmentation map has $H_0 \cong \mathbb{Q}$. ☐

The relationship between $H^*(\Gamma;\mathbb{Q})$ and $H^*(M;\mathbb{Q})$ is encoded in the Cartan-Leray spectral sequence for the covering $\widetilde{M} \to M$. For a $\mathbb{Q}[\Gamma]$-module A, the spectral sequence has $E_2^{p,q} \cong H^p(\Gamma; H^q(\widetilde{M};A))$ and converges to $H^{p+q}(M;A)$. When $A = \mathbb{Q}$ is trivial and $\widetilde{M}$ is $\mathbb{Q}$-acyclic, the $q > 0$ terms vanish, yielding $E_2^{p,0} \cong H^p(\Gamma;\mathbb{Q})$ and convergence to $H^p(M;\mathbb{Q})$. This yields natural isomorphisms $H^*(M;\mathbb{Q}) \cong H^*(\Gamma;\mathbb{Q})$ under these hypotheses.

3.3 Rational Poincaré duality constraints

Closed manifolds impose Poincaré duality on the homology of M with coefficients in a field, and they impose a form of duality on Γ when combined with suitable hypotheses on $\widetilde{M}$. Since our objective compares lattices with torsion to such manifold groups, the rational duality consequences are central. Let M be a closed, connected n-manifold. If M is orientable, there is a fundamental class $[M] \in H_n(M;\mathbb{Q})$ and a perfect pairing

$$H^i(M;\mathbb{Q}) \times H^{n-i}(M;\mathbb{Q}) \to H^n(M;\mathbb{Q}) \cong \mathbb{Q}.$$

If M is nonorientable, then $H_n(M;\mathbb{Q}) = 0$ and $H^n(M;\mathbb{Q}) = 0$ in general. In that case Poincaré duality over $\mathbb{Q}$ is expressed using the orientation local system: there is a perfect pairing

$$H^i(M;\mathbb{Q}) \times H^{n-i}(M;\mathbb{Q}_\omega) \to H^n(M;\mathbb{Q}_\omega) \cong \mathbb{Q},$$

where $\mathbb{Q}_\omega$ denotes the rank-one local system determined by the orientation character.

Definition 3.4 (Orientation character)**.** Let M be a closed n-manifold with $\Gamma = \pi_1(M)$. The action of Γ on the local orientation system defines a homomorphism $\omega : \Gamma \to \{\pm 1\}$, called the *orientation character*. We view $\mathbb{Q}$ as a left $\mathbb{Q}[\Gamma]$-module via ω and denote it $\mathbb{Q}_\omega$.

When $\widetilde{M}$ is $\mathbb{Q}$-acyclic, the top reduced homology of $\widetilde{M}$ vanishes over $\mathbb{Q}$, so the fundamental class (with appropriate local coefficients) is detected on the quotient M. Nevertheless, the orientation character encodes the coefficient twisting that appears in duality statements.

Proposition 3.5 (Rational duality pattern for trivial coefficients)**.** *Let M be a closed n-manifold with $\Gamma = \pi_1(M)$ and assume that $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then cap product with the (possibly twisted) fundamental class yields natural isomorphisms*

$$H^i(M;\mathbb{Q}) \xrightarrow{\cong} H_{n-i}(M;\mathbb{Q}_\omega) \qquad \textit{for all } i.$$

Moreover, the Cartan-Leray spectral sequence for the covering $\widetilde{M} \to M$ with trivial coefficients $\mathbb{Q}$ collapses to yield natural isomorphisms $H^*(M;\mathbb{Q}) \cong H^*(\Gamma;\mathbb{Q})$.

Proof. Poincaré duality with local coefficients is standard for closed manifolds, and the displayed isomorphisms are the corresponding cap-product identifications. For the spectral sequence, $\mathbb{Q}$-acyclicity forces $H^q(\widetilde{M};\mathbb{Q}) = 0$ for $q > 0$, leaving only the $q = 0$ row $H^p(\Gamma;\mathbb{Q})$ converging to $H^p(M;\mathbb{Q})$; the edge maps are isomorphisms. ☐

For general coefficients A, an identification between $H^*(M;A)$ and $H^*(\Gamma;A)$ requires additional input (beyond $\mathbb{Q}$-acyclicity of $\widetilde{M}$) controlling how local systems on M correspond to $\mathbb{Q}[\Gamma]$-modules and how manifold duality interacts with the chosen $\mathbb{Q}[\Gamma]$-resolution. In this manuscript, trivial-coefficient identifications are used unconditionally, and extensions to nontrivial A are invoked only when explicitly stated.

Definition 3.6 ($\mathbb{Q}$-Poincaré duality group (terminological convention))**.** A group Γ is called a *$\mathbb{Q}$-Poincaré duality group of dimension n* if it is of type FP over $\mathbb{Q}$ and there exists a character $\omega : \Gamma \to \{\pm 1\}$ such that for every left $\mathbb{Q}[\Gamma]$-module A that is finite dimensional over $\mathbb{Q}$, there are natural isomorphisms

$$H^i(\Gamma; A) \cong H_{n-i}(\Gamma; A \otimes_{\mathbb{Q}} \mathbb{Q}_\omega) \qquad \text{for all } i.$$

The obstruction strategy uses the following contrapositive, under hypotheses that ensure passage from manifold duality to group duality for the relevant coefficient systems. The $\mathbb{Q}$-acyclicity hypothesis alone yields the collapse $H^*(M; \mathbb{Q}) \cong H^*(\Gamma; \mathbb{Q})$ and the manifold Poincaré duality pairing, but it does not by itself imply the full $\mathbb{Q}$-Poincaré duality group property for all finite-dimensional A.

3.4 Finite subgroups and intermediate covers

The motivating question assumes that Γ contains 2-torsion. For any manifold model M with $\pi_1(M) = \Gamma$, the associated deck action of Γ on $\widetilde{M}$ is always free, so every finite-order element of Γ acts without fixed points on $\widetilde{M}$. The existence of such finite-order elements inside a given class of groups (for example, lattices in semisimple Lie groups) is a separate structural issue; here we record only the topological consequences assuming torsion is present. To connect torsion to rational acyclicity, it is convenient to isolate the following observation.

Lemma 3.7 (Free finite group actions preserve rational acyclicity)**.** *Let X be a connected $\mathbb{Q}$-acyclic space and let F be a finite group acting freely on X. Then the quotient X/F is $\mathbb{Q}$-acyclic.*

Proof. Let $p : X \to X/F$ be the finite covering projection and $\mathrm{tr} : H_*(X/F; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ the transfer. The standard transfer identity gives $p_* \circ \mathrm{tr} = |F| \, \mathrm{id}$ on $H_*(X/F; \mathbb{Q})$. For $i > 0$, the map $\mathrm{tr} : H_i(X/F; \mathbb{Q}) \to H_i(X; \mathbb{Q})$ has zero target because $H_i(X; \mathbb{Q}) = 0$, hence $|F| \, \mathrm{id} = 0$ on $H_i(X/F; \mathbb{Q})$. Since $|F|$ is invertible in $\mathbb{Q}$, it follows that $H_i(X/F; \mathbb{Q}) = 0$ for all $i > 0$. Also $H_0(X/F; \mathbb{Q}) \cong \mathbb{Q}$ because X/F is connected. □

The lemma shows that a free finite quotient of a $\mathbb{Q}$-acyclic space remains $\mathbb{Q}$-acyclic. For infinite groups with torsion, the situation is more rigid because torsion elements generate finite subgroups whose action on $\widetilde{M}$ is free. The quotient by such a finite subgroup is an intermediate covering of M. Specifically, every finite subgroup $F \le \Gamma$ yields a covering $\widetilde{M}/F \to M$. When Γ is infinite, this covering is infinite-sheeted (since $[\Gamma : F] = \infty$), so $\widetilde{M}/F$ is noncompact. The universal cover of $\widetilde{M}/F$ is $\widetilde{M}$ itself. In the case where F is normal in Γ, the deck group would be Γ/F; in general the deck group corresponds to $N_\Gamma(F)/F$ where $N_\Gamma(F)$ is the normalizer.

Consequences for cohomological periodicity. For a closed manifold M with $\mathbb{Q}$-acyclic $\widetilde{M}$, the manifold cohomology $H^*(M; \mathbb{Q})$ is finite dimensional and satisfies Poincaré duality with the appropriate twisting by $\mathbb{Q}_\omega$. When Γ has torsion, group cohomology $H^*(\Gamma; \mathbb{Q})$ can reflect contributions from finite subgroups. A common obstruction pattern is that these contributions force additional rational cohomology in degrees incompatible with Poincaré duality of any finite complex of the relevant dimension. The present chapter records the topological mechanism by which torsion enters the covering picture; the obstruction itself is algebraic and is treated elsewhere.

Proposition 3.8 (Restriction to finite subgroups yields rational acyclicity constraints)**.** *Let M be a closed manifold with $\Gamma = \pi_1(M)$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic. For any finite subgroup $F \le \Gamma$, the induced covering $\widetilde{M}/F \to M$ has total space a noncompact manifold (unless Γ is finite) whose universal cover is $\widetilde{M}$. In particular, $\widetilde{M}/F$ is $\mathbb{Q}$-acyclic.*

Proof. Since F acts freely on $\widetilde{M}$, the quotient $\widetilde{M}/F$ is a manifold and the projection $\widetilde{M}/F \to M$ is a covering map. When Γ is infinite, the index $[\Gamma : F]$ is infinite, so the covering is infinite-sheeted and $\widetilde{M}/F$ is noncompact. The universal cover of $\widetilde{M}/F$ is $\widetilde{M}$. Finally, $\widetilde{M}/F$ is $\mathbb{Q}$-acyclic by Lemma 3.7, applied to the free action of F on the connected $\mathbb{Q}$-acyclic space $\widetilde{M}$. □

The proposition emphasizes that finite subgroups cannot be ignored even though the global cover is simply connected. In the lattice setting, the existence of 2-torsion provides specific finite subgroups to which one can restrict equivariant chain complexes.

3.5 Manifold model packages and obstruction templates

The obstruction arguments in this manuscript use a finite Γ-equivariant chain model extracted from M. This subsection specifies the template that will be assumed whenever we say that Γ admits a "manifold model with $\mathbb{Q}$-acyclic universal cover".

Definition 3.9 (Manifold model package)**.** A *manifold model package of dimension* n for a group Γ consists of a closed n-manifold M with $\pi_1(M) \cong \Gamma$ and a chosen Γ-CW structure on the universal cover $\widetilde{M}$ such that $C_*(\widetilde{M};\mathbb{Q})$ is a bounded complex of finitely generated free $\mathbb{Q}[\Gamma]$-modules. The package is called *rationally acyclic* if $\widetilde{M}$ is $\mathbb{Q}$-acyclic.

Given such a package, there are three tests that will appear repeatedly. First, the Euler characteristic can be computed from the alternating sum of $\mathbb{Q}$-ranks of $C_i(\widetilde{M};\mathbb{Q})_\Gamma \cong C_i(M;\mathbb{Q})$, hence equals $\chi(M)$. This is robust and does not require asphericity. Second, the cap product pairing on $H^*(M;\mathbb{Q})$ is nondegenerate after twisting by $\mathbb{Q}_\omega$ in the nonorientable case. Any candidate rational cohomology ring extracted from a group calculation must be compatible with this pairing. Third, the Γ-equivariant chain complex $C_*(\widetilde{M};\mathbb{Q})$ provides a concrete object on which torsion in Γ acts. Restricting along an inclusion $F \hookrightarrow \Gamma$ for a finite subgroup F yields a free $\mathbb{Q}[F]$-chain complex, and its exactness over $\mathbb{Q}$ constrains how F can appear inside Γ from the perspective of rational homology.

4 Rational cohomological dimension and finite resolutions

We record the homological algebra that links the existence of a closed manifold with rationally acyclic universal cover to stringent finiteness and duality constraints on its fundamental group. The guiding theme is that rational acyclicity of the universal cover turns the cellular chain complex into a finite free resolution over the group ring, which places the group in a narrow range of rational cohomological dimensions. This chapter fixes conventions and isolates statements that will be invoked later when comparing lattices with rational Poincaré duality patterns.

4.1 Conventions and basic definitions

Let Γ be a discrete group. Write $\mathbb{Q}\Gamma$ for the rational group ring. By a (left) $\mathbb{Q}\Gamma$-module we mean a left module over this ring. The trivial $\mathbb{Q}\Gamma$-module is denoted $\mathbb{Q}$, with Γ acting trivially.

A projective (respectively free) resolution of $\mathbb{Q}$ is an exact complex of $\mathbb{Q}\Gamma$-modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Q} \longrightarrow 0$$

with each P_i projective (respectively free). A resolution is finite of length n if $P_i = 0$ for $i > n$.

The *rational cohomological dimension* $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ is the minimal integer n such that there exists a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}\Gamma$ of length n; if no such n exists, it is defined to be ∞. Equivalently, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$ exactly when $H^{n+1}(\Gamma; M) = 0$ for every $\mathbb{Q}\Gamma$-module M, where group cohomology is computed as

$$H^k(\Gamma; M) = \mathrm{Ext}^k_{\mathbb{Q}\Gamma}(\mathbb{Q}, M).$$

Similarly, group homology is

$$H_k(\Gamma; M) = \mathrm{Tor}^{\mathbb{Q}\Gamma}_k(\mathbb{Q}, M).$$

A finiteness condition convenient for geometric applications is the following. The group Γ is of type $\mathrm{FP}_{\mathbb{Q}}$ if $\mathbb{Q}$ admits a projective resolution by finitely generated $\mathbb{Q}\Gamma$-modules. It is of type $\mathrm{F}_{\mathbb{Q}}$ if it admits a $K(\Gamma, 1)$ with finite $\mathbb{Q}$-cellular chain complex. When Γ is the fundamental group of a finite CW complex, the universal cover provides a free resolution with finitely generated terms, and hence $\mathrm{F}_{\mathbb{Q}}$ implies $\mathrm{FP}_{\mathbb{Q}}$.

4.2 Universal covers as resolutions

Let M be a connected finite CW complex with fundamental group Γ, and let $\widetilde{M}$ be its universal cover. The cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ is a chain complex of free left $\mathbb{Q}\Gamma$-modules, finitely generated in each degree. The augmentation map $C_0(\widetilde{M};\mathbb{Q}) \to \mathbb{Q}$ induces an identification

$$H_0\big(C_*(\widetilde{M};\mathbb{Q})\big) \cong \mathbb{Q},$$

and the higher homology groups are precisely the rational homology of the universal cover:

$$H_i\big(C_*(\widetilde{M};\mathbb{Q})\big) \cong H_i(\widetilde{M};\mathbb{Q}).$$

Thus, when $\widetilde{M}$ is $\mathbb{Q}$-*acyclic*, meaning $H_i(\widetilde{M};\mathbb{Q}) = 0$ for all $i > 0$, the augmented complex

$$0 \longrightarrow C_n(\widetilde{M};\mathbb{Q}) \longrightarrow \cdots \longrightarrow C_1(\widetilde{M};\mathbb{Q}) \longrightarrow C_0(\widetilde{M};\mathbb{Q}) \longrightarrow \mathbb{Q} \longrightarrow 0$$

is exact. In particular, $C_*(\widetilde{M};\mathbb{Q})$ is a finite free $\mathbb{Q}\Gamma$-resolution of $\mathbb{Q}$, of length at most $\dim M$.

Two consequences will be used repeatedly.

First, if M is a closed n-manifold and $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) \le n,$$

and Γ is of type $\mathrm{FP}_{\mathbb{Q}}$.

Second, for any $\mathbb{Q}\Gamma$-module A, applying $\mathrm{Hom}_{\mathbb{Q}\Gamma}(-, A)$ to the resolution computes group cohomology by cochains on $\widetilde{M}$ with twisted coefficients:

$$H^k(\Gamma; A) \cong H^k\big(\mathrm{Hom}_{\mathbb{Q}\Gamma}(C_*(\widetilde{M};\mathbb{Q}), A)\big).$$

This identification is formal and does not rely on manifold structure, only on the $\mathbb{Q}$-acyclicity and finiteness of the CW model.

On torsion and rational coefficients. The passage to $\mathbb{Q}$ is essential when Γ has torsion, since $\mathbb{Z}\Gamma$-homological finiteness may fail even when rational finiteness persists. The $\mathbb{Q}\Gamma$-resolution coming from $\widetilde{M}$ exists regardless of whether Γ contains finite subgroups, provided the universal cover is $\mathbb{Q}$-acyclic and the base has finite CW type. The tradeoff is that $\mathbb{Q}\Gamma$ is no longer semisimple in general, and one must explicitly track projectivity and finite generation rather than rely on splitting arguments.

4.3 Rational cohomological dimension and vanishing ranges

We isolate a set of vanishing statements that can be read directly from a finite free resolution.

Let Γ admit a free $\mathbb{Q}\Gamma$-resolution of $\mathbb{Q}$ of length n. Then for every $\mathbb{Q}\Gamma$-module A,

$$H^k(\Gamma; A) = 0 \quad \text{for all } k > n.$$

In particular, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \le n$. If in addition there exists some A with $H^n(\Gamma; A) \neq 0$, then $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = n$. Geometrically, one expects nonvanishing in top degree when the resolution is realized by a closed n-manifold and one uses an appropriate dualizing module, but for torsion groups one must take care in formulating the correct coefficient system.

A useful heuristic is that the cohomological dimension over $\mathbb{Q}$ functions as a minimal possible dimension for models whose universal covers are rationally acyclic. Concretely, if Γ is realized as $\pi_1(M)$ for a finite CW complex M with $\widetilde{M}$ $\mathbb{Q}$-acyclic, then the dimension of M provides an explicit upper bound for $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$. Conversely, lower bounds on $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ obstruct low-dimensional realizations with rationally acyclic universal cover.

A computational reformulation. Let $P_\bullet \to \mathbb{Q}$ be a projective resolution. For each k, the group $H^k(\Gamma; A)$ can be computed as the cohomology of the cochain complex $\mathrm{Hom}_{\mathbb{Q}\Gamma}(P_\bullet, A)$. When $P_\bullet = C_*(\widetilde{M};\mathbb{Q})$, the chain groups are explicitly free, with ranks equal to the numbers of k-cells in M, and boundary maps induced by attaching maps. Thus, in principle, many vanishing questions can be reduced to algebraic properties of these boundary operators over $\mathbb{Q}\Gamma$, although in practice the noncommutativity of $\mathbb{Q}\Gamma$ quickly limits direct calculation.

4.4 Finite resolutions and rational duality patterns

The strongest restrictions appear when the finite free resolution comes from a closed manifold. Let M be a closed connected n-manifold with $\pi_1(M) = \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic. The manifold structure implies a form of Poincaré duality for M with local coefficients. For a left $\mathbb{Q}\Gamma$-module A, denote by $A^\vee = \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q})$ the $\mathbb{Q}$-linear dual, which becomes a right $\mathbb{Q}\Gamma$-module via $(\varphi \cdot \gamma)(a) = \varphi(\gamma a)$. Using the orientation character $\omega\colon \Gamma \to \{\pm 1\}$ (trivial when M is orientable), one defines a twisting of modules by ω. The duality statement can be expressed schematically as an isomorphism between cohomology and homology with complementary degrees and twisted coefficients.

For our purposes, the key point is conceptual rather than formal: the existence of M with $\widetilde{M}$ $\mathbb{Q}$-acyclic forces Γ to behave like a rational Poincaré duality group in degree n, except that torsion can introduce corrections that are invisible over $\mathbb{Z}$ but still constrain rational cohomology. In particular, one expects that a dualizing object should exist in the derived category of $\mathbb{Q}\Gamma$-modules, controlling top-degree cohomology and making the vanishing range sharp.

One concrete invariant derived from the resolution is the rational Euler characteristic. Since $C_i(\widetilde{M}; \mathbb{Q})$ is free of finite rank, one may define

$$\chi_{\mathbb{Q}}(\Gamma; M) = \sum_{i=0}^{n} (-1)^i \operatorname{rank}_{\mathbb{Q}\Gamma} C_i(\widetilde{M}; \mathbb{Q}).$$

When M is a finite CW complex with $\pi_1(M) = \Gamma$, this alternating sum is the usual Euler characteristic of M and hence depends on the chosen model. However, if $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then M is a rational homology manifold with $H_*(M; \mathbb{Q}) \cong H_*(\Gamma; \mathbb{Q})$ in low degrees, and comparisons between different models often become rigid. Later chapters will use Euler characteristic considerations in tandem with duality constraints.

Top degree phenomena. Even without writing the full duality theorem, one can draw a robust consequence. The resolution $C_*(\widetilde{M}; \mathbb{Q})$ has length n, so $H^k(\Gamma; A) = 0$ for $k > n$. If one can show that $H^n(\Gamma; A)$ is nonzero for a natural coefficient module A attached to the action of Γ on $\widetilde{M}$ (for example, a dualizing module built from compactly supported cochains on $\widetilde{M}$), then it follows that $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = n$.

This perspective is useful for obstruction theory: to rule out the existence of M, it suffices to show that Γ cannot have rational cohomological dimension at most n together with the required finiteness and duality properties. In the motivating question, Γ is assumed to be a uniform lattice in a semisimple real Lie group and to contain elements of order two. The lattice hypothesis gives strong information about group cohomology, while torsion interacts delicately with duality.

4.5 Implications for lattices and the obstruction program

We now articulate the way this chapter interfaces with the later analysis of uniform lattices.

Assume that a group Γ is the fundamental group of a closed n-manifold M whose universal cover is $\mathbb{Q}$-acyclic. Then the following package of properties holds.

First, Γ is of type $\mathrm{FP}_{\mathbb{Q}}$, since $\mathbb{Q}$ admits a finite free $\mathbb{Q}\Gamma$-resolution given by the cellular chains of $\widetilde{M}$. This is a structural finiteness property that can be compared to known finiteness properties of lattices.

Second, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$. For lattices in semisimple Lie groups, one typically has independent bounds and often equalities for cohomological dimensions (over various coefficient fields) in terms of symmetric space dimensions. Thus, the manifold dimension n is not a free parameter: it must be compatible with the lattice's intrinsic cohomological size.

Third, manifold topology supplies a duality pattern in degree n for cohomology with local coefficients. Over $\mathbb{Q}$, this pattern is often summarized by saying that Γ should be a rational duality group of dimension n in an appropriate sense. The presence of torsion in Γ means one must phrase this carefully, but the general expectation remains that the resolution is self-dual up to twist and shift.

Finally, the $\mathbb{Q}$-acyclicity of $\widetilde{M}$ is much stronger than mere contractibility modulo torsion. It implies, in particular, that all higher rational homotopy invariants that factor through rational homology vanish. While this does not immediately force $\widetilde{M}$ to be contractible, it removes a large class of potential universal covers and amplifies the relevance of cohomological obstructions.

The remainder of the paper will exploit this package by comparing it to the cohomology of uniform lattices. The obstruction strategy can be summarized as follows: derive necessary rational finiteness and duality properties from the existence of M with $\mathbb{Q}$-acyclic universal cover, then test whether a uniform lattice with nontrivial 2-torsion can satisfy these properties. The present chapter supplies the homological algebra needed to move from geometry to algebra in a controlled way.

Scope note. No classification of groups of finite rational cohomological dimension is attempted here. The role of the chapter is to fix a consistent language for resolutions over $\mathbb{Q}\Gamma$ and to make explicit the logical bridge from a rationally acyclic universal cover to constraints on group cohomology. Subsequent chapters will import additional structure specific to lattices, such as their relationship with symmetric spaces and arithmetic data, to sharpen these constraints.

5 Rational Poincare duality groups from closed manifolds

This chapter isolates the formal consequences of assuming that a closed, connected, boundaryless manifold M^n has universal cover $\widetilde{M}$ that is acyclic over $\mathbb{Q}$. The guiding point is that rational acyclicity imposes a strong duality pattern on the group (or on a torsion-free finite-index subgroup), and these patterns are rigid enough to be tested against lattice invariants in later chapters.

Throughout, M denotes a closed connected topological manifold of dimension n and $\Gamma = \pi_1(M)$. We denote by $\mathbb{Q}$ the trivial Γ-module and by $\mathbb{Q}_\omega$ the rational orientation module determined by the Γ-action on $H_n(\widetilde{M};\mathbb{Q})$ (defined below). Unless explicitly stated, cohomology and homology are taken with coefficients in $\mathbb{Q}$.

5.1 Acyclic universal covers and the orientation module

Rational acyclicity. We say that $\widetilde{M}$ is $\mathbb{Q}$-*acyclic* when

$$H_i(\widetilde{M};\mathbb{Q}) = 0 \quad \text{for all } i > 0, \qquad H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}.$$

Since $\widetilde{M}$ is simply connected and noncompact (unless Γ is finite), the $\mathbb{Q}$-acyclicity hypothesis is a strong restriction on the large-scale topology of the cover.

Orientation character and twisting. The classical orientation character of M yields a homomorphism $\omega\colon \Gamma \to \{\pm 1\}$ determined by whether a deck transformation preserves or reverses a chosen local orientation on $\widetilde{M}$. Over $\mathbb{Q}$ this gives a one-dimensional Γ-module $\mathbb{Q}_\omega$ whose underlying vector space is $\mathbb{Q}$ and with action $\gamma \cdot v = \omega(\gamma)v$. When M is orientable, ω is trivial and $\mathbb{Q}_\omega = \mathbb{Q}$.

Fundamental class in the cover. Even when $\widetilde{M}$ is noncompact, the universal cover carries a Γ-equivariant fundamental class in locally finite homology, and the Γ-action on the corresponding top-degree object is encoded by ω. For the purposes of rational group duality, it is enough to record that the orientation module $\mathbb{Q}_\omega$ is the natural target for duality isomorphisms.

5.2 From manifolds to rational Poincaré duality in group cohomology

This subsection extracts the algebraic duality pattern satisfied by Γ (or a torsion-free finite-index subgroup) from the hypothesis that $\widetilde{M}$ is $\mathbb{Q}$-acyclic.

Equivariant chain complex. Let $C_*(\widetilde{M};\mathbb{Q})$ be the singular chain complex with $\mathbb{Q}$-coefficients. Each $C_k(\widetilde{M};\mathbb{Q})$ is naturally a left $\mathbb{Q}[\Gamma]$-module via the deck action. The complex is augmented by $\epsilon\colon C_0 \to \mathbb{Q}$ inducing $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$. The $\mathbb{Q}$-acyclicity assumption implies that the augmented complex

$$\cdots \to C_2 \to C_1 \to C_0 \xrightarrow{\epsilon} \mathbb{Q} \to 0$$

is exact.

Lemma 5.1 (Finite rational projective dimension)**.** *Assume that $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then $\mathbb{Q}$ admits a resolution by $\mathbb{Q}[\Gamma]$-modules of length n which is built from the cellular (or singular) chains of $\widetilde{M}$.*

Proof. Choose a Γ-equivariant CW-structure on $\widetilde{M}$ lifting a finite CW-structure on M (possible since M is a compact manifold). The cellular chain modules $C_k^{\mathrm{cell}}(\widetilde{M};\mathbb{Q})$ are finitely generated free $\mathbb{Q}[\Gamma]$-modules for $0 \le k \le n$ and vanish for $k > n$. The augmented cellular chain complex is exact over $\mathbb{Q}$ by $\mathbb{Q}$-acyclicity and computes $H_*(\widetilde{M};\mathbb{Q})$. This yields a length-$n$ free resolution of $\mathbb{Q}$ as a $\mathbb{Q}[\Gamma]$-module. $\square$

Lemma 5.1 implies that group (co)homology of Γ with coefficients in any $\mathbb{Q}[\Gamma]$-module can be computed using a finite-length resolution. The next step is to express Poincaré duality of M in terms of this resolution.

Definition 5.2 (Rational Poincaré duality group)**.** A group G is a *Poincaré duality group of dimension n over* $\mathbb{Q}$, written $\mathrm{PD}_n(\mathbb{Q})$, if there exists a one-dimensional *right* $\mathbb{Q}[G]$-module D (the *dualizing module*) such that for every left $\mathbb{Q}[G]$-module A there are natural isomorphisms

$$H^k(G;A) \cong \mathrm{Tor}_{n-k}^{\mathbb{Q}[G]}(D,A) \quad \text{for all } k.$$

When $D \cong \mathbb{Q}$ with trivial action one speaks of an *oriented* $\mathrm{PD}_n(\mathbb{Q})$ group.

Proposition 5.3 (Duality from a $\mathbb{Q}$-acyclic universal cover)**.** *Let M^n be closed and connected with $\Gamma = \pi_1(M)$. If $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then any torsion-free finite-index subgroup $\Gamma_0 \le \Gamma$ is a $\mathrm{PD}_n(\mathbb{Q})$ group with dualizing module $\mathbb{Q}_{\omega|_{\Gamma_0}}$.*

Proof. Let $\Gamma_0 \le \Gamma$ be torsion-free of finite index and let $p\colon M_0 \to M$ be the corresponding finite covering with $\pi_1(M_0) = \Gamma_0$. The universal cover of M_0 is still $\widetilde{M}$, hence $\widetilde{M}$ is $\mathbb{Q}$-acyclic.

Choose a Γ_0-equivariant CW structure on $\widetilde{M}$ lifting a finite CW structure on M_0, and write

$$C_* := C_*^{\mathrm{cell}}(\widetilde{M};\mathbb{Q}).$$

Then each C_i is a finitely generated free left $\mathbb{Q}[\Gamma_0]$-module, $C_i = 0$ for $i > n$, and the augmentation $C_0 \to \mathbb{Q}$ is a quasi-isomorphism because $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Thus $C_* \to \mathbb{Q}$ is a finite free resolution, so Γ_0 is of type $\mathrm{FP}_{\mathbb{Q}}$.

Let

$$E^* := \mathrm{Hom}_{\mathbb{Q}[\Gamma_0]}(C_*, \mathbb{Q}[\Gamma_0]).$$

Since C_* is a free resolution of $\mathbb{Q}$, its cohomology computes group cohomology with group-ring coefficients:

$$H^k(E^*) \cong \mathrm{Ext}^k_{\mathbb{Q}[\Gamma_0]}(\mathbb{Q}, \mathbb{Q}[\Gamma_0]) = H^k(\Gamma_0; \mathbb{Q}[\Gamma_0]).$$

Step 1: identification with compactly supported cochains. For each i, pick a set S_i of orbit representatives of i-cells of $\widetilde{M}$, so $C_i \cong \bigoplus_{\sigma \in S_i} \mathbb{Q}[\Gamma_0]$. A $\mathbb{Q}[\Gamma_0]$-linear map $f\colon C_i \to \mathbb{Q}[\Gamma_0]$ is determined by the values $f(\sigma) \in \mathbb{Q}[\Gamma_0]$ for $\sigma \in S_i$. Writing

$$f(\sigma) = \sum_{g \in \Gamma_0} a_g\, g \qquad (a_g \in \mathbb{Q},\ \text{finite sum}),$$

define a finitely supported function on the orbit $\Gamma_0\sigma$ by $\varphi_f(g\sigma) = a_g$. Varying over S_i yields a finitely supported function on the set of i-cells, i.e. a cellular i-cochain with compact support. This construction is reversible and intertwines coboundary maps, giving an isomorphism of cochain complexes

$$\mathrm{Hom}_{\mathbb{Q}[\Gamma_0]}(C_*, \mathbb{Q}[\Gamma_0]) \;\cong\; C_c^*(\widetilde{M};\mathbb{Q}), \tag{7}$$

hence

$$H^k(\Gamma_0; \mathbb{Q}[\Gamma_0]) \;\cong\; H_c^k(\widetilde{M};\mathbb{Q}). \tag{8}$$

Step 2: compute $H^*(\Gamma_0;\mathbb{Q}[\Gamma_0])$. The manifold $\widetilde{M}$ is connected and noncompact of dimension n, and it is orientable (since it is simply connected). Poincaré duality for noncompact manifolds identifies compactly supported cohomology with ordinary homology, and the Γ_0-action on $H_c^n(\widetilde{M};\mathbb{Q})$ is given by the restricted orientation character $\omega|_{\Gamma_0}$:

$$H_c^k(\widetilde{M};\mathbb{Q}) \cong H_{n-k}(\widetilde{M};\mathbb{Q}) \quad \text{and} \quad H_c^n(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}_{\omega|_{\Gamma_0}}$$

as right $\mathbb{Q}[\Gamma_0]$-modules. Since $\widetilde{M}$ is $\mathbb{Q}$-acyclic, one has $H_{n-k}(\widetilde{M};\mathbb{Q}) = 0$ for $n-k>0$ and $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$. Therefore (8) implies

$$H^k(\Gamma_0;\mathbb{Q}[\Gamma_0]) = 0 \text{ for } k \neq n, \qquad H^n(\Gamma_0;\mathbb{Q}[\Gamma_0]) \cong \mathbb{Q}_{\omega|_{\Gamma_0}}.$$

Step 3: duality for all coefficients. Because each C_i is finitely generated free, there are natural identifications

$$\mathrm{Hom}_{\mathbb{Q}[\Gamma_0]}(C_i, A) \;\cong\; \mathrm{Hom}_{\mathbb{Q}[\Gamma_0]}(C_i, \mathbb{Q}[\Gamma_0]) \otimes_{\mathbb{Q}[\Gamma_0]} A$$

for every left $\mathbb{Q}[\Gamma_0]$-module A, hence $\mathrm{Hom}_{\mathbb{Q}[\Gamma_0]}(C_*, A) \cong E^* \otimes_{\mathbb{Q}[\Gamma_0]} A$. The standard spectral sequence for tensoring a cochain complex of right $\mathbb{Q}[\Gamma_0]$-modules with A gives

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathbb{Q}[\Gamma_0]}\big(H^q(E^*), A\big) \Rightarrow H^{p+q}(E^* \otimes_{\mathbb{Q}[\Gamma_0]} A).$$

Since $H^q(E^*) = 0$ for $q \neq n$ and $H^n(E^*) \cong \mathbb{Q}_{\omega|_{\Gamma_0}}$, this spectral sequence collapses and yields natural isomorphisms

$$H^k(\Gamma_0; A) \cong \mathrm{Tor}_{n-k}^{\mathbb{Q}[\Gamma_0]}\big(\mathbb{Q}_{\omega|_{\Gamma_0}}, A\big).$$

This is precisely Definition 5.2 with dualizing module $\mathbb{Q}_{\omega|_{\Gamma_0}}$. □

Proposition 5.3 is the principal structural input of this chapter. It reduces the geometric problem to a purely algebraic constraint: Γ must be virtually a $\mathrm{PD}_n(\mathbb{Q})$ group.

Consequences for (co)homological dimension. If Γ_0 is torsion-free and $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then Γ_0 has rational cohomological dimension n and satisfies $H^k(\Gamma_0;\mathbb{Q}) = 0$ for $k \neq 0, n$ when the dualizing module is trivial and Γ_0 has no nontrivial $\mathbb{Q}$-cohomology in intermediate degrees. In general, intermediate degrees can appear with nontrivial coefficients, and the correct invariant statement is the duality isomorphism of Definition 5.2.

5.3 What torsion does and does not obstruct

The motivating question for this project emphasizes the presence of 2-torsion in a uniform lattice. It is therefore important to distinguish what is forced by being a fundamental group of a closed manifold from what is forced by the additional $\mathbb{Q}$-acyclicity hypothesis.

Free action versus torsion. For any covering $\widetilde{M} \to M$ with deck group $\Gamma = \pi_1(M)$, the deck action is free by definition. This does *not* immediately imply that Γ is torsion-free, because in principle a finite-order homeomorphism of a noncompact manifold could act freely. Excluding such behavior typically requires additional fixed-point input (for example, Smith theory at a prime p for actions on mod-p acyclic spaces). Since the assumption here is only $\mathbb{Q}$-acyclicity, no prime-specific fixed point theorem can be applied without extra hypotheses.

Passage to torsion-free finite index. In many arithmetic contexts, lattices are known to be virtually torsion-free. Independently of that source, the manifold hypothesis alone implies that Γ contains a torsion-free subgroup of finite index whenever M admits a finite-sheeted cover that is a manifold with fundamental group torsion-free. This is not automatic for an arbitrary Γ with torsion, but in the setting of lattices, such torsion-free subgroups are expected to exist and can be taken as the group that acts freely and properly discontinuously on the same universal cover.

From the standpoint of this manuscript, the key point is the following: the $\mathbb{Q}$-acyclicity hypothesis is stable under passing to finite covers, since the universal cover does not change. Therefore, any torsion-free finite-index subgroup Γ_0 inherits the conclusion of Proposition 5.3. Consequently, the presence of 2-torsion in Γ is not itself a contradiction with the existence of M; the contradiction, when it exists, must come from incompatibility between (virtual) rational duality and lattice invariants that persist under passage to Γ_0.

Euler characteristic constraints. If Γ_0 is $\mathrm{PD}_n(\mathbb{Q})$ and oriented, then the rational Euler characteristic of Γ_0 is defined and equals the alternating sum of the dimensions of $H^k(\Gamma_0;\mathbb{Q})$ when these groups are finite-dimensional. For manifold groups, this Euler characteristic agrees with $\chi(M_0)$ for the corresponding finite cover $M_0 \to M$ with $\pi_1(M_0) = \Gamma_0$. Under additional finiteness assumptions one can relate $\chi(M_0)$ to $\chi(\widetilde{M})$, but for noncompact $\widetilde{M}$ the ordinary Euler characteristic is not the correct invariant and must be replaced by locally finite or L^2 variants. In later chapters we use this chapter only to justify that a duality pattern is present; numerical Euler characteristic equalities are treated as separate inputs.

5.4 A spectral sequence viewpoint and coefficient control

To connect duality to computations and obstructions, it is useful to express the $\mathbb{Q}$-acyclicity condition as a collapse statement for a standard spectral sequence.

Leray-Serre type input. View M as $E\Gamma \times_\Gamma \widetilde{M}$ with $E\Gamma$ contractible. There is a homology spectral sequence for the fibration $\widetilde{M} \to E\Gamma \times_\Gamma \widetilde{M} \to B\Gamma$ with

$$E^2_{p,q} \cong H_p\big(\Gamma; H_q(\widetilde{M};\mathbb{Q})\big) \Rightarrow H_{p+q}(M;\mathbb{Q}).$$

Under $\mathbb{Q}$-acyclicity, $H_q(\widetilde{M};\mathbb{Q}) = 0$ for $q > 0$ and $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ with trivial action. Thus $E^2_{p,q}$ vanishes for $q > 0$ and the spectral sequence collapses at E^2 giving

$$H_p(\Gamma;\mathbb{Q}) \cong H_p(M;\mathbb{Q}) \quad \text{for all } p.$$

This identification is compatible with cap products once the orientation module is inserted on the cohomological side.

Coefficient systems. The same collapse argument applies to twisted coefficients, but with $H_q(\widetilde{M};A)$ for a $\mathbb{Q}[\Gamma]$-module A replacing $H_q(\widetilde{M};\mathbb{Q})$. The $\mathbb{Q}$-acyclicity of $\widetilde{M}$ does not force $H_q(\widetilde{M};A)$ to vanish for $q > 0$ when A has nontrivial Γ-action, so one should not expect the same collapse. This is one reason to formulate duality as in Definition 5.2, which quantifies over coefficients and identifies the correct twisting.

Practical obstruction principle. For obstruction arguments later, the operational takeaway is:

(1) untwisted rational group homology of Γ agrees with rational homology of M when $\widetilde{M}$ is $\mathbb{Q}$-acyclic;

(2) the Γ-module $\mathbb{Q}_\omega$ controls the twisting needed for duality, and the presence of a nontrivial orientation character changes which coefficient system should be used to compare top-degree cohomology to $\mathbb{Q}$.

This is the point where lattice invariants become relevant: many invariants of uniform lattices are stable under passage to finite index and can be formulated in terms of (co)homology, duality modules, and Euler characteristics.

5.5 Compute-backed sanity checks on toy chain models

The previous subsections are conceptual and do not require computation. Nevertheless, in this project we maintain compute-backed checks on simplified algebraic models that mimic the duality and acyclicity constraints at the chain level. The purpose is not to validate the theorems above, which are standard consequences of manifold duality, but to validate that our later automated obstruction checks implement the intended algebraic conditions.

Toy model. A finite free $\mathbb{Q}[G]$-chain complex C_* concentrated in degrees 0 through n with an augmentation $C_0 \to \mathbb{Q}$ is a proxy for cellular chains of a G-cover of an n-complex. The condition "$\mathbb{Q}$-acyclic" becomes exactness of the augmented complex after forgetting the G-action. The condition "duality with twist" becomes the existence of a chain-level pairing identifying $\mathrm{Hom}_{\mathbb{Q}[G]}(C_*, \mathbb{Q}[G])$ with a shift of C_* tensored by a one-dimensional module.

Recorded datasets. The repository includes local artifacts used to test these conditions on families of randomly generated small chain complexes and on hand-constructed examples meant to exhibit edge cases. The datasets used for this chapter's sanity checks are: equivariant v2 dimension sweep samples[4], sample homology certificates[5], toy fixedpoint counterexamples summary[6], and obstruction evidence ledger[7]. The tests recorded there check, for each candidate complex, that (i) the augmented homology over $\mathbb{Q}$ matches the intended "acyclic" pattern, and (ii) the inferred dualizing module (when detected) is one-dimensional and produces the predicted symmetry of Betti numbers under the duality transform.

Interpretation and limitations. These toy computations do not address geometric realizability of the chain complexes by manifolds, and they do not address the existence of free actions of groups with torsion on $\mathbb{Q}$-acyclic manifolds. Their role is narrower: they ensure that the obstruction predicates used later (for example, rational duality feasibility in finite resolutions, and compatibility of twisting characters) behave correctly on controlled inputs.

Protocol deviation. Some exploratory scans that generated the above toy datasets were run with multiple random seeds, whereas the planned protocol specified a single seed. This deviation was used to assess stability of the obstruction predicates across randomized instances. Cross-section numeric comparisons that mix different seeds are therefore provisional, and the executed results should be interpreted as qualitative validation of predicate behavior rather than as final quantitative summaries.

Transition to the lattice setting. The remainder of the manuscript uses this chapter in the following way: if a uniform lattice Γ (possibly with 2-torsion) is the fundamental group of a closed manifold with $\mathbb{Q}$-acyclic universal cover, then for any torsion-free finite-index subgroup Γ_0 the group Γ_0 must satisfy $\mathrm{PD}_n(\mathbb{Q})$ with the appropriate orientation twist. Later chapters test this requirement against lattice cohomology, Euler characteristic constraints, and additional group-theoretic restrictions that survive passage to finite index.

6 Uniform lattices and virtual cohomological dimension

This chapter fixes notation for uniform lattices and records the dimension invariants that will be compared later to the rational duality constraints forced by a closed manifold whose universal cover is $\mathbb{Q}$-acyclic. Because the available bibliography for this project does not include standard sources on lattices or virtual cohomological dimension, the statements below are presented either as direct derivations from basic definitions, or as explicit standing assumptions (clearly separated).

6.1 Uniform lattices and cocompact actions

Ambient Lie data. Let G be a real Lie group and let $K \leq G$ be a maximal compact subgroup when such a subgroup is fixed. In the motivating setting, G is assumed to be connected, semisimple, and with finite center, and $X := G/K$ is the associated Riemannian symmetric space of noncompact type. In this paper we will use only the following structural features of X: (i) X is a smooth, connected, finite-dimensional manifold; (ii) G acts smoothly and properly on X; (iii) X is contractible. These properties are standard in the symmetric space setting, but we treat them as hypotheses whenever they are invoked.

Definition 6.1 (Uniform lattice)**.** A subgroup $\Gamma \leq G$ is a *uniform lattice* if Γ is discrete and the quotient $\Gamma\backslash G$ is compact. Equivalently (under mild regularity assumptions on G), Γ is discrete and $\Gamma\backslash X$ is compact for $X = G/K$.

Proper discontinuity on X. Let $\Gamma \leq G$ be discrete. The induced Γ-action on X is properly discontinuous in the following sense: for each compact $C \subseteq X$, the set $\{\gamma \in \Gamma : \gamma C \cap C \neq \varnothing\}$ is finite. When Γ is a uniform lattice, the quotient space $\Gamma\backslash X$ is compact by definition.

[4]`equivariant_v2_dimension_sweep_samples.jsonl`
[5]`sample_homology_certificates.json`
[6]`toy_fixedpoint_counterexamples_summary.json`
[7]`obstruction_evidence_ledger.json`

Lemma 6.2 (Finite CW models from cocompact proper actions)**.** *Let X be a contractible, finite-dimensional, paracompact manifold. Suppose a discrete group Γ acts properly discontinuously and cocompactly on X by homeomorphisms. Then $\Gamma\backslash X$ has the homotopy type of a finite CW complex. In particular, Γ is finitely generated and of type* F *(finite classifying space) when the action is free.*

Proof sketch. Cocompactness yields a compact quotient orbifold-like space. Under standard triangulability assumptions for manifolds and proper actions, one obtains a Γ-invariant CW decomposition of X with finitely many Γ-orbits of cells; the quotient inherits a finite CW structure. When the action is free, $\Gamma\backslash X$ is a (compact) manifold, hence a finite CW complex, and is a model for $B\Gamma$. □

Torsion and orbifold quotients. If Γ has torsion, the Γ-action on X cannot be free. In that case $\Gamma\backslash X$ should be regarded as having orbifold singularities. The chapter focuses on group-theoretic dimension invariants that remain meaningful even when torsion is present.

6.2 Cohomological dimension and vcd

Algebraic dimension invariants. Let Γ be a discrete group. The (integral) cohomological dimension $\mathrm{cd}(\Gamma)$ is the smallest integer n such that $H^k(\Gamma; M) = 0$ for all $\mathbb{Z}\Gamma$-modules M and all $k > n$, provided such an n exists; otherwise $\mathrm{cd}(\Gamma) = \infty$. We will use the rational version $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ defined analogously using $\mathbb{Q}\Gamma$-modules.

Definition 6.3 (Virtual cohomological dimension)**.** If Γ has a torsion-free subgroup $\Gamma_0 \leq \Gamma$ of finite index, define the *virtual cohomological dimension* by

$$\mathrm{vcd}(\Gamma) := \mathrm{cd}(\Gamma_0).$$

This is well-defined when any two torsion-free finite-index subgroups have the same cohomological dimension.

Standing assumption on torsion-free finite-index subgroups. The existence of torsion-free finite-index subgroups for lattices in semisimple Lie groups is a standard fact in the linear-group literature.

Assumption 6.4 (Virtual torsion-freeness for the lattices under study)**.** For the uniform lattices $\Gamma \leq G$ considered in the motivating question, assume that Γ admits a torsion-free subgroup of finite index.

Remark on scope. Assumption 6.4 is used only to make $\mathrm{vcd}(\Gamma)$ available as an invariant. None of the later fixed-point obstruction arguments will require this assumption, since they apply directly to torsion elements acting as deck transformations on a universal cover. However, the comparison between vcd and manifold dimension is most naturally stated under torsion-freeness.

Lemma 6.5 (Cohomological dimension bounded by geometric dimension)**.** *Let X be a contractible CW complex of dimension n with a free cellular action of Γ such that $\Gamma\backslash X$ is a CW complex. Then $\mathrm{cd}(\Gamma) \leq n$. The same inequality holds with $\mathrm{cd}_{\mathbb{Q}}$ in place of cd.*

Proof sketch. The quotient $\Gamma\backslash X$ is a model for $B\Gamma$. Cellular cochains on $\Gamma\backslash X$ compute group cohomology, and vanish above degree n. □

6.3 Dimension of cocompact torsion-free lattices

Aspherical manifold models. Assume X is a contractible n-manifold and Γ_0 acts freely, properly discontinuously, and cocompactly on X. Then $M := \Gamma_0\backslash X$ is a closed n-manifold with universal cover X. In particular M is aspherical.

Proposition 6.6 (Cohomological dimension equals manifold dimension)**.** *Let M be a closed, connected, aspherical n-manifold with $\pi_1(M) \cong \Gamma_0$. Then $\mathrm{cd}(\Gamma_0) = n$. Consequently, under Assumption 6.4, if $\Gamma \leq G$ is a uniform lattice acting cocompactly on an n-dimensional contractible manifold X and $\Gamma_0 \leq \Gamma$ is torsion-free of finite index, then $\mathrm{vcd}(\Gamma) = n$.*

Proof sketch. Since M is a closed n-manifold, M has the homotopy type of an n-dimensional finite CW complex and therefore $\mathrm{cd}(\Gamma_0) = \mathrm{cd}(\pi_1(M)) \leq n$ by Lemma 6.5. On the other hand, the top-dimensional cohomology of M with local coefficients in the orientation module is nonzero; translating to group cohomology shows that cohomology in degree n does not vanish for an appropriate coefficient system, hence $\mathrm{cd}(\Gamma_0) \geq n$. □

What survives with torsion. When Γ has torsion, the quotient $\Gamma\backslash X$ need not be a manifold, and $\mathrm{cd}(\Gamma)$ can behave differently from $\dim X$. Virtual cohomological dimension is designed to recover the manifold-like dimension by passing to torsion-free finite-index subgroups.

6.4 Relevance to $\mathbb{Q}$-acyclic universal covers

The comparison principle to be used later. The motivating question asks whether a uniform lattice Γ containing an involution can arise as $\pi_1(M)$ for a closed manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Two independent dimension notions then appear: (a) the geometric dimension stemming from the lattice action on $X = G/K$, which in favorable torsion-free situations equals $\mathrm{vcd}(\Gamma)$ by Proposition 6.6; (b) the manifold dimension $\dim M$, which controls the range in which duality-type statements and fixed-point invariants are defined. The subsequent chapters will use the $\mathbb{Q}$-acyclicity of $\widetilde{M}$ to propose cohomological constraints on Γ that resemble rational Poincaré duality. This chapter isolates the lattice-side dimension invariant (vcd) as the quantity to compare against those constraints.

A conditional bridge to rational duality. The following statement is recorded as a conditional template; it will be invoked later only when its hypotheses are verified in the specific setting under discussion.

Proposition 6.7 (Template: $\mathbb{Q}$-acyclic universal cover forces rational duality)**.** *Let M be a closed connected n-manifold with $\pi_1(M) \cong \Gamma$, and let $\widetilde{M}$ be the universal cover. Suppose that $H_i(\widetilde{M};\mathbb{Q}) = 0$ for all $i > 0$. Assume in addition that the standard identifications between group cohomology and cohomology with local coefficients on M apply in the needed range (for example, M is aspherical, or more generally the cellular chain complex of $\widetilde{M}$ gives a finite-length projective resolution over $\mathbb{Q}\Gamma$). Under these hypotheses, Γ behaves as a rational duality group of formal dimension n (in the sense that its cohomology satisfies a duality pattern with respect to a $\mathbb{Q}\Gamma$-module playing the role of an orientation module).*

Proof sketch. The cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ is a chain complex of $\mathbb{Q}\Gamma$-modules of length n. $\mathbb{Q}$-acyclicity implies that this complex is exact in positive degrees, leaving a resolution of $\mathbb{Q}$ up to degree n. Dualizing and comparing with cochains with local coefficients yields the expected duality behavior, provided the needed finiteness and identification hypotheses hold. □

Limitations acknowledged up front. Proposition 6.7 is not used as a black-box theorem in this manuscript. It is a protocol-like template identifying where one needs additional geometric input (asphericity, finiteness of resolutions, or other tameness assumptions) before deducing duality consequences from $\mathbb{Q}$-acyclicity. This separation is important because later fixed-point obstructions will be formulated conditionally, and because the present project does not import external theorems on lattices or duality groups beyond what can be justified directly from the stated hypotheses.

Summary. A uniform lattice $\Gamma \leq G$ carries a canonical geometric dimension through its cocompact action on the contractible manifold $X = G/K$. Under torsion-freeness this dimension coincides with $\mathrm{cd}(\Gamma)$; under virtual torsion-freeness it is encoded in $\mathrm{vcd}(\Gamma)$. Later chapters compare these lattice-side invariants with the dimension and rational (co)homological constraints that would follow from $\Gamma \cong \pi_1(M)$ for a closed manifold with $\mathbb{Q}$-acyclic universal cover.

7 Dimension and Euler characteristic constraints

This chapter records constraints on the putative manifold dimension and on Euler characteristics that follow from the hypothesis that a group Γ occurs as $\pi_1(M)$ for a closed manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. The intent is obstruction-oriented: we isolate numerical equalities forced by (rational) Poincaré duality, then compare them to (a) what is known abstractly for lattices and (b) what is seen in small computational models used in this project.

7.1 Standing setup and the basic Euler characteristic identity

Let M be a connected compact n-manifold without boundary, and let $\Gamma = \pi_1(M)$. Assume that $\widetilde{M}$ is $\mathbb{Q}$-acyclic, meaning $\widetilde{H}_i(\widetilde{M};\mathbb{Q}) = 0$ for all i. The universal cover is contractible in the classical aspherical case, but the present hypothesis is weaker and is only used through its consequences for homology with $\mathbb{Q}$-coefficients.

A first observation is that the cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ is a chain complex of free $\mathbb{Q}\Gamma$-modules with $H_0 \cong \mathbb{Q}$ and $H_i = 0$ for $i > 0$. When M is a finite CW complex (in particular for smooth closed manifolds), this makes $\mathbb{Q}$ a finite $\mathbb{Q}\Gamma$-projective resolution, so Γ is of type FP over $\mathbb{Q}$. This is one of the points where the manifold structure, not merely the group, enters.

Proposition 7.1. *Under the standing setup, $\chi(M) = \chi(\Gamma;\mathbb{Q})$, where $\chi(\Gamma;\mathbb{Q})$ denotes the Euler characteristic of Γ computed from any finite $\mathbb{Q}\Gamma$-projective resolution of $\mathbb{Q}$.*

Proof. Let $C_*(\widetilde{M};\mathbb{Q})$ be the cellular chain complex of $\widetilde{M}$ induced from a finite CW structure on M. Each C_i is a finite rank free $\mathbb{Q}\Gamma$-module and the homology is $\mathbb{Q}$ in degree 0 and 0 otherwise by $\mathbb{Q}$-acyclicity. Thus C_* is a finite free resolution of $\mathbb{Q}$ over $\mathbb{Q}\Gamma$, and its alternating sum of ranks equals $\chi(M)$. By definition this alternating sum is $\chi(\Gamma;\mathbb{Q})$, and it is independent of the chosen finite projective resolution. □

A second observation is that a closed n-manifold satisfies Poincaré duality with coefficients in $\mathbb{Q}$ twisted by the orientation character $\omega\colon \Gamma \to \{\pm 1\}$. When $\widetilde{M}$ is $\mathbb{Q}$-acyclic, the usual argument shows that Γ is a rational Poincaré duality group of dimension n (with possible twist by ω). We record this as a guiding constraint rather than a fully expanded proof, since later chapters will isolate the precise group-cohomological formulation needed.

Research note. The statement "Γ is a rational PD_n group" is used in this project only through consequences for Euler characteristics, cohomological dimension bounds, and (in later chapters) compatibility with torsion. In particular, the present chapter does not assume that Γ is torsion-free.

7.2 Parity constraints from Poincaré duality and consequences for χ

Poincaré duality yields strong restrictions on Betti numbers and hence on Euler characteristics.

Lemma 7.2. *If M is a closed* orientable *manifold of odd dimension n, then $\chi(M) = 0$.*

Proof. Over $\mathbb{Q}$, Poincaré duality gives $b_i = b_{n-i}$ for all i. When n is odd, the terms in $\chi(M) = \sum_i (-1)^i b_i$ cancel in pairs $(i, n-i)$ because $(-1)^i + (-1)^{n-i} = 0$. □

For non-orientable M, a weaker statement holds: the Euler characteristic still satisfies restrictions, but the cancellation argument requires additional care due to the orientation local system. For obstruction purposes, we isolate the orientable case since a uniform lattice in a semisimple Lie group is frequently studied through finite index subgroups, and passing to an orientable cover is a common step at the manifold level.

Proposition 7.3. *Let $p\colon M' \to M$ be a finite covering of degree d between connected finite CW complexes. Then $\chi(M') = d\,\chi(M)$.*

Proof. Lift a finite CW structure from M to M'. Each cell of M lifts to exactly d cells of the same dimension in M'. The Euler characteristic is the alternating sum of cell counts. □

Combining Proposition 7.1 with Proposition 7.3, an obstruction strategy emerges: if Γ has torsion, one can look for a torsion-free finite index subgroup $\Gamma' \leq \Gamma$ (when it exists) that would correspond to a manifold cover $M' \to M$. Then $\chi(M')$ must be divisible by $[\Gamma : \Gamma']$, and in odd dimensions it must vanish when M' is orientable.

Constraint package used later. Fix a torsion-free finite index subgroup $\Gamma' \leq \Gamma$. If $\Gamma \cong \pi_1(M)$ with $\widetilde{M}$ $\mathbb{Q}$-acyclic, and $M' \to M$ is the induced cover with $\pi_1(M') \cong \Gamma'$, then: (i) $\chi(\Gamma';\mathbb{Q}) = \chi(M')$, (ii) $\chi(M') = [\Gamma : \Gamma']\,\chi(M)$, (iii) if M' is orientable and $\dim M'$ is odd then $\chi(\Gamma';\mathbb{Q}) = 0$.

These equalities are elementary but serve as a consistency check against any proposed value of $\chi(\Gamma)$ coming from lattice invariants.

7.3 Lattice torsion, orbifold Euler characteristics, and what is not settled here

Let G be a real semisimple Lie group and $\Gamma \leq G$ a uniform lattice containing elements of order 2. Even when Γ is not torsion-free, one can form the compact orbifold $\Gamma\backslash X$, where X is the associated symmetric space of noncompact type. There is an "orbifold Euler characteristic" $\chi_{\mathrm{orb}}(\Gamma\backslash X)$ which can be computed as a weighted sum over cell stabilizers in an orbifold cell structure.

The obstruction program in this manuscript requires a careful comparison between: (a) $\chi(\Gamma;\mathbb{Q})$ defined algebraically, when it is defined, (b) $\chi(M)$ for a genuine manifold model with $\pi_1(M) \cong \Gamma$, and (c) $\chi_{\mathrm{orb}}(\Gamma\backslash X)$ for the locally symmetric orbifold.

At present, this chapter does *not* prove a general equality among these three quantities in the presence of torsion. Instead, we treat the following as a research direction with explicit failure modes.

Conjecture 7.4 (Manifold versus orbifold Euler characteristic compatibility)**.** *Suppose Γ is a uniform lattice in G and $\Gamma \cong \pi_1(M)$ for a closed n-manifold M with $\widetilde{M}$ $\mathbb{Q}$-acyclic. Then any natural candidate for $\chi(\Gamma;\mathbb{Q})$ that is stable under passage to torsion-free finite index subgroups coincides with $\chi(M)$, and after passing to a torsion-free finite index subgroup Γ' it agrees with $\chi(\Gamma'\backslash X)$.*

Why this is nontrivial. The main subtlety is torsion. If Γ has torsion then $B\Gamma$ has no finite manifold model in general, and Euler characteristics depend on the chosen finiteness and coefficient hypotheses. Even defining $\chi(\Gamma;\mathbb{Q})$ requires a choice of finiteness condition (for example, type FP over $\mathbb{Q}$), which is automatic for groups arising from finite CW complexes but not automatic for arbitrary lattices without additional work. The chapter therefore treats Euler characteristic comparisons as conditional statements, designed to be plugged into later sections once the relevant finiteness properties are established for the class of lattices under consideration.

7.4 Compute-backed consistency checks and observed patterns

This project includes several computational scans over finite chain models intended to emulate constraints coming from equivariant cell structures, Smith-type fixed point constraints for 2-torsion actions, and Euler characteristic bookkeeping under quotients and covers. The goal here is modest: to record what has been checked and how it informs the obstruction narrative, without presenting these checks as proofs about lattices.

Data artifacts used in this chapter. The following repository datasets were consulted for the summaries below: equivariant v2 dimension sweep samples[8], cycle7 fullscan consistency report[9], toy fixedpoint counterexamples summary[10], recertify fixed obstruction example and euler chars compute result (compute report)[11], and sample homology certificates[12]. These artifacts store discrete chain complexes with group actions and derived invariants (ranks, mod p homology summaries, and Euler characteristics). The computations were performed in the planned framework `numpy`.

Protocol deviation. The planned protocol specifies a single seed (7) for randomized components. Some executed scans recorded in the above artifacts used multiple seeds, as reflected in the project protocol manifest. This deviation was used to increase robustness of negative findings (searching for counterexamples across randomized instances). Cross-section numeric comparisons should be treated as provisional because they aggregate across heterogeneous seeds.

(1) Dimension parity checks in toy models. The dimension sweep artifact equivariant v2 dimension sweep samples[13] contains families of finite $\mathbb{Q}$-chain complexes meant to satisfy a Poincaré duality pattern in formal dimension n, together with $\mathbb{Z}/2$-actions and fixed-subcomplex summaries. Across this sweep, the parity rule from Lemma 7.2 is consistently reflected: whenever the sampled complex is symmetric in the sense $b_i = b_{n-i}$ (over $\mathbb{Q}$) and the formal dimension n is odd, the computed Euler characteristic is zero. This is an internal validation of the bookkeeping in the pipeline.

[8] `equivariant_v2_dimension_sweep_samples.jsonl`
[9] `cycle7_fullscan_consistency_report.json`
[10] `toy_fixedpoint_counterexamples_summary.json`
[11] `evidence_recertify_fixed_obstruction_example_and_euler_chars_compute_result.json`
[12] `sample_homology_certificates.json`
[13] `equivariant_v2_dimension_sweep_samples.jsonl`

(2) Fixed point obstructions as Euler characteristic mismatches. The artifact toy fixedpoint counterexamples summary[14] records instances where a $\mathbb{Z}/2$-action on a finite complex satisfies acyclicity over $\mathbb{Q}$ at the total space level but exhibits fixed-subcomplex patterns incompatible with being the universal cover of a manifold with a free Γ-action extending the 2-torsion element. In these instances, the obstruction is witnessed by an inconsistency among three Euler characteristics computed on the model: (i) the total complex, (ii) the fixed subcomplex, and (iii) the quotient complex, under the naive identities expected from free actions.

This is not a theorem about manifolds or lattices, since the complexes are synthetic. The relevance is methodological: it supports the approach of searching for obstructions that can be phrased as equalities among Euler characteristics of fixed sets and quotients, which can then be compared to constraints coming from Poincaré duality for manifolds.

(3) Recertified obstruction example and invariance checks. The artifact recertify fixed obstruction example and euler chars compute result (compute report)[15], together with cycle7 fullscan consistency report[16] and sample homology certificates[17], documents recomputation of a representative obstruction instance under multiple internal consistency checks (Euler characteristic computed from cell counts versus from alternating homology ranks, and agreement of homology summaries with stored certificates). In all recertified cases, the reported Euler characteristic values were consistent across independent calculation paths within the pipeline.

Interpretation for the main problem. The computations do not directly address whether a uniform lattice with 2-torsion can occur as $\pi_1(M)$ with $\widetilde{M}$ $\mathbb{Q}$-acyclic. They do, however, support a concrete obstruction template used later: attempt to realize a 2-torsion element as a deck transformation on $\widetilde{M}$, derive constraints on fixed-point data (at least at the level of induced chain complexes), and translate these into numerical contradictions involving Euler characteristics and duality.

7.5 Summary of constraints to be propagated to later chapters

For later use, we isolate the constraints from this chapter in a form that can be combined with lattice-specific input.

Proposition 7.5. *Assume $\Gamma \cong \pi_1(M)$ for a closed n-manifold M whose universal cover is $\mathbb{Q}$-acyclic.*

(1) $\chi(M) = \chi(\Gamma; \mathbb{Q})$ and this number is multiplicative under finite covers of M.

(2) If M is orientable and n is odd then $\chi(M) = 0$.

(3) If $\Gamma' \leq \Gamma$ is torsion-free of finite index and corresponds to a finite cover $M' \to M$, then $\chi(\Gamma'; \mathbb{Q}) = [\Gamma : \Gamma']\, \chi(\Gamma; \mathbb{Q})$, and in odd dimensions with M' orientable one has $\chi(\Gamma'; \mathbb{Q}) = 0$.

Proof. Parts (1) and (3) follow from Propositions 7.1 and 7.3. Part (2) is Lemma 7.2. □

Operational consequence. Any lattice-theoretic or orbifold-theoretic computation producing a nonzero Euler characteristic for a torsion-free finite index subgroup forces n to be even in any manifold model with $\mathbb{Q}$-acyclic universal cover. Conversely, if all torsion-free finite index subgroups have Euler characteristic zero, then Euler characteristic alone does not obstruct odd dimensions, and later chapters must rely on finer invariants (for example, fixed-point constraints for 2-torsion or compatibility with rational duality modules).

8 Involutions in lattices and centralizer structure

This chapter records the structural input about involutions in uniform lattices that is used repeatedly in later obstruction arguments. The focus is on organizing what is needed, separating (i) statements proved in full generality within this manuscript from (ii) standard structural assertions from Lie theory and arithmetic groups that are used as assumptions because the present project does not yet include vetted citations for them.

[14] `toy_fixedpoint_counterexamples_summary.json`
[15] `evidence_recertify_fixed_obstruction_example_and_euler_chars_compute_result.json`
[16] `cycle7_fullscan_consistency_report.json`
[17] `sample_homology_certificates.json`

8.1 Involutions in a uniform lattice and ambient extensions

Let G be a connected real semisimple Lie group with finite center, and let $K \leq G$ be a maximal compact subgroup. Write $X = G/K$ for the associated Riemannian symmetric space of noncompact type. Let $\Gamma \leq G$ be a uniform lattice. Throughout, we assume Γ contains nontrivial 2-torsion.

Definition 8.1 (Involutions and centralizers)**.** An *involution* in Γ is an element $\tau \in \Gamma$ with $\tau^2 = e$ and $\tau \neq e$. For any subgroup $H \leq \Gamma$, the *centralizer* of H in Γ is

$$C_\Gamma(H) = \{\gamma \in \Gamma : \gamma h = h\gamma \text{ for all } h \in H\}.$$

For an element $\tau \in \Gamma$ we write $C_\Gamma(\tau) = C_\Gamma(\langle\tau\rangle)$.

The motivating geometric situation for this project is a hypothetical closed manifold M^n with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic. In that setting, the subgroup $\langle\tau\rangle$ acts on $\widetilde{M}$ by deck transformations, and understanding $C_\Gamma(\tau)$ becomes a proxy for understanding the stabilizer of the fixed-point data associated to τ in geometric and cohomological arguments.

Two ambient extensions are relevant.

(A) Inner conjugation on G and X. Any $\tau \in \Gamma \leq G$ defines an inner automorphism $\mathrm{Ad}(\tau) : G \to G$, $g \mapsto \tau g \tau^{-1}$. This induces an isometric action on $X = G/K$ by $gK \mapsto \tau gK$ (left translation). In addition, $\mathrm{Ad}(\tau)$ defines an isometry of G with respect to any bi-invariant Riemannian metric on compact factors, but G is noncompact; for the present chapter, only the induced action on X is used.

(B) Symmetric-space involutions from order-two automorphisms. Independently of $\tau \in \Gamma$, one may consider involutive automorphisms $\sigma : G \to G$ (with $\sigma^2 = \mathrm{id}$). These define symmetric subgroups G^σ and fixed-point symmetric subspaces X^σ in the standard sense. The later sections of the manuscript require comparing the fixed-point geometry coming from an *element* $\tau \in G$ with that coming from an *automorphism* σ. The two notions agree when σ is inner and given by conjugation by an involution in G.

Definition 8.2 (Inner involutive automorphism)**.** For $\tau \in G$ with $\tau^2 = e$, define $\sigma_\tau : G \to G$ by $\sigma_\tau(g) = \tau g \tau^{-1}$. Then $\sigma_\tau^2 = \mathrm{id}$. The fixed-point subgroup is $G^{\sigma_\tau} = C_G(\tau)$.

Assumption (structure of $C_G(\tau)$). For $\tau \in G$ of finite order, the centralizer $C_G(\tau)$ is a closed subgroup of G with finitely many connected components and reductive identity component. This is a standard fact in Lie theory, but it is not proved here and is recorded as an assumption pending incorporation of vetted references.

The distinction between $C_G(\tau)$ and $C_\Gamma(\tau)$ is crucial: the former is a Lie group, while the latter is a discrete group. The central question for later obstruction arguments is whether $C_\Gamma(\tau)$ behaves as a uniform lattice in $C_G(\tau)$, in a sense strong enough to import dimension and duality constraints.

8.2 Centralizers of involutions in uniform lattices

This subsection isolates the structural statement that would allow one to pass between lattice centralizers and ambient centralizers.

Definition 8.3 (Lattice centralizer inclusion)**.** Let $\Gamma \leq G$ be discrete and let $\tau \in \Gamma$. Then

$$C_\Gamma(\tau) = \Gamma \cap C_G(\tau),$$

as an equality of subsets of G.

The equality is formal. What is not formal is the extent to which $C_\Gamma(\tau)$ is large inside $C_G(\tau)$.

Assumption (centralizer lattice property). Let $\Gamma \leq G$ be a uniform lattice and let $\tau \in \Gamma$ have finite order. Then $C_\Gamma(\tau)$ is a uniform lattice in $C_G(\tau)$. Equivalently, $C_\Gamma(\tau)$ is discrete in $C_G(\tau)$ and $C_G(\tau)/C_\Gamma(\tau)$ is compact.

This assertion is used in many parts of the classical literature on lattices and periodic elements, often via the interpretation of $C_\Gamma(\tau)$ as a stabilizer of a totally geodesic subspace of X and compactness of quotients in the cocompact case. The present project does not yet include an admissible bibliographic key for this result, so it is explicitly treated as an assumption.

The following consequence is the main reason the assumption is useful for the obstruction strategy.

Proposition 8.4 (Compact quotient for centralizer action under the assumption)**.** *Assume the centralizer lattice property. Let $Y = C_G(\tau)/(C_G(\tau) \cap K)$. Then $C_\Gamma(\tau)$ acts properly discontinuously and cocompactly on Y.*

Proof. Proper discontinuity is immediate from discreteness of $C_\Gamma(\tau)$ in $C_G(\tau)$ and the fact that Y is a smooth homogeneous space. Cocompactness follows from compactness of $C_G(\tau)/C_\Gamma(\tau)$ and compactness of $(C_G(\tau) \cap K)$, since the natural map $C_G(\tau)/C_\Gamma(\tau) \to Y/C_\Gamma(\tau)$ has compact fibers. $\square$

Even without the centralizer lattice property, one can still extract some information that is purely algebraic.

Lemma 8.5 (Finite index in normalizer for an involution)**.** *Let $\tau \in \Gamma$ be an involution. Then $C_\Gamma(\tau)$ has index at most 2 in the normalizer $N_\Gamma(\langle \tau \rangle)$.*

Proof. The conjugation action of $N_\Gamma(\langle \tau \rangle)$ on the cyclic group $\langle \tau \rangle \cong \mathbb{Z}/2$ gives a homomorphism $N_\Gamma(\langle \tau \rangle) \to \mathrm{Aut}(\mathbb{Z}/2)$. Since $\mathrm{Aut}(\mathbb{Z}/2)$ is trivial, this action is trivial, hence every normalizer element commutes with τ, so $N_\Gamma(\langle \tau \rangle) = C_\Gamma(\tau)$. $\square$

In particular, for involutions there is no difference between normalizer and centralizer. This simplifies later arguments where one compares fixed sets and stabilizers.

8.3 Fixed-point geometry on the symmetric space

The project uses two related fixed-point constructions: fixed points of the action of τ on X (by left translation) and fixed points of the involutive automorphism σ_τ on X (by induced action). Only the second has a direct symmetric-subspace interpretation.

Fixed points of left translation. Left translation by τ on X is $x = gK \mapsto \tau g K$. A point gK is fixed precisely when $g^{-1}\tau g \in K$. Thus the fixed set is

$$X^{\langle \tau \rangle} = \{gK \in G/K : g^{-1}\tau g \in K\}.$$

This set can be empty in general, and its geometry depends on whether τ is conjugate into K.

Fixed points of the induced involution from conjugation. The automorphism σ_τ acts on X via $gK \mapsto \sigma_\tau(g)K = \tau g \tau^{-1} K$. The fixed-point set of this action is

$$X^{\sigma_\tau} = \{gK \in X : \tau g \tau^{-1} K = gK\}.$$

Equivalently, $g^{-1}\tau g \in N_G(K)$, and under standard identifications one expects X^{σ_τ} to be a totally geodesic submanifold (possibly with multiple components) modeled on the symmetric space of $C_G(\tau)$.

Assumption (totally geodesic fixed-point subspace). For $\tau \in G$ of order 2, the fixed-point set X^{σ_τ} is a (possibly disconnected) totally geodesic submanifold of X, each component of which is isometric to $C_G(\tau)/(C_G(\tau) \cap K)$.

Under this assumption and Proposition 8.4, one obtains a compact locally symmetric quotient

$$C_\Gamma(\tau) \backslash X^{\sigma_\tau},$$

which can be viewed as a geometric avatar of the centralizer subgroup. This is the bridge used later when comparing cohomological dimensions.

A computable toy proxy (finite models). Although the present chapter is Lie-theoretic, later obstruction heuristics sometimes reduce to finite, combinatorial fixed-point calculations (for example, Smith-type constraints over finite fields, or Euler characteristic comparisons for fixed subcomplexes). In this project, we have generated toy examples of finite $\mathbb{Z}/2$-actions on small cell complexes to stress-test the fixed-point bookkeeping and to detect edge cases in Euler characteristic computations. A summary of such toy scans is recorded in toy fixedpoint counterexamples summary[18]. These computations are not evidence about uniform lattices, but they serve as regression tests for the formal fixed-point identities used in later sections.

Protocol deviation. Some compute artifacts referenced in this manuscript were generated using multiple pseudorandom seeds recorded in the repository metadata, while the planned protocol specifies a single seed. This affects reproducibility bookkeeping rather than mathematical correctness. Cross-section numeric comparisons based on these artifacts are provisional; the intended remediation is to rerun the affected scans under the planned single-seed setting and then treat that output as the canonical record.

8.4 Centralizers, duality constraints, and the obstruction template

This subsection records the way centralizer structure enters the overarching obstruction strategy, without asserting a completed contradiction.

Duality expectations from $\mathbb{Q}$-acyclic universal covers. Suppose M^n is closed, connected, and $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then M is a rational homology n-manifold and the group $\Gamma = \pi_1(M)$ is expected to satisfy a rational Poincaré duality pattern in group cohomology, at least after imposing standard finiteness hypotheses (for example, Γ of type $\mathrm{FP}_{\mathbb{Q}}$). The project uses this as an organizing heuristic: if Γ is a uniform lattice, then Γ is already of strong finiteness type, and the tension is shifted to compatibility with 2-torsion.

Centralizer obstruction template. Fix an involution $\tau \in \Gamma$. The deck transformation action of $\langle\tau\rangle$ on $\widetilde{M}$ is free, hence $\widetilde{M}^{\langle\tau\rangle} = \varnothing$ for $\tau \neq 1$. Consequently, Smith-type fixed-point arguments cannot be applied *directly* to the deck action under the hypothesis of $\mathbb{Q}$-acyclicity alone. Any torsion-sensitive mechanism must instead come from an auxiliary $\langle\tau\rangle$-action on a space functorially attached to M (for example, a boundary or compactification model, or a prime-specific acyclic model where fixed-point tools apply), or from purely group-theoretic input about centralizers.

Independently, under the centralizer lattice property and the totally geodesic fixed-point subspace assumption, $C_\Gamma(\tau)$ acts cocompactly on the symmetric space $Y = C_G(\tau)/(C_G(\tau) \cap K)$, giving an upper bound

$$\mathrm{cd}_{\mathbb{Q}}(C_\Gamma(\tau)) \leq \dim Y.$$

The obstruction template used later can be summarized as follows: combine the geometric upper bound above with a *separate* lower bound on $\mathrm{cd}_{\mathbb{Q}}(C_\Gamma(\tau))$ obtained from such an auxiliary torsion-sensitive framework. A contradiction would follow if these bounds are incompatible (for example, if the auxiliary lower bound forces $\mathrm{cd}_{\mathbb{Q}}(C_\Gamma(\tau)) > \dim Y$), or if the inferred duality properties of $C_\Gamma(\tau)$ conflict with those forced by the rational Poincaré duality structure of Γ_0 from Proposition 5.3.

Status of the required inputs. At present, the manuscript has not validated, with admissible citations, the Lie-theoretic structure assertions needed to treat $C_\Gamma(\tau)$ as a uniform lattice in $C_G(\tau)$, nor the geometric identification of X^{σ_τ} as a symmetric subspace. Accordingly, the later contradiction arguments that rely on these inputs are stated conditionally.

Separately, any torsion-sensitive fixed-point input would have to come from an auxiliary $\langle\tau\rangle$-action on a space functorially attached to M (not from the free deck action on $\widetilde{M}$). Classical Smith theory is formulated over $\mathbb{F}_p$ coefficients, and with only $\mathbb{Q}$-acyclicity available one cannot apply it directly; translating mod-p constraints into rational constraints requires additional hypotheses or alternative arguments. The computational toy scans (for example, group action smith coeff field consistency scan compute result (compute report)[19]) are used only to verify internal consistency of coefficient-field bookkeeping in small finite complexes; they do not by themselves resolve the coefficient-field gap in the manifold setting.

[18] `toy_fixedpoint_counterexamples_summary.json`
[19] `micro_group_action_smith_coeff_field_consistency_scan_compute_result.json`

Research directions tied to this chapter. The centralizer lattice property and the identification of fixed-point symmetric subspaces are key infrastructure items. Progress on the main problem would benefit from (i) incorporating vetted references for these standard structural facts and (ii) isolating precisely which parts of later obstruction arguments can be proved without them. A second direction is to develop a rational fixed-point theory tailored to the $\mathbb{Q}$-acyclic hypothesis, or to replace rational fixed-point arguments by integral or mod-p arguments plus comparison tools that remain valid for manifolds with uniform lattice fundamental group.

9 Centralizers and cohomological dimension inequalities

This section records the cohomological dimension constraints that follow from the existence of a closed manifold M^n with fundamental group Γ and $\mathbb{Q}$-acyclic universal cover $\widetilde{M}$, and it isolates the additional inputs needed to convert these constraints into obstructions for uniform lattices with 2-torsion. The central theme is that torsion forces one to compare global cohomological dimension data for Γ with the cohomological dimensions of centralizers of finite subgroups, and that comparison is the point at which Lie-theoretic structure enters.

9.1 From $\mathbb{Q}$-acyclic universal covers to cohomological dimension bounds

Assume M is a compact, connected, boundaryless topological manifold of dimension n with $\pi_1(M) \cong \Gamma$, and let $\widetilde{M}$ be the universal cover. The hypothesis that $\widetilde{M}$ is $\mathbb{Q}$-acyclic means that the reduced rational homology satisfies

$$\widetilde{H}_i(\widetilde{M};\mathbb{Q}) = 0 \quad \text{for all } i \geq 0.$$

Fix a Γ-equivariant CW structure on $\widetilde{M}$ arising from a triangulation of M. Then the cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ is a chain complex of free $\mathbb{Q}\Gamma$-modules of length n, and $\mathbb{Q}$-acyclicity implies that its augmentation to $\mathbb{Q}$ is a quasi-isomorphism. In particular, $C_*(\widetilde{M};\mathbb{Q}) \to \mathbb{Q}$ supplies a finite free $\mathbb{Q}\Gamma$-resolution of the trivial module $\mathbb{Q}$ of length n. Consequently,

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n. \tag{9}$$

This inequality is unconditional given the existence of $(M, \widetilde{M})$ as above.

The manifold structure yields additional duality information at the level of chain complexes. Let $\omega\colon \Gamma \to \{\pm 1\}$ be the orientation character. There is a Γ-equivariant cap product pairing on chains inducing, at the cochain level, a quasi-isomorphism of $\mathbb{Q}\Gamma$-complexes

$$C^*(\widetilde{M};\mathbb{Q}) \simeq \mathrm{Hom}_{\mathbb{Q}\Gamma}(C_{n-*}(\widetilde{M};\mathbb{Q}), \mathbb{Q}\Gamma_\omega),$$

where $\mathbb{Q}\Gamma_\omega$ denotes the right $\mathbb{Q}\Gamma$-module twisted by ω. When Γ is torsion-free, this is a standard route to the statement that Γ is a Poincaré duality group over $\mathbb{Q}$ of dimension n. With torsion present, one should not expect group cohomology $H^*(\Gamma;\mathbb{Q}\Gamma)$ to satisfy Poincaré duality in the usual sense; nevertheless, the existence of a finite free $\mathbb{Q}\Gamma$-resolution and the induced duality on the chain level provide a constrained environment for any attempt to realize a given group with torsion as such a fundamental group.

Two remarks help to orient later comparisons.

First, the bound (9) depends only on rational acyclicity and compactness, not on smoothness, nonpositive curvature, or asphericity.

Second, the presence of torsion does not contradict freeness of the deck action. Every nontrivial element of Γ acts freely on $\widetilde{M}$ by definition of universal covering transformations. For finite-order elements, this implies that the associated quotient $\widetilde{M}/\langle g\rangle$ is again a manifold, rather than an orbifold, so fixed-point arguments that are typical for linear or isometric actions on Euclidean space are not directly available.

9.2 Centralizers of finite subgroups and a conditional inequality schema

Let $g \in \Gamma$ have finite order. The basic group-theoretic object is the centralizer $C_\Gamma(g) = \{\gamma \in \Gamma : \gamma g = g\gamma\}$. The relevance of centralizers is that they control the equivariant algebra of the conjugacy class of g, and for lattices they are expected to reflect the structure of the corresponding centralizer in the ambient Lie group.

A general obstruction strategy for torsion in geometric group actions compares the global dimension of Γ with the dimensions of these centralizers. One expects an inequality of the following form.

Conditional centralizer inequality schema. Suppose Γ admits a finite-dimensional model for the classifying space for proper actions $\underline{E}\Gamma$, and suppose that for each finite subgroup $F \leq \Gamma$ there is a relation

$$\mathrm{cd}_{\mathbb{Q}}(C_{\Gamma}(F)) \leq \mathrm{cd}_{\mathbb{Q}}(\Gamma) - \delta(F) \tag{10}$$

with $\delta(F) \geq 1$ whenever F is nontrivial.

Here $C_{\Gamma}(F)$ is the centralizer of F and $\delta(F)$ is intended to measure, in some manner, the amount by which the presence of F lowers the relevant cohomological dimension. In settings where $\underline{E}\Gamma$ can be built from fixed-point data, $\delta(F)$ is often related to codimension of fixed sets, to ranks of compact tori, or to dimensions of associated symmetric subspaces.

In the present manuscript, (10) is treated as a conditional input because it depends on a detailed comparison between algebraic and geometric models for proper actions and on finiteness properties of Γ and its centralizers. The point is that, once a strict drop $\delta(F) \geq 1$ is available uniformly for nontrivial finite F, the coexistence of duality constraints coming from M and the structure constraints coming from lattice centralizers becomes much tighter.

Even without committing to (10), there are two unconditional monotonicity facts that remain useful.

First, since $C_{\Gamma}(F) \leq \Gamma$, one always has

$$\mathrm{cd}_{\mathbb{Q}}(C_{\Gamma}(F)) \leq \mathrm{cd}_{\mathbb{Q}}(\Gamma)$$

provided both cohomological dimensions are finite.

Second, for any subgroup $H \leq \Gamma$ of finite index, rational cohomological dimensions satisfy $\mathrm{cd}_{\mathbb{Q}}(H) = \mathrm{cd}_{\mathbb{Q}}(\Gamma)$. This allows passage to torsion-free finite index subgroups when such subgroups exist, and it is one motivation for phrasing the objective in terms of uniform lattices, which are known to be virtually torsion-free under standard hypotheses.

9.3 Uniform lattices with involutions: expected structure of centralizers

Assume from now on that Γ is a uniform lattice in a real semisimple Lie group G with finite center, and that Γ contains an involution τ.

The Lie-theoretic expectation is that the centralizer $C_G(\tau)$ is a closed reductive subgroup of G, and that $C_{\Gamma}(\tau) = \Gamma \cap C_G(\tau)$ is a uniform lattice in $C_G(\tau)$, perhaps up to finite index. In favorable cases, the symmetric space of $C_G(\tau)$ embeds as a totally geodesic submanifold of the symmetric space of G. If this picture holds with sufficient uniformity, it suggests that the virtual cohomological dimension of $C_{\Gamma}(\tau)$ is strictly smaller than that of Γ by an amount governed by the codimension of this symmetric subspace.

This is precisely where the obstruction program meets classification. To make the expectation effective, one needs inputs of the following kind.

First, one needs a description, for each relevant semisimple G, of the possible conjugacy classes of involutions and of the corresponding symmetric subspaces associated to $C_G(\tau)$. These are structural results about real semisimple groups.

Second, one needs an identification of the virtual cohomological dimensions (or rational cohomological dimensions, where appropriate) of the lattices involved. In arithmetic cases, such dimensions are computable from the associated symmetric spaces and $\mathbb{Q}$-ranks.

Third, one needs a mechanism connecting these Lie-theoretic dimensions to the manifold dimension n in the existence problem. The only unconditional bridge available from the manifold hypothesis alone is (9), namely $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$. Any strengthening, such as an equality $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = n$ or a Poincaré duality property at the group level, requires additional assumptions beyond $\mathbb{Q}$-acyclicity.

Because the deck action of Γ on $\widetilde{M}$ is free, the involution τ has no fixed points on $\widetilde{M}$. Therefore, Smith-theoretic fixed-set constraints for a $\mathbb{Z}/2$-action on an acyclic space do not apply directly to the deck action. Instead, the use of centralizers is algebraic: involutions generate finite subgroups, and centralizers govern the behavior of these subgroups inside resolutions of $\mathbb{Q}$ by $\mathbb{Q}\Gamma$-modules. Any attempt to assemble a duality-type chain complex compatible with both the lattice structure and the presence of torsion must control how restriction to $C_{\Gamma}(\tau)$ interacts with the top-degree class induced by M.

A particularly transparent conditional obstruction takes the following shape. Assume that Γ behaves, over $\mathbb{Q}$, like a duality group of dimension n in the sense that $H^i(\Gamma; \mathbb{Q}\Gamma) = 0$ for $i \neq n$ and $H^n(\Gamma; \mathbb{Q}\Gamma) \cong \mathbb{Q}$ up to twist. If, in addition, one has a strict centralizer drop as in (10) for involutions, then centralizers cannot carry cohomology in degree n, and the restriction maps to $C_\Gamma(\tau)$ become severely constrained. Converting this heuristic into a proof requires a precise framework for duality with torsion, which is outside the unconditional scope of the present section.

9.4 Computational probes, protocol note, and current status

The obstruction strategy above involves two distinct layers: an algebraic layer dealing with chain complexes of $\mathbb{Q}\Gamma$-modules and their behavior under restriction to centralizers, and a Lie-theoretic layer predicting the size and structure of centralizers of involutions in lattices. The algebraic layer admits limited computational probing on finite approximations, for example by constructing equivariant chain complexes for toy models and testing homology vanishing patterns that would be consistent with $\mathbb{Q}$-acyclicity and with candidate restriction behavior.

The accompanying computational artifacts relevant to these probes include the datasets regenerate equivariant exact complex sweep v2 compute result (compute report)[20], summarize equivariant v2 dimension sweep rates compute result (compute report)[21], validate sample homology certificates distribution compute result (compute report)[22], equivariant v2 dimension sweep samples[23], toy fixedpoint counterexamples summary[24], group action smith coeff field consistency scan compute result (compute report)[25], and cycle7 fullscan consistency report[26]. These artifacts document systematic checks of internal consistency for families of finite equivariant chain complexes and field-coefficient comparisons. They should be read as exploratory evidence that helps identify plausible patterns and counterexamples in small models, not as verification of any general theorem about lattices.

Protocol deviation. The planned random seed for the computations is 7. Some of the cited artifact datasets were generated using additional seeds 20260213, 123456, 424242, 777, 20240213, 11, 41, 77, and 2025. This is a deviation from the planned protocol, and any conclusions drawn from these probes are treated as heuristic guidance rather than definitive evidence.

The main mathematical deliverable of this section is therefore a clear separation between what follows unconditionally from the manifold hypothesis and what requires additional structure theorems about centralizers in lattices.

Proposition 9.1 (Current status for centralizer-based obstructions)**.** *Let Γ be a group that is the fundamental group of a closed n-manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then* $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$.

Assume further that Γ is a uniform lattice in a real semisimple Lie group and that Γ contains an involution τ. At present, within the scope of this manuscript, no unconditional contradiction is derived from these hypotheses alone.

A negative answer for the existence of such an M for a given Γ would follow from an additional input establishing a strict centralizer cohomological dimension drop for involutions, together with a compatible duality formalism that forces the top-degree rational duality class associated to M to interact nontrivially with restriction to $C_\Gamma(\tau)$. These additional inputs are treated here as conditional.

Accordingly, the objective question remains open under the inputs currently established in the manuscript. The remainder of the argument, beyond the present section, is organized around making the conditional centralizer drop precise for uniform lattices, and around identifying whether the resulting conditions are compatible with the chain-level duality constraints forced by the existence of M with $\mathbb{Q}$-acyclic universal cover.

[20] `evidence_regenerate_equivariant_exact_complex_sweep_v2_compute_result.json`
[21] `evidence_summarize_equivariant_v2_dimension_sweep_rates_compute_result.json`
[22] `evidence_validate_sample_homology_certificates_distribution_compute_result.json`
[23] `equivariant_v2_dimension_sweep_samples.jsonl`
[24] `toy_fixedpoint_counterexamples_summary.json`
[25] `micro_group_action_smith_coeff_field_consistency_scan_compute_result.json`
[26] `cycle7_fullscan_consistency_report.json`

10 Case studies in representative Lie types

This section provides an illustrative discussion of how a conjectural obstruction template involving elements of order two would interact with concrete families of uniform lattices in representative real semisimple Lie groups. The manuscript does not resolve the realizability question in either direction. Instead, this section derives conditional constraints by assuming a topological template (whose validity hinges on resolving specific technical gaps) and comparing it with known and conjectural properties of lattices. The intent is to separate two logically distinct tasks: first, a topological statement that would forbid involutions in fundamental groups of closed manifolds with finite-dimensional $\mathbb{Q}$-acyclic universal covers (conditional on resolving the free deck action obstacle and upgrading rational duality); and second, a Lie-theoretic statement asserting the existence of uniform lattices containing involutions in particular real forms. This section highlights precise obstacles that block current templates from yielding unconditional obstructions, notably the freeness of deck actions preventing direct fixed-point arguments and the limited upgrade to full $\mathrm{PD}_n(\mathbb{Q})$.

10.1 The free action obstacle and conditional status of the template

The primary technical gap preventing an unconditional proof of the involution obstruction template is the freeness of deck transformations on the universal cover $\widetilde{M}$. For a closed manifold M with $\pi_1(M) \cong \Gamma$ and $\mathbb{Q}$-acyclic universal cover, the action of Γ on $\widetilde{M}$ is properly discontinuous and fixed-point-free for all non-identity elements. Consequently, for any involution $\tau \in \Gamma$ of order two, the induced action on $\widetilde{M}$ has no fixed points. This freeness blocks the direct application of Smith theory, which requires fixed-point sets to relate the cohomology of the group action to that of the underlying space. Similarly, Borel localization techniques for equivariant cohomology are inapplicable in their standard form because the fixed-point set is empty [**?**].

Without fixed points, one cannot directly compare the cohomological dimension of the centralizer $C_\Gamma(\tau)$ with the dimension of a fixed-point manifold. This disconnect prevents standard centralizer-based templates from producing an unconditional contradiction with the virtual cohomological dimension of Γ. The obstruction template therefore remains conditional, pending resolution of this free action obstacle.

Potential circumventions discussed in the literature include passing to auxiliary actions on compactifications of the associated symmetric space $X = G/K$, such as the Borel–Serre bordification [**?**]. In this setting, the group action extends to the boundary, which is a compact stratified space where fixed points might appear, allowing Smith-theoretic or localization arguments. Another approach is to study equivariant objects associated to the action of Γ on $\widetilde{M}$, for example via Borel constructions, in settings where the resulting spaces admit tractable finiteness and duality properties. Additionally, L^2-invariants might detect the presence of torsion in Γ without requiring fixed points on $\widetilde{M}$. These alternatives require additional structure, such as finite domination over $\mathbb{Q}$ of the universal cover or controlled behavior at the boundary, which has not been verified for general $\mathbb{Q}$-acyclic covers. Consequently, any conclusion in the following sections invoking the involution obstruction template is explicitly marked as conditional.

10.2 Setup and conditional reduction

Let G be a connected real semisimple Lie group with finite center and let $\Gamma < G$ be a uniform lattice. The motivating question asks whether there exists a closed manifold M with $\pi_1(M) \cong \Gamma$ and such that the universal cover $\widetilde{M}$ is finite-dimensional and $\mathbb{Q}$-acyclic. The following definition isolates the conditional verification used throughout this section.

Definition 10.1 (Involution obstruction template)**.** Fix a discrete group Γ. We say that Γ satisfies the involution obstruction template if the following implication holds: whenever Γ is realized as $\pi_1(M)$ for a closed manifold M whose universal cover $\widetilde{M}$ is finite-dimensional and $\mathbb{Q}$-acyclic, then Γ contains no element of order 2.

Remark 10.2 (Conditional status). The validity of this template for arbitrary uniform lattices is conjectural. It holds only modulo the resolution of the free action obstacle described above and the upgrade from trivial-coefficient rational Poincaré duality (which is unconditional from Cartan–Leray) to full $\mathrm{PD}_n(\mathbb{Q})$ for arbitrary $\mathbb{Q}\Gamma$-modules. The latter requires extra hypotheses such as asphericity of M or finite domination of $\widetilde{M}$ over $\mathbb{Q}$, which are not established here. Any comparison between centralizer dimensions and cohomological degrees in the sections below assumes this template holds and is therefore conditional.

A standing conjectural input used for cross-referencing. The chapter refers to Conjecture 10.3, whose content is used elsewhere in the manuscript as an additional hypothesis in centralizer-based comparisons.

Conjecture 10.3 (Centralizer rigidity input)**.** *Let Γ be a uniform lattice in a connected real semisimple Lie group with finite center, and let $\tau \in \Gamma$ have order 2. Then the centralizer $C_\Gamma(\tau)$ satisfies a cohomological finiteness and dimension property strong enough to support the centralizer-based obstruction template when combined with a suitable rational Poincaré duality upgrade for Γ.*

Proposition 10.4 (Conditional reduction to existence of 2-torsion)**.** *Assume Conjecture 10.3 and assume the involution obstruction template holds for Γ (Definition 10.1). If Γ is a uniform lattice in some real semisimple G and Γ contains an element of order 2, then there is no closed manifold M with $\pi_1(M) \cong \Gamma$ whose universal cover is finite-dimensional and $\mathbb{Q}$-acyclic.*

Proof. By the template, the existence of such an M forces Γ to have no involution. This contradicts the assumption that Γ contains an element of order 2. ☐

Thus the case studies in Lie types serve primarily to identify natural sources of involutions in uniform lattices, conditional on the template. We document what is standard in broad terms in the theory of arithmetic groups [**?**], what requires additional verification, and what remains conjectural. The obstruction framework provides only conditional incompatibility statements, illustrating a template that would obstruct realizability assuming that the free action obstacle is resolved and that the requisite lattices with involutions exist.

10.3 Type A_n: special linear groups and projective quotients

We consider $G = \mathrm{SL}(n+1, \mathbb{R})$ and also its adjoint form $\mathrm{PSL}(n+1, \mathbb{R})$. The most visible involution in the linear group is the scalar matrix $-I$, which belongs to $\mathrm{SL}(n+1, \mathbb{R})$ exactly when $n+1$ is even. In the adjoint quotient, the image of $-I$ is trivial, so involutions must be produced by other elements when working modulo the center.

Central involution when $n+1$ is even. Assume $n+1$ is even so that $-I \in \mathrm{SL}(n+1, \mathbb{R})$ has order 2. If a uniform lattice $\Gamma < \mathrm{SL}(n+1, \mathbb{R})$ contains $-I$, then assuming Conjecture 10.3 and conditional on the involution obstruction template, Γ fails to be a fundamental group as in the motivating question by Proposition 10.4.

The salient point is that $-I$ is central, so its presence in Γ is equivalent to the condition that the projection of Γ to the adjoint group has kernel containing the order-2 center. In the absence of a fixed arithmetic model in this section, we record the needed existence statement as an input.

Assumption 10.5 (Existence input for central torsion in type A_n)**.** For some even $n+1$, there exists a uniform lattice $\Gamma < \mathrm{SL}(n+1, \mathbb{R})$ such that $-I \in \Gamma$.

This assumption is nontrivial in higher rank: cocompact arithmetic lattices in split groups are constrained by rational-rank considerations, and for $\mathrm{SL}(n+1, \mathbb{R})$ (with $n+1 \geq 3$) one typically needs to pass to suitable inner forms or other real forms to obtain cocompact arithmetic lattices. This chapter does not attempt to resolve the existence question in each rank; it isolates how such an input would interact with the conditional obstruction template.

Noncentral involutions in $\mathrm{SL}(n+1, \mathbb{R})$ and $\mathrm{PSL}(n+1, \mathbb{R})$. When $n+1$ is odd, $-I$ is not available. A standard linear-algebraic source of involutions is a diagonal matrix with entries ± 1 and determinant 1. For instance, for $n+1 \geq 3$, one can consider

$$J_k := \mathrm{diag}(\underbrace{-1, \ldots, -1}_{2k}, \underbrace{1, \ldots, 1}_{n+1-2k}) \in \mathrm{SL}(n+1, \mathbb{R}),$$

which has order 2 for $1 \leq k \leq \lfloor (n+1)/2 \rfloor$. The question becomes whether a given uniform lattice intersects the conjugacy class of some J_k.

A conceptual approach is the following: when a lattice arises as the integer points of an algebraic group over a number field, rational involutions in the ambient algebraic group that satisfy the relevant integrality and level conditions may land in the lattice. In the absence of an explicit construction (and corresponding bibliography) in this section, we refrain from asserting that this occurs uniformly across A_n and instead record only the formal conditional implication.

Lemma 10.6 (Conditional obstruction for A_n via a diagonal involution)**.** *Let* $\Gamma < \mathrm{SL}(n+1, \mathbb{R})$ *be a uniform lattice. Suppose Conjecture 10.3 holds. If* Γ *contains an element conjugate in* G *to some* J_k *above, then assuming the involution obstruction template (Definition 10.1),* Γ *cannot be* $\pi_1(M)$ *for a closed manifold* M *with finite-dimensional* $\mathbb{Q}$*-acyclic universal cover.*

Proof. Any conjugate of J_k has order 2, so Γ contains an involution. Apply Proposition 10.4. □

Outcome for the type A_n case study. The obstruction step suggests potential incompatibility: any involution in a candidate lattice would forbid the desired manifold realization when the template holds. What requires further documentation is a fully cited existence theorem guaranteeing a uniform lattice with such an involution in a specified family of real forms. The centralizer structure of involutions in the ambient linear group is explicit at the Lie-group level, but promoting this to a statement about centralizers in a specific cocompact lattice is a separate group-theoretic and arithmetic input that is not verified here.

10.4 Orthogonal types B_n and D_n: quadratic forms and reflections

We next discuss $G = \mathrm{SO}(p, q)$ with $p + q \geq 3$, focusing on the classical types B_n and D_n (depending on parity of $p + q$ and connected components). Uniform lattices in these groups often arise from anisotropic quadratic forms over $\mathbb{Q}$ [**?**]. This provides a broad source of cocompact arithmetic lattices in many real forms, but this section does not attempt a signature-by-signature existence classification.

Involutions from reflections and sign changes. At the level of the ambient real Lie group, involutions arise as orthogonal reflections. For a real quadratic form of signature (p, q), a reflection across a nonisotropic vector v is given by

$$R_v(x) = x - 2\frac{\langle x, v\rangle}{\langle v, v\rangle}v,$$

which has order 2 and preserves the form. In matrix terms, various sign-change diagonal matrices have order 2 and determinant 1 (for example, sign changes in an even number of coordinates). These elements belong to $\mathrm{SO}(p, q)$ and can be realized over $\mathbb{Q}$ when the quadratic form is defined over $\mathbb{Q}$.

The key arithmetic question is whether one can arrange for a uniform lattice Γ to contain such a reflection or sign-change. Since this section does not provide the needed integrality analysis, we formulate the relevant implication as conditional.

Proposition 10.7 (Conditional obstruction for orthogonal lattices)**.** *Let* $\Gamma < \mathrm{SO}(p, q)$ *be a uniform lattice. Suppose Conjecture 10.3 holds. If* Γ *contains a nontrivial element of order* 2 *(for example a reflection* R_v *that lies in* Γ*), then assuming the involution obstruction template (Definition 10.1), there is no closed manifold* M *with* $\pi_1(M) \cong \Gamma$ *whose universal cover is finite-dimensional and* $\mathbb{Q}$*-acyclic.*

Proof. This is Proposition 10.4 applied to Γ. □

Spin covers and spinor norm constructions. A common technique for producing torsion in arithmetic subgroups of orthogonal groups passes to $\mathrm{Spin}(p, q)$ and uses elements of order 2 coming from the Clifford algebra, sometimes tracked via the spinor norm. In this manuscript segment, we do not provide source material verifying such a construction in the cocompact setting for the specific signatures required. Accordingly, the discussion is presented as a proposed approach.

Remark 10.8 (Proposed method, not executed)*.* A plausible path to explicit examples is to start from an anisotropic rational quadratic form f of signature (p, q) over $\mathbb{Q}$, form the integral points of the associated algebraic group, and then search for a rational reflection or a Clifford unit of order 2 that is integral at the chosen level. If successful, the resulting torsion element would lie in a uniform arithmetic lattice. This manuscript does not certify the existence of such an element without a detailed integrality analysis and corresponding citations.

Outcome for the orthogonal case study. As with type A_n, the obstruction framework suggests potential incompatibility once 2-torsion is present, contingent on the template and Conjecture 10.3. The remaining work is to supply an explicit family of cocompact lattices in these groups containing an involution, together with cited sources for any centralizer and cohomological-dimension comparisons used elsewhere under additional hypotheses.

10.5 Symplectic type C_n: central elements and nontrivial 2-torsion

For completeness among classical families, consider $G = \mathrm{Sp}(2n, \mathbb{R})$. Here $-I$ lies in the center and has order 2 for all $n \geq 1$. The case-study question becomes whether there exist uniform lattices $\Gamma < \mathrm{Sp}(2n, \mathbb{R})$ containing $-I$. The centralizer of $-I$ in $\mathrm{Sp}(2n, \mathbb{R})$ is the full group, so a centralizer-based comparison would involve the whole group and remains subject to the same conditional hypotheses as above.

Conditional statement. In contrast to $\mathrm{SL}(n+1, \mathbb{R})$ where $-I$ may fail to have determinant 1, the symplectic group always contains $-I$. Therefore any lattice that intersects the center nontrivially provides the desired 2-torsion.

Assumption 10.9 (Existence input for type C_n)**.** For some $n \geq 1$, there exists a uniform lattice $\Gamma < \mathrm{Sp}(2n, \mathbb{R})$ such that $-I \in \Gamma$.

Under this assumption, the template would apply immediately, contingent on resolution of the free action obstacle, Conjecture 10.3, and the $\mathrm{PD}_n(\mathbb{Q})$ upgrade. No additional structure of $\mathrm{Sp}(2n, \mathbb{R})$ is needed beyond the presence of the central involution.

Lemma 10.10 (Immediate conditional obstruction from central 2-torsion in C_n)**.** *Suppose Conjecture 10.3 holds. If $\Gamma < \mathrm{Sp}(2n, \mathbb{R})$ is a uniform lattice and $-I \in \Gamma$, then assuming the involution obstruction template (Definition 10.1), Γ cannot occur as $\pi_1(M)$ for a closed manifold M with finite-dimensional $\mathbb{Q}$-acyclic universal cover.*

Proof. The element $-I$ has order 2, so Proposition 10.4 applies. □

Outcome for the symplectic case study. The missing ingredient here is the existence of cocompact lattices that contain the central involution, stated with proper references. This chapter records the logical role played by that input and does not claim it as established.

10.6 Exceptional types and the status of existence inputs

Exceptional real forms introduce two distinct uncertainties: the existence of cocompact lattices of a given type, and the existence of involutions inside those lattices. Without a curated bibliography of arithmetic constructions, this section cannot certify either for G of types G_2, F_4, E_6, E_7, or E_8. We therefore restrict to a systematic description of what would suffice to apply the conditional template in each exceptional type.

Uniform lattices and torsion: what would be needed. Fix an exceptional real semisimple Lie group G. To apply the obstruction to the motivating question, it suffices to exhibit a uniform lattice $\Gamma < G$ and an element $\tau \in \Gamma$ with $\tau^2 = e$ and $\tau \neq e$, together with the topological hypotheses packaged in Definition 10.1 and Conjecture 10.3. The burden is thus shifted to an arithmetic or geometric construction of Γ with prescribed torsion.

Conjecture 10.11 (Existence of 2-torsion in exceptional uniform lattices)**.** *For each exceptional Lie type and at least one real form admitting cocompact lattices, there exists a uniform lattice containing an element of order* 2*.*

Remark 10.12 (Status and limitations)*.* This conjecture is recorded as an organizing hypothesis for the overall program. This chapter does not prove it and does not present explicit constructions of uniform lattices with 2-torsion in exceptional types. Any conclusion about the motivating question in exceptional families remains conditional on the conjecture (or on alternative existence results) together with the involution obstruction template and Conjecture 10.3. The verification of Conjecture 10.11 is independent of the topological obstruction and represents a distinct line of work in the theory of lattices.

Why the exceptional case is not purely formal. In the classical families, involutions are easily written down at the level of matrix groups, and it is plausible (though still requiring citations) that lattices can be arranged to contain them. In exceptional groups, even writing down concrete elements of order 2 in a convenient model may require choosing a specific realization. More importantly, controlling integrality and cocompactness simultaneously is subtle. This section therefore treats exceptional types as a bookkeeping device for inputs needed by the conditional obstruction template.

Outcome for exceptional types. At present, the manuscript records a conditional obstruction template and isolates the exceptional-type existence input as explicitly open, via Conjecture 10.11.

10.7 Synthesis and research directions within the obstruction framework

The case studies above support a consistent narrative: once Γ contains 2-torsion, the obstruction template suggests potential incompatibility with any realization of Γ as the fundamental group of a closed manifold with finite-dimensional $\mathbb{Q}$-acyclic universal cover. However, the application of this template to concrete Lie types remains conditional on resolving several technical gaps identified below.

What has been achieved in this section. This section has organized a uniform logical interface between topology and Lie theory, contingent on the involution obstruction template and Conjecture 10.3. The topological interface is the template itself (Definition 10.1), which converts the presence of 2-torsion into a negative answer to the motivating question, assuming that the template holds and that the free action obstacle can be resolved in the required form. The Lie-theoretic interface is a set of conditional existence inputs in each representative family, phrased so that future additions can be localized to a small number of assertions regarding the presence of involutions in cocompact lattices. For classical types A_n, B_n, D_n, and C_n, the required inputs concern the existence of uniform lattices containing specific diagonal, reflection, or central involutions. For exceptional types, the input is the existence of any uniform lattice with 2-torsion.

What remains unresolved and how it should be addressed next. The primary unresolved components are threefold. First, the free action obstacle must be circumvented, potentially via boundary actions on compactifications or via equivariant constructions that admit suitable finiteness properties. Second, the upgrade from trivial-coefficient rational Poincaré duality to full $\mathrm{PD}_n(\mathbb{Q})$ must be verified or bypassed for the specific comparison involving centralizers. Third, bibliographic and constructive work is required to provide cited sources or explicit constructions (with full details) for cocompact lattices with 2-torsion in representative classical families, and to decide the status of Conjecture 10.11 in exceptional types. From the obstruction viewpoint, these additions can be made without modifying the formal conditional implications recorded here, provided one verifies that the torsion element is genuinely present in the lattice under discussion and that the topological hypotheses needed for the template are met.

Relationship to the motivating question. The motivating question is answered negatively for any Γ that both contains an involution and satisfies the involution obstruction template (together with Conjecture 10.3). Insofar as one can produce uniform lattices with involutions in a given Lie type, the problem reduces to checking the template hypotheses for the corresponding manifold class. This section has not introduced any new topological hypothesis beyond those already developed earlier, and therefore does not claim additional obstructions beyond the 2-torsion mechanism. The exploratory value of this analysis lies in delineating the precise technical gaps (free deck actions, $\mathrm{PD}_n(\mathbb{Q})$ upgrade, centralizer verification) that must be bridged to transform the conditional template into an unconditional obstruction.

11 Comparison with alternative obstruction strategies

11.1 Scope and comparison criteria

This chapter compares several obstruction strategies for the following existence problem. Let Γ be a uniform lattice in a real semisimple Lie group, and suppose that Γ contains an element of order 2. Can Γ occur as the fundamental group of a closed manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic?

The aim here is classificatory. We separate what can be extracted unconditionally from the hypotheses from what would require additional geometric or equivariant structure. The guiding comparison criteria are the following: sensitivity to 2-torsion, compatibility with the fact that the Γ-action on $\widetilde{M}$ is a deck action, reliance on extra structure such as smoothness or curvature bounds, and the extent to which the approach produces group-theoretic consequences that can be checked for uniform lattices.

11.2 Smith theory and fixed-point templates

Smith theory is, in principle, the most direct tool for extracting information from the presence of 2-torsion, because it compares mod 2 homology of a space with the mod 2 homology of fixed-point sets of involutions. In typical applications one combines such fixed-point information with manifold topology in order to constrain the global structure.

11.2.1 The free deck action obstacle and possible circumventions

In the present setting the most immediate C_2-action is the action of an involution $\tau \in \Gamma$ on $\widetilde{M}$ by deck transformations. This action is free for every nontrivial element of Γ. Consequently the fixed-point set $\widetilde{M}^\tau$ is empty whenever $\tau \neq 1$. Standard Smith-theoretic conclusions therefore become vacuous for the deck action, since they concern the homology of $\widetilde{M}^\tau$.

A fixed-point obstruction strategy requires an involution acting with nonempty fixed set on a space that is related in a controlled way to $\widetilde{M}$ or to a canonical Γ-space. Several possible circumventions can be stated, each of which demands additional input.

First, one may seek an auxiliary C_2-action on a space built from $\widetilde{M}$ that is not the deck action, for example a geometric involution coming from an isometric symmetry of a chosen metric. In this direction the missing ingredient is a principled reason that such a symmetry exists and interacts suitably with the Γ-structure.

Second, one may replace $\widetilde{M}$ by a compactification or boundary object on which Γ acts and on which torsion elements can have fixed points. For uniform lattices, natural candidates include boundaries associated to symmetric spaces. Turning this idea into an obstruction requires a comparison theorem linking the boundary action to the topology of M and to the rational acyclicity of $\widetilde{M}$.

Third, one may attempt to use equivariant constructions that do not require fixed points, such as Borel equivariant cohomology or analytic invariants. In such approaches, the role of fixed-point sets is replaced by spectral sequences or trace formulas that detect torsion through other means. At present, a general mechanism that produces contradictions from $\mathbb{Q}$-acyclicity in the absence of fixed points is not established in this manuscript.

11.2.2 What remains usable from Smith-theoretic intuition

Although the deck action itself carries no fixed points, the fixed-point template remains valuable as an organizing principle. It identifies precisely where 2-torsion could enter an obstruction argument, namely through a non-free involution action on some auxiliary object that is functorially associated to M and compatible with the Γ-action. Any future development of a torsion-sensitive obstruction in this direction should explicitly specify the relevant C_2-action and the comparison map back to M or $B\Gamma$.

11.3 Localization and universal acyclicity constructions

A different philosophy treats $\mathbb{Q}$-acyclicity of $\widetilde{M}$ as evidence that the classifying map $c\colon M \to B\Gamma$ exhibits strong homological constraints after rationalization (or after inverting a set of primes). One then seeks functorial constructions on classifying spaces that force a contradiction with known properties of lattices.

The difficulty is that such constructions are typically formulated in homotopy-theoretic categories in which their functoriality and finiteness hypotheses are delicate. In particular, this chapter does not assume the availability of a universal acyclic target for $B\Gamma$ that is known to exist and to be compatible with uniform lattices in real semisimple groups. Without such verification, localization-based strategies function here as heuristic comparisons rather than as an established obstruction method.

Even with a suitable functor in hand, a conceptual tension remains. Arguments that invert 2 tend to erase the most torsion-sensitive information, while the motivating group-theoretic hypothesis is the existence of

2-torsion. For this reason, localization and universal acyclicity methods appear best suited as complementary tools that control the part of the problem visible away from the prime 2. They may help constrain rational duality phenomena or low-degree homology, but they are not naturally aligned with detecting involutions.

11.4 Cobordism and characteristic class constraints

Because M is a closed manifold, it determines characteristic numbers and cobordism invariants. A potential obstruction program uses the classifying map $c\colon M \to B\Gamma$ to pull back cohomology classes from $B\Gamma$ and pair them with characteristic classes of TM.

This program requires two bridges that are not currently available in the generality of the existence question. The first bridge relates the rational acyclicity of $\widetilde{M}$ to the cohomology of M in a way that yields concrete restrictions on characteristic classes or characteristic numbers. The second bridge provides computable information about cohomology classes on $B\Gamma$ for uniform lattices that can be detected through evaluation on $[M]$. In the absence of these bridges, cobordism and characteristic classes are best regarded as conditional refinements that become effective once additional structure is assumed, for example strong finiteness properties for $B\Gamma$ together with geometric control on M.

A useful guiding formulation is as follows. One seeks a cohomology class $\beta \in H^d(B\Gamma;\mathbb{F}_2)$ for which $c^*(\beta)$ is forced to be nontrivial by group-theoretic considerations, while manifold topology forces vanishing of all pairings

$$\langle c^*(\beta) \smile P(TM),\, [M]_{\mathbb{F}_2}\rangle$$

for a suitable family of characteristic polynomials P. Establishing either side of this dichotomy in the present setting remains open.

11.5 Geometric and analytic approaches

Several geometric approaches aim to exploit the ambient Lie-theoretic nature of Γ, for example through metric rigidity, curvature bounds, or analytic invariants such as L^2-cohomology. Their potential strength is that lattices come with canonical actions on symmetric spaces, and torsion often has strong geometric signatures in those actions.

The limitation is structural. The existence problem asks for some closed manifold with fundamental group Γ, without assuming that M inherits the locally symmetric metric or any curvature bounds. Consequently, purely geometric obstructions tend to require additional hypotheses that force a geometric realization of Γ on M or on $\widetilde{M}$. Analytic invariants might be more flexible, since they can sometimes be defined from group data alone, but the challenge is to connect them to the topological requirement that $\widetilde{M}$ is $\mathbb{Q}$-acyclic.

From the perspective of this manuscript, geometric and analytic strategies are therefore best viewed as conditional. They can be decisive in restricted classes of realizations, while the general topological existence question remains beyond their current reach.

11.6 Synthesis and working hierarchy

The comparisons above suggest a practical hierarchy for future work.

First, the most torsion-sensitive template is Smith theory, but the deck action of Γ on $\widetilde{M}$ is free, so fixed-point arguments require a different involution action on an auxiliary space together with a controlled comparison to M. Making this precise is a central technical gap.

Second, localization and universal acyclicity methods may organize rational constraints that hold away from 2, and they may help separate the rational duality consequences of $\mathbb{Q}$-acyclicity from the specifically 2-primary phenomena. Their relevance to uniform lattices in the required functorial form is presently unverified.

Third, characteristic classes and cobordism invariants become plausible once one can relate $\mathbb{Q}$-acyclicity to restrictions on the stable tangent bundle, or once one has computable classes in $H^*(B\Gamma)$ whose pullbacks can be evaluated on $[M]$. Neither ingredient is currently available at the level of generality of the main question.

Fourth, geometric and analytic approaches can provide strong conditional obstructions when M is assumed to carry additional structure related to the ambient Lie group. As a contribution to the general existence question, they currently serve mainly to identify regimes in which a positive answer would force unexpected geometric consequences.

In the remainder of the manuscript, the obstruction discussion is therefore organized around unconditional rational duality constraints and cohomological dimension bounds, with torsion-sensitive arguments presented as conditional templates whose decisive technical inputs are stated explicitly.

12 Summary of implications and open cases

Scope and reading guide. This manuscript addresses the following existence problem. Let Γ be a uniform lattice in a real semisimple Lie group, and assume that Γ contains an element of order two. Can Γ occur as the fundamental group of a closed manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic? The discussion developed in the preceding chapters produces several necessary conditions, but it does not resolve the realizability question in either direction. The results established unconditionally are constraints coming from the interaction between manifold Poincaré duality and the Cartan–Leray spectral sequence. The most torsion-sensitive obstruction ideas, especially those based on involutions and centralizers, remain conditional because they encounter a basic obstacle: deck transformations act freely on $\widetilde{M}$, so direct fixed-point arguments for the deck action do not apply.

12.1 Unconditional implications from $\mathbb{Q}$-acyclicity and manifold structure

Let M be a closed n-manifold with $\pi_1(M) \cong \Gamma$, and assume that $\widetilde{M}$ is $\mathbb{Q}$-acyclic. The free and properly discontinuous action of Γ on $\widetilde{M}$ yields the Cartan–Leray spectral sequence

$$E_2^{p,q} = H^p\big(\Gamma; H^q(\widetilde{M}; \mathbb{Q})\big) \Rightarrow H^{p+q}(M; \mathbb{Q}).$$

Since $H^q(\widetilde{M}; \mathbb{Q}) = 0$ for $q > 0$ and $H^0(\widetilde{M}; \mathbb{Q}) = \mathbb{Q}$, the edge map identifies the rational cohomology of M with group cohomology with trivial coefficients:

$$H^*(M; \mathbb{Q}) \cong H^*(\Gamma; \mathbb{Q}). \tag{11}$$

In particular, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$.

The preceding inequality is the only dimension relation available without extra hypotheses. An equality $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = n$ requires a nonvanishing top-degree class in group cohomology, which is automatic in common situations but is not purely formal from $\mathbb{Q}$-acyclicity alone. In the orientable case, the fundamental class gives $H_n(M; \mathbb{Q}) \neq 0$, hence by (11) and universal coefficients one obtains $H^n(\Gamma; \mathbb{Q}) \neq 0$ and therefore $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \geq n$. Combining both inequalities yields

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) = n \qquad \text{when } M \text{ is orientable.} \tag{12}$$

When M is not orientable, the correct top-degree statement uses the orientation character $\omega\colon \Gamma \to \{\pm 1\}$ and the associated Γ-module $\mathbb{Q}_\omega$. In that setting one has $H_n(M; \mathbb{Q}_\omega) \neq 0$ and a parallel argument yields $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \geq n$ only after establishing the appropriate comparison between $H^n(\Gamma; \mathbb{Q}_\omega)$ and $H^n(M; \mathbb{Q}_\omega)$. Without such bookkeeping, it is safest to record the unconditional estimate $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$ and to treat (12) as an orientable refinement.

A second unconditional consequence concerns Euler characteristics. Under (11), one obtains $\chi(M) = \chi(\Gamma; \mathbb{Q})$ whenever both sides are defined in the usual alternating-sum sense, for example when M is closed and Γ has a finite classifying space. This identity is compatible with known formulas for lattices, but it does not by itself provide a contradiction in the presence of torsion. Any stronger torsion-sensitive conclusion, such as a forced vanishing derived from fixed-point data, requires additional geometric input not supplied by the deck action.

12.2 What remains conditional: duality upgrades and torsion-sensitive templates

Trivial-coefficient duality versus $\mathrm{PD}_n(\mathbb{Q})$. The manifold M satisfies Poincaré duality over $\mathbb{Q}$ (or over $\mathbb{Q}_\omega$ in the nonorientable case). Via (11), this transfers to a duality statement for $H^*(\Gamma; \mathbb{Q})$ with trivial coefficients, and similarly for the orientation-twisted coefficients when that comparison is set up. This is substantially weaker than asserting that a torsion-free finite-index subgroup $\Gamma_0 \leq \Gamma$ is a Poincaré duality group over $\mathbb{Q}$ of dimension n in the standard sense, namely duality for all $\mathbb{Q}\Gamma_0$-modules. Establishing a full $\mathrm{PD}_n(\mathbb{Q})$ conclusion generally requires additional finiteness and domination hypotheses on the universal cover, or asphericity of M, neither of which is implied by $\mathbb{Q}$-acyclicity alone. Accordingly, any subsequent argument that needs duality with arbitrary coefficients must be treated as conditional.

The free action obstacle and possible circumventions. Several obstruction templates in the literature use Smith theory, Borel localization, or related fixed-point tools to relate the cohomology of a space to that of the fixed-point set of a finite subgroup action. For the present problem, the natural action available a priori is the deck action of Γ on $\widetilde{M}$. This action is free, hence for any nontrivial finite subgroup $H \leq \Gamma$ one has $\widetilde{M}^H = \varnothing$. Consequently, direct Smith-theoretic deductions from the deck action cannot produce information about nonempty fixed-point sets, and cannot be used to force contradictions by comparing fixed-set cohomology to ambient cohomology.

A torsion-sensitive approach therefore requires an auxiliary action where fixed points are not excluded by definition. One plausible direction is to seek an action of Γ or of a finite subgroup of Γ on a compactification or boundary associated to the symmetric space $X = G/K$, and to relate that auxiliary action to the topology of M by naturality or by controlled topology. Another direction is to replace fixed-point arguments by invariants designed for free actions, such as L^2-invariants or other measured-group invariants, provided one can connect them to $\mathbb{Q}$-acyclicity and to manifold constraints. At present these ideas delineate potential circumventions, but they are not implemented here in a form strong enough to yield unconditional obstructions.

12.3 Centralizers of involutions and the main conjectural input

Even after replacing fixed-point arguments by an appropriate auxiliary equivariant framework, centralizers of involutions remain a key structural invariant of lattices with 2-torsion. The obstruction heuristics explored earlier require a lattice-theoretic mechanism that controls centralizers inside Γ.

Conjecture 12.1 (Centralizer lattice property)**.** *Let G be a connected real semisimple Lie group and let $\Gamma \leq G$ be a uniform lattice. For an involution $\tau \in \Gamma$, let $G_\tau = C_G(\tau)$ be its centralizer in G and let $\Gamma_\tau = C_\Gamma(\tau)$. Then Γ_τ is a uniform lattice in a finite-index subgroup of G_τ.*

This conjecture holds in many well-understood situations, and it is compatible with the standard picture of arithmetic lattices and algebraic centralizers. For the present problem, its role is to justify comparisons between cohomological dimensions of Γ and of centralizers Γ_τ, which is the algebraic input needed for any dimension-reduction scheme.

12.4 Illustrative open regimes and next steps

The summary above clarifies the present boundary of the argument. The unconditional statements consist of the Cartan–Leray identification (11) and the associated cohomological dimension bounds, together with orientability-dependent refinements such as (12). Beyond these, the obstruction narrative is incomplete for two independent reasons. First, upgrading trivial-coefficient duality to a full group-theoretic $\mathrm{PD}_n(\mathbb{Q})$ statement requires additional hypotheses that are not provided by $\mathbb{Q}$-acyclicity. Second, torsion-sensitive methods based on involutions require a setting where fixed points are not ruled out by freeness.

The remaining open cases should therefore be interpreted as genuine gaps rather than contradictions. In higher rank, many uniform lattices contain involutions whose centralizers are large, and Conjecture 10.3 suggests that their cohomological dimensions are governed by the ambient centralizer geometry. What is currently missing is a rigorous bridge from these centralizer calculations to a topological obstruction for a closed manifold with $\mathbb{Q}$-acyclic universal cover.

A concrete agenda for future work is suggested by the two gaps. One line of inquiry is to identify conditions under which $\mathbb{Q}$-acyclicity of a free Γ-space forces a suitable finiteness or domination property over $\mathbb{Q}$, strong enough to justify the $\mathrm{PD}_n(\mathbb{Q})$ upgrade for torsion-free subgroups. A second line of inquiry is to develop an equivariant framework for torsion in Γ that is compatible with free deck actions, for example by working with actions on boundaries, compactifications, or controlled objects attached to $X = G/K$. A third line is to verify Conjecture 10.3 in the generality required for uniform lattices, or to replace it with a weaker but provable substitute that still supports inductive comparisons of cohomological dimensions.

12.5 Conclusion

The manuscript provides a set of necessary conditions and obstruction templates for rationally acyclic manifold realizations of uniform lattices with 2-torsion. The established implications are rigorous and unconditional at the level of trivial-coefficient cohomology and cohomological dimension bounds, with an orientability refinement that yields equality of dimensions in the standard case. The torsion-sensitive part of the story remains exploratory. It highlights where centralizer structure could create tension with manifold duality, while also recording precisely which missing ingredients prevent these tensions from becoming unconditional obstructions.

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