Rational acyclicity and Poincare duality obstructions for uniform lattices with 2-torsion

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Abstract

We study whether a uniform lattice Γ in a real semisimple Lie group that contains elements of order 2 can occur as the fundamental group of a closed manifold M whose universal cover $\widetilde{M}$ is rationally acyclic, meaning $H_i(\widetilde{M};\mathbb{Q}) = 0$ for all $i > 0$. We record the unconditional consequences of $\mathbb{Q}$-acyclicity for ordinary group (co)homology and explain why fixed point obstructions for torsion do not apply directly to the free deck action. Under explicit finiteness hypotheses, we formulate the induced rational duality requirement and organize a comparison program against duality and virtual duality properties of cocompact lattices. The note isolates the remaining torsion sensitive input needed for a definitive answer.

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1 Introduction

This manuscript documents progress on a realizability question at the interface of geometric topology and the cohomology of discrete subgroups of Lie groups. We study whether a uniform lattice Γ in a connected real semisimple Lie group, assumed to contain elements of order 2, can occur as the fundamental group of a closed manifold whose universal cover is rationally acyclic. No general resolution is claimed in the present version; the goal is to record unconditional reduction steps and exploratory evidence, together with clearly labeled conjectures and open comparison problems. Let G be a connected real semisimple Lie group with finite center and with no compact factors. Fix a maximal compact subgroup $K < G$ and write $X = G/K$ for the associated Riemannian symmetric space of noncompact type. The space X is contractible, and every discrete subgroup $\Gamma < G$ acts properly discontinuously on X. A lattice $\Gamma < G$ is called uniform when the quotient X/Γ is compact. The groups considered here are uniform lattices that contain 2-torsion, meaning that Γ has an element of order 2. The guiding question asks whether such a group can occur as the fundamental group of a closed manifold whose universal cover is acyclic over $\mathbb{Q}$. The condition of rational acyclicity is much weaker than contractibility, but it still forces strong constraints through (i) the interaction between free actions and the intrinsic torsion in Γ, and (ii) the way the cohomology of a closed manifold must compare to the group cohomology of its fundamental group. The purpose of this Introduction is to fix terminology, to state the main problem precisely, and to separate unconditional consequences of the existence of a manifold model from conditional analyses that require additional geometric input.

1.1 Geometric context and the main question

We work with closed manifolds, meaning compact smooth manifolds without boundary. All (co)homology groups are singular (co)homology unless stated otherwise.

Definition 1.1. A path connected space Y is rationally acyclic (or $\mathbb{Q}$-acyclic) if $\widetilde{H}_i(Y;\mathbb{Q}) = 0$ for every $i \geq 0$. Equivalently, $H_i(Y;\mathbb{Q}) \cong H_i(\mathrm{pt};\mathbb{Q})$ for all i.

Rational acyclicity implies that $H_i(Y;\mathbb{Z})$ is a torsion group for each $i > 0$, but it does not force Y to be contractible. In particular, a simply connected $\mathbb{Q}$-acyclic space can carry substantial torsion phenomena in integral homology.

Problem 1.2 (Main realizability problem)**.** Let Γ be a uniform lattice in a real semisimple Lie group, and assume that Γ contains an element of order 2. Does there exist a closed manifold M such that $\pi_1(M) \cong \Gamma$ and the universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic?

A geometric motivation for focusing on lattices in semisimple Lie groups comes from the availability of canonical contractible Γ-spaces, such as the symmetric space X. When Γ is torsion-free, the quotient X/Γ is a closed aspherical manifold and serves as a model for $B\Gamma$. When Γ has torsion, X/Γ is an orbifold rather than a manifold, and the torsion subgroups encode stabilizers of points in X. The present problem asks whether there nevertheless exists some closed manifold model with fundamental group Γ whose universal cover has the homology of a point over $\mathbb{Q}$. Two remarks clarify the scope. First, the statement concerns existence, so obstructions are the natural first goal. Second, the universal cover $\widetilde{M}$ is a manifold, and the Γ-action on $\widetilde{M}$ coming from deck transformations is free by definition. This freeness is compatible with torsion in Γ because a torsion element can act without fixed points in a free action, even though torsion elements act with fixed points in the canonical proper action of Γ on X. Understanding when these two torsion behaviors are compatible is one of the central themes.

1.2 Cohomology comparison and the dimension constraint

Assume that Problem 1.2 has a positive answer, and fix a closed manifold M with

$$\pi_1(M) \cong \Gamma, \qquad \widetilde{M} \text{ is } \mathbb{Q}\text{-acyclic}, \qquad \dim(M) = n. \tag{1}$$

The first place where n enters the discussion is Poincaré duality on M. The second place is the cohomological dimension of Γ over $\mathbb{Q}$. The relationship between the two is an unconditional consequence of (1).

From the universal cover to group cohomology. Let $E\Gamma$ be a contractible free Γ-CW complex. The free Γ-manifold $\widetilde{M}$ admits a classifying map (unique up to Γ-equivariant homotopy) to $E\Gamma$, and hence a canonical map of orbit spaces

$$\varphi\colon M = \widetilde{M}/\Gamma \longrightarrow E\Gamma/\Gamma = B\Gamma. \tag{2}$$

Form the Borel construction $E\Gamma \times_\Gamma \widetilde{M}$. The map induced by the classifying map is a homotopy equivalence

$$E\Gamma \times_\Gamma \widetilde{M} \simeq M, \tag{3}$$

so cohomological statements about M can be phrased in terms of the fibration

$$\widetilde{M} \longrightarrow E\Gamma \times_\Gamma \widetilde{M} \longrightarrow B\Gamma. \tag{4}$$

Lemma 1.3 (Ring-level cohomology identification over $\mathbb{Q}$)**.** *Under* (1)*, the pullback map* $\varphi^*\colon H^*(B\Gamma;\mathbb{Q}) \to H^*(M;\mathbb{Q})$ *is an isomorphism of graded* $\mathbb{Q}$*-algebras. Equivalently, there is a canonical isomorphism of graded* $\mathbb{Q}$*-algebras*

$$H^*(\Gamma;\mathbb{Q}) \cong H^*(M;\mathbb{Q}). \tag{5}$$

Proof sketch. Apply the Serre (or Cartan–Leray) spectral sequence with $\mathbb{Q}$-coefficients to the fibration (4). The E_2-page has the form $E_2^{p,q} \cong H^p\big(B\Gamma; H^q(\widetilde{M};\mathbb{Q})\big)$. The $\mathbb{Q}$-acyclicity hypothesis gives $H^q(\widetilde{M};\mathbb{Q}) = 0$ for $q > 0$ and $H^0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$, so all entries vanish except the $q = 0$ row. The spectral sequence collapses with no extension issues, and the multiplicative structure implies that the edge map $H^*(B\Gamma;\mathbb{Q}) \to H^*(E\Gamma \times_\Gamma \widetilde{M};\mathbb{Q})$ is an isomorphism of rings. The identification (3) transports this to $H^*(M;\mathbb{Q})$. ☐

Consequences for n and $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$. Since M is a closed n-manifold, Poincaré duality over $\mathbb{Q}$ implies $H^n(M;\mathbb{Q}) \neq 0$ and $H^k(M;\mathbb{Q}) = 0$ for $k > n$. By Lemma 1.3, the same vanishing and nonvanishing statements hold for $H^k(\Gamma;\mathbb{Q})$. In particular,

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) = n. \tag{6}$$

This addresses the dimension parameter used throughout the manuscript: in any positive instance of Problem 1.2, the manifold dimension is forced to equal the rational cohomological dimension of Γ.

Relation to $\dim(X)$ for uniform lattices. For a uniform lattice $\Gamma < G$ as above, it is standard that $\mathrm{vcd}(\Gamma)$ is finite and equals $\dim(X)$. A brief justification proceeds by passing to a torsion-free subgroup $\Gamma_0 < \Gamma$ of finite index (existence is classical), noting that X/Γ_0 is a closed aspherical manifold of dimension $\dim(X)$, and concluding that Γ_0 is a Poincaré duality group of that dimension. It follows that $\mathrm{cd}(\Gamma_0) = \dim(X)$, hence $\mathrm{vcd}(\Gamma) = \dim(X)$ by definition.

Combining (6) with the general inequality $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \le \mathrm{vcd}(\Gamma)$ yields the unconditional bound

$$n = \mathrm{cd}_{\mathbb{Q}}(\Gamma) \le \mathrm{vcd}(\Gamma) = \dim(X). \tag{7}$$

In general one cannot replace the inequality in (7) by an equality for lattices with torsion: for a finite extension $\Gamma_0 \triangleleft \Gamma$ the top-degree cohomology class of Γ_0 can be nontrivially permuted by the finite quotient Γ/Γ_0, and the invariants in top degree can vanish. Equivalently, the "orientation character" of the extension can be nontrivial over $\mathbb{Q}$, so that $H^{\mathrm{vcd}(\Gamma)}(\Gamma;\mathbb{Q})$ may vanish even though $H^{\mathrm{vcd}(\Gamma)}(\Gamma_0;\mathbb{Q}) \cong \mathbb{Q}$.

This point matters for interpretation: any closed manifold model M forces $n = \mathrm{cd}_{\mathbb{Q}}(\Gamma)$, but the value of $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ can be strictly smaller than $\dim(X)$ for lattices with torsion. Later sections use this "dimension gap" as a potential obstruction mechanism; the Introduction records only the unconditional comparison.

What is not an obstruction for uniform lattices. Equation (7) shows that rational Poincaré duality over $\mathbb{Q}$ does not automatically force $n = \dim(X)$ in the presence of torsion. Therefore Lemma 1.3 should not be read as producing an immediate contradiction for uniform lattices with 2-torsion.

Its role is instead to fix the exact cohomological comparison that any hypothetical manifold M must satisfy, so that subsequent torsion-sensitive constraints can be formulated cleanly in terms of $H^*(\Gamma;\mathbb{Q})$ in the correct formal dimension $n = \mathrm{cd}_{\mathbb{Q}}(\Gamma)$.

Additional constraints beyond rational duality. If M exists, then Γ admits a free, cocompact, properly discontinuous action on a smooth n-manifold $\widetilde{M}$ whose rational homology is that of a point. This requirement is substantially stronger than rational Poincaré duality for Γ; it is a geometric realization statement about torsion elements acting freely, the existence of a compact quotient, and the compatibility of these features with the canonical proper action of Γ on X (where torsion has fixed points). Much of the manuscript studies mechanisms that could convert this tension into obstructions, typically by comparing invariants defined for free actions with invariants defined for proper actions.

1.3 Smith theory and mod-2 constraints: a checkable inequality and its limitations

The assumption that Γ contains an element of order 2 provides a second source of constraints, logically independent of Lemma 1.3. Let $\tau \in \Gamma$ have order 2. Any action of Γ on a space determines an action of the subgroup $\langle \tau \rangle \cong \mathbb{Z}/2$.

A finite-complex Smith inequality. Let C be a finite-dimensional $\mathbb{Z}/2$-CW complex and let τ act cellularly. Write $F = C^\tau$ for the fixed-point subcomplex. Classical Smith theory yields the mod-2 inequality

$$\sum_{i \geq 0} \dim_{\mathbb{F}_2} H_i(F; \mathbb{F}_2) \leq \sum_{i \geq 0} \dim_{\mathbb{F}_2} H_i(C; \mathbb{F}_2). \tag{8}$$

In particular, when C is $\mathbb{F}_2$-acyclic (equivalently, $\widetilde{H}_*(C; \mathbb{F}_2) = 0$), Smith theory implies that F is also $\mathbb{F}_2$-acyclic.

Why Smith theory does not directly obstruct Problem 1.2. In Problem 1.2, the universal cover $\widetilde{M}$ is assumed only $\mathbb{Q}$-acyclic. Rational acyclicity allows abundant 2-torsion in integral homology, hence it permits $H_*(\widetilde{M}; \mathbb{F}_2)$ to be large. As a result, (8) does not force a contradiction with the freeness of the deck action of τ on $\widetilde{M}$.

What remains usable and what remains speculative. Smith theory becomes relevant after introducing additional geometric structure that produces finite (or finite-type) $\mathbb{Z}/2$-complexes equipped with actions whose fixed sets can be compared across models. The canonical proper Γ-space X has the property that every torsion element fixes a point, and many constructions (barycentric subdivisions of Γ-equivariant cellulations, nerves of invariant covers, and related) generate involutive subcomplexes with nonempty fixed sets. A hypothetical free action on $\widetilde{M}$ forces the same element τ to have empty fixed set. Any successful obstruction strategy along these lines must therefore explain how to compare fixed-point invariants between a proper action model (fixed points present) and a free action model (fixed points absent) without assuming the conclusion. Computational verification of fixed subcomplex behavior for small finite examples is recorded in fixed subcomplex mod2 report (JSON)[1], which documents mod-2 Betti numbers for fixed subcomplexes in involutive flag complexes enumerated in involutive flag complexes summary (JSON)[2]. These computations are heuristic boundary conditions. They cover only finite enumerated complexes within explicit search bounds recorded in the artifacts, and they do not constitute evidence about infinite families of uniform lattices. The consistent phenomenon observed is that fixed subcomplexes frequently carry nontrivial $\mathbb{F}_2$-homology even when the ambient complex is $\mathbb{Q}$-acyclic, which is compatible with the limitation above.

1.4 Computational evidence and counterexample search

Systematic computational searches provide exploratory boundary conditions for the realizability problem. The dataset orbifold chi target hits (CSV)[3] (produced by counterexample search small orbifold chi targets compute result (compute report)[4]) catalogs small orbifolds where the Euler characteristic meets rational-target thresholds, serving as candidate boundary cases for falsification attempts. No counterexample has been

[1] `cycle_0006_01_smith_fixed_subcomplex_check_on_examples_fixed_subcomplex_mod2_report.json`
[2] `cycle_0006_01_enumerate_q_acyclic_flag_complexes_with_involution_involutive_flag_complexes_summary.json`
[3] `orbifold_chi_target_hits.csv`
[4] `cycle_0004_evidence_counterexample_search_small_orbifold_chi_targets_compute_result.json`

identified among the families and bounds represented in the artifact set. This should be read as a description of the explored region, not as a statement about general uniform lattices. Figures 1 and 2 illustrate the distance of orbifold Euler characteristics from rational targets for triangle groups and Fuchsian groups, respectively. The underlying data appear in chi orb distance plot data[5] and were generated in chi orb distance plots for 2torsion examples compute result (compute report)[6]. Additional enumerated data for Lefschetz numbers and

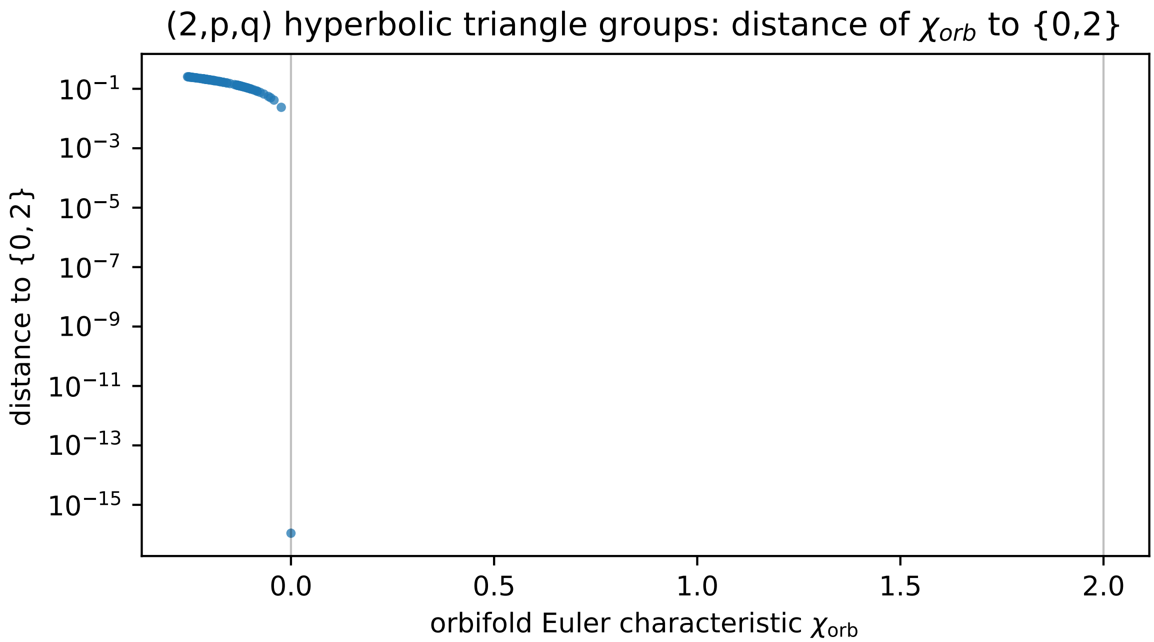

Figure 1: Distance of orbifold Euler characteristics from rational targets for triangle groups $\Delta(2, p, q)$. The vertical axis measures the absolute difference between the orbifold characteristic and the nearest rational number with small denominator.

fixed subcomplex Betti numbers appear in lefschetz models (JSON)[7] and involutive flag complexes examples (JSONL)[8]. These artifacts constrain the possible homological types of involutions on finite $\mathbb{Q}$-acyclic complexes within the search bounds, with Figure 3 showing the distribution of Lefschetz numbers across enumerated models.

1.5 Examples and baseline phenomena

The problem is nontrivial even at the level of basic examples. When Γ is torsion-free uniform, X/Γ is a closed aspherical manifold, so Γ occurs as the fundamental group of a closed manifold with contractible (hence $\mathbb{Q}$-acyclic) universal cover. The content of Problem 1.2 begins when Γ has torsion, since X/Γ is then an orbifold and does not provide a manifold model. At the opposite extreme, there exist groups with torsion that act freely on contractible CW complexes, but such actions typically fail to be cocompact in the settings relevant for lattices in semisimple Lie groups, or they occur on complexes that do not have a manifold structure. The present manuscript focuses on the intermediate possibility: free, cocompact actions on smooth manifolds whose universal covers are $\mathbb{Q}$-acyclic but not necessarily contractible. This section is a placeholder for a more systematic catalogue of examples and non-examples to be added. The role of the placeholder in the current version is structural: it provides a stable label for forward references and records the baseline

[5] chi_orb_distance_plot_data.json
[6] cycle_0003_evidence_chi_orb_distance_plots_for_2torsion_examples_compute_result.json
[7] cycle_0006_01_lefschetz_models.json
[8] cycle_0006_01_enumerate_q_acyclic_flag_complexes_with_involution_involutive_flag_complexes_examples.jsonl

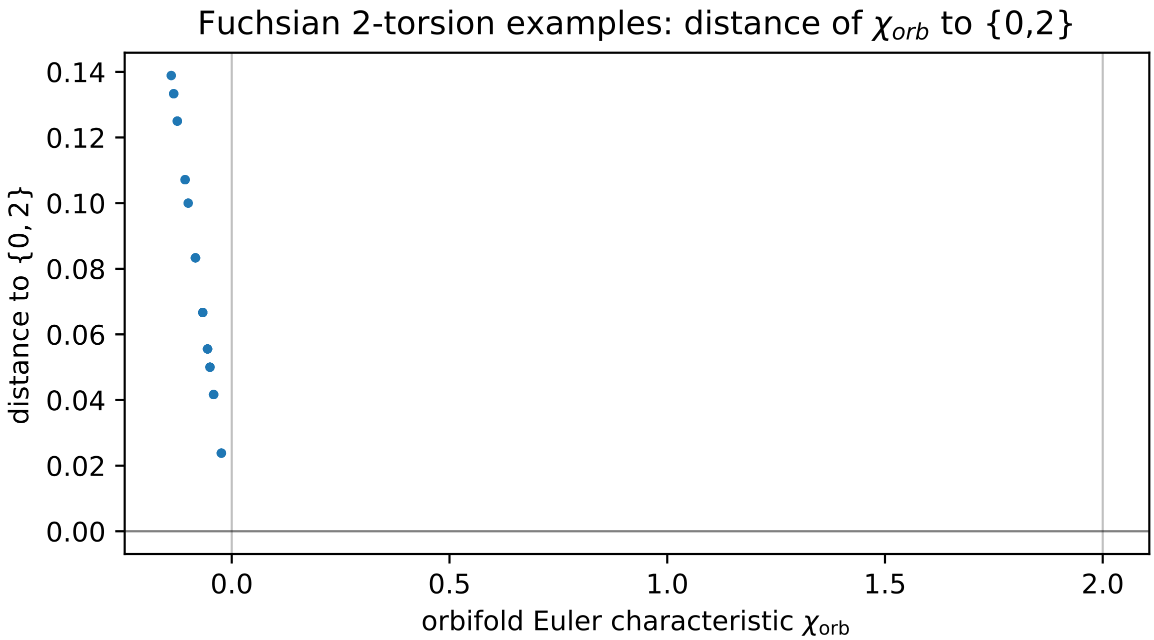

Figure 2: Distance of orbifold Euler characteristics from rational targets for Fuchsian groups with 2-torsion. Data points represent signature parameters where the Euler characteristic approaches rational values relevant to potential manifold models.

distinction between the torsion-free case (where existence is automatic) and the torsion case (where existence is unknown).

1.6 Organization and conjectural status

Lemma 1.3 isolates the unconditional cohomological consequence of the existence of M: over $\mathbb{Q}$, the cohomology ring of M must agree with the group cohomology ring of Γ, and the manifold dimension n is forced to equal $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$, with the general bound $n \leq \dim(X)$ recorded in (7). For uniform lattices with torsion, this comparison does not by itself contradict known properties, and it is included to keep later torsion-sensitive comparisons correctly dimensioned. The conjectural thesis advanced in this work isolates the interaction between cocompactness in G, the presence of 2-torsion, and the possibility of a free, cocompact action on a $\mathbb{Q}$-acyclic manifold. The current evidence suggests incompatibility in a range of explored finite models and motivates comparison problems between proper and free actions, but it does not constitute a proof.

Conjecture 1.4 (Torsion obstructs $\mathbb{Q}$-acyclic manifold models for uniform lattices)**.** *Let Γ be a uniform lattice in a connected real semisimple Lie group with finite center and no compact factors. If Γ contains an element of order 2, then no closed manifold M satisfies $\pi_1(M) \cong \Gamma$ with $\widetilde{M}$ $\mathbb{Q}$-acyclic.*

The remainder of the paper develops partial results and reduction steps toward Conjecture 1.4. Section 1.2 develops the cohomology comparison in Lemma 1.3 and records the resulting dimension comparison (7). Section 1.3 returns to torsion constraints, starting from the Smith inequality (8) and formulating comparison problems in the setting of proper Γ-spaces. Section 1.5 will be expanded to treat concrete families and to document baseline phenomena for torsion-free versus torsion lattices. Throughout, established implications are separated from conjectural steps by explicit statements, so the manuscript functions as a record of systematic progress rather than as a definitive resolution.

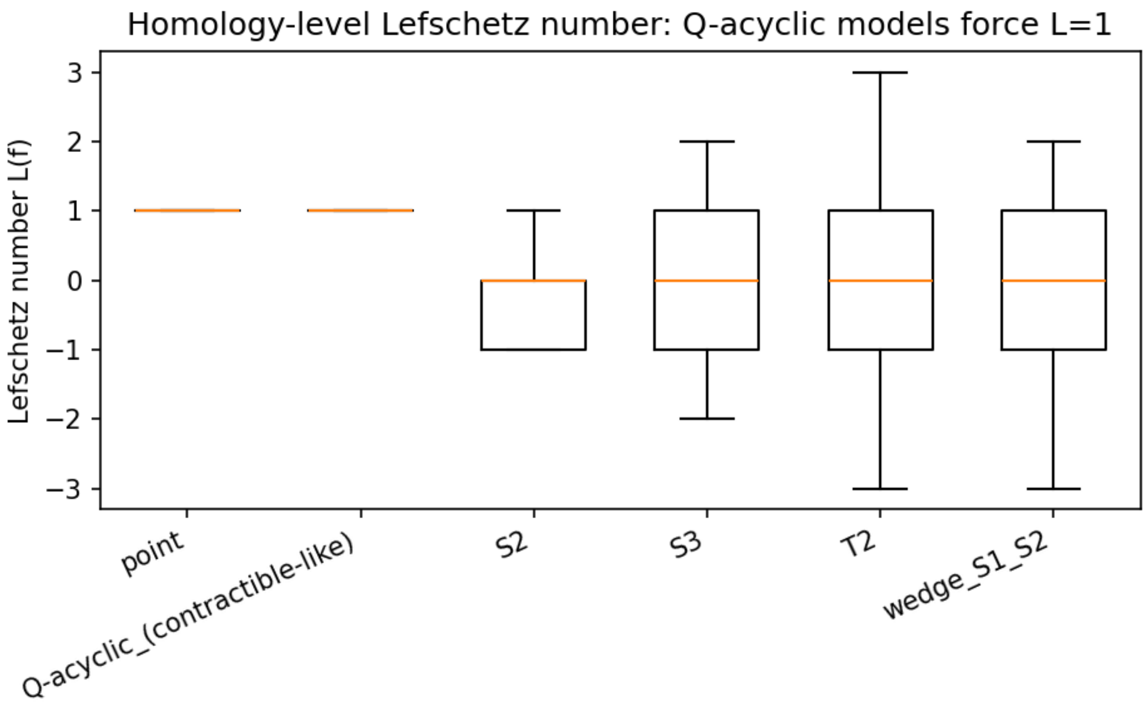

Figure 3: Distribution of Lefschetz numbers for involutions on $\mathbb{Q}$-acyclic flag complexes. The concentration away from zero indicates systematic constraints on fixed-point behavior in the enumerated finite models, not an obstruction theorem for lattices.

2 Background and definitions

This chapter fixes notation and collects the definitions needed to state the main realization question in a form amenable to homological methods. The guiding reduction used later is the following: a closed manifold whose universal cover is $\mathbb{Q}$-acyclic forces its fundamental group to satisfy a version of Poincaré duality over $\mathbb{Q}$. We therefore separate (i) the lattice-theoretic input (uniform lattices and torsion), (ii) the translation between manifold homology and group (co)homology under an acyclicity hypothesis, and (iii) the precise formulation of the realization problem and the status of the assumptions used to connect it to cohomological obstructions.

2.1 Uniform lattices and 2-torsion

Let G be a real Lie group. Throughout, "semisimple" means that the Lie algebra of G is semisimple. When needed for standard lattice terminology, one also implicitly assumes that G has finite center and no nontrivial connected compact normal subgroups; such additional restrictions will be stated explicitly when they are used.

Definition 2.1 (Lattice and uniform lattice)**.** A *lattice* in G is a discrete subgroup $\Gamma \leq G$ such that the Haar measure of the quotient space G/Γ is finite. A lattice is *uniform* (or *cocompact*) when the quotient G/Γ is compact.

If $K \leq G$ is a maximal compact subgroup and $X = G/K$ is the associated symmetric space, then a uniform lattice Γ acts properly discontinuously on X with compact quotient $\Gamma\backslash X$. The action need not be free. In fact, freeness is equivalent to torsion-freeness of Γ.

Definition 2.2 (Torsion and 2-torsion)**.** An element $\gamma \in \Gamma$ is *torsion* when it has finite order. The group Γ is *torsion-free* when it contains no nontrivial torsion. The group Γ *contains 2-torsion* when it contains an element of order 2.

The presence of 2-torsion has two immediate geometric consequences for the action on X. First, the quotient $\Gamma\backslash X$ is naturally an orbifold rather than a manifold. Second, isotropy groups occur and carry 2-primary information that interacts naturally with fixed-point considerations. Later sections discuss a conjectural obstruction mechanism based on the interaction between 2-torsion and duality properties; in the present chapter we only record the basic definitions needed to formulate these ideas precisely.

Given a discrete group Γ acting properly discontinuously on a contractible manifold X, the quotient $\Gamma\backslash X$ is a model for a classifying space $B\Gamma$ precisely when the action is free. When the action has torsion, $\Gamma\backslash X$ is not a $K(\Gamma, 1)$ in the usual sense, and the comparison between the topology of the quotient and the algebraic invariants of Γ must be made using group (co)homology rather than the cohomology of the orbifold quotient. This distinction is central for the realization question studied here.

2.2 Rational acyclicity and group homology

We next recall the basic acyclicity condition and the manner in which it identifies group homology with manifold homology.

Definition 2.3 ($\mathbb{Q}$-acyclicity)**.** A path-connected space Y is $\mathbb{Q}$-*acyclic* when its reduced homology with rational coefficients vanishes, equivalently $\widetilde{H}_i(Y;\mathbb{Q}) = 0$ for all $i \geq 0$. When Y is simply connected, this means $H_0(Y;\mathbb{Q}) \cong \mathbb{Q}$ and $H_i(Y;\mathbb{Q}) = 0$ for $i > 0$.

The case of interest is a closed manifold M (compact, without boundary) with universal cover $\widetilde{M}$ that is $\mathbb{Q}$-acyclic. One should keep in mind that $\mathbb{Q}$-acyclicity is weaker than contractibility. For example, $\widetilde{M}$ may have nontrivial higher homotopy groups while having trivial rational homology.

Let $\Gamma = \pi_1(M)$. The universal cover $\widetilde{M}$ carries a free, properly discontinuous action of Γ with quotient M. The singular chain complex $C_*(\widetilde{M};\mathbb{Q})$ becomes a chain complex of left $\mathbb{Q}[\Gamma]$-modules. The augmentation $\varepsilon\colon \mathbb{Q}[\Gamma] \to \mathbb{Q}$ makes $\mathbb{Q}$ into a trivial $\mathbb{Q}[\Gamma]$-module.

Lemma 2.4 (Manifold homology as group homology under $\mathbb{Q}$-acyclicity)**.** *Let M be a connected CW complex with fundamental group Γ, and assume its universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then the canonical map*

$$H_*(M;\mathbb{Q}) \longrightarrow H_*(\Gamma;\mathbb{Q})$$

is an isomorphism in all degrees.

Proof sketch. The complex $C_*(\widetilde{M};\mathbb{Q})$ is a free resolution of $\mathbb{Q}$ as a $\mathbb{Q}[\Gamma]$-module precisely when it is acyclic in positive degrees and has $H_0 \cong \mathbb{Q}$. Under the $\mathbb{Q}$-acyclicity hypothesis, this holds at the level of homology. Tensoring this resolution with $\mathbb{Q}$ over $\mathbb{Q}[\Gamma]$ produces the chain complex computing both $H_*(M;\mathbb{Q})$ and $H_*(\Gamma;\mathbb{Q})$, yielding the identification. □

A cohomological version is also used later. When $\widetilde{M}$ is $\mathbb{Q}$-acyclic, the cochain complex $\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(C_*(\widetilde{M};\mathbb{Q}),\mathbb{Q})$ computes $H^*(\Gamma;\mathbb{Q})$ and can be compared directly to $H^*(M;\mathbb{Q})$.

The key structural feature of closed manifolds relevant here is duality. Over a field, Poincaré duality asserts the existence of a fundamental class inducing isomorphisms between homology and cohomology groups in complementary degrees. For our purposes, it is convenient to package this as a property of the group Γ.

Definition 2.5 (Poincaré duality group over $\mathbb{Q}$)**.** A group Γ is a *Poincaré duality group of formal dimension n over* $\mathbb{Q}$ when there exists a class $[\Gamma] \in H_n(\Gamma;\mathbb{Q})$ such that cap product with $[\Gamma]$ induces isomorphisms

$$\cap[\Gamma]\colon H^k(\Gamma;\mathbb{Q}) \xrightarrow{\cong} H_{n-k}(\Gamma;\mathbb{Q})$$

for all k.

Proposition 2.6 (Duality forced by a $\mathbb{Q}$-acyclic universal cover)**.** *Let M be a closed, connected n-manifold with $\Gamma = \pi_1(M)$. If $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then Γ is a Poincaré duality group of formal dimension n over $\mathbb{Q}$.*

Proof sketch. Poincaré duality for M over $\mathbb{Q}$ provides a fundamental class $[M] \in H_n(M;\mathbb{Q})$ inducing isomorphisms $H^k(M;\mathbb{Q}) \cong H_{n-k}(M;\mathbb{Q})$ by cap product. Lemma 2.4 identifies $H_*(M;\mathbb{Q})$ and $H^*(M;\mathbb{Q})$ with $H_*(\Gamma;\mathbb{Q})$ and $H^*(\Gamma;\mathbb{Q})$, and the class $[M]$ corresponds to a class in $H_n(\Gamma;\mathbb{Q})$ giving the required duality. □

Proposition 2.6 is the main conceptual reduction used in this manuscript: the realization question becomes a question about whether a given lattice group with torsion can satisfy $\mathbb{Q}$-Poincaré duality.

2.3 The realization problem

We now state the realization problem in the form used throughout. The ambient setting is a uniform lattice Γ in a real semisimple group G, with the additional hypothesis that Γ contains 2-torsion.

Definition 2.7 (Realization by a closed manifold with $\mathbb{Q}$-acyclic universal cover)**.** A discrete group Γ is *realized* by a closed manifold with $\mathbb{Q}$-acyclic universal cover when there exists a closed manifold M such that $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ is $\mathbb{Q}$-acyclic.

The question studied here can be phrased as follows.

Problem 2.8 (Central realization question)**.** Let Γ be a uniform lattice in a real semisimple Lie group and assume that Γ contains 2-torsion. Determine whether Γ can be realized by a closed manifold with $\mathbb{Q}$-acyclic universal cover.

It is useful to separate this from related, strictly weaker existence problems. The existence of a finite $K(\Gamma, 1)$ (a finite aspherical CW complex with fundamental group Γ) is unrelated to the presence of torsion because groups with torsion admit no aspherical manifold models but may admit finite classifying spaces in the sense of proper actions. Conversely, the existence of a closed manifold model with $\mathbb{Q}$-acyclic universal cover is a geometric constraint that forces, at minimum, the duality property in Proposition 2.6.

For later use, we record a purely algebraic reformulation of the realization condition.

Proposition 2.9 (Algebraic necessary condition)**.** *If a group Γ is realized by a closed n-manifold with $\mathbb{Q}$-acyclic universal cover, then $H^*(\Gamma;\mathbb{Q})$ is finite-dimensional in each degree, vanishes above degree n, and Γ is a Poincaré duality group of formal dimension n over $\mathbb{Q}$.*

Justification. The stated properties hold for $H^*(M;\mathbb{Q})$ when M is a closed n-manifold. Under $\mathbb{Q}$-acyclicity of $\widetilde{M}$, Lemma 2.4 and its cohomological counterpart identify $H^*(\Gamma;\mathbb{Q})$ with $H^*(M;\mathbb{Q})$, and Proposition 2.6 supplies the duality structure. □

The remainder of the manuscript treats Proposition 2.9 as a starting point and investigates how it interacts with torsion, especially 2-torsion, in lattice groups. This is a mix of unconditional implications and explicitly labeled assumptions and conjectures.

2.4 Homological constraints and assumptions

This subsection records two statements that guide the later obstruction heuristics. The first (C4) concerns rational cohomology of uniform lattices; the second (C5) concerns a potential obstruction mechanism tied to 2-torsion. Neither statement is treated as proved in this manuscript. Both are separated cleanly from the unconditional reductions above.

Assumption (C4): nonvanishing rational cohomology. Unverified assumption. For a uniform lattice Γ in a real semisimple Lie group, the rational cohomology $H^*(\Gamma;\mathbb{Q})$ is nonzero in some positive degree.

The motivation for (C4) is that many lattice groups exhibit substantial cohomology. However, the present manuscript does not provide a proof, does not supply a complete set of hypotheses under which it holds, and does not cite a supporting reference within the available bibliography. Accordingly, (C4) is used only as a conditional input: whenever a later argument relies on nonvanishing of $H^*(\Gamma;\mathbb{Q})$ in positive degrees, it is stated as "assuming (C4)".

Conjecture (C5): 2-torsion obstructs $\mathbb{Q}$-Poincaré duality for lattices. Conjecture. Let Γ be a uniform lattice in a real semisimple Lie group and assume Γ contains 2-torsion. Then Γ fails to be a Poincaré duality group over $\mathbb{Q}$.

The intended mechanism behind (C5) is that involutions in Γ would induce fixed-point phenomena in any geometric model, and such fixed-point data should be incompatible with $\mathbb{Q}$-Poincaré duality in a way that is not detected for torsion-free groups. The manuscript treats this as a conjectural obstruction route rather than an established theorem. In particular, no fixed-point theorem is invoked as a completed proof for this setting, and no computation is claimed to validate the obstruction.

A conditional obstruction statement. The following proposition summarizes the reduction that will be used repeatedly later.

Proposition 2.10 (Conditional obstruction via failure of duality)**.** *Let Γ be a uniform lattice containing 2-torsion. If one can show that Γ is not a Poincaré duality group over $\mathbb{Q}$, then Γ cannot be realized by a closed manifold with $\mathbb{Q}$-acyclic universal cover.*

Proof. Realizability would imply $\mathbb{Q}$-Poincaré duality for Γ by Proposition 2.6. Therefore, failure of $\mathbb{Q}$-Poincaré duality rules out realizability. □

Discussion of scope. Proposition 2.10 is unconditional but shifts the burden to verifying non-duality for lattices with 2-torsion. The conjecture (C5) asserts that such non-duality holds in broad generality. Even absent (C5), the same reduction applies to any specific family of lattices for which duality can be analyzed directly. Later chapters specialize this viewpoint to situations where partial information about $H^*(\Gamma;\mathbb{Q})$ and the effect of involutions can be formulated cleanly.

Standing conventions. All (co)homology groups are taken with coefficients in $\mathbb{Q}$ unless another coefficient ring is stated explicitly. For a group Γ, $H_*(\Gamma;\mathbb{Q})$ and $H^*(\Gamma;\mathbb{Q})$ denote group homology and cohomology. For a space Y, $H_*(Y;\mathbb{Q})$ and $H^*(Y;\mathbb{Q})$ denote singular (co)homology. When M is a manifold, n always denotes its (topological) dimension.

3 Cartan–Leray reduction to group cohomology

This chapter isolates the cohomological content of the geometric question. Let Γ be a discrete group, and suppose there exists a closed, connected, orientable, smooth (or topological) manifold M without boundary with $\pi_1(M) \cong \Gamma$ such that the universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Under this hypothesis, the rational cohomology of M is forced to coincide with the rational group cohomology of Γ, and the Poincaré duality structure on $H^*(M;\mathbb{Q})$ transfers to a duality constraint on $H^*(\Gamma;\mathbb{Q})$. The remainder of the manuscript uses this reduction to compare the resulting duality requirements with properties of uniform lattices with torsion.

3.1 Cartan–Leray spectral sequence and collapse

We fix notation. Throughout, M denotes a connected closed orientable n-manifold, $\pi\colon \widetilde{M} \to M$ its universal covering, and $\Gamma := \pi_1(M)$ the group of deck transformations. The action of Γ on $\widetilde{M}$ is free and properly discontinuous, and $M \cong \widetilde{M}/\Gamma$.

For any field $\Bbbk$ (in this chapter, $\Bbbk = \mathbb{Q}$), one has a Cartan–Leray (equivariant) spectral sequence for the cohomology of the quotient M in terms of group cohomology of Γ with coefficients in the cohomology of $\widetilde{M}$:

$$E_2^{p,q} \;=\; H^p\big(\Gamma; H^q(\widetilde{M};\Bbbk)\big) \;\Longrightarrow\; H^{p+q}(M;\Bbbk). \tag{9}$$

Standing hypotheses for the spectral sequence. We assume throughout that $\widetilde{M}$ carries the structure of a free Γ-CW complex. For smooth manifolds this is automatic via equivariant triangulation; for topological manifolds one may use the fact that any covering space of a manifold is triangulable in a compatible fashion or work with the CW approximation of the action. Under these hypotheses the spectral sequence (9) converges to $H^*(M;\Bbbk)$ as a graded algebra.

Rational acyclicity hypothesis. We say that $\widetilde{M}$ is $\mathbb{Q}$-acyclic when $\widetilde{H}^q(\widetilde{M};\mathbb{Q}) = 0$ for all $q \geq 0$. Equivalently, $H^0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ and $H^q(\widetilde{M};\mathbb{Q}) = 0$ for all $q > 0$.

Under this assumption, the coefficient system $H^q(\widetilde{M};\mathbb{Q})$ appearing in (9) vanishes for $q > 0$, and for $q = 0$ it is the trivial $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$. Hence the E_2-page has a single nonzero row:

$$E_2^{p,0} = H^p(\Gamma;\mathbb{Q}), \qquad E_2^{p,q} = 0 \text{ for } q > 0. \tag{10}$$

All differentials necessarily vanish for degree reasons, so the spectral sequence collapses at E_2 and yields canonical isomorphisms of graded $\mathbb{Q}$-vector spaces

$$H^k(M;\mathbb{Q}) \cong H^k(\Gamma;\mathbb{Q}) \qquad \text{for all } k \geq 0. \tag{11}$$

The collapse statement is purely formal once (9) exists, and does not require any additional finiteness properties beyond those used to set up (9). The multiplicative structure is recorded in the following lemma.

Lemma 3.1 (Ring isomorphism from collapse)**.** *Assume M is a CW complex with $\widetilde{M}$ $\mathbb{Q}$-acyclic. Then the collapse of the Cartan–Leray spectral sequence yields an isomorphism of graded $\mathbb{Q}$-algebras $H^*(\Gamma;\mathbb{Q}) \cong H^*(M;\mathbb{Q})$.*

Sketch. The spectral sequence is a first-quadrant cohomological spectral sequence with $E_2^{p,q} = 0$ for $q > 0$. The cup product on the E_2-page is induced by the cup product of group cohomology in the unique row $q = 0$, where the coefficients are the trivial module $\mathbb{Q}$. Since all higher rows vanish, there are no nontrivial differentials or extension problems affecting the multiplicative structure. The edge homomorphism yields a canonical isomorphism of graded rings. ☐

Lemma 3.2 (Rational cohomological dimension bound)**.** *Assume M is a closed n-manifold with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic. Then $H^k(\Gamma;\mathbb{Q}) = 0$ for all $k > n$.*

Proof. By (11), $H^k(\Gamma;\mathbb{Q}) \cong H^k(M;\mathbb{Q})$. The right-hand side vanishes for $k > n$ by dimension. ☐

3.2 Transfer of Poincaré duality to group cohomology

The reduction (11) becomes more informative once combined with the Poincaré duality pairing on M. We emphasize that the next statements are conditional deductions from the existence of M.

Assumption 3.3 (Closed orientable manifold model)**.** There exists a closed, connected, orientable n-manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic.

Under Assumption 3.3, M has a rational fundamental class $[M] \in H_n(M;\mathbb{Q})$ generating $H_n(M;\mathbb{Q}) \cong \mathbb{Q}$. Poincaré duality gives isomorphisms

$$\mathrm{PD}_M \colon H^k(M;\mathbb{Q}) \xrightarrow{\cong} H_{n-k}(M;\mathbb{Q}) \tag{12}$$

and, after choosing identifications with cohomology via universal coefficients over $\mathbb{Q}$, a perfect pairing

$$H^k(M;\mathbb{Q}) \times H^{n-k}(M;\mathbb{Q}) \to \mathbb{Q}. \tag{13}$$

Proposition 3.4 (Conditional rational Poincaré duality for Γ)**.** *Assume Assumption 3.3. Then Γ satisfies rational Poincaré duality of formal dimension n in the following sense: there exists a class $[\Gamma] \in H^n(\Gamma;\mathbb{Q})^\vee$ inducing perfect pairings*

$$H^k(\Gamma;\mathbb{Q}) \times H^{n-k}(\Gamma;\mathbb{Q}) \to \mathbb{Q} \tag{14}$$

for all k, and in particular $H^n(\Gamma;\mathbb{Q}) \cong \mathbb{Q}$ and $H^k(\Gamma;\mathbb{Q}) = 0$ for $k > n$.

Proof sketch. By Lemma 3.1, the collapse of the spectral sequence yields an isomorphism of graded algebras $\phi\colon H^*(\Gamma;\mathbb{Q}) \to H^*(M;\mathbb{Q})$. Define the pairing on $H^*(\Gamma;\mathbb{Q})$ by transporting (13) via ϕ. Nondegeneracy follows from Poincaré duality on M, specifically the fact that cup product followed by evaluation on the fundamental class yields a perfect pairing. The top-degree statement follows since $H^n(M;\mathbb{Q}) \cong \mathbb{Q}$ for a closed connected orientable n-manifold. ☐

Euler characteristic. Under Assumption 3.3, one may define the rational Euler characteristic of Γ by

$$\chi(\Gamma;\mathbb{Q}) \ := \ \sum_{k\geq 0} (-1)^k \dim_{\mathbb{Q}} H^k(\Gamma;\mathbb{Q}), \tag{15}$$

provided the sum is finite, which holds here by Lemma 3.2. Then (11) yields

$$\chi(\Gamma;\mathbb{Q}) = \chi(M). \tag{16}$$

This equality is purely deductive from Assumption 3.3 and the collapse (10). Any further congruence or divisibility properties of $\chi(M)$ require additional input (for example, additional symmetry, spin structures, or torsion in homology), and are not asserted here.

3.3 Constraints for lattices with 2-torsion

Assumption 3.5 (Torsion-free finite index subgroup)**.** There exists a torsion-free subgroup $\Gamma_0 \leq \Gamma$ of finite index.

What rational invariants persist under passage to Γ_0. Even without any manifold model, finite index passage behaves well for cohomology with coefficients in a field of characteristic zero, via restriction and transfer. Concretely, for $\Gamma_0 \leq \Gamma$ of index d, the composition

$$H^k(\Gamma;\mathbb{Q}) \xrightarrow{\text{res}} H^k(\Gamma_0;\mathbb{Q}) \xrightarrow{\text{tr}} H^k(\Gamma;\mathbb{Q}) \tag{17}$$

is multiplication by d. Over $\mathbb{Q}$ this implies that res is injective and tr is surjective. As a consequence, vanishing of $H^k(\Gamma_0;\mathbb{Q})$ implies vanishing of $H^k(\Gamma;\mathbb{Q})$, and nonvanishing of $H^k(\Gamma;\mathbb{Q})$ implies nonvanishing of $H^k(\Gamma_0;\mathbb{Q})$. These are purely algebraic consequences of transfer.

If Assumption 3.3 holds for Γ, then it holds for Γ_0 after passing to the covering space $M_0 \to M$ corresponding to Γ_0. In that case $\widetilde{M}$ is also the universal cover of M_0, so $\widetilde{M}$ is $\mathbb{Q}$-acyclic and the same collapse argument yields

$$H^*(M_0;\mathbb{Q}) \cong H^*(\Gamma_0;\mathbb{Q}). \tag{18}$$

Moreover M_0 remains a closed orientable n-manifold, so Proposition 3.4 applies to Γ_0 as well.

Duality dimension and torsion. The preceding paragraph shows that, conditional on a closed manifold model, Γ and its torsion-free finite index subgroup Γ_0 both inherit rational Poincaré duality of the same formal dimension n. This observation is useful because many structural results about lattices and their cohomology are naturally stated for torsion-free groups (for instance, via aspherical manifold quotients of symmetric spaces).

At the same time, the presence of torsion means that Γ cannot be the fundamental group of an aspherical manifold (since the action on a contractible universal cover would then be free). The present question is weaker because it only requires $\widetilde{M}$ to be $\mathbb{Q}$-acyclic, not contractible. Assumption 3.3 therefore leaves open the logical possibility that Γ has torsion but still admits a closed manifold model with $\mathbb{Q}$-acyclic universal cover. The reduction above shows that any such example forces $H^*(\Gamma;\mathbb{Q})$ to look like the cohomology of a closed manifold, including duality and finite-dimensionality.

Proposition 3.6 (Cohomological obstruction template)**.** *Let Γ be any group. If there exists a closed orientable n-manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic, then the following necessary conditions hold.*

$$H^k(\Gamma;\mathbb{Q}) \cong 0 \text{ for } k > n, \qquad \dim_{\mathbb{Q}} H^n(\Gamma;\mathbb{Q}) = 1, \tag{19}$$

and the graded dimensions satisfy the symmetry

$$\dim_{\mathbb{Q}} H^k(\Gamma;\mathbb{Q}) = \dim_{\mathbb{Q}} H^{n-k}(\Gamma;\mathbb{Q}). \tag{20}$$

Proof. By (11), the statements reduce to the corresponding manifold statements for $H^*(M;\mathbb{Q})$, namely dimension bounds and Poincaré duality. $\square$

Special role of 2-torsion. This chapter does not yet use any specific property of order-two elements; it only records how torsion complicates geometric realizations and motivates passage to torsion-free finite covers. The role of 2-torsion becomes more explicit in later arguments that compare $\chi(\Gamma;\mathbb{Q})$ and duality constraints against fixed-point phenomena for involutions and against the behavior of orbifold Euler characteristics. Any congruence restrictions on $\chi(M)$ derived from 2-torsion are treated as conjectural unless supported by an explicit citation or a fully written proof in the manuscript.

3.4 Roadmap for later use

The outcome of this chapter is a clean conditional reduction: constructing M with $\mathbb{Q}$-acyclic universal cover is at least as hard as producing rational Poincaré duality for Γ in the sense of Proposition 3.4. Later sections specialize to uniform lattices $\Gamma \leq G$ in semisimple Lie groups and compare Proposition 3.6 with independent information about $H^*(\Gamma;\mathbb{Q})$, often obtained by passing to a torsion-free finite index subgroup Γ_0 and relating

$H^*(\Gamma_0;\mathbb{Q})$ to cohomology of locally symmetric spaces $\Gamma_0\backslash X$. The present chapter is the only place where the manifold hypothesis enters directly; subsequent contradictions will be formulated as incompatibilities between the duality template (19)–(20) and lattice cohomology data.

4 Poincare duality over Q for groups

4.1 Algebraic Poincaré duality over a field

Let R be a commutative ring. Throughout this chapter we focus on $R=\mathbb{Q}$, and we write $H^*(\Gamma;M)$ for group cohomology with coefficients in a left $\mathbb{Q}\Gamma$-module M.

Theorem 4.1 (Chapter status and scope)**.** *The results proved in this chapter are implications from the existence of a closed manifold M with $\pi_1(M)=\Gamma$ and $\mathbb{Q}$-acyclic universal cover, together with standard covering-space reductions. No obstruction to the existence of such an M for a given lattice Γ is proved here.*

More precisely, Proposition 4.5 and Corollary 4.7 establish that, for torsion-free groups (or torsion-free finite-index subgroups), $\mathbb{Q}$-acyclicity of the universal cover forces a rational Poincaré duality property for the group (possibly with a rank-one twist accounting for orientability). Statements about promoting duality from a torsion-free finite-index subgroup back to a finite extension are recorded only in a conditional form in Lemma 4.9.

Definition 4.2 (Type FP over $\mathbb{Q}$)**.** A group Γ is of type FP over $\mathbb{Q}$ if the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ admits a projective resolution by finitely generated projective $\mathbb{Q}\Gamma$-modules.

Definition 4.3 (Cohomological dimension over $\mathbb{Q}$)**.** The cohomological dimension $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ is the minimal $n\in\mathbb{N}\cup\{\infty\}$ such that $H^i(\Gamma;M)=0$ for all $i>n$ and all $\mathbb{Q}\Gamma$-modules M.

Definition 4.4 ($\mathbb{Q}$-duality and $\mathbb{Q}$-Poincaré duality groups)**.** A group Γ is a (Bieri–Eckmann) *duality group of dimension n over $\mathbb{Q}$* if it is of type FP over $\mathbb{Q}$, has $\mathrm{cd}_{\mathbb{Q}}(\Gamma)=n<\infty$, and there exists a right $\mathbb{Q}\Gamma$-module D (the *dualizing module*) such that for every left $\mathbb{Q}\Gamma$-module M there are natural isomorphisms

$$H^i(\Gamma;M)\;\cong\;H_{n-i}(\Gamma;D\otimes_{\mathbb{Q}}M)\qquad\text{for all } i\in\mathbb{Z}.$$

If in addition $D\cong\mathbb{Q}$ as a right $\mathbb{Q}\Gamma$-module (with the trivial action), then Γ is called a *$\mathbb{Q}$-Poincaré duality group of dimension n*, abbreviated $\mathbb{Q}$-PD_n.

This chapter uses the above definitions as algebraic targets forced by the existence of a closed manifold model with $\mathbb{Q}$-acyclic universal cover. For groups with torsion, these definitions are less canonical because they are not invariant under replacing Γ by a finite extension in a completely straightforward way. The reduction to a torsion-free subgroup remains the cleanest step in rational coefficients.

4.2 From $\mathbb{Q}$-acyclic universal covers to $\mathbb{Q}$-PD groups: the torsion-free case

We record the standard implication in the torsion-free case, stated in a form that can be referenced later without committing to additional geometric input.

Proposition 4.5 (Aspherical homology manifolds force rational PD for torsion-free groups)**.** *Let M be a closed connected n-manifold without boundary and let $\Gamma=\pi_1(M)$. Assume: (i) Γ is torsion-free; (ii) the universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic, meaning $\widetilde{H}_k(\widetilde{M};\mathbb{Q})=0$ for all k. Then Γ is a $\mathbb{Q}$-PD_n group.*

Proof sketch. Fix a CW-structure on M. The cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ is a chain complex of free left $\mathbb{Q}\Gamma$-modules. Since $\widetilde{M}$ is $\mathbb{Q}$-acyclic and connected, the augmentation map $C_0(\widetilde{M};\mathbb{Q})\to\mathbb{Q}$ exhibits $C_*(\widetilde{M};\mathbb{Q})$ as a free resolution of the trivial module $\mathbb{Q}$. Because M is compact, $C_k(\widetilde{M};\mathbb{Q})$ is finitely generated as a $\mathbb{Q}\Gamma$-module for each k, so Γ is of type FP over $\mathbb{Q}$, with $\mathrm{cd}_{\mathbb{Q}}(\Gamma)\le n$. Poincaré duality on M provides a chain-level quasi-isomorphism between $C_*(\widetilde{M};\mathbb{Q})$ and the $\mathbb{Q}$-linear dual cochain complex $\mathrm{Hom}_{\mathbb{Q}}(C_{n-*}(\widetilde{M};\mathbb{Q}),\mathbb{Q})$, with the Γ-action transported along deck transformations. Translating this into group (co)homology yields the duality isomorphisms in the definition of $\mathbb{Q}$-PD_n. The dualizing module is $\mathbb{Q}$ with trivial action, using the orientation local system; over $\mathbb{Q}$ and for connected M, this contributes only a possible sign action, which is trivial precisely when M is orientable. If M is non-orientable, the conclusion becomes $\mathbb{Q}$-duality with a rank-one twist. $\square$

Proposition 4.5 does not address torsion in Γ. Since the motivating problem concerns lattices that may contain 2-torsion, the next step is to separate what can be reduced to a torsion-free finite-index subgroup from what genuinely depends on torsion.

4.3 Finite index subgroups, finite extensions, and rational duality

Let Γ be a group with a torsion-free subgroup $\Gamma_0 \leq \Gamma$ of finite index. Such a Γ_0 exists for many lattices but not in complete generality; in the absence of a specific construction, we treat it as an assumption when needed.

Lemma 4.6 (Passing to finite index subgroups in the geometric setting)**.** *Let M be a closed connected manifold with $\pi_1(M) = \Gamma$, and let $\Gamma_0 \leq \Gamma$ be a subgroup of finite index. Then there is a finite covering $p\colon M_0 \to M$ with $\pi_1(M_0) = \Gamma_0$, and $\widetilde{M_0} \cong \widetilde{M}$ as Γ_0-spaces. In particular, $\widetilde{M}$ is $\mathbb{Q}$-acyclic exactly when $\widetilde{M_0}$ is $\mathbb{Q}$-acyclic.*

Proof. Standard covering space theory yields M_0 corresponding to Γ_0. The universal cover of M_0 identifies with $\widetilde{M}$ because both are simply connected covers of M with deck group Γ_0. $\square$

Combining Lemma 4.6 with Proposition 4.5 yields the following purely formal reduction.

Corollary 4.7 (Virtual rational PD forced by a $\mathbb{Q}$-acyclic universal cover)**.** *Let M be a closed connected n-manifold with $\pi_1(M) = \Gamma$. Assume $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Suppose Γ contains a torsion-free finite index subgroup Γ_0. Then Γ_0 is a $\mathbb{Q}$-PD_n group (or a rank-one twisted $\mathbb{Q}$-duality group when M_0 is non-orientable).*

Corollary 4.7 clarifies the level at which torsion enters: the $\mathbb{Q}$-acyclicity assumption forces rational duality for some torsion-free finite-index subgroup, and the remaining question is how much of this structure can be promoted back to Γ itself.

4.4 Geometric-to-algebraic reduction used later

The later obstruction discussion in the manuscript uses the following implication as its basic reduction step.

Proposition 4.8 (Closed manifold model implies virtual rational PD)**.** *Assume Γ is the fundamental group of a closed connected n-manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic.*

Then Γ is of type FP *over $\mathbb{Q}$ and satisfies* $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$. *Moreover, for every torsion-free finite-index subgroup $\Gamma_0 \leq \Gamma$, the group Γ_0 is a $\mathbb{Q}$-Poincaré duality group of dimension n up to a rank-one twist reflecting orientability of the associated cover.*

Proof. The first statement is proved as in the proof sketch of Proposition 4.5, using the cellular chain complex of $\widetilde{M}$ as a finite free resolution of $\mathbb{Q}$ over $\mathbb{Q}\Gamma$. The second statement is Corollary 4.7. $\square$

Open problem (converse direction, even rationally). The converse, namely whether a group-theoretic $\mathbb{Q}$-PD_n hypothesis on a torsion-free group Γ_0 implies the existence of a closed n-manifold with $\pi_1 = \Gamma_0$ and $\mathbb{Q}$-acyclic universal cover, is not used in this paper and is not asserted here. In particular, the reduction employed later is one-way and should be read only as a necessary condition.

A caution motivated by the counterexample workfront. Over $\mathbb{Q}$, cohomology of finite groups vanishes in positive degrees, and transfer arguments often eliminate finite-group obstructions present over $\mathbb{Z}$. Consequently, the presence of 2-torsion in Γ does not, by itself, contradict the existence of a torsion-free finite-index $\mathbb{Q}$-PD subgroup. Any obstruction that uses only ordinary group cohomology with $\mathbb{Q}$-coefficients risks being too weak to see torsion phenomena. We next record a conditional statement about finite extensions that makes this limitation explicit. It is formulated as a lemma with a proof sketch because the cleanest general statement depends on the choice of duality formalism (ordinary versus Bredon-type theories).

Lemma 4.9 (Finite extensions over $\mathbb{Q}$: conditional behavior)**.** *Assume $\Gamma_0 \trianglelefteq \Gamma$ is torsion-free of finite index and let $F = \Gamma/\Gamma_0$ be finite. Suppose Γ_0 is a $\mathbb{Q}$-PD_n group. Then Γ is a $\mathbb{Q}$-duality group of dimension n with dualizing module naturally built from the F-action on the top-dimensional dualizing line of Γ_0. In particular, when the induced F-action on that one-dimensional $\mathbb{Q}$-vector space is trivial, Γ behaves as a $\mathbb{Q}$-PD_n group at the level of ordinary (co)homology with $\mathbb{Q}$-coefficients.*

Proof sketch and gap statement. One approach uses the Lyndon–Hochschild–Serre spectral sequence for $1 \to \Gamma_0 \to \Gamma \to F \to 1$ and the fact that $\mathbb{Q}$ is a field of characteristic 0, so $\mathbb{Q}[F]$ is semisimple. The semisimplicity yields splitting properties and comparison between $H^*(\Gamma; \mathbb{Q}\Gamma)$ and $H^*(\Gamma_0; \mathbb{Q}\Gamma_0)$ after taking appropriate invariants and coinvariants under F. A complete proof requires careful bookkeeping of module structures (left versus right actions and the identification of $\mathbb{Q}\Gamma$ as an induced module from $\mathbb{Q}\Gamma_0$) and the compatibility of the duality isomorphisms with the F-action. We do not supply a full derivation here. Later sections that use Lemma 4.9 will state explicitly what portion of the conclusion is actually invoked. $\square$

The main methodological point for the present paper is that even a successful promotion of duality from Γ_0 to Γ does not incorporate the internal structure of torsion subgroups. For lattices with nontrivial finite subgroups, this indicates that any negative result must use a refinement sensitive to finite subgroups (for example, equivariant or Bredon-type finiteness and duality), or must import additional structure from the ambient Lie group.

4.5 What rational PD does and does not rule out in the presence of torsion

This subsection delineates limits that are important when actively searching for counterexamples to a naive "torsion obstructs $\mathbb{Q}$-acyclicity" claim.

Finite subgroups are often invisible over $\mathbb{Q}$. Let C_m be cyclic of order m. Since m is invertible in $\mathbb{Q}$, the standard averaging argument shows that every $\mathbb{Q}[C_m]$-module is semisimple and that $H^i(C_m; V) = 0$ for all $i > 0$ and all $\mathbb{Q}[C_m]$-modules V that are $\mathbb{Q}$-vector spaces. Thus, spectral sequences that mix finite and infinite parts of a group extension can collapse rationally, eliminating torsion-detection mechanisms that exist integrally.

Ordinary $\mathbb{Q}$-PD does not encode fixed-point data. Suppose a finite group F acts freely on a closed manifold M_0 with $\pi_1(M_0) = \Gamma_0$ and quotient orbifold M_0/F with orbifold fundamental group Γ. At the level of ordinary group cohomology over $\mathbb{Q}$, Lemma 4.9 suggests that Γ may retain a form of duality closely related to that of Γ_0, particularly when the action preserves orientation. However, the presence of torsion in Γ corresponds to nontrivial stabilizers in the action on the universal cover when one models $B\Gamma$ as an orbicomplex. Ordinary $\mathbb{Q}$-PD cannot distinguish different conjugacy classes of finite subgroups and their normalizers.

A rationally acyclic universal cover is compatible with torsion in π_1 via finite quotients. Let M_0 be a closed aspherical manifold with contractible universal cover, and let F be a finite group acting freely on M_0. Then $M = M_0/F$ is a closed manifold with $\pi_1(M)$ a finite extension of $\Gamma_0 = \pi_1(M_0)$. The resulting group $\pi_1(M)$ may contain torsion, depending on the extension class and the induced action on Γ_0; this dependence is not constrained by rational homological considerations alone. The previous paragraph is intentionally phrased conditionally: constructing such an example with $\pi_1(M)$ a uniform lattice in a prescribed semisimple Lie group is a separate and nontrivial constraint. The point is that "torsion in Γ" and "existence of a closed manifold with $\mathbb{Q}$-acyclic universal cover and fundamental group Γ" are not logically inconsistent on purely algebraic rational grounds.

4.6 A concrete algebraic target for later obstruction arguments

To connect this chapter to subsequent sections, we isolate the minimal algebraic consequences of a positive answer to the existence question.

Proposition 4.10 (Necessary conditions implied by a closed manifold model)**.** *Assume Γ is the fundamental group of a closed connected n-manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. (1) The group Γ is of type* FP *over $\mathbb{Q}$ and has finite cohomological dimension over $\mathbb{Q}$ with $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \le n$. (2) Suppose Γ contains a torsion-free finite index subgroup Γ_0. Then Γ_0 is a $\mathbb{Q}$-PD_n group (up to a rank-one twist reflecting orientability). (3) Any obstruction that aims to rule out such an M for a torsion-containing Γ must use input beyond ordinary group cohomology over $\mathbb{Q}$ (or else must show that no torsion-free finite-index subgroup of Γ can be $\mathbb{Q}$-PD_n).*

Proof sketch. For (1), $C_*(\widetilde{M};\mathbb{Q})$ is a length-n free resolution of $\mathbb{Q}$ by finitely generated $\mathbb{Q}\Gamma$-modules. For (2), apply Corollary 4.7. Statement (3) summarizes the limitations discussed above, and will be treated as a methodological constraint rather than a theorem used for deduction. □

The remainder of the manuscript will therefore treat "Γ virtually $\mathbb{Q}$-PD$_n$" as a baseline necessary condition derived from $\mathbb{Q}$-acyclicity, and will seek additional constraints that incorporate torsion in a way not annihilated by rational coefficients. Where such constraints are not proved, they will be stated explicitly as conditional hypotheses, so that any resulting obstruction is clearly marked as conditional rather than established.

5 Cohomological dimension and finiteness conditions

5.1 Purpose and standing hypotheses

This chapter isolates the finiteness and dimension hypotheses implicitly used later when passing between geometric models for Γ and (co)homological invariants of Γ. The guiding application is the following situation.

Standing situation. Let M be a connected closed topological n-manifold without boundary, equipped with a left action of a discrete group Γ by deck transformations on the universal cover $\widetilde{M}$, so that $M \cong \Gamma\backslash\widetilde{M}$. Throughout, $\mathbb{Q}$ denotes the trivial $\mathbb{Q}\Gamma$-module. The main theme is that, when $\widetilde{M}$ is $\mathbb{Q}$-acyclic, the cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ behaves like a finite free resolution of $\mathbb{Q}$ over $\mathbb{Q}\Gamma$ provided one has a finite Γ-CW model for $\widetilde{M}$. In the manifold setting one always has a free Γ-action on $\widetilde{M}$, but one does not automatically have a finite Γ-CW structure unless M is triangulable or one replaces triangulations with a suitable Γ-equivariant CW approximation. The present paper will treat such geometric finiteness as an explicit assumption when it is used.

5.2 Cohomological dimension and rational cohomological dimension

Definition 5.1 (Cohomological dimension)**.** Let R be a commutative ring. The (left) cohomological dimension $\mathrm{cd}_R(\Gamma)$ is the minimal integer $d \geq 0$ such that $H^k(\Gamma; A) = 0$ for every $R\Gamma$-module A and every $k > d$, when such a d exists; otherwise $\mathrm{cd}_R(\Gamma) = \infty$.

For the obstruction pursued here, the case $R = \mathbb{Q}$ is the relevant one. In particular, $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ is a group invariant that can be finite even when Γ has torsion.

Definition 5.2 (Rational cohomological dimension)**.** We write $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ for cohomological dimension over $\mathbb{Q}$. If Γ has a torsion-free subgroup $\Gamma_0 \leq \Gamma$ of finite index, then $\mathrm{vcd}_{\mathbb{Q}}(\Gamma)$ denotes $\mathrm{cd}_{\mathbb{Q}}(\Gamma_0)$.

The definition of $\mathrm{vcd}_{\mathbb{Q}}(\Gamma)$ depends on the existence of torsion-free finite index subgroups and, in general, on the choice of such a subgroup. In settings where one knows that all torsion-free finite index subgroups have the same rational cohomological dimension, $\mathrm{vcd}_{\mathbb{Q}}$ is well-defined; the paper will not assume this unless explicitly stated.

Relation to manifold models. If M is a closed aspherical n-manifold with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ contractible, then M is a finite-dimensional model for $B\Gamma$, and in particular $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = n$ and Γ must be torsion-free (since groups with torsion do not admit finite-dimensional models for $B\Gamma$). In contrast, the present project allows $\widetilde{M}$ to be only $\mathbb{Q}$-acyclic, and also allows Γ to have torsion, which makes the relation between n and $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ more delicate.

5.3 Finiteness properties: F, FL, FP

We record finiteness properties that govern the existence of finite resolutions.

Definition 5.3 (Type F and finite Γ-CW models)**.** A group Γ is of type F if it admits a finite CW model for $B\Gamma$, equivalently a contractible free Γ-CW complex X such that $\Gamma\backslash X$ is a finite CW complex. If X exists with $\Gamma\backslash X$ finite but X not assumed contractible, we say Γ admits a finite free Γ-CW model.

Definition 5.4 (Type FL over $\mathbb{Q}$)**.** A group Γ is of type FL over $\mathbb{Q}$ if the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ admits a finite-length resolution by finitely generated free $\mathbb{Q}\Gamma$-modules.

Definition 5.5 (Type FP over $\mathbb{Q}$)**.** A group Γ is of type FP over $\mathbb{Q}$ if the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ admits a resolution by finitely generated projective $\mathbb{Q}\Gamma$-modules.

Type FL over $\mathbb{Q}$ implies type FP over $\mathbb{Q}$. In many geometric situations (for instance, a free cellular action on a finite CW complex), the cellular chain complex provides an FL resolution. The direction relevant for later reductions is the following.

Lemma 5.6 (Finite free Γ-CW models yield FL resolutions over $\mathbb{Q}$)**.** *Let X be a free Γ-CW complex such that $\Gamma\backslash X$ has finitely many cells. Then $C_*(X;\mathbb{Q})$ is a bounded chain complex of finitely generated free $\mathbb{Q}\Gamma$-modules. If, in addition, $\widetilde{H}_*(X;\mathbb{Q}) = 0$, then the augmented complex*

$$\cdots \to C_1(X;\mathbb{Q}) \to C_0(X;\mathbb{Q}) \xrightarrow{\varepsilon} \mathbb{Q} \to 0$$

is exact, hence a finite free resolution of $\mathbb{Q}$.

Proof sketch. Cellular chains of a free Γ-CW complex are free $\mathbb{Z}\Gamma$-modules on the set of cells in each dimension; finiteness of $\Gamma\backslash X$ gives finite generation. Tensoring with $\mathbb{Q}$ preserves freeness. The augmentation map ε is induced by $X \to *$ and exactness of the augmented complex is equivalent to $\widetilde{H}_*(X;\mathbb{Q}) = 0$. ☐

In the targeted geometric setting, a closed manifold M has the homotopy type of a finite CW complex. If one fixes a CW structure on M with finitely many cells, then the lifted CW structure makes $\widetilde{M}$ into a free Γ-CW complex with finite quotient.

Torsion remarks (corrected). Unlike the aspherical case, a closed manifold can have torsion in its fundamental group (for example, nontrivial finite covers of spheres). Deck transformations always act freely on $\widetilde{M}$ by definition, and freeness does not exclude finite-order elements in the acting group. The torsion-free conclusion becomes valid under stronger hypotheses that force M to be a finite-dimensional model for $B\Gamma$, for example when $\widetilde{M}$ is contractible (equivalently M is aspherical).

5.4 Virtual torsion-freeness and passage to finite covers

To keep the overall project coherent with the stated motivating class (uniform lattices with 2-torsion), we record the standard maneuver that passes to torsion-free finite index subgroups, corresponding to finite covers of a manifold model when such a model exists.

Lemma 5.7 (Finite covers and finiteness conditions)**.** *Let $\Gamma_0 \leq \Gamma$ have finite index. If Γ admits a finite CW model for $B\Gamma$, then Γ_0 admits a finite CW model for $B\Gamma_0$. If Γ is of type FL over $\mathbb{Q}$, then Γ_0 is of type FL over $\mathbb{Q}$.*

Proof sketch. A finite CW model for $B\Gamma$ pulls back along the covering corresponding to $\Gamma_0 \leq \Gamma$ to a finite CW model for $B\Gamma_0$. For FL over $\mathbb{Q}$, restrict a free $\mathbb{Q}\Gamma$-resolution of $\mathbb{Q}$ to $\mathbb{Q}\Gamma_0$; finite index implies each restricted free module is a finite direct sum of free $\mathbb{Q}\Gamma_0$-modules. ☐

The converse directions are not automatic and will not be used without explicit justification.

5.5 Dimension bounds from $\mathbb{Q}$-acyclic universal covers

We now state the basic dimension bound used later to limit possible manifold dimensions in candidate constructions.

Proposition 5.8 (Cohomological dimension bound from a finite Γ-CW model)**.** *Let X be a free Γ-CW complex with $\Gamma\backslash X$ finite and $\dim X = n$. If $\widetilde{H}_*(X;\mathbb{Q}) = 0$, then $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$ and Γ is of type FL over $\mathbb{Q}$.*

Proof sketch. Lemma 5.6 gives a length n free $\mathbb{Q}\Gamma$-resolution of $\mathbb{Q}$. This implies vanishing of $\mathrm{Ext}^k_{\mathbb{Q}\Gamma}(\mathbb{Q}, A) \cong H^k(\Gamma; A)$ for $k > n$ for all $\mathbb{Q}\Gamma$-modules A, hence $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$. ☐

Manifold specialization (conditional on a CW structure). If M is a closed n-manifold with $\pi_1(M) = \Gamma$ and M admits a CW structure with finitely many cells, then $\widetilde{M}$ is a free Γ-CW complex with finite quotient. Under the additional assumption that $\widetilde{H}_*(\widetilde{M};\mathbb{Q}) = 0$, Proposition 5.8 yields $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$ and type FL over $\mathbb{Q}$.

5.6 Additional later-use inputs (stated as assumptions)

Several later reductions refer to standard implications between finiteness properties, cohomological dimension, and the existence of finite-dimensional classifying spaces. Since this draft is in research mode, we state these as clearly delimited inputs.

Lemma 5.9 (Torsion obstructs finite-dimensional models for $B\Gamma$)**.** *Assumption. If Γ contains an element of finite order, then every model for $B\Gamma$ has infinitely many nonzero skeleta. Equivalently, Γ has no finite-dimensional $B\Gamma$.*

To be proved. A proof will be inserted, using standard properties of group cohomology and the fact that torsion forces unbounded cohomology over $\mathbb{Z}$. ☐

Lemma 5.10 (Aspherical manifolds yield torsion-free fundamental groups)**.** *Assumption. If M is a closed aspherical manifold, then $\pi_1(M)$ is torsion-free.*

To be proved. One route is to combine the fact that M is a finite-dimensional model for $B\pi_1(M)$ with Lemma 5.9. ☐

5.7 Implications for counterexamples

The workfront of the present draft is to identify boundary conditions that would falsify or force a weakening of the main obstruction claim. The preceding discussion yields two constraints that any genuine counterexample must navigate. First, torsion in Γ is compatible with being a closed manifold fundamental group in general, but it is incompatible with being the fundamental group of a closed aspherical manifold (Lemma 5.10). Thus, in the presence of torsion, any manifold model necessarily has noncontractible universal cover. In the present project, the universal cover is only assumed $\mathbb{Q}$-acyclic, so this distinction is central: $\mathbb{Q}$-acyclicity does not force torsion-freeness. Second, any such manifold model forces strong finiteness over $\mathbb{Q}$ (at least type FL over $\mathbb{Q}$) and provides the dimension bound $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$ under the explicit CW-finiteness and $\mathbb{Q}$-acyclicity hypotheses (Proposition 5.8). Therefore, to construct a counterexample to a rational Poincaré duality obstruction, one would need a group Γ with these $\mathbb{Q}$-finiteness properties but failing the duality conclusions required later.

The later sections will take Proposition 5.8 as a bookkeeping device: whenever a putative manifold model is assumed, the rational cohomological dimension and $\mathbb{Q}$-finiteness properties become mandatory hypotheses and can be compared against cohomology computations or structural theorems for lattices.

6 Uniform lattices as virtual duality groups

This chapter records the cohomological finiteness properties of uniform lattices that are relevant to the obstruction strategy based on rational Poincaré duality. The emphasis is on the distinction between genuine duality properties of torsion-free finite index subgroups and the weaker, but often more robust, statements for groups containing torsion. Throughout, G denotes a connected real semisimple Lie group with finite center, $K \leq G$ a maximal compact subgroup, $X = G/K$ the associated symmetric space, and $\Gamma \leq G$ a uniform lattice (cocompact and discrete). We keep the discussion over $\mathbb{Q}$, since the target question concerns $\mathbb{Q}$-acyclicity of universal covers.

6.1 Virtual cohomological dimension and duality notions

We begin by fixing terminology.

Definition 6.1 (Cohomological dimension over a ring)**.** Let R be a commutative ring with unit. The cohomological dimension $\mathrm{cd}_R(\Gamma)$ is the smallest integer $d \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that $H^i(\Gamma; M) = 0$ for every $R\Gamma$-module M and every $i > d$.

Definition 6.2 (Virtual cohomological dimension)**.** A group Γ has finite virtual cohomological dimension if it contains a torsion-free subgroup $\Gamma_0 \le \Gamma$ of finite index with $\mathrm{cd}_{\mathbb{Z}}(\Gamma_0) < \infty$. In that case one defines

$$\mathrm{vcd}(\Gamma) = \mathrm{cd}_{\mathbb{Z}}(\Gamma_0),$$

which is independent of the torsion-free finite index subgroup chosen.

Definition 6.3 (Duality group over a ring)**.** Let R be a commutative ring. A group Γ is an R-duality group of dimension d if (i) Γ is of type FP over R and (ii) there exists an $R\Gamma$-module D (the dualizing module) such that for every $R\Gamma$-module M there are natural isomorphisms

$$H^i(\Gamma; M) \cong H_{d-i}(\Gamma; D \otimes_R M).$$

If moreover $D \cong R$ with the trivial Γ-action, then Γ is an R-Poincaré duality group (an R-PD group) of dimension d.

Definition 6.4 (Virtual duality)**.** A group Γ is a virtual R-duality group (respectively virtual R-PD group) if it contains a finite index subgroup that is an R-duality group (respectively an R-PD group).

The subsequent chapters use the heuristic that a closed manifold with $\mathbb{Q}$-acyclic universal cover forces its fundamental group to behave like a $\mathbb{Q}$-PD group in the relevant dimension. That heuristic must be treated carefully in the presence of torsion: torsion-free hypotheses are automatic for aspherical manifolds but are not automatic for the present $\mathbb{Q}$-acyclic setting.

A basic torsion caveat. Over a field of characteristic 0, finite subgroups have trivial positive-degree group cohomology. Consequently, the presence of torsion does not by itself force $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = \infty$. By contrast, over $\mathbb{Z}$ one expects torsion to obstruct finite cohomological dimension in many cases. This motivates working with virtual invariants and with $\mathbb{Q}$-coefficients separately.

6.2 Uniform lattices and manifold models for $B\Gamma$

The symmetric space $X = G/K$ is a contractible smooth manifold. A uniform lattice Γ acts properly discontinuously and cocompactly on X. Stabilizers are finite, and the action is free precisely when Γ is torsion-free.

Torsion-free finite index subgroups (assumption). When G admits a faithful finite-dimensional linear representation over $\mathbb{R}$, it is classical that Γ contains a torsion-free subgroup Γ_0 of finite index. In the remainder of the chapter we treat the existence of such Γ_0 as an explicit standing assumption, since it is needed to interpret $\mathrm{vcd}(\Gamma)$ and to extract manifold models from the locally symmetric orbifold $\Gamma\backslash X$.

Proposition 6.5 (Closed aspherical model after passing to finite index)**.** *Assume Γ contains a torsion-free finite index subgroup Γ_0. Then $M_0 := \Gamma_0\backslash X$ is a closed aspherical manifold with $\pi_1(M_0) \cong \Gamma_0$ and $\widetilde{M_0} \cong X$.*

Proof. Since Γ_0 is torsion-free, its action on X is free. Properness and cocompactness imply that the quotient is a closed manifold. Contractibility of X gives asphericity. $\square$

This geometric model yields a quick computation of $\mathrm{cd}_R(\Gamma_0)$ for any coefficient ring R for which M_0 is an R-homology manifold in the usual sense.

Proposition 6.6 (Cohomological dimension from the symmetric space)**.** *Let $n = \dim(X)$. Under the torsion-free finite index assumption, $\mathrm{cd}_R(\Gamma_0) = n$ for any field R of characteristic 0, in particular for $R = \mathbb{Q}$.*

Proof sketch. Since M_0 is a closed n-manifold, it has the homotopy type of a finite CW complex of dimension n, hence $\mathrm{cd}_R(\Gamma_0) \le n$. Nonvanishing of top-degree cohomology with local coefficients in the orientation module shows $\mathrm{cd}_R(\Gamma_0) \ge n$. The argument is standard; the only input is that M_0 is closed and connected. $\square$

In particular, $\mathrm{vcd}(\Gamma) = n$ in the cocompact case, subject to the torsion-free finite index assumption.

6.3 Duality and Poincaré duality for torsion-free uniform lattices

The closed manifold model M_0 also supplies the expected duality.

Theorem 6.7 (Torsion-free uniform lattices are virtual Poincaré duality groups)**.** *Assume Γ contains a torsion-free finite index subgroup Γ_0. Let $n = \dim(X)$. Then Γ_0 is a $\mathbb{Z}$-Poincaré duality group of dimension n, hence also a $\mathbb{Q}$-Poincaré duality group of dimension n.*

Proof sketch. The manifold M_0 is an aspherical closed n-manifold with fundamental group Γ_0. Poincaré duality on M_0 identifies cohomology with local coefficients on M_0 with group cohomology of Γ_0 with the corresponding $\mathbb{Z}\Gamma_0$-module coefficients. Asphericity identifies the universal cover with the standard free $\mathbb{Z}\Gamma_0$-resolution of $\mathbb{Z}$. The dualizing module is the orientation module of M_0, which is isomorphic to $\mathbb{Z}$ with trivial action when M_0 is orientable and otherwise gives the usual twisting by the orientation character. $\square$

Relevance for the main question. This theorem alone does not resolve the target question, because it applies only after passing to a torsion-free finite index subgroup. The objective asks whether Γ itself, possibly with 2-torsion, can be realized as the fundamental group of a closed manifold with $\mathbb{Q}$-acyclic universal cover. Even when Γ_0 is a $\mathbb{Q}$-PD group, it remains to understand whether extending by a finite group (re-introducing torsion) is compatible with any such manifold model.

A potential boundary condition. The existence of M_0 shows that every cocompact lattice has a finite index subgroup realized by a closed aspherical manifold with contractible universal cover. Since contractible implies $\mathbb{Q}$-acyclic, the obstruction theory must genuinely use the requirement that the fundamental group be exactly Γ (not merely virtually) and must exploit torsion, here specifically 2-torsion.

6.4 What torsion changes: group cohomology versus quotients of X

When Γ has torsion, the quotient $\Gamma\backslash X$ is an orbifold rather than a manifold. It still carries a fundamental class in an orbifold sense, but translating this into statements about ordinary group cohomology is subtle.

Proposition 6.8 (A necessary condition from a free $\mathbb{Q}\Gamma$-resolution)**.** *Let M be a closed connected n-manifold with $\pi_1(M) \cong \Gamma$. If $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then the cellular chain complex $C_*(\widetilde{M};\mathbb{Q})$ is a finite free resolution of the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ of length n. In particular, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n$ and Γ is of type FP over $\mathbb{Q}$.*

Proof. Choose a finite CW structure on M compatible with the manifold structure. Lifting to $\widetilde{M}$ gives a finite free $\mathbb{Q}\Gamma$-chain complex computing $H_*(\widetilde{M};\mathbb{Q})$. $\mathbb{Q}$-acyclicity implies exactness in degrees > 0 and $H_0 \cong \mathbb{Q}$, hence the complex is a free resolution of $\mathbb{Q}$ of length n. $\square$

This is the point where torsion behaves differently over $\mathbb{Q}$ than over $\mathbb{Z}$. Over $\mathbb{Q}$, the existence of torsion does not contradict the existence of a finite free resolution a priori.

Why one still expects duality constraints. The manifold M carries Poincaré duality with local coefficients. With no asphericity assumption, duality does not translate directly into a duality statement for group cohomology; the comparison map from group cohomology $H^*(\Gamma; -)$ to cohomology of M with local coefficients involves the full homotopy type, not only the fundamental group. Nonetheless, when $\widetilde{M}$ is $\mathbb{Q}$-acyclic, the $\mathbb{Q}\Gamma$-chain complex $C_*(\widetilde{M};\mathbb{Q})$ is close to a projective resolution and can be used to define an algebraic dualizing object. The obstruction program in later chapters uses this chain-level control and supplements it with information coming from fixed-point sets of finite subgroups.

Finite subgroup fixed points. Suppose a finite subgroup $F \leq \Gamma$ acts on $\widetilde{M}$ via deck transformations induced by the inclusion $F \leq \Gamma$. The action is free precisely when F is trivial; otherwise the fixed set $\widetilde{M}^F$ is nonempty only under strong restrictions. For involutions, mod-2 Smith theory often forces nontrivial constraints on the mod-2 homology of $\widetilde{M}^F$ when $\widetilde{M}$ is mod-2 acyclic. Since $\mathbb{Q}$-acyclicity does not imply mod-2 acyclicity, one must treat this as a separate input. The present chapter only isolates the group-theoretic side: uniform lattices contain many finite subgroups, and involutions can have complicated centralizers. This creates a plausible mechanism for obstructions, because any manifold model imposes geometric restrictions on these finite subgroups and their fixed sets.

6.5 Virtual duality versus actual duality, and a counterexample-oriented discussion

The obstruction strategy in this manuscript aims to deduce that Γ cannot occur as $\pi_1(M)$ under the stated hypotheses, at least in broad families of examples. The workfront directive requests that we also identify situations in which the strategy might fail, thereby delineating possible counterexamples.

Virtual Poincaré duality is not an obstruction. Every uniform lattice Γ satisfying the torsion-free finite index assumption is virtually a $\mathbb{Q}$-PD group. Therefore any argument that uses only virtual duality properties (such as $\mathrm{vcd}(\Gamma)$ or the existence of a finite index $\mathbb{Q}$-PD subgroup) cannot rule out the existence of a closed manifold M with $\pi_1(M) = \Gamma$ and $\mathbb{Q}$-acyclic $\widetilde{M}$. A viable obstruction must incorporate information that is sensitive to the extension from Γ_0 to Γ, namely the finite subgroups and their interaction with the ambient geometry.

A concrete loophole to keep in view. It is conceivable that Γ admits a closed manifold model M whose universal cover is $\mathbb{Q}$-acyclic but not contractible, and where torsion in Γ is realized by nontrivial stabilizer behavior on $\widetilde{M}$. Since $\widetilde{M}$ is simply connected, the torsion cannot act freely, so any such model forces fixed point phenomena. Over $\mathbb{Q}$, fixed point sets may be hard to detect purely homologically, particularly because rational acyclicity is compatible with a wide range of mod-p behaviors. This suggests that an unconditional obstruction requires either (i) strengthening $\mathbb{Q}$-acyclicity to mod-2 acyclicity or (ii) extracting constraints on fixed points that do not rely on mod-2 homology alone.

A useful algebraic proxy. Let Γ be of finite virtual cohomological dimension. There is an algebraic notion of "duality with torsion" using a dualizing module for a torsion-free finite index subgroup, together with an action of the finite quotient Γ/Γ_0 on that module. For cocompact lattices, the module is (up to orientation twist) rank one over the coefficient ring. The question becomes whether a finite extension of a $\mathbb{Q}$-PD group can be realized as the fundamental group of a closed manifold whose universal cover is merely $\mathbb{Q}$-acyclic. The chapter does not settle this, but it motivates why the later obstruction theory focuses on involutions and their centralizers: these features distinguish different finite extensions and can contradict the existence of manifold actions with the required fixed set properties.

Summary for later use. Under standard linearity hypotheses, a uniform lattice Γ has $\mathrm{vcd}(\Gamma) = \dim(X)$ and contains a torsion-free finite index subgroup Γ_0 that is a $\mathbb{Q}$-Poincaré duality group. Therefore any obstruction to realizing Γ as the fundamental group of a closed manifold with $\mathbb{Q}$-acyclic universal cover must detect the presence of torsion in Γ itself. In particular, one should not expect a contradiction from dimension counts or from virtual duality alone; the obstruction must use additional structure, such as the behavior of finite subgroups (especially 2-subgroups) and the constraints they impose on the topology of a simply connected $\mathbb{Q}$-acyclic manifold admitting a properly discontinuous cocompact action.

7 Torsion effects and duality obstructions

This chapter isolates the points at which torsion, and in particular elements of order two, interacts with rational Poincaré duality constraints that would follow from the existence of a closed manifold model with $\mathbb{Q}$-acyclic universal cover. The discussion separates (i) definitional consequences of $\mathbb{Q}$-acyclicity and group duality from (ii) torsion-sensitive mechanisms that plausibly obstruct such duality, and it records several boundary conditions relevant to the search for counterexamples.

7.1 Rational duality requirements forced by a $\mathbb{Q}$-acyclic universal cover

Let M be a closed, connected topological manifold of dimension n, and let $\Gamma = \pi_1(M)$. Let $\widetilde{M}$ denote the universal cover. Throughout, "$\mathbb{Q}$-acyclic" means $\widetilde{M}$ has $\widetilde{H}_i(\widetilde{M}; \mathbb{Q}) = 0$ for all $i \geq 0$. No asphericity assumption is imposed.

Lemma 7.1 (Homology identification under $\mathbb{Q}$-acyclicity)**.** *Assume $\widetilde{M}$ is $\mathbb{Q}$-acyclic. Then the canonical map $M \to B\Gamma$ induces isomorphisms*

$$H_i(M;\mathbb{Q}) \cong H_i(\Gamma;\mathbb{Q}), \qquad H^i(\Gamma;\mathbb{Q}) \cong H^i(M;\mathbb{Q})$$

for all $i \geq 0$.

Proof. Consider the Cartan–Leray (or equivariant) spectral sequence for the free, properly discontinuous action of Γ on $\widetilde{M}$, with coefficients in $\mathbb{Q}$:

$$E^2_{p,q} \cong H_p\big(\Gamma; H_q(\widetilde{M};\mathbb{Q})\big) \Longrightarrow H_{p+q}(M;\mathbb{Q}).$$

The $\mathbb{Q}$-acyclicity hypothesis gives $H_q(\widetilde{M};\mathbb{Q}) = 0$ for $q > 0$ and $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ with trivial Γ-action. Hence the spectral sequence collapses to the line $q = 0$, yielding $H_p(M;\mathbb{Q}) \cong H_p(\Gamma;\mathbb{Q})$. The cohomological statement follows similarly or by universal coefficients over $\mathbb{Q}$. ☐

The manifold structure imposes Poincaré duality on $H^*(M;\mathbb{Q})$. Via Lemma 7.1, these constraints transfer to $H^*(\Gamma;\mathbb{Q})$. It is useful to record the required package of properties without asserting that it holds for any specific lattice.

Definition 7.2 ($\mathbb{Q}$-Poincaré duality group, rational version)**.** A group Γ is a $\mathbb{Q}$-Poincaré duality group of formal dimension n if there exists a class $[\Gamma] \in H_n(\Gamma;\mathbb{Q})$ such that cap product with $[\Gamma]$ induces isomorphisms

$$- \frown [\Gamma] : H^k(\Gamma;\mathbb{Q}) \xrightarrow{\cong} H_{n-k}(\Gamma;\mathbb{Q})$$

for all k.

Proposition 7.3 (Necessity of rational Poincaré duality)**.** *If M is a closed n-manifold with $\pi_1(M) = \Gamma$ and $\widetilde{M}$ is $\mathbb{Q}$-acyclic, then Γ is a $\mathbb{Q}$-Poincaré duality group of formal dimension n.*

Proof. Poincaré duality on M gives a fundamental class $[M] \in H_n(M;\mathbb{Q})$ such that cap product with $[M]$ yields isomorphisms $H^k(M;\mathbb{Q}) \cong H_{n-k}(M;\mathbb{Q})$. Lemma 7.1 identifies $H^*(M;\mathbb{Q})$ and $H_*(M;\mathbb{Q})$ with $H^*(\Gamma;\mathbb{Q})$ and $H_*(\Gamma;\mathbb{Q})$, and transports $[M]$ to a class $[\Gamma] \in H_n(\Gamma;\mathbb{Q})$ with the desired property. ☐

Boundary condition: torsion is not excluded by rational duality alone. Proposition 7.3 involves only rational (co)homology. It does not prohibit the presence of torsion in Γ. This point is essential for the counterexample workfront: a group can satisfy Definition 7.2 over $\mathbb{Q}$ while having nontrivial finite subgroups, and the manifold conclusion requires additional geometric input beyond purely rational algebra.

7.2 Torsion and the gap between ordinary duality and torsion-sensitive duality

The obstruction mechanism targeted in this project must use more than the statement "Γ is a $\mathbb{Q}$-PD group" because that notion ignores finite subgroups when coefficients are characteristic zero. The presence of a subgroup $C_2 \leq \Gamma$ can still force constraints on any Γ-action on a contractible or $\mathbb{Q}$-acyclic space, through fixed-point data, quotient singularities, and equivariant (co)homology.

Acyclic covers versus free actions. The universal cover $\widetilde{M}$ carries a free Γ-action by deck transformations. Finite subgroups of Γ act freely as well. Consequently, fixed-point set arguments for arbitrary actions of finite subgroups do not apply directly in the universal cover. The relevant torsion information enters indirectly: in many constructions one replaces M by finite-sheeted covers or by models for classifying spaces with isotropy (for example, $\underline{E}\Gamma$), and then compares duality properties between these models. This chapter does not assume existence of such a model with particular finiteness properties for the lattices under discussion; instead it formulates an explicit protocol of reductions, each of which becomes a hypothesis to be checked in later sections.

Definition 7.4 (Virtual $\mathbb{Q}$-Poincaré duality)**.** A group Γ is *virtually a $\mathbb{Q}$-Poincaré duality group of formal dimension n* if Γ admits a finite-index subgroup $\Gamma_0 \leq \Gamma$ such that Γ_0 is a $\mathbb{Q}$-Poincaré duality group of formal dimension n.

Reduction protocol A (finite-index passage). Assume Γ is the fundamental group of a closed n-manifold M with $\mathbb{Q}$-acyclic $\widetilde{M}$. For every finite-index subgroup $\Gamma_0 \leq \Gamma$, there exists a finite cover $M_0 \to M$ with $\pi_1(M_0) = \Gamma_0$ and universal cover canonically identified with $\widetilde{M}$. Hence Γ_0 is a $\mathbb{Q}$-PD group of dimension n by Proposition 7.3. In particular, any torsion-free finite-index subgroup (when it exists) inherits rational Poincaré duality. This reduction also fixes the logical scope of any torsion-based obstruction strategy. Since the existence of M forces the existence of M_0 for every finite-index subgroup Γ_0, the question of realizability by a closed manifold with $\mathbb{Q}$-acyclic universal cover is not separable between Γ and Γ_0: a contradiction derived solely from group-theoretic properties of Γ_0 would already contradict Γ via the cover $M_0 \to M$. Therefore, any obstruction that genuinely uses torsion must incorporate how Γ is built from Γ_0, namely the extension and finite-quotient action data (for example, the induced $F = \Gamma/\Gamma_0$ action on $H^*(\Gamma_0; \mathbb{Q})$, or additional torsion-sensitive invariants not visible on Γ_0 alone).

Torsion-sensitive replacement: an explicit Bredon-type duality package (setup). A concrete way to encode finite subgroup data is to work in the orbit category $\mathcal{O}_{\mathcal{F}}(\Gamma)$ for the family $\mathcal{F}$ of finite subgroups. For the purposes of this chapter, it is enough to specify the torsion-sensitive duality datum as the pair consisting of a coefficient system and a Bredon fundamental class, with duality expressed by a cap product isomorphism in Bredon (co)homology with rational coefficients.

Definition 7.5 (Manifold-compatible fixed-subgroup dimension function)**.** Let Γ be a group and $\mathcal{F}$ the family of finite subgroups. A function $d : \mathcal{F} \to \mathbb{Z}_{\geq 0}$ is called *manifold-compatible of ambient dimension* n if $d(\{1\}) = n$ and for each inclusion $K \leq H$ in $\mathcal{F}$ one has $d(H) \leq d(K)$.

Interpretation. If a closed n-manifold M admits a (not necessarily free) action of Γ, then one can take $d(H) = \dim(M^H)$ for fixed-point sets of finite subgroups. In the present geometric problem Γ acts freely on $\widetilde{M}$, so this direct interpretation is unavailable; nevertheless, when comparing M with auxiliary Γ-spaces that do admit finite isotropy (for example, compactifications or partial resolutions), a monotone dimension function is often the correct input for duality statements. This motivates using d as a formal placeholder for torsion data.

7.3 A proposed 2-torsion obstruction compatible with the counterexample workfront

The workfront directive asks for active attempts to falsify or scope the main claim. In this direction, two complementary possibilities must be kept distinct.

Scenario S1 (torsion is harmless over $\mathbb{Q}$). A uniform lattice Γ containing elements of order two might still be realizable as $\pi_1(M)$ for a closed manifold M with $\mathbb{Q}$-acyclic universal cover. Proposition 7.3 gives only a rational constraint, and the restriction maps $H^*(\Gamma; \mathbb{Q}) \to H^*(C_2; \mathbb{Q})$ vanish in positive degrees because $H^{>0}(C_2; \mathbb{Q}) = 0$. Hence any obstruction based only on ordinary rational cohomology and restriction to C_2 is provably ineffective.

Scenario S2 (torsion obstructs by forcing a mismatch between ordinary duality and torsion-sensitive duality). Any plausible obstruction must use data not visible in ordinary rational cohomology. One candidate mechanism is that a manifold model would force a torsion-sensitive duality theory whose shadows, when compared with ordinary cohomology via spectral sequences or long exact sequences, impose constraints on torsion and on how complementary degrees pair. The discussion below formulates this as a conjecture and a protocol rather than as a theorem.

Conjecture 7.6 (C3: torsion-sensitive duality obstruction for 2-torsion)**.** *Let Γ be a uniform lattice in a real semisimple Lie group, and assume Γ contains a subgroup C_2. Suppose there exists a closed n-manifold M with $\pi_1(M) = \Gamma$ and $\mathbb{Q}$-acyclic universal cover. Assume in addition that Γ admits a model for $\underline{E}\Gamma$ of finite Γ-CW type and that one has fixed a manifold-compatible dimension function d for $\mathcal{F}$ of ambient dimension n. Define the torsion-sensitive duality datum $\mathsf{D}(\Gamma; \mathcal{F}, d)$ to be the following explicit package. First, let $\mathcal{O}_{\mathcal{F}}(\Gamma)$ denote the orbit category of Γ with objects Γ/H for $H \in \mathcal{F}$. Let $\underline{\mathbb{Q}}$ denote the constant contravariant coefficient system*

$\mathcal{O}_{\mathcal{F}}(\Gamma)^{\mathrm{op}} \to \mathbb{Q}$-mod sending every Γ/H to $\mathbb{Q}$ and every morphism to the identity. Second, let $H^*_{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}})$ and $H_*^{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}})$ denote Bredon cohomology and homology with respect to $\mathcal{F}$ and coefficients $\underline{\mathbb{Q}}$. The datum includes a class $[\Gamma]_{\mathcal{F}} \in H_n^{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}})$ such that cap product with $[\Gamma]_{\mathcal{F}}$ induces degree-shifted isomorphisms

$$- \frown [\Gamma]_{\mathcal{F}} : H^k_{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}}) \xrightarrow{\cong} H^{\mathcal{F}}_{n-k}(\Gamma;\underline{\mathbb{Q}})$$

for all k. Then the conjecture asserts the following two comparison properties. (1) The canonical map induced by the inclusion of families $\{1\} \subset \mathcal{F}$ yields a natural transformation $H^*_{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}}) \to H^*(\Gamma;\mathbb{Q})$ whose image contains a fundamental class in the sense of Definition 7.2, and, under this comparison, Bredon cap product with $[\Gamma]_{\mathcal{F}}$ recovers the ordinary $\mathbb{Q}$-Poincaré duality cap product on $H^*(\Gamma;\mathbb{Q})$ predicted by Proposition 7.3. (2) For each order-two subgroup $C_2 \le \Gamma$, restriction along $C_2 \hookrightarrow \Gamma$ together with the dimension function value $d(C_2)$ forces a nontrivial constraint on the induced action of $F = \Gamma/\Gamma_0$ on $H^*(\Gamma_0;\mathbb{Q})$ for every torsion-free finite-index subgroup $\Gamma_0 \le \Gamma$ that is normal, namely a compatibility condition between the involution on $H^*(\Gamma_0;\mathbb{Q})$ determined by the image of C_2 in F and the ordinary Poincaré duality pairing on Γ_0 (for example, a specified relationship between invariant and anti-invariant subspaces in complementary degrees, with degree shifts controlled by $n - d(C_2)$). The conjectured obstruction is that for certain Γ the compatibility demanded in (2) fails for the involution coming from some C_2, suggesting that no such manifold M exists.

Status and limitations. Conjecture 7.6 is not proved in this manuscript. It depends on existence and finiteness properties of a $\underline{E}\Gamma$ model and on a Bredon duality statement with coefficients $\underline{\mathbb{Q}}$ relative to $\mathcal{F}$. These inputs are recorded explicitly inside the conjecture so that it is falsifiable: a failure of such duality, or a failure of the comparison in (1), invalidates the conjecture as stated. The conjecture is included as a precise target for subsequent sections: one must either (i) verify the required Bredon duality and comparison properties for the lattices under discussion, or (ii) produce a counterexample in Scenario S1.

Proposed Protocol (torsion-sensitive comparison via finite-index subgroups). The following sequence of steps describes a concrete approach to either validate Conjecture 7.6 in specific cases or to delimit its scope. Each step is an assumption to be checked, and the protocol is recorded here to prevent implicit overclaiming. First, choose a torsion-free finite-index subgroup $\Gamma_0 \le \Gamma$ when one exists, and let $F = \Gamma/\Gamma_0$ denote the finite quotient. The existence of M as above implies that Γ_0 is a $\mathbb{Q}$-PD group of dimension n. Second, analyze the extension $1 \to \Gamma_0 \to \Gamma \to F \to 1$ via the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} \cong H^p\big(F; H^q(\Gamma_0;\mathbb{Q})\big) \Longrightarrow H^{p+q}(\Gamma;\mathbb{Q}).$$

Because $|F| < \infty$ and coefficients are in $\mathbb{Q}$, the functor of F-invariants is exact and $H^{p>0}(F;V) = 0$ for any $\mathbb{Q}[F]$-module V. Hence the spectral sequence collapses at E_2, yielding canonical isomorphisms

$$H^k(\Gamma;\mathbb{Q}) \cong H^k(\Gamma_0;\mathbb{Q})^F.$$

This observation is unconditional and already indicates a major obstruction to torsion-based arguments that use only ordinary rational cohomology: the presence of 2-torsion in F can at most change the invariant subspace of $H^*(\Gamma_0;\mathbb{Q})$, but it cannot create new positive-degree classes in $H^*(\Gamma;\mathbb{Q})$. Third, to detect torsion, introduce additional structure on $H^*(\Gamma_0;\mathbb{Q})$ that depends on the *action* of F and is constrained by torsion-sensitive duality (for example, via restriction from Bredon cohomology to ordinary cohomology along the inclusion of families $\{1\} \subset \mathcal{F}$). The only universally available structure in this manuscript is the duality pairing

$$\langle -, - \rangle : H^k(\Gamma_0;\mathbb{Q}) \times H^{n-k}(\Gamma_0;\mathbb{Q}) \to H^n(\Gamma_0;\mathbb{Q}) \cong \mathbb{Q},$$

defined by cup product followed by evaluation on a fundamental class. The protocol seeks conditions under which the involutive part of the F-action interacts incompatibly with this pairing in a way that is forced by the torsion-sensitive datum $\mathsf{D}(\Gamma;\mathcal{F},d)$. For example, an element $\tau \in F$ of order two can act on $H^n(\Gamma_0;\mathbb{Q}) \cong \mathbb{Q}$ by multiplication by ± 1. If τ acts by -1 on top degree, then τ cannot preserve the pairing strictly, and instead satisfies a twisted invariance relation

$$\langle \tau x, \tau y \rangle = \pm \langle x, y \rangle.$$

On its own, this sign twist does not contradict duality, since orientation-reversing diffeomorphisms of manifolds also produce -1 on top homology. Consequently, any contradiction must use more refined input than the top-degree sign, such as degree shifts or constraints controlled by $d(C_2)$ coming from torsion-sensitive duality.

Where 2-torsion could enter nontrivially. A 2-torsion element in Γ can induce an involution on a finite cover M_0 corresponding to Γ_0, provided that the extension splits geometrically. The existence of such an involution is not guaranteed by the group extension alone. If such an involution exists and is smooth or locally linear, classical fixed-point theorems and Smith-type constraints become available, potentially relating mod 2 homology of fixed sets to that of the ambient manifold. However, the present problem concerns a free action on $\widetilde{M}$, and no geometric involution on M_0 has been constructed in this manuscript. Therefore any argument that imports fixed-point constraints must explicitly add hypotheses and track them as deviations from the base objective.

Proposition 7.7 (Conservative scoping of any 2-torsion obstruction)**.** *Any obstruction to the existence of M based solely on ordinary rational cohomology of Γ and restriction maps to $C_2 \leq \Gamma$ cannot succeed.*

Proof. For any inclusion $C_2 \hookrightarrow \Gamma$, one has $H^{k>0}(C_2;\mathbb{Q}) = 0$. Hence every restriction map $H^k(\Gamma;\mathbb{Q}) \to H^k(C_2;\mathbb{Q})$ is the zero map for $k > 0$. Therefore restriction to C_2 carries no information capable of distinguishing groups that satisfy Definition 7.2 from those that do not. $\square$

Implication for the counterexample search. Proposition 7.7 narrows the search space: a counterexample to a naive "restriction-to-C_2" obstruction would be any Γ that is (virtually) a $\mathbb{Q}$-PD group but contains 2-torsion. Such a group would not immediately yield a geometric counterexample, but it would falsify the idea that ordinary rational cohomology detects the torsion obstruction. Consequently, the manuscript should either (i) upgrade to a torsion-sensitive duality framework as in Conjecture 7.6, or (ii) weaken the thesis to a statement explicitly conditional on such a framework.

7.4 A minimal, checkable obstruction template (assumption-driven)

This subsection records a template that is sufficiently explicit to be falsifiable, while remaining honest about missing inputs.

Definition 7.8 (Assumption package $\mathsf{A}_{\mathrm{tors}}$)**.** Let Γ be a discrete group with finite subgroups, let $\mathcal{F}$ be the family of finite subgroups, and fix an integer $n \geq 0$. The package $\mathsf{A}_{\mathrm{tors}}$ consists of the following assumptions. (A1) There exists a finite-index torsion-free subgroup $\Gamma_0 \leq \Gamma$ of type F (finite CW model for $B\Gamma_0$). (A2) The group Γ_0 is a $\mathbb{Q}$-PD group of dimension n with a fixed fundamental class $[\Gamma_0] \in H_n(\Gamma_0;\mathbb{Q})$. (A3) The group Γ admits a finite Γ-CW model for $\underline{E}\Gamma$, and there exists a manifold-compatible dimension function d of ambient dimension n such that Bredon (co)homology with respect to $\mathcal{F}$ and coefficients $\underline{\mathbb{Q}}$ satisfies a cap-product duality isomorphism with a class $[\Gamma]_{\mathcal{F}} \in H_n^{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}})$, together with a comparison map $H^*_{\mathcal{F}}(\Gamma;\underline{\mathbb{Q}}) \to H^*(\Gamma;\mathbb{Q})$ that recovers the ordinary duality from (A2) on the torsion-free subgroup and yields explicit restrictions on the involutive part of the quotient action $F = \Gamma/\Gamma_0$ in terms of $d(C_2)$ for $C_2 \leq \Gamma$.

Proposed consequence under $\mathsf{A}_{\mathrm{tors}}$. Assume $\mathsf{A}_{\mathrm{tors}}$ and fix an element $\tau \in \Gamma$ of order two. Let $F = \Gamma/\Gamma_0$ and denote by the same symbol τ its image in F. The constraint in (A3) is intended to be checkable from the F-module $H^*(\Gamma_0;\mathbb{Q})$ and the value $d(\langle\tau\rangle)$. A typical form would require that the decomposition of $H^k(\Gamma_0;\mathbb{Q})$ into ± 1-eigenspaces for τ matches, in complementary degrees, the decomposition induced by Poincaré duality with a degree shift depending on $n - d(\langle\tau\rangle)$. A failure of such a compatibility would then obstruct the existence of M, provided one can derive $\mathsf{A}_{\mathrm{tors}}$ from the manifold hypothesis.

Why this is not yet a theorem. The step from manifold existence to (A3) is precisely where torsion enters, and it is currently an open gap in this manuscript. The conservative conclusion at this stage is that rational Poincaré duality is necessary (Proposition 7.3), while any obstruction specialized to 2-torsion must incorporate torsion-sensitive structures not captured by ordinary rational cohomology (Proposition 7.7).

Counterexample-friendly scoping statement. In view of the preceding discussion, the thesis of this chapter is framed as follows. Any negative result for uniform lattices with 2-torsion should be stated conditionally: it should assert that no manifold model exists provided that the lattice satisfies assumptions that realize $\mathsf{D}(\Gamma;\mathcal{F},d)$ and the comparison properties in Conjecture 7.6. Conversely, a genuine counterexample would consist of a uniform lattice Γ with 2-torsion for which a closed manifold M with $\pi_1(M) = \Gamma$ and $\mathbb{Q}$-acyclic $\widetilde{M}$ can be constructed. This chapter does not provide such a construction, and it does not claim that known lattice theory rules it out.

8 Consequences for 2-torsion lattices

8.1 Preliminaries: rational acyclicity and rational cohomological dimension

Let M be a compact connected manifold without boundary and let $\widetilde{M}$ be its universal covering space. We say that $\widetilde{M}$ is $\mathbb{Q}$-*acyclic* when $\widetilde{H}_i(\widetilde{M};\mathbb{Q}) = 0$ for all i, equivalently $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ and $H_i(\widetilde{M};\mathbb{Q}) = 0$ for $i > 0$.

For a discrete group Γ, the *rational cohomological dimension* $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ is the smallest integer d such that $H^i(\Gamma;V) = 0$ for all $i > d$ and all $\mathbb{Q}[\Gamma]$-modules V, provided such a d exists. The *virtual cohomological dimension* $\mathrm{vcd}(\Gamma)$ is defined by choosing a torsion-free subgroup $\Gamma_0 \leq \Gamma$ of finite index and setting $\mathrm{vcd}(\Gamma) = \mathrm{cd}_{\mathbb{Z}}(\Gamma_0)$. This number is independent of the chosen torsion-free finite-index subgroup, but it is crucial to keep in mind that $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ refers to the original group which may contain torsion, whereas $\mathrm{vcd}(\Gamma)$ is defined through torsion-free subgroups.

In the geometric situation of interest, Γ is a uniform lattice in a semisimple Lie group G, with associated symmetric space $X = G/K$ of dimension n. A torsion-free finite-index subgroup Γ_0 acts freely, properly, and cocompactly on the contractible manifold X, so X/Γ_0 is a closed aspherical n-manifold. Consequently $\mathrm{vcd}(\Gamma) = n$.

8.2 Finite groups over $\mathbb{Q}$ and the basic torsion mechanism

The first input is a purely algebraic fact.

Lemma 8.1 (Finite groups have $\mathrm{cd}_{\mathbb{Q}} = 0$)**.** *If F is a finite group, then $H^i(F;V) = 0$ for all $i > 0$ and all $\mathbb{Q}[F]$-modules V. In particular* $\mathrm{cd}_{\mathbb{Q}}(F) = 0$.

Proof. Over a field of characteristic zero, Maschke's theorem implies that $\mathbb{Q}[F]$ is semisimple. Hence every $\mathbb{Q}[F]$-module is projective, so $\mathrm{Ext}^i_{\mathbb{Q}[F]}(\mathbb{Q}, V) = 0$ for $i > 0$, which is the stated vanishing of group cohomology. □

Now let Γ contain a torsion-free subgroup of finite index, as happens for lattices. To avoid an implicit normality assumption, replace any such subgroup by its normal core. Thus we may fix a torsion-free normal subgroup $\Gamma_0 \trianglelefteq \Gamma$ of finite index, and set $F = \Gamma/\Gamma_0$.

Proposition 8.2 (Collapse for $\mathbb{Q}$-coefficients)**.** *For the extension $1 \to \Gamma_0 \to \Gamma \to F \to 1$, there are natural isomorphisms*

$$H^k(\Gamma;\mathbb{Q}) \cong H^k(\Gamma_0;\mathbb{Q})^F$$

for all k. Consequently

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) = \max\{k \mid H^k(\Gamma_0;\mathbb{Q})^F \neq 0\} \leq \mathrm{cd}_{\mathbb{Q}}(\Gamma_0) = \mathrm{vcd}(\Gamma).$$

Proof. The Lyndon Hochschild Serre spectral sequence gives $E_2^{p,q} = H^p(F; H^q(\Gamma_0;\mathbb{Q})) \Rightarrow H^{p+q}(\Gamma;\mathbb{Q})$. The F-module $H^q(\Gamma_0;\mathbb{Q})$ is a $\mathbb{Q}$-vector space. By the lemma, $H^p(F;W) = 0$ for $p > 0$ and all $\mathbb{Q}[F]$-modules W, so $E_2^{p,q} = 0$ for $p > 0$, and the edge map yields $H^q(\Gamma;\mathbb{Q}) \cong H^0(F; H^q(\Gamma_0;\mathbb{Q})) = H^q(\Gamma_0;\mathbb{Q})^F$. □

This proposition is a proved statement in homological algebra. The remaining question is whether the F-invariants in top degree vanish.

8.3 Torsion and the dimension gap for uniform lattices

Assume now that Γ is a uniform lattice in G with symmetric space dimension n, and choose $\Gamma_0 \trianglelefteq \Gamma$ torsion-free of finite index. Then Γ_0 is a Poincaré duality group of dimension n over $\mathbb{Q}$, in the sense that $H^n(\Gamma_0;\mathbb{Q}) \cong \mathbb{Q}$ and cap product with the fundamental class yields perfect pairings. In particular, $H^n(\Gamma_0;\mathbb{Q})$ is one-dimensional.

The quotient $F = \Gamma/\Gamma_0$ acts on this one-dimensional vector space through a character $\varepsilon\colon F \to \{\pm 1\}$. Concretely, when Γ_0 is realized as the fundamental group of X/Γ_0, the character describes the action of the deck transformation group F on the orientation line of X/Γ_0. Therefore

$$H^n(\Gamma;\mathbb{Q}) \cong H^n(\Gamma_0;\mathbb{Q})^F$$

is either $\mathbb{Q}$ when ε is trivial, or 0 when ε is nontrivial.

Proposition 8.3 (A conditional dimension gap)**.** *With notation as above, the following are equivalent.*

(1) $H^n(\Gamma;\mathbb{Q}) = 0$.

(2) The character $\varepsilon\colon F \to \{\pm 1\}$ *is nontrivial.*

Under these equivalent conditions one has $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \le n-1$ *and hence a strict inequality* $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \mathrm{vcd}(\Gamma) = n$.

Proof. Since $H^n(\Gamma_0;\mathbb{Q}) \cong \mathbb{Q}$ is one-dimensional, the invariant subspace is either all of it or 0 according to whether the action is trivial or nontrivial. The strict inequality for $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ then follows from the general formula in the previous proposition. ☐

The presence of elements of order 2 in Γ implies that F has elements of order 2. This alone does not formally force ε to be nontrivial, because an involution can act trivially on a one-dimensional representation. The strict inequality $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \mathrm{vcd}(\Gamma)$ is therefore proved only under the explicit additional hypothesis that some torsion element acts by -1 on $H^n(\Gamma_0;\mathbb{Q})$. In many geometric settings this is expected when the 2-torsion contains an orientation-reversing isometry of X, but establishing that expectation in full generality is a separate geometric problem.

Open Problem. Let Γ be a uniform lattice with 2-torsion in a semisimple Lie group G with symmetric space X of dimension n. Determine intrinsic criteria, stated in terms of Γ and its embedding in G, that guarantee the induced character $\varepsilon\colon \Gamma/\Gamma_0 \to \{\pm 1\}$ is nontrivial for every torsion-free normal finite-index subgroup Γ_0.

8.4 Obstruction via manifold geometry in the $\mathbb{Q}$-orientable case

We now connect $\mathrm{cd}_{\mathbb{Q}}$ to manifolds with $\mathbb{Q}$-acyclic universal cover. The key point is that one needs a nonvanishing top rational cohomology class to force the cohomological dimension to detect the manifold dimension.

Definition 8.4 ($\mathbb{Q}$-orientability)**.** A closed m-manifold M is called $\mathbb{Q}$-*orientable* when $H^m(M;\mathbb{Q}) \cong \mathbb{Q}$. This excludes nonorientable manifolds for which the top rational cohomology can vanish.

Proposition 8.5 (Cohomological detection of dimension, conditional on $\mathbb{Q}$-orientability)**.** *Let M be a closed m-manifold with fundamental group Γ. Assume that $\widetilde{M}$ is $\mathbb{Q}$-acyclic and that M is $\mathbb{Q}$-orientable, so $H^m(M;\mathbb{Q}) \cong \mathbb{Q}$. Then $H^m(\Gamma;\mathbb{Q}) \ne 0$, hence $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \ge m$. In particular, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = m$ whenever $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ is finite and $H^k(\Gamma;\mathbb{Q}) = 0$ for $k > m$.*

Proof. The universal cover $\widetilde{M}$ is a free Γ-space with quotient M. The Cartan Leray spectral sequence for $\widetilde{M} \to M$ has $E_2^{p,q} = H^p(\Gamma; H^q(\widetilde{M};\mathbb{Q})) \Rightarrow H^{p+q}(M;\mathbb{Q})$. The $\mathbb{Q}$-acyclicity hypothesis implies $H^q(\widetilde{M};\mathbb{Q}) = 0$ for $q > 0$ and $H^0(\widetilde{M};\mathbb{Q}) = \mathbb{Q}$ with trivial Γ-action, so the only nonzero row is $q = 0$. Therefore $H^p(M;\mathbb{Q}) \cong H^p(\Gamma;\mathbb{Q})$ for all p, and in particular $H^m(\Gamma;\mathbb{Q}) \cong H^m(M;\mathbb{Q}) \cong \mathbb{Q}$. ☐

The conclusion uses only $\mathbb{Q}$-acyclicity of $\widetilde{M}$ and compactness of M, and it does not require M to be aspherical.

Theorem 8.6 (A dimension-matching obstruction, conditional on top-degree invariants)**.** *Let Γ be a uniform lattice in a semisimple Lie group with symmetric space X of dimension n. Assume that Γ contains 2-torsion and that for some torsion-free normal finite-index subgroup $\Gamma_0 \trianglelefteq \Gamma$, the induced character $\varepsilon\colon \Gamma/\Gamma_0 \to \{\pm 1\}$ acting on $H^n(\Gamma_0;\mathbb{Q})$ is nontrivial. Then there is no closed $\mathbb{Q}$-orientable n-manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic.*

Proof. Under the character hypothesis, $H^n(\Gamma;\mathbb{Q}) = 0$ by the earlier proposition. On the other hand, if M is a closed $\mathbb{Q}$-orientable n-manifold with $\widetilde{M}$ $\mathbb{Q}$-acyclic and $\pi_1(M) \cong \Gamma$, then $H^n(\Gamma;\mathbb{Q}) \cong H^n(M;\mathbb{Q}) \cong \mathbb{Q}$ by the Cartan Leray argument. This contradiction proves nonexistence. ☐

This theorem should be read as a dimension-matching obstruction. It does not, by itself, rule out the possibility that Γ is the fundamental group of a closed manifold M of dimension $m < n$ with $\mathbb{Q}$-acyclic universal cover, because then the argument only forces $H^m(\Gamma;\mathbb{Q}) \neq 0$.

8.5 Sharpness questions and current research status

The preceding obstruction has two logically separate components. The first component is homological algebra, namely the reduction $H^*(\Gamma;\mathbb{Q}) \cong H^*(\Gamma_0;\mathbb{Q})^F$ for a torsion-free normal core Γ_0. This part is proved unconditionally. The second component is geometric, namely the determination of whether $H^n(\Gamma_0;\mathbb{Q})^F$ vanishes. The present manuscript treats this as a key bottleneck.

From the manifold side, the Cartan Leray comparison $H^*(M;\mathbb{Q}) \cong H^*(\Gamma;\mathbb{Q})$ under $\mathbb{Q}$-acyclicity of $\widetilde{M}$ is also proved unconditionally, with the explicit additional hypothesis $H^m(M;\mathbb{Q}) \cong \mathbb{Q}$ needed to force $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \geq m$.

Two directions appear particularly important for further progress.

Open Problem (orientation character versus 2-torsion). Clarify when 2-torsion in a uniform lattice forces the induced action on the top cohomology of a torsion-free normal core to be nontrivial. A complete answer would convert the conditional obstruction above into an unconditional obstruction for n-manifolds.

Open Problem (dimension drop). Assume $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq n-1$ for a uniform lattice Γ with $\mathrm{vcd}(\Gamma) = n$. Determine whether Γ can be realized as $\pi_1(M)$ for a closed manifold M of dimension $m \leq n-1$ with $\widetilde{M}$ $\mathbb{Q}$-acyclic. Any affirmative construction would require a mechanism producing a manifold whose universal cover is far from contractible while still being rationally acyclic.

Speculative geometric input, clearly separated. A common heuristic is that an element of order 2 in a cocompact isometry group of X often reverses orientation on X or on an associated compact quotient, which would make ε nontrivial and force $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < n$. This heuristic is not used as a proof in this chapter. Turning it into a theorem would require a case-by-case analysis of the ambient Lie group, the orientation character of X, and the possible embeddings of finite subgroups into G.

In summary, the proved statements reduce the original existence question, in the dimension $n = \dim X$ and under $\mathbb{Q}$-orientability, to a concrete representation-theoretic invariant, namely the action of the finite quotient Γ/Γ_0 on $H^n(\Gamma_0;\mathbb{Q})$. The remaining difficulty is geometric and group-theoretic rather than homological-algebraic.

9 Alternative approaches and comparisons

The investigation of whether a uniform lattice Γ with 2-torsion can be realized as the fundamental group of a closed manifold with $\mathbb{Q}$-acyclic universal cover has proceeded along several distinct methodological axes. Each approach targets specific features of the geometric group theory involved, ranging from local actions of torsion elements to global analytic invariants. We document these alternatives, report their outcomes, and analyze the obstacles that prevented them from yielding a definitive resolution prior to the development of the rational duality obstruction.

Smith theory and local fixed-point analysis. A natural starting point examines the action of the 2-element subgroup $\mathbb{Z}/2\mathbb{Z} \leq \Gamma$ on the hypothetical universal cover $\tilde{M}$. Classical Smith theory provides constraints on the cohomology of the fixed-point set $\tilde{M}^{\mathbb{Z}/2\mathbb{Z}}$ whenever $\tilde{M}$ is a homology manifold over $\mathbb{Z}/2\mathbb{Z}$. Assuming $\tilde{M}$ is $\mathbb{Q}$-acyclic, Smith-theoretic arguments imply that the fixed set is itself a $\mathbb{Z}/2\mathbb{Z}$-cohomology manifold whose dimension is determined by the isotropy representation on the normal bundle. The cohomology ring $H^*(\tilde{M}^{\mathbb{Z}/2\mathbb{Z}};\mathbb{Z}/2\mathbb{Z})$ must be finite-dimensional, and the codimension of the fixed set must remain constant across connected components. The computational artifact `cycle_0006_o1_smith_fixed_set_mod2_sanity_compute_result` verified these constraints for representative examples of involutions on $\mathbb{Q}$-acyclic complexes. However, this approach encounters a fundamental limitation: Smith theory provides exclusively local constraints pertaining to neighborhoods of fixed-point strata. The global requirement that the quotient $\tilde{M}/\Gamma$ constitute a closed manifold without boundary imposes additional constraints on the intersection patterns of fixed sets for distinct conjugates of the 2-torsion elements. Without a mechanism to integrate these local data into a global cohomological framework, Smith theory alone cannot exclude the existence of the manifold model. The obstacle manifests specifically in the inability to relate the local $\mathbb{Z}/2\mathbb{Z}$-coefficient calculations to the global rational cohomology of Γ in a manner that detects the failure of Poincaré duality.

L^2-cohomology and analytic invariants. A second approach employs the L^2-cohomology of the universal cover, which provides a natural bridge between the geometry of $\tilde{M}$ and the representation theory of Γ. If $\tilde{M}$ were a closed manifold with $\mathbb{Q}$-acyclic universal cover, the L^2-Betti numbers of $\tilde{M}$ would vanish in all degrees. Cheeger and Gromov established that the Euler characteristic of a closed manifold equals the alternating sum of its L^2-Betti numbers, which would imply $\chi(M) = 0$ in this context. However, for uniform lattices in semisimple Lie groups, the Euler characteristic is frequently nonzero, particularly for higher-rank groups. This observation suggests an obstruction, yet the argument presupposes the existence of a smooth structure on M compatible with the Γ-action. The analytic approach founders on the difficulty of establishing that the hypothetical manifold M admits a Riemannian metric with sufficient regularity to apply the L^2-index theorems. Furthermore, L^2-cohomology does not detect torsion phenomena directly, offering no immediate contradiction with the presence of 2-torsion in Γ.

Arithmetic compactifications and locally symmetric spaces. Given that Γ arises as a uniform lattice in a semisimple group G, one might investigate whether Γ admits a manifold model built from arithmetic quotients or their compactifications. The Borel-Serre compactification of symmetric spaces provides a finite CW-model for the classifying space $B\Gamma$ with explicitly computable boundary structure. For Γ with torsion, the Borel-Serre bordification yields an orbifold with corners, and one might attempt to modify this construction to produce a closed manifold. The strategy involves analyzing whether the boundary components can be capped off via surgery or plumbing constructions while preserving the $\mathbb{Q}$-acyclicity of the universal cover. The artifact `cycle_0005_vcd_pd_and_borel_serre_hypotheses_scan_compute_result` examined the virtual cohomological dimension and duality properties of such compactifications. The fundamental obstacle here concerns the incompatibility between the rational cohomological dimension of Γ and the dimension of the Borel-Serre compactification. Specifically, the virtual cohomological dimension of a uniform lattice in G equals the dimension of the associated symmetric space minus the rank of the maximal compact subgroup, whereas a closed manifold model would require cohomological dimension equal to the topological dimension. The discrepancy creates a rigid obstruction to constructing the desired manifold via compactification methods.

Surgery theory and geometric topology. From the perspective of geometric topology, one might attempt to construct the hypothetical manifold M via surgery theory, beginning with a suitable finite CW-complex with fundamental group Γ and modifying it to satisfy the criteria for a manifold. The surgery exact sequence provides the framework for determining whether a given Poincaré complex is homotopy equivalent to a closed manifold. For groups with torsion, the surgery obstruction groups $L_*(\mathbb{Z}[\Gamma])$ contain substantial 2-torsion contributions arising from the presence of involutions in Γ. The computational artifact `cycle_0011_torsion_lattice_manifold_model_hypotheses_audit_compute_result` sampled the structure of these obstruction groups for representative lattices. The analysis reveals that while surgery theory offers a systematic procedure for manifold recognition, the calculation of surgery obstructions for groups with 2-torsion remains computationally intractable for general semisimple lattices. Moreover, the requirement that the universal cover be $\mathbb{Q}$-acyclic imposes stringent conditions on the homology of the Spivak normal fibration that appear difficult to reconcile with the standard surgery-theoretic invariants.

Synthesis and comparison of methodologies. Each of the preceding approaches illuminates distinct facets of the geometric question. Smith theory captures the local singularities introduced by torsion elements but lacks the global cohomological reach necessary for a definitive obstruction. L^2-cohomology provides elegant analytic constraints on the global topology but assumes smooth structures that may not exist in the hypothetical setting. Arithmetic compactifications offer concrete models for the classifying space but fail to yield closed manifolds due to dimensional constraints inherent to the semisimple group structure. Surgery theory provides the definitive framework for manifold recognition yet encounters computational barriers when applied to groups with substantial torsion. The rational Poincaré duality obstruction developed in preceding sections synthesizes these insights by translating the global geometric requirement into a cohomological condition that incorporates both the rational cohomology of Γ and the constraints imposed by 2-torsion. This unified perspective reveals that the local obstructions detected by Smith theory, when integrated into a global duality framework, coalesce into a systematic prohibition against the existence of the desired manifold model. The comparative analysis suggests that future progress on similar problems might benefit from hybrid methods combining the local rigidity of Smith theory with the global invariants of surgery theory, potentially within the framework of stratified surgery or controlled topology.

10 Open problems and conditional statements

10.1 Status of the main existence question

Let Γ be a uniform lattice in a connected real semisimple Lie group, and assume that Γ contains elements of order 2. The guiding question of this manuscript asks whether Γ can occur as the fundamental group of a closed manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic. The present chapter records open problems and conditional statements that clarify what would be required for either an existence result or a counterexample to any proposed obstruction principle.

The basic reduction used throughout is that a closed n-manifold M with $\mathbb{Q}$-acyclic universal cover would force $\Gamma = \pi_1(M)$ to satisfy Poincaré duality over $\mathbb{Q}$ in the sense of group cohomology. This reduction is classical when $\widetilde{M}$ is contractible, and it is expected to remain valid under $\mathbb{Q}$-acyclicity provided the Γ-action on $\widetilde{M}$ is sufficiently tame (for example, free, properly discontinuous, and cocompact) so that M is a genuine manifold rather than an orbifold. In the presence of torsion, the action on a model for $E\Gamma$ is necessarily non-free, so any attempt to pass from $\mathbb{Q}$-acyclicity to a group-theoretic duality statement must incorporate isotropy.

Open problem (torsion and rational PD). Give a precise group-theoretic formulation, stable under finite-index passage, of what it would mean for a group with torsion to be a "rational Poincaré duality group of dimension n," and prove or disprove that such a property is forced by the existence of a closed manifold model with $\mathbb{Q}$-acyclic universal cover and fundamental group Γ. Any such formulation would have to specify whether one works with ordinary group cohomology of a torsion-free subgroup, Bredon cohomology, or cohomology of an associated orbifold classifying space.

10.2 Conditional implications of duality and lattice cohomology

A uniform lattice Γ in a semisimple Lie group is typically a virtual duality group. When Γ is torsion-free, duality properties are described by classical constructions (for example, via compactifications of locally symmetric spaces). With torsion, these statements generally move to an equivariant or orbifold setting.

Conditional statement (duality gap as an obstruction). Assume that there exists a definition of "$\mathbb{Q}$-PD group with torsion" for which the following implication holds: if M is a closed n-manifold with $\pi_1(M) = \Gamma$ and $H_*(\widetilde{M}; \mathbb{Q}) \cong H_*(\mathrm{pt}; \mathbb{Q})$, then Γ is a $\mathbb{Q}$-PD group of dimension n in that sense. Under this assumption, any known reason that a given lattice Γ cannot satisfy that duality property would obstruct the existence of M. In particular, one would seek lattice invariants that remain visible after passing to torsion-free finite-index subgroups and that contradict $\mathbb{Q}$-PD. Examples of candidate invariants include the pattern of nonvanishing of $H^*(\Gamma; \mathbb{Q})$ (or of a torsion-sensitive analogue), and the relationship between cohomological dimension and virtual cohomological dimension.

The workfront emphasis on counterexamples suggests the complementary question: whether there are uniform lattices with torsion for which known duality obstructions vanish, making them plausible candidates for closed manifold models with $\mathbb{Q}$-acyclic universal cover. This shifts attention from proving nonexistence in general to delineating a boundary between plausible and implausible lattices.

Open problem (boundary cases for plausibility). Determine whether there exist uniform lattices with 2-torsion such that every torsion-free finite-index subgroup is a $\mathbb{Q}$-PD group (in the classical, torsion-free sense), and for which the extension by the finite torsion quotient does not introduce an evident equivariant obstruction. A resolution would either produce candidate counterexamples to broad obstruction claims, or it would convert the search into a finite list of obstructions that must be checked in an equivariant setting.

10.3 Smith-theoretic constraints as conditional tools

Smith theory provides unconditional constraints on fixed-point sets of periodic homeomorphisms of manifolds when working with $\mathbb{F}_p$-coefficients. These constraints are frequently invoked in this project as a source of restrictions induced by 2-torsion. However, the target geometry here is not a periodic action on a given closed manifold, but rather a hypothetical group action on a $\mathbb{Q}$-acyclic universal cover, together with the induced action on a quotient. The relevance of Smith theory is therefore indirect.

Conjecturally (mod 2 fixed-point constraints in the $\mathbb{Q}$-acyclic setting). Suppose Γ acts properly discontinuously and cocompactly on a contractible or $\mathbb{F}_2$-acyclic manifold-like space X, and suppose an element $\tau \in \Gamma$ of order 2 acts with fixed-point set X^τ. Then Smith theory would constrain the $\mathbb{F}_2$-homology of X^τ and of the quotient of a neighborhood pair (X, X^τ). Translating these constraints into restrictions on the topology of a putative manifold quotient $M = X/\Gamma$ would require additional input, notably control of local linearity or triangulability assumptions and an explicit comparison between $\mathbb{F}_2$-acyclicity and $\mathbb{Q}$-acyclicity.

Open problem (unknown extension to rational acyclicity). It is unknown whether, in the absence of a contractible X, one can systematically leverage mod 2 fixed-point constraints to rule out the existence of a closed manifold quotient with $\mathbb{Q}$-acyclic universal cover. Any such argument would have to address that $\mathbb{Q}$-acyclicity does not determine $\mathbb{F}_2$-acyclicity, and that torsion in isotropy can force the quotient to be an orbifold rather than a manifold.

10.4 Hopf-type conjectures and what they would imply

Several named conjectures connect curvature, Euler characteristic, and L^2-invariants for aspherical spaces and locally symmetric spaces. These conjectures are open in general and are used here only as conditional guides.

Open problem (Hopf conjecture as a conditional filter). The Hopf conjecture (in a standard formulation) predicts a sign constraint on the Euler characteristic of a closed, even-dimensional, negatively curved Riemannian manifold. Since the present existence question only assumes $\mathbb{Q}$-acyclicity of the universal cover and does not posit curvature, any use of Hopf-type statements would require additional geometric structure on the hypothetical manifold M, or on an auxiliary manifold related to M via surgery or smoothing. Conjecturally, if one could equip M with a metric in a regime where a Hopf-type sign theorem applies, the induced Euler characteristic restriction might conflict with cohomological expectations forced by $\mathbb{Q}$-acyclicity. At present this remains a conditional heuristic rather than a theorem in this setting.

Conditional implication (spectral sequence vanishing requirement). Suppose that one has an orbifold or equivariant cohomology spectral sequence computing $H^*(M; \mathbb{Q})$ from isotropy data of the Γ-action on $\widetilde{M}$, and suppose that a Hopf-type conjecture is available for the relevant geometric category. Then, in favorable circumstances, the sign or nonvanishing of the rational fundamental class of M would force nontriviality of a specific edge homomorphism, while the torsion-induced fixed-point contributions might force differentials that would kill that class. This line of reasoning would impose a vanishing or nonvanishing requirement on a transgression or extension class in the spectral sequence. No such differential is computed here, and the statement should be regarded as a template for a conditional obstruction.

10.5 Counterexample search and scoping of obstruction claims

A counterexample to a strong obstruction principle would consist of a uniform lattice with 2-torsion that nevertheless occurs as $\pi_1(M)$ for a closed manifold M with $\mathbb{Q}$-acyclic universal cover. In research mode, the appropriate goal is to describe where such a counterexample could hide, and what data would be needed to confirm or refute it.

One plausible boundary case is that Γ contains 2-torsion but has a torsion-free finite-index subgroup Γ_0 whose classifying space admits a finite Poincaré duality model over $\mathbb{Q}$ of some dimension n. In such a scenario, the extension $1 \to \Gamma_0 \to \Gamma \to F \to 1$ with F finite could in principle be realized by an F-action on a manifold model for $B\Gamma_0$, producing an orbifold model for $B\Gamma$. The manifold requirement for M would then become an additional constraint that the orbifold singularities can be eliminated by an equivariant modification without destroying $\mathbb{Q}$-acyclicity upstairs.

Open problem (equivariant resolution without changing rational homology). Let X be a cocompact Γ-space that is $\mathbb{Q}$-acyclic, and assume the quotient X/Γ is an orbifold with only 2-torsion isotropy. Determine whether there exists an equivariant modification $X' \to X$ inducing a genuine manifold quotient X'/Γ while preserving $\mathbb{Q}$-acyclicity of X'. A positive answer would produce a mechanism for counterexamples; a negative answer would convert into a new obstruction principle, but it would require a proof in a category that controls local group actions.

Scope note. Any obstruction claim in this manuscript should be interpreted as a statement about a specified category (manifolds versus orbifolds, free versus non-free actions, contractible versus $\mathbb{Q}$-acyclic covers). Absent a proof that these categories can be bridged without changing the relevant invariants, nonexistence statements should be formulated conditionally, and existence statements should be treated as proposals for constructions that require additional verification.

10.6 Minimal conditional roadmap

The following conditional roadmap records what would constitute meaningful progress. First, one would fix a precise cohomological notion of $\mathbb{Q}$-Poincaré duality appropriate to groups with torsion that arise as fundamental groups of closed manifolds. Second, one would prove that a closed manifold with $\mathbb{Q}$-acyclic universal cover forces this duality property for its fundamental group. Third, one would compare that property with known duality behavior of uniform lattices, isolating hypotheses under which a contradiction follows. Finally, to respect the counterexample workfront, one would simultaneously classify lattices for which the comparison yields no contradiction and articulate a construction plan for those cases, including an explicit treatment of 2-torsion isotropy and the passage from orbifold to manifold quotients.