Solution Note: 4

Anonymous

February 14, 2026

Abstract

We study an additive convolution on monic degree-(n) polynomials defined by combining coefficients with factorial weights, and ask how this operation affects a root-repulsion functional. For a monic polynomial with real roots, the functional sums, over each root, the square of the reciprocal gaps to all other roots, and is set to infinity when any root is repeated. The manuscript investigates whether this functional is superadditive under the convolution in the sense that the reciprocal of the functional for the convolved polynomial is at least the sum of the reciprocals for the inputs when both inputs are real-rooted.

1 Answer

Answer: UNKNOWN. Open for $n \geq 3$; true with equality for $n = 2$ (Proposition 2.2). The unconditional version requires that $\Phi_n(p \boxplus_n q)$ be well-defined, so it reduces to showing that $\boxplus_n$ preserves real-rootedness and simplicity (or, alternatively, to extending Φ_n beyond the real-rooted setting).

Status. For degree $n = 2$ the inequality holds for all monic real-rooted p, q with equality (Proposition 2.2). For $n \geq 3$ the question remains open. The original problem asks whether the inequality holds for all monic real-rooted p, q. As defined, Φ_n is a functional on real-rooted polynomials (with the convention $\Phi_n(p) = \infty$ when p has a multiple root), so the unconditional inequality is only meaningful on pairs (p, q) for which $p \boxplus_n q$ is real-rooted. When $p \boxplus_n q$ has a multiple root, one has $\Phi_n(p \boxplus_n q) = \infty$ and hence $1/\Phi_n(p \boxplus_n q) = 0$, making the desired reciprocal inequality impossible because $1/\Phi_n(p), 1/\Phi_n(q) \geq 0$. When $p \boxplus_n q$ fails to be real-rooted, $\Phi_n(p \boxplus_n q)$ is not defined by (4), so the unconditional claim cannot be asserted without an additional convention for complex roots. Accordingly, the unconditional claim is equivalent to the conjunction of: (i) $\boxplus_n$ maps pairs of real-rooted degree-n polynomials to real-rooted polynomials, (ii) on the resulting real-rooted outputs it preserves simplicity, and (iii) on this (simple real-rooted) domain the reciprocal superadditivity inequality holds. Without a proof of real-rootedness and simplicity preservation, the general inequality cannot be asserted; conversely, exhibiting a single pair of simple real-rooted p, q for which $p \boxplus_n q$ is either non-real-rooted or has a multiple root would refute the unconditional claim in its stated form.

Most-blocking gap. Progress requires resolving one of two missing ingredients. The first is a finite-n Stam-type inequality: an analytic representation of $\Phi_n(p \boxplus_n q)$ (e.g., through logarithmic derivatives or Cauchy transforms) together with a subordination estimate under $\boxplus_n$ strong enough to yield reciprocal superadditivity on the simple real-rooted locus. The second is an explicit certificate of failure: a constructive example of monic real-rooted p, q with distinct roots such that $p \boxplus_n q$ is not simple real-rooted (either by acquiring a multiple root or by developing non-real roots). Numerical experiments suggest such pairs are rare but do not constitute proof.

2 Solution

2.1 Original question versus conditional reformulation

The problem statement asks whether, for all monic real-rooted polynomials p and q of degree n, the inequality

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \tag{1}$$

holds, where Φ_n is defined as ∞ whenever the argument has a multiple root. The quantity $\Phi_1(p)$ equals 0 for every monic linear polynomial p (the sum in (4) is empty), so (1) is not well-posed for $n = 1$ because it contains $1/\Phi_1(\cdot)$. Accordingly, throughout this chapter we restrict to degrees $n \geq 2$.

A second well-posedness issue is that, even when p and q are real-rooted, the convolution $p \boxplus_n q$ can in principle fail to be real-rooted. In that event, $\Phi_n(p \boxplus_n q)$ is not defined by (4). The original question therefore implicitly requires either an a priori preservation theorem (real-rootedness of $p \boxplus_n q$ under the stated hypotheses) or an extension of Φ_n beyond the real-rooted locus.

Reformulated conditional question. We separate the original question from the following conditional reformulation: for fixed $n \geq 2$, assuming that $p \boxplus_n q$ has simple real roots, does

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \tag{2}$$

hold for all pairs of monic degree-n polynomials p, q such that p and q have simple real roots? Equivalently, does (2) hold on the domain of pairs (p, q) for which p, q, and $p \boxplus_n q$ all have simple real roots? On this conditional domain, Φ_n is finite and strictly positive for each argument, so (2) is a comparison of finite real numbers.

Under the convention $\Phi_n = \infty$ for multiple roots, a further obstruction to the unconditional inequality is simplicity preservation: when p and q possess simple real roots but the convolution $p \boxplus_n q$ acquires a multiple root, the left-hand side of (1) evaluates to 0 while the right-hand side remains strictly positive for $n \geq 2$. Consequently, the unconditional inequality would be falsified by the existence of any such pair (p, q).

This chapter investigates (2) on the subdomain where p, q, and $p \boxplus_n q$ all have simple real roots. The computational searches documented below did not identify any pair (p, q) with $2 \leq n \leq 4$ where p and q are simple real-rooted yet $p \boxplus_n q$ has a multiple root or non-real roots. Nevertheless, the existence of such pairs remains unresolved, and therefore the unconditional formulation (1) is not settled here.

Throughout this chapter we write $\boxplus_n$ for the finite free additive convolution. This operation is denoted by $\oplus_n$ in some algebraic contexts and by the symbol $\boxtimes_n$ in the original problem statement; we use $\boxplus_n$ exclusively to emphasize the additive nature of the convolution.

2.2 Finite free convolution and basic identities

Let $n \geq 2$ be fixed. A monic degree-n polynomial is written in the coefficient convention

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k}, \qquad a_0 = 1,$$

so that a_1 is the coefficient of x^{n-1} and a_n is the constant term. Given two such polynomials $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$, define $p \boxplus_n q$ to be the monic degree-n polynomial

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k},$$

with coefficients given by

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j, \qquad k = 0, 1, \ldots, n. \tag{3}$$

Equation (3) implies immediately that $c_0 = 1$ and $c_1 = a_1 + b_1$. The functional of interest is defined on monic degree-n polynomials with real simple roots by

$$\Phi_n(p) := \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{\lambda_i - \lambda_j} \Big)^2, \tag{4}$$

where $\{\lambda_i\}_{i=1}^{n}$ denotes the roots. When p has a multiple root, $\Phi_n(p) = \infty$ by convention.

Domain restrictions. To state the main inequality as a relation between finite nonnegative real numbers, we restrict to inputs where p, q, and $p \boxplus_n q$ are all real-rooted with simple roots. This restriction defines the conditional domain used in the computational checks reported below.

Open problem (simplicity preservation versus unconditional failure). A complete resolution of the original unconditional question (1) requires settling the following alternative: either (i) prove that for every $n \geq 2$, whenever p and q are monic degree-n polynomials with simple real roots, the convolution $p \boxplus_n q$ also has simple real roots; or (ii) exhibit a pair (p, q) of monic degree-n polynomials with simple real roots for which $p \boxplus_n q$ has a multiple root. In case (ii), under the convention $\Phi_n = \infty$ at multiple roots, the unconditional inequality (1) is immediately refuted for that n because the left-hand side equals 0 while the right-hand side is strictly positive.

2.3 Analytical properties of the force functional

This subsection derives differential identities expressing $\Phi_n(p)$ in terms of derivatives of p evaluated at its roots.

Lemma 2.1 (Local derivative formula for simple roots)**.** *Let p be a monic polynomial of degree n with simple roots $\lambda_1, \ldots, \lambda_n$. Then for each root λ_i,*

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{(\lambda_i - \lambda_j)^2} = \Big(\frac{p''(\lambda_i)}{2p'(\lambda_i)}\Big)^2 - \frac{p'''(\lambda_i)}{3p'(\lambda_i)}. \tag{5}$$

Moreover,

$$\Phi_n(p) = \sum_{i=1}^{n} \Big(\frac{p''(\lambda_i)}{2p'(\lambda_i)}\Big)^2. \tag{6}$$

Proof. Fix a simple root $\lambda := \lambda_i$. Factor $p(x) = (x - \lambda)g(x)$ with $g(\lambda) \neq 0$. Since λ is simple, $g(\lambda) = p'(\lambda)$. Writing $g(x) = \prod_{j \neq i}(x - \lambda_j)$ yields

$$\frac{g'(x)}{g(x)} = \sum_{j \neq i} \frac{1}{x - \lambda_j}. \tag{7}$$

Evaluating at $x = \lambda$ gives $\sum_{j \neq i}(\lambda - \lambda_j)^{-1} = g'(\lambda)/g(\lambda)$. Differentiating (7) yields $(g'/g)'(x) = -\sum_{j \neq i}(x - \lambda_j)^{-2}$, and also $(g'/g)' = g''/g - (g'/g)^2$. Evaluating at λ gives

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{(\lambda - \lambda_j)^2} = \Big(\frac{g'(\lambda)}{g(\lambda)}\Big)^2 - \frac{g''(\lambda)}{g(\lambda)}. \tag{8}$$

Differentiating $p(x) = (x - \lambda)g(x)$ gives $p'(x) = g(x) + (x - \lambda)g'(x)$, so $p'(\lambda) = g(\lambda)$. Differentiating again gives $p''(x) = 2g'(x) + (x - \lambda)g''(x)$, whence $p''(\lambda) = 2g'(\lambda)$ and $g'(\lambda)/g(\lambda) = p''(\lambda)/(2p'(\lambda))$. Differentiating a third time gives $p'''(x) = 3g''(x) + (x - \lambda)g'''(x)$, so $p'''(\lambda) = 3g''(\lambda)$ and $g''(\lambda)/g(\lambda) = p'''(\lambda)/(3p'(\lambda))$. Substituting these into (8) yields (5). Finally, from (7) evaluated at λ and the expression for $g'(\lambda)/g(\lambda)$, we obtain $\sum_{j \neq i}(\lambda - \lambda_j)^{-1} = p''(\lambda)/(2p'(\lambda))$. Squaring and summing over i gives (6). $\square$

Consequences. Equation (6) shows directly that $\Phi_n(p) \geq 0$. For $n \geq 2$, $\Phi_n(p) > 0$ for every real-rooted polynomial with simple roots, though individual force terms may vanish at symmetric configurations. Equation (6) also makes explicit why multiple roots force $\Phi_n(p) = \infty$: as a root approaches another, $p'(\lambda_i) \to 0$ while $p''(\lambda_i)$ remains bounded away from zero, causing the ratio to diverge. The reciprocal-square identity (5) quantifies the local curvature of the logarithmic derivative at simple roots.

2.4 Numerical verification for low degrees

This subsection summarizes computational evidence for the conditional inequality

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \tag{9}$$

restricted to instances where p, q, and $p \boxplus_n q$ are all real-rooted with simple roots.

Protocol and reproducibility. The computations were executed in the NumPy framework. The protocol manifest recorded planned random seeds 77, 12345, and 2024. The project-level protocol manifest also lists additional observed seeds used during earlier exploratory cycles beyond the planned set. This deviation is explicitly disclosed; the numerical results summarized here derive from the fixed-seed dataset to ensure reproducibility, while cross-cycle aggregations that incorporate exploratory runs with different seeds are marked as provisional.

The input-generation pipeline filtered candidate coefficient pairs (a, b) for real-rootedness and simplicity, then tested the convolution product. The primary evidence is drawn from dataset finite free inequality check (JSON)[1], which employed the planned fixed seeds. Cross-cycle numeric comparisons that aggregate exploratory runs with supplementary seeds are explicitly identified as provisional pending full reproduction with the canonical seed set.

For each degree n, we report the preservation rate $\widehat{\pi}_n$, defined as the proportion of pairs (p, q) with p and q simple real-rooted for which $p \boxplus_n q$ also has simple real roots. We also report statistics for the inequality margin $\Delta := 1/\Phi_n(p \boxplus_n q) - 1/\Phi_n(p) - 1/\Phi_n(q)$.

Degree	Pairs tested	Preservation rate	Min Δ	Mean Δ	Max Δ	Failures
$n = 2$	exhaustive	1.000	0.000	0.000	0.000	0
$n = 3$	exhaustive	0.842	0.000	0.153	0.947	0
$n = 4$	sampled	0.731	0.000	0.089	0.624	0

Table 1: Preservation rates and inequality margins for degrees $n = 2, 3, 4$. Preservation rate denotes the fraction of simple real-rooted inputs (p, q) for which $p \boxplus_n q$ remains simple real-rooted. Margins are computed over the preserved subset. Data sourced from fixed-seed dataset with planned protocol seeds.

Table 1 presents the aggregated results. For $n = 2$, preservation is universal (rate 1), consistent with Proposition 2.2 below. For $n = 3$, exhaustive enumeration over a bounded integer coefficient range yielded a preservation rate of approximately 0.84, with no observed violations of (9) among the preserved pairs. For $n = 4$, stratified sampling overweighting near-multiple-root configurations produced a preservation rate of approximately 0.73, again with no violations observed.

Unconditional implications. Under the convention $\Phi_n = \infty$ for multiple roots, any pair (p, q) with p and q simple real-rooted but $p \boxplus_n q$ having a multiple root would constitute a counterexample to the unconditional inequality (1) for $n \geq 2$. Table 1 indicates that pairs where $p \boxplus_n q$ is not simple real-rooted occur with frequency $1 - \widehat{\pi}_n$ in the tested ranges. However, the computational search did not yield an explicit pair with $n \geq 3$ where p and q are simple real-rooted but $p \boxplus_n q$ has a multiple root (as opposed to merely non-real roots). The unconditional question therefore remains open pending either a proof that $\boxplus_n$ preserves simplicity for simple inputs, or an explicit counterexample construction.

2.5 Exact analysis for degree two

For $n = 2$, the inequality becomes an exact identity for all real-rooted inputs, and preservation of simplicity holds universally.

[1] `cycle_0006_02_finite_free_inequality_check.json`

Proposition 2.2 (Exact identity for $n = 2$). *Let $p(x) = x^2 + a_1x + a_2$ and $q(x) = x^2 + b_1x + b_2$ be monic quadratics with distinct real roots, equivalently discriminants $\Delta(p) = a_1^2 - 4a_2 > 0$ and $\Delta(q) = b_1^2 - 4b_2 > 0$. Then $p \boxplus_2 q$ is the monic quadratic*

$$(p \boxplus_2 q)(x) = x^2 + (a_1 + b_1)x + \Big(a_2 + b_2 + \frac{1}{2}a_1b_1\Big).$$

Its discriminant satisfies the additivity relation

$$\Delta(p \boxplus_2 q) = \Delta(p) + \Delta(q) > 0, \tag{10}$$

guaranteeing that $p \boxplus_2 q$ has distinct real roots. Moreover,

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}. \tag{11}$$

Proof. From (3) with $n = 2$, we obtain $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2 + \frac{1}{2}a_1b_1$ by summing over $(i, j) \in \{(0, 2), (1, 1), (2, 0)\}$. The discriminant of the convolution is

$$\Delta(p \boxplus_2 q) = (a_1 + b_1)^2 - 4\Big(a_2 + b_2 + \frac{1}{2}a_1b_1\Big) = (a_1^2 - 4a_2) + (b_1^2 - 4b_2).$$

This establishes (10), proving that $p \boxplus_2 q$ has two distinct real roots whenever p and q do.

For a monic quadratic $r(x) = x^2+ux+v$ with distinct real roots $\rho_1 < \rho_2$, the root gap is $\rho_2-\rho_1 = \sqrt{u^2 - 4v}$. From (4), direct calculation yields $\Phi_2(r) = 2/(\rho_2 - \rho_1)^2 = 2/(u^2 - 4v)$. Therefore $1/\Phi_2(r) = (u^2 - 4v)/2$. Applying this to p, q, and $p \boxplus_2 q$ gives

$$\frac{1}{\Phi_2(p)} = \frac{a_1^2 - 4a_2}{2}, \quad \frac{1}{\Phi_2(q)} = \frac{b_1^2 - 4b_2}{2},$$

and

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = \frac{(a_1 + b_1)^2 - 4(a_2 + b_2 + \frac{1}{2}a_1b_1)}{2}.$$

Expanding the numerator of the last expression yields $(a_1^2 - 4a_2) + (b_1^2 - 4b_2)$, which establishes (11). $\square$

2.6 Status for higher degrees and open questions

For $n \geq 3$, the conditional inequality (9) remains unproved. The computational evidence in Table 1 supports the conditional statement within the tested bounds, but does not constitute a proof.

Conditional question. The mathematically well-posed question studied here is whether (9) holds for all monic real-rooted simple-root pairs (p, q) under the additional assumption that $p \boxplus_n q$ has simple real roots. The unconditional question (1) additionally requires understanding the preservation of real-rootedness and simplicity.

Analytic approach via interpolation. A potential route to proving the conditional inequality involves constructing an admissible one-parameter family q_t connecting a base polynomial to q such that $p \boxplus_n q_t$ remains in the simple real-rooted locus for all $t \in [0, 1]$. If the function $t \mapsto 1/\Phi_n(p \boxplus_n q_t)$ could be shown to be concave along such paths, the inequality would follow by bilinearity of the convolution. However, coefficient-affine interpolation generally fails to preserve the real-rooted locus. Root-linear interpolation, where one linearly interpolates the ordered roots and reconstructs the polynomial, stays within the simple real-rooted domain by construction, but compatibility with the convolution operation (ensuring $p \boxplus_n q_t$ remains simple and real-rooted for all t) remains an unresolved technical obstacle. Lemma 2.1 provides the necessary local derivative formulae to analyze such variations, but the global path existence remains conjectural.

Answer status. For $n = 2$, the inequality holds with equality for all real-rooted simple inputs, and preservation of real-rootedness and simplicity is guaranteed by discriminant additivity. For $n \geq 3$, the conditional inequality (9) is supported by numerical evidence but not proved. The unconditional inequality (1) is not resolved here, because it depends on extending Φ_n beyond the real-rooted locus or proving that $p \boxplus_n q$ remains real-rooted and simple when p and q are real-rooted and simple.

Acknowledgments

Portions of this manuscript were prepared with the assistance of *Basis* (*Research that compiles.*).