Solution Note: Suppose that Γ is a uniform lattice in a real semi-simple group, and that Γ contains some 2-torsion

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Abstract

We investigate whether a uniform lattice in a real semisimple Lie group that contains elements of order two can arise as the fundamental group of a closed manifold whose universal covering space has trivial reduced homology with rational coefficients. Framing the problem in terms of the interplay between torsion in lattices, virtual torsion-freeness, and constraints from (rational) acyclicity, we outline potential obstructions and criteria that would govern when such a realization is possible.

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1 Introduction

A classical source of closed aspherical manifolds is provided by uniform lattices in real semisimple Lie groups: if G has no compact factors and $\Gamma \leq G$ is torsion-free, then Γ acts freely, properly, and cocompactly on the associated symmetric space X, yielding a closed manifold $\Gamma\backslash X$ with contractible universal cover. The presence of torsion changes this picture in a genuinely geometric way. Although a lattice with torsion still acts properly and cocompactly on X, the quotient becomes an orbifold, and it is no longer automatic that Γ can be realized as the fundamental group of any closed manifold.

This note focuses on a specific realization problem motivated by the interaction between torsion and homological constraints. Suppose that M is a closed manifold with $\pi_1(M) \cong \Gamma$ and that the universal cover $\widetilde{M}$ is acyclic over $\mathbb{Q}$, meaning $H_i(\widetilde{M};\mathbb{Q}) = 0$ for all $i > 0$. Such manifolds behave, rationally, like aspherical manifolds: the cellular chain complex of $\widetilde{M}$ with $\mathbb{Q}$–coefficients furnishes a finite free $\mathbb{Q}[\Gamma]$–resolution of the trivial module $\mathbb{Q}$, and consequently the group (co)homology of Γ with trivial $\mathbb{Q}$–coefficients can be identified with the (co)homology of M. The central question addressed here is whether this rational acyclicity condition can coexist with 2–torsion when Γ is a uniform lattice in a real semisimple group.

The draft does not claim a definitive answer. Its main contribution is to disentangle several tempting but incorrect reductions and to pinpoint where current techniques fail. First, because deck transformations act freely on $\widetilde{M}$, the fixed-point set $\widetilde{M}^H$ is empty for every nontrivial finite subgroup $H \leq \Gamma$; therefore, fixed-point theorems for involutions do not apply directly to the deck action. Second, $\mathbb{Q}$–acyclicity does not control mod-2 (or integral) homology of $\widetilde{M}$, so arguments that require $\mathbb{F}_2$–acyclicity or integral finiteness must be stated with additional hypotheses. Third, while uniform lattices are often virtually torsion-free, passing to a torsion-free finite-index subgroup only yields manifold realizations for the subgroup; it does not, by itself, reconstruct a free cocompact action of the original torsionful group on the same universal cover.

These observations suggest that any genuine obstruction to 2–torsion must involve invariants that detect finite subgroups in a way compatible with free actions, for example via a comparison between ordinary ("trivial-family") information coming from the free action on $\widetilde{M}$ and the proper-action/Bredon invariants associated to $\underline{E}\Gamma$. The note records the rational comparison statements that follow formally from $\mathbb{Q}$–acyclicity, and then explains why turning these into torsion-sensitive constraints requires additional input that is currently missing.

Organization. Section 2 states the current verdict and isolates the most serious conceptual gap preventing a resolution. Section 3 develops the basic homological consequences of the $\mathbb{Q}$–acyclicity hypothesis for group (co)homology with trivial coefficients and indicates where torsion complicates attempts to promote these consequences to torsion-sensitive duality statements. Section 4 collects checks and edge cases, emphasizing coefficient issues, the irrelevance of fixed-point sets for deck transformations, and consistency constraints that any proposed construction would need to satisfy.

2 Answer

Answer: UNKNOWN.

Status. The question remains open. We possess neither a proof that a uniform lattice Γ with 2-torsion cannot arise as the fundamental group of a closed manifold with $\mathbb{Q}$-acyclic universal cover, nor a construction realizing such an example. For torsion-free uniform lattices in real semisimple groups without compact factors, the associated symmetric space provides a contractible (hence $\mathbb{Q}$-acyclic) manifold on which the lattice acts freely, properly and cocompactly, establishing that such groups satisfy rational duality. However, the extension of these duality results to the torsion case encounters fundamental obstacles. Existing cohomological characterizations of fundamental groups of manifolds with acyclic universal covers rely heavily on the absence of torsion to ensure finite global dimension of the rational group ring. The presence of 2-torsion introduces equivariant cohomological constraints through fixed-point subcomplexes that existing theory does not adequately constrain.

Most-blocking gap. The obstruction resides in the absence of a rigorous implication from the topological condition that $\Gamma = \pi_1(M)$ for a closed manifold M with $\mathbb{Q}$-acyclic universal cover $\widetilde{M}$ to a precise cohomological duality property that interacts substantively with finite subgroups. In the manifold setup, the deck action of Γ on $\widetilde{M}$ is free, so the relevant family for Bredon (co)homology associated to this action is the *trivial family* $\mathcal{F}_{\mathrm{tr}} = \{1\}$, not the family of all finite subgroups. Equivalently, Bredon cohomology with respect to $\mathcal{F}_{\mathrm{tr}}$ reduces to ordinary group cohomology, and the $\mathbb{Q}$-acyclicity of $\widetilde{M}$ can at most be expected to force a rational duality statement for Γ with trivial coefficients.

Any constraint that genuinely *involves* finite subgroups of Γ cannot be read off from the free action on $\widetilde{M}$ via fixed-point sets, because $\widetilde{M}^H = \varnothing$ for each nontrivial finite subgroup $H \leq \Gamma$. To bring finite subgroups into the picture one must instead work with a model for $\underline{E}\Gamma$ (the classifying space for proper actions) and formulate a Bredon duality condition for the family $\mathcal{F}$ of finite subgroups at that level. What is missing is a theorem that relates the "free-action" information (equivalently, $\mathcal{F}_{\mathrm{tr}}$-information) coming from $\widetilde{M}$ to a specific $\mathcal{F}$-Bredon Poincaré duality pattern for $\underline{E}\Gamma$ and its fixed-point systems. Without such a comparison principle, it is presently unclear how to deduce constraints on centralizers or normalizers of finite subgroups that might contradict the presence of 2-torsion in a uniform lattice.

Scope of the required theorem. One might hope that the specific arithmetic or Lie-theoretic structure of uniform lattices would furnish a shortcut avoiding the full machinery of Bredon duality. However, the diversity of possible 2-torsion elements across different semisimple groups suggests that any resolution must address the general situation of groups acting on $\mathbb{Q}$-acyclic manifolds. A lattice-specific approach would require case-by-case analysis of fixed-point configurations for involutions, which currently lacks the systematic cohomological characterization needed to rule out all possibilities. Consequently, the most viable path forward involves establishing general Bredon duality constraints for groups acting freely, properly and cocompactly on $\mathbb{Q}$-acyclic manifolds, together with an additional comparison theorem that upgrades this "free-action" information to constraints on proper actions that can detect and obstruct 2-torsion.

Falsifiable conjecture. To render the question empirically and theoretically decidable, we formulate the following precise conjecture. If Γ is a uniform lattice in a real semisimple Lie group and Γ contains elements of order 2, then Γ cannot act freely, properly and cocompactly on any $\mathbb{Q}$-acyclic manifold. Consequently, such a lattice cannot serve as the fundamental group of a closed manifold whose universal cover is $\mathbb{Q}$-acyclic. We emphasize that these conditions are not equivalent without an extension theorem for group actions. Indeed, such a lattice Γ admits a torsion-free finite-index subgroup Γ', which acts freely and cocompactly on the contractible symmetric space of the ambient Lie group; thus the impossibility for Γ does not preclude the existence of such an action for a finite-index subgroup.

What would resolve it. Resolution in the negative direction would require a theorem establishing that any group acting freely, properly and cocompactly on a $\mathbb{Q}$-acyclic manifold is a rational Bredon duality group satisfying specific constraints on centralizers of finite subgroups, together with a comparison principle relating the resulting ordinary (trivial-family) duality information to Bredon cohomology for the family of finite subgroups, and combined with a demonstration that uniform lattices with 2-torsion violate these constraints. Resolution in the positive direction would require an explicit geometric construction of a closed manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic, or alternatively a proof that some specific uniform lattice with 2-torsion admits such a free action on a contractible or acyclic complex.

3 Solution

We fix notation and make explicit the coefficient choices used in the reduction. Let M be a compact, connected, boundaryless n–manifold with fundamental group $\Gamma = \pi_1(M)$. Let $\widetilde{M} \to M$ denote the universal covering space with deck group Γ. The guiding hypothesis for the existence question is

$$H_i(\widetilde{M};\mathbb{Q}) = 0 \text{ for } i > 0, \qquad H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}. \tag{1}$$

We call this $\mathbb{Q}$–*acyclicity*. The goal of this chapter is to push this hypothesis through standard homological tools to obtain constraints on Γ, and then to isolate where, and in what coefficient ring, torsion yields an obstruction.

3.1 Rational Poincaré duality constraints

This subsection records the formal consequences of the $\mathbb{Q}$–acyclicity hypothesis for the (co)homology of Γ. The main point is that, under standard finiteness hypotheses implicit in the closed manifold input, M behaves like a $K(\Gamma, 1)$ with respect to rational (co)homology with trivial coefficients.

Proposition 3.1 (Rational comparison of homology)**.** *Assume M is a finite CW complex (in particular, a closed manifold) and $\widetilde{M}$ is $\mathbb{Q}$–acyclic. Then there are natural isomorphisms*

$$H_*(\Gamma;\mathbb{Q}) \cong H_*(M;\mathbb{Q}), \qquad H^*(\Gamma;\mathbb{Q}) \cong H^*(M;\mathbb{Q}), \tag{2}$$

where group (co)homology is taken with trivial Γ–action on $\mathbb{Q}$.

Proof. Choose a finite CW decomposition of M and lift it to a Γ–CW decomposition of $\widetilde{M}$. Since the deck action is free, every lifted cell has trivial stabilizer, hence each cellular chain group is a free module of the form

$$C_i(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}[\Gamma]^{r_i}. \tag{3}$$

The augmented cellular chain complex $C_*(\widetilde{M};\mathbb{Q}) \to \mathbb{Q} \to 0$ is a complex of free $\mathbb{Q}[\Gamma]$–modules. The $\mathbb{Q}$–acyclicity assumption means this augmented complex is exact. Hence $C_*(\widetilde{M};\mathbb{Q})$ is a free $\mathbb{Q}[\Gamma]$–resolution of the trivial module $\mathbb{Q}$, and by definition

$$H_*(\Gamma;\mathbb{Q}) = H_*(\mathbb{Q} \otimes_{\mathbb{Q}[\Gamma]} C_*(\widetilde{M};\mathbb{Q})). \tag{4}$$

Using $\mathbb{Q} \otimes_{\mathbb{Q}[\Gamma]} \mathbb{Q}[\Gamma]^{r_i} \cong \mathbb{Q}^{r_i}$, one identifies $\mathbb{Q} \otimes_{\mathbb{Q}[\Gamma]} C_*(\widetilde{M};\mathbb{Q})$ with the cellular chain complex $C_*(M;\mathbb{Q})$, giving $H_*(\Gamma;\mathbb{Q}) \cong H_*(M;\mathbb{Q})$. For cohomology, since $C_*(\widetilde{M};\mathbb{Q})$ is a free $\mathbb{Q}[\Gamma]$–resolution of $\mathbb{Q}$,

$$H^*(\Gamma;\mathbb{Q}) = H^*\big(\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(C_*(\widetilde{M};\mathbb{Q}),\mathbb{Q})\big). \tag{5}$$

Moreover $\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(\mathbb{Q}[\Gamma]^{r_i},\mathbb{Q}) \cong \mathbb{Q}^{r_i}$, and under this identification the cochain complex $\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(C_*(\widetilde{M};\mathbb{Q}),\mathbb{Q})$ agrees with the cellular cochain complex $C^*(M;\mathbb{Q})$, yielding $H^*(\Gamma;\mathbb{Q}) \cong H^*(M;\mathbb{Q})$. □

The preceding comparison transfers the Poincaré duality structure of M into a group-theoretic duality statement, but only for the particular coefficient systems forced by the manifold orientation character.

Theorem 3.2 (Poincaré duality for Γ with trivial rational coefficients)**.** *Let M be a closed, connected n–manifold. Assume $\widetilde{M}$ is $\mathbb{Q}$–acyclic. Then there exists a Γ–module $\mathcal{O}_{\mathbb{Q}}$ which is one-dimensional over $\mathbb{Q}$ (the rational orientation module) such that for every k there are natural isomorphisms*

$$H^k(\Gamma;\mathbb{Q}) \cong H_{n-k}(\Gamma;\mathcal{O}_{\mathbb{Q}}). \tag{6}$$

If M is orientable, then $\mathcal{O}_{\mathbb{Q}} \cong \mathbb{Q}$ as a trivial Γ–module.

Proof. Poincaré duality on M gives $H^k(M;\mathbb{Q}) \cong H_{n-k}(M;\mathcal{O}_{\mathbb{Q}})$ where $\mathcal{O}_{\mathbb{Q}}$ is the local system associated to the orientation character. Proposition 3.1 identifies $H^k(M;\mathbb{Q})$ with $H^k(\Gamma;\mathbb{Q})$ and similarly identifies $H_{n-k}(M;\mathcal{O}_{\mathbb{Q}})$ with the corresponding group homology with twisted coefficients, because the same lifted cellular resolution computes both. □

Remark on terminology. Theorem 3.2 is intentionally formulated only for the trivial rational coefficient module $\mathbb{Q}$ (and its associated orientation twist on homology). It should not be conflated with the standard notion of a Poincaré duality group over $\mathbb{Q}$ that quantifies over all $\mathbb{Q}\Gamma$–modules.

3.2 Cohomological dimension obstruction (integral versus rational)

A common obstruction to Poincaré duality group structures is torsion, via cohomological dimension. In this problem, the coefficient ring matters. This subsection separates the integral obstruction (which is robust) from the rational setting (where torsion by itself does not force large cohomological dimension).

Integral obstruction (robust). Let $\mathrm{cd}_{\mathbb{Z}}(\Gamma)$ denote the cohomological dimension over $\mathbb{Z}$. It is a theorem that if Γ contains a nontrivial finite subgroup, then $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = \infty$. For the present existence problem, however, Theorem 3.2 provides only a duality statement with trivial rational coefficients. Consequently, the implication "torsion yields a contradiction" cannot be routed through $\mathrm{cd}_{\mathbb{Z}}(\Gamma)$ without an additional hypothesis that produces an *integral* finiteness statement for group cohomology (for example, a finite-length projective $\mathbb{Z}\Gamma$–resolution of $\mathbb{Z}$ coming from a $\mathbb{Z}$–acyclic universal cover).

Proposition 3.3 (What the integral argument would require)**.** *Assume the stronger hypothesis that $\widetilde{M}$ is $\mathbb{Z}$–acyclic, namely $H_i(\widetilde{M};\mathbb{Z}) = 0$ for $i > 0$. Then there are natural isomorphisms with trivial coefficients*

$$H_*(\Gamma;\mathbb{Z}) \cong H_*(M;\mathbb{Z}), \qquad H^*(\Gamma;\mathbb{Z}) \cong H^*(M;\mathbb{Z}), \tag{7}$$

and in particular $\mathrm{cd}_{\mathbb{Z}}(\Gamma) \le n$. Since groups with torsion have $\mathrm{cd}_{\mathbb{Z}} = \infty$, it follows that Γ is torsion-free. If, in addition, $\widetilde{M}$ is contractible (equivalently, M is aspherical), then M is a finite $K(\Gamma, 1)$ and Γ is a Poincaré duality group of dimension n over $\mathbb{Z}$.

Discussion. Under $\mathbb{Z}$–acyclicity, the augmented cellular chain complex $C_*(\widetilde{M};\mathbb{Z}) \to \mathbb{Z} \to 0$ is exact and consists of finitely generated free $\mathbb{Z}[\Gamma]$–modules in degrees 0 through n. This is exactly a finite free $\mathbb{Z}[\Gamma]$–resolution of the trivial module $\mathbb{Z}$, hence $\mathrm{cd}_{\mathbb{Z}}(\Gamma) \le n$. Since the presence of torsion forces $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = \infty$, $\mathbb{Z}$–acyclicity already rules out torsion in Γ. The additional asphericity hypothesis is not needed for torsion-freeness, but it is the standard route to the stronger conclusion that Γ is a Poincaré duality group over $\mathbb{Z}$ in the sense that duality holds uniformly for $\mathbb{Z}\Gamma$–modules.

Rational cohomological dimension does not detect torsion. Let $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ denote cohomological dimension over $\mathbb{Q}$. The naive claim "torsion implies $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = \infty$" is false: for example, a finite group F satisfies $H^i(F;\mathbb{Q}) = 0$ for $i > 0$, hence $\mathrm{cd}_{\mathbb{Q}}(F) = 0$. More generally, torsion does not by itself force any contradiction with Theorem 3.2. This coefficient issue means that a negative answer to the existence question cannot follow solely from combining Theorem 3.2 with integral cohomological-dimension arguments, because the latter require an integral finiteness input not implied by $\mathbb{Q}$–acyclicity.

3.3 Torsion in deck groups and fixed-point obstructions: what is proved and what remains

This subsection explains what additional input would convert the rational constraint into a torsion obstruction. It also records a precise conditional statement that would settle the question in the desired direction.

Deck actions are free, but freeness alone does not contradict $\mathbb{Q}$–acyclicity. If $\Gamma = \pi_1(M)$, then Γ acts freely and properly discontinuously on $\widetilde{M}$ by deck transformations. Thus every finite subgroup of Γ acts freely on $\widetilde{M}$. A direct contradiction from "finite group acting freely" typically uses fixed-point theorems (for example, Smith theory for p–groups acting on $\mathbb{F}_p$–acyclic spaces). With only $\mathbb{Q}$–acyclicity, there is no immediate route to apply $\mathbb{F}_2$–Smith theory.

A conditional negative result (precise gap). The following statement formalizes a sufficient condition under which 2–torsion would be ruled out.

Fixed-point input from Smith theory (explicit hypotheses). We isolate the fixed-point input needed below as an explicit assumption. The hypotheses are phrased in one standard topological setting for Smith theory; other settings are possible, but this chapter does not attempt to derive them.

Lemma 3.4 (Smith nonemptiness input (assumption))**.** *Let P be a nontrivial finite 2–group acting continuously by homeomorphisms on a finite-dimensional, locally compact, locally contractible space X. Assume that X is an ENR (equivalently, a locally compact ANR), and that X is a mod-2 homology manifold in the sense of Čech homology (equivalently, Čech cohomology) with coefficients in $\mathbb{F}_2$. Assume*

$$\widetilde{H}_*^{\check{C}ech}(X;\mathbb{F}_2) = 0. \tag{8}$$

Assume moreover that a Smith fixed-point theorem is available for this action in the chosen category, with the standard conclusion that X^P is nonempty and inherits the structure of a mod-2 homology manifold. Then X^P is nonempty. Moreover, X^P is itself a mod-2 homology manifold (possibly of smaller dimension).

Verification of hypotheses for universal covers (formalities). Let M be a closed topological manifold and let $X = \widetilde{M}$. Then X is an n–manifold, hence a mod-2 homology manifold; it is also locally compact, Hausdorff, locally contractible, and finite-dimensional. Every deck transformation is a homeomorphism of X, and the deck action is continuous. To apply Lemma 3.4, one needs a Smith-theory statement covering the given action on X and the chosen homology theory (here, Čech with $\mathbb{F}_2$ coefficients). One sufficient route is to assume that the P–action on X admits a compatible finite-dimensional P–CW model or an equivariant triangulation to which a standard Smith theorem applies. This chapter treats that as part of the explicit assumption.

Theorem 3.5 (Conditional torsion obstruction from mod-2 acyclicity)**.** *Let M be a closed manifold with $\Gamma = \pi_1(M)$. Suppose that $\widetilde{M}$ is acyclic over $\mathbb{F}_2$, that is,*

$$H_i(\widetilde{M}; \mathbb{F}_2) = 0 \text{ for } i > 0. \tag{9}$$

Assume further that the Smith fixed-point conclusion in Lemma 3.4 applies to the action of every finite 2–subgroup of Γ on $\widetilde{M}$. Then Γ contains no elements of order 2.

Proof. Assume $g \in \Gamma$ has order 2, and set $P = \langle g \rangle$. The subgroup P acts freely on $\widetilde{M}$ by deck transformations. The mod-2 acyclicity hypothesis is exactly $\widetilde{H}_*(\widetilde{M}; \mathbb{F}_2) = 0$ (for singular homology), and in particular it implies the corresponding Čech vanishing in the manifold setting. By Lemma 3.4, the fixed-point set $\widetilde{M}^P$ is nonempty. This contradicts freeness of the deck action. ☐

Theorem 3.5 gives a clean negative answer under a hypothesis strictly stronger than $\mathbb{Q}$–acyclicity together with a fixed-point input. It does not resolve the question as posed because $\mathbb{Q}$–acyclicity does not imply $\mathbb{F}_2$–acyclicity.

Relevance of "uniform lattice in a real semisimple group". The preceding reductions and conditional fixed-point obstruction use only that Γ is the fundamental group of a closed manifold with a specified universal cover. The additional assumption that Γ is a uniform lattice in a real semisimple group is expected to provide further finiteness and duality properties, and may connect torsion to a more refined form of duality (for example, a duality theory that keeps track of finite subgroups). Establishing such a bridge requires group-theoretic input beyond Theorem 3.2, because duality with trivial rational coefficients alone does not exclude torsion.

Conclusion (status of the objective). From the existence of a closed manifold M with $\pi_1(M) = \Gamma$ and $\mathbb{Q}$–acyclic universal cover, one can deduce that Γ satisfies Poincaré duality with trivial rational coefficients and an orientation twist (Theorem 3.2). This deduction is coefficient-consistent and does not invoke any torsion-related cohomological-dimension claim that fails over $\mathbb{Q}$. A definitive negative answer in the purely rationally acyclic setting requires an additional mechanism that turns the presence of 2–torsion into an obstruction compatible with rational coefficients. One sufficient mechanism is to strengthen $\mathbb{Q}$–acyclicity to $\mathbb{F}_2$–acyclicity together with an applicable Smith fixed-point theorem, which yields a contradiction via Lemma 3.4 (Theorem 3.5). Without such strengthening or a separate group-theoretic duality input, the torsion obstruction remains unresolved in this chapter, and the verdict for the objective under the stated hypotheses is therefore provisional.

4 Checks and Edge Cases

4.1 Integral versus rational acyclicity: a coefficient sanity check

Let M be a closed manifold with fundamental group Γ and universal cover $\widetilde{M}$. The hypothesis in the main question is that $\widetilde{M}$ is $\mathbb{Q}$-acyclic, meaning $H_i(\widetilde{M}; \mathbb{Q}) = 0$ for $i > 0$. A first check is that this condition does not imply any corresponding vanishing with $\mathbb{F}_2$ coefficients, and therefore does not automatically place the action of a subgroup of order 2 in the regime where classical fixed-point constraints are strongest.

Concretely, it is compatible with $\mathbb{Q}$-acyclicity that $H_i(\widetilde{M};\mathbb{Z})$ has 2-torsion for some $i > 0$; this torsion is invisible rationally but controls $H_i(\widetilde{M};\mathbb{F}_2)$. For the present objective, the relevant caution is methodological: any attempted obstruction that uses mod 2 homology (for example via fixed-point arguments for involutions) needs an additional hypothesis or an independent mechanism relating $\mathbb{Q}$-acyclicity to mod 2 acyclicity.

One practical way to keep this distinction explicit in the manuscript is to separate two verification tasks. First, to verify that $\widetilde{M}$ is $\mathbb{Q}$-acyclic, it suffices to check vanishing of rational homology. Second, to bring fixed-point methods into play, one checks whether $\widetilde{M}$ is also $\mathbb{F}_2$-acyclic (or at least mod 2 homologically finite with constraints on the $\mathbb{F}_2$-Euler characteristic). The second task is not implied by the first, and therefore should be stated as an independent assumption whenever it is invoked.

4.2 Obstructions from 2-torsion: fixed points versus freeness

Assume Γ contains an element τ of order 2, and let it act as a deck transformation on $\widetilde{M}$. Since deck transformations act freely, the fixed-point set $\widetilde{M}^{\langle\tau\rangle}$ is empty. Any obstruction based on involutions with nonempty fixed sets is therefore irrelevant for this specific action; the only available fixed-point input is indirect, for example by combining freeness with global invariants (Euler characteristic, Lefschetz number) computed in suitable coefficients.

In a $\mathbb{Z}$-acyclic setting, one expects strong divisibility constraints for free $\mathbb{Z}/2$-actions on finite CW complexes (or manifolds), often phrased in terms of Euler characteristic parity. In the present $\mathbb{Q}$-acyclic setting, the same template becomes conditional: a parity statement for $\chi(\widetilde{M})$ can be formulated over $\mathbb{Q}$ (and hence for $\chi(M)$), but it does not force mod 2 acyclicity of $\widetilde{M}$.

The project currently uses only toy computations as a sanity check for the relevant logical shape, rather than as evidence for a general theorem. For example, the dataset file characteristic free z2 action divisibility compute result (compute report)[1] records checks on small finite chain models illustrating that, in examples where a free $\mathbb{Z}/2$-action exists and integral acyclicity holds, the Euler characteristic behaves as expected (dataset: characteristic free z2 action divisibility compute result (compute report)[2]). Likewise, cycle 0001 lefschetz number involution on acyclic compute result (compute report)[3] and lefschetz number distribution compute result (compute report)[4] record Lefschetz-number behavior for involutions on acyclic toy complexes (datasets: cycle 0001 lefschetz number involution on acyclic compute result (compute report)[5], lefschetz number distribution compute result (compute report)[6]). These files do not establish a new obstruction for lattices; they only support that the manuscript's intended obstruction templates align with standard parity and Lefschetz heuristics in finite models.

Protocol deviation. Several referenced toy datasets were generated under the observed protocol seeds $\{12345, 24680, 314159, 202603\}$ rather than the planned seeds $\{12345, 24680\}$. This deviation is recorded in the protocol manifest; cross-section numeric comparisons are provisional, and any future consolidated statement should either (i) regenerate these datasets under the planned seeds or (ii) update the project-wide protocol to the observed seeds and propagate it consistently.

4.3 Edge cases from automorphism group actions and why they do not settle the lattice question

There are strong restrictions on actions of certain large groups on homology spheres and acyclic manifolds. A relevant point of comparison is the theory of actions of $\mathrm{Aut}(F_n)$ and related groups, where constraints often arise from the interaction of torsion, fixed-point theory, and the structure of finite subgroups. For instance, results and methods in [**?**] show how to derive nontrivial restrictions on smooth actions of automorphism groups of free groups on homology spheres and acyclic manifolds.

[1] `euler_characteristic_free_z2_action_divisibility_compute_result.json`
[2] `euler_characteristic_free_z2_action_divisibility_compute_result.json`
[3] `lefschetz_number_involution_on_acyclic_compute_result.json`
[4] `dataset_lefschetz_number_distribution_compute_result.json`
[5] `lefschetz_number_involution_on_acyclic_compute_result.json`
[6] `dataset_lefschetz_number_distribution_compute_result.json`

This comparison is useful as an edge-case check, but it does not directly answer the present question for uniform lattices in semisimple Lie groups. The underlying reasons are structural.

First, the group class differs: while $\mathrm{Aut}(F_n)$ contains many finite subgroups and admits natural representations by diffeomorphisms in special constructions, it is not a uniform lattice in a semisimple Lie group. Transferring constraints from [?] to lattices would require an argument that the obstruction mechanism depends only on abstract properties shared by both classes of groups. The current manuscript does not have such a bridge, and therefore should not suggest it.

Second, many arguments about actions on acyclic manifolds implicitly exploit compactness or specific homology-sphere boundary behavior of fixed sets. For the deck transformation action of a torsion element on $\widetilde{M}$, the action is free, so fixed-set mechanisms in [?] (when specialized to involutions) are not immediately applicable.

Third, even when one passes to a torsion-free finite-index subgroup $\Gamma' \leq \Gamma$, the resulting manifold questions change. Γ' can be the fundamental group of a manifold cover $M' \to M$, and $\widetilde{M}$ remains the universal cover. This maneuver removes torsion from the acting group but does not remove the original torsion issue for Γ itself, because the objective asks whether Γ (not merely Γ') arises as a closed-manifold fundamental group with $\mathbb{Q}$-acyclic universal cover.

4.4 Finite-index torsion-free subgroups (Selberg-type maneuver) as a consistency check

Uniform lattices in linear semisimple Lie groups often admit torsion-free finite-index subgroups. For the manuscript's logic, the only safe use of this fact is as a consistency check, not as a solution. If Γ were such a manifold group, then any finite-index subgroup Γ' would also be a manifold group (fundamental group of a finite cover). Therefore, any obstruction that applies to torsion-free uniform lattices would also obstruct Γ.

However, the converse direction is invalid for the present objective: the existence of a torsion-free finite-index subgroup Γ' that is realizable as a manifold group does not imply that Γ is realizable as such. The extension problem is nontrivial: one needs a free, properly discontinuous action of Γ on the same universal cover that yields a manifold quotient. The 2-torsion in Γ forces compatibility constraints on how the extension permutes cells (in a CW model) or acts smoothly (in a manifold model), and these constraints are exactly where fixed-point and duality considerations are expected to enter.

Accordingly, in the proof strategy adopted in this note, finite-index torsion removal functions only as a way to align dimensions and duality properties (via vcd-type comparisons) and to isolate the additional obstruction created by reintroducing 2-torsion.

4.5 Dimensional consistency checks: manifold dimension versus group-theoretic dimension

Assume again that M is closed, $\pi_1(M) = \Gamma$, and $\widetilde{M}$ is $\mathbb{Q}$-acyclic. A basic consistency requirement is that Γ behaves like a rational Poincaré duality group of formal dimension $\dim M$; in particular, the cohomology of Γ with $\mathbb{Q}\Gamma$-coefficients should match the pattern expected from a closed-aspherical manifold, at least after restricting to torsion-free finite-index subgroups.

Independently, for a uniform lattice Γ in a semisimple Lie group with associated symmetric space X, one expects that any torsion-free finite-index subgroup Γ' has cohomological dimension comparable to $\dim X$ (the manifold $\Gamma' \backslash X$ provides a geometric model). The manuscript's consistency check is therefore the following conditional statement: if a closed manifold model M with $\pi_1(M) = \Gamma$ and $\mathbb{Q}$-acyclic $\widetilde{M}$ exists, then the dimension of M must be compatible with the group-theoretic dimension of Γ coming from the lattice embedding, and hence with $\dim X$.

This does not yield an immediate contradiction by itself, but it narrows the search space for any counterexample or construction. In particular, any proposed example should specify $\dim M$ and explain why it is compatible with the lattice's geometric dimension. A mismatch here would rule out the example without invoking torsion obstructions.

4.6 A minimal "failure mode" checklist for candidate constructions

The preceding discussion can be condensed into a small number of concrete failure modes that any proposed construction must address.

First, coefficient mismatch: $\mathbb{Q}$-acyclicity must be checked directly and does not control $\mathbb{F}_2$-homology. Therefore, any mod 2 obstruction must state an additional hypothesis about $H_*(\widetilde{M};\mathbb{F}_2)$.

Second, freeness: for $\tau \in \Gamma$ of order 2, the induced action on $\widetilde{M}$ is free. Therefore, proposed obstructions based on nontrivial fixed sets of involutions do not apply unless the action under study is not the deck action (for example, a different action of Γ on another acyclic space).

Third, extension rigidity: passing to a torsion-free finite-index subgroup does not solve the original torsionful realization problem. Any argument that "reduces to the torsion-free case" must include a step that reconstructs the torsionful extension as a free action on the same universal cover.

Fourth, dimension: any candidate M must match the lattice's geometric or cohomological dimension constraints coming from the ambient symmetric space. This is a necessary check even when torsion is ignored.

These checks are designed to prevent incorrect reductions rather than to provide a complete obstruction theorem. Within the committed approach of this note, the only claimed progress is the clarification that 2-torsion obstructions must be formulated in a way that respects (i) the freeness of deck transformations and (ii) the separation between rational and mod 2 acyclicity, and that any putative construction should be dimensionally consistent with the lattice embedding.

5 Conclusion and Outlook

The realization problem considered here remains unresolved: we do not have a proof that a uniform lattice in a real semisimple Lie group with 2–torsion cannot be the fundamental group of a closed manifold whose universal cover is $\mathbb{Q}$–acyclic, and we likewise do not present an explicit construction producing such a manifold. What the present draft does provide is a careful delineation of what the $\mathbb{Q}$–acyclicity hypothesis forces at the level of ordinary group (co)homology with trivial coefficients, together with a diagnosis of why these rational consequences do not automatically interact with the finite subgroups of Γ.

The key obstruction to progress is conceptual rather than computational. The manifold hypothesis supplies "free-action" data: the augmented cellular chain complex of $\widetilde{M}$ over $\mathbb{Q}$ gives a finite free $\mathbb{Q}[\Gamma]$–resolution of $\mathbb{Q}$, yielding rational comparison isomorphisms between the (co)homology of Γ and that of M. However, because the deck action is free, fixed-point sets for nontrivial finite subgroups are empty and do not encode additional structure. Consequently, any argument that aims to rule out 2–torsion must access torsion-sensitive invariants indirectly, for instance by relating the free-action information to proper-action/Bredon phenomena associated to a model for $\underline{E}\Gamma$. The current manuscript does not contain a theorem providing such a comparison, and without it there is no clear route from $\mathbb{Q}$–acyclicity to constraints on centralizers or normalizers of involutions in Γ.

A concrete next step is therefore to establish a torsion-detecting upgrade principle: under suitable finiteness hypotheses, one would like a theorem that promotes the existence of a free, proper, cocompact action on a $\mathbb{Q}$–acyclic manifold to a specific Bredon duality (or Bredon Poincaré duality) pattern for the family of finite subgroups, strong enough to constrain the structure of involution centralizers. In parallel, any constructive approach would need to produce a free cocompact action of the full torsionful lattice (not merely a torsion-free finite-index subgroup) on an acyclic or contractible manifold of dimension compatible with the lattice's geometric/cohomological dimension. Until either such a comparison theorem or such a construction is available, the appropriate conclusion is that the question is open, with the identified gap isolating the precise point at which torsion enters beyond the reach of the current arguments.

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