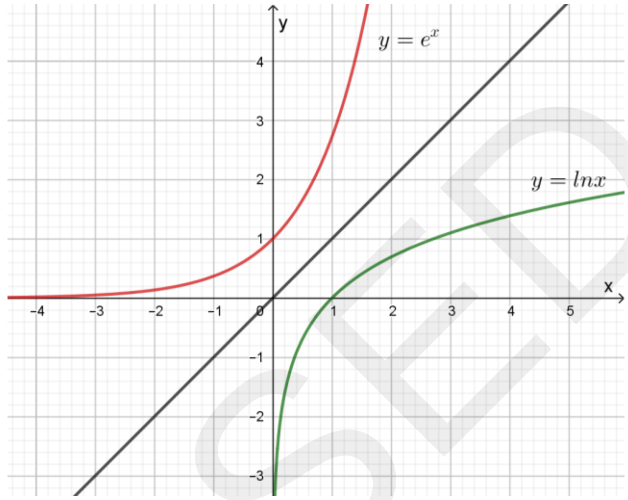
Mathematics: Chapter Summary and rules

ln(*x*) and exp(x) summary:



- $\frac{\mathrm{d}}{\mathrm{dx}}(\ln|x|) = \frac{1}{x}$; $\frac{d}{dx}\ln|u| = \frac{u'}{u}$
- $\ln(a \times b) = \ln a + \ln b$, where $a > 0$ and $b > 0$.
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$, where $a > 0$ and $b > 0$.
- $\ln\left(\frac{1}{a}\right) = -\ln a$, where $a > 0$.
- $\ln a^n = n \ln a$, where $a > 0$ and n is real.
- $\ln 1 = 0$; $\ln e = 1$; $\ln e^2 = 2$
- $\ln e^n = n$; $\ln\frac{1}{e} = -1$.
- $a = \ln e^a = e^{\ln a}$.
- $\ln x = a \Leftrightarrow x = e^a$.
- If $e^x = a$ $(a > 0)$, then $x = \ln a$
- $\ln a = \ln b \Leftrightarrow a = b$, where $a > 0$ and $b > 0$.
- $\ln a < \ln b \Leftrightarrow a < b$, where $a > 0$ and $b > 0$.
- $\ln a > \ln b \Leftrightarrow a > b$, where $a > 0$ and $b > 0$.

- $\lim\limits_{x\to 0^+} \ln x = -\infty$; $\lim\limits_{x\to+\infty} \ln x = +\infty$.
- $\lim\limits_{x\to 0^+} x\ln x = 0^-$; $\lim\limits_{x\to+\infty}\left(\frac{\ln x}{x}\right) = 0$; $\lim\limits_{x\to+\infty}\left(\frac{x}{\ln x}\right) = +\infty$.
- $\lim\limits_{x\to 0^+} x^n \ln x = 0^-$; $\lim\limits_{x\to+\infty}\left(\frac{\ln x}{x^n}\right) = 0^+$ where $n > 0$.

L'Hopital's Rule for limits of indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

If $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ exists, and $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$.

- The function: $x \mapsto \ln x$ has domain $=]0; +\infty[$ and range $=]-\infty; +\infty[$.
- The function: $x \mapsto e^x$ has domain $=]-\infty; +\infty[$ and range $=]0; +\infty[$.
- These two functions are inverses of each other. Their graphs are symmetrical with respect to the first bisector ($y = x$).

- $\ln(e^x) = x; e^{\ln x} = x$; $y = \ln x \Leftrightarrow x = e^y$.
- $e^a = e^b \Leftrightarrow a = b$; $e^a < e^b \Leftrightarrow a < b$; $e^a > e^b \Leftrightarrow a > b$
- $e^a \cdot e^b = e^{a+b}$; $\frac{e^a}{e^b} = e^{a-b}$; $(e^a)^b = e^{ab}$; $e^{-a} = \frac{1}{e^a}$
- $(e^x)' = e^x$; $(e^u)' = e^u \cdot u'$

- $\lim\limits_{x\to-\infty} e^x = 0^+$; $\lim\limits_{x\to+\infty} e^x = +\infty$; $\lim\limits_{x\to+\infty}\frac{e^x}{x} = +\infty$; $\lim\limits_{x\to+\infty}\frac{x}{e^x} = 0$;
  $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = 1$; $\lim\limits_{x\to+\infty} e^{-x} = 0$; $\lim\limits_{x\to+\infty}\frac{e^x}{x^n} = +\infty$; $\lim\limits_{x\to-\infty} x^n e^x = 0$. $(n \in \mathbb{Z}^+)$

Integration and derivatives:

$*\ F(x)$ is an anti-derivative $g(x) \Leftrightarrow F'(x) = g(x)$.

$*\ F(x) = \int g(x)dx \ \Leftrightarrow\ F'(x) = g(x)$.

$*\ \frac{d}{dx}\left(\int f(x)dx\right) = f(x)$ and $\int f'(x)dx = f(x) + C$

$*\ \int x^n dx = \frac{x^{n+1}}{n+1} + C \ \ (n \neq -1)$

$*\ \int (u+v)dx = \int udx + \int vdx \ ; \quad \int kudx = k\int udx \ ; k \in \mathbb{R}$

$*\ \int u'v = uv - \int v'u$ (integration by parts)

$*\ \int u'f(u)dx = \int f(u)du$

$*\ f'_x = f'_u \times u'_x$ (chain rule: $\frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx}$)

$*\ (u \pm v)' = u' \pm v'$

$*\ (uv)' = u'v + v'u$

$*\ \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

$*\ \left(\frac{1}{u}\right)' = -\frac{u'}{u}$

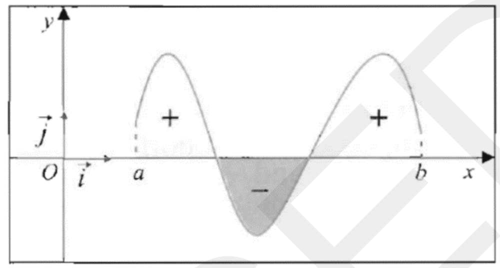
In this table : a and b are real numbers.

$\boldsymbol{f(x)}$	$\boldsymbol{f'(x)}$	$\int \boldsymbol{f(x)dx}$
1	0	x
x	1	x^2
a	0	ax
$ax+b$	a	$\frac{1}{2}ax^2 + bx$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\ln\lvert x\rvert$
$\frac{1}{x^2}$	$-\frac{2}{x^3}$	$-\frac{1}{x}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\frac{2}{3}x\sqrt{x}$
$\frac{1}{\sqrt{x}}$	$-\frac{1}{2x\sqrt{x}}$	$2\sqrt{x}$
$\ln x$	$\frac{1}{x}$	$x\ln x - x$
e^x	e^x	e^x
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$
$\tan x$	$1+\tan^2 x$	$-\ln\lvert\cos x\rvert$

* If F is an anti-derivative of f over I, then $\int_a^b f(x)dx = F(b) - F(a)$.

* $\int_a^b f(x)dx$ = algebraic area of the domain limited by the representative curve of f, the x-axis and the two straight lines of equations $x = a$ and $x = b$.

* Mean value of a function (average value) over an interval $[a; b]$ is $\overline{f} = \frac{1}{b-a}\int_a^b f(x)dx$.



* $\int_a^b \big(f(x) - g(x)\big)dx =$ area of the domain limited by the representative curves of f and g ($f(x) > g(x)$ over $[a, b]$) and the two straight lines of equations $x = a$ and $x = b$.

* $\int_b^a f(x)dx = -\int_a^b f(x)dx$

* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

* $\int_a^b kdx = k(b-a)$ and $\int_a^b dx = b - a$ and $\int_a^a dx = 0$.

* If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$, $a \in \mathbb{R}$.
* If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2 \times \int_0^a f(x)dx$, $a \in \mathbb{R}$.
* If $f(x) \leq g(x)$, then $\int f(x)dx \leq \int g(x)dx$

* If $H(x) = \int_a^{u(x)} f(t)dt$, then $H'(x) = f\big(u(x)\big) \times u'(x)$, $a \in \mathbb{R}$

Special economics:

* Given the cost function : $C_T(x)$, then marginal cost function : $C_m(x) = C_T{}'(x)$

* Given marginal cost function : $C_m(x)$, then total cost function : $C_T(x) = \int C_m(x)dx + K$.

Lines and Planes

1) Scalar product (or dot product):
* Geometric form: $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\vec{u},\vec{v})$.
* Angle between two vectors : $\cos(\vec{u},\vec{v}) = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$
* Basic properties :
    * $\vec{u}$ and $\vec{v}$ are orthogonal iff $\vec{u}\cdot\vec{v} = 0$
    * $\vec{u}^2 = \|\vec{u}\|^2$
    * $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
    * $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})\ ; k \in \mathbb{R}$
    * $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$
* Analytic form :

In an orthonormal system $(O;\vec{\imath},\vec{\jmath},\vec{k})$ where $\vec{u}(x,y,z)$ and $\vec{v}(x',y',z')$:

$$\vec{u}\cdot\vec{v} = xx' + yy' + zz'$$

* Magnitude of a vector : $\|\vec{u}\| = \sqrt{x^2+y^2+z^2}$

* Angle between two vectors :

$$\cos(\vec{u},\vec{v}) = \frac{xx'+yy'+zz'}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{x'^2+y'^2+z'^2}}$$

2) Vector product (or cross product) :
* Geometric form: :

If $\vec{u}$ and $\vec{v}$ are collinear, then, $\vec{u}\times\vec{v} = \vec{0}$.
If $\vec{u}$ and $\vec{v}$ are not collinear, then, $\vec{u}\times\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin(\vec{u},\vec{v})\cdot\vec{n}$
where $\vec{n}$ is a unit vector perpendicular to the plane formed by $\vec{u}$ and $\vec{v}$ and such that $(\vec{u},\vec{v},\vec{n})$ form a direct base (or right handed base).

* Basic properties :
    * $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot|\sin(\vec{u},\vec{v})|$
    * $\vec{v}\times\vec{u} = -\vec{u}\times\vec{v}$
    * $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
    * $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})\ ; k \in \mathbb{R}$
* Analytic form :

In a direct orthonormal system $(O;\vec{\imath},\vec{\jmath},\vec{k})$ where $\vec{u}(x,y,z)$ and $\vec{v}(x',y',z')$:

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix}$$

* Area of parallelogram with sides $\vec{u}$ and $\vec{v}$: $Area = \|\vec{u}\times\vec{v}\|$
* Area of parallelogram $ABCD$: $Area = \left\|\overrightarrow{AB}\times\overrightarrow{AD}\right\|$
* Area of triangle ABC : $Area = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$

This is an alternative method to the classical rule :

$$Area = \frac{1}{2}\times base\times height$$

* Volume of parallelepiped with sides $\vec{u}, \vec{v}$ and $\vec{w}$: $V = (\vec{u} \times \vec{v}) \cdot \vec{w}$
* Volume of parallelepiped $ABCDEFGH$: $V = (\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE}$
* Volume of tetrahedron $SABC$: $V = \frac{1}{6}(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AS}$

  This is an alternative method to the classical rule :

$$Volume = \frac{1}{3} \times area\ of\ base \times height\ of tetrahedron$$

* Proving 3 points A, B and C collinear : $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$
* Proving 3 vectors $\vec{u}, \vec{v}$ and $\vec{w}$ coplanar : $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$
* Proving 4 points A, B, C and D coplanar : $\det(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) = 0$
* Proving 4 points A, B, C and D coplanar : $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = 0$.
* Distance between point A and line (D) of direction vector $\vec{S}$ with $B \in (D)$:

  $d(A, (D)) = \dfrac{\|\overrightarrow{AB} \times \vec{S}\|}{\|\vec{S}\|}$
* Distance between point S and plane (ABC) using volume :

  $d(S, (ABC)) = \dfrac{(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AS}}{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}$
* Equation of plane with normal vector $\vec{n}(u, v, w)$ and passing through point A :

  $\overrightarrow{AM} \cdot \vec{n} = 0$ where $M(x, y, z)$ is a point on the plane.

  or equivalently: $ux + vy + wz + r = 0$
* Parametric equations of a straight line with directing vector $\vec{S}(\alpha, \beta, \gamma)$ and passing through point A :

  $\begin{cases} x = \alpha t + x_A \\ y = \beta t + y_A \\ z = \gamma t + z_A \end{cases} \quad ; \quad t \in \mathbb{R}$