PH100: Mechanics and Thermodynamics

Lecture 5

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Conservation of momentum

The total external force *F* acting on a system is related to the total momentum *P* of the system by

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}. \qquad \Delta\vec{\mathbf{p}}_{\text{system}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} dt \equiv \vec{\mathbf{I}}.$$

Impulse

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings.

$$\mathbf{F} = 0, \qquad d\mathbf{P}/dt = 0.$$

The total momentum is constant.

- No matter how strong the interactions among an isolated system of particles.
- No matter how complicated the motion is.

The change in the total momentum of the system is zero, $\Delta\vec{\mathbf{p}}_{\text{system}} = \vec{\mathbf{0}};$

Initial momentum: $\vec{\mathbf{p}}_0^{\text{total}} = m_1\vec{\mathbf{v}}_{1,0} + m_2\vec{\mathbf{v}}_{2,0} + \cdots.$ Final momentum: $\vec{\mathbf{p}}_f^{\text{total}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f} + \cdots.$

$$\vec{\mathbf{p}}_0^{\text{total}} = \vec{\mathbf{p}}_f^{\text{total}}. \qquad p_{0x}\hat{i} + p_{0y}\hat{j} + p_{0z}\hat{k} = p_{fx}\hat{i} + p_{fy}\hat{j} + p_{fz}\hat{k}$$

Component wise conserved

A few points about the conservation law

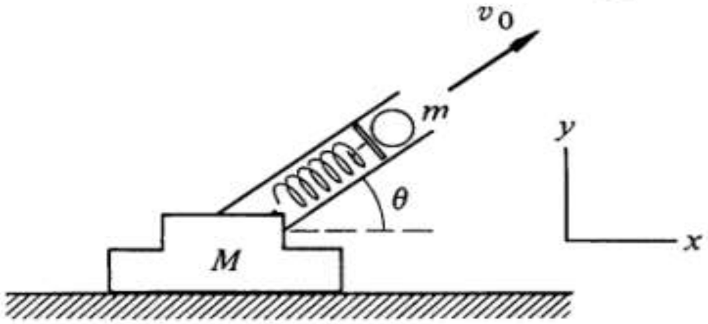
- **Conservation of momentum holds true even in areas where Newtonian mechanics proves inadequate, including the realms of quantum mechanics and relativity. So it is more fundamental.**

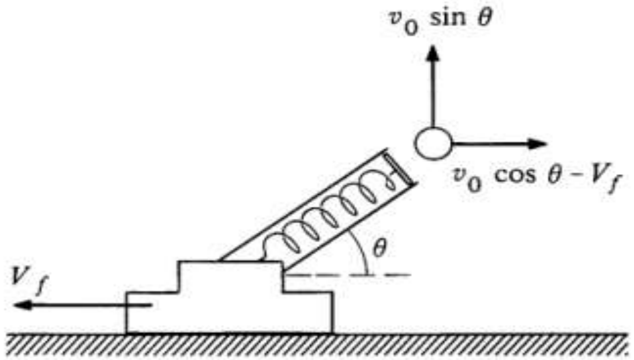
- **Conservation of momentum can be generalized to apply to systems like the electromagnetic field, which possess momentum but not mass.**

The momentum of a system is conserved if the net external force on the system is zero

Spring Gun Recoil

A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble is v_0. What is the final motion of the gun?





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$P_{x,\text{initial}} = 0,$ the system is initially at rest. $0 = \dfrac{dP_x}{dt}.$ $P_{x,\text{initial}} = P_{x,\text{final}}.$

After the marble has left the muzzle, the gun recoils with some speed V_f and its final horizontal momentum is MV_f, to the left. The final horizontal speed of the marble relative to the table is $v_0 \cos\theta - V_f$.

Therefore, $$0 = m(v_0 \cos\theta - V_f) - MV_f \longrightarrow V_f = \frac{mv_0 \cos\theta}{M+m}.$$

Variable Mass Problem

Momentum and the Flow of Mass

External force causes the momentum of a system to change

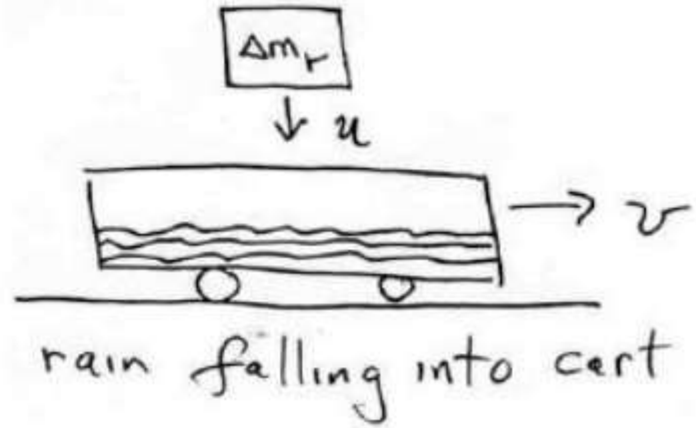
$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt}$$

What if the mass flows between constituent objects and not a constant?

We shall consider four examples of mass flow problems that are characterized by the momentum transfer of the material of mass Δm .

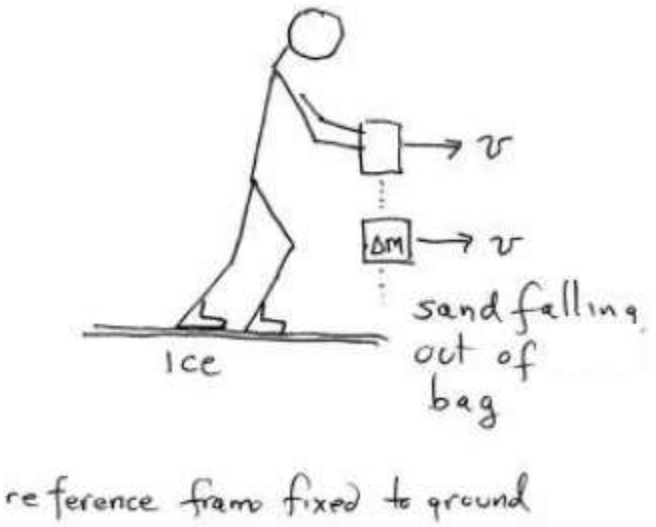
Example 1: **Rain Falling on to cart**

There is a transfer of material into the object but no transfer of momentum in the direction of motion of the object. Consider for example rain falling vertically downward into a moving cart. A small amount of rain Δm has no component of momentum in the direction of motion of the cart.



Example 2: **Leaky Sand Bag**

The material continually leaves the object but it does not transport any momentum away from the object in the direction of motion of the object. For example, consider an ice skater gliding on ice holding a bag of sand that is leaking straight down with respect to the moving skater.

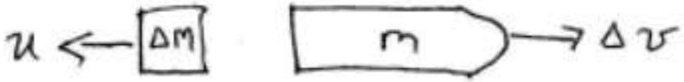


Example 3: Rocket Motion

The material continually is ejected from the object, resulting in a recoil of the object. For example when fuel is ejected from the back of a rocket, the rocket recoils forward.

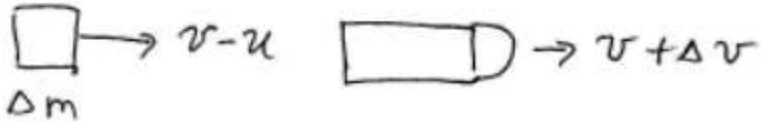
referance frame of rocket

recoil forward speed increases by Δv



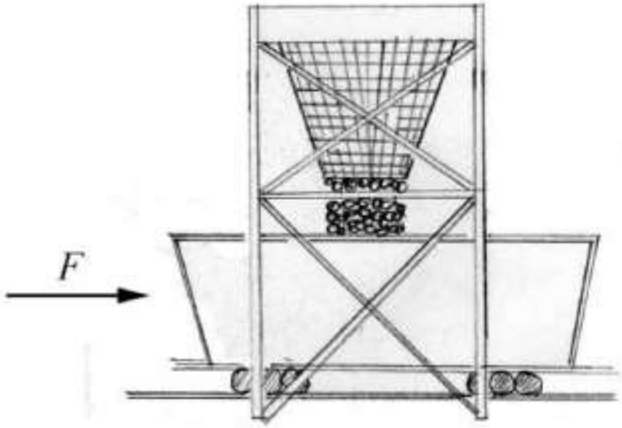
u = speed of ejected fuel relative to rocket

reference frame in which rocket moves with speed v



Coal Car

An empty coal car of mass m_0 starts from rest under an applied force of magnitude F . At the same time coal begins to pour vertically onto the car at a steady rate b from a coal hopper at rest along the track. Find the speed when a mass m_c of coal has been transferred.



Because the falling coal does not have any horizontal velocity, the falling coal is not transferring any momentum in the x-direction to the coal car.

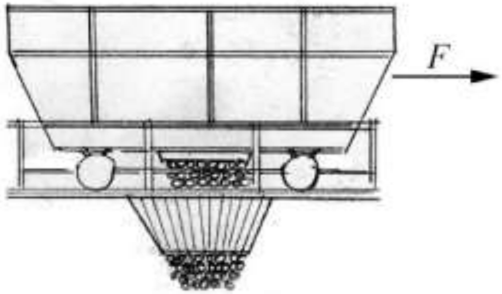
Initial state at $t = 0$ is when the coal car is empty and at rest before any coal has been transferred. Final state at $t = t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car that is now moving at speed v_f .

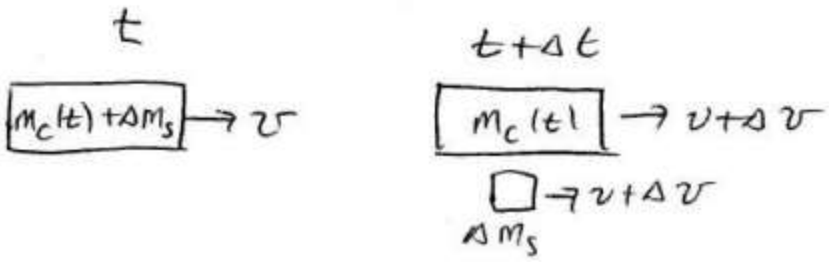
$$p_x(0) = 0. \qquad p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f\,.$$

$$\int_0^{t_f} F_x dt = \Delta p_x = p_x(t_f) - p_x(0). \qquad Ft_f = (m_0 + bt_f)v_f \qquad v_f = \frac{Ft_f}{(m_0 + bt_f)}.$$

Emptying a Freight Car:

A freight car of mass m_c contains a mass of sand m_s. At $t = 0$ a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s/dt$. Find the speed of the freight car when all the sand is gone. Assume that the freight car is at rest at $t = 0$.





Our system is (i) the amount of sand of mass Δm_s that leaves the freight car during the time interval $[t, t + \ t]$, and (ii) the freight car and whatever sand is in it at time t.

$$p_x(t) = (\Delta m_s + m_c(t))v$$

$$p_x(t+\Delta t) = (\Delta m_s + m_c(t))(v+\Delta v)$$

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt$$

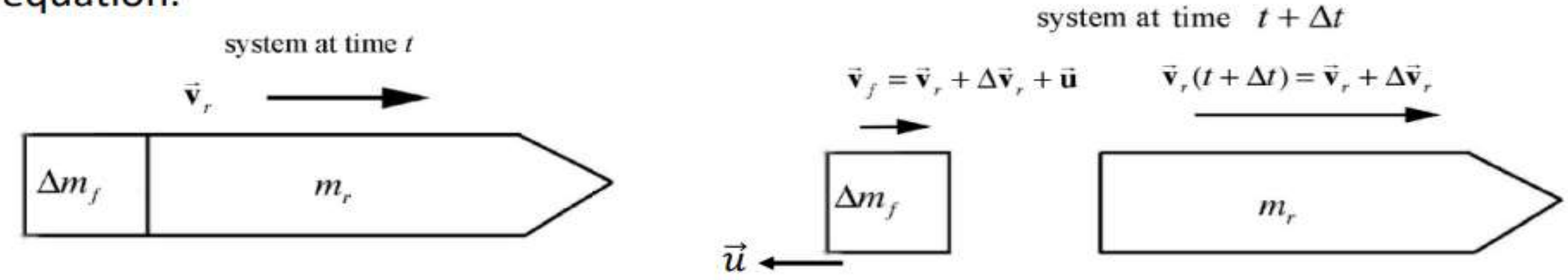
$$F = \lim_{\Delta t \to 0} \frac{p_x(t+\Delta t) - p_x(t)}{\Delta t} \qquad F = \lim_{\Delta t \to 0} m_c(t)\frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_s \Delta v}{\Delta t}$$

The second term vanishes.

$$F = m_c(t)\frac{dv}{dt} \qquad \int_{v=0}^{v(t)} dv = \int_0^t \frac{F dt}{m_c + m_s - bt} \qquad v(t) = -\frac{F}{b}\ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right).$$

Rocket Propulsion

A rocket at time t = 0 is moving with speed $v_{r,0}$ in the positive x-direction in empty space. The rocket burns fuel that is then ejected backward with velocity $\vec{\mathbf{u}}$ relative to the rocket. The rocket velocity is a function of time, $\vec{\mathbf{v}}_r(t)$, and increases at a rate $d\vec{\mathbf{v}}_r / dt$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r/dt. Determine a differential equation that relates $d\vec{\mathbf{v}}_r / dt$, dm_r / dt, $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}_r(t)$, and $\vec{\mathbf{F}}_{\text{ext}}^{\text{total}}$, an equation to be called as the rocket equation.



$$\vec{\mathbf{p}}_{system}(t) = (m_r + \Delta m_f)\vec{\mathbf{v}}_r \qquad \vec{\mathbf{p}}_{system}(t+\Delta t) = m_r(\vec{\mathbf{v}}_r + \Delta\vec{\mathbf{v}}_r) + \Delta m_f(\vec{\mathbf{v}}_r + \Delta\vec{\mathbf{v}}_r + \vec{\mathbf{u}})$$

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{\text{system}}(t+\Delta t) - \vec{\mathbf{p}}_{\text{system}}(t)}{\Delta t} = \lim_{\Delta t \to 0} m_r \frac{\Delta\vec{\mathbf{v}}_r}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_f \Delta\vec{\mathbf{v}}_r}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_f}{\Delta t}\vec{\mathbf{u}} \approx m_r \frac{d\vec{\mathbf{v}}_r}{dt} + \frac{dm_f}{dt}\vec{\mathbf{u}}.$$

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt} \qquad \boxed{\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m_r \frac{d\vec{\mathbf{v}}_r}{dt} - \frac{dm_r}{dt}\vec{\mathbf{u}}}$$

is called the *rocket equation.*

Rocket in Free Space

If there is no external force on a rocket, ***F*** = 0 and its motion is given by

$$M\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dM}{dt} \qquad \text{or} \qquad \frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt}.$$

Generally the exhaust velocity ***u*** is constant,

$$\int_{t_0}^{t_f}\frac{d\mathbf{v}}{dt}\,dt = \mathbf{u}\int_{t_0}^{t_f}\frac{1}{M}\frac{dM}{dt}\,dt = \mathbf{u}\int_{M_0}^{M_f}\frac{dM}{M}$$

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\frac{M_f}{M_0} = -\mathbf{u}\ln\frac{M_0}{M_f}.$$

If $\mathbf{v}_o = 0$, then $\qquad \mathbf{v}_f = -\mathbf{u}\ln\frac{M_0}{M_f}.$

The final velocity is independent of how the mass is released-the fuel can be expended rapidly or slowly without affecting $\mathbf{v}_f$. The only important quantities are the exhaust velocity and the ratio of initial to final mass.

Rocket in a Gravitational Field

If a rocket takes off in a constant gravitational field, the equation of motion becomes

$$M\mathbf{g} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt},$$

where ***u*** and ***g*** are directed down and are assumed to be constant.

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt} + \mathbf{g}.$$

Integrating with respect to time, we obtain

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\left(\frac{M_f}{M_0}\right) + \mathbf{g}(t_f - t_0).$$

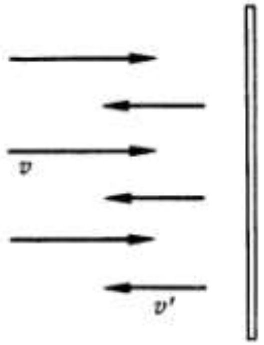
$Let\ \boldsymbol{v}_o = 0, to = 0$, and take velocity positive upward.

$$v_f = u\ln\left(\frac{M_0}{M_f}\right) - gt_f.$$

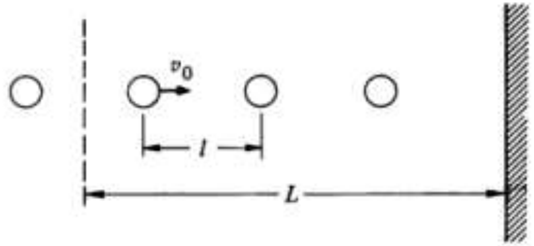
Now there is a premium attached to burning the fuel rapidly. The shorter the burn time, the greater the velocity. This is why the takeoff of a large rocket is so spectacular- it is essential to burn the fuel as quickly as possible.

Stream Bouncing off Wall

A stream of particles each of mass m and separation l hits a perpendicular surface with speed v . The stream rebounds along the original line of motion with speed v'. The mass per unit length of the incident stream is $\lambda = m/l$. What is the magnitude of the force on the surface?



Consider length L of the stream just about to hit the surface. The number of particles in L is L/l, and since each particle has momentum mv_o, the total momentum is



$$\Delta p = \frac{L}{l} m v_0. \qquad \Delta t = \frac{L}{v_0}. \qquad F_{av} = \frac{\Delta p}{\Delta t} = \frac{m}{l} v_0^2.$$

Call $m/l \equiv \lambda$, $\quad \dfrac{dp}{dt} = \lambda v^2.$

Amount of mass transferred:

$$\Delta m = mv\,\Delta t/l. \qquad \frac{dm}{dt} = \frac{m}{l} v = \lambda v.$$

$$F = \frac{dp'}{dt} + \frac{dp}{dt} = \lambda' v'^2 + \lambda v^2 = \lambda v(v' + v). \qquad \lambda' v' = \lambda v,$$

$$v' = v, \quad F = 2\lambda v^2.$$

Rate of mass inflow and outflow must be same

Example: falling raindrop

Suppose that a raindrop falls through a cloud and accumulates mass at a rate kmv where $k > 0$ is a constant, m is the mass of the raindrop, and v its velocity. What is the speed of the raindrop at a given time if it starts from rest, and what is its mass?

Solution: We are taking x as distance fallen and $v = \dot{x}$. Then the external force is its weight mg and so

$$mg = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = m\frac{dv}{dt} + kmv^2,$$

since we are told that $dm/dt = kmv$. Cancelling the mass and rearranging

$$\frac{dv}{dt} = g - kv^2,$$

so that

$$\int_0^v \frac{dv}{g - kv^2} = \int_0^t dt = t.$$

Now set $V^2 = g/k$ and use partial fractions to get

$$t = \int_0^v \frac{dv}{g - kv^2} = \frac{1}{2kV} \int_0^v \frac{1}{V+v} + \frac{1}{V-v}\, dv = \frac{1}{2kV} \log\left(\frac{V+v}{V-v}\right)$$

so $V + v = (V - v)e^{2kVt}$, i.e. $v = V\left(\frac{e^{2kVt}-1}{e^{2kVt}+1}\right) = V\tanh(Vkt)$, so that

$$v = \sqrt{\frac{g}{k}} \tanh(\sqrt{kg}t).$$

Now we may find the mass: We have $\frac{dm}{dt} = kmv = km\sqrt{\frac{g}{k}}\tanh(\sqrt{kg}t) = m\sqrt{kg}\tanh(\sqrt{kg}t)$. Thus

$$\int_0^t \frac{1}{m}\frac{dm}{dt}\, dt = \int_0^t \sqrt{kg}\tanh(\sqrt{kg}t)\, dt$$

$$\int_{m_0}^m \frac{dm}{m} = \int_0^t \sqrt{kg}\tanh(\sqrt{kg}t)\, dt$$

$$\log m - \log m_0 = \log\cosh(\sqrt{kg}t)$$

which gives

$$m = m_0\cosh(\sqrt{kg}t).$$

Thank You