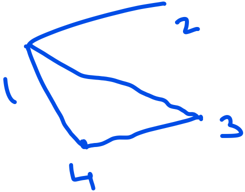
Graph theory

Definition

An undirected graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges (edge is an unordered pair of vertices).

eg. $G_1 = (\underbrace{\{1,2,3,4\}}_{V}, \underbrace{\{\{1,2\},\{3,4\},\{1,4\},\{1,3\}\}}_{E})$



Graph theory

Definition

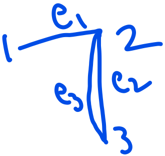
An undirected graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges (edge is an unordered pair of vertices).

Note: If either V or E is infinite then G is called Infinite graph, otherwise finite.

Definition

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

eg Graph on Slide 1 is simple



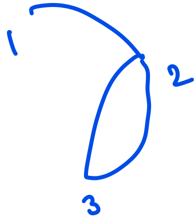
not simple

Definition

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

A graph in which multiple edges connect the same pair of vertices is called a multigraph.

eg



Definition

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.
A graph in which multiple edges connect the same pair of vertices is called a multigraph.

An edge that connect a vertex to itself is called loop.

Definition

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.
A graph in which multiple edges connect the same pair of vertices is called a multigraph.

An edge that connect a vertex to itself is called loop.
Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called pseudographs.

Definition

A directed graph (or digraph) (V ,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.

Definition

A directed graph (or digraph) (V ,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.

A directed graph which has no loops and has no multiple directed edges is called a simple directed graph.

Definition

A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.

A directed graph which has no loops and has no multiple directed edges is called a simple directed graph.
Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex are called directed multigraphs.

Definition

A directed graph (or digraph) (V ,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.

A directed graph which has no loops and has no multiple directed edges is called a simple directed graph.
Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex are called directed multigraphs.
A graph with both directed and undirected edges is called a mixed graph.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Definition

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Definition

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

The set of all neighbors of a vertex v of G = (V ,E), denoted by N(v), is called the neighborhood of v.

Definition

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

The set of all neighbors of a vertex v of G = (V ,E), denoted by N(v), is called the neighborhood of v.

Definition

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Definition

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

The set of all neighbors of a vertex v of G = (V ,E), denoted by N(v), is called the neighborhood of v.

Definition

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Question: Construct a graph with 4 vertices of degree 2, 3, 4, 2.

Theorem (THE HANDSHAKING THEOREM)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg\,(v)$$

Theorem (THE HANDSHAKING THEOREM)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg\,(v)$$

If G is a graph with 10 vertices each of degree six then its number of edges is

Theorem (THE HANDSHAKING THEOREM)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg\ (v)$$

If G is a graph with 10 vertices each of degree six then its number of edges is 30.

Theorem (THE HANDSHAKING THEOREM)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg\,(v)$$

If G is a graph with 10 vertices each of degree six then its number of edges is 30.

Question: How many vertices in a graph with odd degree are there?

Theorem (THE HANDSHAKING THEOREM)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg\,(v)$$

If G is a graph with 10 vertices each of degree six then its number of edges is 30.

Question: How many vertices in a graph with odd degree are there?

Theorem

An undirected graph has an even number of vertices of odd degree.

Theorem (THE HANDSHAKING THEOREM)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg\,(v)$$

If G is a graph with 10 vertices each of degree six then its number of edges is 30.

Question: How many vertices in a graph with odd degree are there?

Theorem

An undirected graph has an even number of vertices of odd degree.

Definition

When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v). The initial vertex and terminal vertex of a loop are the same.

Definition

In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.

Definition

In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of it.)

Definition

In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of it.)

Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

Definition

In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of it.)

Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Some Special Simple Graphs

Complete Graph on n vertices: K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.
Draw complete graphs with 1, 2, 3, 4 vertices.

A cycle $C_n, n \geq 3$, consists of n vertices $v_1, v_2, ..., v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n1}, v_n\}$, and $\{v_n, v_1\}$.
Draw C_n for $n \leq 4$.

A cycle $C_n, n \geq 3$, consists of n vertices $v_1, v_2, ..., v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n1}, v_n\}$, and $\{v_n, v_1\}$.
Draw C_n for $n \leq 4$.

Wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

Definition

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

Definition

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Definition

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Show that C_6 is bipartite.

Definition

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Show that C_6 is bipartite.
Show that K_3 is not bipartite.

Definition

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Show that C_6 is bipartite.
Show that K_3 is not bipartite.

Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Marriages on an Island: Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse.

Marriages on an Island: Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse.A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair.

Marriages on an Island: Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse.A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair.A maximum matching is a largest possible set of married couples, and a complete matching of V_1 is a set of married couples where every man is married, but possibly not all women.

Marriages on an Island: Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse.A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair.A maximum matching is a largest possible set of married couples, and a complete matching of V_1 is a set of married couples where every man is married, but possibly not all women.

Note: Similar question to Job assignment or for simple graphs.

Marriages on an Island: Suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse.A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair.A maximum matching is a largest possible set of married couples, and a complete matching of V_1 is a set of married couples where every man is married, but possibly not all women.

Note: Similar question to Job assignment or for simple graphs.

Theorem (HALL'S MARRIAGE THEOREM)

The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Generating New Graphs from Old

Definition (Subgraph of a Graph)

A subgraph of a Graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V, \; F \subseteq E$

Generating New Graphs from Old

Definition (Subgraph of a Graph)

A subgraph of a Graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V, \ F \subseteq E$

Definition (Union of two Graphs)

Union of two simple graphs $G = (V, E)$ and $H = (W, F)$ is a graph $G \cup H = (W \cup V, \ F \cup E)$

Generating New Graphs from Old

Definition (Subgraph of a Graph)

A subgraph of a Graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V, \ F \subseteq E$

Definition (Union of two Graphs)

Union of two simple graphs $G = (V, E)$ and $H = (W, F)$ is a graph $G \cup H = (W \cup V, \ F \cup E)$

Adjacency lists: For a graph with no multiple edges.

Vertex	Adjacent vertices
x	y, z
y	x
z	x

Adjacency lists: For a graph with no multiple edges.

Vertex	Adjacent vertices
x	y, z
y	x
z	x

Adjacency Matrices: $A = [a_{ij}]$, where $a_{ij} = 1$ if $\{i, j\}$ is an edge in V and 0 else

Adjacency lists: For a graph with no multiple edges.

Vertex	Adjacent vertices
x	y, z
y	x
z	x

Adjacency Matrices: $A = [a_{ij}]$, where $a_{ij} = 1$ if $\{i, j\}$ is an edge in V and 0 else

Incidence Matrix: $A = [a_{ij}]$, where $a_{ij} = 1$ if edge e_j is incident with vertex i and 0 else

Isomorphism of Graphs

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

If $G_1 G_2$ are isomorphic then
$|V_1| = |V_2|$,

Isomorphism of Graphs

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

If $G_1 G_2$ are isomorphic then
$|V_1| = |V_2|$, $|E_1| = |E_2|$,

Isomorphism of Graphs

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

If $G_1 G_2$ are isomorphic then
$|V_1| = |V_2|$, $|E_1| = |E_2|$,
and both graphs will have same no. of vertices of given degree.

Isomorphism of Graphs

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

If $G_1 G_2$ are isomorphic then
$|V_1| = |V_2|$, $|E_1| = |E_2|$,
and both graphs will have same no. of vertices of given degree.
SOFTWARE FOR GRAPH ISOMORPHISM: NAUTY for no of vertices ≤ 100.

Isomorphism of Graphs

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

If $G_1 G_2$ are isomorphic then
$|V_1| = |V_2|$, $|E_1| = |E_2|$,
and both graphs will have same no. of vertices of given degree.
SOFTWARE FOR GRAPH ISOMORPHISM: NAUTY for no of vertices ≤ 100.
Still there is no polynomial time (worst case) algorithm for graph isomorphism.

Connectivity

For an undirected graph G, a path of length n is a sequence of edges $e_1, e_2, ..., e_n$ that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph, where $e_i = \{x_i, x_{i+1}\}$.

Connectivity

For an undirected graph G, a path of length n is a sequence of edges $e_1, e_2, ..., e_n$ that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph, where $e_i = \{x_i, x_{i+1}\}$.
The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.

Connectivity

For an undirected graph G, a path of length n is a sequence of edges $e_1, e_2, ..., e_n$ that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph, where $e_i = \{x_i, x_{i+1}\}$.
The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.
A path or circuit is simple if it does not contain the same edge more than once.

Connectivity

For an undirected graph G, a path of length n is a sequence of edges $e_1, e_2, ..., e_n$ that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph, where
$e_i = \{x_i, x_{i+1}\}$.
The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.
A path or circuit is simple if it does not contain the same edge more than once.
An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

Connectivity

For an undirected graph G, a path of length n is a sequence of edges $e_1, e_2, ..., e_n$ that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph, where $e_i = \{x_i, x_{i+1}\}$.

The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.

A path or circuit is simple if it does not contain the same edge more than once.

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Connectivity

For an undirected graph G, a path of length n is a sequence of edges $e_1, e_2, ..., e_n$ that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph, where $e_i = \{x_i, x_{i+1}\}$.

The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.

A path or circuit is simple if it does not contain the same edge more than once.

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph. An isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2.

Euler Paths and Circuits

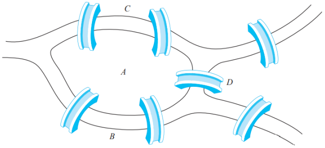


FIGURE 1 The Seven Bridges of Königsberg.

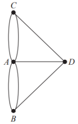


FIGURE 2 Multigraph Model of the Town of Königsberg.

Question: Is it possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

Euler Paths and Circuits

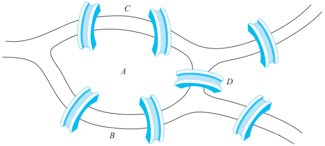


FIGURE 1 The Seven Bridges of Königsberg.

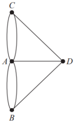


FIGURE 2 Multigraph Model of the Town of Königsberg.

Question: Is it possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

Answer: No (Euler, 1736)

Euler Paths and Circuits

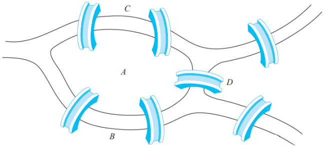


FIGURE 1 The Seven Bridges of Königsberg.

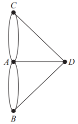


FIGURE 2 Multigraph Model of the Town of Königsberg.

Question: Is it possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

Answer: No (Euler, 1736)

An Euler circuit in a graph G is a simple circuit containing every edge of G.

Euler Paths and Circuits

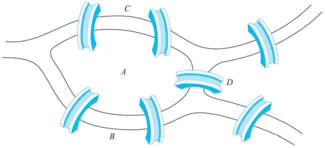


FIGURE 1 **The Seven Bridges of Königsberg.**

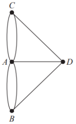


FIGURE 2 **Multigraph Model of the Town of Königsberg.**

Question: Is it possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

Answer: No (Euler, 1736)

An Euler circuit in a graph G is a simple circuit containing every edge of G.

An Euler path in G is a simple path containing every edge of G.

Euler Paths and Circuits

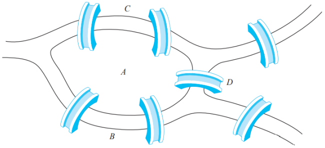


FIGURE 1 The Seven Bridges of Königsberg.

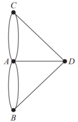


FIGURE 2 Multigraph Model of the Town of Königsberg.

Question: Is it possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point?

Answer: No (Euler, 1736)

An Euler circuit in a graph G is a simple circuit containing every edge of G.

An Euler path in G is a simple path containing every edge of G.

NECESSARY AND SUFFICIENT CONDITIONS FOR EULER CIRCUITS AND PATHS

Theorem

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

NECESSARY AND SUFFICIENT CONDITIONS FOR EULER CIRCUITS AND PATHS

Theorem

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem

A connected multigraph with at least two vertices has an Euler path if and only if each of its vertices has even degree except 2 vertices (i.e., exactly 2 vertices have odd degree).

ALGORITHM 1 Constructing Euler Circuits.

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
H := G with the edges of this circuit removed
while H has edges
  subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of *circuit*
  H := H with edges of *subcircuit* and all isolated vertices removed
  circuit := *circuit* with *subcircuit* inserted at the appropriate vertex
return *circuit* {*circuit* is an Euler circuit}

Hamilton Path and Circuit

A simple path in a graph G=(V, E) that passes through every vertex exactly once is called a **Hamilton path**, i.e. $x_0, x_1, \ldots, x_n$ is a Hamilton path iff V=$\{x_0, x_1, \ldots, x_n\}$ and $x_i \neq x_j \; fori \neq j$

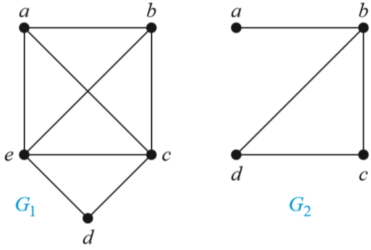
Hamilton Path and Circuit

A simple path in a graph G=(V, E) that passes through every vertex exactly once is called a **Hamilton path**, i.e. $x_0, x_1, \ldots, x_n$ is a Hamilton path iff V=$\{x_0, x_1, \ldots, x_n\}$ and $x_i \neq x_j \ fori \neq j$
A simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**. i.e. $x_0, x_1, \ldots, x_n, x_0$ is a Hamilton circuit iff $x_0, x_1, \ldots, x_n$ is a Hamilton path.

Hamilton Path and Circuit

A simple path in a graph G=(V, E) that passes through every vertex exactly once is called a **Hamilton path**, i.e. $x_0, x_1, \ldots, x_n$ is a Hamilton path iff V=$\{x_0, x_1, \ldots, x_n\}$ and $x_i \neq x_j\ for i \neq j$

A simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**. i.e. $x_0, x_1, \ldots, x_n, x_0$ is a Hamilton circuit iff $x_0, x_1, \ldots, x_n$ is a Hamilton path.



Theorem (Dirac's Theorem)

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Theorem (Ore's Theorem)

If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

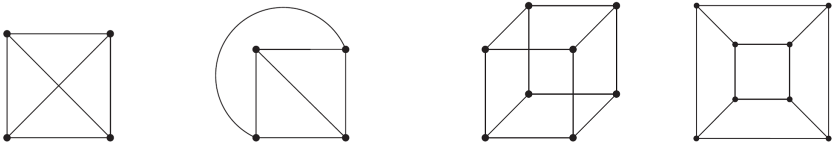
Hamilton Circuits has applications in traveling salesperson problem, coding theory.

Planar Graphs

A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

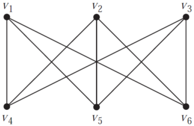
Planar Graphs

A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

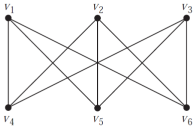


A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

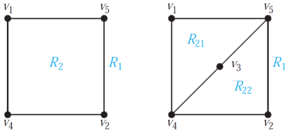
Is $K_{3,3}$ planar?



Is $K_{3,3}$ planar?

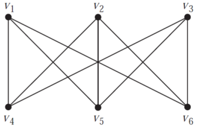


the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2

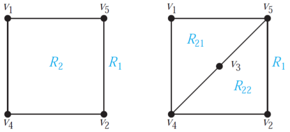


The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two subregions, R_{21} and R_{22}.

Is $K_{3,3}$ planar?



the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2



The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two subregions, R_{21} and R_{22}.
There is no way to place the final vertex v_6 without forcing a crossing.

APPLICATIONS OF PLANAR GRAPHS Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.

APPLICATIONS OF PLANAR GRAPHS Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.

Theorem (EULER'S FORMULA)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

APPLICATIONS OF PLANAR GRAPHS Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.

Theorem (EULER'S FORMULA)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Corollary (1)

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

APPLICATIONS OF PLANAR GRAPHS Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.

Theorem (EULER'S FORMULA)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Corollary (1)

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Corollary (2)

If G is a connected planar simple graph then G has a vertex of degree not exceeding five.

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

Theorem (Kuratowski's Theorem)

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Graph Coloring

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Graph Coloring

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
What is the least number of colors necessary?

Graph Coloring

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

What is the least number of colors necessary?

The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by G.

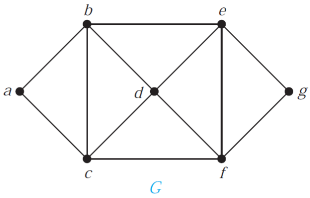
Graph Coloring

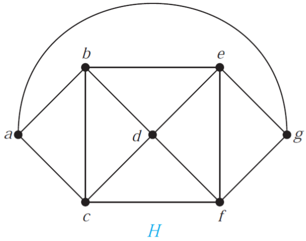
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
What is the least number of colors necessary?
The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by G.

Theorem (THE FOUR COLOR THEOREM)

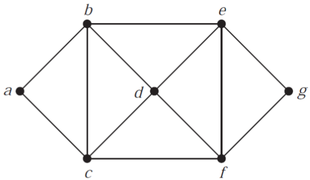
The chromatic number of a planar graph is no greater than four.

What are the chromatic numbers of the graphs G and H:

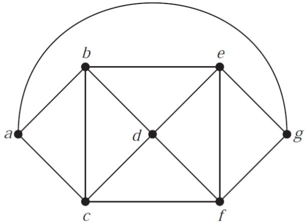




What are the chromatic numbers of the graphs G and H:



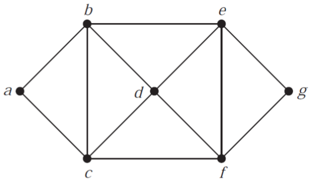
G



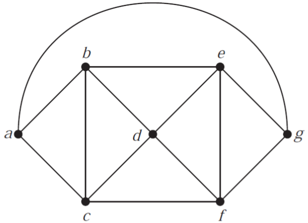
H

What is the chromatic number of K_n?

What are the chromatic numbers of the graphs G and H:



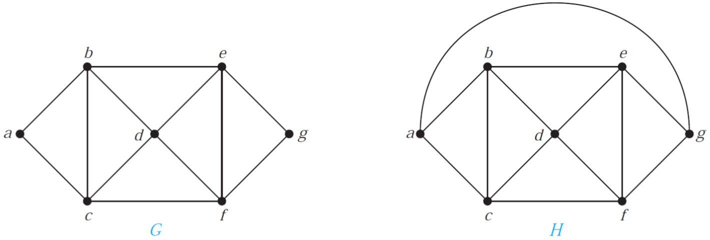
G



H

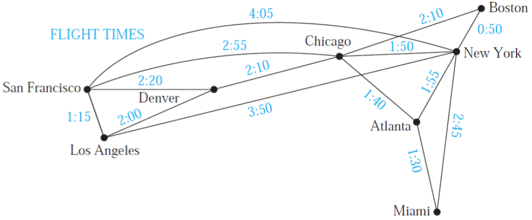
What is the chromatic number of K_n? n

What are the chromatic numbers of the graphs G and H:



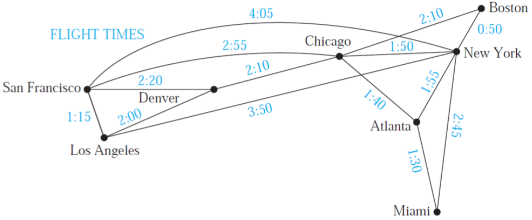
What is the chromatic number of K_n? n

Weighted Graph: Graphs that have a number assigned to each edge are called weighted graphs.



Shortest-Path Problems

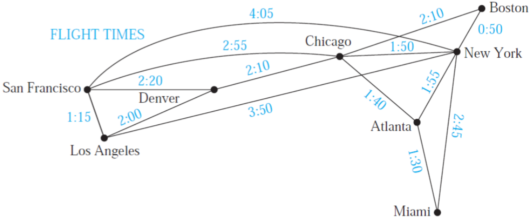
Weighted Graph: Graphs that have a number assigned to each edge are called weighted graphs.



length of a path in a weighted graph be the sum of the weights of the edges of this path.

Shortest-Path Problems

Weighted Graph: Graphs that have a number assigned to each edge are called weighted graphs.

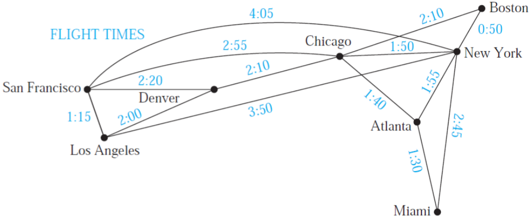


length of a path in a weighted graph be the sum of the weights of the edges of this path.

Question: Find a shortest path from San Francisco to Boston.

Shortest-Path Problems

Weighted Graph: Graphs that have a number assigned to each edge are called weighted graphs.



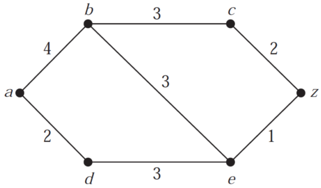
length of a path in a weighted graph be the sum of the weights of the edges of this path.

Question: Find a shortest path from San Francisco to Boston.

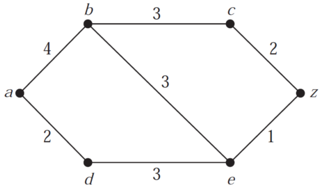
San Francisco, New York, Boston:4:05+0:50=4:55

San Francisco, Chicago, Boston: 2:55+2:10=5:05

Shortest-Path Algorithm: Dijkstra

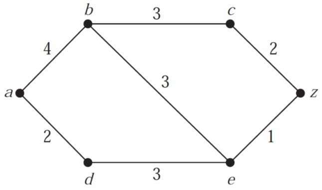


Shortest-Path Algorithm: Dijkstra



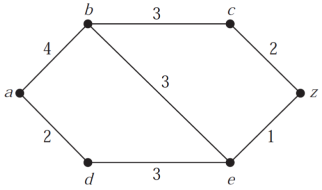
Question: Find a shortest path from a to z.

Shortest-Path Algorithm: Dijkstra



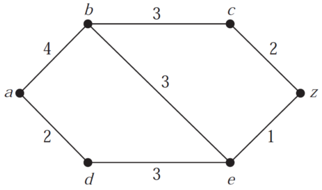
Question: Find a shortest path from a to z.
Idea: find the length of a shortest path from a to successive vertices, until z is reached.

Shortest-Path Algorithm: Dijkstra



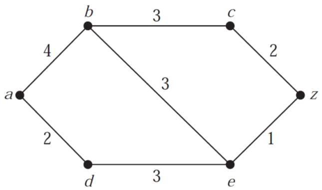
Question: Find a shortest path from a to z.
Idea: find the length of a shortest path from a to successive vertices, until z is reached.
The only paths starting at a that contain no vertex other than a are formed by adding an edge that has a as one endpoint: a, b of length 4 and a, d of length 2.

Shortest-Path Algorithm: Dijkstra



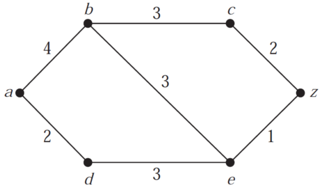
Question: Find a shortest path from a to z.
Idea: find the length of a shortest path from a to successive vertices, until z is reached.
The only paths starting at a that contain no vertex other than a are formed by adding an edge that has a as one endpoint: a, b of length 4 and a, d of length 2.
d is the closest vertex to a and the shortest path from a to d has length 2.

Ctd...



find the second closest vertex by examining all paths that begin with the shortest path from a to a vertex in the set $\{a, d\}$, followed by an edge that has one endpoint in $\{a, d\}$ and its other endpoint is not in this set.

Ctd...



find the second closest vertex by examining all paths that begin with the shortest path from a to a vertex in the set $\{a, d\}$, followed by an edge that has one endpoint in $\{a, d\}$ and its other endpoint is not in this set.

There are two such paths to consider, a, d, e of length 7 and a, b of length 4. Hence, the second closest vertex to a is b and the shortest path from a to b has length 4.

To find the third closest vertex to a, we need to examine only the paths that begin with the shortest path from a to a vertex in the set $\{a, d, b\}$, followed by an edge that has one endpoint in the set $\{a, d, b\}$ and its other endpoint not in this set.

To find the third closest vertex to a, we need to examine only the paths that begin with the shortest path from a to a vertex in the set $\{a, d, b\}$, followed by an edge that has one endpoint in the set $\{a, d, b\}$ and its other endpoint not in this set.
There are three such paths, a, b, c of length 7, a, b, e of length 7, and a, d, e of length 5. Because the shortest of these paths is a, d, e the third closest vertex to a is e.

To find the third closest vertex to a, we need to examine only the paths that begin with the shortest path from a to a vertex in the set $\{a, d, b\}$, followed by an edge that has one endpoint in the set $\{a, d, b\}$ and its other endpoint not in this set.
There are three such paths, a, b, c of length 7, a, b, e of length 7, and a, d, e of length 5. Because the shortest of these paths is a, d, e the third closest vertex to a is e.
To find the fourth closest vertex to a, we need to examine only the paths that begin with the shortest path from a to a vertex in the set $\{a, d, b, e\}$, followed by an edge that has one endpoint in the set $\{a, d, b, e\}$ and its other endpoint not in this set.

To find the third closest vertex to a, we need to examine only the paths that begin with the shortest path from a to a vertex in the set $\{a, d, b\}$, followed by an edge that has one endpoint in the set $\{a, d, b\}$ and its other endpoint not in this set.

There are three such paths, a, b, c of length 7, a, b, e of length 7, and a, d, e of length 5. Because the shortest of these paths is a, d, e the third closest vertex to a is e.

To find the fourth closest vertex to a, we need to examine only the paths that begin with the shortest path from a to a vertex in the set $\{a, d, b, e\}$, followed by an edge that has one endpoint in the set $\{a, d, b, e\}$ and its other endpoint not in this set.There are two such paths, a, b, c of length 7 and a, d, e, z of length 6. Because the shorter of these paths is a, d, e, z, the fourth closest vertex to a is z and the length of the shortest path from a to z is 6

ALGORITHM 1 Dijkstra's Algorithm.

procedure *Dijkstra*(G: weighted connected simple graph, with
all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
other labels are ∞, and S is the empty set}
while $z \notin S$
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

The traveling salesperson problem asks for the circuit of minimum total weight in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point. This is equivalent to asking for a Hamilton circuit with minimum total weight in the complete graph, because each vertex is visited exactly once in the circuit.

The traveling salesperson problem asks for the circuit of minimum total weight in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point. This is equivalent to asking for a Hamilton circuit with minimum total weight in the complete graph, because each vertex is visited exactly once in the circuit.

The most straightforward way to solve an instance of the traveling salesperson problem is to examine all possible Hamilton circuits and select one of minimum total length: need only to examine $(n-1)!/2$ circuits to find our answer.

The traveling salesperson problem asks for the circuit of minimum total weight in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point. This is equivalent to asking for a Hamilton circuit with minimum total weight in the complete graph, because each vertex is visited exactly once in the circuit.

The most straightforward way to solve an instance of the traveling salesperson problem is to examine all possible Hamilton circuits and select one of minimum total length: need only to examine $(n-1)!/2$ circuits to find our answer.

No algorithm with polynomial worst-case time complexity.