PH100: Mechanics and Thermodynamics (3-1-0:4)

Lecture 2

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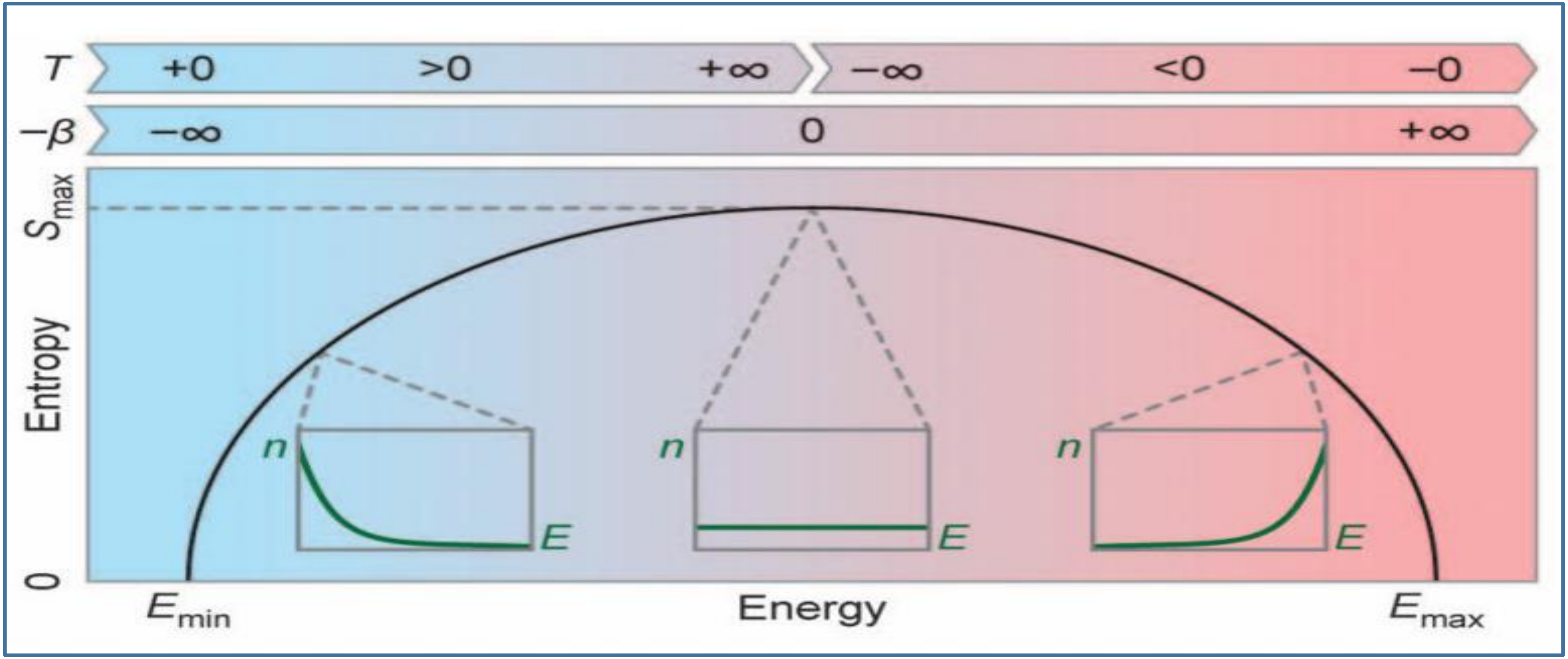
Negative Temperature



https://www.science.org/doi/10.1126/science.1227831

Vectors

- **Definition of vector:**

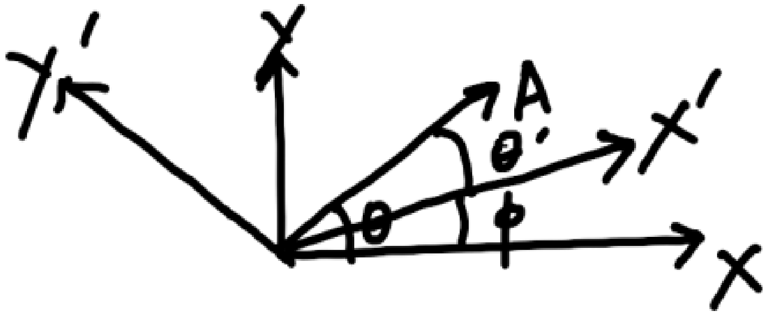
A vector is defined by its invariance properties under certain operations --

- **Translation**
- **Rotation**
- **Inversion etc**

Invariance Under Rotation

$A_x = A\cos\theta$

$A_y = A\sin\theta$



$A_x' = A\cos\theta'$

$A_y' = A\sin\theta'$

$A_x' = A\cos(\theta - \phi) = A\cos\theta\cos\phi + A\sin\theta\sin\phi$

$A_y' = A\sin(\theta - \phi) = A\sin\theta\cos\phi - A\cos\theta\sin\phi$

Simplifying

$A_x' = A_x\cos\phi + A_y\sin\phi$

$A_y' = -A_x\sin\phi + A_y\cos\phi$

- **In a compact form**

$$\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Transformation equations for the components of a vector can be written as,

$$\overline{A'} = R\overline{A}$$

GENERALISATION TO 3 DIMENSIONS

- Consider the Rotation Matrix in 3D,

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation about *z*-axis by an angle θ

With the help of this we shall prove that $\vec{A} \times \vec{B}$ is a vector i.e. it is invariant under rotation.

- Since $\vec{A}$ is a vector its component transform as,

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$A_x' = A_x\cos\theta + A_y\sin\theta$

$A_y' = - A_x\sin\theta + A_y\cos\theta$

$A_z' = A_z$

(because of rotation about ***z***-axis, the ***z***-component remains invariant.)

Similarly

$B_x' = B_x \cos\theta + B_y \sin\theta,$

$B_y' = - B_x \sin\theta + B_y \cos\theta$

$B_z' = B_z$

Now, consider the vector,

$$\bar{\mathbf{C}}' = \bar{\mathbf{A}}' \times \bar{\mathbf{B}}'$$

$$\bar{A}' \times \bar{B}' = (\bar{A}' \times \bar{B}')_x + (\bar{A}' \times \bar{B}')_y + (\bar{A}' \times \bar{B}')_z$$

Consider only x- component (for a moment)

$$\overrightarrow{(A \times \vec{B})}_x = (-\sin\theta A_x + \cos\theta A_y)B_z' - (-\sin\theta\, B_x + \cos\theta B_y)\, A_z'$$

Since $\quad A_z' = A_z$

$B_z' = B_z$

$$\overrightarrow{(A \times \vec{B})}_x = \sin\theta(B_x A_z - A_x B_z) + \cos\theta(A_y B_z - B_y A_z)$$

$$\overrightarrow{(A' \times \vec{B}')}_x = R_x \overrightarrow{(A \times \vec{B})}_x$$

Similarly we can prove it for the other components also.

$$\overrightarrow{(A' \times \vec{B}')}_y = R_y \overrightarrow{(A \times \vec{B})}_y ; \overrightarrow{(A' \times \vec{B}')}_z = R_z \overrightarrow{(A \times \vec{B})}_z$$

Hence, $\overrightarrow{(A \times \vec{B})}$ is invariant under rotation and transforms like a vector.

Vector Calculus

- Gradient: To know the direction along which a scalar function changes the fastest
- φ(x, y, z) is scalar function in cartesian coordinates

$$\bar{\nabla}\Phi = \hat{x}\frac{\partial \Phi}{\partial x} + \hat{y}\frac{\partial \Phi}{\partial y} + \hat{z}\frac{\partial \Phi}{\partial z}$$

Gradient operator

$$\bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Find $\bar{\nabla}$φ for φ $(x, y, z) = r = \sqrt{x^2 + y^2 + z^2}$

Divergence

- It quantifies how much a vector function diverges.It is scalar.

$$\bar{\nabla}.\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- Example: $\bar{A} = x\hat{x} + y\,\hat{y} + z\hat{z}$

$$\bar{\nabla}.\bar{A} = 3$$

Curl

- Circulation of a vector field,

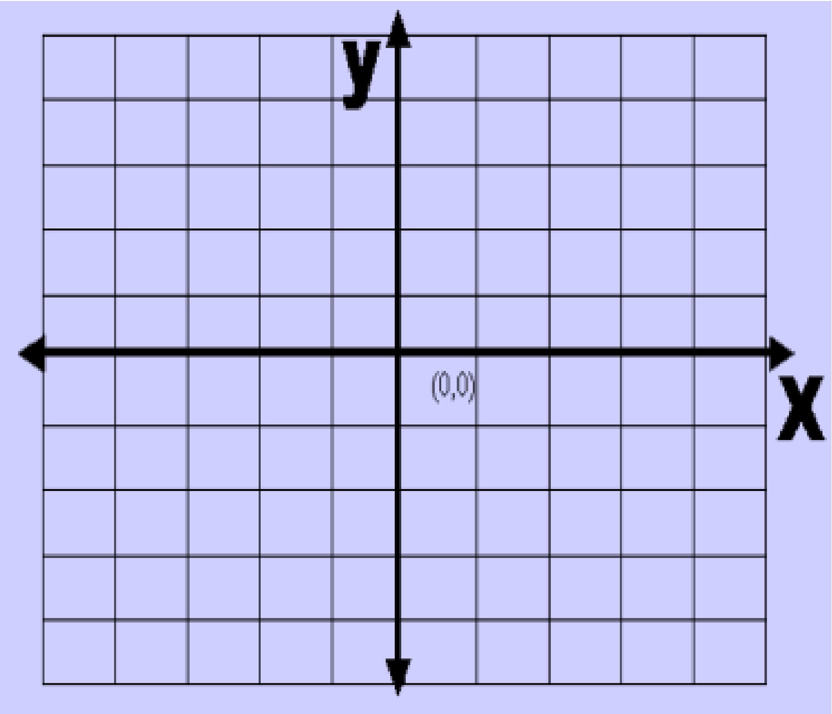
$$\bar{\nabla} \times \bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\bar{A} = \bar{r}$$

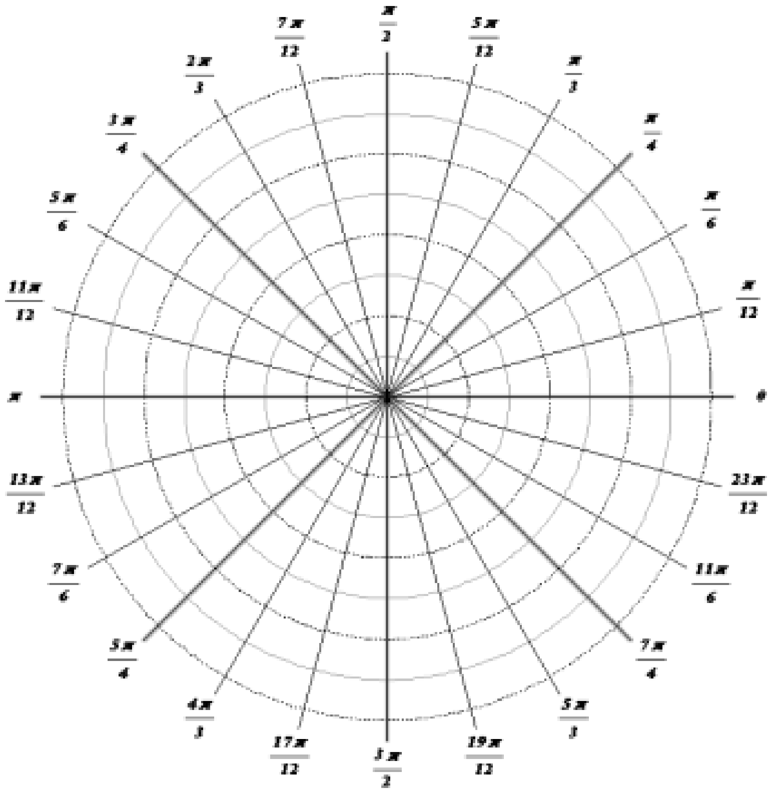
$$\bar{\nabla} \times \bar{r} = 0$$

Polar Coordinates

You are familiar with plotting with a rectangular coordinate system.

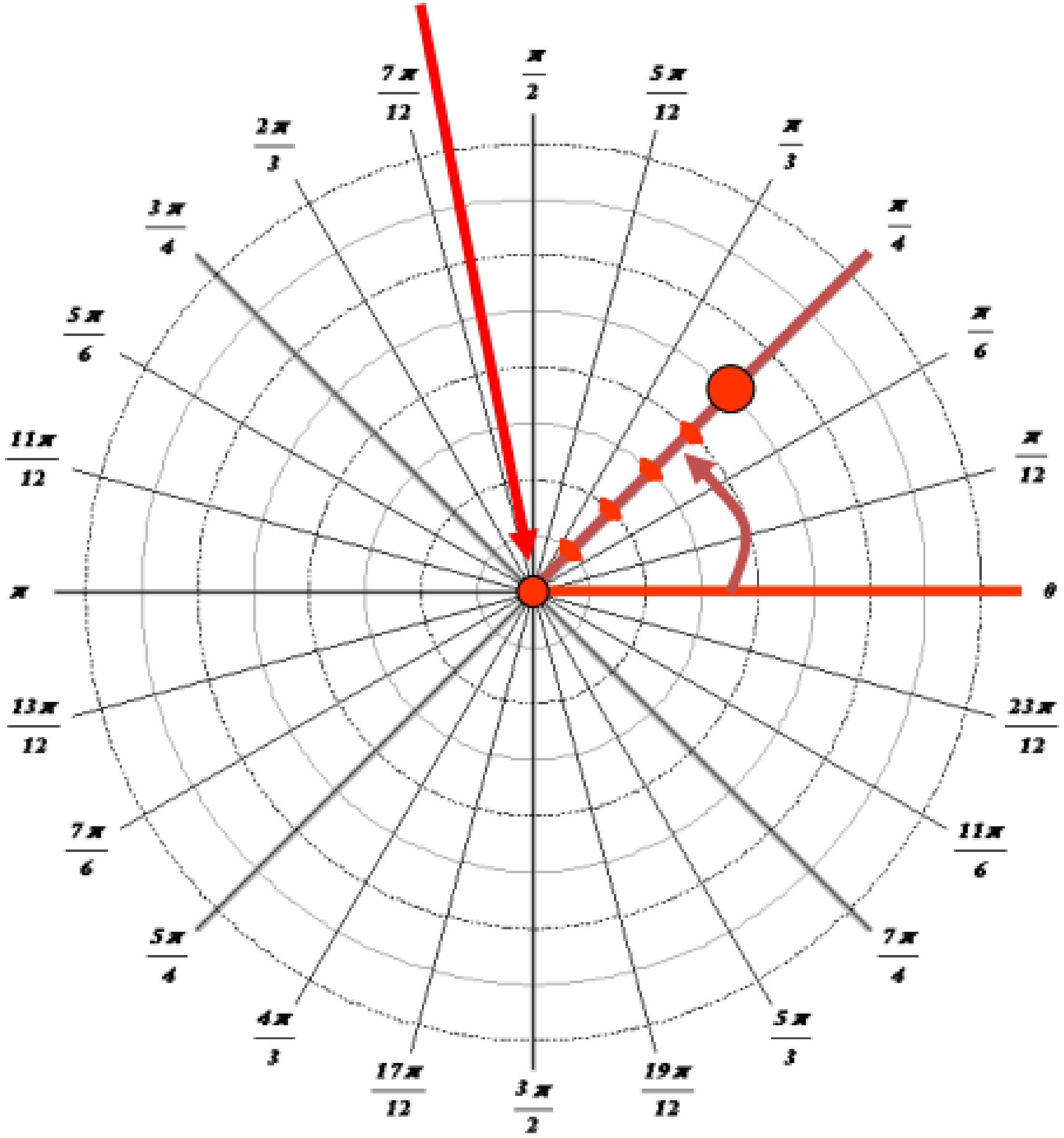


We are going to look at a new coordinate system called the polar coordinate system.



The center of the graph is called the pole.

Angles are measured from the positive *x* axis.



Points are represented by a radius and an angle

$$(r, \theta)$$

To plot the point

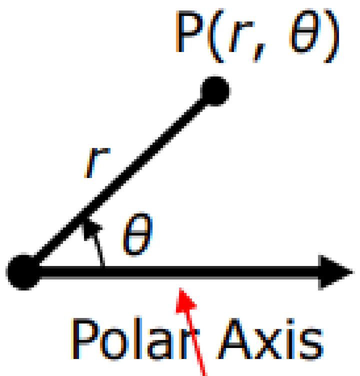
$$\left(5, \frac{\pi}{4}\right)$$

First find the angle $\pi/4$

Then move out along the terminal side 5

Polar Coordinates

To define the Polar Coordinates of a plane we need first to fix a point which will be called the Pole (or the origin) and a half-line starting from the pole. This half-line is called the Polar Axis.

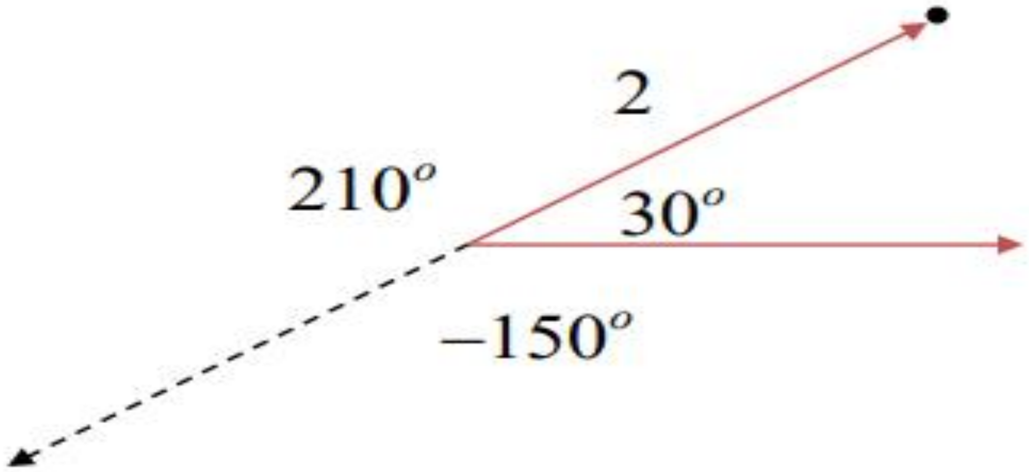


A positive angle.

Polar Angles

The Polar Angle θ of a point P, P ≠ pole, is the angle between the Polar Axis and the line connecting the point P to the pole. Positive values of the angle indicate angles measured in the counterclockwise direction from the Polar Axis.

More than one coordinate pair can refer to the same point.



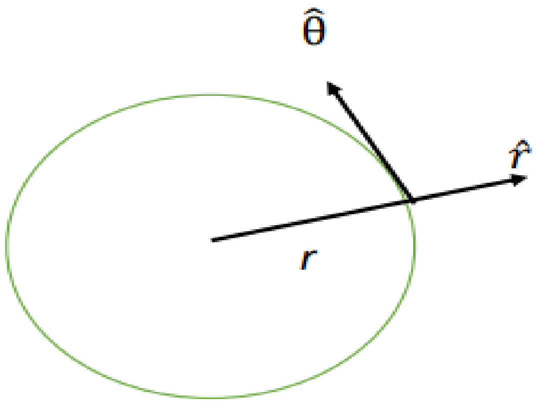
$$\left(2, 30^o\right)$$
$$= \left(-2, 210^o\right)$$
$$= \left(-2, -150^o\right)$$

All of the polar coordinates of this point are:

$$\left(2, 30^o + n \cdot 360^o\right)$$
$$\left(-2, -150^o + n \cdot 360^o\right) \qquad n = 0,\ \pm 1,\ \pm 2\ \ldots$$

Velocity and acceleration in polar coordinates

Velocity in polar coordinate:



The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

And the unit vectors are: $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ & $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$

Since the unit vectors are not constant and changes with time, they should have finite time derivatives:

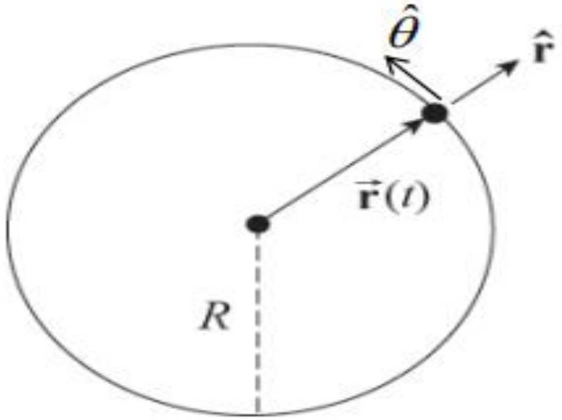
$$\dot{\hat{r}} = \dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = \dot{\theta}\hat{\theta} \quad \text{and} \quad \dot{\hat{\theta}} = \dot{\theta}\left(-\cos\theta\hat{i} - \sin\theta\hat{j}\right) = -\dot{\theta}\hat{r}$$

Therefore the velocity is given by: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Radial velocity + tangential velocity

In Cartesian coordinates $= \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$

Example-1: Uniform Circular Motion



$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Since $\dot{r} = \dfrac{dR}{dt} = 0$ and $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\,\omega\,\hat{\boldsymbol{\theta}}(t)$$

Since $\vec{v}$ is along $\hat{\theta}$ it must be perpendicular to the radius vector $\vec{r}$ and it can be shown easily

$$R^2 = \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} \qquad \frac{d}{dt}R^2 = \frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 0, \qquad \vec{\mathbf{r}} \perp \vec{\mathbf{v}}$$

Acceleration in Polar coordinate:

$$\begin{aligned}
\mathbf{a} &= \frac{d}{dt}\mathbf{v} \\
&= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.
\end{aligned}$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta},\ \dot{\hat{\theta}} = -\dot{\theta}\hat{r}}$$

$$\begin{aligned}
\mathbf{a} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.
\end{aligned}$$

The term $\ddot{r}\hat{\mathbf{r}}$ is a linear acceleration in the radial direction due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$ is a linear acceleration in the tangential direction due to change in the magnitude of the angular velocity.

The term $-r\dot{\theta}^2\hat{\mathbf{r}}$ is the centripetal acceleration

Finally, the Coriolis acceleration $2\dot{r}\dot{\theta}\hat{\theta}$

Usually, Coriolis force appears as a fictitious force in a rotating coordinate system. However, the Coriolis acceleration we are discussing here is a real acceleration and which is present when *r* and θ both change with time.

Thank You