Linear Recurrence Relations

Homo. L. R. R.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$c_k \neq 0$

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0$$

$$(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2} \cdots (x-\alpha_r)^{m_r}$$

$\alpha_1, \alpha_2, \ldots \alpha_r$

$\downarrow \quad \downarrow \quad \downarrow$

$m_1 \quad m_2 \quad m_r$

$$a_n = (n^0 c_{11} + c_{12} n + c_{13} n^2 + \cdots + c_{1m_1} n^{m_1 - 1})\alpha_1^n$$

$+$

$\vdots$

$+$

$$(\quad)\alpha_r^n$$

$$\boxed{a_n = \frac{5}{2}2^n - n - 2} \leftarrow a_n = 2a_{n-1} + n \quad \forall n \geq 2$$

non-homo

$\cancel{a_1 = 2}$

$a_n = -n-2$

does not satisfy $a_1 = 2$

$\cancel{a_n^h + a_n^p = 2(a_{n-1}^h + a_{n-1}^p) + n}$

$$a_n = a_n^h + a_n^p$$

Sol^n of associated homo. R.R.

particular sol^n

$$a_n^h = \alpha . 2^n$$

Guess: Try $a_n^p = an+b$

$$a_n = \alpha 2^n - n - 2$$

$$a_1 = 2 = 2\alpha - 1 - 2 \Rightarrow \alpha = 5/2$$

$$an+b = 2(a(n-1)+b) + n$$

$$n(a-2a-1) + b + 2a - 2b = 0 \quad \forall n \geq 2$$

$$n(-a-1) + 2a - b = 0$$

$$(-a-1)x + 2a - b = 0$$

poly eq^n of deg 1

$a=-1$, $b=-2$

If $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k} + \sum_{i=0}^{r} d_i n^i$

Then $a_n^p = \sum_{i=0}^{r} p_i n^i$

$a_n = \underline{5 a_{n-1}} - 6 \underline{a_{n-2}} + 7^n \qquad a_1 = 2, \; a_2 = 3$

$x^2 - 5x + 6 = 0$ ch. eq^n

$a_n^h = \alpha_1 2^n + \alpha_2 3^n$

Try $a_n^p = c \cdot 7^n$

$c \cdot 7^n = 5c \cdot 7^{n-1} - 6c\, 7^{n-2} + 7^n \quad \forall n \geq 3$

$49c = 35c - 6c + 49 \qquad (50-1)^2$

$c = 49/20$

$a_n = \alpha_1 2^n + \alpha_2 3^n + \frac{49}{20} \cdot 7^n$

$a_1 = 2 = 2\alpha_1 + 3\alpha_2 + \frac{343}{20}$

$a_2 = 3 = 4\alpha_1 + 9\alpha_2 + 2401/20$

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$$

$a_0 = 1,\ a_1 = 2$

Try $a_n^p = c \cdot 2^n$

$$c2^n = 5c \cdot 2^{n-1} - 6c2^{n-2} + 2^n$$

$$\cancel{4c} = 10\cancel{c} - \cancel{6c} + 4 \Rightarrow \underline{0 = 4} \Rightarrow \Leftarrow$$

Try $a_n^p = cn \cdot 2^n$

$$cn2^n = 5\left(c(n-1)2^{n-1}\right) - 6\left((n-2)c2^{n-2}\right) + 2^n$$

$$4\cancel{c}n = 10\cancel{c}n - 10c - 6\cancel{c}n + 12c + 4$$

$$0 = 2c + 4 \Rightarrow c = -2$$

$$a_n = \underset{-3}{\alpha_1} 2^n + \underset{4}{\alpha_2} 3^n - 2n2^n$$

$$\left.\begin{aligned} a_0 &= 1 = \alpha_1 + \alpha_2 \\ a_1 &= 2 = 2\alpha_1 + 3\alpha_2 - 4 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_2 &= 4 \\ \alpha_1 &= -3 \end{aligned}$$

Let $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-1} + b^n$

If b is a characteristic root with multiplicity r

then $a_n^p = c \cdot n^r \cdot b^n$

$$a_n = c_1 a_{n-1} - - - - + c_k a_{n-k} + \left(\sum_{i=0}^{r} d_i n^i \right) \cdot b^n$$

$$a_n^p = \left(\sum_{i=0}^{r} p_i n^i \right) \cdot n^s \cdot b^n \quad \text{wh mult. of } b = s$$

Ex. $a_n = 5a_{n-1} - 6a_{n-2} + \underline{n}\,\underline{2}^n$

$$a_n = \underline{\alpha_1} 2^n + \underline{\alpha_2} 3^n + \left((\underline{c} n + \underline{d}) 2^n \cdot n \qquad \right)$$

$$a_n^2 - 2a_{n-1}^2 = 1 \qquad a_0 = 1$$

Put $b_n = a_n^2$

$$b_n = 2b_{n-1} + 1 \qquad \underline{b_0 = 1}$$

$$b_n^p = c$$

$$c = 2c + 1$$

$$c = -1$$

$$b_n = \alpha \cdot 2^n - 1$$

$$b_n = 2^{n+1} - 1$$

$$a_n = \pm\sqrt{2^{n+1} - 1}$$

$a_{2^k} = b_k = 2k+2$

$a_n = 2\cdot\log_2 n + 2$

$k = \log_2 n$

$n = 2^k$

$a_n = a_{2^k} = b_k$

$b_k = b_{k-1} + 2\cdot 1^n$

$b_k = c\cdot 1^k + 2\cancel{n}k$

$\overset{k}{\cancel{n}}c = c(\overset{k}{\cancel{n}}-1)+2$

$c = 2$

$2 = c + 2\cdot 0$

$c = 2$

$b_k = 2k+2$

$x_1, x_2, \ldots x_{2^{k-1}} - x_{2^k}$ $\quad x_i < x_{i+1}$

compare x & $x_{2^{k-1}}$

Let a_n = no. of comp.

$a_n = a_{n/2} + 2$

a

$a_1 = 2$

$b_0 = 2$

$x_1, x_2 x_3 x_4 x_5 x_6 x_7 x_8$

Generating functions

$$\frac{1-x^{n+1}}{1-x} = 1+x+\cdots\cdots x^n$$

Let $a_0, a_1, a_2, \text{---}$ be a seq. of real numbers

$$a_0 + a_1 x + a_2 x^2 + \text{------} + a_n x^n + \text{---}$$

is called generating function corresponding to seq. $\{a_i\}_{i=0}^{\infty}$

$$(1+x)^{\pi} =$$

$$\frac{1}{1-x}(1-x) = 1$$

$$\left(\sum_{i=0}^{\infty} x^i\right)(1-x) = 1$$

$$\left(1+(-1)\right)x^2$$

$$(1-x)^{-1} = \frac{1}{1-x} = 1+x+x^2+x^3+\text{---} = \sum_{i=0}^{\infty} x^i \quad \text{if } |x|<1$$

$$(1+x)^{-1} = \sum_{i=0}^{\infty}(-1)^i x^i = 1-x+x^2-x^3+x^4 \text{---}$$

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = (1-x)^{-1}(1-x)^{-1} = \left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^i\right)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)x^n$$

$${}^nC_i = \cancel{\frac{n!}{(n-i)!\,i!}}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{i!}$$

Extended binomial theorem

For $|x| < 1$, $u \in \mathbb{R}$

$$(1-x)^u = \sum_{n=0}^{\infty} {}^uC_n\, x^n$$

where ${}^uC_n = \dfrac{u(u-1)\cdots(u-n+1)}{n!}$

$$(1-x)^u \cdot (1-x)^{-u} = 1$$

$$\left(\sum {}^uC_i\, x^i\right)\left(\sum {}^{-u}C_i\, x^i\right)$$

coeff of x^n in LHS $= \sum_{i=0}^{n} {}^uC_i \cdot {}^{-u}C_{n-i} = 0$

$n \geq 1$

$${}^{-n}C_i = \frac{(-n)(-n-1)(-n-2)\cdots(-n-i+1)}{i!}$$

$$= (-1)^i \frac{n(n+1)(n+2)\cdots(n+i-1)}{i!}$$

$$= (-1)^i \, {}^{n+i-1}C_i$$

Sol^n of $x_1 + x_2 + \cdots + x_n = i$

$x_1 + x_2 + \cdots + x_{i+1} = n$

$$(1-x)^u = \sum_{i=0}^{\infty} {}^uC_i \, x^i$$

$$1 = (1-x)^u \cdot (1-x)^{-u} = \left(\sum_{i=0}^{\infty} {}^uC_i x^i\right)\left(\sum_{i=0}^{\infty} {}^{-u}C_i x^i\right)$$

$$\sum_{i=0}^{n} {}^uC_i \, {}^{-u}C_{n-i} = 0 \quad \forall\, n \geq 1$$

Find no. of solutions of

$$x_1 + x_2 + x_3 = 17$$ wh

$2 \le x_1 \le 5$

$3 \le x_2 \le 6$

$4 \le x_3 \le 7$

$$(x^2+x^3+x^4+x^5)\,(x^3+x^4+x^5+x^6)\,(x^4+x^5+x^6+x^7)$$

Ans = coeff of x^{17} =

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} 2^n x^n \Rightarrow a_i = 2^i$$

$$x^n \times a_n = 5a_{n-1} - 6a_{n-2} \times x^n$$

$$a_0 = 1 \quad a_1 = 2$$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 5a_{n-1}x^n - 6a_{n-2}x^n$$

$$\left(G(x) - a_0 - a_1 x\right) = 5x\left(\sum_{n=2}^{\infty} a_{n-1}x^{n-1}\right) - 6x^2\left(\sum_{n=2}^{\infty} a_{n-2}x^{n-2}\right)$$

$$\left(G(x) - 1 - 2x\right) = 5x\left(G(x) - 1\right) - 6x^2 G(x)$$

$$G(x)\left[1 - 5x + 6x^2\right] = -5x + 1 + 2x = 1 - 3x$$

$$G(x) = \frac{1-3x}{1-5x+6x^2} = \frac{1-\cancel{3x}}{(1-\cancel{3x})(1-2x)} = \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$$