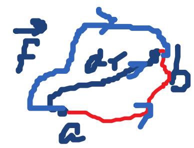
Lecture 9: Section 2

Thursday, January 20, 2022 8:59 AM

Work: $\underline{W} = \int_a^b \vec{F}\cdot d\vec{r}$ → Path dependent

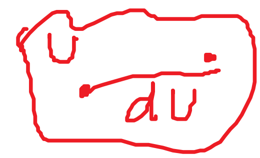


*. $\underline{W} = \oint_a^b \vec{F}\cdot d\vec{r} = \underline{K.E_b - K.E_a} = \Delta K.E$

*. 1D case / Central force } $\underline{\text{path independent}}$: Conservative forces.

*. $W = \int_a^b \vec{F}\cdot d\vec{r} = -U_b + U_a = K.E_b - K.E_a$.

*. $\vec{F} = -\nabla \underline{U}$ Conservative



$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

$$= \left[\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$\boxed{dU = \nabla U \cdot d\vec{r}} \longrightarrow \underline{1D} \quad -\frac{dU}{dx} = F$$

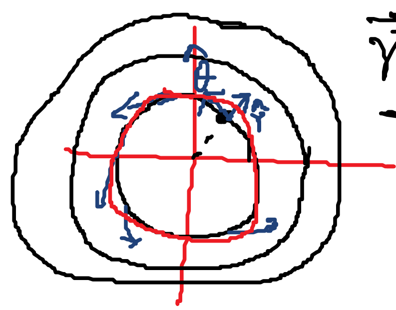
$$\boxed{\vec{F} = -\nabla U}$$

path independent

Gradient of U.

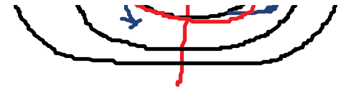
Simplest Ex → $U = x^2 + y^2$

$\vec{r} = r\hat{r}$



$\underline{dU} = \underline{\nabla} U \cdot d\vec{r}$ $\quad \nabla U \cdot \hat{\theta}$

$\nabla U = \hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} = 2x\,\hat{i} + 2y\,\hat{j}$



$$\nabla U = \hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} = 2x\hat{i} + 2y\hat{j}$$

$$= 2(x\hat{i} + y\hat{j}) = 2\vec{r} = 2r\hat{r}$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

$$\nabla U \cdot \cancel{d\vec{r}}\;\hat{\theta} = 2(x\hat{i} + y\hat{j}) \cdot \hat{\theta}$$

$$= 2(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}) \cdot (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) = 0$$

$$\nabla U \cdot \hat{r} = 2r\hat{r} \cdot \hat{r} = 2r$$

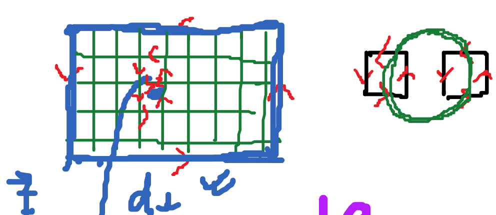
$$\nabla = \hat{i}\,\partial_x + \hat{j}\,\partial_y + \hat{k}\,\partial_z$$

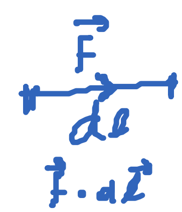
Stoke's Theorem:

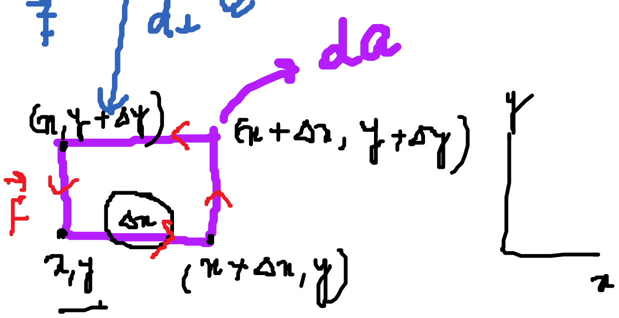
$$\boxed{\oint \vec{F} \cdot d\vec{l} = \iint_A (\nabla \times \vec{F}) \cdot da}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}, \qquad \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$\rightarrow \frac{\text{Circulation}}{\text{Area}} \cdot \text{Area}$







$$\vec{F} = F_x\hat{i} + F_y\hat{j}$$

Top: $-F_x(x, y+\Delta y)\Delta x$

Bottom: $F_x(x, y)\Delta x$ $\Rightarrow -\frac{\partial F_x}{\partial x}\Delta x\Delta y$

Left: $-F_y(x, y)\Delta y$

Right: $F_y(x+\Delta x, y)\Delta y$

$$F_y(x+\Delta x, y)\Delta y - F_y(x, y)\Delta y \Leftarrow$$

$\ldots F_y(x+\Delta x, y)\,\Delta y$

$$\simeq \left(\frac{F_y(x+\Delta x, y) - F_y(x,y)}{\Delta x}\right) \Delta y\, \Delta x$$

$$\frac{dF}{dx} = \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \frac{\partial F_y}{\partial x} \cdot \overset{\Delta A}{(\Delta x \Delta y)}$$

$$= \frac{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \cancel{\Delta x \Delta y}}{\cancel{\Delta A}} = \frac{\partial F_x}{\partial y} - \frac{\partial F_x}{\partial y} = (\nabla \times F)_z$$

Circulation

$$\frac{\text{Circulation}}{dA}$$

$$\vec{F} = -\nabla U$$

$$\oint_a^b \vec{F} \cdot d\vec{l} = \iint \nabla \times \vec{F} \cdot d\vec{a}$$

$$\vec{F} = -\nabla U = \hat{i} \frac{\partial}{\partial x} U + \hat{j} \frac{\partial U}{\partial y} + \hat{k} \frac{\partial U}{\partial z}$$

$$\frac{\partial}{\partial x \partial y} xy = 1 = \frac{\partial xy}{\partial y \partial x}$$

$$\nabla \times \vec{F} = \nabla \times (\nabla U) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

$$= 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z}\right)$$

$$= \hat{i}\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 U}{\partial x \partial z} - \frac{\partial^2 U}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial x \partial z}\right) \to 0$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F}$$

$$\nabla = i \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}, \quad \vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$\nabla \times \vec{F} = \nabla \times (\nabla U) = 0$$

$$U_b - U_a = -\int \vec{F} \cdot d\vec{l}$$

Stokes $\oint \vec{F}\cdot d\vec{l} = \int_A (\nabla\times\vec{F})\cdot d\vec{a}$

$\nabla\times\vec{F} = 0$

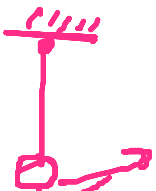
$\oint \vec{F}\cdot d\vec{l} = 0$

$\int_a^b \vec{F}\cdot d\vec{l}$

$\vec{F} \rightarrow \nabla U$

Conservative force :-

$\vec{F} = -\nabla U = 0$



Stable equilibrium

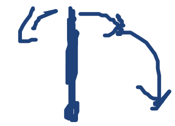
$U = x^2 + y^2, \ (0,0)$

$\nabla U = 2x\hat{i} + 2y\hat{j}$

$\vec{F} = -2(x\hat{i} + y\hat{j})$

$(0,0) = 0$

Unstable



$U = -(x^2+y^2), \ (0,0)$

$\nabla U = -(2x\hat{i} + 2y\hat{j})$

$F = 2(x\hat{i} + y\hat{j})$

Saddle point ?

$U = x^2 - y^2, \ (0,0), \ \nabla U = 2x\hat{i} - 2y\hat{j}$

$F = -\nabla U = -2x\hat{i} + 2y\hat{j}$?