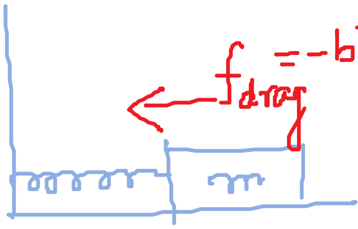
Lecture 14: Section 1

Tuesday, February 1, 2022 8:50 AM

Harmonic Oscillatory



Undamped

$\vec{F} = -\vec{f}_s$

$\ddot{x} + \frac{K}{m}x = 0$

$x(t) = C\cos(\omega_0 t + \phi)$

$\omega_0 = \sqrt{K/m}$

Damped — Lightly, Critically, Heavily

$\vec{F} = -\vec{f}_s - \vec{f}_{drag}$

$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0}$

$\gamma = \frac{b}{m}$, $\omega_0 = \sqrt{\frac{k}{m}}$

Forced Oscillation — Forced damped osc.

Damped HO: $\ddot{Z} + \gamma\dot{Z} + \omega_0^2 Z = 0$, $Z = x + iy$

$x(t) = Re[Z]$

$Z = Z_0 e^{i\alpha t}$

$\Downarrow$

$[(i\alpha)^2 + i\gamma\alpha + \omega_0^2]\underbrace{Z_0 e^{i\alpha t}}_{\neq 0} = 0$

$\boxed{\alpha^2 - i\gamma\alpha - \omega_0^2 = 0}$

$\omega_0^2 < \frac{\gamma^2}{4}$, $\omega_0^2 = \frac{\gamma^2}{4}$

$\alpha = \frac{i\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}}{2} = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

$\omega_0^2 = \frac{k}{m}$, $\gamma^2 = \frac{b^2}{m^2}$ → damping

1. Lightly: $\omega_0^2 > \frac{\gamma^2}{4}$, $\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} > 0$, $\left.\begin{matrix} Z_+ \\ Z_- \end{matrix}\right\} \rightarrow$

$Z(t) = \underline{C}\, e^{-\frac{\gamma}{2}t}[\cos(\omega t + \phi)]$

$$Z_{\text{initial}} = C\cos\phi\ , \quad \tan\phi = \frac{-\gamma}{2\omega}.$$

2. Critically damped: $\omega_0^2 = \frac{\gamma^2}{4}$, $\alpha = \frac{i\gamma}{2}$

Lightly damp

Critically

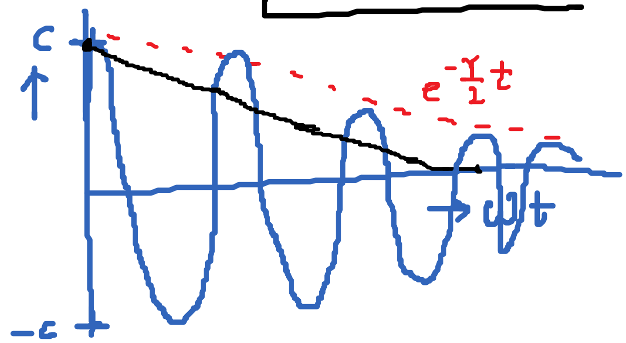
$$Z_1(t) = c\,e^{-\frac{\gamma}{2}t}\cos\omega t = c\,e^{-\frac{\gamma}{2}t}$$

$$\boxed{\begin{gathered}\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\\ \omega \to 0\\ \omega = 0\end{gathered}}$$

$$\lim_{\omega\to0} Z_2(t) = \lim_{\omega\to0} C_1 e^{-\gamma/2\, t}\sin\omega t = 0$$

$$\lim_{\omega\to0}\frac{Z_2(t)}{\omega} = \lim_{\omega\to0} C e^{-\frac{\gamma}{2}t}\cdot\underbrace{\frac{\sin\omega t}{\omega}}_{t}$$

$$= C_2\, t\, e^{-\frac{\gamma}{2}t}.$$



$$\boxed{Z(t) = (C_1 + C_2 t)\,e^{-\frac{\gamma}{2}t}} \rightarrow \text{Critically damped}$$

3. Heavily damped :-

$$\alpha = \frac{i\gamma}{2} \pm \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}_{\Downarrow\ \text{Imaginary}}\ , \quad \omega_0^2 < \frac{\gamma^2}{4}$$

$$\Gamma = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

$$\rightarrow \alpha_\pm = \frac{i\gamma}{2} \pm i\Gamma$$

$$Z(t) = Z_0 e^{i\alpha t} = Z_0 e^{i\left(i\frac{\gamma}{2} \pm i\Gamma\right)t}$$

$$\boxed{Z(t) = Z_0 e^{-\left(\frac{\gamma}{2}\pm\Gamma\right)t}}$$

$$z(y) = z_0 e$$

1. $\omega_0^2 > \frac{\gamma^2}{4} \Rightarrow x(t) = C e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi)$, $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$.

2. $\omega_0^2 = \frac{\gamma^2}{4} \Rightarrow \boxed{x(t) = (A + Bt) e^{-\frac{\gamma}{2}t}}$.

3. $\omega_0^2 < \frac{\gamma^2}{4} \Rightarrow x(t) = C_1 e^{-\left(\frac{\gamma}{2} \pm r\right)t}$

Energy of Lightly damped HO $\Rightarrow$ $\omega_0^2 > \frac{\gamma^2}{4}$

$$x(t) = C e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi)$$

$$E(t) = K.E(t) + P.E(t) = \frac{1}{2} m v^2(t) + \frac{1}{2} K x^2$$

$$v = -\frac{\gamma}{2} C e^{-\frac{\gamma}{2}t} \cdot \cos(\omega t + \phi) - C e^{-\frac{\gamma}{2}t} \cdot \omega \sin(\omega t + \phi)$$

$$= -\omega C e^{-\frac{\gamma}{2}t} \left[\sin(\omega t + \phi) + \underbrace{\left(\frac{\gamma}{2\omega}\right) \cos(\omega t + \phi)}_{\text{small and negligible}} \right]$$

$$v = -\omega C e^{-\frac{\gamma}{2}t} \sin(\omega t + \phi)$$

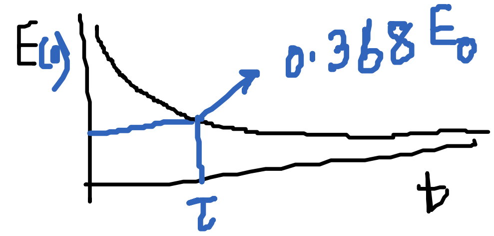
Undamped
$$x(t) = C \cos(\omega_0 t + \phi)$$
$$v(t) = -C\omega_0 \sin(\omega_0 t + \phi)$$
$$K.E(t) = \frac{1}{2} m C^2 \omega_0^2 \sin^2(\omega_0 t + \phi)$$

$$E(t) = \frac{1}{2} m \omega^2 C^2 e^{-\gamma t} \sin^2(\omega t + \phi)$$
$$+ \frac{1}{2} K C^2 e^{-\gamma t} \cos^2(\omega t + \phi)$$

$$+\frac{1}{2}KC^2 e^{-\gamma t}\cos^2(\omega t+\phi)$$

$$\Rightarrow \boxed{E(t)=\frac{1}{2}KC^2 e^{-\gamma t}} \Rightarrow E(0)=\frac{1}{2}KC^2$$

$$E(t)=E(0)\,\underbrace{e^{-\gamma t}}_{e^{-1}}$$



$\gamma\tau=1$, $\quad E(t)=0.368\,E_0$

$$\frac{b}{m}\tau=1 \Rightarrow \boxed{\tau=\frac{m}{b}}$$

$$E(t)=E_0 e^{-\gamma t}. \Rightarrow \frac{dE}{dt}=-\gamma E_0 e^{-\gamma t}=-\gamma E$$

Q - factor of the oscillator :-

$$Q=\frac{\text{Energy Stored in the oscillator}}{\text{Energy dissipated per radian}}=\frac{E}{\frac{\gamma E}{\omega}}=\frac{\omega}{\gamma}\approx\frac{\omega_0}{\gamma}$$

$$\boxed{\frac{dE}{dt}=-\gamma E} \Rightarrow \Delta E\simeq\left|\frac{dE}{dt}\right|\Delta t\simeq\gamma E\Delta t\simeq\frac{\gamma E}{\omega}$$

$$\omega\Delta t=1. \Rightarrow \boxed{\Delta t=\frac{1}{\omega}}$$

$$\omega=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}$$

γ very small

$\omega\approx\omega_0$

$\gamma=b/m$

$$\omega=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}$$

Lightly

$$Q=\frac{\omega}{\gamma}\approx\frac{\omega_0}{\gamma}\gg 1$$

Lightly

$Q = \frac{[illegible]}{\gamma} = \frac{[illegible]}{\gamma} \gg 1$

$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

x