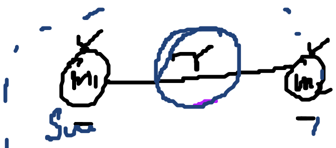
Lecture 17: Section 2

Tuesday, February 15, 2022 11:12 AM

Central force problem: Two body



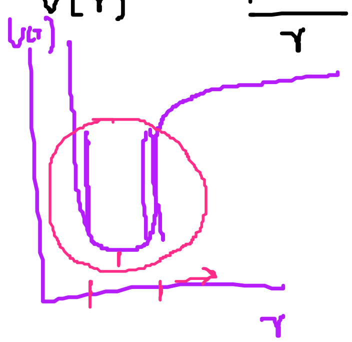
$$F_G = \frac{G m_1 m_2}{r^2}$$

$$E = \underline{K.E} + \underline{P.E} \Rightarrow \underline{P}.E = E - K.E$$

- $E > 0$: $E = \underline{K.E} + \underline{P.E}$ Unbound
- $E < 0$:
- $E = 0$

$$V(r) = -\frac{G m_1 m_2}{r}$$

$r = r_1 - r_2$



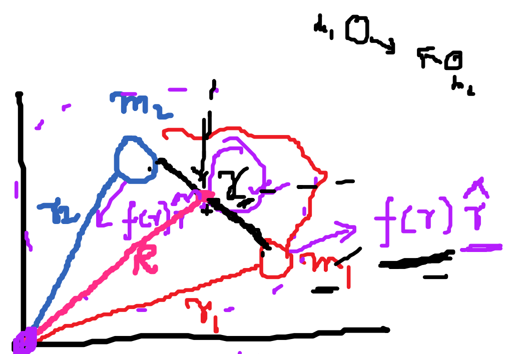
m_1 = Sun

m_2 = Earth

Problem: $\boxed{r(t) \quad \theta(r) \quad \theta(t)}$

$$\boxed{\vec{r} = \vec{r}_1 - \vec{r}_2}$$

$r(t)$ ↓ V_{eff} E, μ l



Equation of Motion:

$$m_1 \ddot{r}_1 = f(r)\,\hat{r}$$

$$m_2 \ddot{r}_2 = -f(r)\,\hat{r}$$

$$\Rightarrow \ddot{r}_1 - \ddot{r}_2 = \ddot{r} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\,\hat{r}$$

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}} \Rightarrow \boxed{\mu \ddot{r} = f(r)\,\hat{r}} \to r = ?$$

$\boxed{r_1 - r_2 = r}$ $\Rightarrow r_2 = r_1 - r$, $\boxed{r_1 = r + r_2}$

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}, \quad \ddot{R} = 0, \quad R = R_0 + v t$$

$r + r_2$; $(r_1 - r)$; 0 ; 0

$$R = \frac{(m_1 + m_2) r_1}{m_1 + m_2} - \frac{m_2 r}{m_1 + m_2} \qquad R = 0$$

$$\boxed{r_1 = R + \frac{m_2 r}{m_1 + m_2}}$$

$$r_1 = R + \frac{m_2 r}{m_1+m_2}$$

$$r_2 = R - \frac{m_1 r}{m_1+m_2}$$

$$\frac{\cancel{m_1+m_2}}{} - \frac{m_2 r}{m_1+m_2}$$

$$r = ?$$

General property of Central force Motion :→

$$\vec{F}_{ext} = \frac{d\vec{p}}{dt}, \quad \vec{\tau} = \frac{d\vec{L}}{dt}, \quad \vec{L} = \vec{r}\times\vec{p} \qquad |L|$$

p = constant $\qquad \boxed{\vec{L} = \text{Constant}}$

$$p = mv = m[v_r + v_\theta]$$

$$f(r)\,\hat{r}$$

$$\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times f(r)\hat{r} = r\hat{r}\times f(r)\hat{r} = 0$$

Radial : $\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$

Tangential : $\mu[r\ddot{\theta} + 2\dot{r}\dot{\theta}] = 0$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}, \quad \vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$v^2 = \vec{v}\cdot\vec{v}$$

r^2 ?

$$\vec{E} = K.E + P.E$$

$$= \frac{1}{2}\mu v^2 + U(r) = \frac{1}{2}\mu[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}]^2 + U(r)$$

$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r), \qquad \dot{\theta} = \frac{d\theta}{dt} = \omega$$

$\vec{L}$, θ $\qquad$ l

$$\vec{L} = r\times\vec{p}$$

$$E|L| = \vec{r}\times\vec{p} = \vec{r}\times\mu\vec{V}_\theta = \mu r^2\dot{\theta} \leftarrow$$

$$|L| = \vec{r} \times \vec{p} = r \times \mu v_\theta = \mu r^2 \dot{\theta}$$

$L = r \times p$

$\mu [v_r + v_\theta]$

$$\Rightarrow \boxed{\dot{\theta} = \frac{l}{\mu r^2}}$$

$$\vec{E} = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\cdot \mu r^2 \cdot \frac{l^2}{\mu^2 r^4} + U(r)$$

$(+ve)$

$r = r_1 - r_2$

$U(r)$

$$+ \underbrace{\frac{1}{2}\frac{l^2}{\mu r^2} + U(r)}$$

$$\vec{E} = \frac{1}{2}\mu \dot{r}^2 + U_{eff}$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff})} \quad \checkmark$$

$$\Rightarrow \boxed{\int \frac{dr}{\sqrt{\frac{2}{\mu}(E - V_{eff})}} = \int_{t_0}^{t} dt = t - t_0}$$

$r(t)$

$$\dot{\theta} = \frac{l^2}{\mu r^2} \Rightarrow \frac{d\theta}{dt} = \frac{l^2}{\mu r^2}$$

$$\Rightarrow \boxed{\int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = \int_{t_0}^{t} \frac{l^2}{\mu r^2} dt}$$

$\theta(t)$

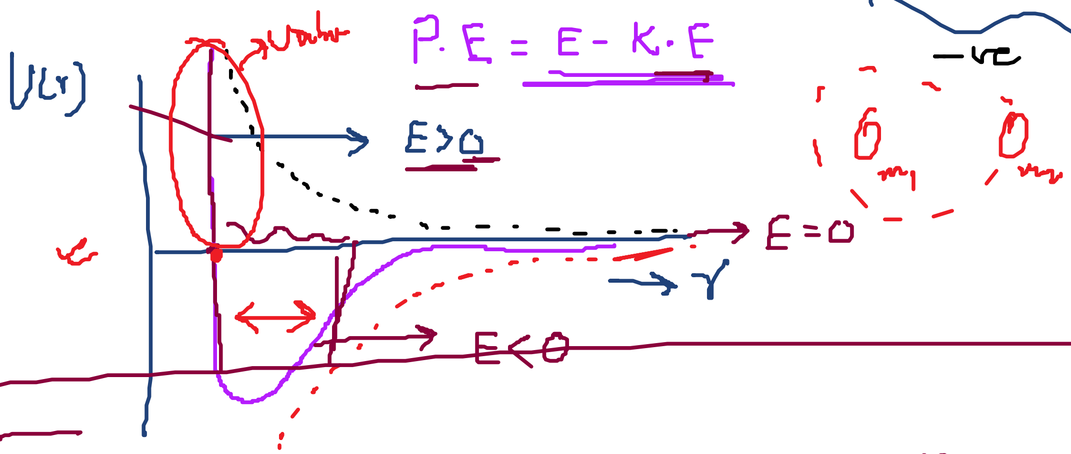
$$\frac{d\theta}{dr} = \frac{d\theta}{dt} \cdot \frac{1}{\frac{dr}{dt}} = \frac{l^2}{\mu r^2} \cdot \frac{1}{\sqrt{\frac{2}{\mu}(E - U_{eff})}}$$

$\theta(r)$

$$\boxed{r(t),\ \theta(t),\ \theta(r)}$$

$$U_{eff} = \underline{U(r)} + \frac{\ell^2}{2\mu r^2} = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

$-ve$ $+ve$



$r \to$ small

$r \to$ large.

$$-\frac{dU}{dr} = F$$

Ex: $F = -\frac{A}{r^2}$, $U = \frac{A}{r}$

$F = -\frac{dU}{dr}$

$F = -kx$, $U = -\int F\,dx$

$U = \frac{1}{2}kx^2$

$$dU = -\int F\,dr$$

$$= -\int A r^{-2}\,dr$$

$$= (-A)\cdot\left(-\frac{1}{r}\right) = \frac{A}{r}.$$