# Mathematical Induction

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Sum of first $n$ positive integers is $n(n+1)/2$.
Sum of squares first $n$ positive integers is $n(n+1)(2n+1)/6$.

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*Inductive step: For every $k$, if we assume that $P(k)$ is true, then $P(k+1)$ is also true.*

$$\forall n(P(n)) \text{ iff } (P(1) \wedge \forall k(P(k) \Rightarrow P(k+1))$$

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Remark: To use mathematical induction, we need correct statement/conjecture.

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Let $2(n+1)+1$ people be standing at mutually distinct distances.

Let $A$ and $B$ be the people standing closest.

Hence $A$ and $B$ will throw ball at each other.

If either $A$ or $B$ is closer to some other people, then $A$ and $B$ will get hit by at least 3 ball.
No. of balls thrown at remaining people $2n+1$ is at most $2n$.
This guarantees that there is at least one survivor.
What if nobody throws ball at $A$ or $B$?

$$\text{Let } H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

Show that $H_{2^n} \geq 1 + \frac{n}{2}$ for each $n \in \mathbb{N}$.

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Show that $H_{2^n} \geq 1 + \frac{n}{2}$ for each $n \in \mathbb{N}$.
Show that if a finite set $S$ has $n$ elements then $P(S)$ has $2^n$ elements.

# Strong Form of Mathematical Induction

Equivalent formulation of principle of mathematical induction:

**Theorem (Strong induction)**

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1. Show that if $n$ is a natural number greater than 1, then $n$ can be written as a product of primes.

2. Consider a game in which two players take turns removing any positive number of matchsticks they want from one of two piles of matchsticks. The player who removes the last matchsticks wins the game. Show that if the two piles contain the same number of matchsticks initially, the second player can always guarantee a win.

3. Use strong induction to show that $\sqrt{2}$ is irrational.

For a natural number $n$, $n/m \neq \sqrt{2}$ for any natural number $m$.

# Recursively Defined Functions

Consider a function $f : \mathbb{N} \to \mathbb{N}$ as $f(n) = n!$.
Redefine $f$ as $f(1) = 1$ and $f(n) = (n) * f(n-1)$. This is called recursive definition of $f$.

**Definition**

A function $f$ with domain $\mathbb{N}$ is called recursively defined if $f(n)$ can be expressed in terms of $f(n-1), f(n-2), ..., f(1)$ and some function of $n$, for each $n \geq 2$.

Identify the following function:
1. $f(1) = 1, f(n) = f(n-1) + n$.
2. $f(1) = 5, f(n) = 5 * f(n-1)$
3. $f(1) = 9, f(n) = 2 * f(n-1) + 3$

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2. $f(1) = 5, f(n) = 5 * f(n-1)$
3. $f(1) = 9, f(n) = 2 * f(n-1) + 3$ Verify $f(n) = 3.2^{n+1} - 3$.

**Theorem (Lame's theorem)**

*The number of division steps in Euclid's algorithm with entries $a, b$ with $a \geq b$ is less than 5 times the number of decimal digits of $b$.*

# Recursively Defined Sets

Consider $A = \{1, 3, 5, 7, ....\}$.
We can define $A$ as follows: $1 \in A$. Whenever $x \in A$, $x + 2 \in A$.
Let $B$ be the set of all binary strings. Then $B$ can be defined as:
$0, 1 \in B$. If $X \in B$ then $X0, X1 \in B$.
OR
$0, 1 \in B$. If $X, Y \in B$ then $XY, YX \in B$.