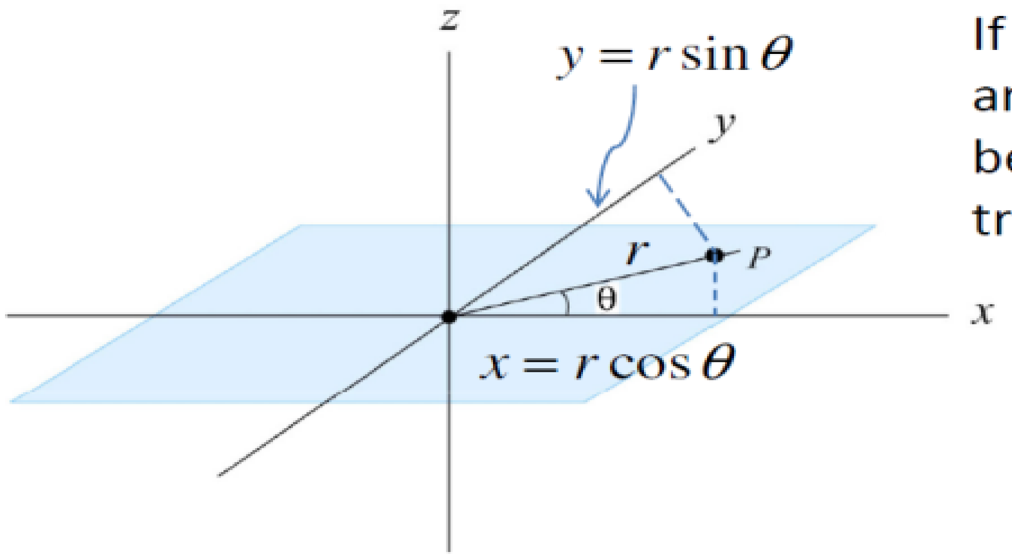
PH100: Mechanics and Thermodynamics

Lecture 3

Ajay Nath

Polar and Cartesian coordinates:



If polar coordinates (r, θ) of a point in the plane are given, the Cartesian coordinates (x, y) can be determined from the coordinate transformations

$$x = r\cos\theta \qquad r = +(x^2 + y^2)^{1/2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

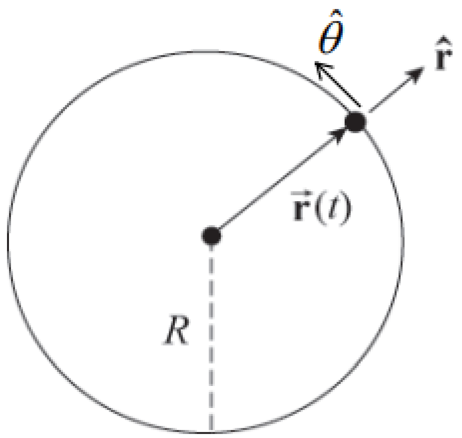
Note: $r \geq 0$ so take the positive square root only.

Since $\tan\theta = \tan(\theta + \pi)$

For $0 \leq \theta \leq \pi/2$ $x \geq 0$ and $y \geq 0$

For $(-x, -y)$ take $\theta + \pi$

Unit Vectors in Polar coordinates



The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

In Cartesian coordinate: $\vec{r} = x\hat{i} + y\hat{j}$

By coordinate transformations:
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore: $\vec{r} = r\cos\theta\hat{i} + r\sin\theta\,\hat{j}$

The unit vectors are defined as :

$$\hat{r} = \frac{\partial\vec{r}/\partial r}{\left|\partial\vec{r}/\partial r\right|} = \cos\theta\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{r}\times\hat{\theta} = \hat{k}$$

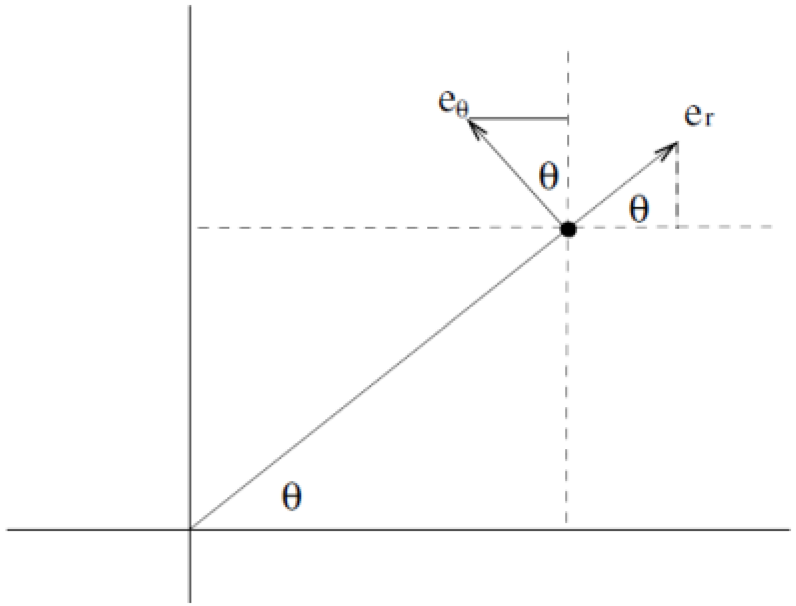
$$\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$
$$\hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{\left|\partial\vec{r}/\partial\theta\right|} = -\sin\theta\hat{i} + \cos\theta\,\hat{j}$$

Unit Vectors in Polar coordinates

Define at each
point, a set of two unit vectors
$\hat{\mathbf{r}}$ and $\hat{\theta}$ as shown in the figure.

$$\hat{\mathbf{r}} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$
$$\hat{\theta} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$



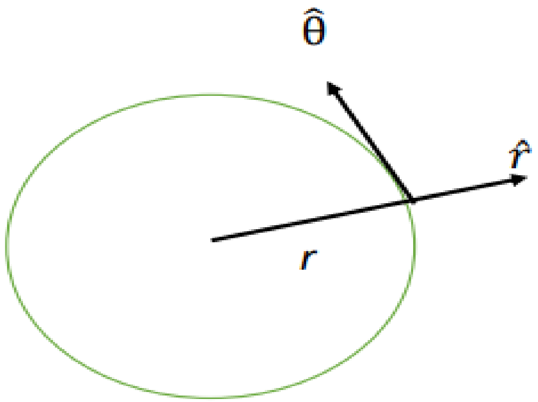
Unit vectors only depend on θ

unit vectors are functions of the polar coordinates

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\theta}$$
$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{\mathbf{r}}$$

Velocity and acceleration in polar coordinates

Velocity in polar coordinate:



The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

And the unit vectors are: $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ & $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$

Since the unit vectors are not constant and changes with time, they should have finite time derivatives:

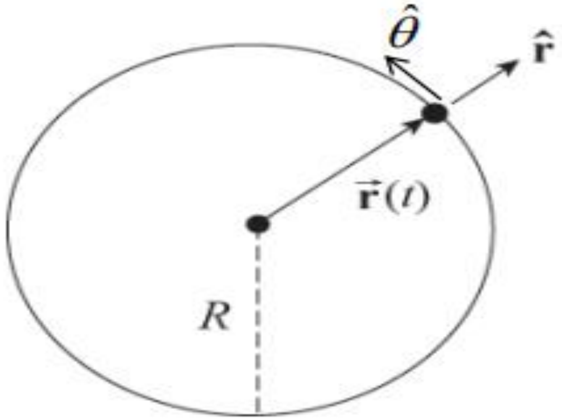
$$\dot{\hat{r}} = \dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = \dot{\theta}\hat{\theta} \quad \text{and} \quad \dot{\hat{\theta}} = \dot{\theta}\left(-\cos\theta\hat{i} - \sin\theta\hat{j}\right) = -\dot{\theta}\hat{r}$$

Therefore the velocity is given by: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Radial velocity + tangential velocity

In Cartesian coordinates $= \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$

Example-1: Uniform Circular Motion



$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Since $\dot{r} = \dfrac{dR}{dt} = 0$ and $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$

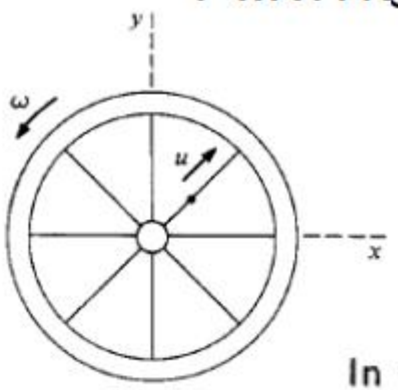
$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\,\omega\,\hat{\boldsymbol{\theta}}(t)$$

Since $\vec{v}$ is along $\hat{\theta}$ it must be perpendicular to the radius vector $\vec{r}$ and it can be shown easily

$$R^2 = \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} \qquad \frac{d}{dt}R^2 = \frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 0, \qquad \vec{\mathbf{r}} \perp \vec{\mathbf{v}}$$

Why polar coordinates?

Example-2: **Velocity of a Bead on a Spoke**



A bead moves along the spoke of a wheel at constant speed u meters per second. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the x axis, and the bead is at the origin. Find the velocity at time t

In polar coordinates : $r = ut$, $\dot{r} = u$, $\dot{\theta} = \omega$. Hence

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

To specify the velocity completely, we need to know the direction of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. This is obtained from $\mathbf{r} = (r,\theta) = (ut,\omega t)$.

In cartesian coordinates. : $v_x = v_r \cos\theta - v_\theta \sin\theta \qquad v_y = v_r \sin\theta + v_\theta \cos\theta.$

Since $v_r = u$, $v_\theta = r\omega = ut\omega$, $\theta = \omega t$,

$$\mathbf{v} = (u\cos\omega t - ut\omega\sin\omega t)\hat{\mathbf{i}} + (u\sin\omega t + ut\omega\cos\omega t)\hat{\mathbf{j}}$$

Note how much simpler the result is in plane polar coordinates.

Symmetry is important.

Acceleration in Polar coordinate:

$$\mathbf{a} = \frac{d}{dt}\mathbf{v}$$

$$= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta},\ \dot{\hat{\theta}} = -\dot{\theta}\hat{r}}$$

$$\mathbf{a} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

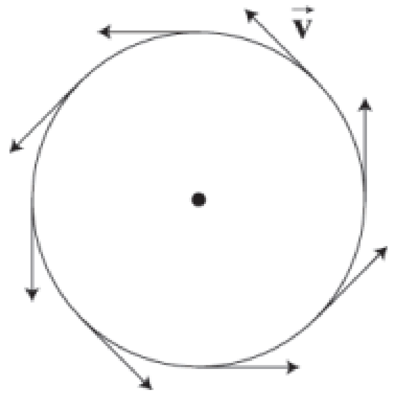
The term $\ddot{r}\hat{\mathbf{r}}$ is a linear acceleration in the radial direction due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$ is a linear acceleration in the tangential direction due to change in the magnitude of the angular velocity.

The term $-r\dot{\theta}^2\hat{\mathbf{r}}$ is the centripetal acceleration

Finally, the Coriolis acceleration $2\dot{r}\dot{\theta}\hat{\theta}$

Usually, Coriolis force appears as a fictitious force in a rotating coordinate system. However, the Coriolis acceleration we are discussing here is a real acceleration and which is present when *r* and θ both change with time.

Example-1: Circular motion



$$\vec{a} = a_r \hat{r} + a_\theta \hat{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

For a circular motion, $r = R$, the radius of the circle.

Hence, $\dot{r} = \ddot{r} = 0$

So, $a_\theta = R\ddot{\theta}$ and $a_r = -R\dot{\theta}^2$

For uniform circular motion, $\dot{\theta} = \omega =$ constant. Hence, $a_\theta = R\dfrac{d\omega}{dt} = 0$

For non-uniform circular motion, ω is function of time. Hence, $a_\theta = R\dfrac{d\omega}{dt} = R\alpha$,

where $\alpha = \dfrac{d\omega}{dt}$ is the angular acceleration.

However, the radial acceleration is always $a_r = -R\dot{\theta}^2 = -R\omega^2$

Therefore, an object traveling in a circular orbit with a constant speed is always accelerating towards the center. Though the magnitude of the velocity is a constant, the direction of it is constantly varying. Because the velocity changes direction, the object has a nonzero acceleration.

Example-2: **Acceleration of a Bead on a Spoke**

A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is a constant. Find the velocity and acceleration.

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

We are given that $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position is given by $r = ut$, and we have

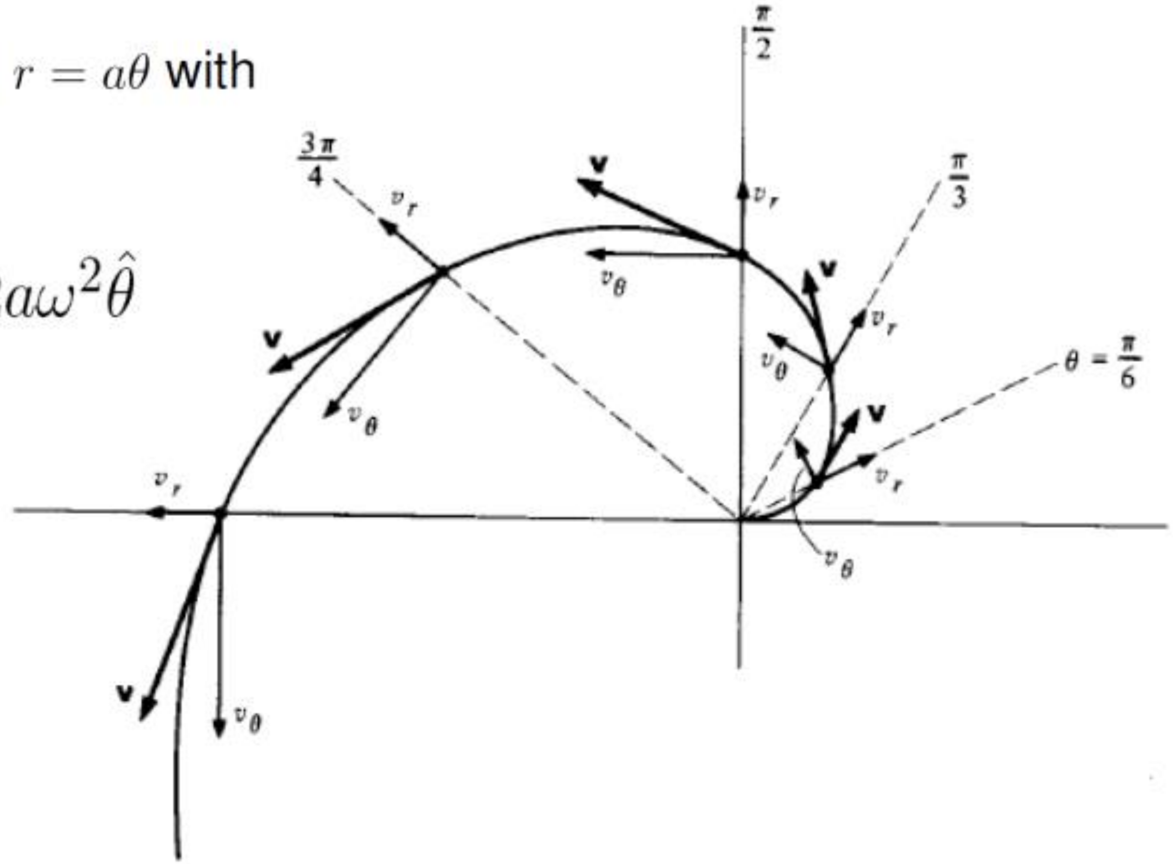
$$\mathbf{v} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned} \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}. \end{aligned}$$

Consider a particle moving on a spiral given by $r = a\theta$ with a uniform angular speed ω. Then $\dot{r} = a\dot{\theta} = a\omega$.

$$\mathbf{v} = a\omega\hat{\mathbf{r}} + a\omega^2 t\hat{\theta} \text{ and } \mathbf{a} = -a\omega^3 t\hat{\mathbf{r}} + 2a\omega^2\hat{\theta}$$



The velocity is shown in the sketch for several different positions of the wheel. Note that the radial velocity is constant. The tangential acceleration is also constant—can you visualize this?

Though the magnitude of radial velocity is constant there is a radial acceleration.

Motion: Kinematics in 1D

The motion of the particle is described specifying the position as a function of time, say, $x(t)$.

The instantaneous velocity is defined as

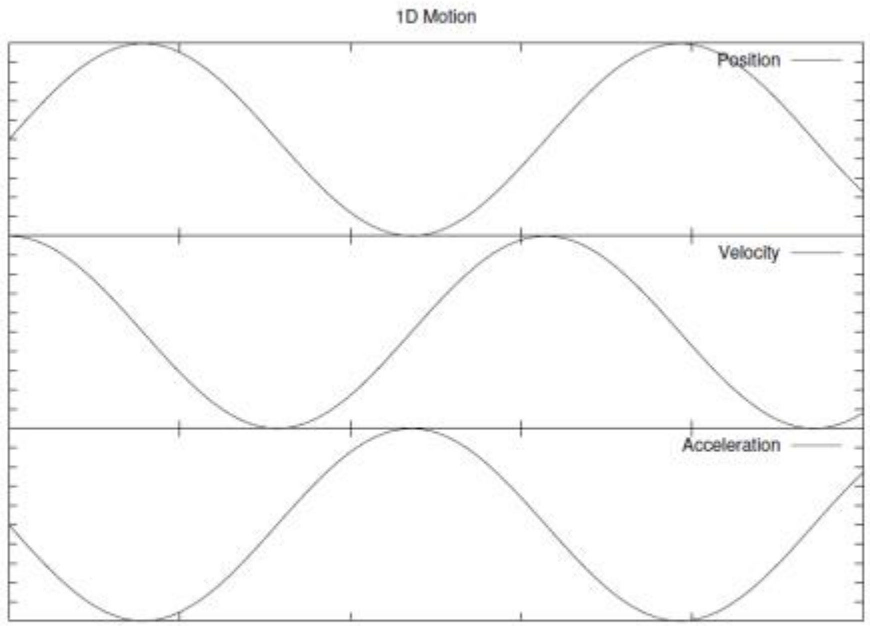
(1) $$v(t) = \frac{dx}{dt}$$

and instantaneous acceleration, as

(2) $$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example

If $x(t) = \sin(t)$, then $v(t) = \cos(t)$ and $a(t) = -sin(t)$.



Usually the $x(t)$ is not known in advance!

But the acceleration $a(t)$ is known and at some given time, say t_0, position $x(t_0)$ and velocity $v(t_0)$ are known.

The formal solution to this problem is

$$\begin{aligned} v(t) &= v(t_0) + \int_{t_0}^{t} a(t')\,dt' \\ x(t) &= x(t_0) + \int_{t_0}^{t} v(t')\,dt' \end{aligned}$$

Let the acceleration of a particle be a_0, a constant at all times. If, at $t = 0$ velocity of the particle is v_0, then

$$\begin{aligned} v(t) &= v_0 + \int_0^t a_0 dt \\ &= v_0 + a_0 t \end{aligned}$$

And if the position at $t = 0$ is x_0,

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2$$

More complex situations may arise, where an acceleration is specified as a function of position, velocity and time. $a(x, \dot{x}, t)$. In this case, we need to solve a differential equation

$$\frac{d^2x}{dt^2} = a(x, \dot{x}, t)$$

which may or may not be simple.

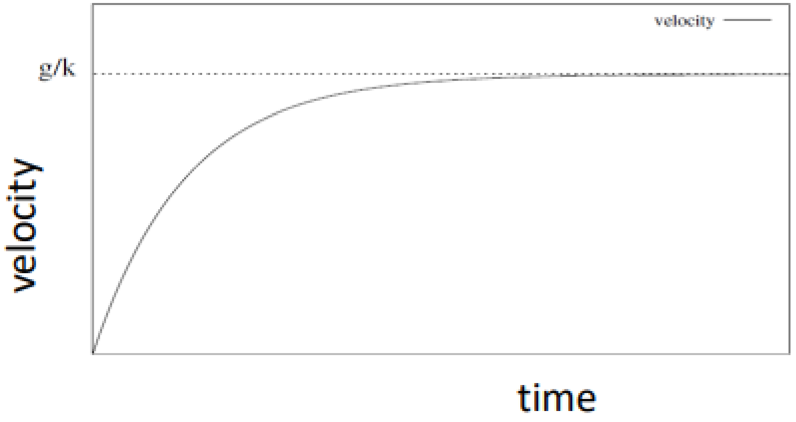
Example

Suppose a ball is falling under gravity in air, resistance of which is proportional to the velocity of the ball.

$$a(\dot{y}) = -g - k\dot{y}$$

If the ball was just dropped, velocity of the ball after time then

$$v(t) = -\frac{g}{k}\left(1 - e^{-kt}\right)$$



Kinematics in 2D

The instantaneous velocity vector is defined as

$$\begin{aligned}
\mathbf{v}(t) &= \frac{d}{dt}\mathbf{r} \\
&= \lim_{dt\to 0}\frac{\mathbf{r}(t+dt)-\mathbf{r}(t)}{dt} \\
&= \lim_{dt\to 0}\frac{x(t+dt)-x(t)}{dt}\mathbf{i} + \lim_{dt\to 0}\frac{y(t+dt)-y(t)}{dt}\mathbf{j} \\
&= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}
\end{aligned}$$

The instantaneous acceleration is given by:

$$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

Kinematics in 2D

In this case, we have to solve two differential equations

$$\frac{d^2x}{dt^2} = a_x$$
$$\frac{d^2y}{dt^2} = a_y$$

Example

A ball is projected at an angle θ with a speed u. The net acceleration is in downward direction. Then $a_x = 0$ and $a_y = -g$. The equations are

$$\frac{d^2x}{dt^2} = 0$$
$$\frac{d^2y}{dt^2} = -g$$

Charge particle in a magnetic field

A particle has a velocity v in XY plane. Magnetic field is in z direction The acceleration is given by $\frac{q}{m}\mathbf{v} \times \mathbf{B}$

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{qB}{m}v_y \\ \frac{d^2y}{dt^2} &= -\frac{qB}{m}v_x\end{aligned}$$

Solution is rather simple, that is circular motion in xy plane.

Thank You