Introduction to Quantum Mechanics

Lecture II

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Blackbody Radiation Problem:

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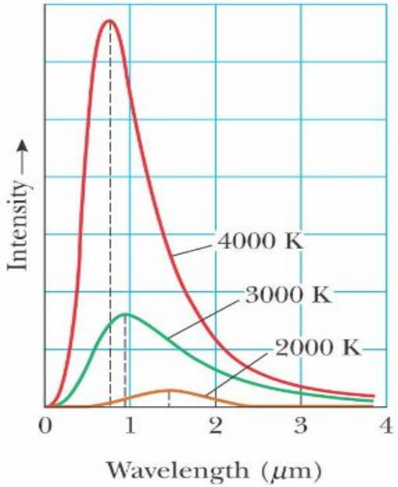
- Hot filament glows.
- Classical physics cant explain the observed wavelength distribution of EM radiation from such a hot object.
- This problem is historically the problem that leads to the rise of quantum physics during the turn of 20th century

- The intensity increases with increasing temperature
- The amount of radiation emitted increases with increasing temperature
  - The area under the curve
- The peak wavelength decreases with increasing temperature



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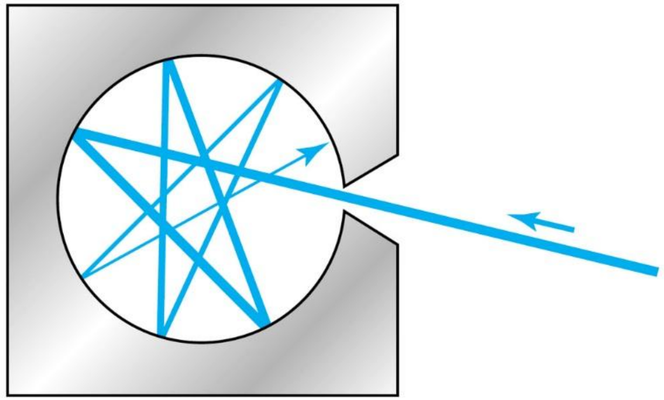
- An object at any temperature is known to emit thermal radiation
  - Characteristics depend on the temperature and surface properties
  - The thermal radiation consists of a continuous distribution of wavelengths from all portions of the em spectrum

- At room temperature, the wavelengths of the thermal radiation are mainly in the infrared region
- As the surface temperature increases, the wavelength changes
  - It will glow red and eventually white
- The basic problem was in understanding the observed distribution in the radiation emitted by a black body
  - Classical physics didn't adequately describe the observed distribution

Blackbody Radiation Problem

- When matter is heated, it emits radiation.
- A blackbody is a cavity in a material that only emits thermal radiation. Incoming radiation is absorbed in the cavity. Material is dense, so we expect a continuous spectrum



emissivity ϵ ($\epsilon = 1$ for idealized black body)

- Blackbody radiation is interesting because the radiation properties of the blackbody are independent of the particular material! Physicists can study the distribution of intensity versus wavelength (or frequency) at fixed temperatures. Principle of a pyrometer to measure temperatures remotely.

Stefan-Boltzmann Law

- Empirically, total power radiated (per unit area and unit wavelength = m^{-3}) increases with temperature to power of 4:

$$R(T) = \int_0^\infty \mathcal{I}(\lambda, T)\, d\lambda = \epsilon\, \sigma T^4$$

Watt per m^2

- This is known as the **Stefan-Boltzmann law**, with the constant
- $\sigma = 5.6705 \times 10^{-8}$ W / (m^2 · K^4)
- The **emissivity** ϵ (ϵ = 1 for an idealized black body) is simply the ratio of the emissive power of an object to that of an ideal blackbody and is always less than 1.

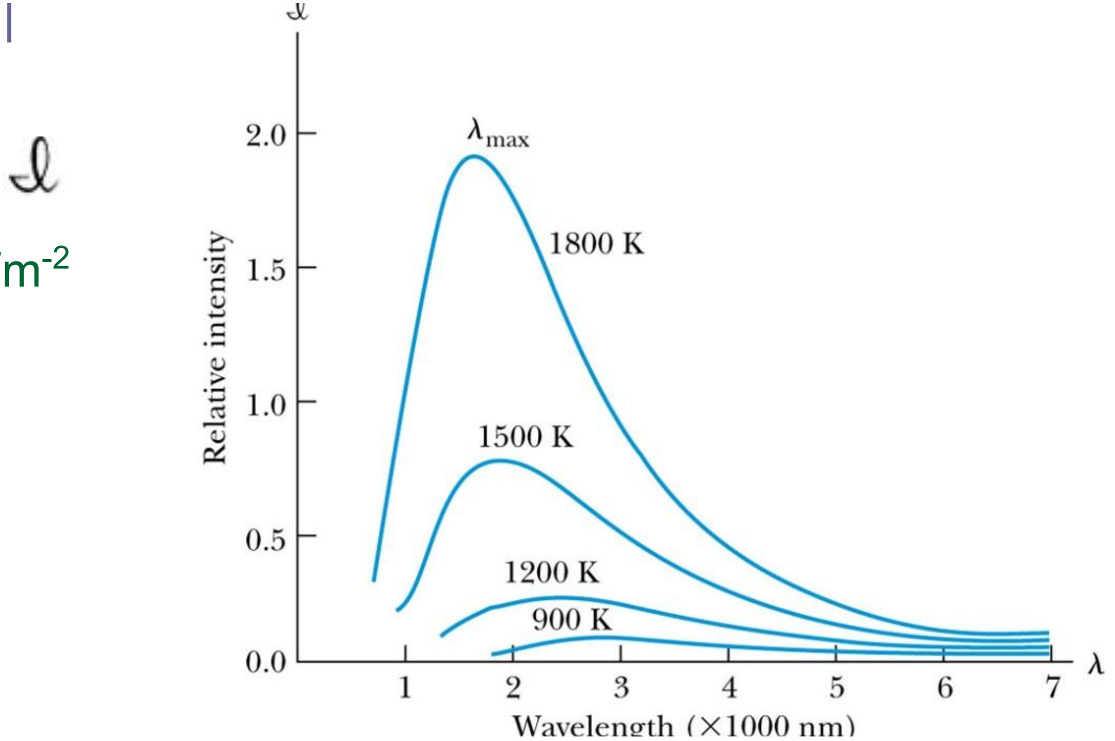
Wien's Displacement Law

- The intensity (λ, *T*) is the total power radiated per unit area per unit wavelength at a given temperature.
- **Wien's displacement law**: The maximum of the distribution shifts to smaller wavelengths as the temperature is increased.

$$\lambda_{max} T = 2.898 \times 10^{-3}\ \mathrm{m \cdot K}$$

Most interesting, what is the mathematical function that describes all of these curves??

Wm^{-2}



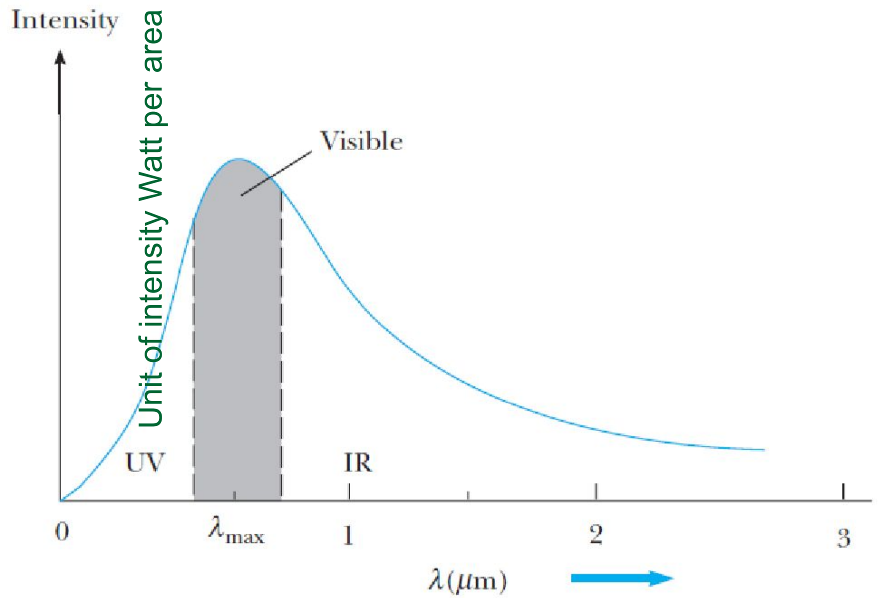


Figure 4.14 Intensity versus wavelength for a body heated to 6000 K.

So that must be approximately the black body radiation from the sun.

Why are our eyes particularly sensitive to green light of 550 nm? Because life evolved on earth receiving radiation from the sun that peaks at this particular wavelength.

https://phet.colorado.edu/sims/html/blackbody-spectrum/latest/blackbody-spectrum_en.html

Rayleigh-Jeans Formula

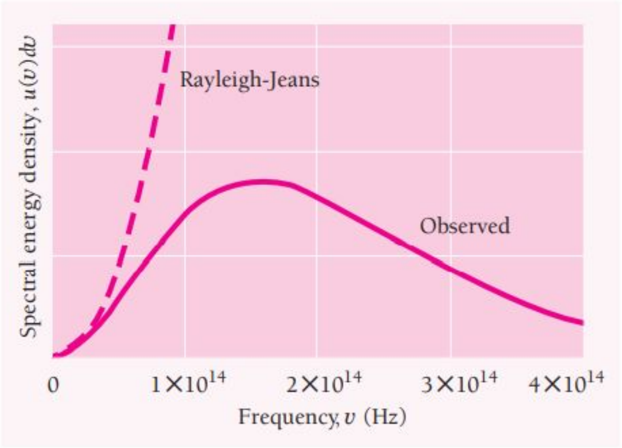
- Lord Rayleigh (John Strutt) and James Jeans used the classical theories of electromagnetism and thermodynamics to show that the blackbody spectral distribution should be, 1905

Rayleigh-Jeans formula

$$u(\nu)\, d\nu = \overline{\epsilon} G(\nu)\, d\nu = \frac{8\pi kT}{c^3} \nu^2\, d\nu$$

With k as Boltzmann constant: 1.38065 10^{-23} J/K

kT at room temperature (293 K) ≈ 25 meV



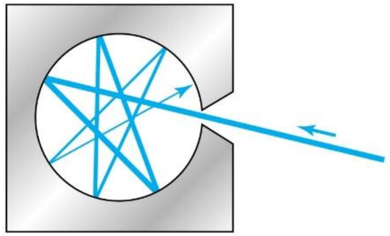
- It approaches the data at longer wavelengths, but it deviates very badly at short wavelengths. This problem for small wavelengths became known as “the ultraviolet catastrophe” and was one of the outstanding problems that classical physics could not explain around 1900s.

How Rayleigh-Jeans Formula calculated?

Consider a blackbody as a radiation-filled cavity at the temperature T. Because the cavity walls are assumed to be perfect reflectors, the radiation must consist of standing em waves.

In order for a node to occur at each wall, the path length from wall to wall, in any direction, must be an integral number *j* of half wavelengths.

If the cavity is a cube L long on each edge, this condition means that for standing waves in the *x, y,* and *z* directions respectively, the possible wavelengths are such that



$$j_x = \frac{2L}{\lambda} = 1, 2, 3, \ldots = \text{number of half-wavelengths in } x \text{ direction}$$

$$j_y = \frac{2L}{\lambda} = 1, 2, 3, \ldots = \text{number of half-wavelengths in } y \text{ direction}$$

$$j_z = \frac{2L}{\lambda} = 1, 2, 3, \ldots = \text{number of half-wavelengths in } z \text{ direction}$$

For a standing wave in any arbitrary direction, it must be true that

Standing waves in a cubic cavity

$$j_x^2 + j_y^2 + j_z^2 = \left(\frac{2L}{\lambda}\right)^2 \qquad \begin{aligned} j_x &= 0, 1, 2, \ldots \\ j_y &= 0, 1, 2, \ldots \\ j_z &= 0, 1, 2, \ldots \end{aligned}$$

To count the number of standing waves $g(\lambda)\, d\lambda$ within the cavity whose wavelengths lie between λ and $\lambda + d\lambda$, what we have to do is count the number of permissible sets of j_x, j_y, j_z values that yield wavelengths in this interval. Let us imagine a j-space whose coordinate axes are j_x, j_y, *and* j_z; Fig shows part of the j_x-j_y plane of such a space. Each point in the j-space corresponds to a permissible set of j_x, j_y, j_z values and thus to a standing wave. If $\mathbf{j}$ is a vector from the origin to a particular point j_x, j_y, j_z, its magnitude is

$$j = \sqrt{j_x^2 + j_y^2 + j_z^2}$$

The total number of wavelengths between λ and $\lambda + d\lambda$ is the same as the number of points in j space whose distances from the origin lie between j and $j + dj$. The volume of a spherical shell of radius j and thickness dj is $4\pi j^2\, dj$, but we are only interested in the octant of this shell that includes non-negative values of j_x, j_y, and j_z. Also, for each standing wave counted in this way, there are two perpendicular

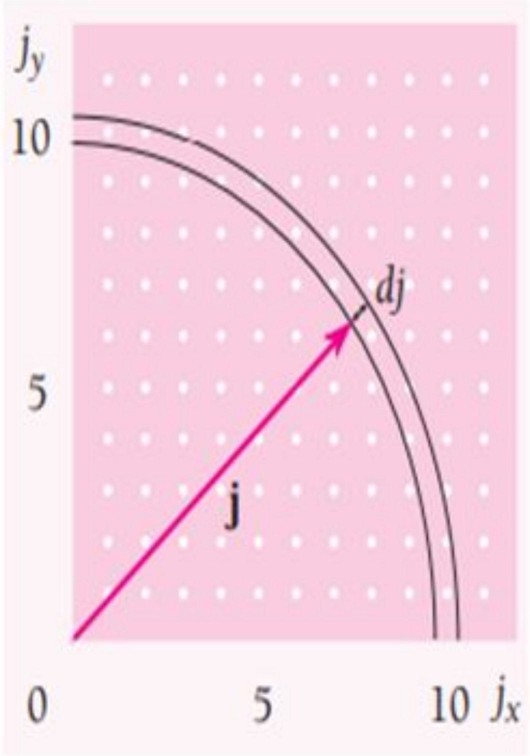


Figure Each point in j space corresponds to a possible standing wave.

directions of polarization. Hence the number of independent standing waves in the cavity is

Number of standing waves

$$g(j)\, dj = (2)(\tfrac{1}{8})(4\pi j^2\, dj) = \pi j^2\, dj$$

What we really want is the number of standing waves in the cavity as a function of their frequency ν instead of as a function of j. From Eqs. (9.31) and (9.32) we have

$$j = \frac{2L}{\lambda} = \frac{2L\nu}{c} \qquad dj = \frac{2L}{c}\, d\nu$$

and so

Number of standing waves

$$g(\nu)\, d\nu = \pi\left(\frac{2L\nu}{c}\right)^2 \frac{2L}{c}\, d\nu = \frac{8\pi L^3}{c^3}\nu^2\, d\nu$$

The cavity volume is L^3, which means that the number of independent standing waves per unit volume is

Density of standing waves in a cavity

$$G(\nu)\, d\nu = \frac{1}{L^3}\, g(\nu)\, d\nu = \frac{8\pi\nu^2\, d\nu}{c^3}$$

The next step is to find the average energy per standing wave. According to the **theorem of equipartition of energy,** a mainstay of classical physics, the average energy per degree of freedom of an entity (such as a molecule of an ideal gas) that is a member of a system of such entities in thermal equilibrium at the temperature T is $\frac{1}{2}kT$. Here k is **Boltzmann's constant:**

Boltzmann's constant
$$k = 1.381 \times 10^{-23} \text{ J/K}$$

A degree of freedom is a mode of energy possession. Thus a monatomic ideal gas molecule has three degrees of freedom, corresponding to kinetic energy of motion in three independent directions, for an average total energy of $\frac{3}{2}kT$.

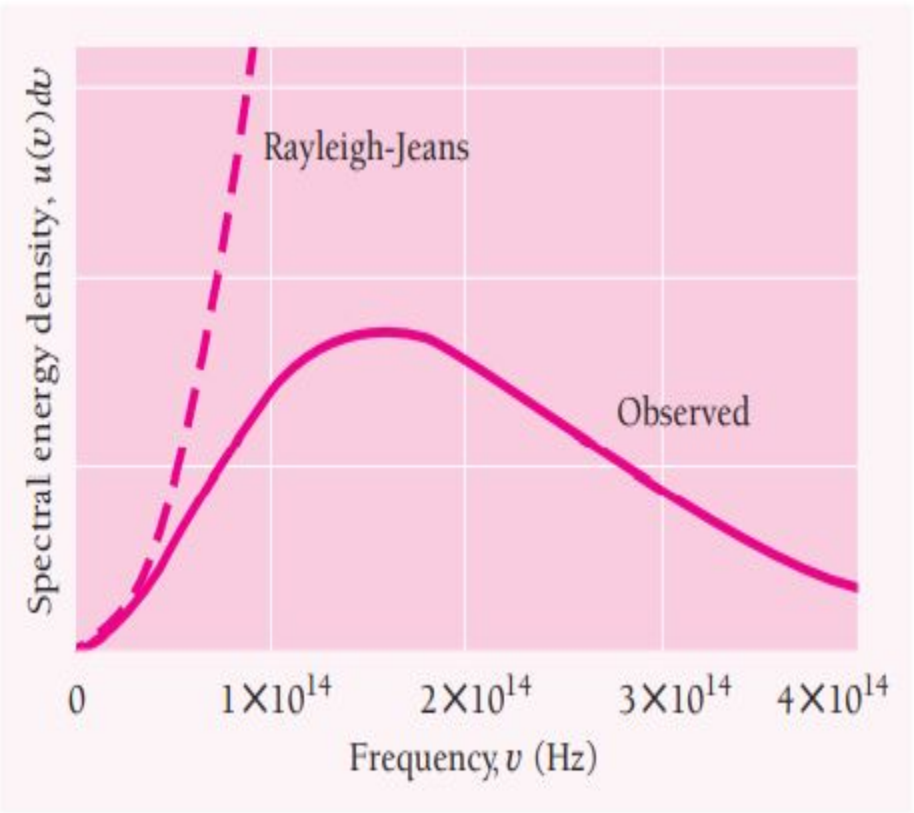
A one-dimensional harmonic oscillator has two degrees of freedom, one that corresponds to its kinetic energy and one that corresponds to its potential energy. Because each standing wave in a cavity originates in an oscillating electric charge in the cavity wall, two degrees of freedom are associated with the wave and it should have an average energy of $2(\frac{1}{2})kT$:

Classical average energy per standing wave
$$\overline{\epsilon} = kT$$

The total energy $u(\nu)\,d\nu$ per unit volume in the cavity in the frequency interval from ν to $\nu + d\nu$ is therefore

Rayleigh-Jeans formula
$$u(\nu)\,d\nu = \overline{\epsilon}G(\nu)\,d\nu = \frac{8\pi kT}{c^3}\nu^2\,d\nu$$

Ultraviolet Catastrophe



Planck Radiation Formula

In 1900 the German physicist Max Planck used "lucky guesswork" (as he later called it) to come up with a formula for the spectral energy density of blackbody radiation:

Planck radiation formula

$$u(\nu)\,d\nu = \frac{8\pi h}{c^3}\frac{\nu^3\,d\nu}{e^{h\nu/kT}-1}$$

Here h is a constant whose value is

Planck's constant

$$h = 6.626 \times 10^{-34}\ \mathrm{J \cdot s}$$

At high frequencies, $h\nu \gg kT$ and $e^{h\nu/kT} \to \infty$, which means that $u(\nu)\, d\nu \to 0$ as observed. No more ultraviolet catastrophe. At low frequencies, where the Rayleigh-Jeans formula is a good approximation to the data , $h\nu \ll kT$ and $h\nu/kT \ll 1$. In general,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

If x is small, $e^x \approx 1 + x$, and so for $h\nu/kT \ll 1$ we have

$$\frac{1}{e^{h\nu/kT}-1} \approx \frac{1}{1 + \frac{h\nu}{kT} - 1} \approx \frac{kT}{h\nu} \qquad h\nu \ll kT$$

Thus at low frequencies Planck's formula becomes

$$u(\nu)\, d\nu \approx \frac{8\pi h}{c^3}\nu^3\left(\frac{kT}{h\nu}\right) d\nu \approx \frac{8\pi kT}{c^3}\nu^2\, d\nu$$

which is the Rayleigh-Jeans formula. Planck's formula is clearly at least on the right track; in fact, it has turned out to be completely correct.