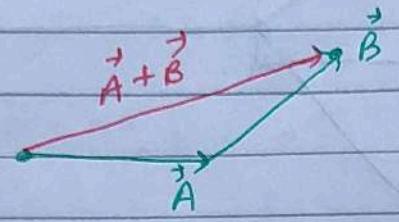
Mechanics

$\vec{F} = m\vec{a}$

$$\begin{pmatrix} F_x = ma_x \\ F_y = ma_y \\ F_z = ma_z \end{pmatrix}$$

unit vectors $\left(\hat{A} = \frac{\vec{A}}{|\vec{A}|} \right)$

Addition / subtraction of vectors

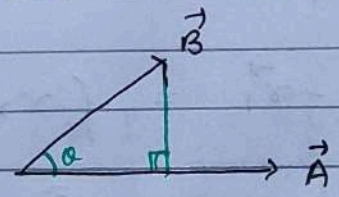


multiplication of vectors

↳ scalar ⇒ $\vec{A} \cdot \vec{B}$ ⇒ gives scalar quantity

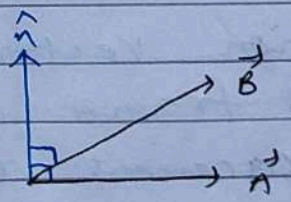
↳ vector ⇒ $\vec{A} \times \vec{B}$ ⇒ gives vector quantity

- $\vec{A} \cdot \vec{B} = AB\cos\theta$

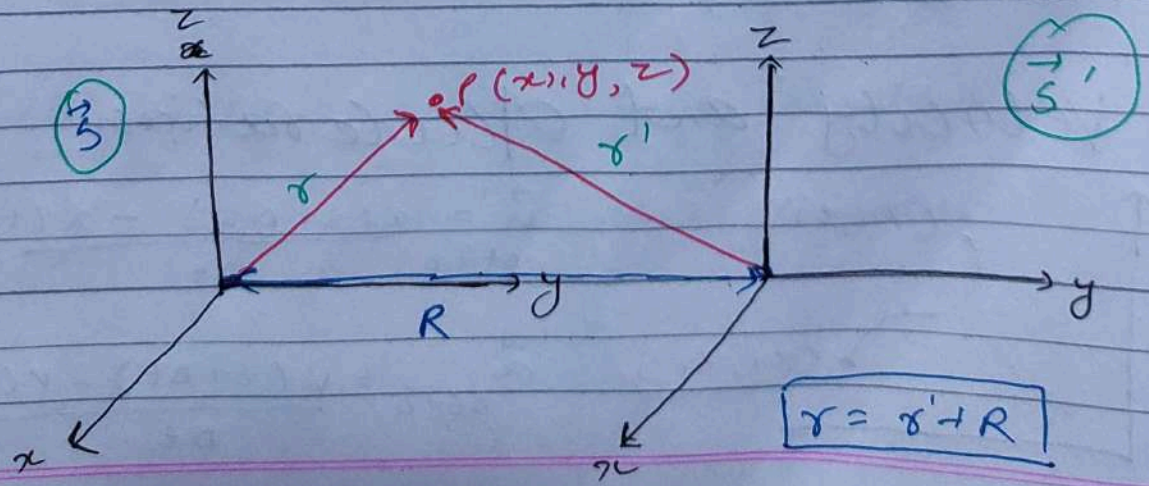


- $\vec{A} \times \vec{B} = AB\sin\theta\ \hat{n}$

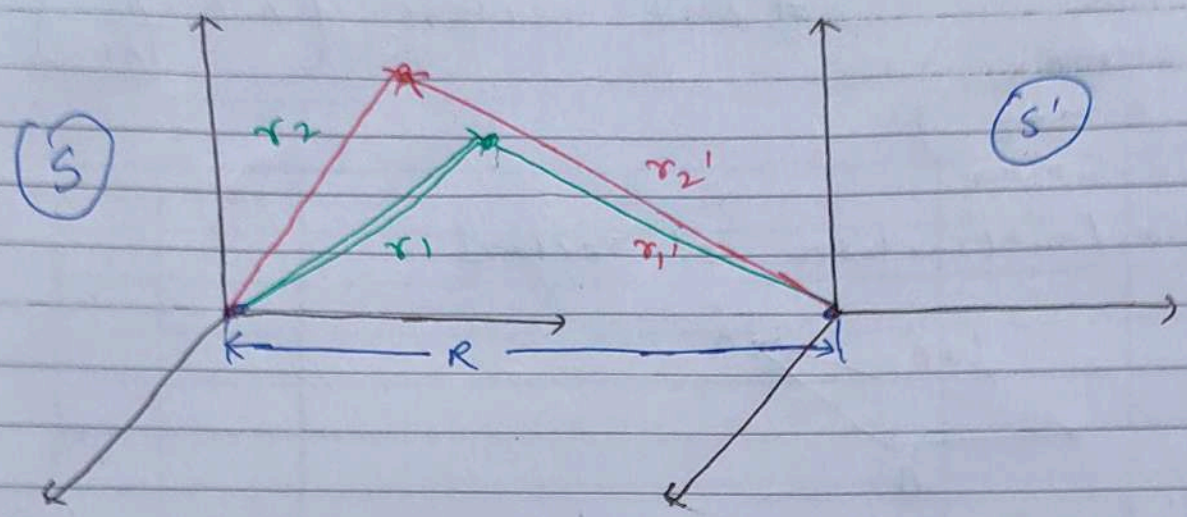
  $\hat{n}$ → ⊥r to the plane



cartesian co-ordinate system



$$\boxed{r = r' + R}$$



(S) $s = \vec{r_2} - \vec{r_1}$

(S') $s = \vec{r_2}' - \vec{r_1}'$

(S) $s = \vec{r_2} - \vec{r_1} = (r_2' + R) - (r_1' + R)$

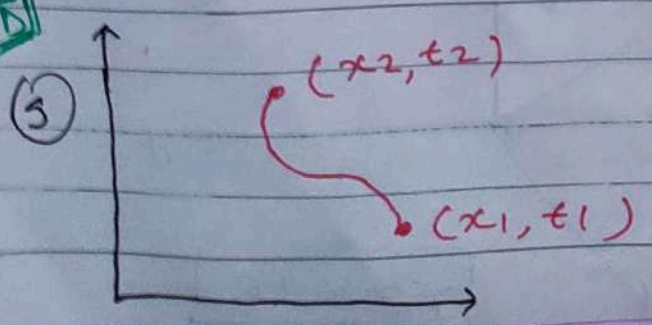
$= \vec{r_2}' - \vec{r_1}'$

↳ The position vector will be different for different frame of reference.

↳ The displacement vector will be same for different frame of reference

(displacement vector is Independent of frame of Reference)

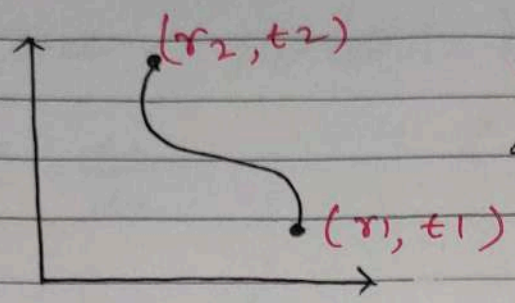
Velocity and Acceleration



$$\vec{v}_{\Delta t \to 0} = \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$\vec{a}_{\Delta t \to 0} = \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

3D



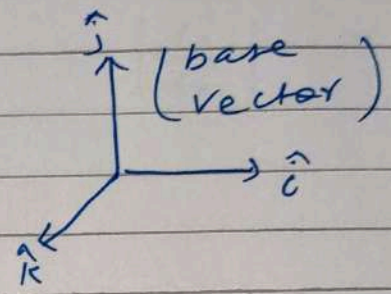
$$\vec{v} \underset{\Delta t \to 0}{=} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

$$\vec{a} \underset{\Delta t \to 0}{=} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

$r_i \Rightarrow (x_i, y_i, z_i)$

Ex:

$\vec{r} = A\left[e^{\alpha t}\hat{i} + e^{-\alpha t}\right]\hat{j}$



$$\vec{v} = \frac{d\vec{r}}{dt} = A\alpha\left[e^{\alpha t}\hat{i} - e^{\alpha t}\hat{j}\right]$$

$\hat{i}\cdot\hat{i} = 1$

$\hat{i}\cdot\hat{j} = 0$

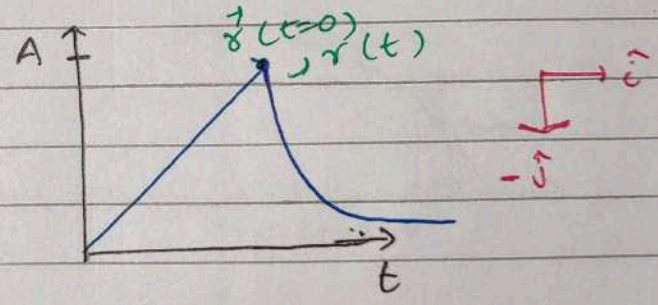
$\hat{i}\times\hat{j} = \hat{k}$

$\vec{v}(t=0) = A\alpha\left[\hat{i} - \hat{j}\right]$

$\vec{r}(t=0) = A\left[\hat{i} + \hat{j}\right]$

$\vec{r}(t=\infty) \cong Ae^{\alpha t}\hat{i}$

$\vec{v}(t=\infty) \cong A\alpha e^{\alpha t}\hat{i}$



speed $= \sqrt{v_x^2 + v_y^2}$

$\vec{r} = A\cos t\,\hat{i} + B\sin t\,\hat{j}$, $\vec{r} = At + Bt^2$

↳ $v = \frac{dx}{dt}$ = function of time / constant

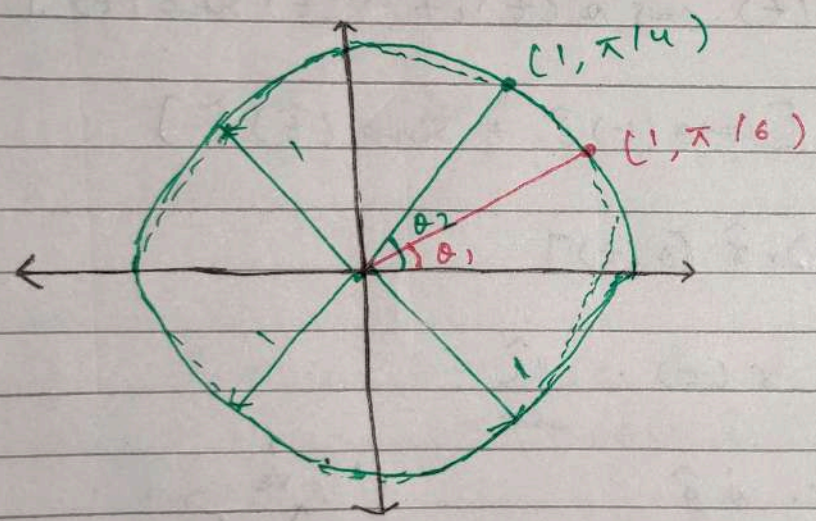
↳ $a = \frac{dv}{dt}$ = function of time / constant

P. $\hat{r} = \cos\frac{\pi}{6}\,\hat{i} + \sin\frac{\pi}{6}\,\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

$\hat{\theta} = -\sin\frac{\pi}{6}\,\hat{i} + \cos\frac{\pi}{6}\,\hat{j} = \frac{-1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$

Q. $\hat{r} = \cos\frac{\pi}{4}\,\hat{i} + \sin\frac{\pi}{4}\,\hat{j} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$\hat{\theta} = -\sin\frac{\pi}{4}\,\hat{i} + \cos\frac{\pi}{4}\,\hat{j} = \frac{-1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$



$\theta_1 \to \theta_2$

$\left(\frac{\pi}{6}\right) \to \left(\frac{\pi}{4}\right)$

$\underbrace{\hat{r}(\theta),\ \hat{\theta}(\theta)}_{\text{changed}}$

$a(t) \to v(t) \to r(t)$

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$

$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$

$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta}\,\hat{\theta}$

$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta} \cdot \frac{d\theta}{dt} = -\dot{\theta}\,\hat{r}$

$$\hat{r} = \frac{d\vec{r}}{dr} \cdot \frac{1}{\left|\frac{d\vec{r}}{dr}\right|} = \frac{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = \frac{d\vec{r}}{d\theta} \cdot \frac{1}{\left|\frac{d\vec{r}}{d\theta}\right|} = \frac{-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}}{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}}$$

$$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{r}\cdot\hat{\theta} = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta$$

$$= 0$$

$(\hat{i}\cdot\hat{j} = 0,\ \hat{i}\cdot\hat{i} = 1,\ \hat{j}\cdot\hat{j} = 1)$

$\hat{i} = \frac{d\vec{r}}{dx} \cdot \frac{1}{\left|\frac{d\vec{r}}{dx}\right|}$ # $\hat{j} = \frac{d\vec{r}}{dy} \cdot \frac{1}{\left|\frac{d\vec{r}}{dy}\right|}$

$\hat{k} = \frac{d\vec{r}}{dz} \cdot \frac{1}{\left|\frac{d\vec{r}}{dz}\right|}$ # $\hat{r} = \frac{d\vec{r}}{dr} \cdot \frac{1}{\left|\frac{d\vec{r}}{dr}\right|}$

$\hat{\theta} = \frac{d\vec{r}}{d\theta} \cdot \frac{1}{\left|\frac{d\vec{r}}{d\theta}\right|}$

- $\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$$\frac{d\vec{r}}{dr} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

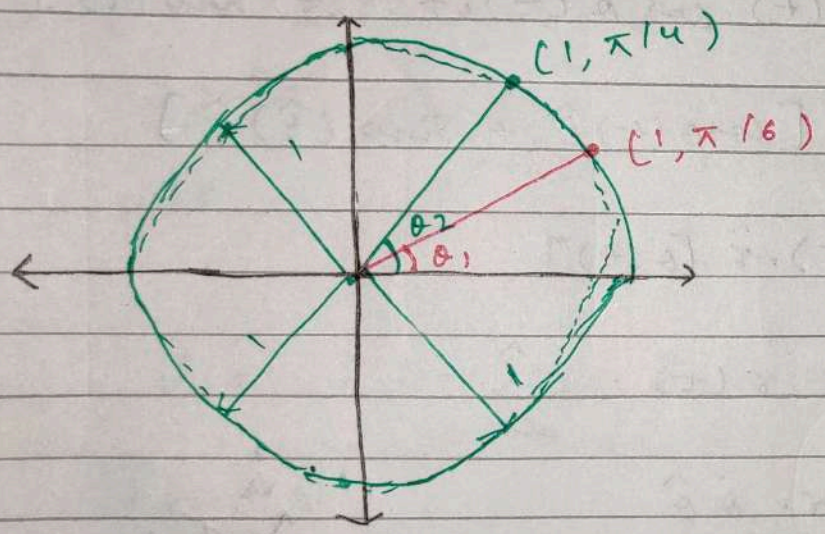
$$\frac{d\vec{r}}{d\theta} = r(-\sin\theta)\,\hat{i} + r\cos\theta\,\hat{j}$$

P. $\hat{r} = \cos\frac{\pi}{6}\,\hat{i} + \sin\frac{\pi}{6}\,\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

$\hat{\theta} = -\sin\frac{\pi}{6}\,\hat{i} + \cos\frac{\pi}{6}\,\hat{j} = \frac{-1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$

Q. $\hat{r} = \cos\frac{\pi}{4}\,\hat{i} + \sin\frac{\pi}{4}\,\hat{j} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$\hat{\theta} = -\sin\frac{\pi}{4}\,\hat{i} + \cos\frac{\pi}{4}\,\hat{j} = \frac{-1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$



$\theta_1 \to \theta_2$

$\left(\frac{\pi}{6}\right) \to \left(\frac{\pi}{4}\right)$

$\underbrace{\hat{r}(\theta),\ \hat{\theta}(\theta)}_{\text{changed}}$

$a(t) \to v(t) \to r(t)$

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

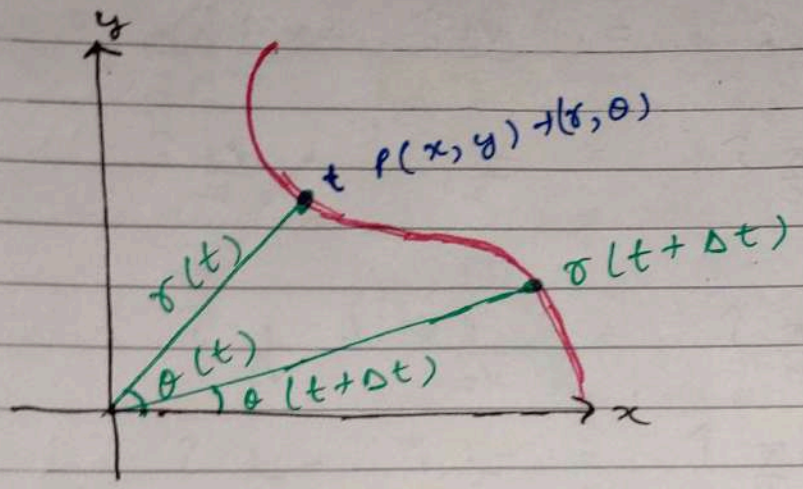
$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$

$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$

$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\,\hat{\theta}$

$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\,\hat{r}$



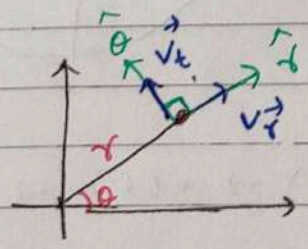
$$\vec{r}(t) = x\hat{i} + y\hat{j} = r(t)\cos\theta(t)\,\hat{i} + r(t)\sin\theta(t)\,\hat{j}$$

$$= r(t)\left[\cos\theta(t)\,\hat{i} + \sin\theta(\hat{t})\,\hat{j}\right]$$

$$\Rightarrow r(t)\cdot\hat{r}\left[\theta(t)\right]$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\cdot\hat{r} + r(t)\cdot\frac{d\hat{r}}{dt}$$

$$= \underbrace{\dot{r}\,\hat{r}}_{\text{Radial}} + \underbrace{r\cdot\dot{\theta}\,\hat{\theta}}_{\text{tangential}}$$



$$\vec{v} = \frac{d}{dt}(r\cdot\hat{r}) = \dot{r}\,\hat{r} + r\frac{d\hat{r}}{dt} \rightarrow \dot{\theta}\,\hat{\theta}$$

$$\boxed{\vec{v}(t) = \dot{r}\,\hat{r} + r\times\dot{\theta}\,\hat{\theta}}$$

$$\boxed{\vec{r}(t) = r\cdot\hat{r}}$$

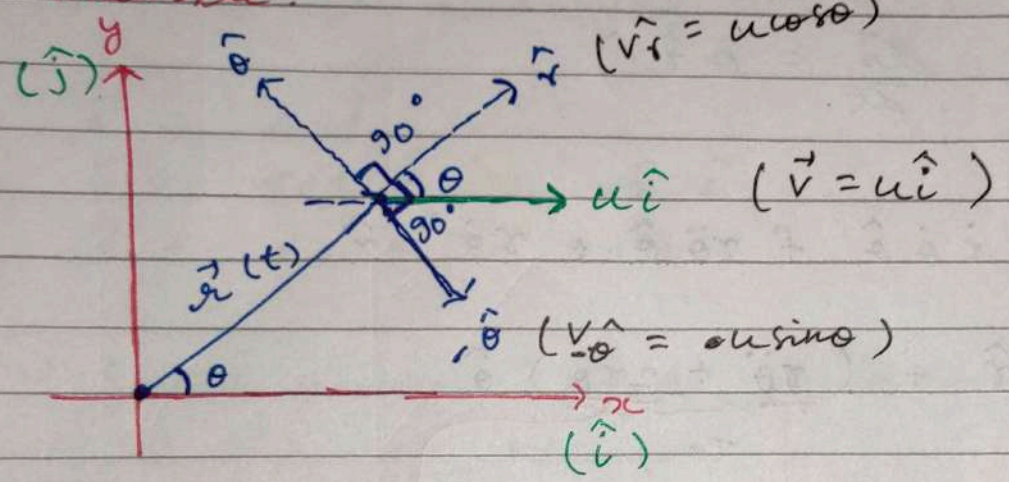
$$\boxed{\vec{a}(t) = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}}$$

$\vec{v}(t) = \underbrace{\dot{r}\,\hat{r}}_{\text{Radial}} + \underbrace{r\dot{\theta}\,\hat{\theta}}_{\text{Tangential}}$

(i) θ = constant :- Radial = $\dot{r}\,\hat{r}$

(ii) r = constant :- Tangential = $r\dot{\theta}\,\hat{\theta}$

Example :-

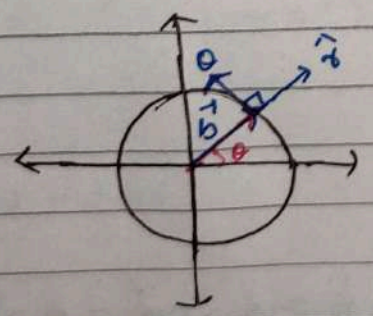


$$\vec{v} = u\hat{i} = v_r\,\hat{r} + v_\theta\,\hat{\theta}$$
$$= u\cos\theta\,\hat{r} - u\sin\theta\,\hat{\theta}$$

$\theta \to$ small $\quad \sin\theta \rightsquigarrow 0$, $\cos\theta \rightsquigarrow 1$, $\tan\theta \rightsquigarrow 1$

$$\vec{v} = u\cos\theta\,\hat{r}$$

Example :-



$\dot{\theta} = \alpha t \quad \alpha \to$ constant

radius = b = constant

$\dot{r} = 0$,

$V_\theta = r\dot{\theta}\,\hat{\theta} = r\alpha t\,\hat{\theta}$

$$\theta = \theta_0 + \int_0^t \dot{\theta}\,dt = \theta_0 + \frac{\alpha t^2}{2}$$

$\vec{r} = r(t)\,\hat{r}$ $\qquad$ $\vec{v}(t) = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right]$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\left(\frac{d\vec{r}}{dt} = \dot{\theta}\times\hat{\theta}\ ,\ \frac{d\vec{\theta}}{dt} = \dot{\theta}\hat{r}\right)$$

$$\Rightarrow \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}^2\hat{r}$$

$$\vec{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{Radial}}\hat{r} + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{\text{Tangential}}\hat{\theta}$$

Linear Radial accleration

Centripetal acceleration

Linear Tangential

Coriolis accleration

(i) $r =$ constant $(\dot{r} = 0,\ \ddot{r} = 0)$

$$\vec{a} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}$$

(ii) $\theta =$ constant $(\dot{\theta} = 0,\ \ddot{\theta} = 0)$

$$\vec{a} = \ddot{r}\hat{r}$$

Special cases:-

(1) circular motion $(r =$ constant $,\ \dot{r} = 0,\ \ddot{r} = 0)$

$$\vec{r} = r\cdot\hat{r}$$

$$\vec{v} = r\dot{\theta}\,\hat{\theta}$$

$$\vec{a} = r\ddot{\theta}\,\hat{\theta} - r\dot{\theta}^2\hat{r}$$

(a) uniform circular motion ($\alpha = 0$, $\ddot{\theta} = 0$)

($\dot{\theta}$ = constant)

$$\vec{r} = r \cdot \hat{r}$$
$$\vec{v} = r\dot{\theta}\ \hat{\theta}$$
$$\vec{a} = \cancel{r\ddot{\theta}} - r\dot{\theta}^2\ \hat{r}$$

(b) Non-uniform circular motion ($\ddot{\theta} = \alpha$, $\dot{\theta} = \alpha t$)

$$\vec{r} = r \cdot \hat{r}$$
$$\vec{v} = r\alpha t\ \hat{\theta} = r\dot{\theta}\ \hat{\theta}$$
$$\vec{a} = \alpha r\ \hat{\theta} - r\alpha^2 t^2\ \hat{r}$$

(2) Spiral motion ($r = a\theta$, uniform angular speed ω)

($A = a$)

$$\boxed{\dot{r} = a\dot{\theta} = a\omega}$$

($\theta = \omega t$)

$$\boxed{\vec{r} = a\theta \cdot \hat{r}}$$

($r = a\theta$, $\dot{\theta} = \omega$)

($r = A\omega t$)

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\ \hat{\theta}$$

($r\dot{\theta} = A\omega t \cdot \omega = a\omega^2 t$)

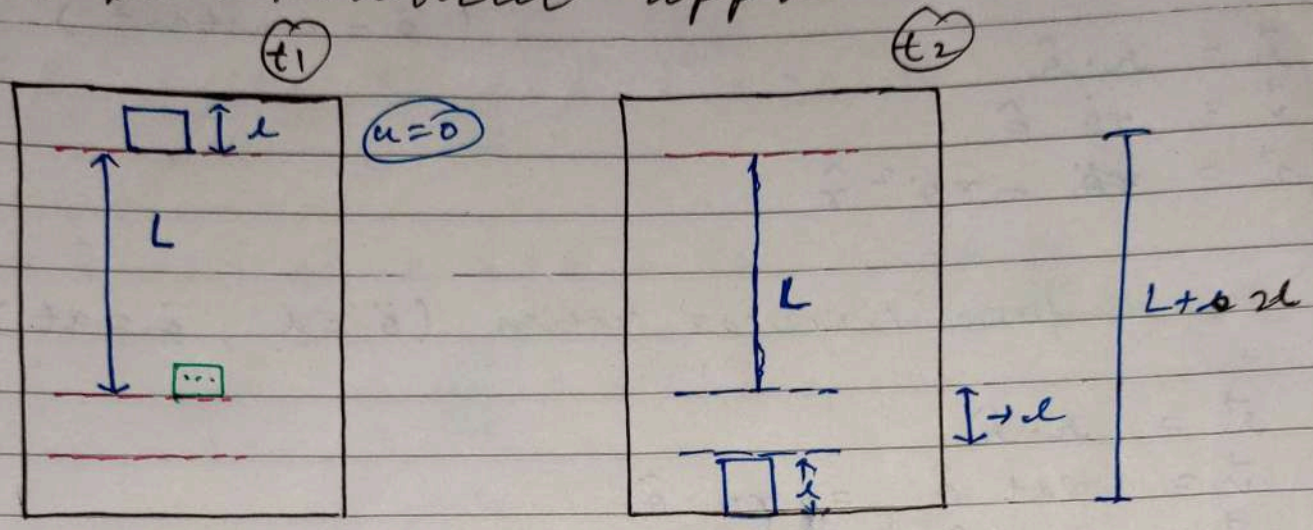
$$\boxed{\vec{v} = a\omega\ \hat{r} + a\omega^2 t\ \hat{\theta}}$$

$$\vec{a} = (\underbrace{\ddot{r}}_{0} - r\dot{\theta}^2)\ \hat{r} + (\underbrace{r\ddot{\theta}}_{0} + 2\dot{r}\dot{\theta})\ \hat{\theta}$$

$$\Rightarrow -A\omega t(\omega^2)\ \hat{r} + 2(A\omega)(\omega)\ \hat{\theta}$$

$$\boxed{\vec{a} = -a\omega^3 t\ \hat{r} + 2a\omega^2\ \hat{\theta}}$$

Mathematical approximations

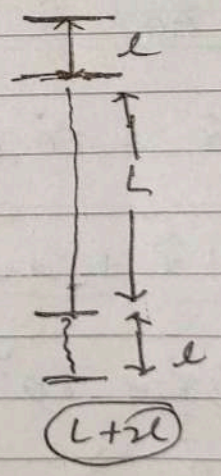


$$s = \underset{\substack{\downarrow \\ 0}}{ut} + \frac{1}{2}gt^2 \qquad s = \frac{1}{2}gt^2 \qquad g = \frac{2s}{t^2}$$

$$\boxed{t = \sqrt{\frac{2s}{g}}}$$

$$t_2 = \sqrt{\frac{2}{g}(s_2)} = \sqrt{\frac{2}{g}(L+l)}$$

$$t_1 = \sqrt{\frac{2}{g}(s_1)} = \sqrt{\frac{2}{g}(L)}$$



$$t_2 - t_1 = \sqrt{\frac{2}{g}(L+l)} - \sqrt{\frac{2}{g}(L)}$$

$$\Rightarrow \sqrt{\frac{2}{g}}\left(\sqrt{L+l} - \sqrt{L}\right)$$

$$\hookrightarrow \sqrt{g} = \sqrt{2}\,\frac{\left(\sqrt{L+l} - \sqrt{L}\right)}{t_2 - t_1}$$

$$g = 2\left[\frac{\sqrt{L+l} - \sqrt{L}}{(t_2 - t_1)}\right]^2$$

$\therefore (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$

where, $|x| < 1$

b) $(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$

$\sqrt{L+l} - \sqrt{L} = \sqrt{L}\left(\sqrt{1+\frac{l}{L}} - 1\right]$

taking $x = \frac{l}{L}$, we get

$\Rightarrow \sqrt{L}\left(\cancel{1} + \frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 \cdots - \cancel{1}\right)$

$\Rightarrow \cancel{\sqrt{L}} \times \frac{l}{2\cancel{L}\sqrt{L}}\left[1 - \frac{1}{4}\frac{l}{L} + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \cdots\right)$

$\Rightarrow \frac{l}{2\sqrt{L}}\left[\underbrace{1 - \frac{l}{4L} + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \cdots}_{1}\right] = \frac{l}{2\sqrt{L}}$

$\boxed{\frac{l}{L} \rightarrow \text{very small}}$, then further terms can be neglected

$g = \frac{x}{(t_2-t_1)^2} \times \frac{l^2}{2\cancel{4}L\cancel{2}} \Rightarrow \frac{l^2}{2L \times (t_2-t_1)^2}$

Taylor Series:-

$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$

$f(0) = a_0$, $f'(x) = a_1 + 2a_2 x + \cdots$

$f'(0) = a_1$

$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + \cdots$

Laws of Motion

↳ fundamental Laws

Aristotle:- Whenever you are allowing a body to move, so it require a certain amount of force, otherwise body will not maintain its state of uniform motion. (for Aristotle System):

for uniform motion → external force.

Newton's Law → Inertial co-ordinate system | Isolated Bodies

↳ Although newton's Law is not valid in all the system, system in which newton's Law of system become Invalid, but for most of the system in which bodies velocity is very very less than speed of light ($c >>>> v$)

↳ The system which at having Atomic level quantum effects doesn't follows newton's law of motion.

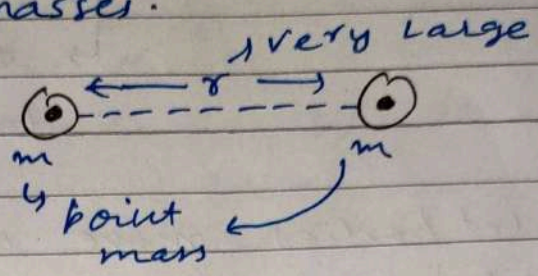
In Real life → ↳ $c >>> v$

↳ we dont study at microscopic levels

↓

So for our daily life, we have to study Newton's Law of motion.

↳ Newton's law are valid, where you are dealing with point masses.



classical mechanics
⇓

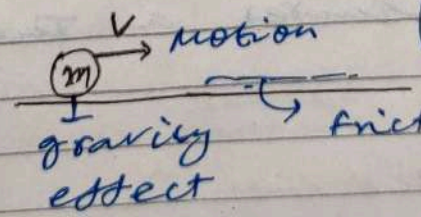
classical system → Force (Newtonian)
→ Energy

→ Lagrangian (K.E − P.E)
→ Hamiltonian (K.E + P.E)
⇓
~~waldstein~~

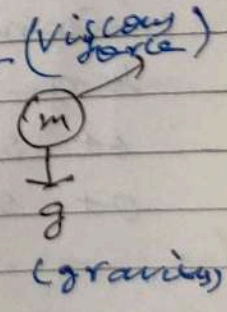
these terms are calculated to understand the dynamics of classical system.

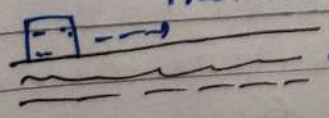
Laws of Motion:- there are two big hurdles (problems), when analyzing any system.

↳ friction forces
↳ gravity



(a type of frictional force)



- 

→ a device, when air passes through this, does not face a substantial amount of frictional force

linear Air track:

1st Law of motion:- (Inertial frames)

- Theoritical defination
- Experimental idea how to construct an Inertial frame.

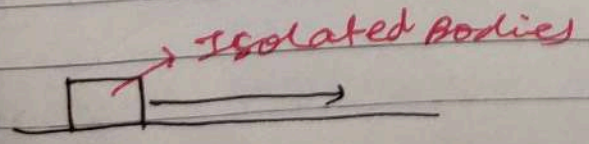
↳ "Isolated bodies move with a uniform velocity".

If in any system, "Isolated bodies move with a uniform velocity" are known as Inertial frame or Inertial co-ordinate system.

Isolated Bodies:-

Common notion is that, whenever a Body is moving come to hault (body stops), because u have the frictional forces acting on it.

↳ But if you have a system in which zero frictional forces are present over that system. and Body is given Inertial Jerk / Push, body will ~~to~~ remain in uniform motion, if there are no-frictional force acting on that system and constructing that type of system requires a Inertial frame of system.



$F \propto \frac{1}{r^2}$

- → ultravitational
- → electrostatic

} → dependent on Inverse square law

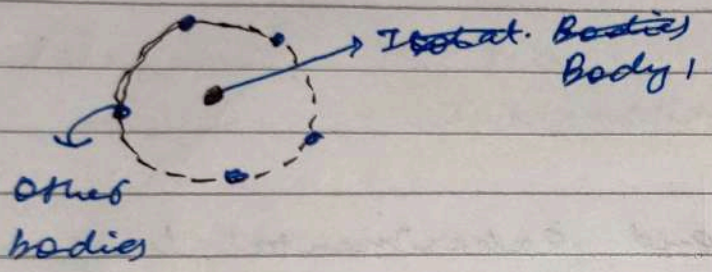
- → strong forces
- → weak forces

} are predominant over Nuclear region

↳ Isolated system :- A physical system ~~that~~ so far removed from other system that it doesn't Interact with them.

↳ cannot exchange either energy or matter outside the boundaries of system.

↳ Free from Influence of net external Force



⇒ there will be Interaction b/w them

⇓

so it is not Isolated body.

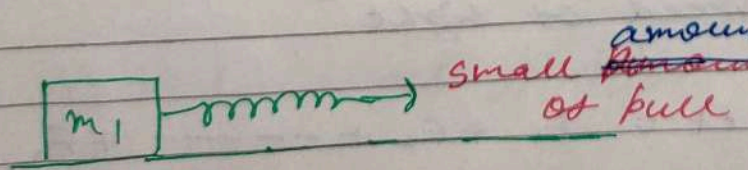
↳ When we removed other bodies very far from Body 1, the net external force become very negligible or zero ($F \propto 1/r^2$), when this situation is created, this type of system is called Isolated Body.

2nd Law of motion :-

↳ Idea of calculation of Force

↳ Net force whichever acting on particular system

↳ mass is additive

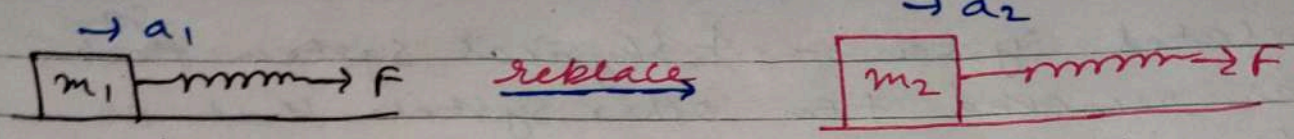


(motion depends on Force, mass, direction of Force)

↳ we can measure change of position of body, Time in which the particular displacement has happened.

⇓

give ~~of~~ Idea of velocity, acceleration of system.



(Changing → mass, Force → Constant)

$\boxed{m_1 a_1 = m_2 a_2}$ (Force Applied → Same)

↳ $a_2 = \frac{m_1}{m_2} a_1$

$\boxed{a_1 = \frac{F}{m_1}}$ $\boxed{a_2 = \frac{F}{m_2}}$

↳ It is an observation and Experimental law, which further was converted into ~~theor~~ theory.

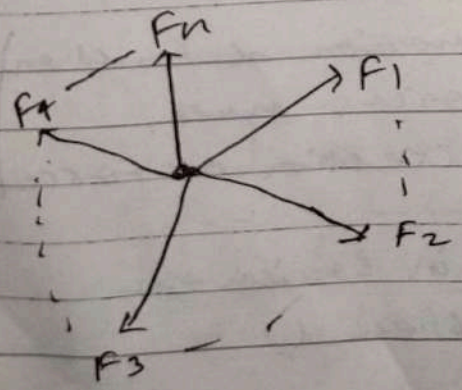
• $\boxed{F = m_1 a_1 = m_2 a_2}$

$\boxed{F = ma = \frac{d\vec{p}}{dt}}$

↳ when mass is not a Rest mass. (used generally)

$\boxed{M = \frac{M_0}{\sqrt{1 - \frac{v^2}{c^2}}}}$

M_0 → Rest mass
M → mass of Body
v → velocity of Body
c → Speed of light



$F_{net} = F_1 + F_2 + \text{------} + F_n$

$= \Sigma F_i$

↳ If you have no net external force is acting in over that particular system, the system will be completely Isolated bodies, for that type of system, u have no any type of Interaction, because there are no other bodies are present.

Isolated Bodies ⇒ No Interaction ⇒ No force ⇒ Motion with uniform velocity

↳ Forces are coming from Interaction (body/charge)

$F_{gravitational} = \frac{G \cdot m_1 m_2}{r^2}$ $F_{spring} = -kx$

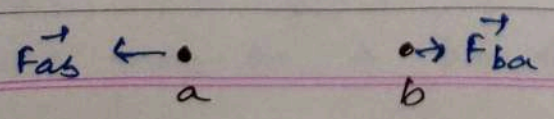
$F_{electric} = \frac{K q_1 q_2}{r^2}$

Motion of three bodies or n number of bodies ⇒ calculating F ↓ requires lot of calculation, approximation

(Molecular Force calculation) ↓ tough task

3rd Law of Motion :-

↳ whenever two bodies are Interacting up, u need to maintain the conservations Idea and this was the concept of 3rd law.



Every action ⇒ equal reaction.

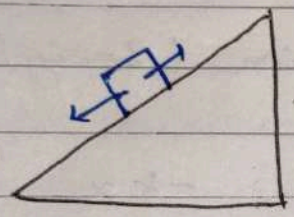
$(\vec{F}_{ba} = -\vec{F}_{ab})$ ⇒ Conservation of Momentum.

Force acting on body (b) ~~body~~ by (a)

Force acting on body (a) by body (b)

↳ working in non-relativistic domain ($c >>> v$)

↳ working not in microscopic domain



(i) Divide system in small units

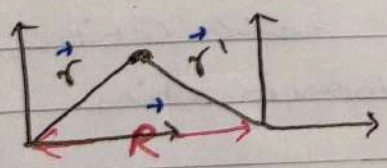
(ii) Make FBD (~~Force~~ Free Body diagram)

↳ can always apply 3rd Law of motion

(iii) co-ordinate system

(iv) constraints which are applied for system

(v) for dynamics, use second law



$\vec{r} = \vec{R} + \vec{r}'$

$\ddot{r} = \ddot{R} + \ddot{r}'$

↳ Inertial frame ($\ddot{R} = 0$)

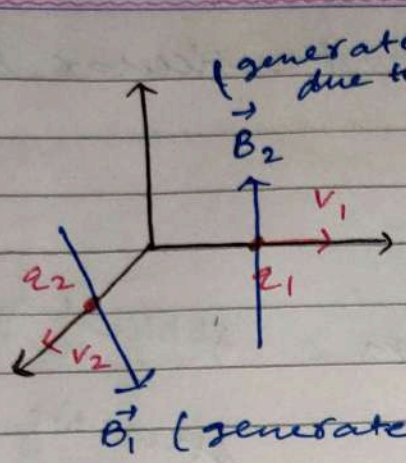
↳ non-Inertial frame ($\ddot{R} \neq 0$)

Galillean transformation

(Lorentz transformation)

$\boxed{\dot{r} = \dot{R} + \dot{r}'}$ (t = same)

#



$q_1, q_2 \rightarrow$ charges

$F = q(\vec{v} \times \vec{B})$ (Biot-Savart Law)

check. $\vec{F}_{12} = -\vec{F}_{21}$? (True or False)

Here, newton's 3rd law or conservation of momentum fails because the direction of forces are not in opposite direction

Reason → Beyond our Syallabus.

~~Everyday~~ total Forces

↳ Gravitational
↳ Electrostatic
↳ Strong
↳ Weak

Everyday Forces (forces due to contact)

↳ Contact Forces → Push, Pull → microscopic properties of Atomic molecules.

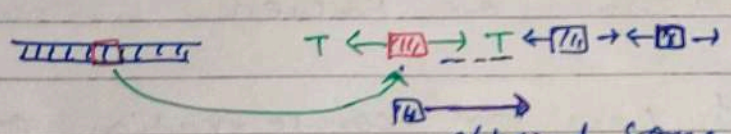
↳ Tension of string
↳ Normal forces
↳ Frictional force

↳ Whenever 2 Bodies come in contact with each other, so due to Intermolecular Interaction there may be a bonding, then body will

always have to overcome that particular bonding to execute the motion.

Tension in a string

mass of string = m



applied some amount of force and moved towards this direction

↓

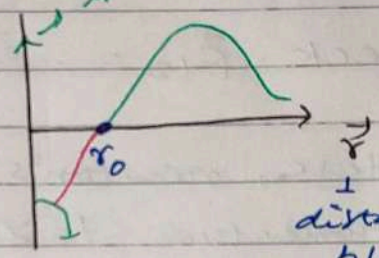
neighbouring atom will apply attractive forces

maximum attractive forces will be applied so that this atom should return back to its equilibrium position.

Applied by neighbouring atoms

↑

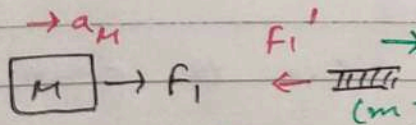
Attractive force



distance b/w 2 atoms

repulsive force applied by neighbouring atoms

FBD



(string is inextensible)

↓

$a_s = a_M$

$(\vec{F_1} = \vec{F_1'})$ (in magnitude)

$F_1 = Ma_M$ $\quad F - F_1' = ma_s$ $\quad (a = a_s = a_M)$

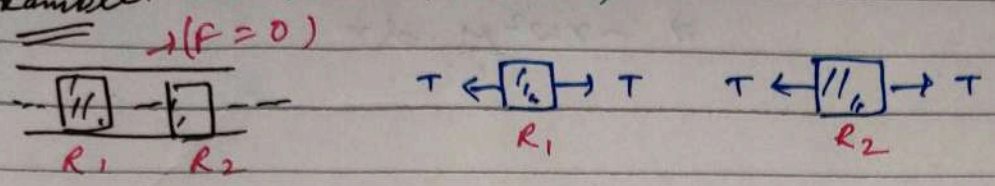
$F = (M+m)a$

$\Rightarrow \boxed{a = \frac{F}{M+m}}$

$F_1 = \frac{M}{M+m} \times F$ $\quad (F > F_1)$

• $m \to$ small $F_1 \approx F$ • $m \to$ big $F > F_1$

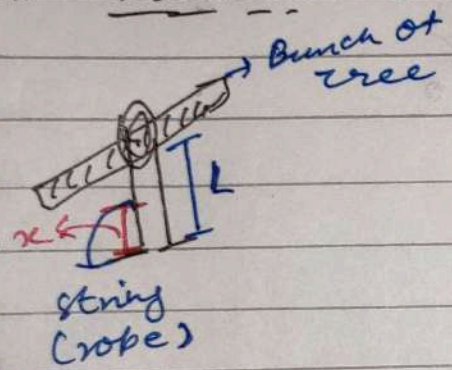
Example 1 (Constant Tension)



We considered tension between each region of string is uniform.

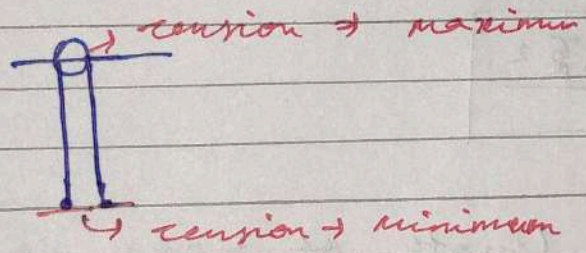
$(T_{R_1} = T_{R_2})$

Example 2 (Variable Tension)



$x \in [0, L]$

Find the tension at x distance from the bottom.

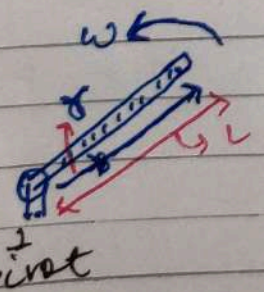


$T(x) = mg\left(\frac{x}{L}\right)$

↳ depending upon the distance, tension varies.

$T(x_1) \neq T(x_2)$

Example 3 :- Whirling String (Rotating String)



Find the tension at distance r from the pivot.

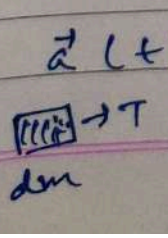
$$\vec{a}(t) = (\underbrace{\ddot{r}}_{\to 0} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + \underbrace{2\dot{r}\dot{\theta}}_{\to 0})\hat{\theta}$$

(r = constant $\Rightarrow \dot{r} = 0, \ddot{r} = 0$)

(ω = constant $\Rightarrow \ddot{\theta} = 0$)

$(\dot{\theta} = \omega)$

$$\vec{a}(t) = -r\omega^2\hat{r}$$



$dt(r)$

$$\underbrace{dT(r)}_{\text{Small tension at distance } r.} = -r\omega^2 dm \Rightarrow -r\omega^2 \rho dr$$

$$\Rightarrow -r\omega^2 \frac{M}{L} \cdot dr$$

$$\rho = \frac{M}{L}$$

$$\int dT(r) = -\int r\omega^2 \frac{M}{L} \cdot dr$$

$$T(r) = \frac{-\omega^2 M}{L} \int r\,dr = \frac{-\omega^2 M}{L} \times \frac{r^2}{2}$$

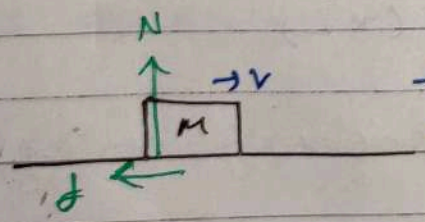
$$= \frac{-\omega^2 M}{L} \times \frac{r^2}{2} \Big|_0^L$$

$$\Rightarrow \frac{-\omega^2 M}{L} \times$$

$$\boxed{T(r) = \frac{-\omega^2 M \cdot r^2}{2L}}$$

when my $r=0$, length of rope is L
when my $r=r$, length of rope is 0.

#



→ Body is in Rest
→ Body is in motion.

Force exerted by a surface on the Body.

Perpendicular force (Normal Forces)

Tangential force (Frictional force)

(contact force)

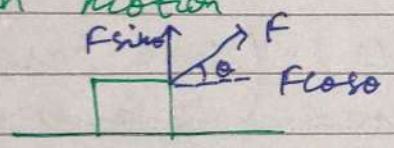
Body ⇒ Rest ⇒ only normal forces acting
Body ⇒ motion ⇒ Normal and friction force acting.

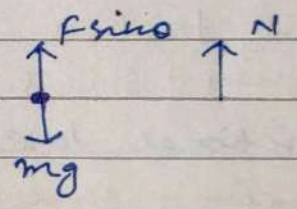
Normal Force

↳ All the Resultant forces which is acting along perpendicular direction of a Body.
↳ direction will be equal ~~to~~ and opposite of Resultant forces, which are acting along perpendicular direction of surface.

Body in Rest $N = mg$

Body in motion





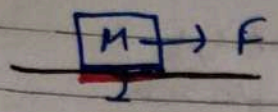
$N = mg - F\sin\theta$ (Here Normal fluctuates)

Frictional forces

↳ complicated & difficult to measure. at a particular point of time over a particular surface.

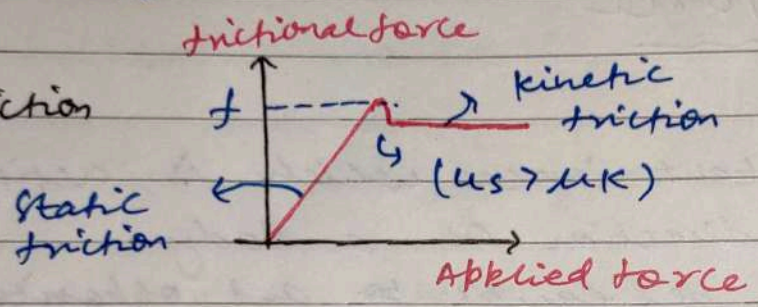
friction :- force which is going to oppose the motion of Body

- sliding friction
- Fluid / Air friction



Intermolecular attraction between atoms try to oppose the motion

Sliding friction



$f \equiv \mu N$

Static friction → tendency to relative motion.

Kinetic friction → thier is relative motion b/w them.

Fluid / Air friction

example :-

→ velocity → small ($v = 10^{-5}$ m/s) (dust particle falling towards ground)

Frictional force → Neglected

→ velocity → moderate (viscous fluid like Honey)

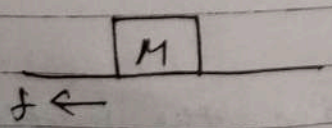
→ Frictional force → linearly proportional to velocity.

→ velocity → Large (velocity of Aircraft)

Frictional force → v^{α}

($\alpha = 2, 3, ---$)

#



• when body is started to moving, (During Motion)

$f = \mu N$

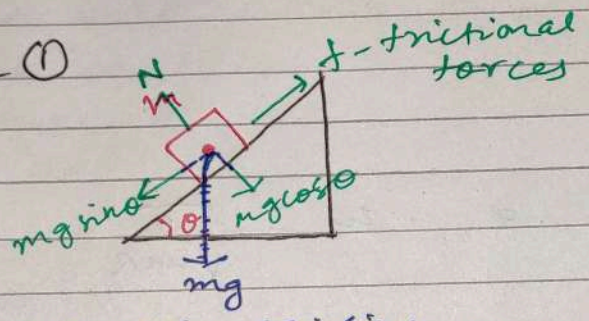
• when body is at Rest: (till motion started)

$0 < f < \mu N$

↳ Frictional forces are Empirical forces.
↳ Newton's Law is a more generic concept and connected with broader Idea, where as Frictional forces are Empirical forces.

Empirical forces → Based on practical experience or experiments.
⇓
(cv^{α})

Example ①



$$\left(\begin{array}{l} N = mg\cos\theta \\ f = mg\sin\theta \end{array} \right)$$

At the verge of sliding,

$$\boxed{f = \mu N}$$

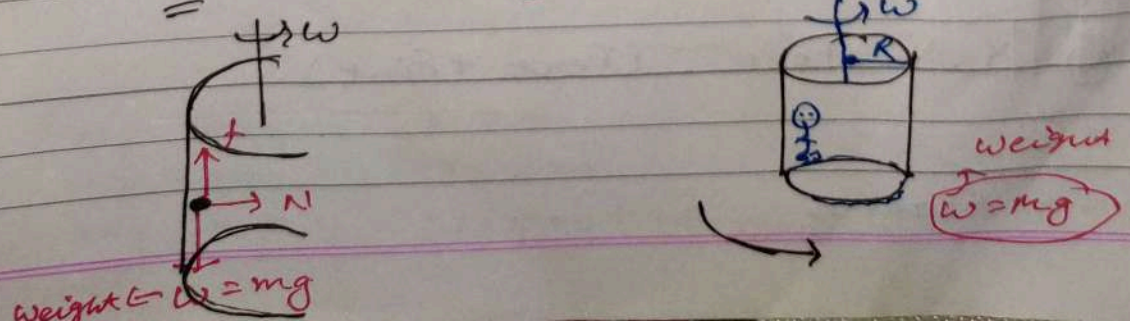
$$mg\sin\theta = \mu \cdot mg\cos\theta \Rightarrow \boxed{\mu = \tan\theta}$$

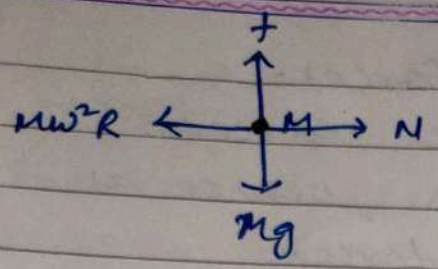
(Temperature, pressure, surface forces, Intermolecular forces → Neglected)

$\mu_{steel-steel} \Rightarrow 0.58$
$\mu_{wood-brick} \Rightarrow 0.60$
$\mu_{rock-masonry} \Rightarrow 0.70$

{ approximate values }
↳ under certain approximations.

Example 2 Spinning Tennor





(R = constant , ω = uniform)

$$\vec{a} = -R\omega^2 \hat{r}$$

$$M\vec{a} = -MR\omega^2 \hat{r}$$

$N = M\omega^2 R$

At the static state, $\boxed{\mu N \geq f}$

$$\mu N \geq f$$

$$\mu (M\omega^2 R) \geq Mg$$

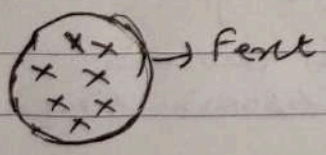
$$\omega^2 \geq \frac{g}{\mu R} \Rightarrow \omega \geq \sqrt{\frac{g}{\mu R}}$$

$$\omega_{min} = \sqrt{\frac{g}{\mu R}}$$

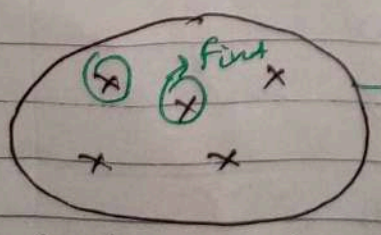
minimum Angular velocity floor drops / person stands safely.

$\boxed{\sum F_j^{ext} = \sum m_j \ddot{r}_j}$

$$\boxed{M\ddot{R} = \sum F_j^{ext} = \sum m_j \ddot{r}_j}$$



⇒ extension of Newton's Law of motion ⇒ for system of particles (dynamic of Body)



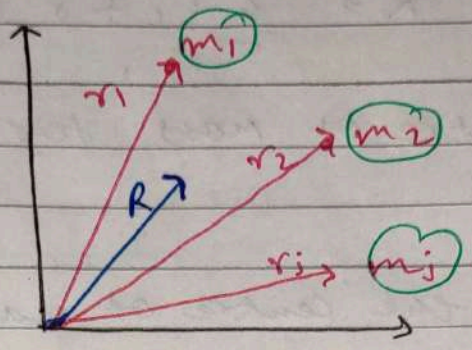
(Fext + Fint)

System of particles

Consider a system represented by particular frame of Reference

$F_{ext} + F_{int}$

$F_{int} \to 0$



F_{int} → certain no. of particles present Inside the System cancels Internal force due to each other.

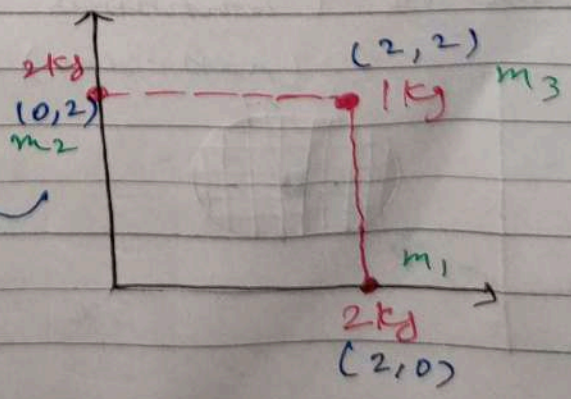
$$\boxed{\sum F_j^{ext} = \sum M_j \ddot{r}_j}$$ (For system of particles)

$$\boxed{M\ddot{R} = \sum F_j^{ext} = \sum M_j r_j}$$ $\ddot{R}$ → acceleration of Centre of mass

↳ center of mass is the point where whole mass system is Active and you can utilise ~~use~~ that particular point as a single point system over which dynamic of the systems could be studied

$$\boxed{R = \frac{1}{M} \sum M_j r_j}$$

Centre of Mass.



$m_1 = 2kg$ $\quad r_1 = \hat{x}$

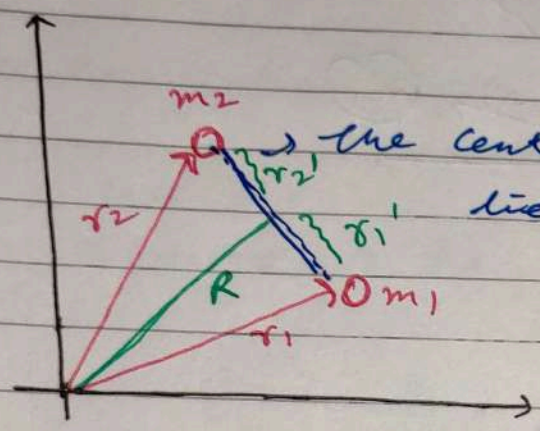
$m_2 = 2kg$ $\quad r_2 = 2\hat{j}$

$m_3 = 1kg$ $\quad r_3 = 2\hat{i} + 2\hat{j}$

$$R = \frac{1}{M} \sum m_i r_i = \frac{1}{5}\left[4\hat{i} + 4\hat{j} + 1(2\hat{i} + 2\hat{j})\right]$$

$$R \ni \frac{6}{5}\left[\hat{i} + \hat{j}\right]$$

Case - I (Centre of mass for two bodies)



the centre of mass ~~lies~~ of both mass lies on the given line.

$$r_1' = r_1 - R$$
$$r_2' = r_2 - R$$

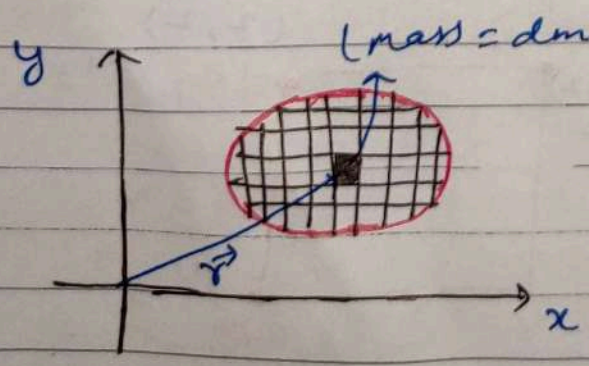
$$R_{CM} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

$$r_1' = r_1 - \left(\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}\right) = \frac{m_2}{m_1 + m_2}(r_1 - r_2)$$

$$r_2' = r_2 - \left(\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}\right) = \frac{-m_1}{m_1 + m_2}(r_1 - r_2)$$

Both r_1' and r_2' are function of $(r_1 - r_2)$

Case - II (Continous Body)



$$M = \int dM$$

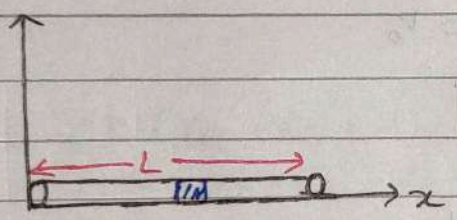
$$R_{CM} = \frac{\int dm \cdot \vec{r}}{\int dm}$$

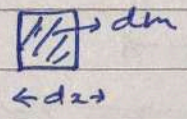
$$\vec{R}_{CM} = \frac{\int \rho\, dx\, dy \cdot \vec{r}}{\int dm} = \frac{\int \lambda\, dx \cdot \vec{r}}{\int dm}$$

$\lambda = \frac{\text{Mass}}{\text{Length}}$ ⊥ Linear density	$\sigma = \frac{\text{Mass}}{\text{Area}}$ ⊥ Areal density	$\rho = \frac{\text{Mass}}{\text{Volume}}$ ⊥ Volume density

Example:-

$\lambda = \lambda_0 \cdot \frac{x}{L}$ $\boxed{\text{CM of this Rod}}$



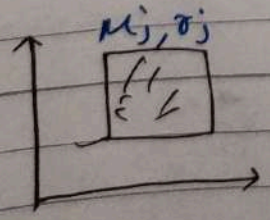


$$M = \int dm = \int \lambda_0 \cdot \frac{x}{L}\, dx \qquad \vec{r} = x\hat{i}$$

$$\vec{r}_{COM} = \frac{\int d\vec{m} \cdot \vec{r}}{\int d\vec{m}} = \frac{\int_0^L \lambda_0 \cdot \frac{x^2}{L} \cdot dx\, \hat{i}}{\int_0^L \lambda_0 \cdot \frac{x}{L}\, dx}$$

$$\vec{r}_{COM} = \frac{\frac{x^3}{3}\Big|_0^L}{\frac{x^2}{2}\Big|_0^L} \Rightarrow \frac{L^3}{3} \cdot \frac{2}{L^2} \Rightarrow \frac{2L}{3}$$

Equation of motion → Translational Motion



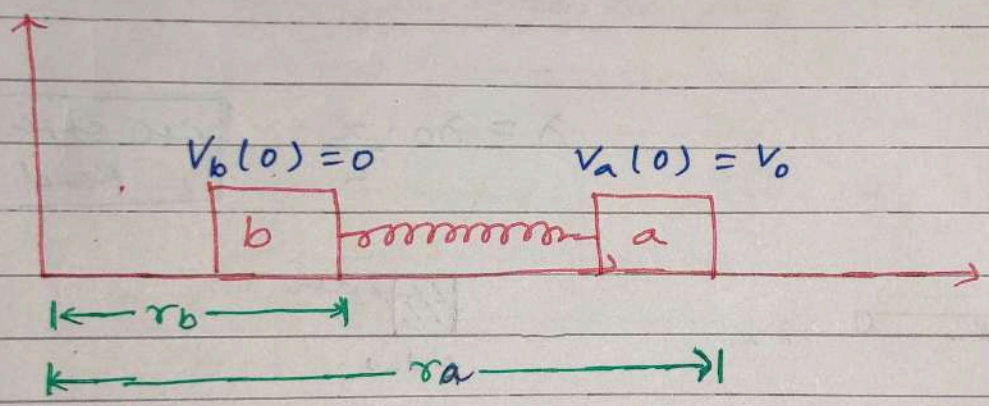
$$M\ddot{R} = \sum m_j r_j = \sum F_{j\,ext}$$

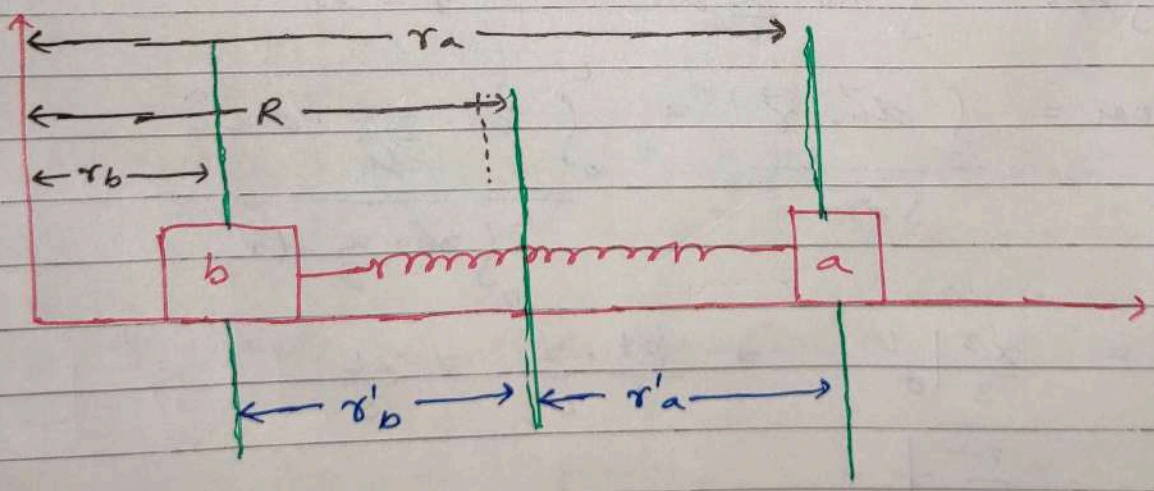
↳ * Shape of object → (Rotational dynamics)

↳ * point of application of force (acting on COM)

Example:- Two Identical blocks a and b both of mass m slide without friction on a straight track. They are attatched by a spring of length l and spring constant k. Initially, they are at rest. At t=0, block a is hit sharply, giving it an instantaneous velocity v_0 to the right. Find the velocity for subsequent times.

⇒





$$R = \frac{m r_a + m r_b}{m+m} = \frac{1}{2}(r_a + r_b)$$

$$r_a' = r_a - R = \frac{1}{2}(r_a - r_b)$$

$$r_b' = r_b - R = \frac{-1}{2}(r_a - r_b) = -r_a'$$

$$r_a - r_b - l = r_a' - r_b' - l$$

$$m\ddot{r}'_a = -k(r_a' - r_b' - l)$$ (Equation of motion in COM Frame)

$$m\ddot{r}'_b = +k(r_a' - r_b' - l)$$

$$m(\ddot{r}'_a - \ddot{r}'_b) = -2k(r_a' - r_b' - l)$$

$$u = r_a' - r_b' - l \qquad m\ddot{u} = -2ku$$

$$\boxed{m\ddot{u} + 2ku = 0}$$

$$m_1\ddot{r}_a = -k(r_a - r_b + l)$$
$$m_2\ddot{r}_b = -k(r_b - r_a - l)$$

$u = A\sin\omega t + B\cos\omega t$, where $\omega = \sqrt{2k/m}$

(Difference in COM co-ordinates executes SHM)

Applying initial conditions

$t = 0$, $u(0) = 0$, $B = 0$

Since, $u = r_a' - r_b' - l = r_a - r_b - l$

At $t = 0$, $\dot{u}(0) = v_a(0) - v_b(0) = A\omega\cos(0) = v_0$

$$\boxed{A = \frac{v_0}{\omega}}$$

Therefore, $u = \left(\frac{v_0}{\omega}\right)\sin\omega t$

$v_a' - v_b' = \dot{u}$ & $v_a' = -v_b'$, $\boxed{v_a' = -v_b' = \frac{1}{2}v_0\cos\omega t}$

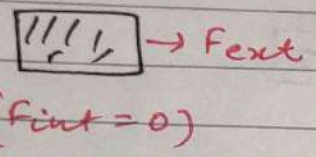
$V_a = \dot{R} + V_a'$ $\qquad \dot{R} = \frac{1}{2}\left[V_a(0) + V_b(0)\right] = \frac{1}{2}V_0$

$V_b = \dot{R} + V_b'$

$V_a = \frac{V_0}{2}(1 + \cos\omega t)$

$V_b = \frac{V_0}{2}(1 - \cos\omega t)$

#



$F_{ext} = \frac{d\vec{p}}{dt}$

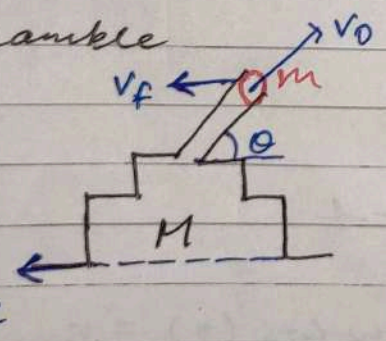
$\underbrace{\vec{F}_{ext} \cdot \vec{dt}}_{\text{Impulse}} = d\vec{p}$

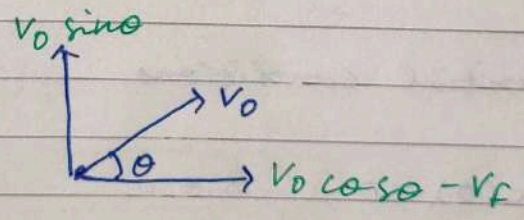
Isolated system :- $(F_{ext} = 0)$

↳ $\Delta p = 0$

↳ $p_i(t) = p_f(t')$

Example





$t = 0$ $\quad P_i = 0$

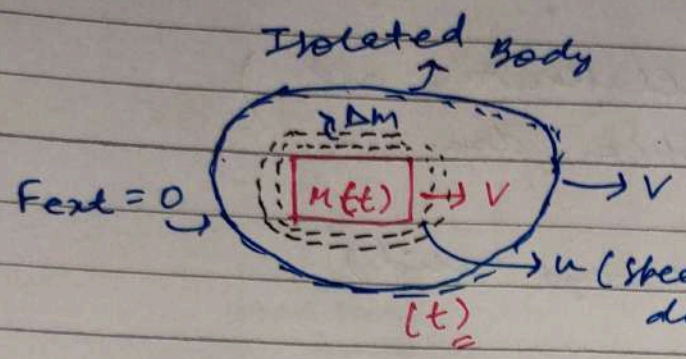
$P_i = P_f \quad (F_x = 0)$

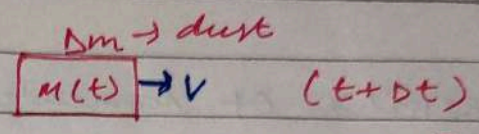
$0 = m(V_0 \cos\theta - V_f) - MV_f$

$$\boxed{V_f = \frac{m V_0 \cos\theta}{m + M}}$$

Example:-

↳ dust particles start to depositing on mass m, so mass is changing





(t) $\quad P(t) = M(t)V + \Delta m u$

(t+Δt) $\quad P(t+\Delta t) = M(t)V + \Delta m \cdot V$

$$F = \frac{P(t+\Delta t) - P(t)}{\Delta t}$$

$$= \frac{M(t)\cdot V + \Delta M \cdot V - M(t)V - \Delta m u}{\Delta t}$$

$$\boxed{F = \frac{\Delta M (V-u)}{\Delta t}}$$ → for a system of particle

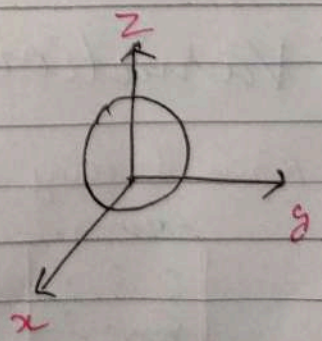
$$F = \frac{d}{dt}(mv) = \frac{v\,dm}{dt} + m\frac{dv}{dt}^{\to 0} = \frac{v\,dm}{dt}$$

$$\boxed{F = \frac{v\,dm}{dt}}$$ → single particle

$P_i(t) = P_f(t)$

$$P_{ini,x}\,\hat{i} + P_{ini,y}\,\hat{j} + P_{ini,z}\,\hat{k}$$

$$= P_{f,x}\,\hat{i} + P_{f,y}\,\hat{j} + P_{f,z}\,\hat{k}$$



Fundamental ⇒ Newton's Law of motion
↓
(Fails in relativistic and microscopic domain)

Conservation of momentum (always valid)
↓
(also valid in relativistic and microscopic domain)

Electromagnetic Radiations (massless system)
↳ combrises of photons (massless)
↳ Conservation of momentum is valid

$$\boxed{F_{ext} = 0, \quad \Delta p = 0}$$

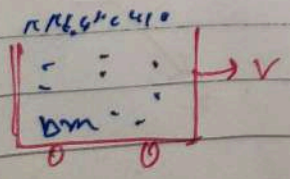
difference b/w $F = \frac{v\,dm}{dt}$ & $F = \frac{\Delta m (v-u)}{\Delta t}$

$\boxed{F = \frac{v\,dm}{dt}}$ ⇒ derived from Newton's 2nd Law considering a single particle

$\boxed{F = \frac{\Delta m (v-u)}{\Delta t}}$ ⇒ derived from conservation of momentum considering a system of particles.

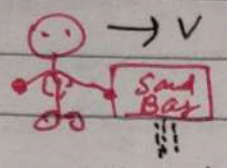
Variation of Mass :-

(i) Rain falling on a cart



↳ no horizontal Force generated on cart

(ii) Sand Bag Problem

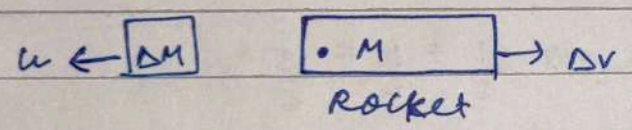


Sand is falling from the hole

↳ no momentum contribution on object

(iii) Rocket Propogation

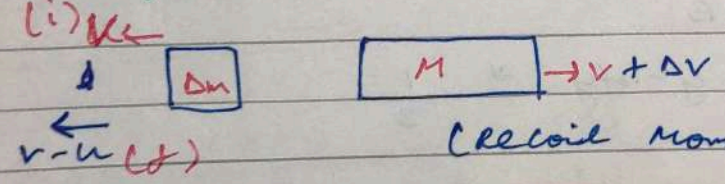
(Reference frame is in rocket)



(iii) Rocket Propogation

(Reference frame in which observer has v velocity for rocket)

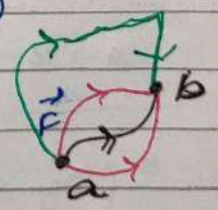
(i)



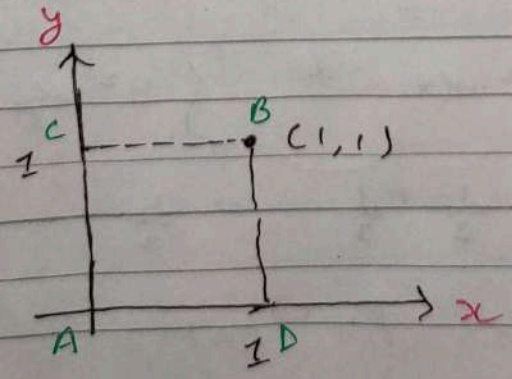
(Recoil Momentum)

L-14

$$\boxed{\text{work} = \int_a^b \vec{F} \cdot d\vec{r}}$$



Example:



A → C → B path ①

A → D → B path ②

A → B path ③

$\vec{F} = xy\hat{i} + y^2\hat{j}$

$$W_{BA} = \int \vec{F}\cdot d\vec{r} \Rightarrow \int_A^B (xy\hat{i} + y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$\Rightarrow \int_A^B xy\,dx + \int_A^B y^2 dy$$

path 1 $\quad A \to C \to B$

$(x=0, y=\text{variable})$ $\quad (y=1, x\in[0,1])$

$(dy=0)$

$$W_{BA} = W_{CA} + W_{CB}$$

$$\int_0^1 y^2\cdot dy + \int_0^1 x\,dx \Rightarrow \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

path 2 $\quad A \to D \to B$

$(y=0, x\in[0,1])$ $\quad (x=1, y\in[0,1])$

$dx = 0$

$$W_{BA} = W_{DA} + W_{DB}$$

$$= 0 + \int_0^1 y^2\cdot dy \Rightarrow \frac{1}{3}$$

path 3 $\quad A \to B \quad$ (path $y = x$)

$$W_{BA} = \int_A^B xy\cdot dx + \int_A^B y^2\cdot dy$$

$$\Rightarrow \int_0^1 x^2\cdot dx + \int_0^1 y^2\cdot dy$$

$$\Rightarrow \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$W_{path\,1} \neq W_{path\,2} \neq W_{path\,3}$

$\Downarrow$

path matters

↳ The path which is taken to reach from initial position to final position has important contribution over calculation of work.

Work - Energy Theorem

$$W_{ba} = \int_a^b \vec{F} \cdot d\vec{r} \Rightarrow \int_a^b m\frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$$

$$\left(\because \vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} \;,\; \frac{d\vec{r}}{dt} = \vec{v} \right)$$

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(v \cdot v) = v \cdot \frac{dv}{dt} + \frac{dv}{dt} \cdot v$$

$$= 2v \cdot \frac{dv}{dt} = 2\frac{dv}{dt} \cdot v$$

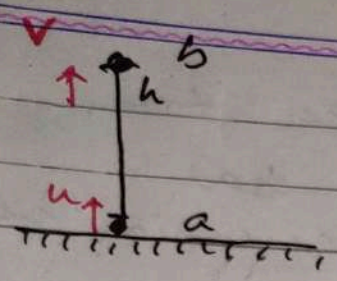
$$W_{ba} = \int_a^b m \cdot \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt \Rightarrow \frac{1}{2}\int_a^b m\left(\frac{d}{dt}(v^2)\right) dt$$

$$\Rightarrow \frac{1}{2} mv^2 \Big|_a^b$$

$$\boxed{W_{ba} = \frac{mv_b^2}{2} - \frac{mv_a^2}{2} = K.E_b - KE_a}$$

↳ for all types & kinds of forces.

(conservative + Non-Conservative forces)



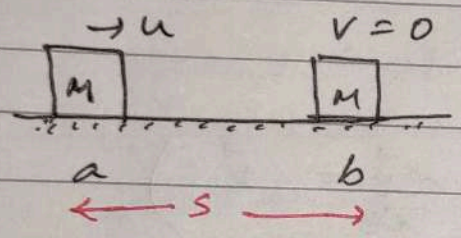
$$W_{ba} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$W_{ba} = \int_a^b -mg \cdot dr$$

$$\Rightarrow -mgh$$

- $mgh = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$

Example 2



$$W_{ba} = \frac{1}{2}m(0)^2 - \frac{1}{2}mu^2$$

$$W_{ba} = \int_0^s -f_s \cdot ds$$

$$\Rightarrow -f_s \cdot s$$

$$\cancel{-}f_s \cdot s = \cancel{+}\frac{1}{2}mu^2$$

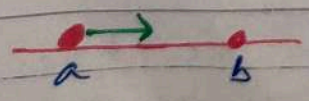
$$\boxed{\mu mg \cdot s = \frac{1}{2}mu^2}$$

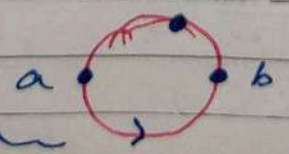
$W_{ba} = \int F^{ext} \cdot dr = \frac{1}{2}M\dot{x}^2_{(b)} - \frac{1}{2}M\dot{x}^2_{(a)}$

($x \rightarrow$ centre of mass of body)

↳ work is path dependent. ($W_{BA}^{(1)} \neq W_{BA}^{(2)}$)

1D





$$\underset{\text{central forces}}{\vec{F}} = f(r)\,\hat{r}$$

conservative system

$$\vec{W} = \int_a^b \vec{F} \cdot d\vec{r} = \int f(r) \cdot \hat{r}\,(dr\,\hat{r} + r\,d\theta\,\hat{\theta})$$

$\nabla \rightarrow$ delta Operator

$$\vec{W} = \int_a^b f(r)\cdot\hat{r} \cdot (dr\cdot\hat{r} + r\,d\theta\,\hat{\theta})$$

$$\vec{W} = \int_a^b f(r)\cdot dr = f(b) - f(a)$$

Conservative system

↳ work is path Independent.

↳ work in closed loop = 0. $\oint_a^b \vec{F}\cdot d\vec{r} = 0$

* For conservative system,
Potential energy is defined as

$$W_{ba} = \int_a^b \vec{F}\cdot d\vec{r} = \underbrace{-U_b}_{\text{PE associated at } b} + \underbrace{U_a}_{\text{PE associated at } a}$$

$$\Rightarrow U_b - U_a = -\int_a^b \vec{F}\cdot d\vec{r} = -[KE_b - KE_a]$$

$$\Rightarrow \boxed{U_b + K_b = U_a + K_a}$$ (only for conservative system)

$\underbrace{U_b - U_a}_{dU} = -\int \vec{F}\cdot d\vec{r}$ $\Rightarrow$ (1-D) $d\vec{r} = x\hat{i}$

$$\boxed{-\frac{dU}{dx} = F_x}$$

(change in potential energy)

(3-D) $U(x, y, z)$

$$\boxed{dU = -\vec{F}\cdot d\vec{r}}$$

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}\cdot dy + \frac{\partial U}{\partial z}\cdot dz$$

$$dU = \underbrace{\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)}_{-\vec{F}} \underbrace{(dx\hat{i} + dy\hat{j} + dz\hat{k})}_{d\vec{r}}$$

$$\vec{F} = -\left[\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right]$$

$$\Rightarrow \underbrace{-\Delta U}_{\text{Gradient}} \quad \therefore \left(\Delta = i \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z}\right)$$

Example $\vec{F} = -\Delta U$

$$U = \frac{1}{2}K\left[x^2 + y^2 + z^2\right]$$

$$-\Delta U = -i \cdot \frac{\partial}{\partial x}\left(\frac{1}{2}Kx^2\right) - \hat{j}\ \frac{\partial}{\partial y}\left(\frac{1}{2}Ky^2\right)$$

$$- \hat{k}\ \frac{\partial}{\partial y}\left(\frac{1}{2}Kz^2\right)$$

$$= -Kx\,\hat{i} - Ky\,\hat{j} - Kz\,\hat{k}$$

$$\Rightarrow -K\left[x\hat{i} + y\hat{j} + z\hat{k}\right] \Rightarrow -K\hat{r}$$

Conservative System conditions:-

(i) $\int_a^b \vec{F} \cdot d\vec{r} = W_{ba} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$$W_{ba}^{(1)} = W_{ba}^{(2)} = W_{ba}^{(3)} \quad (\text{path } 1, 2, 3)$$

(ii) $\oint_a^b \vec{F} \cdot d\vec{r} = 0$ (closed path)

$$\int_a^b \vec{F} \cdot d\vec{r} = \iint \underbrace{(\Delta \times F)}_{\text{Rot}} \cdot dA \quad (\text{Stokes's theorem})$$

$F = -\underline{\underline{\Delta U}}$ • $(\Delta \times \nabla U = 0)$

Signature features of conservative forces :-

(i) $\int_a^b \vec{F} \cdot d\vec{r} = W_{ba}$ ⇒ Independent of path (depends on only end points)

(ii) $\oint_a^a \vec{F} \cdot d\vec{r} = 0$ ⇒ for closed loop.

(iii) $\nabla \times F = 0$ (stokes's theorem)

$\vec{F} = -\nabla U$

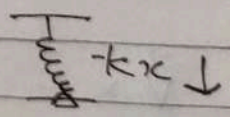
$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} \cdot dy + \frac{\partial U}{\partial z} \cdot dz$$

$$= \underbrace{\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)}_{-\vec{F}} \underbrace{(dx\hat{i} + dy\hat{j} + dz\hat{k})}_{d\vec{r}}$$

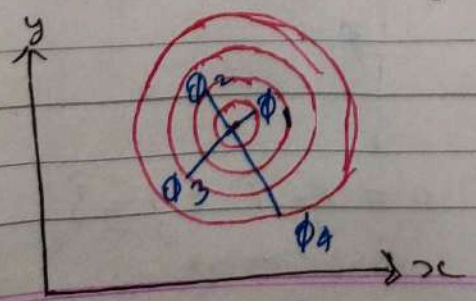
$$dU = -\vec{F} \cdot d\vec{r}$$

$\vec{F} = -\boxed{\nabla U}$ → physical meaning

$F = -kx$



potential $\phi = x^2 + y^2$



$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}$$

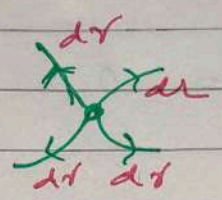
$$= (x\hat{i} + y\hat{j})\,2$$

$$\boxed{dU = \nabla U \cdot d\vec{r}}$$

$$dV = \nabla\phi \cdot d\vec{r} = 2\,[x\hat{i} + y\hat{j}]\cdot dr$$
$$= 2\,[r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}]\cdot d\vec{r}$$

$\Rightarrow$ $(\Delta V \parallel dr)$, that will be the direction of maximum change in which will be observed in dV.

$(dr = \text{constant})$



$\Rightarrow$ $dV = \nabla V \cdot \hat{\theta}$ $\quad (\Delta V = \Delta\phi)$

$$= 2\,[r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}]\cdot[-\sin\theta\,\hat{\theta} + \cos\theta\,\hat{\theta}]$$

$\Rightarrow$ 0

$(\underbrace{\nabla\phi \perp^{r} \hat{\theta}}_{\text{Equipotential surface}}, \quad \Delta\phi \parallel \hat{r})$

Conservative Forces $(\nabla \times F = 0)$

del × curl (rot) → Rotation of Field (3D)

$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & 0 \end{vmatrix}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \qquad F = \underbrace{F_x}_{\text{function of } (x,y)}\hat{i} + \underbrace{F_y}_{\text{function of } (x,y)}\hat{j}$$

$$\Rightarrow \hat{i}\left(0 - \overbrace{\frac{\partial F_y}{\partial z}}^{0}\right) - \hat{j}\left(0 - \overbrace{\frac{\partial F_x}{\partial z}}^{0}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

F_y is function of x and y,

$$\frac{\partial F_y}{\partial z} = 0$$

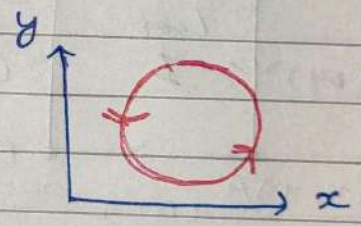
F_x is function of x and y

$$\frac{\partial F_x}{\partial z} = 0$$

F_x and F_y is function of x and y.

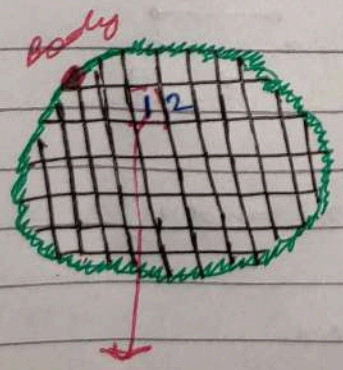
$$\frac{\partial F_y}{\partial x} \neq 0 \quad \& \quad \frac{\partial F_x}{\partial y} \neq 0$$

$$\vec{\nabla} \times \vec{F} = \hat{k} \cdot \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$



$\vec{F}$ depends on co-ordinate of x and y

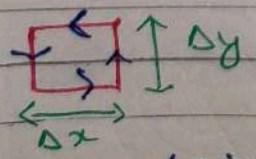
Stoke's theorem



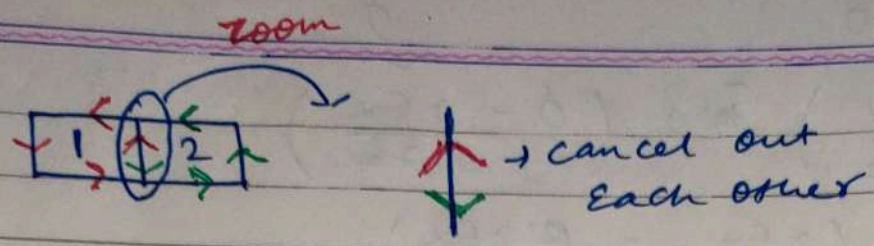
$$\underbrace{\oint \vec{F} \cdot \vec{dl}}_{\text{L.H.S}} = \underbrace{\iint_{A} (\nabla \times \vec{F}) \cdot \vec{dA}}_{\text{RHS}}$$

Areal Integral

$$\sum_{\text{loop-1}}^{\text{all loops}} \int \vec{F} \cdot \vec{dl} = \oint \vec{F} \cdot \vec{dl}$$



divide Loops in equal rectangular parts;



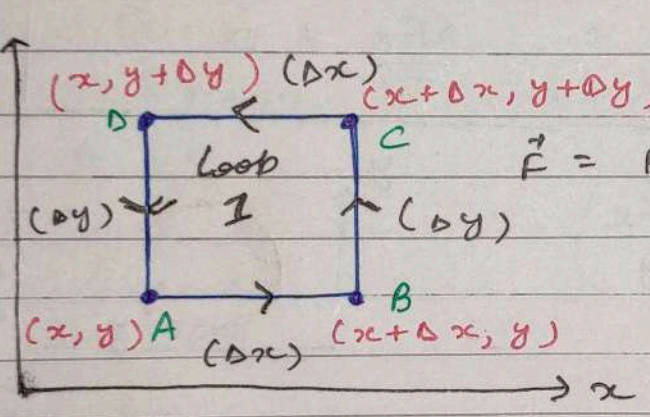
↳ Internally, all the loops circulation are going out to cancel each other and only the circulation of work done contribution will come over the surface of this particular body.

(LHS)

$\oint \vec{F} \cdot d\vec{l}$ ⇒ work due to surface of Body.

(Internal circulation cancels out)

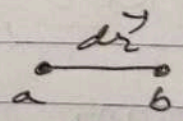
↳ (RHS)



$\vec{F} = F_x \hat{i} + F_y \hat{j}$

(circulation of Field / Force)
⇓
(Field lines)

(1) $\oint \vec{F} \cdot d\vec{l}$ ⇒ (A → B → C → D → A)



$\vec{F} \cdot d\vec{l}$

A-B $\quad F_x(x,y) \cdot \Delta x \, \hat{i} = F_x \cdot \Delta x$

∴ ($F_x(x,y)$ ⇒ F_x is function of x and y)

C-D $\quad F_x(x,y) \cdot \Delta x$

B-C $\quad -F_x(x, y+\Delta y)\, \Delta x$

B-C $\quad F_y(x+\Delta x, y)\, \Delta y$

D-A $\quad -F_y(x,y)\, \Delta y$

$$\oint \vec{F}\cdot d\vec{l} = A\to B + B\to C + C\to D + D\to A$$

$$\therefore\ F_x(x,y)\cdot\Delta x + F_y(x+\Delta x, y)\cdot\Delta y - F_x(x, y+\Delta y)\cdot\Delta x - F_y(x,y)\,\Delta y$$

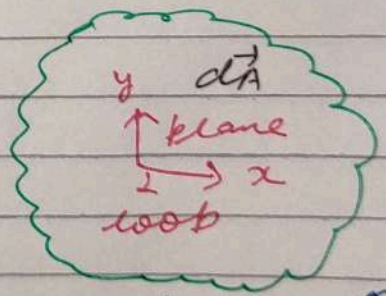
$$\boxed{\begin{aligned}\frac{dF}{dx} &= \frac{F(x+\Delta x) - F(x)}{\Delta x}\\ \frac{\partial F}{\partial x} &= \frac{F(x+\Delta x, y) - F(x,y)}{\Delta x}\end{aligned}}$$

$$\Rightarrow\ \frac{-\partial F_x}{\partial y}\cdot\Delta y\,\Delta x + \frac{\partial F_y}{\partial x}\cdot\Delta x\cdot\Delta y$$

$$\boxed{\oint_{1\,loop} \vec{F}\cdot d\vec{l} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\cdot\Delta y}$$

$$\oint_{\text{all loop}} \vec{F}\cdot d\vec{l} = \sum_{\text{all loop}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\cdot\Delta y$$

$$\underline{\underline{RHS}} = \iint_A (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$



$$\vec{F} = F_x\hat{i} + F_y\hat{j}$$

$$\boxed{(\nabla\times F) = \hat{k}\left(\frac{\partial F_y}{\partial z} - \frac{\partial F_x}{\partial y}\right)}$$

$$\boxed{d\vec{A} = dA\,\hat{k}}$$

$d\vec{A} = |dA|\,\hat{n} \to \hat{k}$

for open surface, direction of Area ⊥ ⊥r to plane of 2-D bodies

$$(\nabla \times \vec{F}) \cdot d\vec{A} = \sum_{\substack{\text{for} \\ \text{all loops}}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \cdot dx\,dy = LHS$$

2D's stokes law → Green's Theorem

$$\oint \vec{F} \cdot d\vec{l} = \iint_A (\nabla \times F) \cdot d\vec{A}$$

conservative forces, $\oint \vec{F} \cdot d\vec{l} = 0$

$$\Downarrow$$

$$(\underbrace{\nabla \times \vec{F}}_{\text{curl of that field is zero}} = 0)$$

Conservative Forces:-

(i) $W_{ba} = \int_a^b \vec{F} \cdot d\vec{l}$ (path Independent)

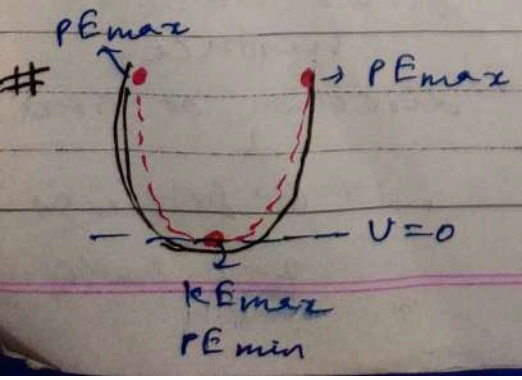
(ii) $\vec{\nabla} \times \vec{F} = 0$ (~~bad~~ Irrotational)

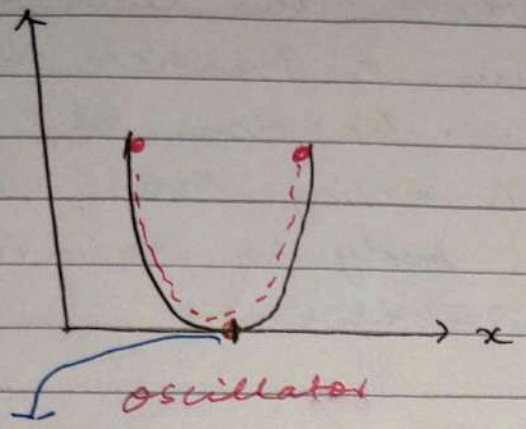
(iii) $\vec{F} = -\nabla U$

Concept of potential Energy in conservative system:

- $\vec{F} = -\nabla U$ $U \Rightarrow$ potential energy (at rest) of the system.

#





It will be the point when the whole system will be in state of equilibrium:

↳ at equilibrium condition, $\vec{F} = 0$

1D $\boxed{\frac{\partial u}{\partial x} = 0}$

Equilibrium :- (i) stable
(ii) unstable
(iii) Neutral

Example :- $U = x^2 + y^2$ $U'' = +2$

$\vec{F} = -\nabla U = -2(x\hat{i} + y\hat{j})$ (stable equilibrium)

↳ at $(0,0)$ $U = 0$, $\vec{F} = 0$

↳ As you move away from Equilibrium position $\to (0,0)$, then the force $\vec{F}$ will be acting such that particle is experiencing a force to return back to its equilibrium position.

↳ those kind of system where this kind of behaviour is present when a particle moves away from the equilibirium position and Resultant forces which is acting over the system is comparing the body to return back to its equilibirium position.

↳ when a Body is oscillating up, then always a restoring force acting so that it returns back to its equilibirium position, so those kind of system are called stable equilibirium.

Example $U = -(x^2+y^2)$ $U'' = -2$

(unstable Equilibirium)

$\vec{F} = -\nabla U = (2x\hat{i} + 2y\hat{j})$

↳ when you have the force which is acting over the system is such that you are moving away from the equilibirium position, then those kind of systems are known as unstable equilibirium.

$\boxed{2D}$ $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

Example 3 $V = x^2 - y^2$ $U'' = 0$

$\vec{F} = -\nabla V = 2(x\hat{i} - y\hat{j})$

(Neutral Equilibirium)

Stable

(1D) $\frac{\partial u}{\partial x} = 0$ $\qquad$ $v'' = +ve$ $(v'' > 0)$

(2D) $\frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0$ $\qquad$ $v'' = +ve$ $(v'' > 0)$

(3D) $\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0 \quad \frac{\partial v}{\partial z} = 0$ $\qquad$ $(v'' = +ve)$ $(v'' > 0)$

unstable

(1D) $\frac{\partial v}{\partial z} = 0$ $\qquad$ $v'' = -ve$ $(v'' < 0)$

(2D) $\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$ $\qquad$ $v'' = -ve$ $(v'' < 0)$

(3D) $\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial z} = 0$ $\qquad$ $v'' = -ve$ $(v'' < 0)$

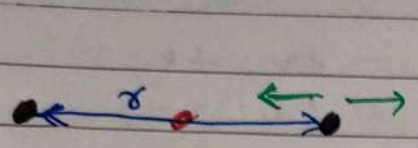
Neutral

(1D) $\frac{\partial u}{\partial x} = 0$ $\qquad$ $v'' = 0$

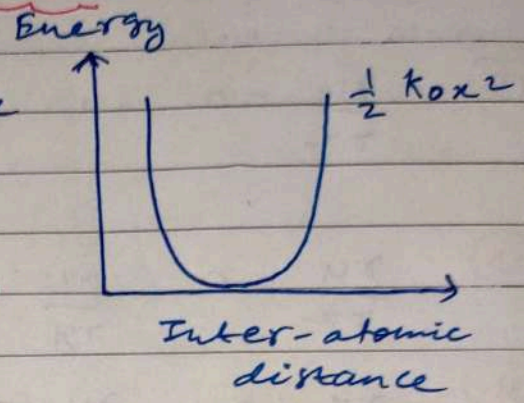
(2D) $\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \cancel{\frac{\partial v}{\partial z} = 0}$ $\qquad$ $v'' = 0$

(3D) $\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial z} = 0$ $\qquad$ $v'' = 0$

Harmonic Approximation

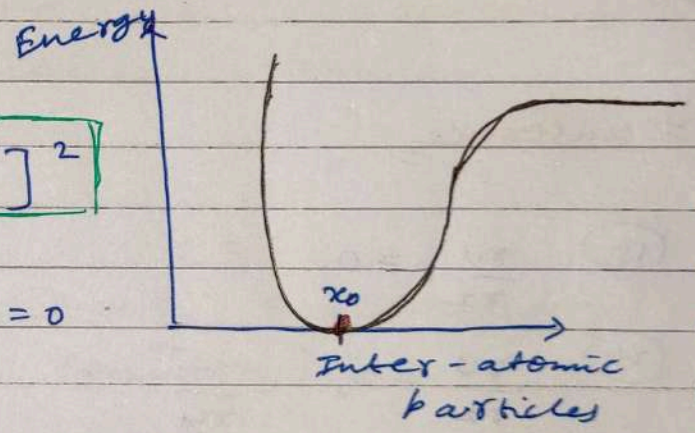


(i) for single particle (oscillator system)



(ii) for two or more particles

• morse potential form



$$\boxed{V(x) = D\left[1 - e^{-\alpha(x-x_0)}\right]^2}$$

at $x = x_0$, $\frac{\partial V}{\partial x} = 0$ $V(x_0) = 0$

at $x = \infty$, $V(x) = D$

$V(x) = V(x_0) + V'(x_0)(x-x_0)$

$$+ \frac{V''(x_0)}{2!}(x-x_0)^2 + \text{----}$$

$$V'(x) = \left.\frac{\partial V}{\partial x}\right|_{x=x_0}$$

for oscillator system,

$V(x_0) = 0$ $V'(x_0) = 0$ $V(x) = \frac{V''(x_0)}{2}(x-x_0)^2$

$\boxed{(K = V''(x_0))}$ $= \frac{1}{2} K x^2$

$F = -kx$

$ma = -kx$

$a + \frac{kx}{m} = 0$

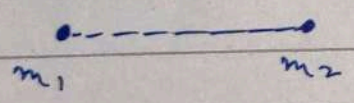
$x = A\sin\omega t + B\cos\omega t$

$\omega = \sqrt{k/m}$

Morse potential form

$$V(x) = D\left[1 - e^{-\alpha(x-x_0)}\right]^2$$

- $\underbrace{V(x_0) = 0}_{(PE=0)}$, $\underbrace{\frac{\partial V(x_0)}{\partial x} = 0}_{\text{Force}=0}$, $k = \frac{\partial^2 V}{\partial x^2}$

- 

$\mu \Rightarrow$ reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

- $v'(x) = 2D\alpha\, e^{-\alpha(x-x_0)}\left[1 - e^{-\alpha(x-x_0)}\right]$

$v'(x_0) = 2D\alpha\, e^{-0}\left[1-1\right] = 0$

- $v''(x) = 2D\alpha\left[-\alpha e^{-\alpha(x-x_0)} + 2\alpha e^{-\alpha(x-x_0)}\right]$

at equilibirium $x = x_0$, $v''(x) = 2D\alpha^2 = k$

$$k = \sqrt{\frac{\omega}{\mu}} = 2D\alpha^2$$

↳ In case of oscillator System, at very large distance, potential Energy tends to Infinity. There is no breakdown in potential energy.

↳ In case of morse potential form, at very large distance, potential Energy tends to constant value. There is breakdown in potential Energy.

Poesh - Teller potential

$$V(x) = -\frac{\lambda(\lambda+1)}{2} \operatorname{sech}^2(x)$$

$\lambda = 1, 2, ------$

$n = 1, 2, ---- \lambda-1, \lambda$

↳ Eigen values

Lennard - Jones potential

$$V_{LJ} = 4\epsilon \left[\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6} \right]$$

$$K = 2\epsilon \left(\frac{m}{r_0}\right)^2$$

#

Conservation on laws and collisions

↳ $W = \int_a^b \vec{F} \cdot d\vec{r} = \int_a^b (F_c + F_{NC}) \cdot dr$

$= \int F^c \cdot d\vec{r} + W_{NC}$

$K_b - K_a = -U_b + U_a + W_{NC}$

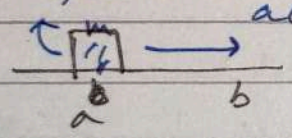
$\Downarrow$

$\boxed{E_b - E_a = W^{NC}}$

Example of non-conservative forces
- ↳ Viscous force
- ↳ Frictional force

(path dependent)

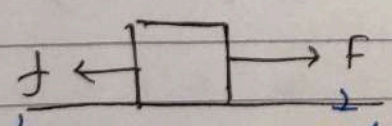
1D



↳ mechanic conservation Energy is not valid:

⇒ It is conservative system, because it It does not depends on path, because in 1D, body can travel from point A to point B in one direction:

#

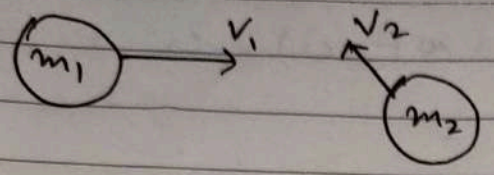


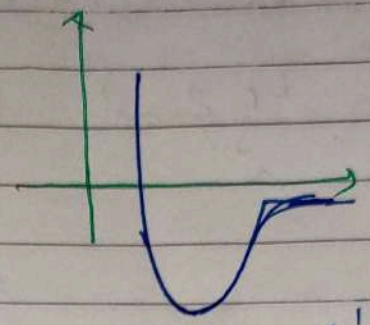
frictional force

external force:

$E_b - E_a = W^{NC}$ } Conservation of Energy:

Collision

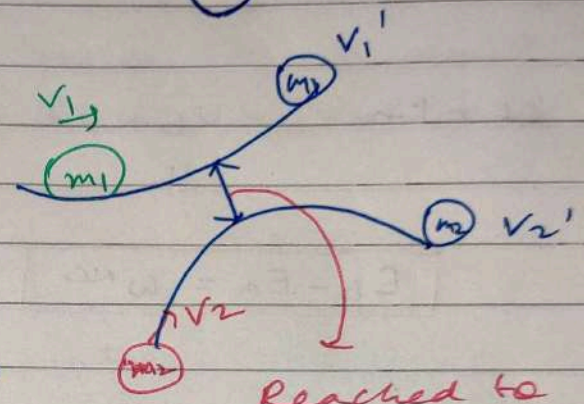




↳ ~~In Actual~~

↳ If you are working in day-to-day life scenario, so you have the actual physical collisions happen

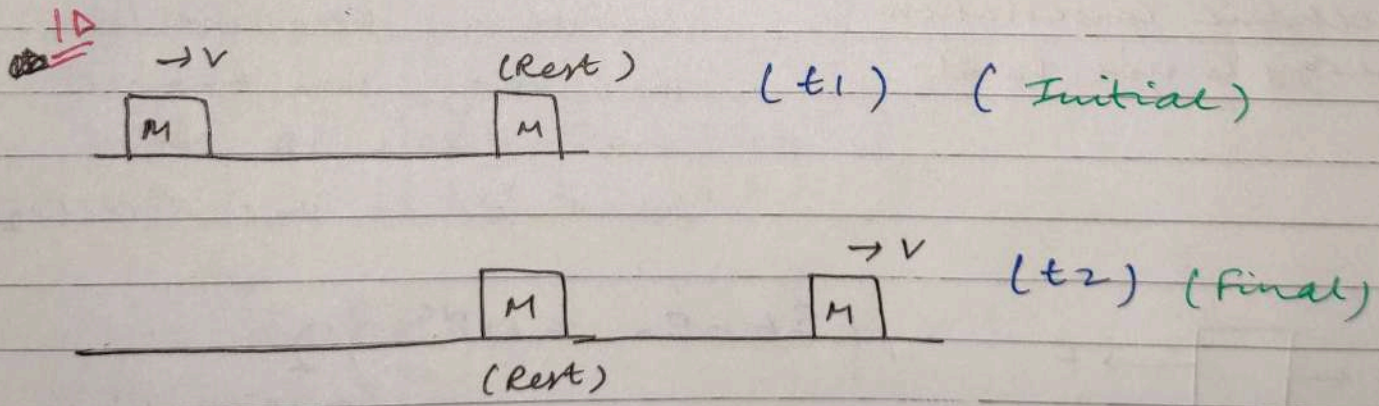
↳ when you are working in the atomic dimensions, the actual physical collision don't take place.



Reached to Impact parameter distance.

(i) Elastic collision :- (P, KE = constant)

~~(ii) Inelastic collision~~

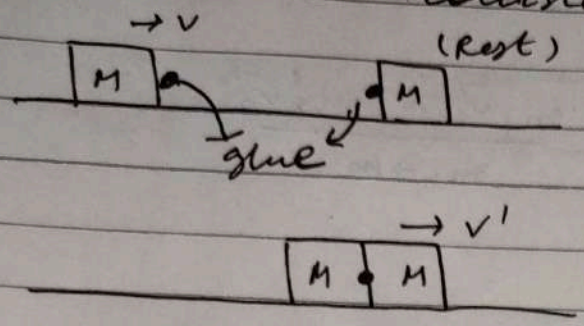


	(i)	(f)
Momentum	MV	MV
Kinetic Energy	$\frac{1}{2}MV^2$	$\frac{1}{2}MV^2$

→ Momentum and Kinetic energy remains unchanged

⇒ Elastic collision.

(ii) Inelastic collision (P = constant, KE = variable)



(t1) (Initial)

(t2) (Final)

	initial	final
momentum	Mv	$2Mv'$
Kinetic Energy	$\frac{1}{2}Mv^2$	$\frac{1}{2}(2M)v'^2$

• as $F_{ext} = 0$, momentum is conserved

$$Mv = 2Mv'$$

$$\boxed{v' = \frac{v}{2}}$$

$K_i = \frac{1}{2}Mv^2$ # $K_f = \frac{1}{2}(2M) \times \frac{v^2}{4} \Rightarrow \frac{Mv^2}{4}$

$$\boxed{K_i \neq K_f \quad , \quad P_i = P_f}$$

$E_b - E_a = W_{NC}$, work associated with glue.

• $H + H \rightarrow H_2 + E$

→ Energy loss (heat & Thermal / dissociation / combination Energy)

↓

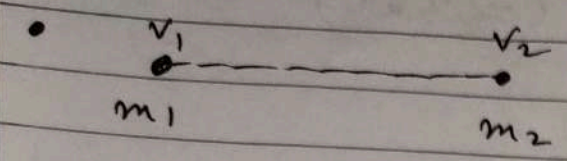
work done is non-conservative in Nature.

Momentum remains unchanged and kinetic energy changed

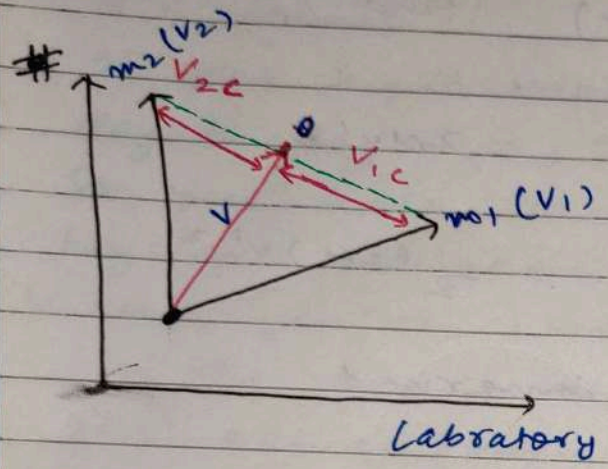
⇓

Inelastic collision

Labratory / COM frame of reference for collision :-



$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$



$v_{1c} \Rightarrow$ velocity of mass 1 wrt centre of mass

$v_{2c} \Rightarrow$ velocity of mass 2 wrt centre of mass.

$\vec{v}_c \Rightarrow$ velocity of centre of mass

$$\vec{v}_{1c} = \vec{v}_1 - \vec{v}_c = \vec{v}_1 - \left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}\right)$$

$$= \frac{m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$$

$$\vec{v}_{2c} = \vec{v}_2 - \vec{v}_c = \vec{v}_2 - \left(\frac{m_1 v_1 + m_2 v_2}{m_1+m_2}\right)$$

$$\Rightarrow \frac{-m_1}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$$

$\vec{P}_{1c} = m_1\vec{v}_{1c} = \frac{m_1 m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$

$\vec{P}_{2c} = m_2\vec{v}_{2c} = \frac{-m_1 m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$

$\boxed{\vec{P}_{1c} + \vec{P}_{2c} = 0}$ Total Momentum of system wrt centre of mass initially is coming out to be zero.

CM frame

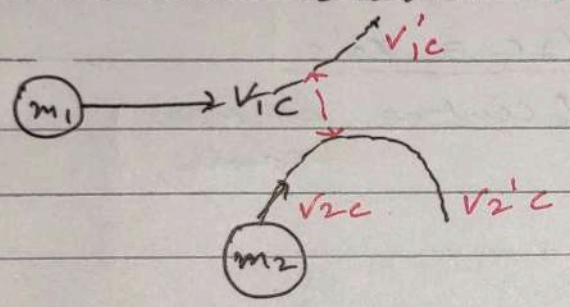
Labratory Frame

$$\boxed{m_1\vec{v_1} + m_2\vec{v_2} = (m_1+m_2)\vec{v}}$$

COM Frame

$$\boxed{\begin{array}{l} \vec{P}_{1c} + \vec{P}_{2c} = 0 \text{ (initial)} \\ \vec{P}'_{1c} + \vec{P}'_{2c} = 0 \text{ (Final)} \end{array}}$$

Elastic collision (WRt COM)



Momentum

- $m_1 v_{1c} - m_2 v_{2c} = 0$ (before collision)

$$\Rightarrow v_{2c} = \frac{m_1}{m_2} v_{1c}$$

- $m_1 v'_{1c} - m_2 v'_{2c}$ (After collision)

$$\Rightarrow v'_{2c} = \frac{m_1}{m_2} v'_{1c}$$

Kinetic Energy

$$\underbrace{\frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2}_{\text{before collision}} = \underbrace{\frac{1}{2} m_1 v'^2_{1c} + \frac{1}{2} m_2 v'^2_{2c}}_{\text{After collision}}$$

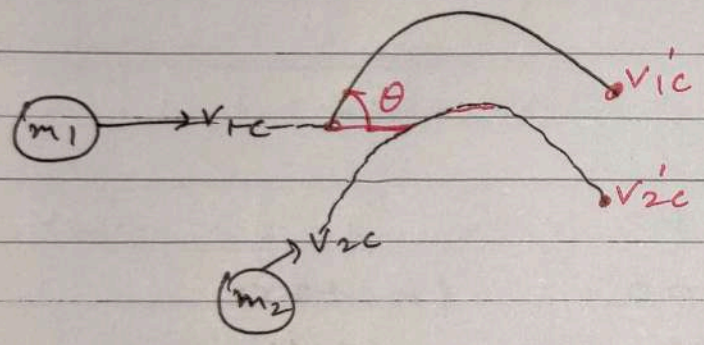
- $V_{2c} = \frac{m_1}{m_2} V_{1c} \Rightarrow V_{2c}^2 = \frac{m_1^2}{m_2^2} V_{1c}^2$
- $V_{2c}' = \frac{m_1}{m_2} V_{1c}' \Rightarrow V_{2c}'^2 = \frac{m_1^2}{m_2^2} V_{1c}'^2$

KE

$$\frac{1}{2}\left[m_1 + \frac{m_1^2}{m_2}\right] V_{1c}^2 = \frac{1}{2}\left[m_1 + \frac{m_1^2}{m_2}\right] V_{1c}'^2$$

$$V_{1c}^2 = V_{1c}'^2 \Rightarrow \boxed{\begin{array}{l} V_{1c} = V_{1c}' \\ V_{2c} = V_{2c}' \end{array}}$$

(centre of mass frame)



↳ After collision, these velocity are going out to be the same, only thing which is going to change it up here is the angle in between there. ($\theta \to$ angle of reflection)

$\boxed{\begin{array}{l} V_1 \neq V_1' \\ V_2 \neq V_2' \end{array}}$ ← (Labarotary Frame) ↳ $\boxed{P_1 + P_2 \neq 0}$

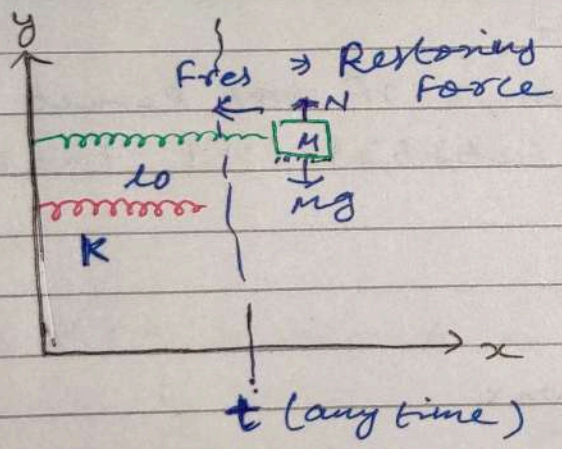
Harmonic Oscillation :-

- → Restoring Forces acting on system.
- → depending on system drag forces and other parameters
- → Nature of this kind of oscillations
- → behaviour of displacement will come with respect to time.

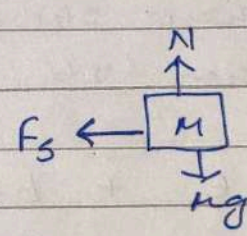
Harmonic Oscillation

- → Undamped
- → Damped
- → Driven
- → Damped driven.

Undamped :-



$F_s \rightarrow$ spring force



** (Friction less surface) **

$(N = mg)$

$(N.\hat{y} = mg.\hat{y})$

using Newton's 2nd Law

$$\boxed{\vec{F} = -kx\,\hat{i}}$$

$m\ddot{x} = -kx$ # $x(t) = a\cos\omega_0 t$

$$\boxed{\ddot{x} = -\frac{k}{m}x}$$

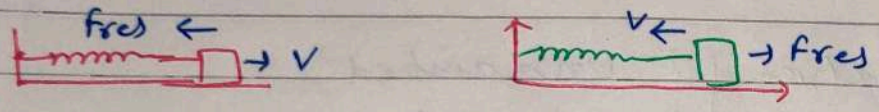
$$\left.\begin{aligned} LHS &= -\omega_0^2 x \\ RHS &= \frac{-kx}{m} \end{aligned}\right\} \rightarrow \omega_0^2 = \frac{k}{m}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$x = b\sin\omega_0 t$ $\quad \dot{x} = b\omega_0 \cos\omega t$ $\quad \ddot{x} = -b\omega_0^2 \sin\omega t$

$$\left.\begin{aligned} \underline{\underline{LHS}} &= -\omega_0^2 x \\ RHS &= -\frac{kx}{m} \end{aligned}\right\} \rightarrow \omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

Here $x(t) = a\cos\omega_0 t$ and $x(t) = b\sin\omega_0 t$ are periodic functions



(Fres → Restoring Force)

↳ we can choose any periodic function for satisfying the eqn $\boxed{\ddot{x} = -\frac{k}{m}x}$, If the Result comes true, then It satisfies, If ~~not~~ Result comes false, then It not satisfies (Hit - trial)

$x = a\cos\omega_0 t + b\sin\omega_0 t$

$\underline{\underline{LHS}}$

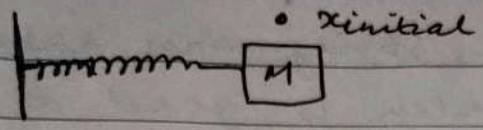
$$\dot{x} = -a\omega_0 \sin\omega t + b\omega_0 \cos\omega_0 t$$

$$\ddot{x} = -a\omega_0^2 \cos\omega t - b\omega_0^2 \sin\omega_0 t$$

$$\left.\begin{aligned} &\Rightarrow -\omega_0^2 x \\ \underline{\underline{RHS}} \quad &-\frac{kx}{m} \end{aligned}\right\} \rightarrow \omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

↳ this is also a solution of this system.

#



$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$

Initial condition (i) $x(0) = x_{initial}$

(ii) $\dot{x}(0) = 0$

at $\boxed{t=0}$ $x(0) = x_{inital} = a \cdot 1 + b \cdot 0 = \underline{\underline{a}}$

$\Downarrow$

$x_{inital} = a$

$\dot{x}(0) = 0$

$\Downarrow$

$-\omega_0 a \sin\omega_0 t + b\omega_0 \cos\omega_0 t = 0$

$\boxed{\cancel{\omega_0 a \sin\omega_0 t + b\omega_0 \cos\omega_0 t}}$

$0 + b\omega_0 \Rightarrow b\omega_0 = 0$

$\Downarrow$

$b = 0$

$x(t) = a\cos\omega_0 t = \underbrace{x_i}_{\text{Amplitude}} \cos \underbrace{\omega_0}_{\sqrt{\frac{k}{m}}} t$

$\cos x \in [-1, 1]$

$\downarrow$

never go beyond your Amplitude

Physical situation $\rightarrow$ Mathematical Problem $\rightarrow$ solution

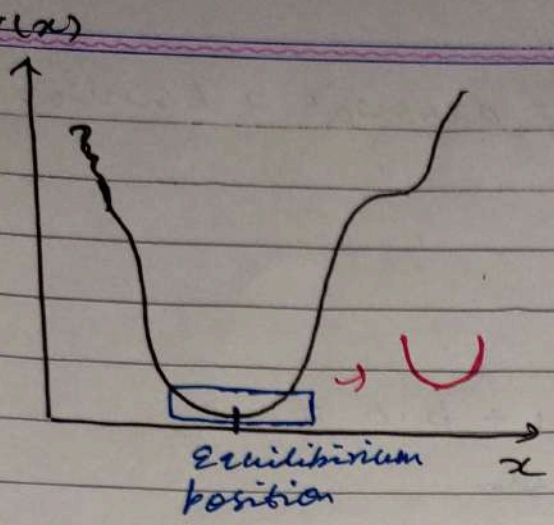
$\Downarrow$

Physical condition

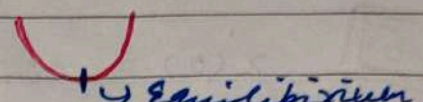
Ideal spring mass :-

↳ used for small Amplitude vibration.

#



↳ The spring-mass ~~approximation~~ problem is good ~~of~~ approximation for this type of problem.



$$V(x) = V(0) + V'(0)\cdot\frac{x}{1} + V''(0)\cdot\frac{x^2}{2!} + V'''(0)\cdot\frac{x^3}{3!} + \cdots$$

$$\vec{F} = -\frac{d\vec{V}}{dx} = -\left[\underbrace{V'(0)}_{0} + V''(0)\cdot x + V'''(0)\cdot\frac{x^2}{2} + \cdots\right]$$

(equilibrium position, where net Force is acting zero.)

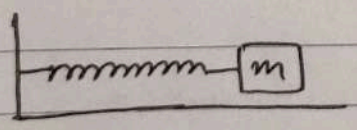
$$\boxed{\vec{F} = -V''(0)\,x}$$

($x \to$ very very small)

$|x V''(0)| \ll |V''|$

Ideal Spring mass Problem (undamped)

VVI #

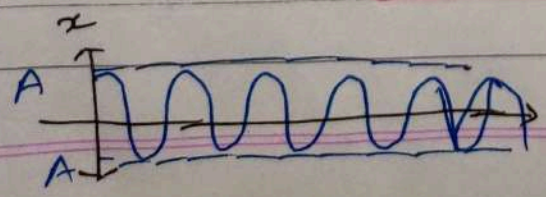


$$\ddot{x} + \frac{k}{m}x = 0$$

$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t \quad \text{--- (i)}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

↳ Body will move for Infinite time.



(Amplitude → remains same)

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$ —(i)

* $x(t) = c\cdot\cos(\omega_0 t + \phi)$ —(ii)

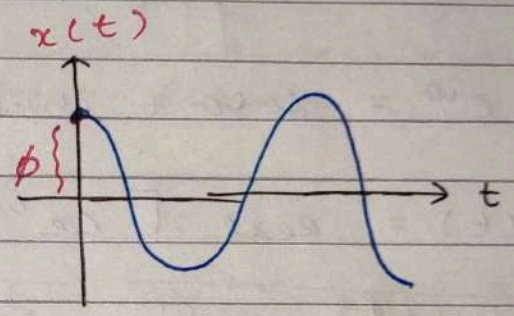
} → Both are of same form.

$\Rightarrow c\cdot\cos\phi\cos\omega_0 t - c\sin\phi\sin\omega_0 t$

$(\because \cos(A+B) = \cos A\cos B - \sin A\sin B)$

$(c\cdot\cos\phi = a)$ & $(-c\sin\phi = b)$

$\boxed{c = \sqrt{a^2+b^2}}$, $\boxed{\tan\phi = -\frac{b}{a}}$



Time translation invariance :-

$t' = t + a$

$$\frac{dx(t+a)}{dt} = \frac{dx(t+a)}{dt'}\cdot\underbrace{\frac{d(t+a)}{dt}}_{1} = \frac{dx(t+a)}{dt'}$$

$$= \frac{dx(t')}{dt'}$$

$f(x)$, $\xi = x + a$ → say (any arbitary func^n)

$$\frac{\partial f(\xi)}{\partial x} = \frac{\partial f(\xi)}{\partial \xi}\cdot\underbrace{\frac{\partial \xi}{\partial x}}_{1} \Rightarrow f'(\xi)$$

If time is ~~no~~ changing

⇓

the eq^n for $x(t)$ remains same.

(time invariance not loses)

If ω_0 is changing with time or Restoring force is changing with time:

$\Downarrow$

the eq^n for $x(t)$ will be different
(Time invariance loses)

- $x(t) = a\cos\omega_0 t + b\sin\omega_0 t$
  $x(t) = c\cdot\cos(\omega t + \phi)$ } → similar functions.

$e^{i\theta} = \cos\theta + i\sin\theta$

$x(t) = \text{Real}\left[ce^{i(\omega_0 t + \phi)}\right]$

$e^{i(\omega t + \phi)} = R\left[c\cos(\omega t + \phi) + ic\cdot\sin(\omega t + \phi)\right]$

$\rightarrow \left[c\cos(\omega t + \phi) + ci\sin(\omega t + \phi)\right]$

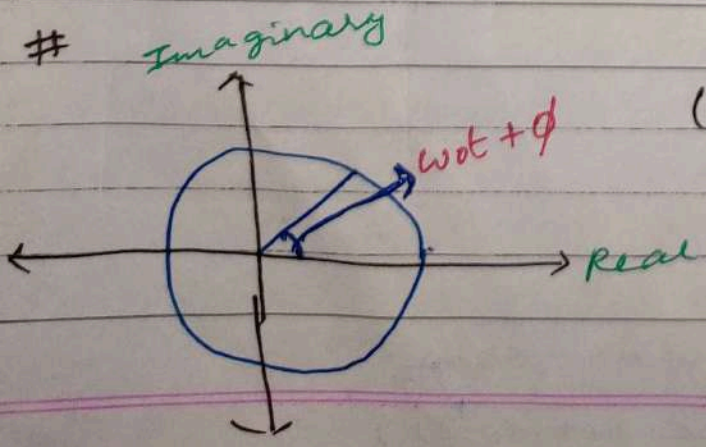
(R → Real part)

$x(t) = \text{Real}\left[\underbrace{c\cdot\cos(\omega_0 t + \phi) + i\cdot c\sin(\omega_0 t + \phi)}\right]$

Real part of this

$\Downarrow$

$c\cdot\cos(\omega_0 t + \phi)$ → Similar functions or solution.



(Argand diagram)

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$
$x(t) = c \cdot \cos(\omega t + \phi)$
$x(t) = \text{Real}\left[ce^{i(\omega t + \phi)}\right]$

→ similar solutions

Advantage of $x(t) = \text{Real}\left[c \cdot e^{i(\omega t + \phi)}\right]$

↳ $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

↳ Pheonix function (Function is not going to be zero, when we find its derivative)

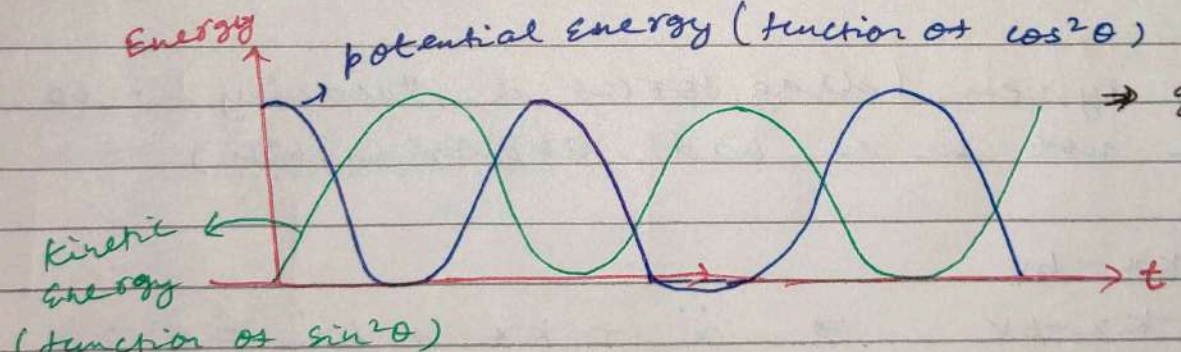
↳ Involve both Real & Imaginary Plane.

* $x(t) = c \cdot \cos(\omega_0 t + \phi)$

$F = -\frac{dU}{dx}$

- $U = \frac{1}{2}kx^2 = \frac{1}{2}kc^2 \cos^2(\omega_0 t + \phi)$
- $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 c^2 \sin^2(\omega_0 t + \phi)$

$(K = m\omega_0^2)$



→ graph for $(\phi = 0)$

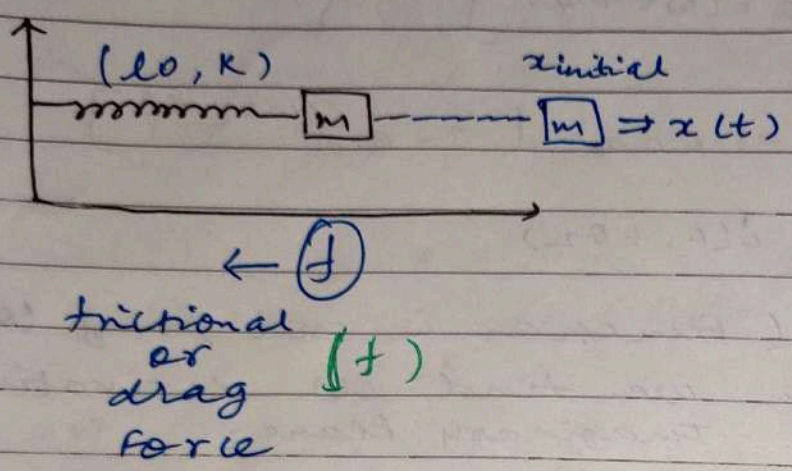
(PE + KE = constant) → at any point of time

$$\boxed{\langle U \rangle = \langle K \rangle = \frac{1}{4}kc^2}$$ (neglecting friction)

$\langle U \rangle$ → average potential energy
$\langle K \rangle$ → average kinetic energy

$$\boxed{\langle \cos^2\theta \rangle = \frac{1}{2}, \quad \langle \sin^2\theta \rangle = \frac{1}{2}}$$

Damped Harmonic oscillation:-



(viscous) frictional Force

$(f = -b\vec{v})$

↳ we taking frictional force as linear function of velocity, because Quadratic and higher velocity of functions $[-b\vec{v}^2, -bv^3, bv^K, (K>1)]$ are not solvable by analytical method for numerical equation.

↳ In many system, (drag forces $\propto$ velocity). so it will not be a bad approximation)

$$\vec{F} = -Kx - bv$$

$$m\ddot{x} = -Kx - bv \Rightarrow \ddot{x} + \frac{Kx}{m} + \frac{b\dot{x}}{m} = 0$$

$$\Downarrow$$

$$\boxed{\ddot{x} + \frac{b\dot{x}}{m} + \frac{Kx}{m} = 0}$$

$\boxed{\frac{b}{m} = \gamma \qquad \frac{K}{m} = \omega_0^2}$

(Hit-trial)

$x(t) = Re[z(t)]$, $z(t) = e^{i\alpha t}$

$$\ddot{x}(t) = -\alpha^2 e^{i\alpha t} \qquad \therefore (i^2 = -1)$$
$$\dot{x}(t) = i\alpha\gamma e^{i\alpha t}$$
$$x(t) = \omega_0^2 e^{i\alpha t}$$

$$\ddot{x}(t) = i^2\alpha^2 e^{i\alpha t} = -\alpha^2 e^{i\alpha t} =$$

$$\Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \underbrace{[-\alpha^2 + i\gamma\alpha + \omega_0^2]}_{0} e^{i\alpha t} = 0$$

$$\Downarrow$$

$$\alpha^2 - i\gamma\alpha - \omega_0^2 = 0$$

$$\Downarrow$$

$$\alpha = \frac{i\gamma \pm \sqrt{i^2\gamma^2 + 4\omega_0^2}}{2}$$

$$\boxed{\begin{array}{c} ax^2 + bx + c = 0 \\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array}}$$

$$\alpha = \frac{i\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}}{2}$$

$$\alpha = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$x(t) = Re[z(t)] = Re[e^{i\alpha t}]$

$$\Rightarrow Re\left[\underbrace{e^{-\frac{\gamma t}{2}}}_{1} \cdot e^{\pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}\right]$$

Amplitude will decrease $\Leftarrow$ exponentially decreasing time

3 cases :- (i) light damped
(ii) critical damped
(iii) over damped

Light damping ($\omega_0^2 > \frac{\gamma^2}{4}$)

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} ,$$

$$z_+(t) = e^{-\frac{\gamma t}{2}} \cdot e^{i\omega t}$$ (positive solution)

$$z_-(t) = e^{-\frac{\gamma t}{2}} \cdot e^{-i\omega t}$$ (Negative solution)

For real solution, we take $[z_+(t) \oplus z_-(t)]$
↳ sum

$$\Rightarrow z_1(t) = z_+(t) + z_-(t)$$

$$\Rightarrow e^{-\frac{\gamma t}{2}} [e^{i\omega t} + e^{-i\omega t}]$$

$$\Rightarrow e^{-\frac{\gamma t}{2}} [\cos\omega t + i\sin\omega t + \cos\omega t - i\sin\omega t]$$

$$\Rightarrow e^{-\frac{\gamma t}{2}} [2\cos\omega t]$$

difference

For real solution, we take $\left[\frac{z_+(t) \ominus z_-(t)}{2i}\right]$

$$\Rightarrow z_2(t) = \frac{z_+(t) - z_-(t)}{2i}$$

$$\Rightarrow e^{-\frac{\gamma t}{2}} \left[\frac{e^{i\omega t} - e^{i\omega t}}{2i}\right]$$

$$\Rightarrow e^{-\frac{\gamma t}{2}} \left[\frac{\cos\omega t + i\sin\omega t - \cos\omega t + i\sin\omega t}{2i}\right]$$

$$\Rightarrow \quad e^{-\frac{\gamma t}{2}} \left[\frac{2i \sin \omega t}{2i} \right] \quad \Rightarrow \quad e^{-\frac{\gamma t}{2}} \sin \omega t$$

$$\boxed{\begin{aligned} z(t) &= e^{-\frac{\gamma t}{2}} \left[a \cos \omega t + b \sin \omega t \right] \\ z(t) &= A e^{-\frac{\gamma t}{2}} \cos(\omega t + \phi) \\ \omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \qquad \gamma = \frac{b}{m}, \quad \omega_0^2 = \frac{k}{m} \end{aligned}}$$

Initial Conditions:-

(i) $z(0) = z_{initial}$ (ii) $\dot{z}(0) = 0$

(i) $z(0) = A e^0 \cdot \cos(0 + \phi) = A \cos \phi$

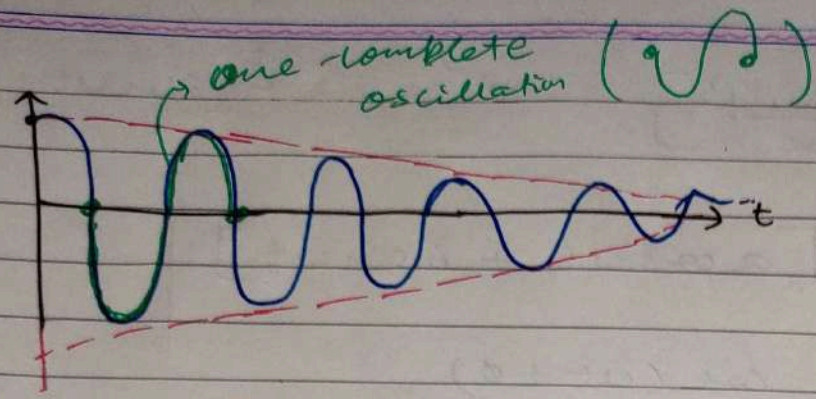
$z_{inital} = A \cos \phi \quad \Rightarrow \quad \boxed{A = \frac{z_{initial}}{\cos \phi}}$

(ii) $\dot{z}(t) = -\frac{A \gamma}{2} e^{-\frac{\gamma t}{2}} \cos(\omega t + \phi) - A e^{-\frac{\gamma t}{2}} \cdot \omega \cdot \sin(\omega t + \phi)$

$\dot{z}(0) = -\frac{A \gamma}{2} e^0 \cdot \cos(\phi) - A \cdot e^0 \cdot \omega \sin(\phi)$

$\Rightarrow \quad -\frac{A \gamma}{2} \cos \phi - A \omega \sin \phi = 0$

$\Rightarrow \quad \boxed{\tan \phi = \frac{-\gamma}{2\omega}}$



Amplitude
↓
decreasing

(Reason ⇒ In function of x, you have a exponentially decreasing function $e^{-\gamma t/2}$)

$(\omega T = 2\pi)$

$(\omega_0 > \omega)$

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$\boxed{Q = \frac{\omega}{\gamma}}$$

ω → omega

Q → Q-factor

γ → gamma $\left(\frac{b}{m}\right)$

Critical damping oscillator $\left(\omega_0^2 = \frac{\gamma^2}{4}\right)$

$$\alpha = i\gamma \pm \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}_{0} \Rightarrow \alpha = \frac{i\gamma}{2}$$

$$z_1(t) = \lim_{\omega \to 0} e^{-\frac{\gamma t}{2}} \cos\omega t \rightarrow e^{-\frac{\gamma}{2}t}$$

$$z_2(t) = \lim_{\omega \to 0} e^{-\frac{\gamma t}{2}} \sin\omega t \rightarrow 0$$

→ not quite helpful

$\frac{z_2(t)}{\omega} = e^{-\gamma t} \frac{\sin\omega t}{\omega}$

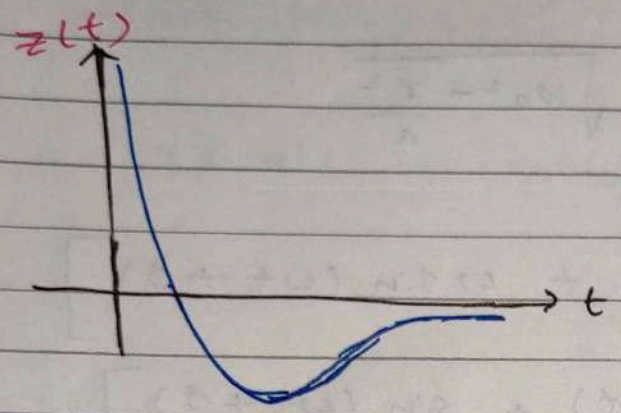
$$\lim_{\omega \to 0} e^{-\gamma t} \frac{\sin\omega t}{\omega t} \times t \Rightarrow t \cdot e^{-\frac{\gamma t}{2}}$$

$$z(t) = (1+t)\, e^{-\frac{\gamma t}{2}}$$

$$\boxed{z(t) = (A+Bt)\, e^{-\frac{\gamma t}{2}}}$$ → door stopper system

$$\boxed{\gamma = \frac{b}{m}}$$

controlling factor



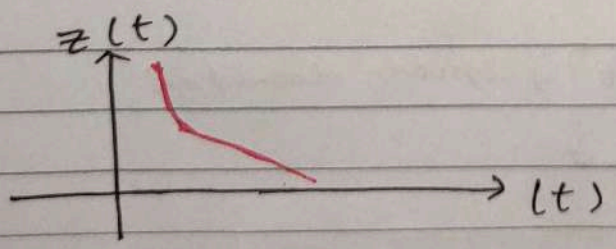
over damped oscillator $\left(\frac{\gamma^2}{4} > \omega_0^2\right)$

$$\alpha = i\gamma \pm \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}_{\text{imaginary}}$$

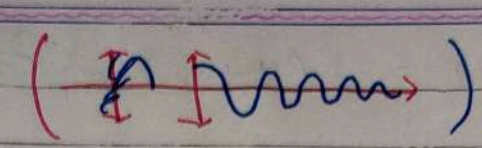
$$\alpha = i\underbrace{\left[\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]}_{\Gamma_\pm}$$

$(i^2 = -1)$

$$z(t) = e^{i\alpha t} = \bar{\alpha}\Gamma_\pm t$$



lightly (Energy) damped



$$z(t) = Ae^{-\frac{\gamma t}{2}} \cos(\omega t + \phi)$$

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$v = -Ae^{-\frac{\gamma t}{2}} \left[\frac{\gamma}{2} \cos(\omega t + \phi) + \omega \sin(\omega t + \phi) \right]$$

$$\Rightarrow -A\omega e^{-\frac{\gamma t}{2}} \left[\underbrace{\frac{\gamma}{2\omega}}_{\text{negligible}} \cos(\omega t + \phi) + \sin(\omega t + \phi) \right]$$

(because daming is small)

$$v = -A\omega e^{-\frac{\gamma t}{2}} \cdot \sin(\omega t + \phi)$$

$$\boxed{K = \frac{1}{2} m v^2 = \frac{1}{2} m A^2 \omega^2 e^{-\gamma t} \sin^2(\omega t + \phi)}$$

→ lightly damped

$$\boxed{U = \frac{1}{2} K x^2 = \frac{1}{2} K z(t)^2 = \frac{1}{2} K A^2 e^{-\gamma t} \cos(\omega t + \phi)}$$

$$(\omega \approx \omega_0) \Rightarrow (m\omega_0^2 = K)$$

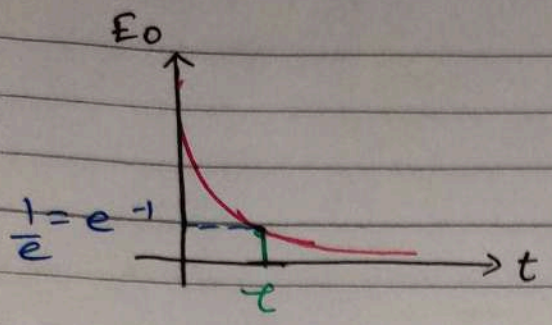
$$\boxed{E = K + U = \frac{1}{2} K A^2 e^{-\gamma t}}$$ → lightly damped

$$E(t=0) = \frac{1}{2} K A^2$$

$$\boxed{E = E_0 \cdot e^{-\gamma t}}$$

↓

Energy is continousoly decreasing.



$$\gamma\tau = 1 \Rightarrow \tau = \frac{1}{\gamma} = \frac{m}{b}$$

$$\gamma\tau = 1 \Rightarrow \tau = \frac{1}{\gamma} = \frac{m}{b}$$

Q-factor = $\dfrac{\text{Energy stored}}{\text{Energy dissipiated per radian}}$

$$\Delta E = \left|\frac{dE}{dt}\right| \cdot \Delta t = \gamma E \Delta t = \frac{\gamma E}{\omega} \quad (\omega t = 2\pi)$$

$$\pi \left(\frac{\gamma E}{\omega}\right)$$

$$Q\text{-factor} = \frac{\gamma E \Delta t}{E \cdot 2\pi} \quad \frac{\gamma E}{\omega \cdot E} = \frac{\gamma}{\omega}$$