PH100: Mechanics and Thermodynamics (3-1-0:4)

Lecture 3

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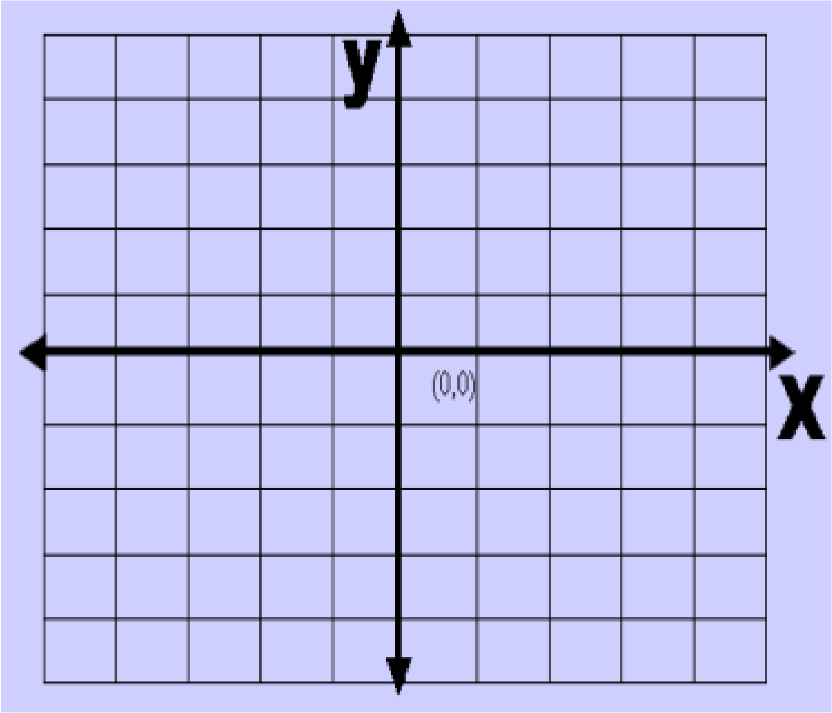
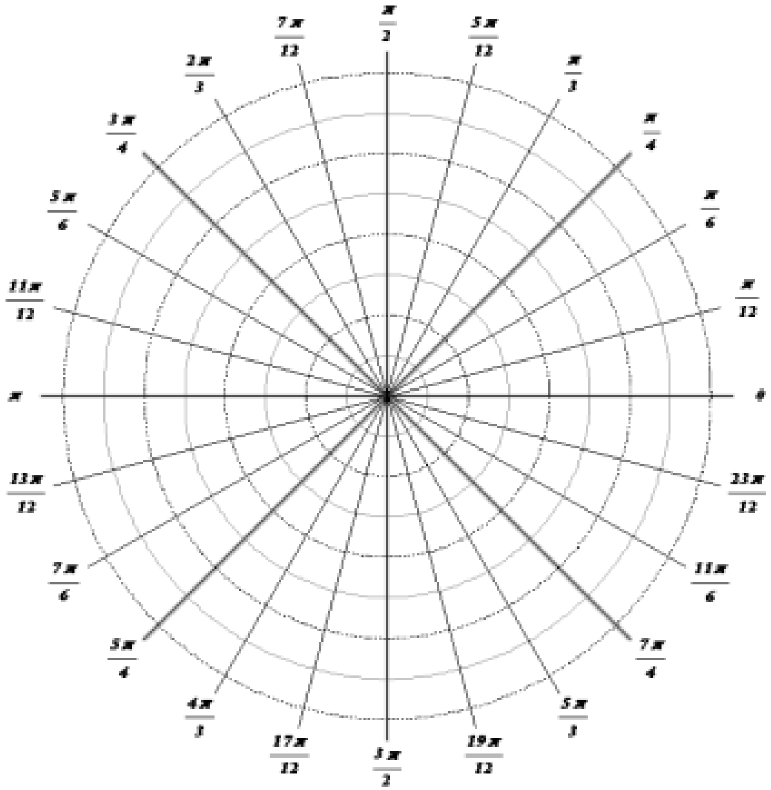
Polar Coordinates

You are familiar with plotting with a rectangular coordinate system.

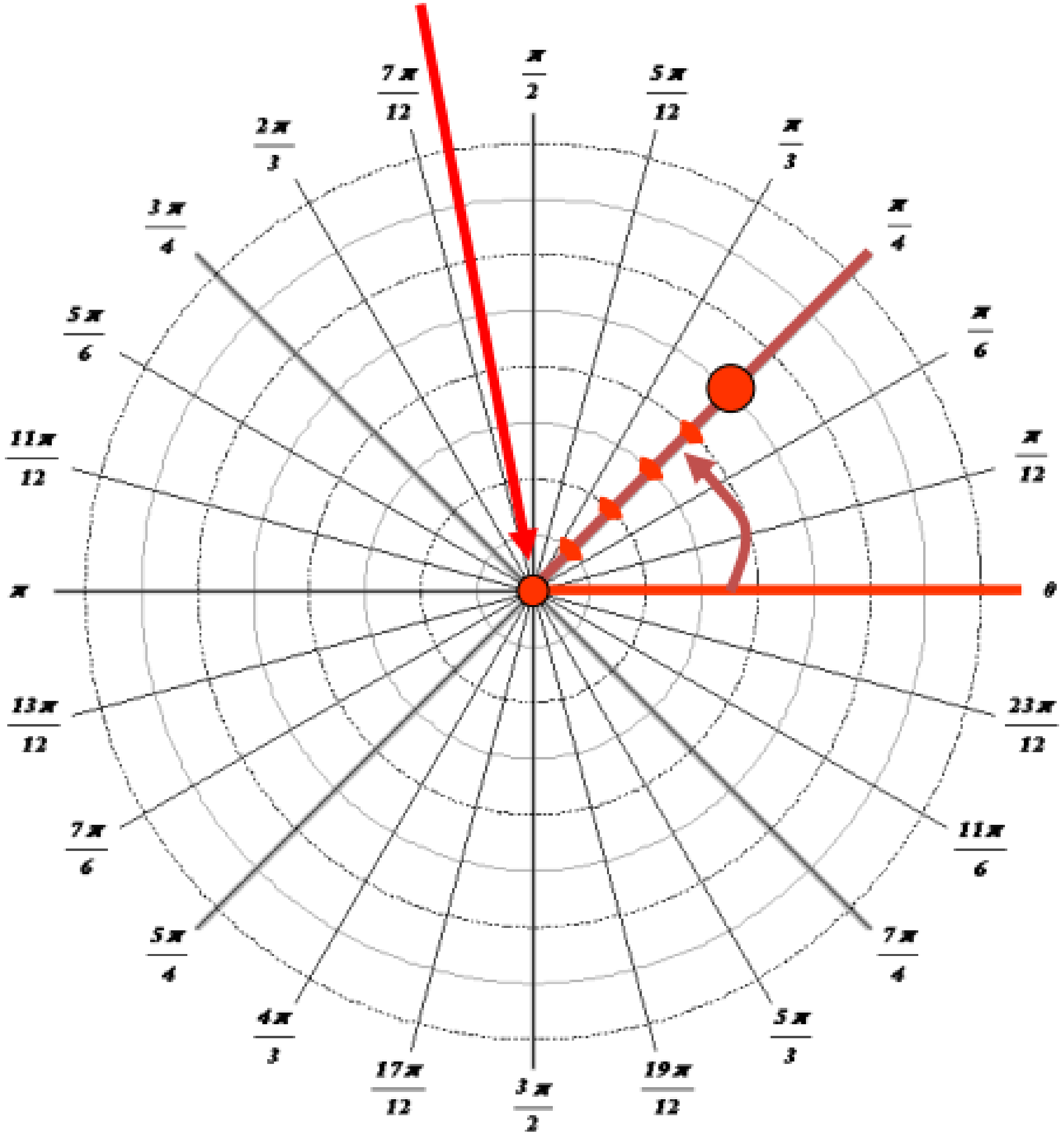
We are going to look at a new coordinate system called the polar coordinate system.





The center of the graph is called the **pole**.

Angles are measured from the positive *x* axis.



Points are represented by a radius and an angle

$$(r, \theta)$$

To plot the point

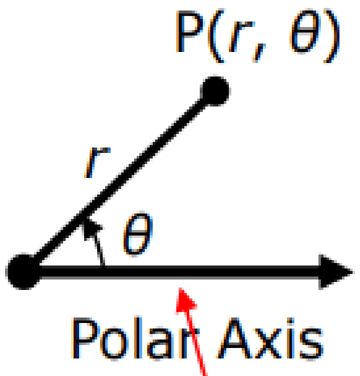
$$\left(5, \frac{\pi}{4}\right)$$

First find the angle $\pi/4$

Then move out along the terminal side 5

Polar Coordinates

To define the Polar Coordinates of a plane we need first to fix a point which will be called the Pole (or the origin) and a half-line starting from the pole. This half-line is called the Polar Axis.

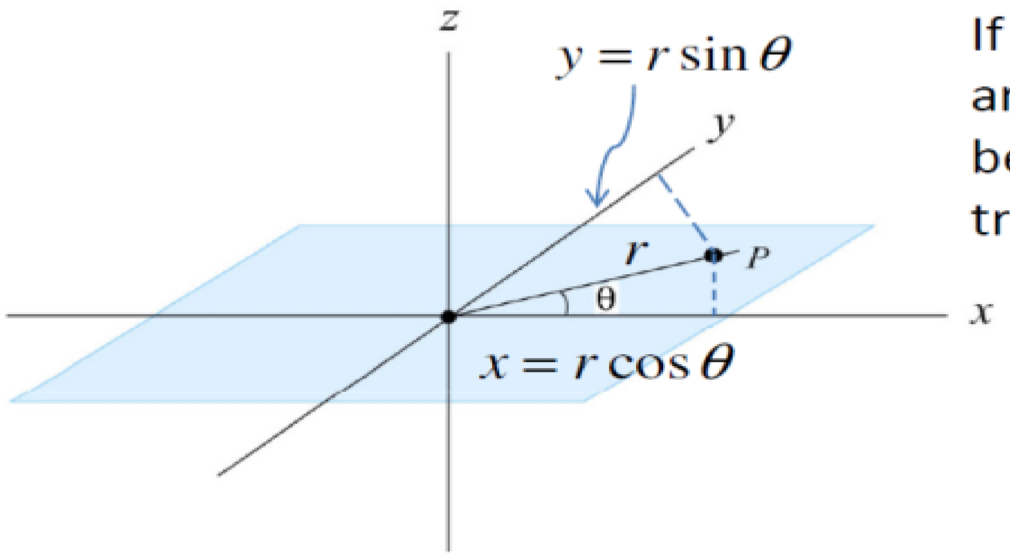


A positive angle.

Polar Angles

The Polar Angle θ of a point P, P $\neq$ pole, is the angle between the Polar Axis and the line connecting the point P to the pole. Positive values of the angle indicate angles measured in the counterclockwise direction from the Polar Axis.

Polar and Cartesian coordinates:



If polar coordinates (r, θ) of a point in the plane are given, the Cartesian coordinates (x, y) can be determined from the coordinate transformations

$$x = r\cos\theta \qquad r = +(x^2 + y^2)^{1/2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

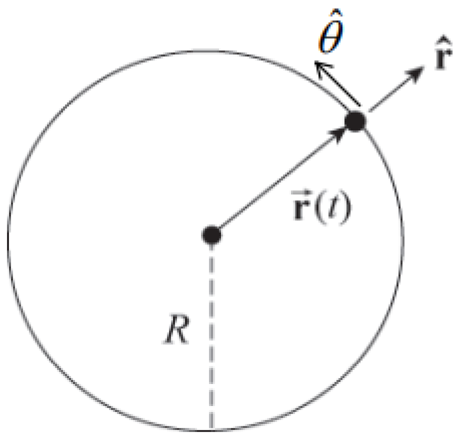
Note: $r \geq 0$ so take the positive square root only.

Since $\tan\theta = \tan(\theta + \pi)$

For $0 \leq \theta \leq \pi/2$ $x \geq 0$ and $y \geq 0$

For $(-x, -y)$ take $\theta + \pi$

Unit Vectors in Polar coordinates



The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

In Cartesian coordinate: $\vec{r} = x\hat{i} + y\hat{j}$

By coordinate transformations:
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore: $\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$

The unit vectors are defined as : $$\hat{r} = \frac{\partial\vec{r}/\partial r}{|\partial\vec{r}/\partial r|} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{r}\times\hat{\theta} = \hat{k}$$

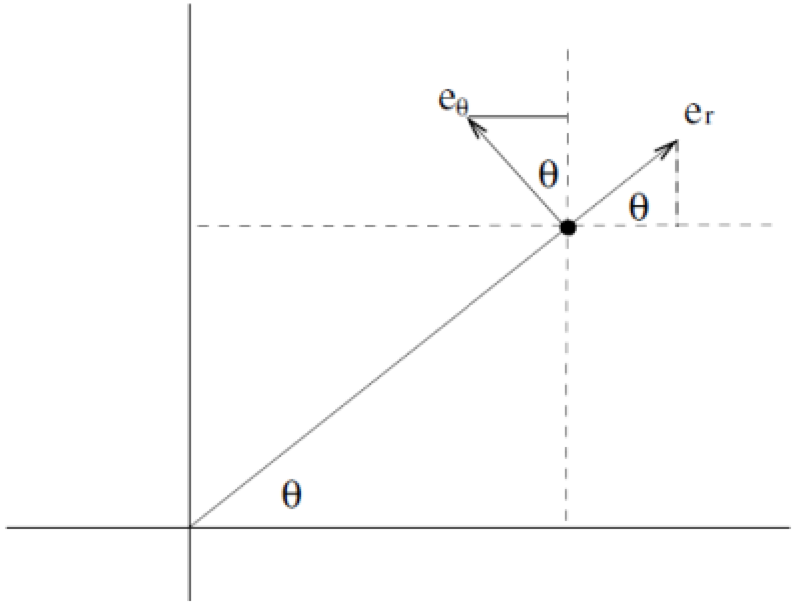
$$\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$
$$\hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

Unit Vectors in Polar coordinates

Define at each
point, a set of two unit vectors
$\hat{\mathbf{r}}$ and $\hat{\theta}$ as shown in the figure.

$$
\begin{aligned}
\hat{\mathbf{r}} &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\
\hat{\theta} &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta
\end{aligned}
$$



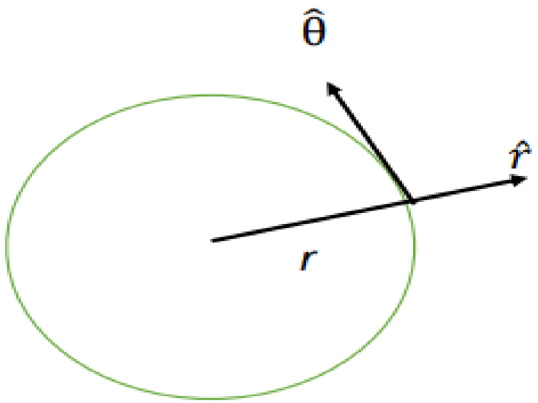
Unit vectors only depend on θ

unit vectors are functions of the polar coordinates

$$
\begin{aligned}
\frac{\partial \hat{\mathbf{r}}}{\partial \theta} &= \hat{\theta} \\
\frac{\partial \hat{\theta}}{\partial \theta} &= -\hat{\mathbf{r}}
\end{aligned}
$$

Velocity and acceleration in polar coordinates

Velocity in polar coordinate:



The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

And the unit vectors are: $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ & $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$

Since the unit vectors are not constant and changes with time, they should have finite time derivatives:

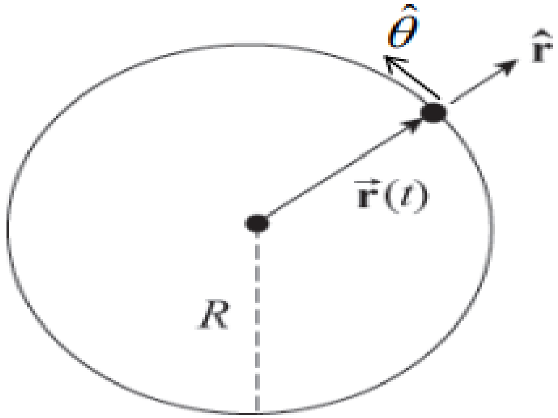
$$\dot{\hat{r}} = \dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = \dot{\theta}\hat{\theta} \quad \text{and} \quad \dot{\hat{\theta}} = \dot{\theta}\left(-\cos\theta\hat{i} - \sin\theta\hat{j}\right) = -\dot{\theta}\hat{r}$$

Therefore the velocity is given by: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Radial velocity + tangential velocity

In Cartesian coordinates $= \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$

Example-1: Uniform Circular Motion



$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Since $\dot{r} = \dfrac{dR}{dt} = 0$ and $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\omega\hat{\boldsymbol{\theta}}(t)$$

Since $\vec{v}$ is along $\hat{\theta}$ it must be perpendicular to the radius vector $\vec{r}$ and it can be shown easily

$$R^2 = \vec{\mathbf{r}}\cdot\vec{\mathbf{r}} \qquad \frac{d}{dt}R^2 = \frac{d}{dt}(\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}) = 2\vec{\mathbf{r}}\cdot\vec{\mathbf{v}} = 0, \qquad \vec{\mathbf{r}} \perp \vec{\mathbf{v}}$$

Acceleration in Polar coordinate:

$$\mathbf{a} = \frac{d}{dt}\mathbf{v}$$

$$= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta},\ \dot{\hat{\theta}} = -\dot{\theta}\hat{r}}$$

$$\mathbf{a} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

The term $\ddot{r}\hat{\mathbf{r}}$ is a linear acceleration in the radial direction due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$ is a linear acceleration in the tangential direction due to change in the magnitude of the angular velocity.

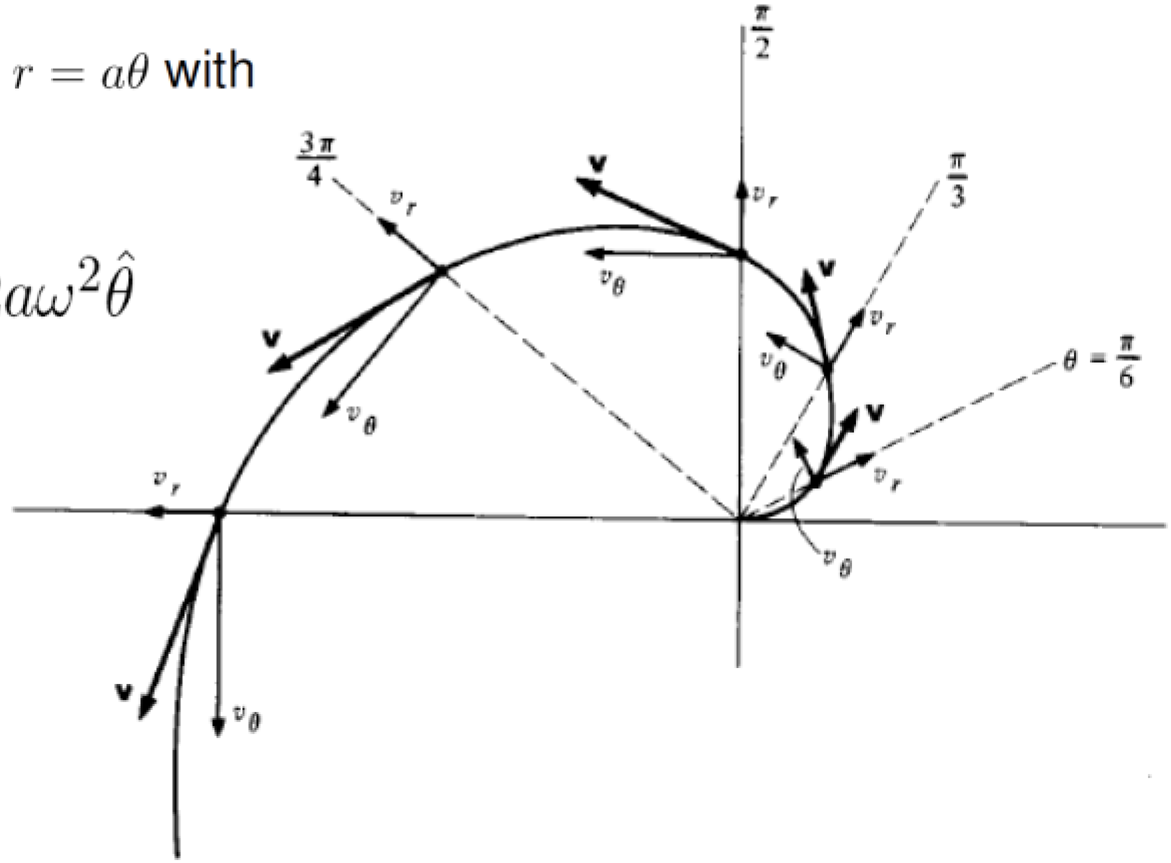
The term $-r\dot{\theta}^2\hat{\mathbf{r}}$ is the centripetal acceleration

Finally, the Coriolis acceleration $2\dot{r}\dot{\theta}\hat{\theta}$

Usually, Coriolis force appears as a fictitious force in a rotating coordinate system. However, the Coriolis acceleration we are discussing here is a real acceleration and which is present when *r* and θ both change with time.

Consider a particle moving on a spiral given by $r = a\theta$ with a uniform angular speed ω. Then $\dot{r} = a\dot{\theta} = a\omega$.

$\mathbf{v} = a\omega\hat{\mathbf{r}} + a\omega^2 t\hat{\theta}$ and $\mathbf{a} = -a\omega^3 t\hat{\mathbf{r}} + 2a\omega^2\hat{\theta}$



Motion: Kinematics in 1D

The motion of the particle is described specifying the position as a function of time, say, $x(t)$.

The instantaneous velocity is defined as

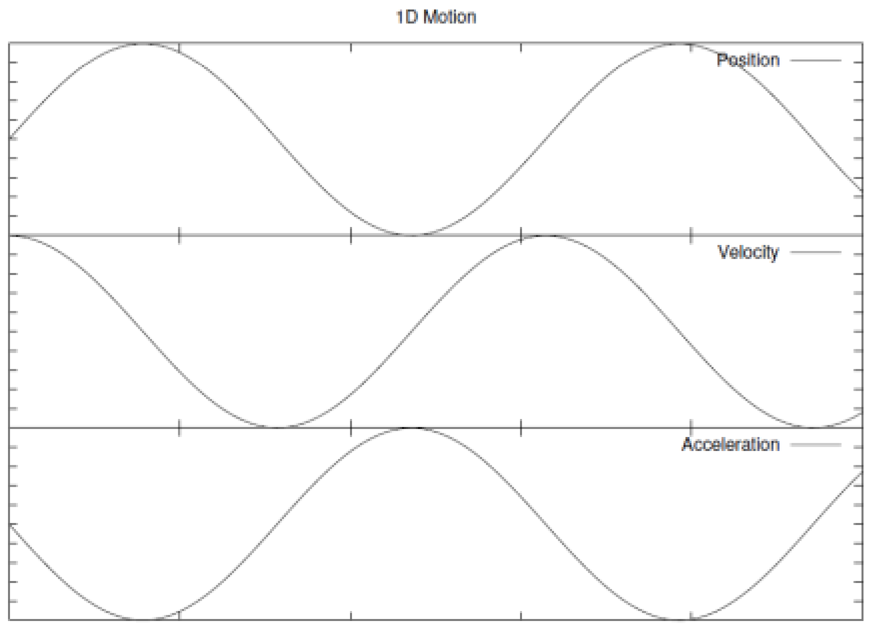
(1) $$v(t) = \frac{dx}{dt}$$

and instantaneous acceleration, as

(2) $$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example

If $x(t) = \sin(t)$, then $v(t) = \cos(t)$ and $a(t) = -sin(t)$.



Usually the $x(t)$ is not known in advance!

But the acceleration $a(t)$ is known and at some given time, say t_0, position $x(t_0)$ and velocity $v(t_0)$ are known.

The formal solution to this problem is

$$\begin{aligned} v(t) &= v(t_0) + \int_{t_0}^{t} a(t')\, dt' \\ x(t) &= x(t_0) + \int_{t_0}^{t} v(t')\, dt' \end{aligned}$$

Let the acceleration of a particle be a_0, a constant at all times. If, at $t = 0$ velocity of the particle is v_0, then

$$\begin{aligned} v(t) &= v_0 + \int_0^t a_0 dt \\ &= v_0 + a_0 t \end{aligned}$$

And if the position at $t = 0$ is x_0,

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2$$

More complex situations may arise, where an acceleration is specified as a function of position, velocity and time. $a\left(x, \dot{x}, t\right)$. In this case, we need to solve a differential equation

$$\frac{d^2x}{dt^2} = a\left(x, \dot{x}, t\right)$$

which may or may not be simple.

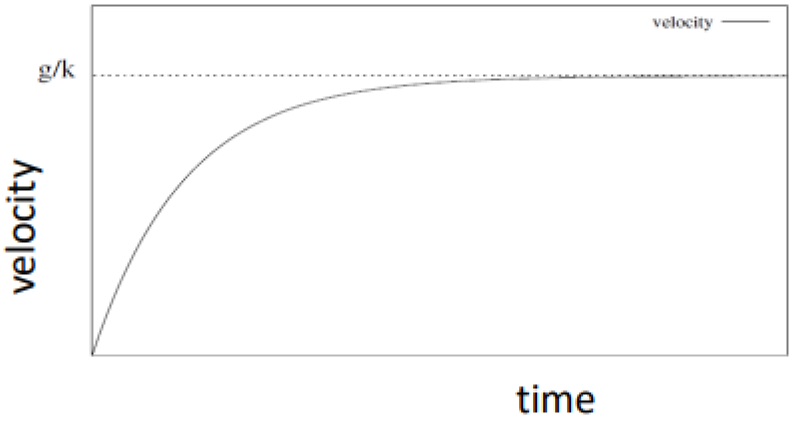
Example

Suppose a ball is falling under gravity in air, resistance of which is proportional to the velocity of the ball.

$$a\left(\dot{y}\right) = -g - k\dot{y}$$

If the ball was just dropped, velocity of the ball after time then

$$v(t) = -\frac{g}{k}\left(1 - e^{-kt}\right)$$



Kinematics in 2D

The instantaneous velocity vector is defined as

$$\begin{aligned}
\mathbf{v}(t) &= \frac{d}{dt}\mathbf{r} \\
&= \lim_{dt\to 0}\frac{\mathbf{r}(t+dt)-\mathbf{r}(t)}{dt} \\
&= \lim_{dt\to 0}\frac{x(t+dt)-x(t)}{dt}\mathbf{i} + \lim_{dt\to 0}\frac{y(t+dt)-y(t)}{dt}\mathbf{j} \\
&= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}
\end{aligned}$$

The instantaneous acceleration is given by:

$$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

Kinematics in 2D

In this case, we have to solve two differential equations

$$\frac{d^2x}{dt^2} = a_x$$
$$\frac{d^2y}{dt^2} = a_y$$

Example

A ball is projected at an angle θ with a speed u. The net acceleration is in downward direction. Then $a_x = 0$ and $a_y = -g$. The equations are

$$\frac{d^2x}{dt^2} = 0$$
$$\frac{d^2y}{dt^2} = -g$$

Charge particle in a magnetic field

A particle has a velocity v in XY plane. Magnetic field is in z direction The acceleration is given by $\frac{q}{m}\mathbf{v} \times \mathbf{B}$

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{qB}{m}v_y \\
\frac{d^2y}{dt^2} &= -\frac{qB}{m}v_x
\end{aligned}
$$

Solution is rather simple, that is circular motion in xy plane.

Foundations of Newtonian Mechanics

Three fundamental quantities:

(i) Mass,

(ii) Motion &

(iii) Force

Excerpts from Newton's Principia (Book 1)

Mass

The quantity of matter is the measure of the same arising from it's density and bulk conjointly.

Motion

The quantity of motion is the measure of the same arising from the velocity and quantity of matter conjointly.

Force (Definiton # 1)

The *vis insita:* an innate forces of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be of rest, or of moving uniformly forward in a right line."

Force (Definition # 2)

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line.

These definitions gave rise to the famous three laws: known as Newton's laws of motion.

Law 1

Every body continues in it's state of rest or of uniform rectilinear motion except if it is compelled by forces acting on it to change that sate.

Law 2

The change of motion is proportional to the applied force and takes place in the direction of the straight line along which that force acts.

Laws 3

To every action there is always an equal and contrary reaction; or the mutual actions of any two bodies are always equal and oppositely directed along the same straight line.

By solving Newton's laws we shall find $\boldsymbol{r}(t)$.

$r(t) = 0$: implies that the body is in rest for all time.

In general,

$$r(t)=(x(t),\ y(t),\ z(t)) \text{ or } (r(t),\theta(t),\ \phi(t))$$

Example

$$r(t) = (v_x t + x_0;\ 0;\ v_z t + z_0 - gt^2/2)$$

represents uniform motion in the x-direction with v_x as the velocity, in a state of rest in the y-direction and is having a uniform velocity v_z and a free fall in the gravitational field.

Mechanics of particles

Classical Mechanics	Non-relativistic (Newton's Laws)	Relativistic (Special Theory of Relativity)
Quantum Mechanics	Non-Relativistic (Schroedinger Equation)	Relativistic (Dirac Equation)

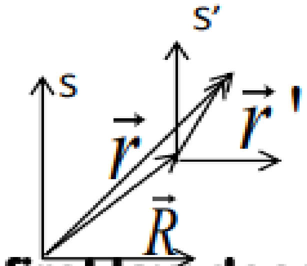
Newton's first law of motion

1. Gives a definition of (zero) force

2. Defines an *inertial frame*.

Zero Force: When a body moves with *constant velocity in a straight line,* either there are no forces present or the net force acting on the body is zero $\sum_i \vec{F}_i = 0$ If the body changes its velocity, then there must be an acceleration, and hence a total non-zero force must be present. Velocity can change due to change in its magnitude or due to change in its direction or change in both.

Inertial frame: If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero, $\vec{A} = \frac{d\vec{V}}{dt} = \vec{0}$ and the reference frames are considered to be *inertial reference frames.* The inertial frame is then simply a frame of reference in which the first law holds.



$$\vec{r}\,' = \vec{r} - \vec{v}t, \quad \vec{v} = \frac{d\vec{R}}{dt}$$

Galilean transformation

Is Earth an inertial frame?

The first law does *not hold in an arbitrary frame. For example, it fails in the frame of a rotating* turntable.

Newton's Second law of motion:

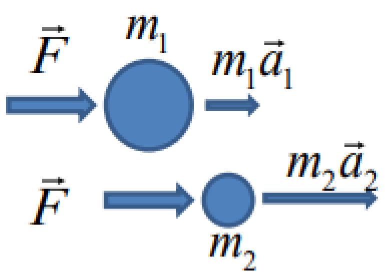
If any force generates a change in motion, a double force will generate double change in the motion, a triple force will correspond to triple change in the motion, whether that force is impressed altogether and at once or gradually or successively.

Change of motion is described by the change in momentum of body. For a point mass particle, the momentum is defined as $\vec{p} = m\vec{v}$

Suppose that a force is applied to a body for a time interval Δt . *The impressed force or impulse produces a change in the momentum of the body,*

$$\vec{\bar{\mathbf{I}}} = \vec{\bar{\mathbf{F}}}\,\Delta t = \Delta\vec{\mathbf{p}}$$

The instantaneous action of the total force acting on a body at a time *t is defined by taking the mathematical limit as the time interval Δt becomes smaller and smaller,*



$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

$$\vec{\mathbf{F}}^{\text{total}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{p}}}{\Delta t} \equiv \frac{d\vec{\mathbf{p}}}{dt} \qquad \vec{F}^{\text{total}} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt}$$

Inertial mass

$$\vec{\mathbf{F}}^{\text{total}} = m\vec{\mathbf{a}}$$

Inertial mass $\equiv$ Gravitational mass

Newton’s third law of motion:

Consider two bodies engaged in a mutual interaction. Label the bodies 1 and 2 respectively. Let $\vec{F}_{1,2}$ be the force on body 1 due to the interaction with body 2, and $\vec{F}_{2,1}$ be the force on body 2 due to the interaction with body 1.

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}$$

$$\text{Gravitational force: } \vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{12} \qquad \hat{r}_{12} = -\hat{r}_{21}$$

$$\text{Coulomb force: } \vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r}_{12} \qquad \vec{F}_{12} = -\vec{F}_{21}$$

All real Forces arise due to interaction!

If the acceleration of a body is the result of an outside force, then somewhere in the universe there must be an equal and opposite force acting on another body. The interaction may be a complicated one, but as long as the forces are equal and opposite, Newton's laws are satisfied.

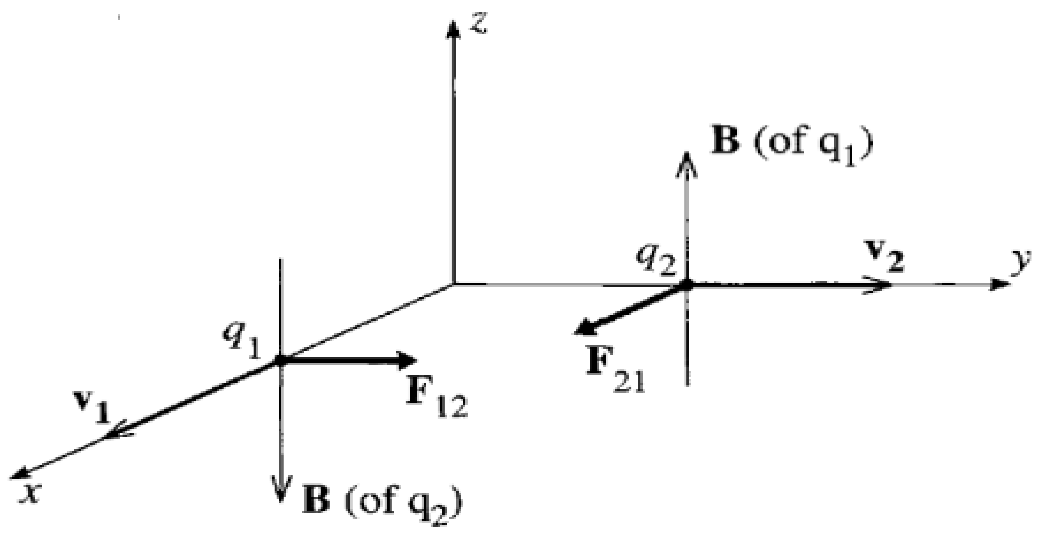
Newton’s 3rd law emphasizes Conservation of Momentum

Validity of Newton's laws

- **Validity of the first two laws**
  - The first law is always valid (add a pseudo force).
  - The second law $\mathbf{F} = \dot{p}$ holds but $\mathbf{F}$ and p have different expressions in the relativistic limit.

- **The 3rd law is not valid in the relativistic limit. Why????**

Consider two positive charges



Each of the positive charges q_1 and q_2 produces a magnetic field that exerts a force on the other charge. The resulting magnetic forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ do not obey Newton's third law.

Momentum conservation is not valid

Application of Newton's laws: Prescription

Step 1: Divide a composite system into constituent systems each of which can be treated as a point mass.

Step 2: Draw free body force diagrams for each point mass.

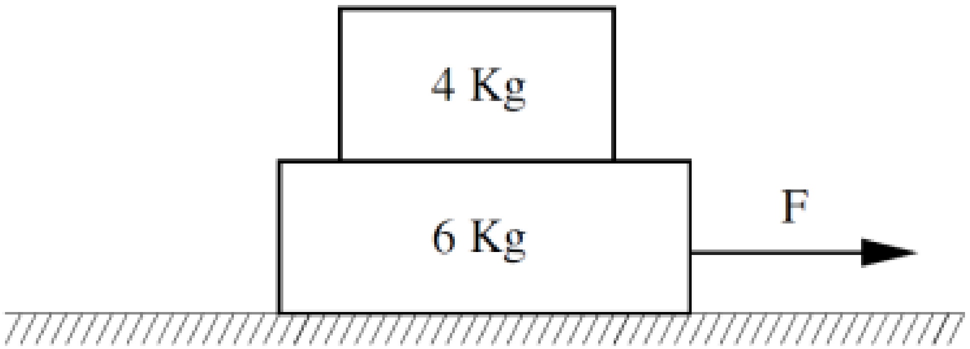
Step 3: Introduce a coordinate system, the inertial frame, and write the equations of motion.

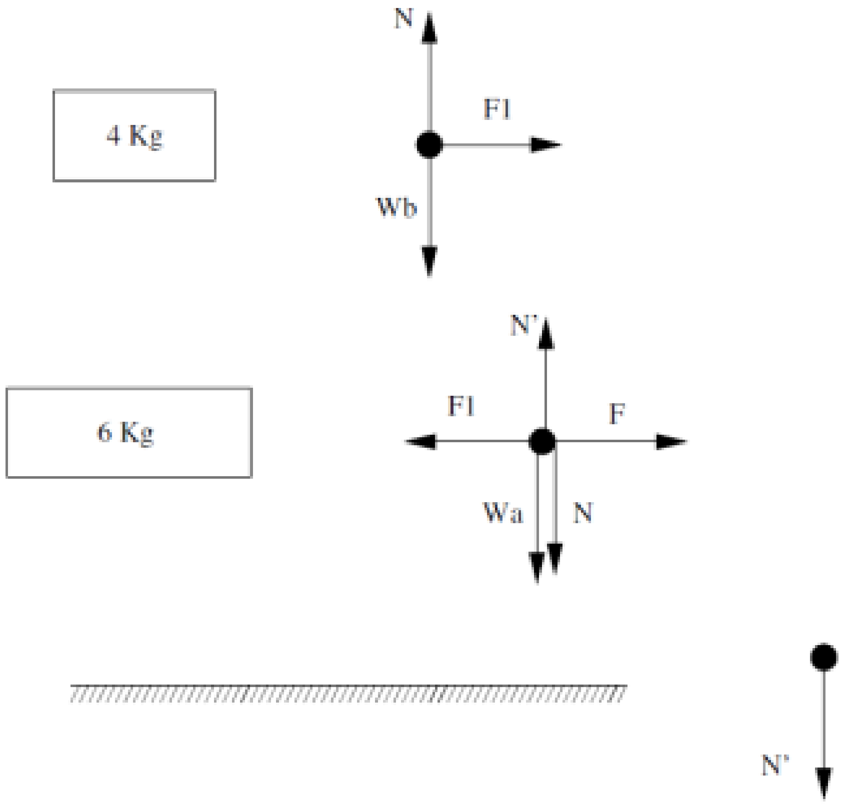
Step 4: Motion of a body may be constrained to move along certain path or plane. Express each constraint by an equation called constraint equation.

Step 6: Identify the number of unknown quantities. There must be enough number of equations (Equations of motion + constraint equations) to solve for all the unknown quantities.

Example 1

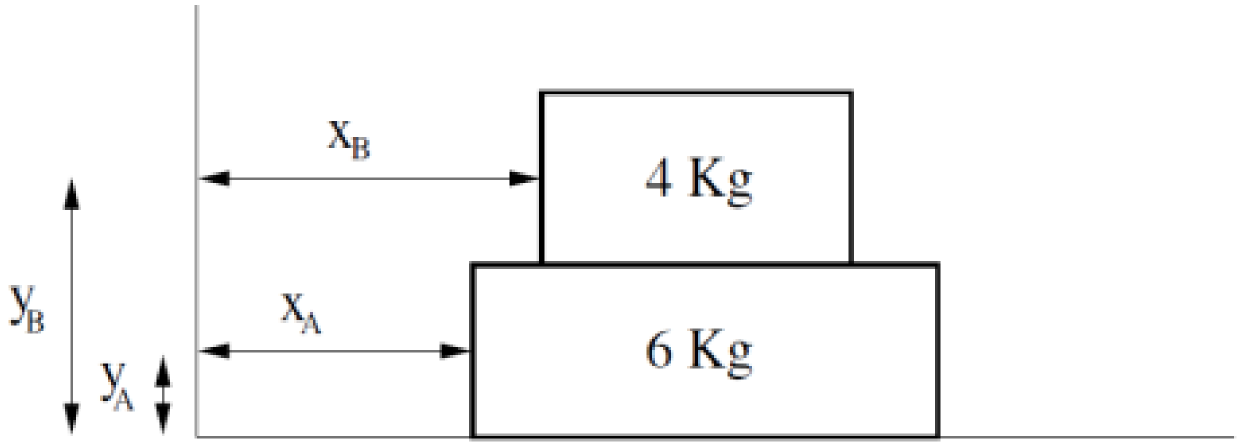
A 4 Kg block rests on top of a 6 Kg block, which rests on a frictionless table. Coefficient of friction between blocks is 0.25. A force $F = 10N$ is applied to the lower block.





Identify the constraints

Fix the coordinate system to the table.



$$
\begin{aligned}
y_A &= const \\
y_B &= const \\
x_A &= x_B + const
\end{aligned}
$$

EOM in x and y-directions

Equations of Motion in Y direction.

$$\begin{aligned} m_A \ddot{y_A} &= N' - W_A - N \\ m_B \ddot{y_B} &= N - W_B \end{aligned}$$

Constraints

$$\begin{aligned} \ddot{y_A} &= 0 \\ \ddot{y_B} &= 0 \end{aligned}$$

Solution

$$\begin{aligned} N' &= W_A + W_B \\ N &= W_B \end{aligned}$$

Equations of Motion in X direction.

$$\begin{aligned} m_A \ddot{x_A} &= F - F_1 \\ m_B \ddot{x_B} &= F_1 \end{aligned}$$

Constraints

$$\ddot{x_A} = \ddot{x_B}$$

Solution

$$\begin{aligned} \ddot{x_A} = \ddot{x_B} &= \frac{F}{m_A + M_B} = 1\mathrm{m/s}^2 \\ F_1 &= m_B \ddot{x_B} = 4N \end{aligned}$$

The force $F_1 < \mu N = 10$ N, the maximum frictional force between the blocks. Hence the solution is consistent with assumption.