MA101- Mathematics I(Introduction to Discrete Mathematics)

Set Theory and functions: set operations, set identities and functions, inverse and composition functions, graph of functions.
Number theory: Division operator, prime factorization, properties of prime numbers, prime number theorem, GCD and LCM, modular arithmetic and applications, sequences and summations.
Counting:Permutation & combinations, pigeonhole principle, inclusion-exclusion principle, binomial theorem, Pascal identity & triangle.
Relations and Recurrence: Relations and their properties, applications and representations, equivalence relations, partial ordering, Hasse diagram, recursive algorithm, recurrence relations, solving recurrence relations.
Logic and Mathematical Reasoning: Rules of inference, direct proof, proof by contradiction, proof by contrapositive, mathematical induction and second law of mathematical induction.
Graph Theory: Introduction and terminology, representation, isomorphism, connectivity, Warshall's algorithm, Euler and Hamilton path, shortest path.

Evaluation and Grading policy:

Assignments: 10%

Tutorial: 10%

Surprise Quizzes: 5%

Mid-semester Exam.: 30%

End-semester Exam.: 45%

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Mid-semester Exam.: 30%
End-semester Exam.: 45%

Text/Reference books:

0) Discrete Mathematics and its Applications, 7th Ed, K. Rosen, Tata McGraw Hill, 2011.
1) Elements of Discrete Mathematics: A Computer Oriented Approach. 4th Edition, C Liu, D. Mohapatra, McGraw Hill, 2017.
2) Discrete Mathematical Structures, 6th Ed, B. Kolman, R.C. Busby and S. C. Ross, PHI, 2000.
3) Discrete Mathematics, Richard Johnsonbaugh, Prentice Hall, 2007.

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- Set of all English Capital letters: $\{A, B, C, \cdots, Z\}$
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Question 3: Is the collection of all sets a set?

History

Set theory was initiated by the German mathematicians Richard Dedekind and Georg Cantor in the 1870s. Georg Cantor is commonly considered the founder of set theory. The non-formalized systems investigated during this early stage go under the name of **naive set theory**. After the discovery of paradoxes within naive set theory (such as Russell's paradox, Cantor's paradox and the Burali-Forti paradox) various axiomatic systems were proposed in the early twentieth century, of which **Zermelo–Fraenkel set theory** is still the best-known and most studied.

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If R is not a member of itself, then its definition dictates that it must contain itself, and if it contains itself, then it contradicts its own definition as the set of all sets that are not members of themselves.

Axiom of Specification

To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.

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Question: Does $B \in A$?
If $B \in A$ then either $B \in B$ or $B \notin B$.
Both are not possible. Why?
There exists a set that does not belong to A.
Nothing contains everything.

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Axiom of Choice (Ernst Zermelo): Cartesian product of a collection of non-empty sets is non-empty. i.e.,
For every indexed family $(S_i)_{i \in I}$ of non-empty sets, there exists an indexed set $(x_i)_{i \in I}$ such that $x_i \in S_i$ for every $i \in I$.

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Note: $\{4, 1, 2, 3\}$ is same as

$$\{1, 2, 3, 4\} \text{ (Roster method)}$$
$$\{x | x \text{ is a natural number less than } 5\} \text{ (set builder notation)}$$

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$\mathbb{Z} = \{x|x \text{ is an integer }\} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
$\mathbb{N} = \{x|x \text{ is a natural number}\} = \{1, 2, 3, ...\}$
$\mathbb{Q} = \{x|x \text{ is a rational number}\} = \{\frac{p}{q}|p, q \in \mathbb{Z}, q \neq 0\}$
$\mathbb{R} =$ set of all real numbers.

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$\phi = \{\ \ \}$: an empty set

Definition (Subset)

Let A and B be two sets. If every element of A is an element of B, then A is said to be a subset of B and written as

$$A \subseteq B.$$

Examples: $\mathbb{N} \subseteq \mathbb{Z}$; $\quad \mathbb{Z} \subseteq \mathbb{Q}$; $\quad \mathbb{Q} \subseteq \mathbb{R}$.
Remark: By definition ϕ and A are subsets of A.

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

Definition (Finite set)

A set A is called finite if it has exactly n distinct elements, for some $n \in \mathbb{N}$;
and n is called the cardinality of A, denoted by $|A|$.
For $A = \{1, 2, 4, 5\}$, $|A| = 4$.
What if a set is not finite? Such set is called infinite set.

Definition (Power set)

The set of all subsets of a set A is called power set of A, denoted as $P(A)$.
For $A = \{1, 2, 4\}$, $P(A) = \{\phi, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, A\}$

Set operations

Let A, B be two sets.
Union of two sets: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
Intersection of two sets: $A \cap B = \{x | x \in A \text{ and } x \in B\}$

$$\cup_{i=1}^{n} A_i = \{x | x \in A_j \text{ for } \textit{some } j\}$$
$$\cap_{i=1}^{n} A_i = \{x | x \in A_j \text{ for } \textit{all } j\}$$

Example: $A_i = \{1, 2, \ldots, i\}$ for $i = 1, 2, \ldots, 10$. Then
$\cup_{i=1}^{10} A_i = \{1, 2, \ldots, 10\}$.
$\cap_{i=1}^{10} A_i = \{1\}$.
Disjoint sets: Sets A and B are said to be disjoint if $A \cap B = \phi$.
Complement of a set: The complement of B with respect to A is

$$A - B = \{x | x \in A \text{ and } x \notin B\}.$$

If U is a universal set containing A, then complement of A is

$$A^c = U - A.$$

Symmetric difference: $A \oplus B = (A - B) \cup (B - A)$.

Cartesian product

Let A, B be two sets. Then we define another set called cartesian product of A and B as

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Axiom of choice(E. Zermelo, 1904): A Cartesian product of a collection of non-empty sets is non-empty.

Algebraic properties of set operations

Let A, B, C be sets.

$$A \cup B = B \cup A$$

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$$A \cup (B \cup C) = (A \cup B) \cup C$$

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If A, B and C are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$