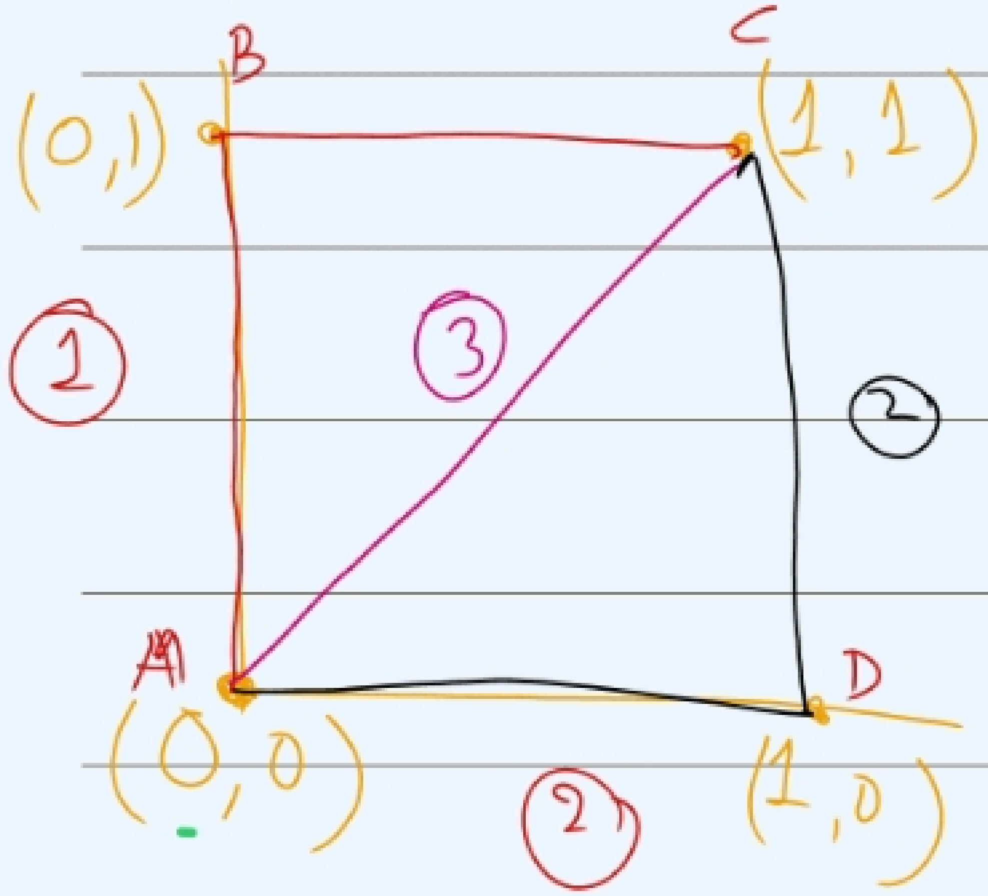
Conservation of Momentum: $F_{ext} = \frac{d\vec{P}}{dt}$.

$$W_{ab} = \int_a^b \vec{F}\cdot d\vec{r} = -W_{ba}$$

Work & Energy:



$$\vec{F} = xy\,\hat{i} + y^2\,\hat{j} \quad , \quad d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

① $x=0$

$$\int_A^B (xy\,\hat{i} + y^2\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j}) = \int_{A,0}^{B,1} \overset{0}{xy\,dx} + y^2\,dy = \frac{1}{3}$$

$$\int_B^C xy\,dx + \overset{0}{y^2\,dy} = \frac{1}{2}$$

$y=1,\ dy=0$

$W_{AC} = 5/6$

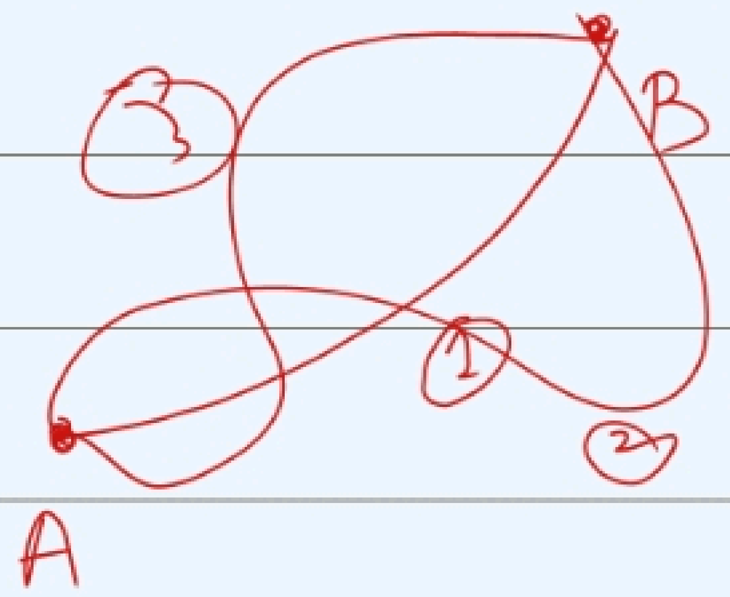
06/10/2023

Work & Energy :-

$F = \frac{d\vec{p}}{dt} = m\vec{a}$

$$W = \int_A^B \vec{F} \cdot \vec{dr}$$

$W_{1} \neq W_{2} \neq W_{3}$



$$W = \int_A^B m\left(\frac{d\vec{v}}{dt} \cdot \vec{V}\right) dt$$

[m = constant]

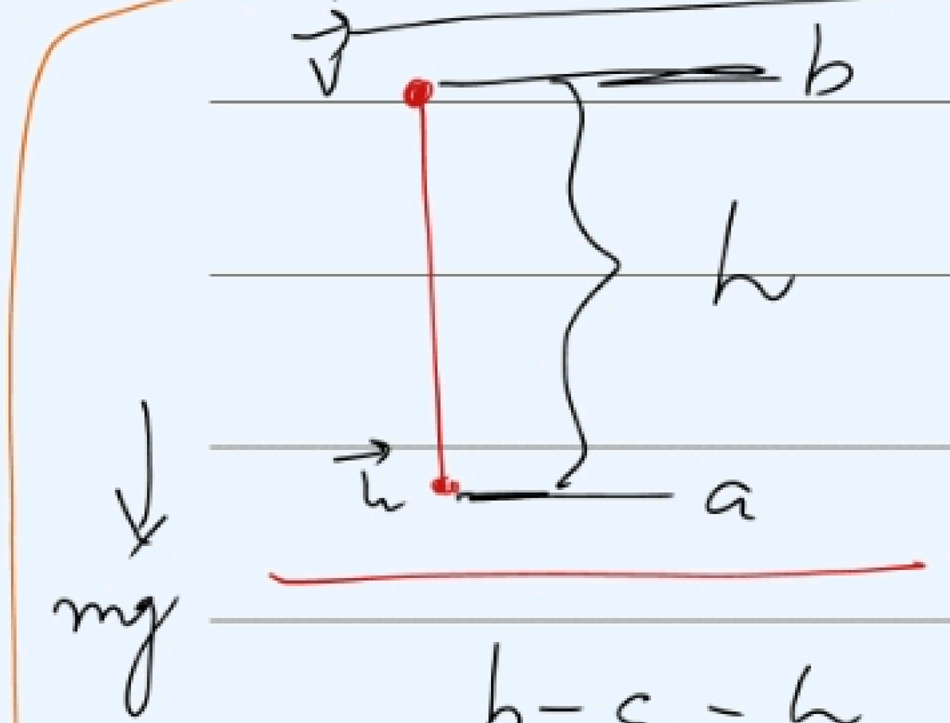
$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\vec{V} \cdot \vec{V}) = 2\,\frac{d\vec{v}}{dt} \cdot \vec{V}$$

Work Energy Theorem

$$W = \int_A^B \frac{m}{2}\left(\frac{d}{dt}v^2\right) dt = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = K.E(B) - K.E(A)$$

one-particle

$$\boxed{W = K\cdot E(B) - K\cdot E(A)}$$



$$W = \int \vec{F}\cdot d\vec{r} = -\int_a^b (mg)\, dr = -mgh$$

$$\frac{1}{2} m\left[v^2 - u^2\right] = \frac{1}{2}m\left[2gh\right]$$

$b - a = h$

$$K\cdot E(B) - K\cdot E(A) = -mgh$$

System of particles $W = \frac{1}{2} M\dot{X}_B^2 - \frac{1}{2} M\dot{X}_A^2$

$X_{A,B}$ = C.M of system

Conservative force $\Leftrightarrow$ 1D   Central force

Central force

$$W = \int_A^B f(r)\,\hat{r}\cdot \underbrace{d\vec{r}}_{(dr\,\hat{r} + r\,d\theta\,\hat{\theta})} = \int_A^B f(r)\,dr$$

$$W\Big|_{ba} = K.E(b) - K.E(a) = -\underbrace{U(b) + U(a)}_{dU} = \int \vec{F}\cdot d\vec{r}$$

$$U(b) - U(a) = -\int \vec{F}\cdot d\vec{r}$$

1D: $d\vec{r} = dx\,\hat{i}$

$\vec{F} = f\,\hat{i}$

$$\Rightarrow \boxed{F = -\frac{dU}{dx}}$$

② : (0,0)−(1,0)−(1,1)

$W_{AC} \to W_{AD} - W_{DC}$

$$W_{AD} = \int_A^D xy\,dx + y^2\,dy \to 0$$

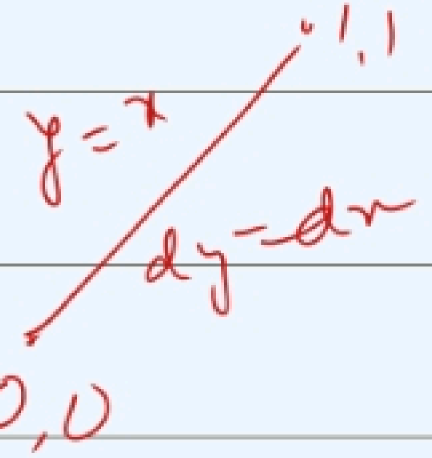
$y = 0,\ dy \to 0$

$$W_{DC} = \int_D^C \overset{0}{xy\,dx} + y^2\,dy \to 1/3$$

$x = 1$

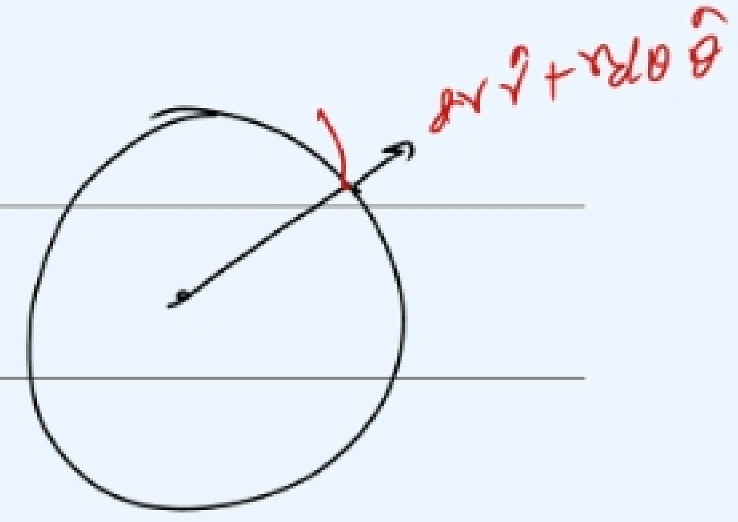
$W_{①} \neq W_{②}$

③



$$W_3 = \int_0^1 xy\,dx + y^2\,dy = 2/3$$

Central forces:



$$\vec{F}_{grav} = -G\frac{m_1 m_2}{r^2}\hat{r} = f(r)\hat{r}$$

$$\vec{F} = q\vec{E} = K\frac{q_1 q_2}{r^2}\hat{r}$$

$$W = \int_a^b \vec{F}\cdot d\vec{r}$$

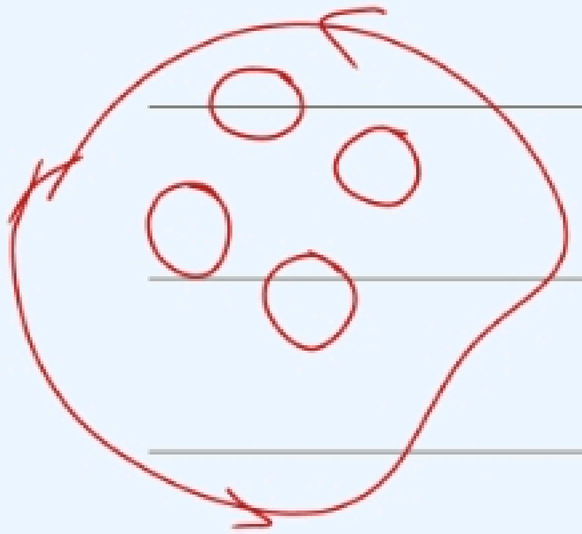
$$= \int_a^b f(r)\,\hat{r}\cdot\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta}\right)$$

$$= \int_a^b f(r)\,dr = f(b) - f(a)$$

$$\int_a^a \vec{F}\cdot d\vec{r} = 0$$

$$U_b - U_a = -\int_a^b \vec{F}\cdot d\vec{r}$$

Stokes Theorem:



$$\oint \vec{F}\cdot d\vec{l} = \iint (\nabla\times\vec{F})\cdot d\vec{a}$$

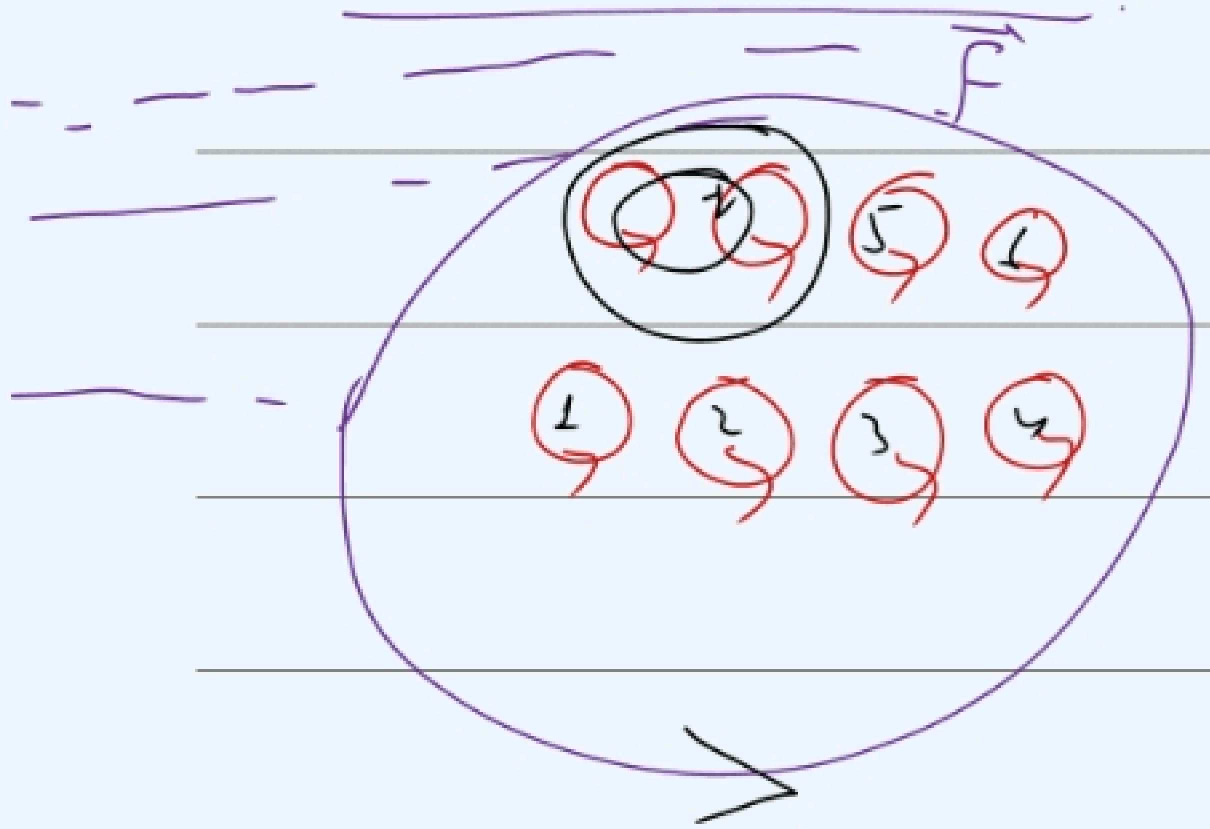
$$0$$

$$\nabla\times\nabla f$$

$$0$$

Conservative force is Path Independent
work done

Stoke's Theorem is



$$\oint \vec{F} \cdot d\vec{l} = \underbrace{\oiint (\nabla \times \vec{F}) \cdot d\vec{a}}_{0}$$

$$\vec{F} = -\nabla f$$

$$= -\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

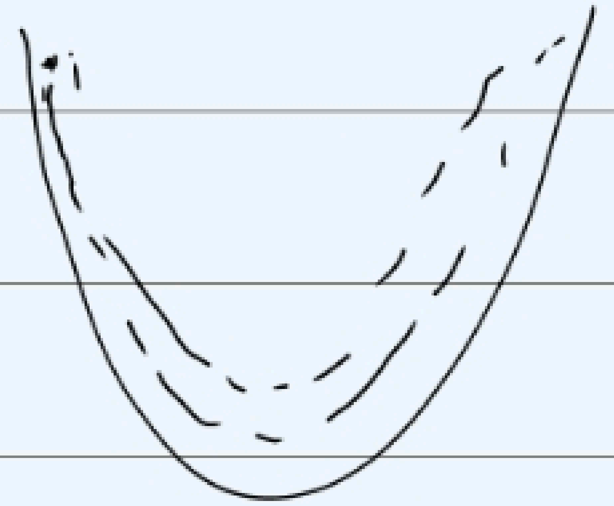
1D: $\vec{F} = -\hat{i}\frac{\partial f}{\partial x}$

$$U(r) = \frac{1}{2}K\left[x^2 + y^2 + z^2\right]$$

1D

$y = 0$
$z = 0$

$U(r)$

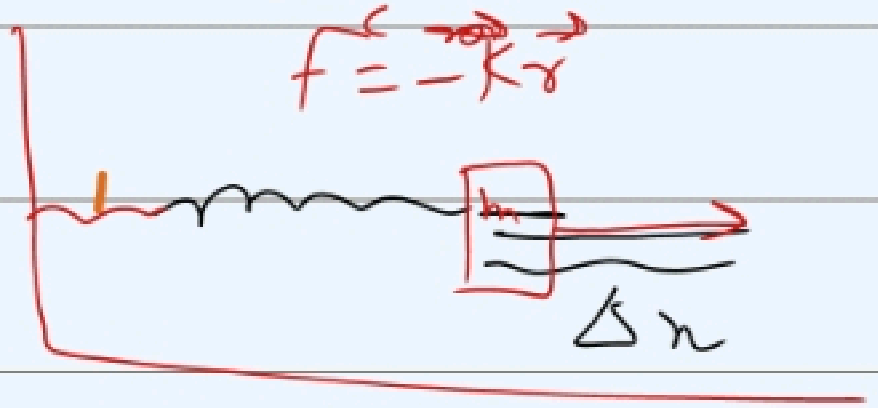


$$\vec{F} = -\nabla U = -K\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$\boxed{\vec{F} = -K\vec{r}}$$

r

$\vec{F} = -K\vec{r}$



Δx

Task $\Rightarrow$ $U(r) = -\frac{G m_1 m_2}{r}$, $\vec{F}$

$$W = \int \vec{F}\cdot d\vec{r}, \quad \vec{F} = F^{C} + F^{NC}$$

$$W = K.E(b) - K.E(a) = -U(b) + U(a)$$

$$W = \underline{E(b) - E(a)} + Q \rightarrow \text{Heat}$$

Particle collisions and conservation Laws:

$K.E_i = K.E_f$

① Elastic: $\Delta p = 0 \Rightarrow p_i = p_f$

② Inelastic: $K.E_i \neq K.E_f$

$K.E_i = K.E_f + Q$

Elastic collision

m_1

$m_2 = 3m_1$ || Both have ($\vec{v}$) equal & opposite velocities || Final velocity

Momentum: $m_1 v - 3m_1 v = m_1 v_1' + 3m_1 v_2'$

$\Rightarrow \boxed{v_1' = -2v - 3v_2'}$

K.E: $K.E_i = K.E_f$

K.E:

$$\cancel{\tfrac{1}{2}}\cancel{m_1}v^2 + \cancel{\tfrac{1}{2}}3\cancel{m_1}v^2 = \cancel{\tfrac{1}{2}}\cancel{m_1}v_1'^2 + \cancel{\tfrac{1}{2}}3\cancel{m_1}v_2'^2$$

$m_1 \parallel v \parallel v_1'$

$3m_1 \parallel v \parallel v_2'$

$v_1' = -2v - 3v_2'$

m_1 $\xrightarrow{v}$ $\xleftarrow{}$ $3m_1$

$$4v^2 = v_1'^2 + 3v_2'^2$$

$$= [-2v - 3v_2']^2 + 3v_2'^2$$

$$\cancel{4v^2} = \cancel{4v^2} + 9v_2'^2 + 12vv_2' + 3v_2'^2$$

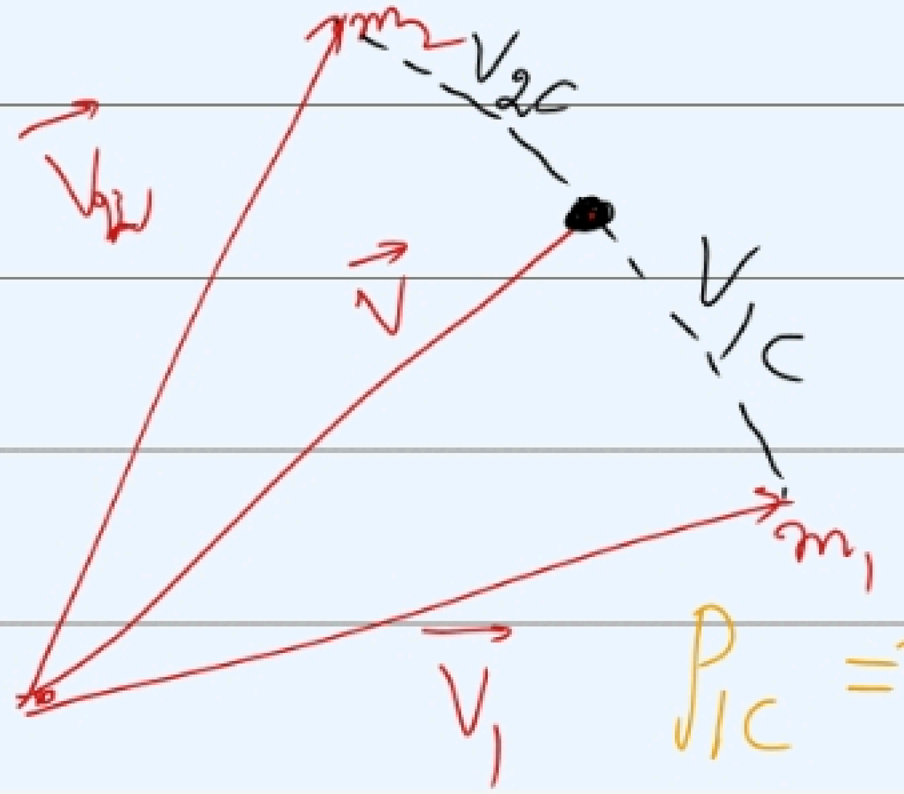
$$12[v_2'^2 + vv_2'] = 0$$

$v_2' = 0$

$v_2' = -v$

Collisions and C.M. Coordinates:→

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$



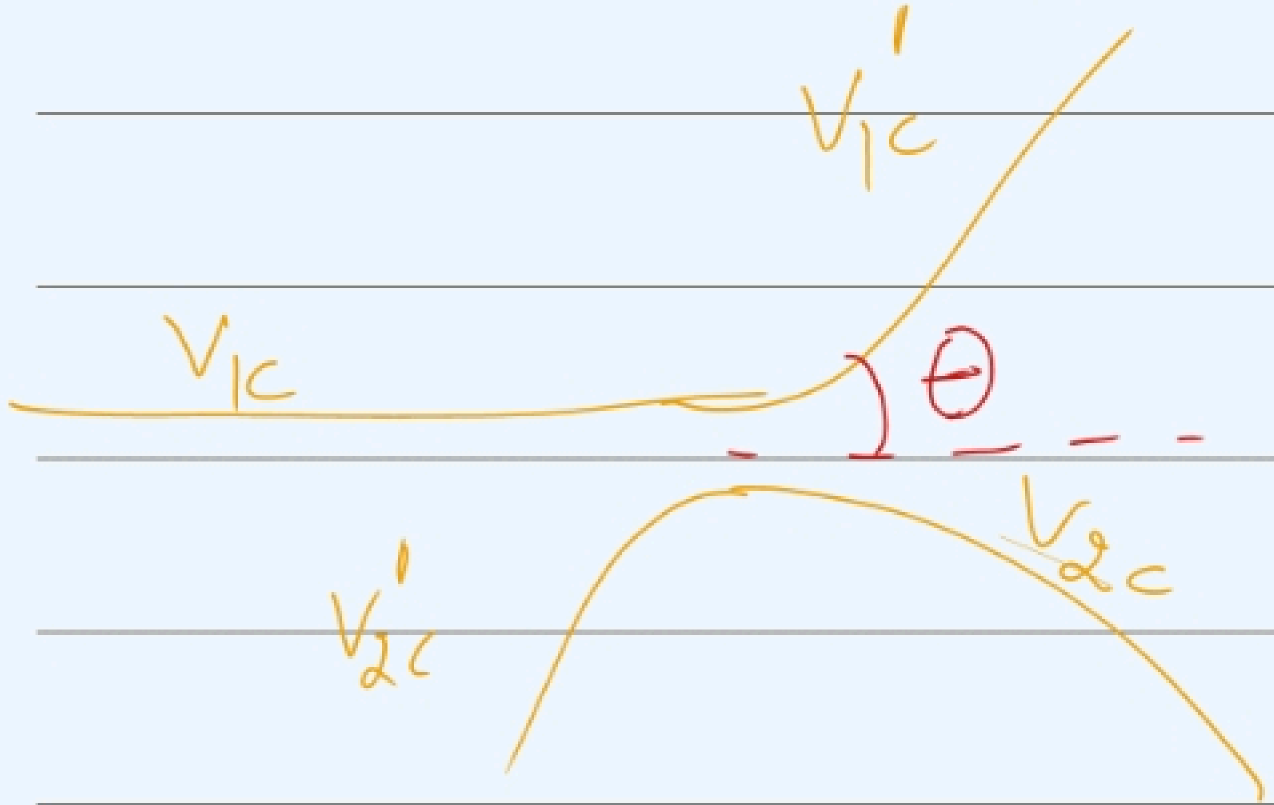
$$\vec{V}_{1c} = \vec{V}_1 - \vec{V} = \vec{V}_1 - \left[\frac{m_1\vec{V}_1 + m_2\vec{v}_2}{m_1+m_2}\right]$$

$$P_{1c} = m_1\vec{V}_{1c} = m_1\left[\frac{\cancel{m_1\vec{V}_1} + m_2\vec{V}_1 - (\cancel{m_1\vec{V}_1} + m_2\vec{V}_2)}{m_1+m_2}\right] = \frac{m_1 m_2(\vec{V}_1 - \vec{V}_2)}{m_1+m_2}$$

$$\vec{P}_{2c} = m_2\vec{V}_{2c} = \vec{V}_2 - \vec{V} = -\frac{m_1 m_2}{m_1+m_2}(\vec{V}_1 - \vec{V}_2)$$

$$\vec{P} = 0$$

Elastic Collision & C.M. frame of reference →



$m_1 V_{1c} - m_2 V_{2c} = 0 \Rightarrow V_{1c} = \frac{m_2}{m_1} V_{2c}$

$m_1 V_{1c}' - m_2 V_{2c}' = 0 \Rightarrow V_{1c}' = \frac{m_2}{m_1} V_{2c}'$

$K.E_i = K.E_f$

$$\Rightarrow \frac{1}{2} m_1 \underline{V_{1c}^2} + \frac{1}{2} m_2 V_{2c}^2 = \frac{1}{2} m_1 V_{1c}'^2 + \frac{1}{2} m_2 V_{2c}'^2$$

$$\Rightarrow \cancel{\frac{1}{2}} m_1 \left[\frac{m_2^2}{m_1^2} V_{2c}^2\right] + \cancel{\frac{1}{2}} m_2 V_{2c}^2 = \cancel{\frac{1}{2}} m_1 \left[\frac{m_2^2}{m_1^2} V_{2c}'^2\right] + \cancel{\frac{1}{2}} m_2 V_{2c}'^2$$

$$\Rightarrow \left[m_2 + \cancel{\frac{m_2^2}{m_1}}\right] \underline{V_{2c}^2} = \left[\cancel{\frac{m_2^2}{m_1}} + m_2\right] V_{2c}'^2 \Rightarrow \boxed{V_{2c} = \underline{V_{2c}'}}$$

$V_{2c}^2 = V_{2c}'^2$

Laboratory & C.M frames scattering Angle :→

Correlation scattering angle
in L-frame & c.m.-frame.

L

09.10.2023

Oscillations :→

10^{-18} sec

$\nu = 2\pi$

$\lambda \nu = v$

$\nu = \frac{v}{\lambda} \simeq \frac{10^8 \, m/s \times}{400 \times 10^{-9} \, m}$

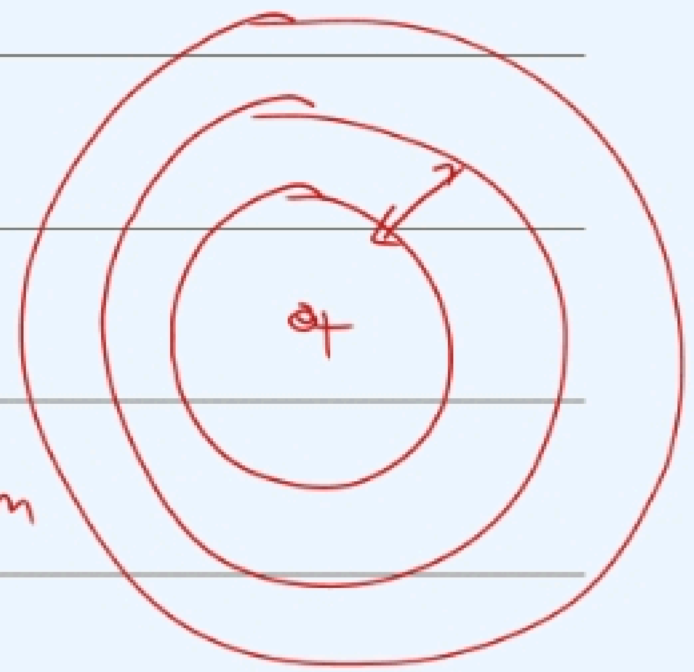
$\approx 10^{15}$ Hz

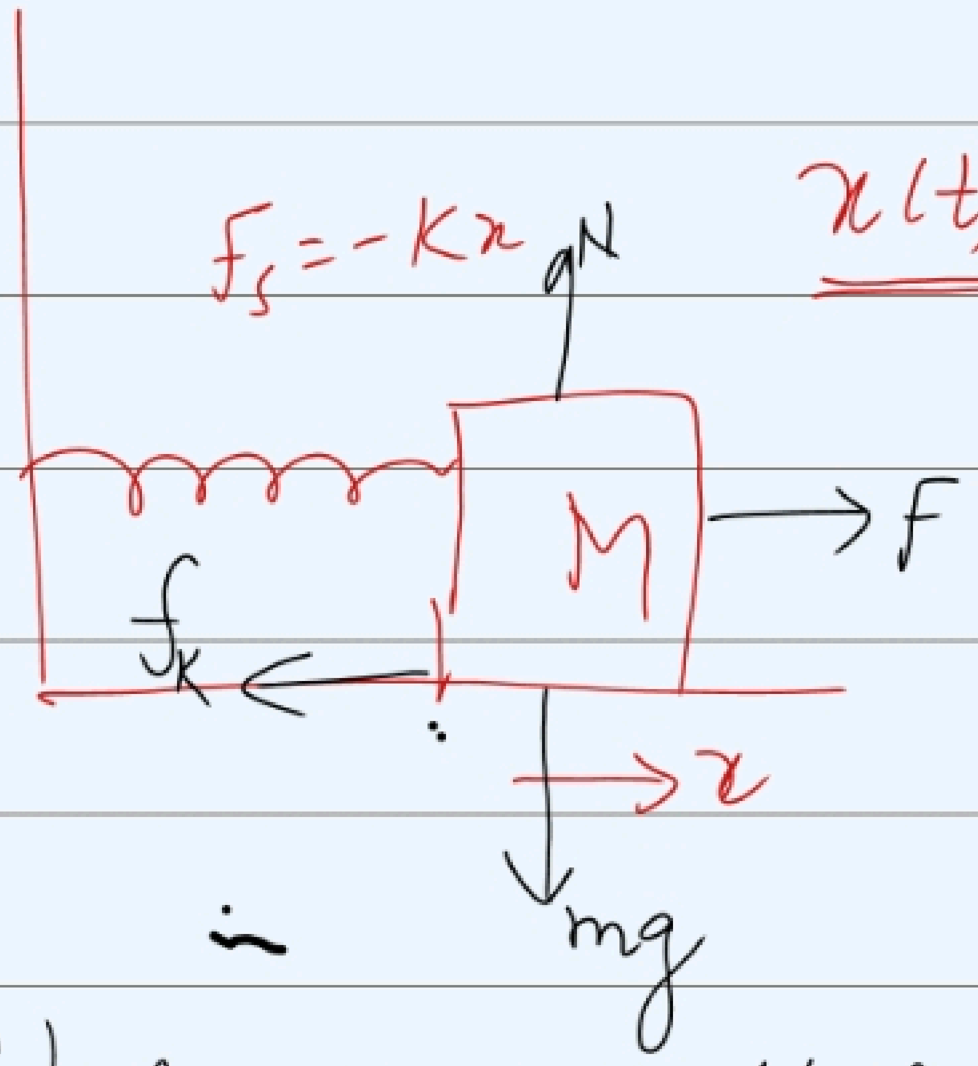
$\omega T = 2\pi$

$T = \frac{2\pi}{\omega}$

$E_m - E_n = h\nu$

$\lambda : \sim 400 \, nm - 700 \, nm$





$\underline{x(t) = ?}$

Oscillations

1. Undamped
2. Damped
3. Forced
4. Forced damped oscillation.

Undamped oscillation →

$$\vec{F} = -Kx = m\ddot{x} \qquad \left[\ddot{x} = \frac{d^2x}{dt^2}\right]$$

$$\Rightarrow \ddot{x} + \frac{K}{m}x = 0$$

Let $x(t) = A\sin\omega t \,/\, \cos\omega_0 t$

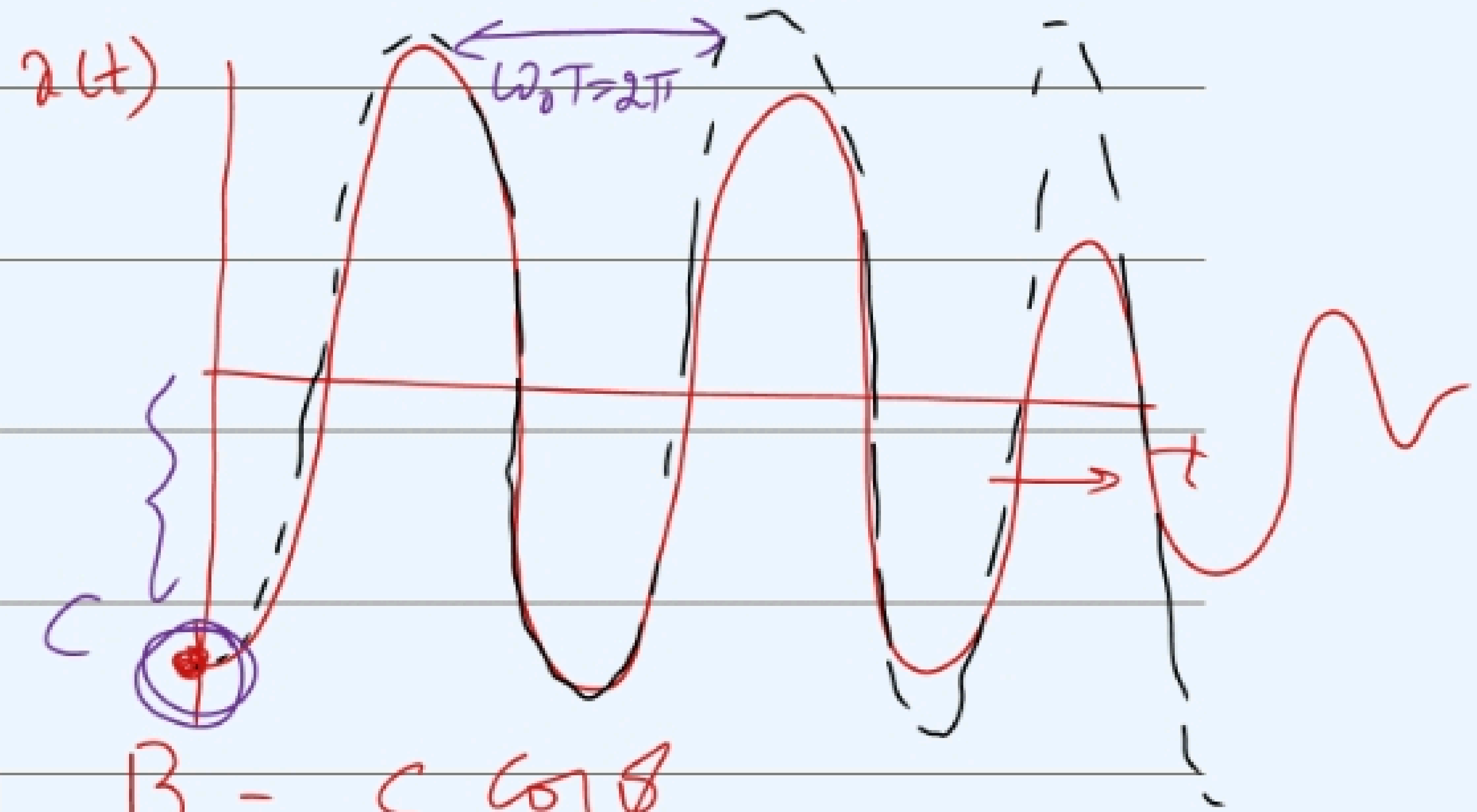
$$-\omega^2 \sin\omega t + \frac{K}{m}\sin\omega t = 0 \qquad (\cos\omega_0 t)$$

$$\omega_0 = \sqrt{K/m}$$

$$x(t) = A\sin\omega_0 t + B\cos\omega_0 t$$

$$\ddot{x} + \frac{K}{m} x = 0 \qquad x(t) = A\sin\omega_0 t + B\cos\omega_0 t \quad , \omega_0 = \sqrt{\frac{K}{m}}$$

$$-\omega_0^2\left[A\sin\omega_0 t + B\cos\omega_0 t\right] + \frac{K}{m}\left[A\sin\omega_0 t + B\cos\omega_0 t\right] = 0$$



$$\rightarrow x(t) = \underline{C}\,\underline{\cos(\omega_0 t + \phi)}$$

$$= C\left[\cos\omega_0 t \cos\phi - \sin\omega_0 t \sin\phi\right]$$

$$= B\cos\omega_0 t + A\sin\omega_0 t$$

$$B = C\cos\phi$$

$$A = -C\sin\phi$$

$$\Rightarrow C^2 = A^2 + B^2$$

$$\tan\phi = -\frac{A}{B}$$

$$x(t) = \left[C\cos\left[\omega_0 t + \phi\right] \right] = \mathrm{Re}\left[C e^{i(\omega t + \phi)} \right]$$

$$= C\cos(\omega t + \phi) + C\sin(\omega_0 t + \phi)$$

$$= C e^{i(\omega t + \phi)}$$

$$z = x + iy$$

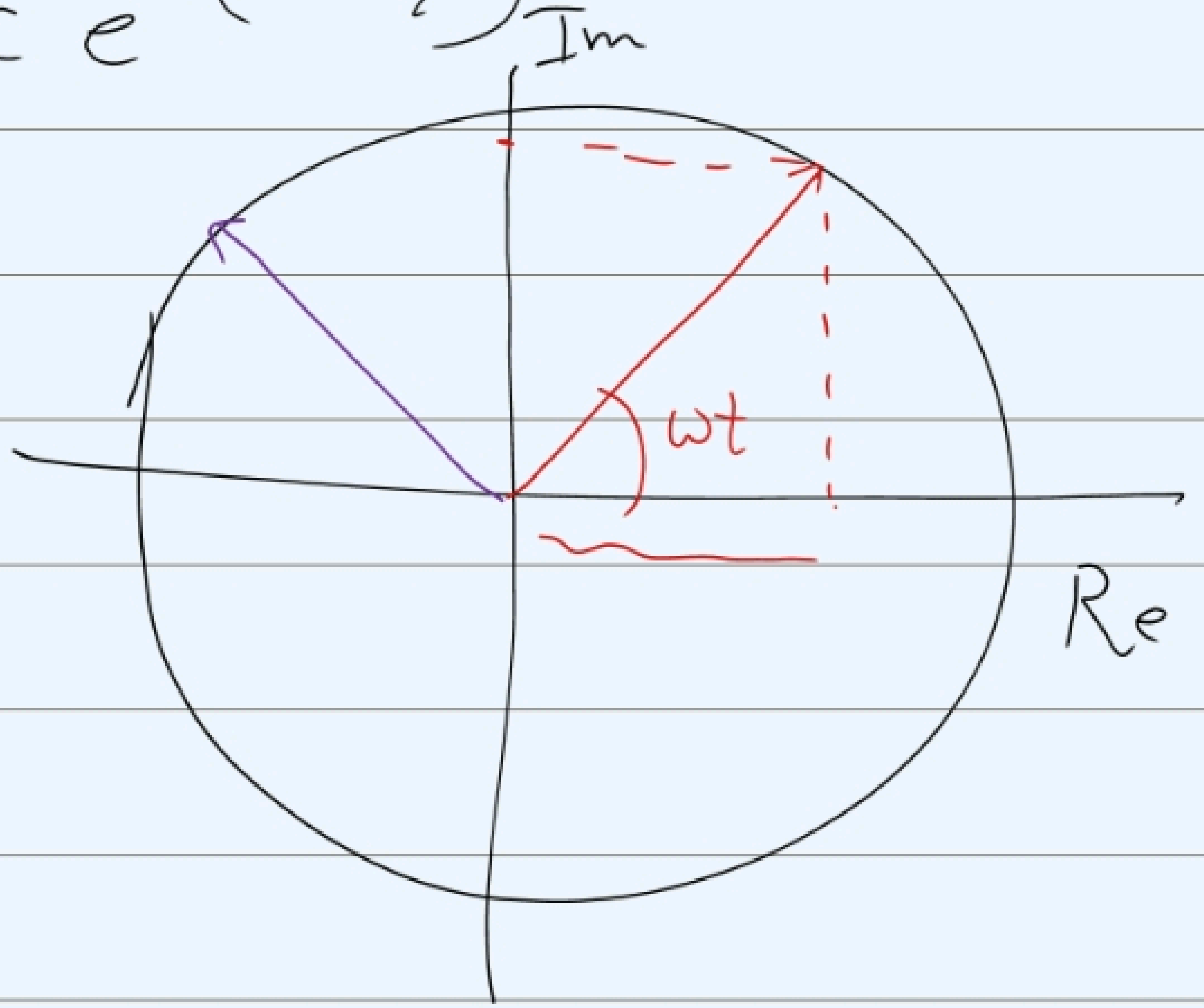
$$\bar{z} = x - iy$$

$$|z|^2 = x^2 + y^2$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$x(t) \sim e^{i\omega_0 t}$$

$$\dot{x} \rightarrow i\omega_0 e^{i\omega_0 t}$$



$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

Energy $\Rightarrow$ $K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$x(t) = C\cos(\omega_0 t + \phi)$

$= A\sin\omega_0 t + B\cos\omega_0 t$

$\vec{F} = -K\vec{x}$

$x(t)$

$-\frac{dU}{dx} = F$

$U = \frac{1}{2}Kx^2$

$= \frac{1}{2}m\left[-C\omega_0 \sin(\omega_0 t + \phi)\right]^2$

$K.E = \frac{1}{2}\underbrace{m\omega_0^2}_{K}\left[C^2 \sin^2(\omega_0 t + \phi)\right]$

$P.E = \frac{1}{2}Kx^2 = \frac{1}{2}KC^2\cos^2(\omega_0 t + \phi)$

$\langle KE \rangle = \frac{1}{4}KC^2$

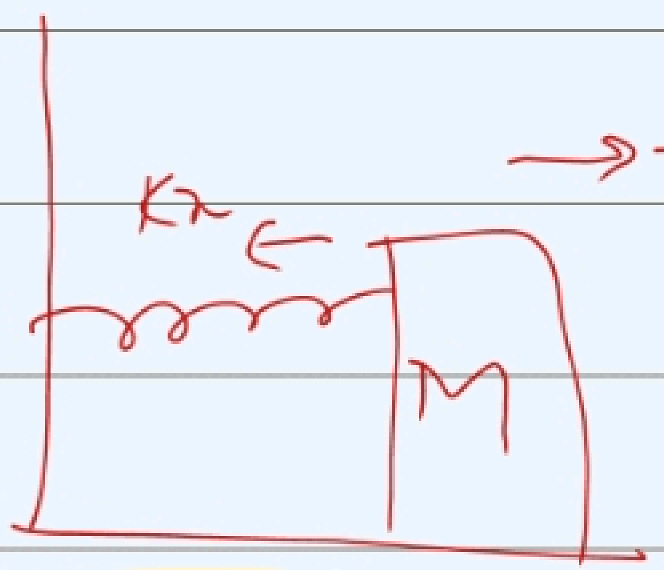
$\langle P.E \rangle = \frac{1}{4}KC^2$

$\langle E \rangle = \frac{1}{2}KC^2$

$\omega_0^2 = K/m$

$$\boxed{E = \frac{1}{2}KC^2}$$

2. Damped HO :→



$$\vec{F} = -kx + b\dot{x}$$

$$x(t) = C e^{i(\omega_0 t + \phi)}$$

Real + Img

$$\ddot{x} = -\frac{k}{m}x + \frac{b}{m}\dot{x}$$

Real

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Imaginary

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0$$

$$\left[-\alpha^2 + i\gamma\alpha + \omega_0^2\right] z_0 e^{i\alpha t} = 0 \longleftarrow \ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0 \quad — (1)$$

$$x(t) = \mathrm{Re}\left[z(t)\right] \quad — (2)$$

(3) into (1)

$$z(t) = z_0 e^{i\alpha t} \quad — (3)$$

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0 \quad , \quad z(t) = z_0 e^{i\alpha t}$$

$$(i\alpha)^2 z_0 e^{i\alpha t} + \gamma(i\alpha) z_0 e^{i\alpha t} + \omega_0^2 z = 0$$

$\gamma = \frac{b}{m}$

$$\Rightarrow \left[-\alpha^2 + i\alpha\gamma + \omega_0^2\right] z_0 e^{i\alpha t} = 0$$

$$\alpha^2 - (i\gamma)\alpha - \omega_0^2 = 0 \Rightarrow \alpha = \frac{i\gamma \pm \sqrt{(i\gamma)^2 + 4\omega_0^2}}{2}$$

- → ~~S~~light
- → Critical
- → Heavy

$\}\ \alpha$

11/10/2023

Damped harmonic oscillation:

$$\vec{F} = -K\vec{x} + b\vec{v}$$

$$\ddot{x} + \overset{\gamma}{\frac{b}{m}}\dot{x} + \overset{\omega_0^2}{\frac{K}{m}}x = 0$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad \Big\| \quad \ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0$$

$z = x + iy$

$x = Re(z)$

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0 \quad x(t)$$

$$z = z_0 e^{i\alpha t}$$

Undamped

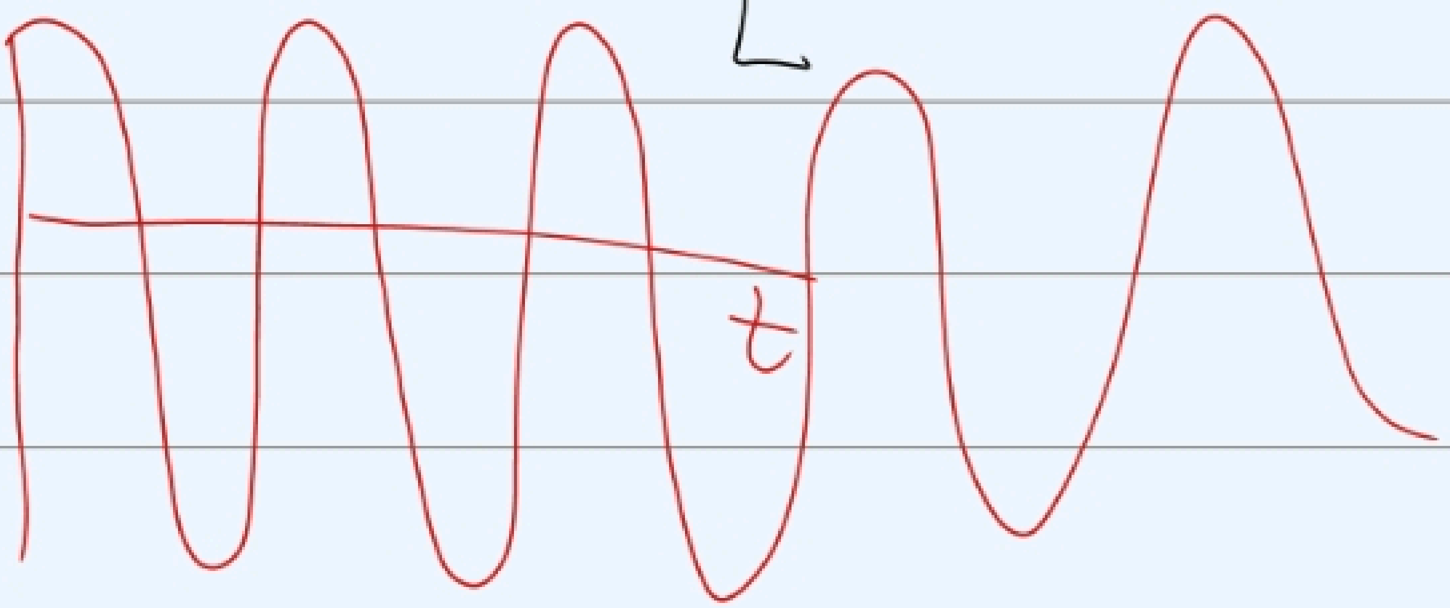
$$\vec{F} = -K\vec{x}$$

$$\ddot{x} + \frac{K}{m}x = 0$$

$$x(t) = C\cos(\omega_0 t + \phi)$$

$$= A\sin(\omega_0 t + \cancel{\phi}) + B\cos(\omega_0 t)$$

$$= Re\left[C e^{i(\omega_0 t + \phi)}\right]$$



$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = 0$, $\underline{z} = z_0 e^{i\alpha t}$ $\Rightarrow \left[(i\alpha)^2 + i\gamma\alpha + \omega_0^2 \right] \overset{\neq 0}{z} = 0$

$\alpha^2 - (i\gamma)\alpha - \omega_0^2 = 0$ $\Rightarrow$ $\alpha = \dfrac{(i\gamma) \pm \sqrt{(i\gamma)^2 + 4\omega_0^2}}{2}$

Lightly $\left(\frac{k}{m}\right) \gg \frac{b^2}{4m^2}$

① $\omega_0^2 \gg \frac{\gamma^2}{4}$

Critically

② $\omega_0^2 = \frac{\gamma^2}{4}$

$\alpha = \frac{(i\gamma)}{2} \pm \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}_{\omega}$

Heavily

③ $\omega_0^2 \ll \frac{\gamma^2}{4}$

① Lightly damped HO :-

$z_1 = z_{01} e^{i\alpha_1 t} = z_{01} e^{-\frac{\gamma}{2}t} e^{i\omega t}$

$z_2 = z_{02} e^{i\alpha_2 t} = z_{02} e^{-\frac{\gamma}{2}t} e^{-i\omega t}$

$z_+ = \frac{1}{2}\left[z_1 + z_2 \right]$

$z_{01} = z_{02}$

$$Z_+ = \frac{1}{2}\left[Z_1 + Z_2 \right]$$

$$= \frac{z_{01}}{2} e^{-\frac{\gamma}{2}t} \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right]$$

$$\frac{\cos\omega t + i \cancel{\sin\omega t} + \cos\omega t + i \cancel{\sin\omega t}}{2i}$$

$$Z_+ = z_{01} e^{-\frac{\gamma}{2}t} \cos\omega t$$

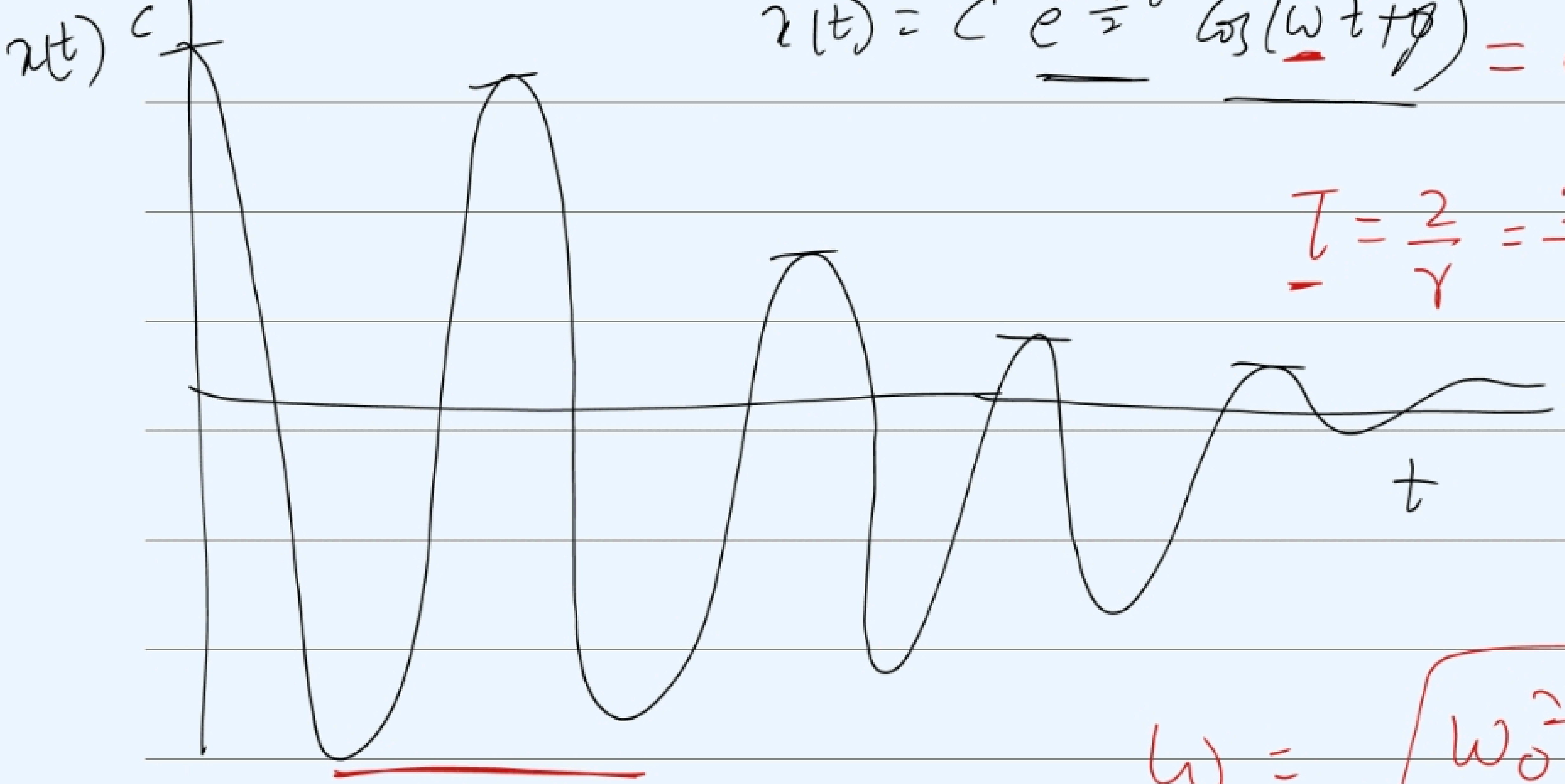
$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

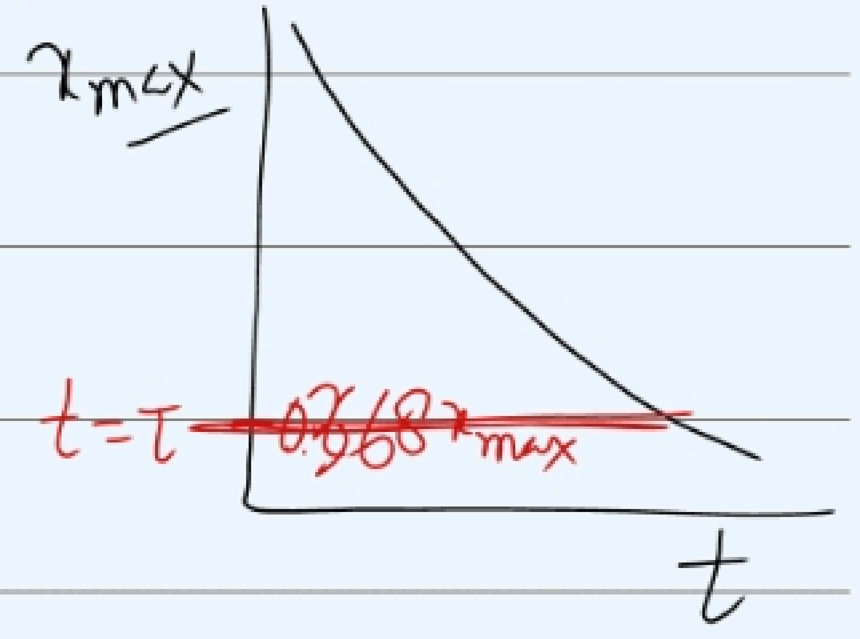
$$Z_- = \frac{1}{2i}\left[Z_1 + Z_2 \right] = Z_{01} e^{-\frac{\gamma}{2}t} \sin\omega t$$

$$x(t) = C\cos(\omega_0 t + \phi)$$

$$x(t) = \mathrm{Re}\left[Z_+ + Z_- \right] = \underline{C e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi)}$$

$$x(t) = C\, e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi) = C\, e^{-1} \cos(\omega \tau + \phi)$$





$$\tau = \frac{2}{\gamma} = \frac{2m}{b}$$

ω

$$\omega T = 2\pi$$

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

Critically damped case $\rightarrow$ $\quad \omega_0^2 = \dfrac{\gamma^2}{4}$ $\qquad \alpha = \dfrac{(i\gamma)}{2} \pm \sqrt{\omega_0^2 - \dfrac{\gamma^2}{4}}$

$\alpha = \dfrac{i\gamma}{2}, \quad \omega = 0$ $\qquad z = z_0 e^{i\alpha t}$

$z_1 = z_{01} e^{-\frac{\gamma}{2}t} \Rightarrow \boxed{x_1(t) = x_{01} e^{-\frac{\gamma}{2}t}}$

$x_2(t) = \underline{u(t)}\, e^{-\frac{\gamma}{2}t}$ $\qquad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

$\Rightarrow \dot{x}_2(t) = \dot{u}\, e^{-\frac{\gamma}{2}t} + u(t)\left(-\frac{\gamma}{2}\right) e^{-\frac{\gamma}{2}t}$

$\ddot{x}_2(t) = \ddot{u}\, e^{-\frac{\gamma}{2}t} + 2\dot{u}\left(-\frac{\gamma}{2}\right) e^{-\frac{\gamma}{2}t} + u(t)\,\frac{\gamma^2}{4}\, e^{-\frac{\gamma}{2}t}$

$\gamma\dot{x}_2(t) = \gamma\left[\dot{u}\, e^{-\frac{\gamma}{2}t} + u(t)\left(-\frac{\gamma}{2}\right) e^{-\frac{\gamma}{2}t}\right]$

$\omega_0^2 x_2 = \frac{\gamma^2}{4} u\, e^{-\gamma/2\, t}$

$\ddot{u} - \gamma\dot{u} + \frac{\gamma^2}{4}u + \gamma\dot{u} - \frac{\gamma^2}{2}u + \frac{\gamma^2}{4}u = 0$

$\ddot{u} = 0, \quad u(t) = A + Bt$

$x_2(t) = (A + Bt)\, e^{-\frac{\gamma}{2}t}$, $x_1(t) = x_0\, e^{-\frac{\gamma}{2}t}$

Heavily damped case.

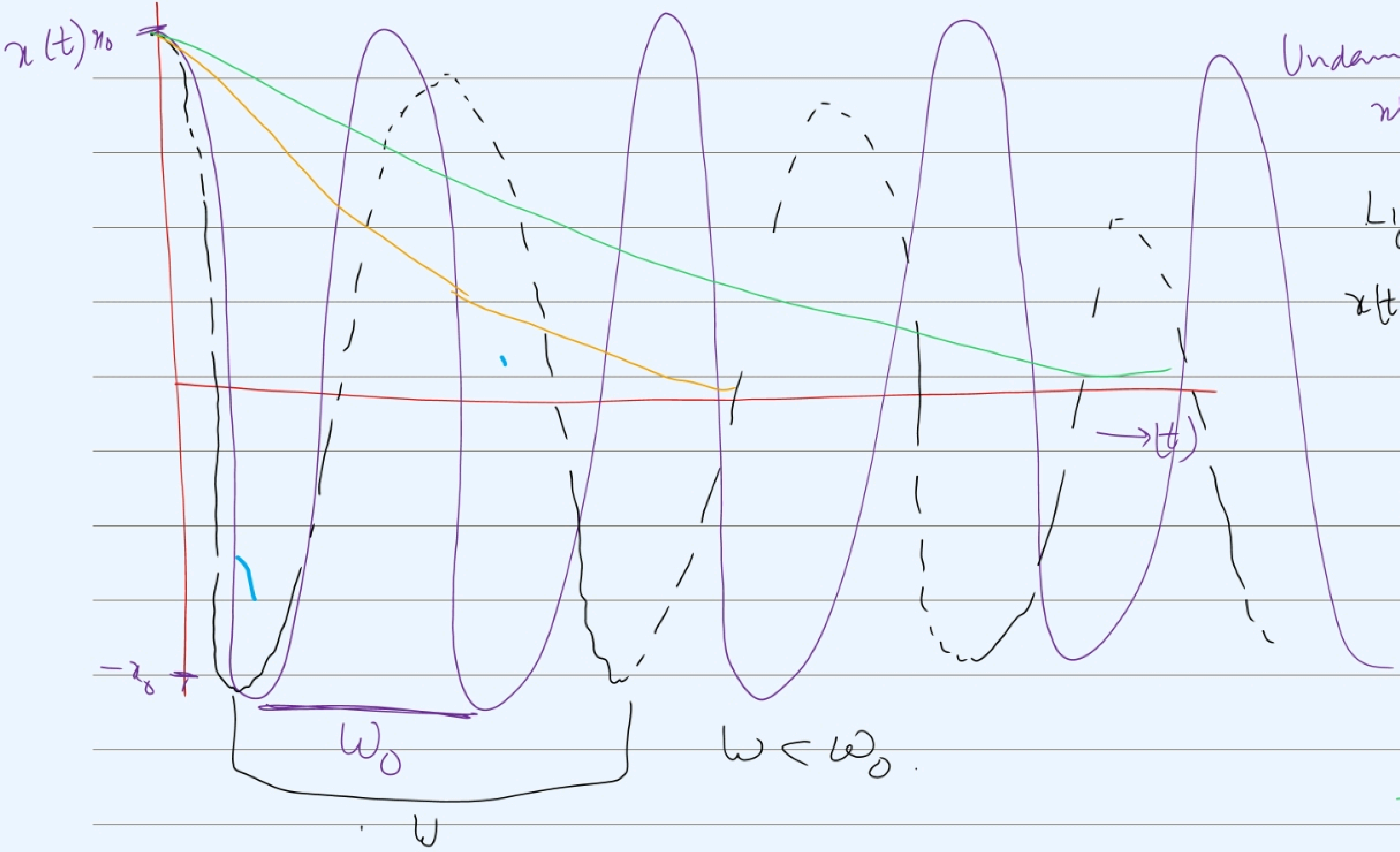
(3) $\omega_0^2 < \frac{\gamma^2}{4}$.

$\underline{\underline{\omega}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, Imaginary.

$\alpha = \frac{i\gamma}{2} \pm i\Gamma$

$\Gamma = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

$$\boxed{z_{\pm} = z_0\, e^{-\frac{\gamma}{2}t}\, e^{-(\pm\Gamma)t}}$$



Undamped

$x(t) = x_0 \cos(\omega_0 t + \phi)$

Light damped case.

$x(t) = x_0 e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi)$

$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

Critically damped

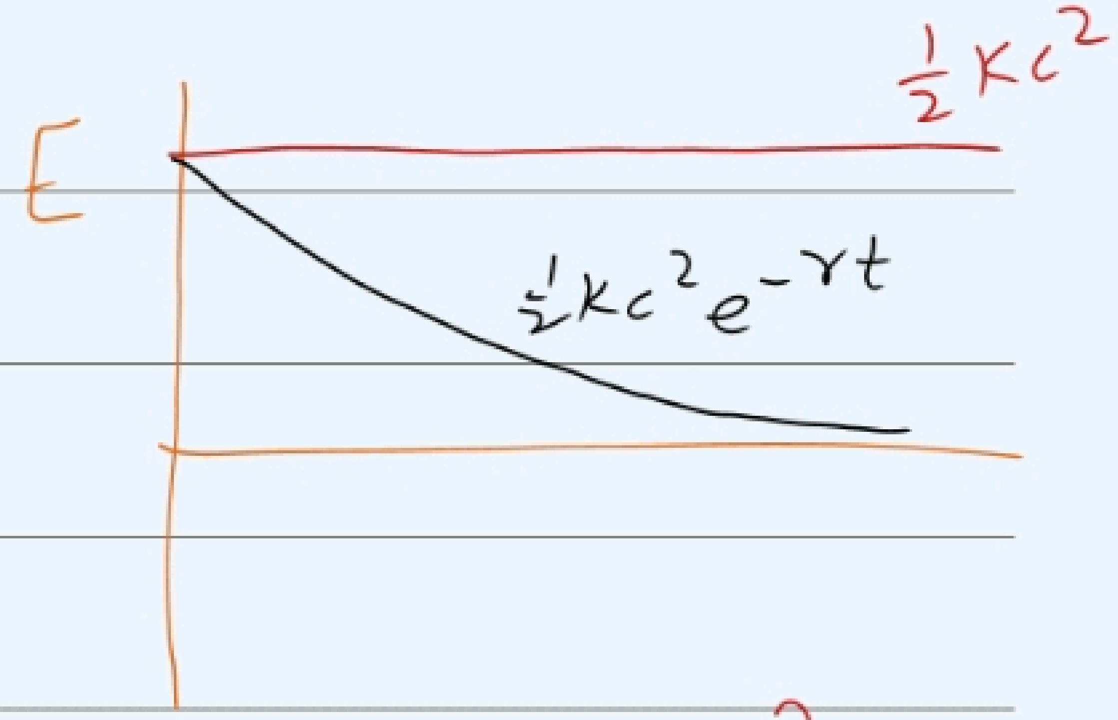
$x(t) = e^{-\frac{\gamma}{2}t} [A + Bt]$

Heavily damped.

$x(t) = x_0 [e^{-\frac{\gamma}{2}t}] e^{\pm \Gamma t}$

Energy:

Undamped: $E = \frac{1}{2}Kc^2$, $K.E = \frac{1}{4}Kc^2$, $P.E = \frac{1}{4}Kc^2$



Lightly damped:

$$x(t) = c\, e^{-\frac{\gamma}{2}t}\cos(\omega t+\phi)$$

$$P.E = \frac{1}{2}Kx^2 = \frac{1}{2}K c^2 e^{-\gamma t}\cos^2(\omega t+\phi)$$

$$K.E = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mc^2\left\{\left[\underbrace{\left(-\frac{\gamma}{2\omega}\right)\cos(\omega_0 t+\phi)}_{\text{small} \ll 1} - \sin(\omega t+\phi)\right]\omega e^{-\frac{\gamma}{2}t}\right\}^2$$

$$\boxed{\frac{\gamma}{2\omega} = \frac{b}{2m\left[\sqrt{\omega_0^2 - \frac{b^2}{4m}}\right]} \ll 1}$$

$$\approx \frac{1}{2}mc^2\omega^2\sin^2(\omega t+\phi)\,e^{-\gamma t}$$

$$E = \frac{1}{2}kc^2e^{-\gamma t} = E_0 e^{-\gamma t}$$

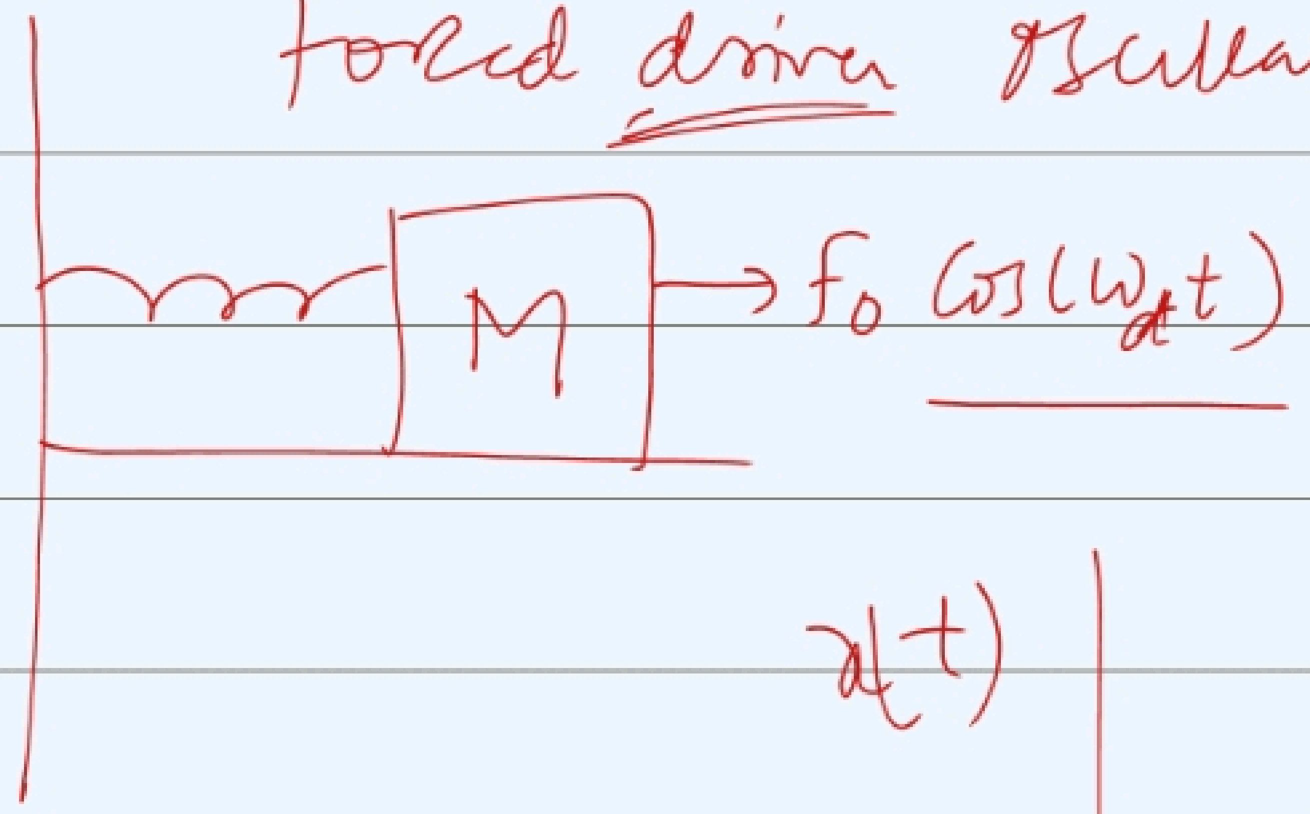
Q-factor: $= \dfrac{\text{Energy Stored in the oscillator}}{\text{Energy dissipated per radian}} = \dfrac{E}{\left(\frac{\gamma E}{\Delta\omega}\right)} = \dfrac{\Delta\omega}{\gamma}$.

$$E = E_0 e^{-\gamma t}$$

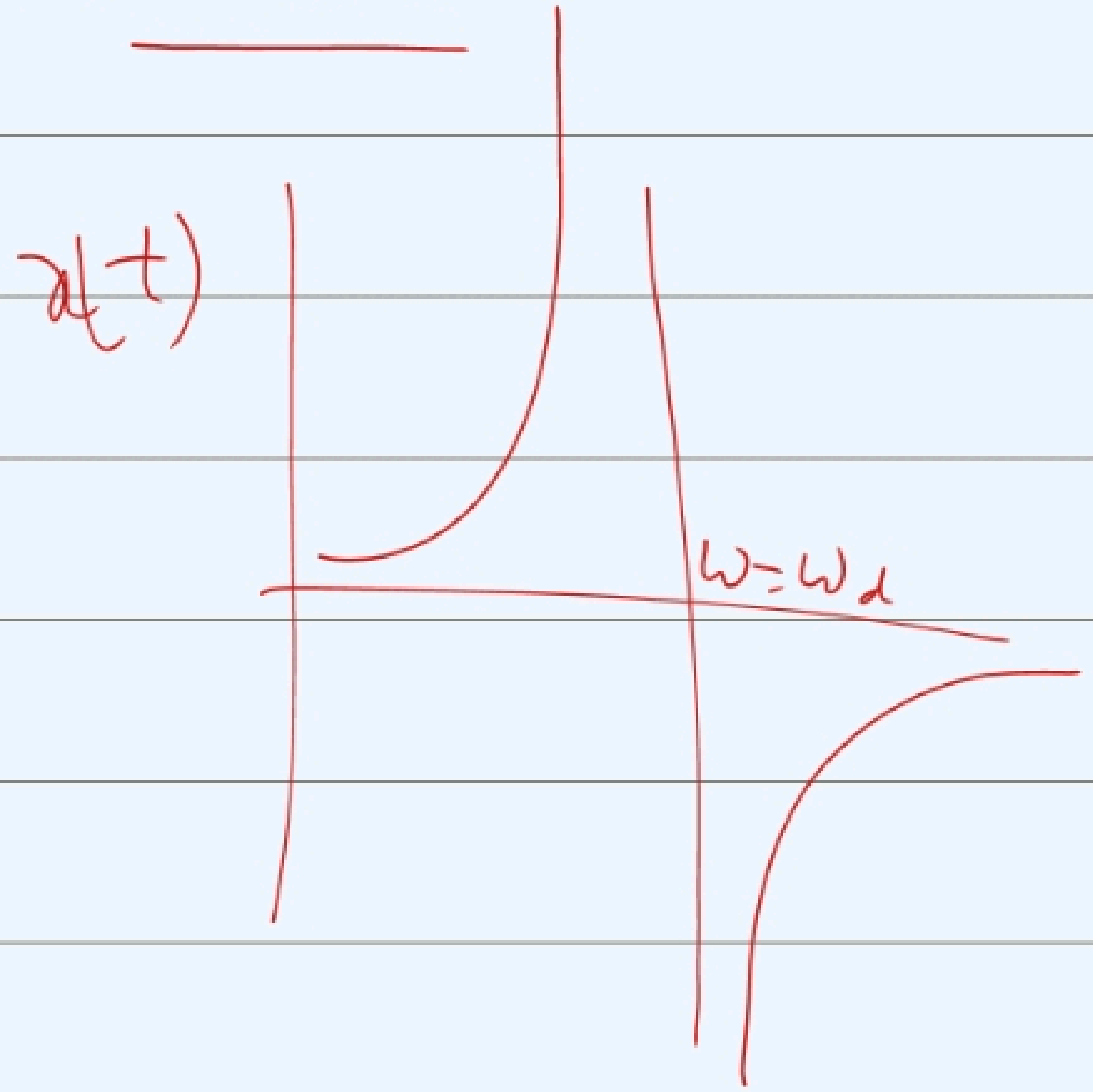
$$\frac{dE}{dt} = -\gamma E \qquad \Delta E \approx \left|\frac{dE}{dt}\right| \Delta t = \frac{\gamma E}{\Delta\omega}$$

1 radian : $\Delta t \propto \dfrac{1}{\Delta\omega}$

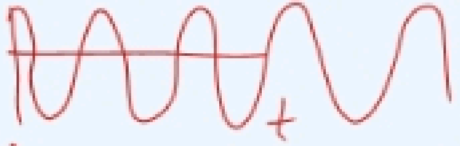
Forced driven oscillation



$$\vec{F} = -K\vec{x} + F_0 \cos(\omega_d t)$$



16/10/2023

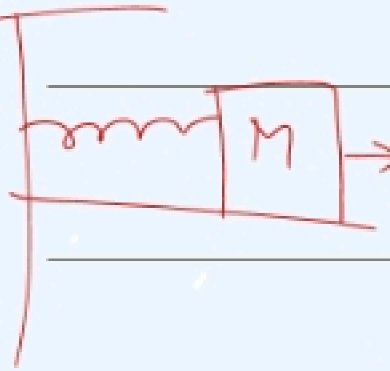
$A\cos(\omega_0 t+\phi) = x(t)$  Undamped

$A\cos(\omega t+\phi) = x(t)$  Damped

$\Big\}\ x(t)$

$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

Forced HO:



$\to F_0\cos\omega_d t$

↳ driving frequency

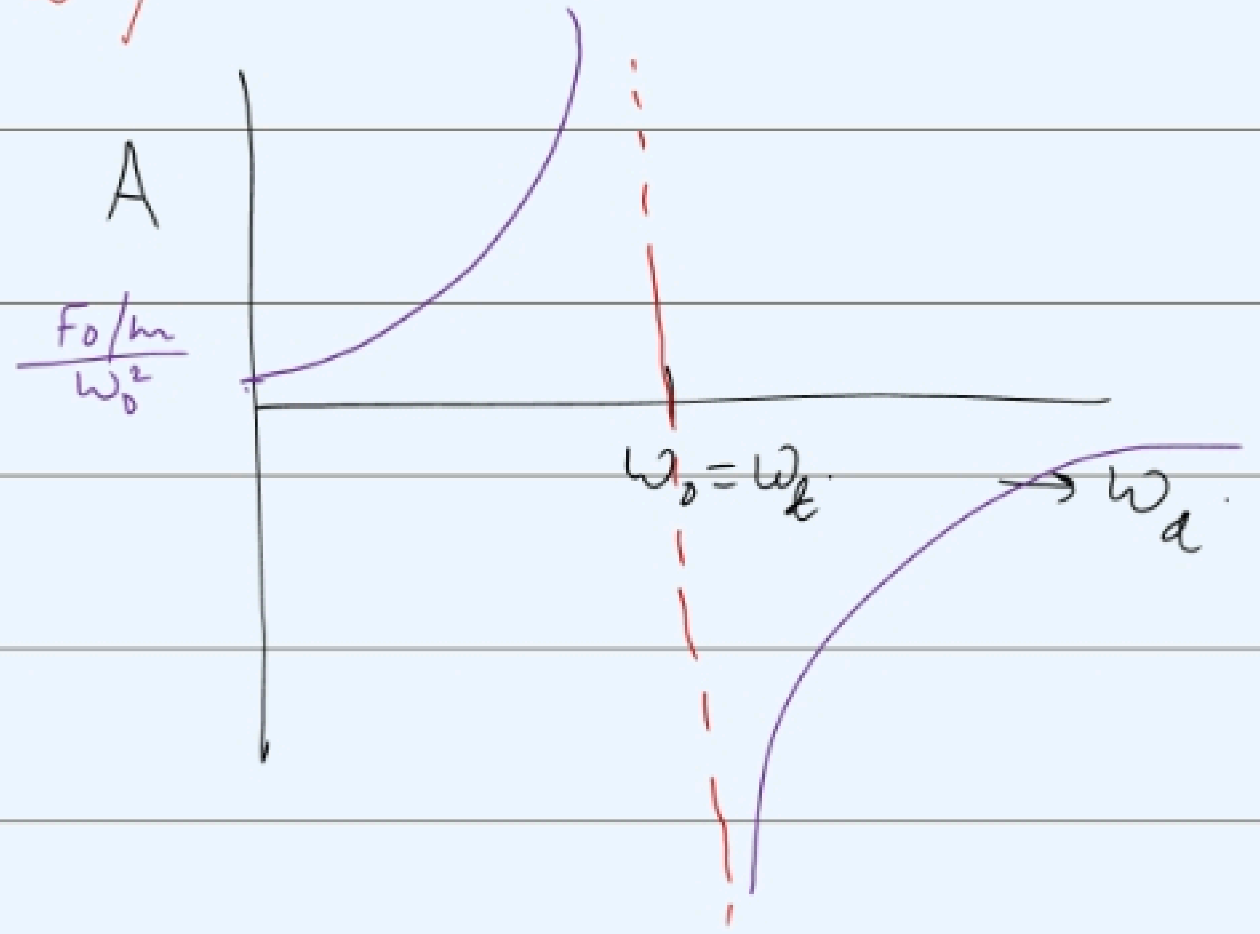
ω_d is not same $\omega_0 = \sqrt{\frac{K}{m}}$

$\vec{F} = -K\vec{x} + F_0\cos\omega_d t \Rightarrow \ddot{x} + \overbrace{\frac{K}{m}}^{\omega_0^2}x = \frac{F_0}{m}\cos\omega_d t$

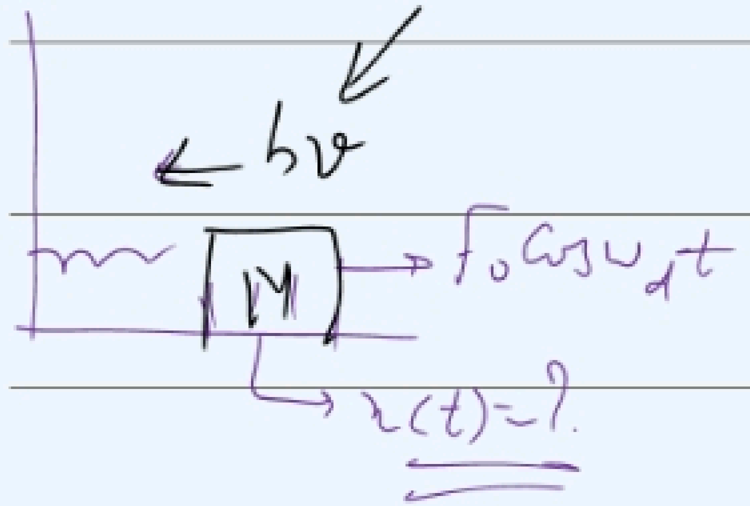
Guess: $x(t) = A\cos\omega_d t$

$\Rightarrow \left[-\omega_d^2 + \omega_0^2\right]A\cancel{\cos\omega_d t} = \frac{F_0}{m}\cancel{\cos\omega_d t}$

$\Rightarrow \boxed{A = \frac{F_0/m}{\omega_0^2 - \omega_d^2}}$



Forced Damped oscillation :-



$$\vec{F} = -kx - bv + f_0 \cos \omega_d t$$

$z = x + iy$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos \omega_d t \,,$$

y same

$$\boxed{\ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega_d t}} \quad — (1)$$

$\mathrm{Re}\left[e^{i\omega_d t}\right]$

Im $\quad \ddot{y} + \gamma \dot{y} + \omega_0^2 y = \frac{F_0}{m} \sin \omega_d t$

Guess: $\boxed{Z(t) = A e^{i(\omega_d t - \delta)}}$ — (2) $\qquad \cos(\omega_d t - \delta)$

$e^{i\delta}$

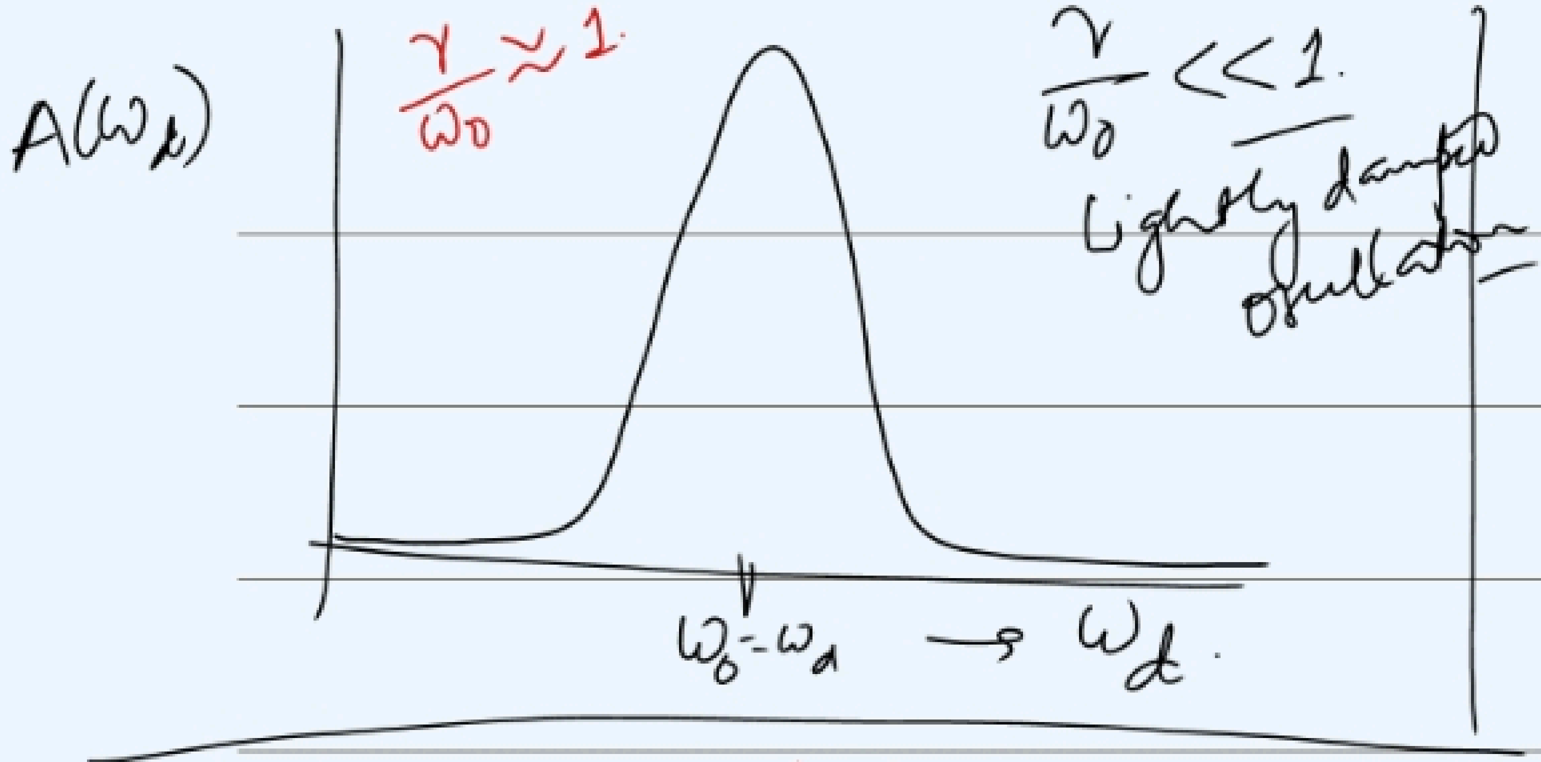
$$A e^{i(\omega_d t - \delta)} \left[-\omega_d^2 + i\gamma\omega_d + \omega_0^2\right] = \frac{F_0}{m} e^{i\omega_d t}$$

$$\rightarrow \left[(\omega_0^2 - \omega_d^2) + i\gamma\omega_d\right] A = \frac{F_0}{m}\left[\cos\delta + i \sin\delta\right] \Rightarrow$$

Real: $A(\omega_0^2 - \omega_d^2) = \frac{F_0}{m} \cos\delta$

Im: $A\gamma\omega_d = \frac{F_0}{m} \sin\delta$

$$A^2 = \frac{\left(\frac{F_0}{m}\right)^2}{(\omega_0^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}, \qquad \tan\delta = \frac{\gamma\omega_d}{\omega_0^2 - \omega_d^2}$$



3. $\omega_d \to \infty$, $A(\omega_d) \Rightarrow 0$, δ

$$\delta = \tan^{-1}\left[\frac{\infty}{-\infty}\right]$$

1. $$A(\omega_d) = \frac{(F_0/m)}{\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]^{1/2}}$$

$$\tan\delta(\omega_d) = \frac{\gamma\omega_d}{\omega_0^2-\omega_d^2}$$

$$z = A e^{i(\omega_d t-\delta)}$$

forced damped oscillation

$$x(t) = \frac{F_0}{m}\frac{\cos(\omega_d t-\delta)}{\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]^{1/2}}$$

2. If $\overset{b/m}{\gamma} \to 0$, $x(t)$: forced oscillation $A(\omega_d) = \frac{F_0/m}{(\omega_0^2-\omega_d^2)}$, $\delta = \pm\pi$

3. When $\omega_0 = \omega_d$, $A(\omega_d) = \frac{F_0/m}{\gamma\omega_d}$, $\delta = \pm\pi/2$

Energy for forced Damped HO → $x(t) = A\cos[\omega_d t - \delta]$

$E = K.E + P.E.$ $\quad A = \dfrac{F_0/m}{\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]^{1/2}}$, $\tan\delta = \dfrac{\gamma\omega_d}{\omega_0^2-\omega_d^2}$

$K.E = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\left[\sin^2(\omega_d t-\delta)\,\omega_d^2\right] \Rightarrow \langle K.E\rangle = \frac{1}{4}mA^2\omega_d^2$

$P.E = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2\left[\cos^2(\omega_d t-\delta)\right] \Rightarrow \langle P.E\rangle = \frac{1}{4}KA^2$

$$\langle E\rangle = \frac{1}{4}A^2\left[m\omega_d^2+K\right] = \frac{1}{4}A^2 m\left[\omega_d^2+\omega_0^2\right] = \frac{1}{4}\frac{(F_0^2/m)\overbrace{(\omega_0^2+\omega_d^2)}^{2\omega_0^2}}{\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]}$$

Special Case:

Lightly damped: $\gamma \ll \omega_0$

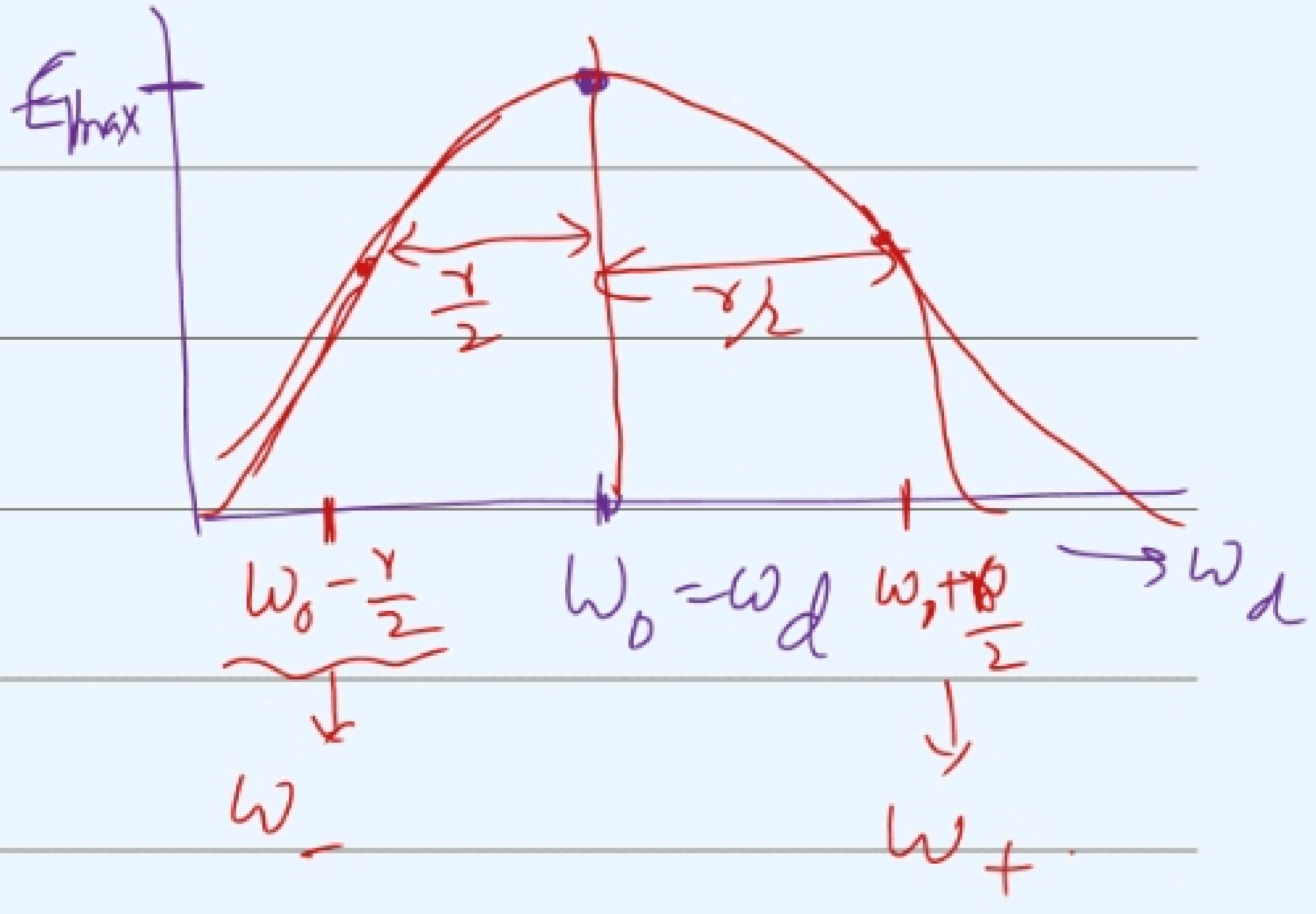
we consider $\omega_0 \approx \omega_d$

$$\omega_0^2-\omega_d^2 = (\omega_0+\omega_d)(\omega_0-\omega_d) \approx 2\omega_0(\omega_0-\omega_d)$$

$$\langle E(t)\rangle = \frac{1}{2}\frac{F_0^2}{m}\,\frac{\approx\omega_0^2}{4\omega_0^2(\omega_0-\omega_d)^2+\gamma^2\omega_0^2}$$

<1

$$E(t) = \frac{1}{\cancel{4}_{2}} \cdot \frac{F_0^2}{m} \frac{\cancel{2} \omega_0^2}{\cancel{4}\omega_0^2 (\omega_0 - \omega_d)^2 + \gamma^2 \omega_0^2}$$



$$E(t) = \frac{1}{8} \frac{F_0^2}{m} \cdot \frac{1}{(\omega_0 - \omega_d)^2 + \frac{\gamma^2}{4}}$$

c_1

$$\langle E(t) \rangle_{max} = c_1 \left(\frac{4}{\gamma^2} \right) \qquad \left[\omega_0 = \omega_d \right]$$

$$\boxed{\omega_+ - \omega_- = \underline{\gamma} = \Delta\omega}$$

Half-Max $\langle E(t) \rangle_{max}$: $(\omega_0 - \omega_d)^2 = \left(\frac{\gamma}{2}\right)^2$

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega}$$

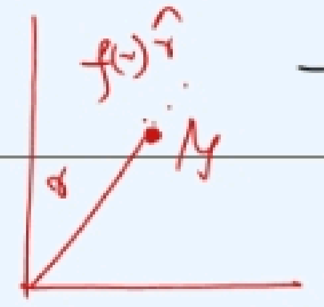
18/10/2023 Central force Motion $\to$ $f(r)\,\hat{r}$

EOM: $m_1 \ddot{r}_1 = f(r)\,\hat{r}$ — (1)

$m_2 \ddot{r}_2 = -f(r)\,\hat{r}$ — (2)

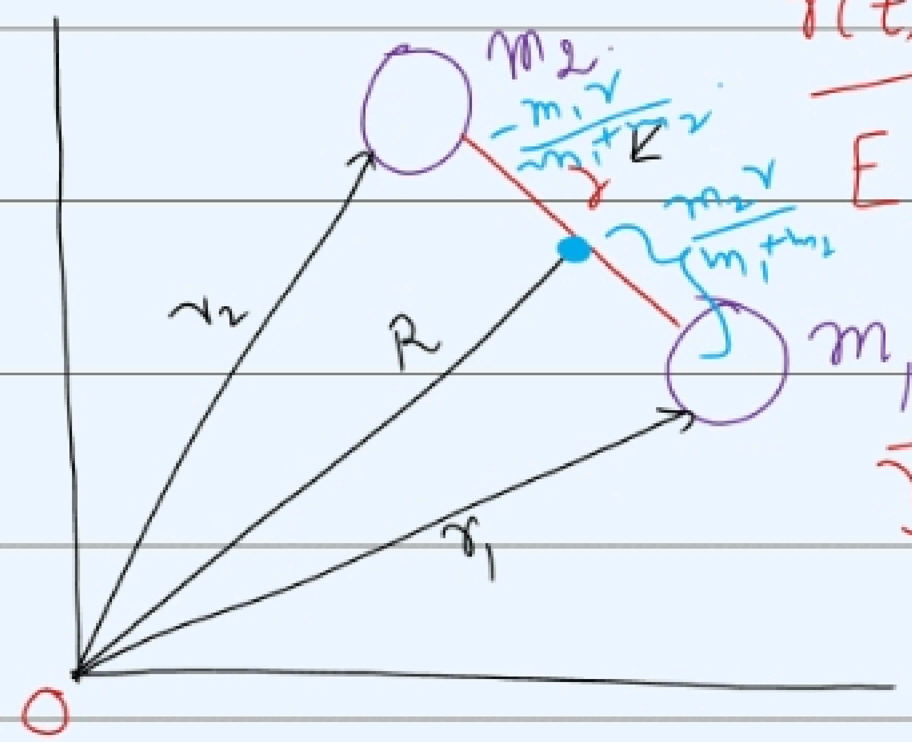
(1) − (2)

$$\ddot{r}_1 - \ddot{r}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\,\hat{r}$$



$$\frac{m_1 m_2}{m_1 + m_2}\left(\ddot{r}\right) = f(r)\,\hat{r}$$

$$\Rightarrow \boxed{M \ddot{r} = f(r)\,\hat{r}}$$ — 1-body problem



$r(t) = ?$

$E > 0, < 0?$

Angle (t)

$\vec{r}(\theta) = ?$

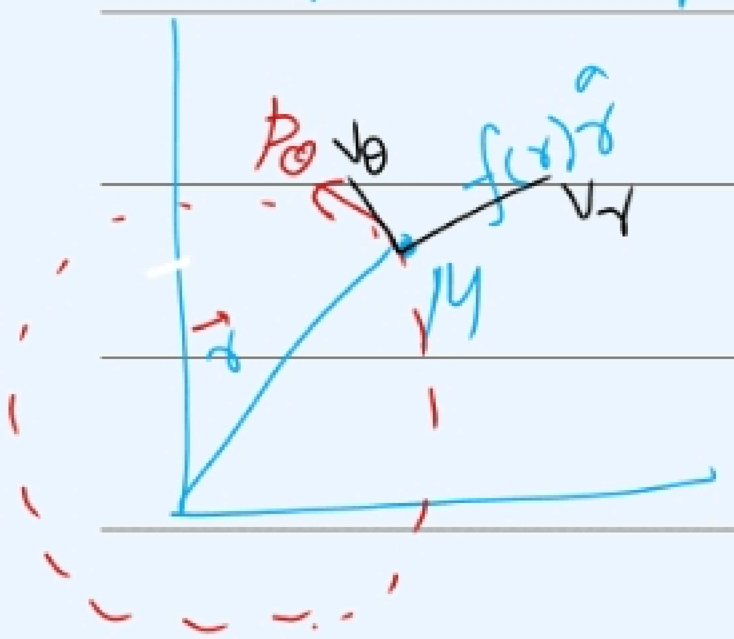
$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m_1 r_1 + m_2\left[r_1 - r\right]}{m_1 + m_2}$$

$$\to \boxed{r = r_1 - r_2}$$

$$= \frac{(m_1 + m_2) r_1}{m_1 + m_2} - \frac{m_2}{m_1 + m_2} r$$

$$r_1 = R + \frac{m_2 r}{m_1 + m_2}$$

General property of Central force field Motion :-



$$\vec{L} = \text{Constant} = \vec{r} \times \vec{p}$$

$$\boxed{|L| = l = \mu r v_\theta = \mu r^2 \dot{\theta}}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}(\hat{r}) = 0$$

$$\hat{r} \times \hat{r} = 0$$

$$\vec{p} = \mu \left[\overset{\dot{r}\hat{r}\ +\ r\dot{\theta}\hat{\theta}}{v_r \hat{r} + v_\theta \hat{\theta}} \right]$$

$$\vec{r} \times \vec{p} = 0 \quad \mu r v_\theta$$

Energy :

$$E = K.E + P.E$$

$$= \frac{1}{2} \mu v^2 + U(r)$$

$$= \frac{1}{2} \mu \left[\dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \right]^2 + U(r)$$

$$E(t) = \frac{1}{2} \mu \dot{r}^2 + \underbrace{\frac{1}{2} \mu r^2 \dot{\theta}^2}_{\frac{1}{2} \frac{l^2}{\mu r^2}} + U(r)$$

$$\underbrace{\frac{1}{2} \frac{l^2}{\mu r^2} + U(r)}_{U_{eff}}$$

$$\frac{1}{2} \mu r^2 \dot{\theta}^2 = \frac{1}{2} \mu r^2 \left(\frac{l^2}{\mu^2 r^4} \right)$$

$$= \frac{1}{2} \frac{l^2}{\mu r^2}$$

$$E(t) = \frac{1}{2}\mu \dot{r}^2 + \underbrace{\frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)}_{U_{eff.}}$$

$$U_{eff.} = \frac{\ell^2}{2\mu r^2} + U(r)$$

$$\boxed{\frac{d\vec{r}}{dt} = \sqrt{\frac{2}{\mu}\left[E - U_{eff.}\right]}} \quad , \quad \dot{\theta} = \frac{d\theta}{dt} = \frac{\ell}{\mu r^2} \rightarrow \theta(t)$$

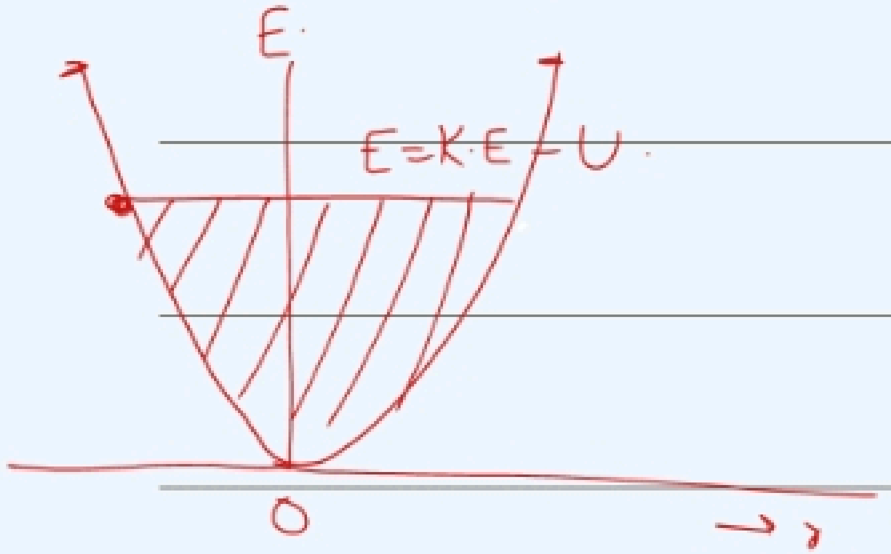
$r(t)$?

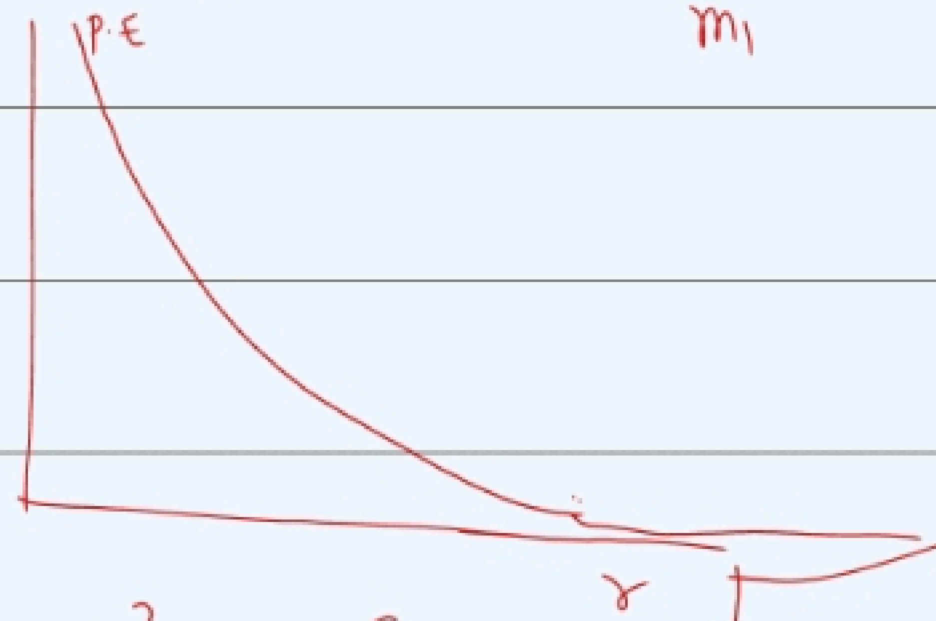
$$\int \frac{dr}{\sqrt{\frac{2}{\mu}\left[E - U_{eff}\right]}} = \int dt$$

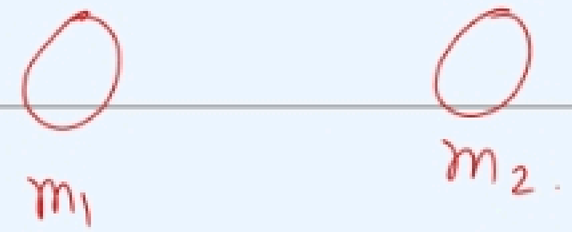
$$\frac{d\theta}{dr} = \frac{d\theta}{dt} \cdot \frac{1}{\frac{dr}{dt}} \rightarrow \theta(r)$$

Recall: $U(x) = \frac{1}{2}Kx^2$ $\qquad$ $\vec{F} = -\frac{A}{r^2}\hat{r}$

$F = -Kx$ $\qquad$ $U(r) = \frac{A}{r}$



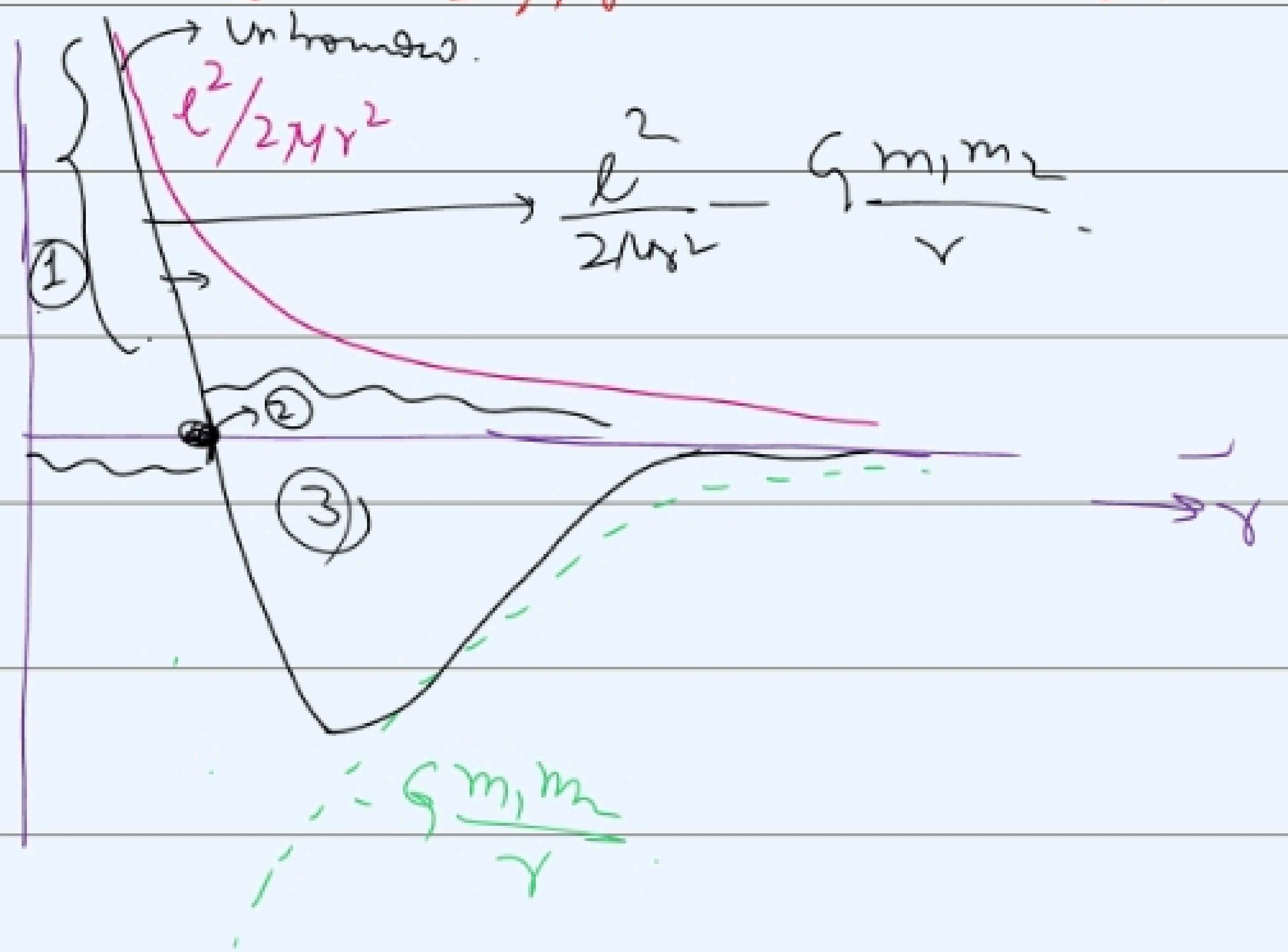




Case: $U_{eff} = \frac{l^2}{2\mu r^2} + U(r) = \frac{l^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$

m_1, m_2

$$\boxed{U = -\frac{Gm_1m_2}{r}}$$



1. $E > 0$, Unbounded.
2. → Boundary of bounded & unbound region
3. $E < 0$

$l = \mu V_\theta$

$\sin(\theta - \theta_0)$

Planetary Motion:→ $\frac{d\theta}{dr} = \frac{d\theta}{dt} \cdot \frac{1}{\frac{dr}{dt}} = \frac{l}{\mu r^2} \cdot \frac{1}{\sqrt{\frac{2}{\mu}\left[E - V_{eff}\right]}}$

$U(r) = -\frac{G m_1 m_2}{r} = -\frac{c}{r}$, $V_{eff} = \frac{l^2}{2\mu r^2} + U(r)$

$\int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = l \int \frac{dr}{r^2\left[\frac{2\mu^2}{\mu}\left\{E - \left(\frac{l^2}{2\mu r^2} - \frac{c}{r}\right)\right\}\right]^{1/2}}$

$\theta - \theta_0 = l \int \frac{dr}{r\left[2\mu E r^2 + 2\mu c r - l^2\right]^{1/2}} = \arcsin\left[\frac{\mu c r - l^2}{r\left(\sqrt{\mu c^2 + 2\mu E l^2}\right)}\right]$

$$\epsilon = \sqrt{1+\frac{2El^2}{Mc^2}}$$

hyperbola

1. $\epsilon > 1$: $y^2 - Ax^2 - Bx = \text{constant}$

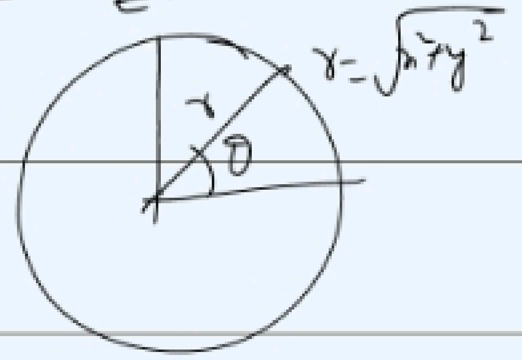
2. $\underline{\epsilon = 1}$: $y^2 - Bx = \text{constant}$ parabola

$$\theta - \theta_0 = \arcsin\left[\frac{Mcr - l^2}{r\sqrt{M^2c^2 + 2MEl^2}}\right]$$

$$Mcr - l^2 = r\left(\sqrt{M^2c^2 + 2MEl^2}\right)\underbrace{\sin(\theta - \theta_0)}_{\cos\theta}, \quad \theta_0 = \frac{\pi}{2}$$

$$\Rightarrow r = \frac{l^2/Mc}{1 - \underbrace{\sqrt{1+\frac{2El^2}{Mc^2}}}_{\epsilon}\cos\theta}$$

3. $\underline{0 < \epsilon < 1}$ → Ellipses



$$r = \frac{r_0}{1-\epsilon\cos\theta} \Rightarrow r - r\epsilon\cos\theta = r_0$$

$$\sqrt{x^2+y^2} - \epsilon x = r_0$$

$$x^2 + y^2 = r_0^2 + \epsilon^2 x^2 + 2\epsilon x r_0$$

$$\boxed{(1-\epsilon^2)x^2 + y^2 - (2\epsilon r_0)x = r_0^2}$$