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Theorem

Let $R \subseteq A \times B$ be a relation and $A_1, \; A_2 \subseteq A$. Then

$$R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$$

Definition (n-ary relation)

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Relations and database

Name	*Roll No*	*Branch*	*Date of birth*
Sunil K	202352001	*IT*	1/1/2005
Ramesh K	202351002	*CS*	26/1/2005
Sachin T	202352002	*IT*	1/1/2005

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Let $R = \{(1,1), (1,2), (2,1), (2,2), (2,4), (2,3), (3,4), (4,1)\}$ be a relation on $\{1, 2, 3, 4\}$.

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Theorem

A relation R on set A is transitive if and only if $R^n \subseteq R$ for all $n \geq 1$.

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Then A is a disjoint union of $R(\{a\})$ for $a \in A$. Relation R partitions A into disjoint sets called equivalence classes.

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Let $A = [a_{ij}], B = [b_{ij}]$ be two $m \times n$ zero-one matrices.
Define join of A and B: $A \vee B$, a matrix $[c_{ij}]$, where $c_{ij} = a_{ij} \vee b_{ij}$.
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Define the Boolean product as $A \odot B = [c_{ij}]$, where
$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

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$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

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Can we find it using Directed graph of R?

Transitive closure: Warshall's Algorithm

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Algorithm: War(M_R)
{ Set $W = M_R$
for $k = 1$ to n
    for $i = 1$ to n
        for $j = 1$ to n
            $w_{ij} = w_{ij} \vee (w_{ik} \wedge w_{kj})$
end
Return($W = [w_{ij}]$)
}

Partially Ordered Set/Poset

Definition: A relation R on a set A is called partial order if R is reflexive, antisymmetric and transitive.
A set A with partial order $\preceq$ is called partially ordered set or Poset.
Examples:

1. Let S be a set and on power set- $P(S)$ define $S_1 \preceq S_2$ if $S_1 \subseteq S_2$. Then $(P(S), \subseteq)$ is a Poset.
2. $(\mathbb{N}, |)$, where $a|b$ iff a divides b, is also a Poset.
3. $(\mathbb{N}, \leq)$ is a Poset.

Remark: In a Poset, every pair of elements a, b need not be related.
e.g. in (2), neither $2 \mid 3$ nor $3 \mid 2$.
Definition: In a Poset $(A, \preceq)$, $a, b \in A$ are said to be comparable if either $a \preceq b$ or $b \preceq a$.
Definition: If every pair (a, b) of a Poset $(A, \preceq)$ is comparable, then $(A, \preceq)$ is said to be linearly/totally ordered;
and partial order is called linear order/chain.

Properties of Poset

- Let $(A, \preceq_1)$ and $(B, \preceq_2)$ be Posets. Then $(A \times B, \preceq)$ is also a Poset, where $(a_1, b_1) \preceq (a_2, b_2)$ iff $a_1 \preceq_1 a_2$ and $b_1 \preceq_2 b_2$.
- Let $(A, \preceq_1)$ and $(B, \preceq_2)$ be Posets. Then $(A \times B)$ with lexicographic/dictionary order is also a Poset, lexicographic order$=(a, b) \preceq (c, d)$ iff $a \prec_1 c$ or if $a = c$ then $b \preceq_2 d$. (Note $a \prec_1 c$ means that $a \neq c$ and $a \preceq_1 c$)
  Let $G = (V, E)$ be a directed graph.
  A cycle of length n in G is a sequence $a_1, \cdots, a_n$ of vertices such that $(a_i, a_{i+1}) \in E$ (i.e., an edge) for each $1 \leq i \leq n-1$ and $(a_n, a_1) \in E$.
- Directed graph of a partial order has no cycle of length > 1.

Definition

A Poset $(A, \preceq)$ is said to be well ordered if $\preceq$ is a linear order and every non-empty subset of A has a least element.

($a \in A_1$ is said to be least element of A_1 if $a_1 \preceq x$ for all $x \in A_1$.)

$(\mathbb{N}, \leq)$ is a well-ordered but $(\mathbb{Z}, \leq)$ is not well-ordered.

$\mathbb{N} \times \mathbb{N}$ with lexicographic ordering is well ordered.

Theorem (Principle of Well-Ordered Induction)

Let A be a well-ordered set and $P(x)$ a statement for each element $x \in A$. Then $P(x)$ is true for all $x \in A$ if

for every $y \in A$, if $P(x)$ is true for every $x \prec y$, then $P(y)$ is true. *OR*

Every non-empty subset of a well ordered set has least element.

Application of WOI: Show that any payment of Rs at least 8 can be made using notes of Rs 3 and 5.

$P(x): \ x = 3a + 5b$ for some $a, b \in \mathbb{N} \cup \{0\}$

$A = \{8, 9, 10, \ldots\}$ is well ordered with $\leq$ relation.

Pick n from A. Assume $P(x)$ is true for every $x < n$.

$n = n - 3 + 3 \Rightarrow \ P(n) = P(n-3) + 1.3$ (except $n = 8, , 9, 10$

$\sqrt{2}$ is irrational.

Theorem (Fundamental theorem of Arithmetic)

Every positive integer greater than one can be factored as a product of primes.

Hasse diagram

Let $A = \{2, 3, 4, 6, 8\}$. Then $(A, |)$ is a Poset. Find graph representation of it.

Construction of Hasse diagram:

1. Let $G = (V, E)$ be a directed graph of partial order $\preceq$ on a finite set A.
2. Remove all edges corresponding to reflexive relations (i.e., (a,a) for all $a \in A$).
3. Remove all the edges corresponding to transitive relations, i.e., remove (a, c) if $(a, b), (b, c) \in E$.
4. Arrange/Draw the graph in such a way that arrows of the edges are pointing upward. Then drop the arrows.

Definition

An element x of a Poset $(A, \preceq)$ is said to be maximal if there is no $b \in A$ such that $x \prec b$.
An element y of a Poset $(A, \preceq)$ is said to be minimal if there is no $b \in A$ such that $b \prec y$.

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minimal elements are 2,3.

Definition

An element x of a Poset $(A, \preceq)$ is said to be greatest element if $a \prec x$ **for all** $a \in A$.

An element y of a Poset $(A, \preceq)$ is said to be least element if $y \prec a$ **for all** $a \in A$.

Find greatest and least elements for $A = \{2, 3, 4, 6, 8\}$ with divisibility relation.

Least Upper Bound (LUB): Let $(A, \preceq)$ a Poset and A_1 be a subset of A.

$u \in A$ is called upper bound of $A_1 \subseteq A$ if $a \preceq u$ for all $a \in A_1$.
$u \in A$ is called least upper bound of $A_1 \subseteq A$ if u is upper bound and $u \preceq u_1$ for all upper bounds u_1.

Greatest Lower Bound (GLB):
$l \in A$ is called lower bound of A_1 if $l \preceq a$ for all $a \in A_1$
l is called greatest lower bound if l is lower bound and $l_1 \preceq l$ for every lower bound.

Lattice: A Poset in which every pair of elements has GLB and LUB is called lattice.

Examples: $(\mathbb{N}, |)$ ✓ ,
$(\{1, 2, 3, 4, 5\}, |)$ $\times$,
$(\{1, 2, 4, 8, 16\}, |)$ ✓
$(P(S), \subseteq)$✓

Notations: Given $a, b \in (A, \preceq)$-Lattice,
$a \vee b :=$ LUB of a and b (join of a and b).
$a \wedge b :=$ GLB of a and b (meet of a and b).

Theorem

L1. $a \vee a = a$ *and* $a \wedge a = a$

L2. $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$

L3. $(a \vee b) \vee c = a \vee (b \vee c)$ *and* $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

L4. $a \wedge (a \vee b) = a$ *and* $a \vee (a \wedge b) = a$

L5. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

L6. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Topological sorting

Suppose that a project is made up of 10 different tasks and only one person to complete it. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

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Question: Does there always exists total ordering on a Poset?
Definition: We say total ordering $\preceq$ is compatible with partial ordering R on a set A if for all $a, b \in A$

$$a \; R \; b \; \Rightarrow \; a \preceq b$$

Lemma: Every non-empty finite poset $(A, \preceq)$ has atleast one minimal element.
Choose $a_0 \in A$.
If a_0 is minimal then we are done.
If a_0 is not minimal then $\exists a_1 \in A$ such that $a_1 \preceq a_0$.
If a_1 is minimal then we are done.
If a_1 is not minimal then $\exists a_2 \in A$ such that $a_2 \preceq a_1$.
Since A is finite, this process stops.

Algorithm: Topological sorting Topsort$((A, R))$

$\{k = 1$

while $(A \neq \phi)$

$\{a_k =$ minimal element of A

$S = S - \{a_k\}$

$k = k + 1$

$\}\}$ Output$=a_1, a_2, \ldots, a_n$ which gives total ordering of A.

$$a_1 \preceq a_2 \preceq a_3 \preceq \cdots \preceq a_n$$