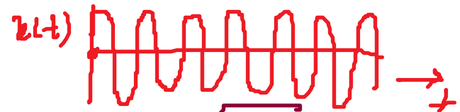
Lecture 13: Section 2

Friday, January 28, 2022 8:50 AM

Harmonic Oscillator



Undamped HO:

$$x(t) = C\cos(\omega_0 t + \phi)$$

$$\omega_0 = \sqrt{\frac{K}{m}}$$

$$C = \sqrt{A^2+B^2}$$

$$\phi = \tan^{-1}\left[\frac{-B}{A}\right]$$

Damping { air, Water, Honey }

* Energy :→ $E(t) = K.E(t) + P.E(t)$

$$= \frac{1}{2}mv^2 + \frac{1}{2}Kx^2$$

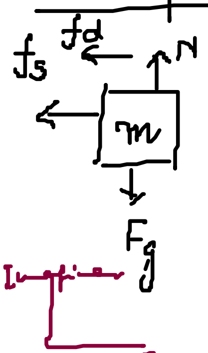
$$= \frac{1}{2}m\left[-C\omega_0 \sin(\omega_0 t+\phi)\right]^2 + \frac{1}{2}K\left[C\cos(\omega_0 t+\phi)\right]^2$$

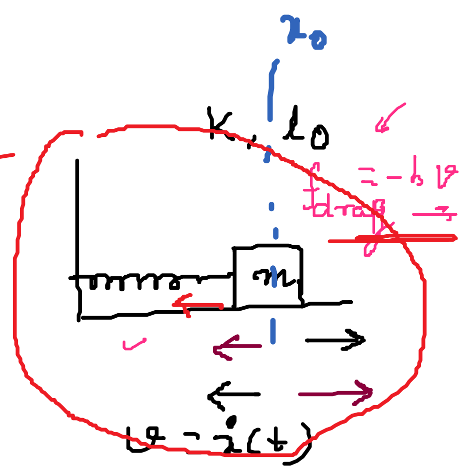
$$\boxed{E(t) = \frac{1}{2}mC^2\omega_0^2}$$

$$\langle K.E\rangle = \frac{1}{4}mc^2\omega_0^2 = \langle P.E\rangle$$

$x(t) =$

Damped HO :→





$$\vec{F} = f_s + f_d + N + F_g + \text{More forces}$$

$N + F_g = 0$

$$m\ddot{x} = -Kx(t) - b\dot{x}(t)$$

$$\Rightarrow \boxed{\ddot{x} + \frac{b}{m}\dot{x}(t) + \frac{K}{m}x = 0}$$

$$\gamma = \frac{b}{m} \qquad \omega_0^2 = \frac{K}{m}$$

$f_d = -bv$

$b =$ damping Constant

Imaginary / Real

$z = x + iy$

$\ddot{x} + i\ddot{y} = \ddot{z}$

$$x(t) = Re[z(t)] = Re\left[z_0 e^{i\alpha t}\right].$$

$$x(t) = Re\left[C e^{i(\omega_0 t+\phi)}\right]$$

$$= C\cos(\omega_0 t+\phi)$$

$$\Rightarrow \ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0$$

$$\rightarrow \left[\alpha^2 + i\gamma\alpha + \omega_0^2\right] z_0 e^{i\alpha t} = 0$$

$\alpha \rightarrow \gamma, \omega_0$
m, K

$$\rightarrow \left[-\alpha^2 + i\gamma\alpha + \omega_0^2\right] z_0 e^{i\alpha t} = 0$$

m, k

$$\Rightarrow \boxed{\alpha^2 - i\gamma\alpha - \omega_0^2 = 0}$$

$$ax^2 + bx + c = 0$$

$$\Downarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2}$$

$$\boxed{\alpha = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}$$

$$\omega_0^2 = \frac{K}{m}, \quad \gamma = \frac{b}{m}$$

$$= \frac{i\gamma}{2} \pm \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}$$

b : damping constant.

(1) $\omega_0^2 > \frac{\gamma^2}{4}$: Lightly damped oscillator

(2) $\omega_0^2 = \frac{\gamma^2}{4}$: critically damped oscillator

(3) $\omega_0^2 < \frac{\gamma^2}{4}$: Heavily damped oscillator

Lightly damped oscillator :- $\omega_0^2 > \frac{\gamma^2}{4}$, $\alpha = \frac{i\gamma}{2} \pm \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}_{+ve\ number}$

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\rightarrow z_+(t) = z_{0+} e^{i[\alpha t]} = z_{0+} e^{-\frac{\gamma}{2}t} e^{i\omega t}$$

$$z_-(t) = z_{0-} e^{-\frac{\gamma}{2}t} e^{-i\omega t}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$+ \quad e^{-i\omega t} = \cos\omega t - i\sin\omega t$$

$$z_1(t) = \frac{1}{2}\left[z_{+0}(t) + z_{-0}(t)\right]$$

$$= \frac{1}{2}\left[z_{0+} e^{-\frac{\gamma}{2}t} e^{i\omega t} + z_{0-} e^{-\frac{\gamma}{2}t} e^{-i\omega t}\right]$$

$$[z_{0+} = z_{0-}]$$

$$-\gamma t \left[e^{i\omega t} \quad e^{-i\omega t} \right]$$

$$= z_0 e^{-\frac{\gamma}{2}t}\left[\underbrace{\frac{e^{i\omega t}+e^{-i\omega t}}{2}}_{\cos\omega t}\right]$$

$z_{0+} = z_{0-}$

$$z_1(t) = z_0 e^{-\gamma/2\, t}\cos\omega t$$

$$z_2(t) = \frac{1}{2}\left[z_{0+} - z_{0-}\right] = \frac{1}{2} z_0\left[\underbrace{\frac{e^{i\omega t} - e^{-i\omega t}}{2}}_{i\sin\omega t}\right]$$

$$z(t) = z_0 e^{-\frac{\gamma}{2}t}\cos\omega t + i z_{0_b} e^{-\frac{\gamma}{2}t}\sin\omega t$$

$$z(t) = e^{-\gamma/2\, t}\left[z_0\cos\omega t + i z_0\sin\omega t\right]$$

$$x(t) = Re\left[z(t)\right] = C e^{-\frac{\gamma}{2}t}\cos(\omega t+\phi)$$

$$x(t) = C e^{-\frac{\gamma}{2}t}\cos(\omega t+\phi) \qquad \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

Initial Condⁿ ⇒ 1. $z(0) = z_{initial} = z_0$

$$x(0) = x_{initial} = C\cos\phi$$

2. $\dot{x}(t_0) = 0$, $\dot{x}(t) = C\left(-\frac{\gamma}{2}\right)\cos(\omega t+\phi)\, e^{-\gamma/2\, t}$

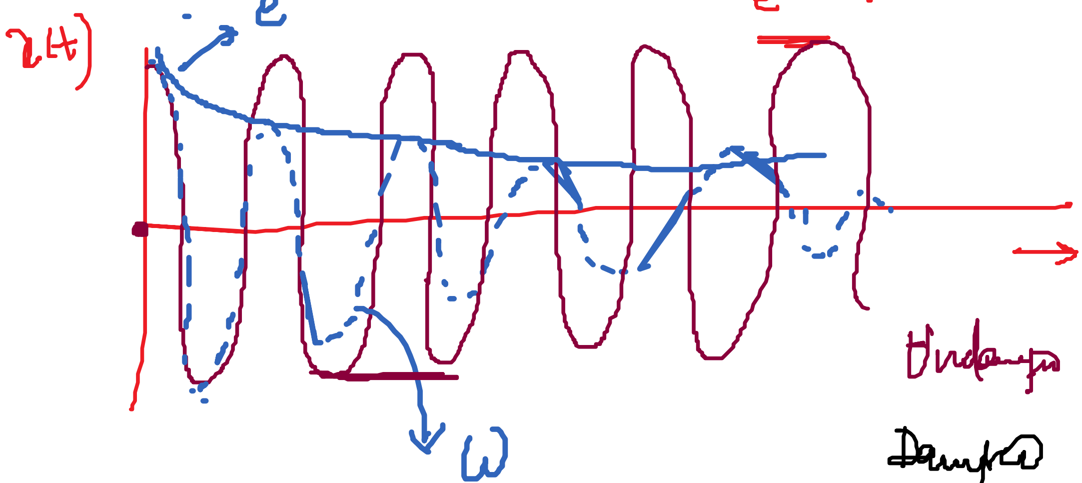
$$- C e^{-\gamma/2\, t}\cdot\omega\cdot\sin(\omega t+\phi)$$

$$\dot{x}(0) = -C\frac{\gamma}{2}\cos\phi - C\omega\sin\phi = 0$$

$$\Rightarrow \boxed{\tan\phi = -\frac{\gamma}{2\omega}}$$

$$\boxed{x(t) = x_{initial}\, e^{-\gamma/2} \cos(\omega t + \phi)}$$

$e^{-\frac{b}{2m}}$ $\qquad$ $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$



$\omega_0 > \omega$

Undamped $\omega_0 T = 2\pi$

Damped $\omega T = 2\pi$