PH110: Waves and Electromagnetics

Lecture 10

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Laplace's Equation

Q: How to find electric field $\mathbf{E}$?

Ans: $\mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{dq}{\mathscr{r}^2}\,\hat{\mathscr{r}}$ (Coulomb's Law)

Very difficult to calculate the integral except for very simple situation

Alternative: First calculate the electric potential

$$\mathrm{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}}$$

This integral is relatively easier but in general still difficult to handle

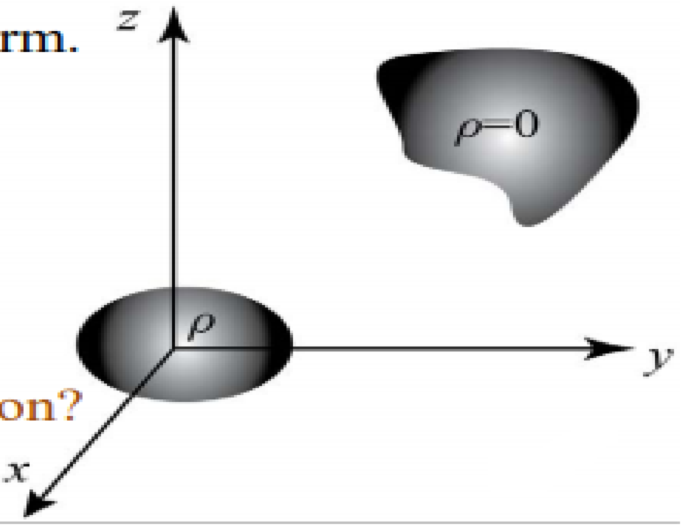
Alternative: Express the above equation in the different form.

$$\nabla^2 \mathrm{V} = -\frac{\rho}{\epsilon_0} \qquad \text{(Poisson's Equation)}$$

When $\rho = 0$ $\quad \nabla^2 \mathrm{V} = 0$ (Laplace's Equation)

If $\rho = 0$ everywhere, $V = 0$ everywhere

If ρ is localized, what is V away from the charge distribution?



Laplace's Equation in One Dimension

$$\nabla^2 V = 0 \qquad \text{(Laplace's Equation)}$$

In Cartesian coordinates,

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

If $V(x, y, z)$ depends on only one variable, x, We have

$$\frac{d^2}{dx^2}V = 0$$

(One-dimensional Laplace's Equation, ordinary differential equation)

General Solution: $V(x) = mx + b$

How to calculate the constants m and b ?

Using boundary conditions

What decides the boundary condition?

The charge distribution

Laplace's Equation in one dimension

If the potential V(x) is a solution to the Laplace's equation then V(x) is the average of the potential at $x + a$ and $x - a$

$$V(x,y) = \frac{1}{2}[V(x+a) + V(x-a)]$$

As a result, V(x) cannot have local maxima or minima; the extreme values of V(x) must occur at the end points.

Laplace's Equation in two dimensions

If the potential V(x, y) is a solution to the Laplace's equation then V(x, y) is the average value of potential over a circle of radius R centered at (x, y).

$$V(x,y) = \frac{1}{2\pi R}\oint_{circle} V\,dl$$

As a result, V(x, y) cannot have local maxima or minima; the extreme values of V(x, y) must occur at the boundaries.

Laplace's Equation

$$\nabla^2 V = 0$$

(Laplace's Equation)

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

(In Cartesian coordinates)

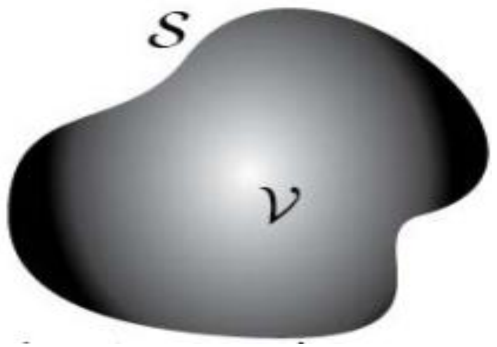
If the potential $V(\mathbf{r})$ is a solution to the Laplace's equation then $V(\mathbf{r})$ is the average value of potential over a spherical surface of radius R centered at $\mathbf{r}$.

$$V(\mathbf{r}) = \frac{1}{4\pi R^2}\oint_{sphere} V da$$

As a result, $V(\mathbf{r})$ cannot have local maxima or minima; the extreme values of $V(\mathbf{r})$ must occur at the boundaries.

Why? Because if the potential has a maximum value $V_{max}(\mathbf{r})$ at $\mathbf{r}$ then one could draw a small sphere around $\mathbf{r}$ such that every value of potential on that sphere and thus the average would be smaller than $V_{max}(\mathbf{r})$

Laplace's Equation (Solutions without solving it)



First Uniqueness Theorem: The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Proof: Suppose V_1 and V_2 are two distinct solutions to Laplace's equation within volume $\mathcal{V}$ with the same value on the boundary surface $\mathcal{S}$.

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

$$V_3 \equiv V_1 - V_2 \qquad \nabla^2 V_3 = \nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

V_3 also satisfies Laplaces's equation.

What is the value of V_3 at the boundary surface $\mathcal{S}$?

0 (Because at the boundary, $V_1 = V_2$. Hence, $V_3 = V_1 - V_2 = 0$)

But Laplace's equation does not allow for any local extrema.

So, since $V_3 = 0$ at the boundary, V_3 must be 0 everywhere.

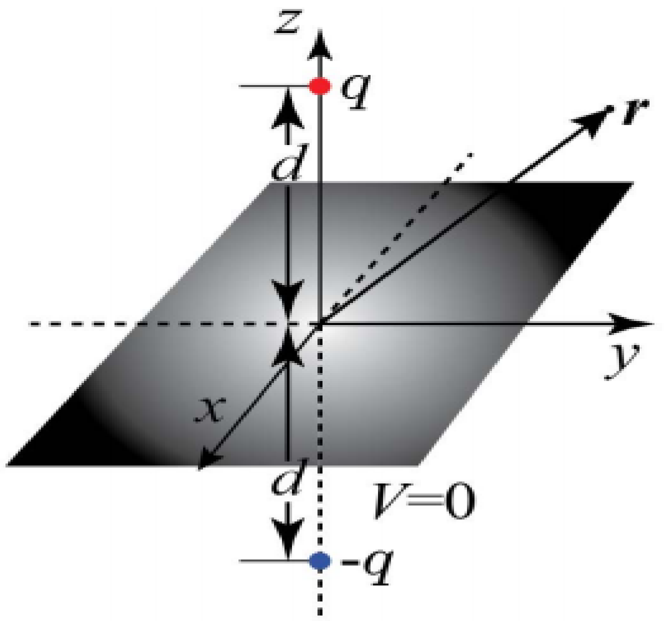
Hence $V_1 = V_2$ **QED**

Applications of Uniqueness theorem:
(The method of images)

Q: What is the potential in the region above the infinite grounded conducting plane?

Ans: $V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}$ × **No** **Why?**

Because the point charge will induce some charges in the conducting plane which will also contribute to the total potential



We have to solve Poisson's equation $\nabla^2 V = -\rho/\epsilon_0$ with the following boundary conditions:

(i) $V = 0$ when $z = 0$

(ii) $V \to 0$ when $x^2 + y^2 + z^2 \gg d^2$

Trick: Let's first solve the problem by removing the plate and adding $-q$ at $z = -d$

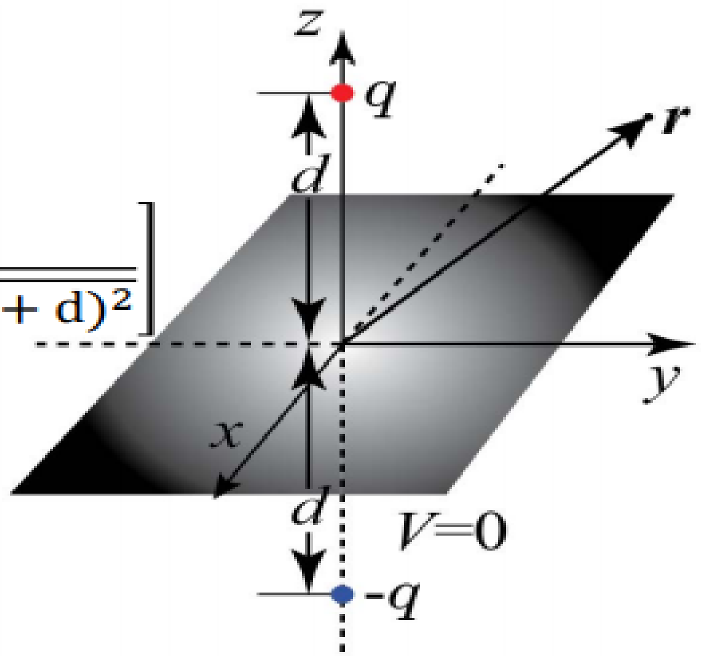
$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

Does this satisfy the boundary conditions? **(i) ✓Yes** **(ii) ✓Yes**

So, the first uniqueness theorem says that this is ***the*** solution of the problem.

Applications of Uniqueness theorem:
(The method of images)

Q: What is the induced surface charge?



$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Just near conductor, we know that

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \Rightarrow -\frac{\partial V}{\partial n}\hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \Rightarrow \sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

In the present problem, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$

And, $\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$

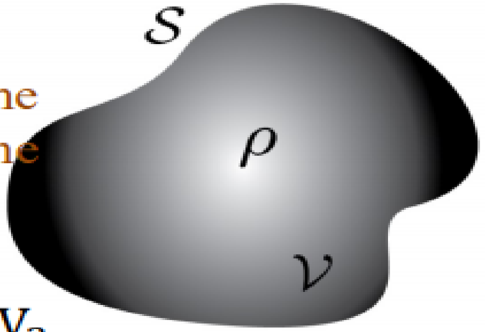
$$= -\epsilon_0 \quad \frac{q}{4\pi\epsilon_0}\left[\frac{-(z-d)}{[x^2+y^2+(z-d)^2]^{\frac{3}{2}}} + \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{\frac{3}{2}}}\right]_{z=0}$$

So, $\sigma = -\dfrac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$ (induced charge is –ve since q is +ve)

Total induced charge: $Q = \int \sigma da = \int_{s=0}^{\infty}\int_{\phi=0}^{2\pi} \dfrac{qd}{2\pi(s^2+d^2)^{3/2}} s\,ds\,d\phi \;= -q$

(correct prediction)

Laplace's Equation (Solutions without solving it)



Corollary to First Uniqueness Theorem: The potential in a volume is uniquely determined if (a) the charge desity throughout the region and (b) the value of V at all boundaries, are specified.

Proof: Suppose V_1 and V_2 are two distinct solutions to Poisson's equation in a region with volume $\mathcal{V}$ and charge density ρ. V_1 and V_2 have the same value at the boundary surface $\mathcal{S}$.

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$$

$$V_3 \equiv V_1 - V_2 \qquad \nabla^2 V_3 = \nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

V_3 satisfies Laplaces's equation.

What is the value of V_3 at the boundary surface $\mathcal{S}$?

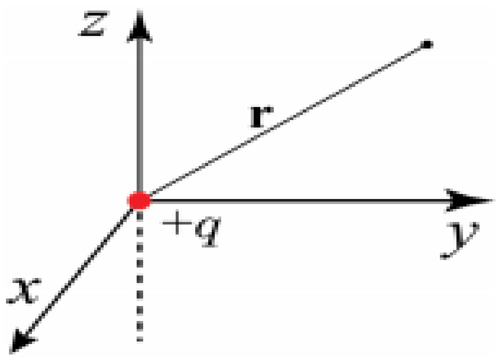
0 (Because at the boundary, $V_1 = V_2$. Hence, $V_3 = V_1 - V_2$=0)

But Laplace's equation does not allow for any local extrema.

So, if $V_3 = 0$ at the boundary, it must be 0 everywhere.

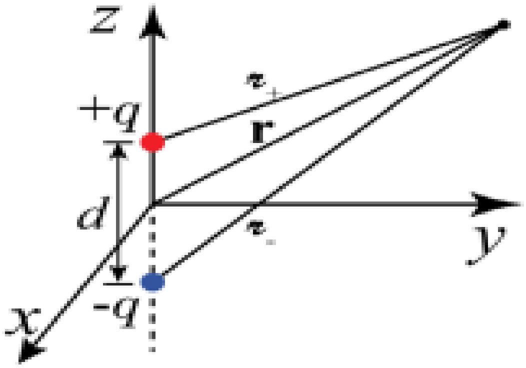
Hence $V_1 = V_2$ **QED**

Multipole Expansion (Potentials at large distances)



- What is the potential due to a point charge (monopole)?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \qquad \text{(goes like } 1/r\text{)}$$



- What is the potential due to a dipole at large distance?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

$$\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp r d\cos\theta \ = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$\approx r^2\left(1 \mp \frac{d}{r}\cos\theta\right) \quad (\text{for } r \gg d)$$

for $r \gg d$, and using binomial expansion

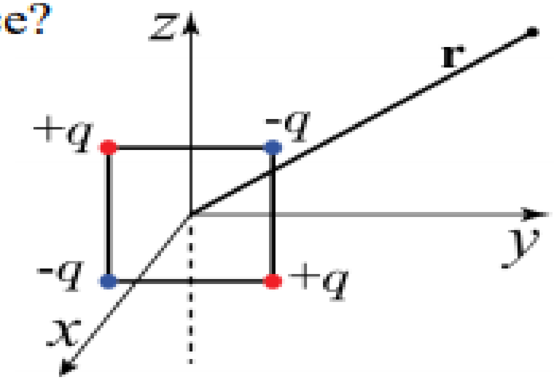
$$\frac{1}{\mathscr{r}_\pm} = \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \qquad \text{So, } \frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} = \frac{d}{r^2}\cos\theta$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q d\cos\theta}{r^2} \qquad \text{(goes like } 1/r^2 \text{ at large } r\text{)}$$

Multipole Expansion (Potentials at large distances)

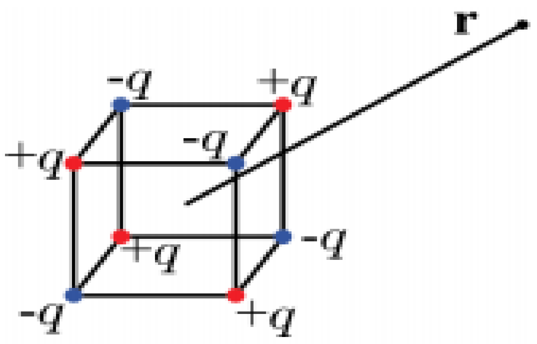
- What is the potential due to a quadrupole at large distance?

(goes like $1/r^3$ at large r)



- What is the potential due to a octopole at large distance?

(goes like $1/r^4$ at large r)



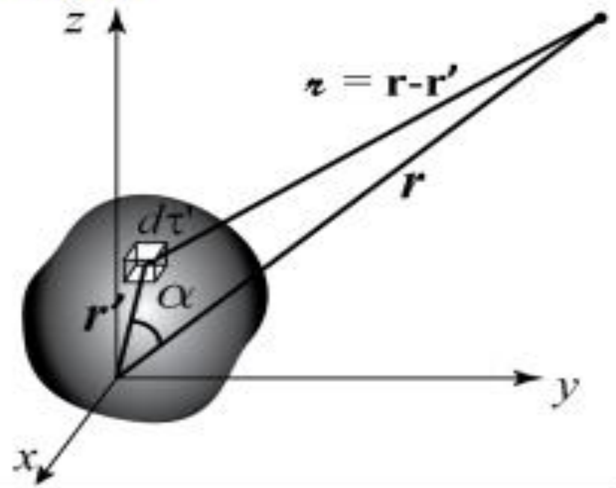
Note1: Multipole terms are defined in terms of their r dependence, not in terms of the number of charges.

Note 2: The dipole potential need not be produced by a two-charge system only . A general n-charge system can have any multipole contribution.

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$



Using the cosine rule,

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\mathscr{r} = r\sqrt{1 + \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)}$$

$$\mathscr{r} = r\sqrt{1+\epsilon}$$

Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between $\mathbf{r}$ and $\mathbf{r}'$: α

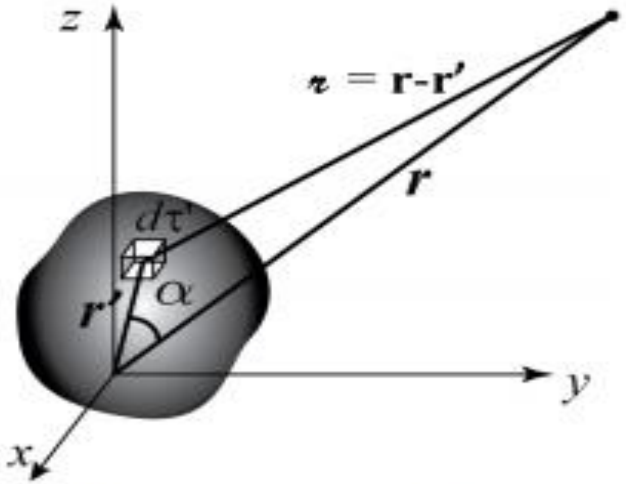
Define: $\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

So, $\dfrac{1}{\mathscr{r}} = \dfrac{1}{r}(1+\epsilon)^{-1/2}$

Or, $\dfrac{1}{\mathscr{r}} = \dfrac{1}{r}\left(1 - \dfrac{1}{2}\epsilon + \dfrac{3}{8}\epsilon^2 - \dfrac{5}{16}\epsilon^3 + \cdots\right)$ (using binomial expansion)

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2(3\cos^2\alpha - 1)/2 - \left(\frac{r'}{r}\right)^3(5\cos^3\alpha - 3\cos\alpha)/2 + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

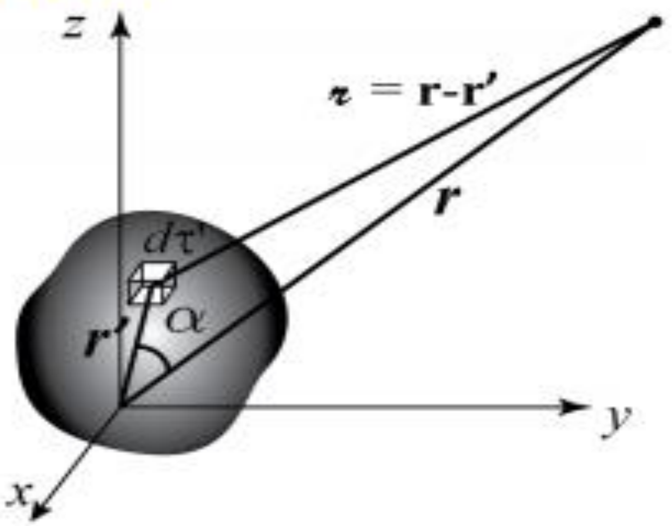
$P_n(\cos\alpha)$ are Legendre polynomials

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$$

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

$$= \underbrace{\frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau'}_{\substack{\text{Monopole potential} \\ (1/r \text{ dependence})}} + \underbrace{\frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau'}_{\substack{\text{Dipole potential} \\ (1/r^2 \text{ dependence})}} + \underbrace{\frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau'}_{\substack{\text{Quadrupole potential} \\ (1/r^3 \text{ dependence})}} + \cdots$$

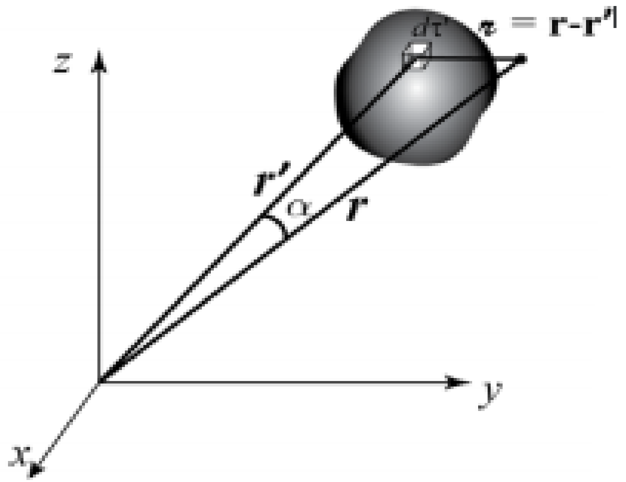
Multipole Expansion of V(r)

Multipole Expansion (Few comments)

$$
\begin{aligned}
V(\mathbf{r}) &= \underbrace{\frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau'}_{\substack{\text{Monopole potential} \\ (1/r \text{ dependence})}} + \underbrace{\frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau'}_{\substack{\text{Dipole potential} \\ (1/r^2 \text{ dependence})}} + \underbrace{\frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau'}_{\substack{\text{Quadrupole potential} \\ (1/r^3 \text{ dependence})}} \\
&\quad + \cdots
\end{aligned}
$$

- **It is an exact expression, not an approximation.**
- **A particular term in the expansion is defined by its r dependence**
- **At large r, the potential can be approximated by the first non-zero term.**
- **More terms can be added if greater accuracy is required**

Questions 1:



Q: In this following configuration, is the "large $\mathbf{r}$" limit valid, since the source dimensions are much smaller than $\mathbf{r}$?

Ans: No. The "large $\mathbf{r}$" limit essentially mean $|\mathbf{r}| \gg |\mathbf{r}'|$. In majority of the situations, the charge distribution is centered at the origin and therefore the "large $\mathbf{r}$" limit is the same as source dimension being smaller than $\mathbf{r}$.

Multipole Expansion (Monopole and Dipole terms)

Monopole term:

$$V_{mono}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')d\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

- $Q = \int \rho(\mathbf{r}')d\tau'$ is the total charge
- If $Q = 0$, monopole term is zero.
- For a collection of point charges

$$Q = \sum_{i=1}^{n} q_i$$

Dipole term:

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

α is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

So, $r'(\cos\alpha) = \hat{\mathbf{r}} \cdot \mathbf{r}'$

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

- $\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$ is called the dipole moment of a charge distribution
- If $\mathbf{p} = 0$, dipole term is zero.
- For a collection of point charges.

$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i$$

Multipole Expansion (Monopole and Dipole terms)

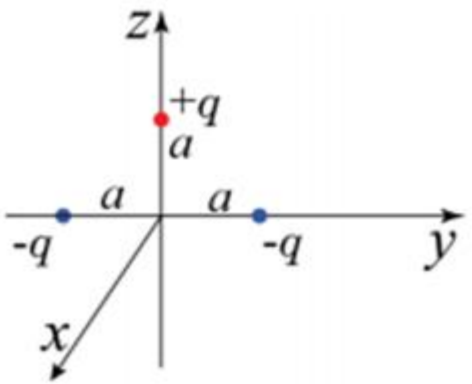
Monopole term:

$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \text{ (for point charges)} \qquad Q = \sum_{i=1}^{n} q_i$$

Dipole term:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \text{ (for point charges)} \quad \mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i$$

Example: A three-charge system



$$Q = \sum_{i=1}^{n} q_i = -q$$

$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i = qa\,\hat{\mathbf{z}} + [-qa - q(-a)]\hat{\mathbf{y}} = qa\,\hat{\mathbf{z}}$$

Therefore the system will have both monopole and dipole contributions

Multipole Expansion (Monopole and Dipole terms)

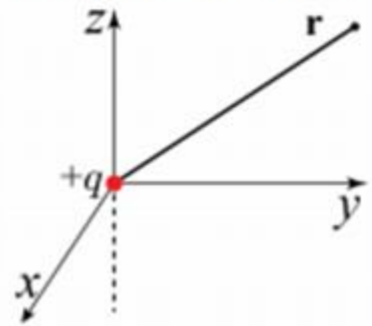
Monopole term:

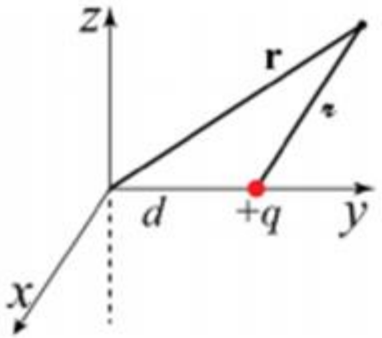
$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \text{ (for point charges)} \qquad Q = \sum_{i=1}^{n} q_i$$

Dipole term:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \text{ (for point charges)} \qquad \mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i$$

Example: Origin of Coordinates





$Q = q \qquad V_{\text{mono}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r}$

$\mathbf{p} = 0 \qquad V_{\text{dip}}(\mathbf{r}) = 0$

$Q = q \qquad V_{\text{mono}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r} \neq \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\mathscr{r}}$

$\mathbf{p} = qd\hat{\mathbf{y}} \qquad V_{\text{dip}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$

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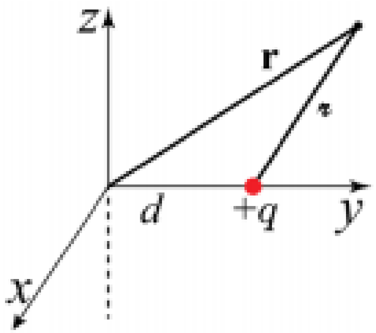
Questions 2:



$$Q = q \qquad V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \neq \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

$$\mathbf{p} = qd\hat{\mathbf{y}} \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Q: Why not calculate the potential directly ??

Ans: Yes, that is what should be done. For a point charge, we don't need a multipole expansion to find the potential. This is only for illustrating the connection.

The electric field of pure dipole ($Q = 0$)

$Q = 0$ And $\mathbf{p} \neq 0$ Assume $\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathrm{V_{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\mathrm{V}$$

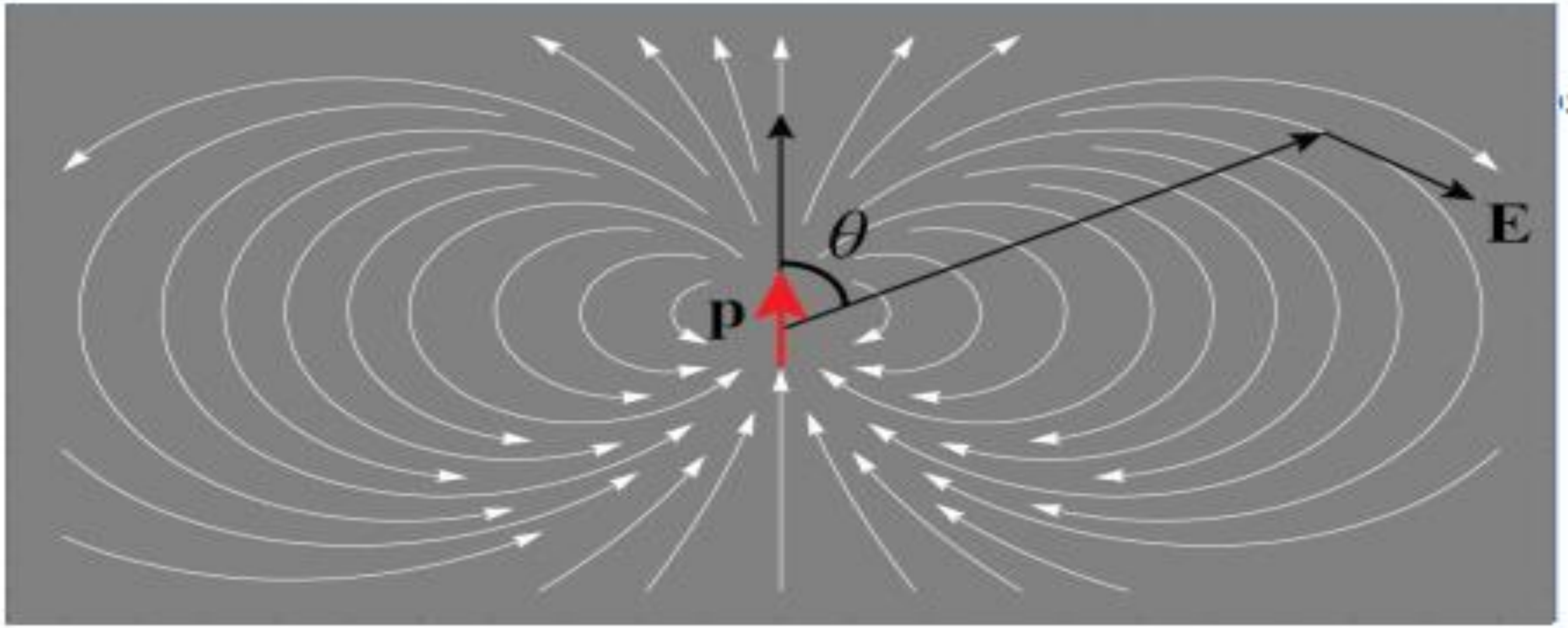
$$\left.\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\ E_\phi &= -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0 \end{aligned}\right\} \quad \mathbf{E_{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

The electric field of pure dipole ($Q = 0$)

$Q = 0$ And $\mathbf{p} \neq 0$ Assume $\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathrm{V}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\mathrm{V}$$



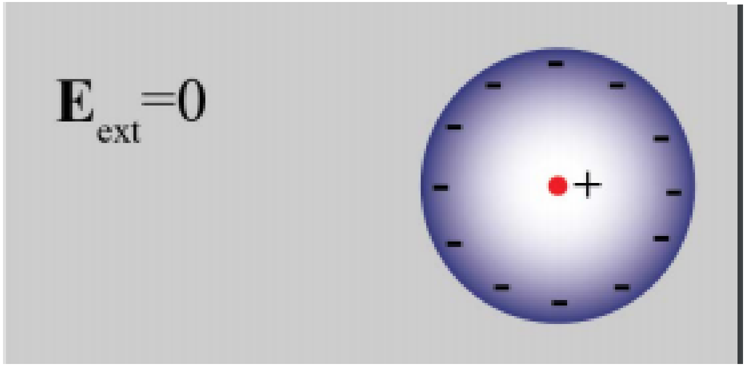
$s\theta\ \hat{\mathbf{r}} + \sin\theta\ \hat{\theta})$

Electrostatics in Matter (Electric Fields in Matter)

Polarization (Induced Dipoles)

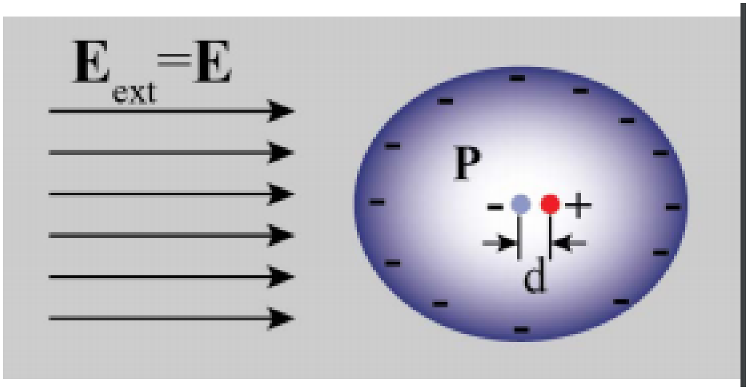
A neutral atom

Total charge $Q = 0$; Dipole moment $\mathbf{p} = 0$;



Atom in an electric field

Total charge $Q = 0$; Dipole moment $\mathbf{p} \neq 0$;



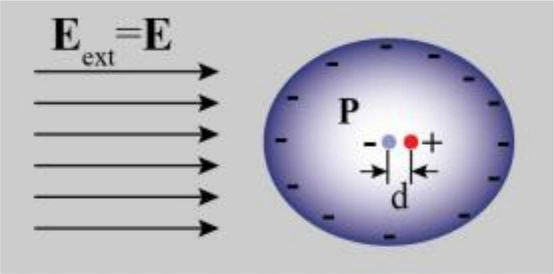
Atom has become polarized

$$\boxed{\mathbf{p} = \alpha\,\mathbf{E}}$$

- α is called the atomic polarizability
- α depends on the detailed structure of the atom
- α is determined experimentally

Polarization (Induced Dipoles)

Ex. 4.1 (Griffiths, 3rd Ed.): An atom can be considered as a point nucleus of charge $+q$ surrounded by a uniformly charged spherical electron cloud of charge $-q$ and radius a ? Find the atomic polarizability.



Here, we are assuming that the electron cloud remains spherical in shape even in the presence of the external electric field.

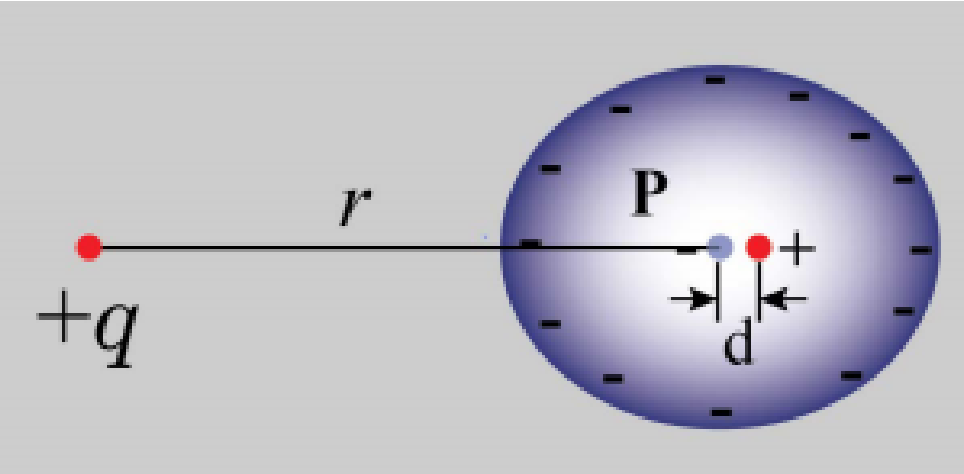
In equilibrium, the field at the nucleus due to the electron cloud is

$$E_{\text{elec}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} = E \quad \Rightarrow \quad d = \frac{4\pi\epsilon_0\, a^3 E}{q}$$

So the atomic polarizability $\alpha = \dfrac{p}{E} = \dfrac{qd}{E} = \dfrac{4\pi\epsilon_0\, a^3 E}{E} = 4\pi\epsilon_0\, a^3$

Polarization (Induced Dipoles)

Ex. 4.4 (Griffiths, 3rd Ed.): A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.



The field due to charge q is

$$\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The induced dipole moment is

$$\mathbf{p} = \alpha\,\mathbf{E} = \frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The field due to this dipole at the location ($\theta = \pi$) of the charge is

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi\epsilon_0 r^3}\frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}(-2\,\hat{\mathbf{r}})$$

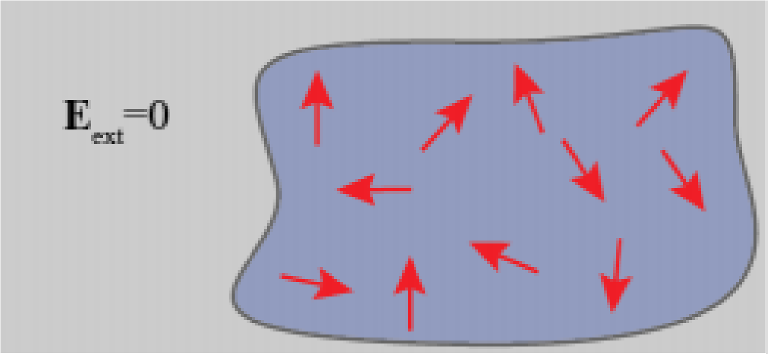
Therefore, the force is $\mathbf{F} = q\mathbf{E}_{\text{dip}}(\mathbf{r}) = -2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{r^5}\,\hat{\mathbf{r}}$

Polarization (Permanent Dipoles)

A neutral atom has no dipole moment to begin with but some molecules (polar molecules) have permanent dipole moment, even without the external electric field.

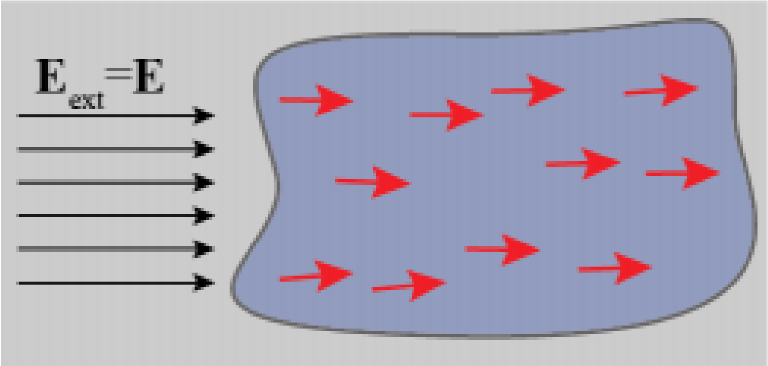
<u>Polar molecules</u>

Total charge $Q = 0$;
Net dipole moment $\mathbf{p} = 0$;



<u>Polar molecules in an electric field</u>

Total charge $Q = 0$;
Net dipole moment $\mathbf{p} \neq 0$;



Molecules already had dipole moments
The dipole moments have now become aligned

Quick Summary:

- We studied electrostatics in vacuum as well as in conductors
- Calculating Electric field **E** given a charge ρ is one of main aims of electrostatics
- Electric field can be calculated using Coulomb's law but usually it is very difficult
- The easier way is to first calculate the electric potential and then $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}V$
- Electric potential can be calculated in two different ways
  - ❑ Using the differential (Poisson's) equation $\boldsymbol{\nabla}^2 V = -\dfrac{\rho}{\epsilon_0}$
    - ✓ Uniqueness theorems guarantee that a solution is unique if it is found
    - ✓ We studied method of images to be able to guess a solution in some cases
    - ✓ Poisson's equation can sometimes be analytically solved
  - ❑ Using the integral equation $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{dq}{\mathscr{r}}$
    - ✓ Multipole expansion method is based on using this form.
    - ✓ Using multipole expansion, approximate potential at large **r** are calculated

Thank You