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$A(cu) = c.Au$ for any real number c.
This motivates to define

Definition (Linear Transformation)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Linear Transformation if
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$$T(u) = Au \qquad u \to Au$$

$$\{T : \mathbb{R}^n \to \mathbb{R}^m,\ T \text{ lin}\} \longleftrightarrow \{A_{m \times n}\}$$

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$T(cu) = c.T(u)$ for any real number c.

Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

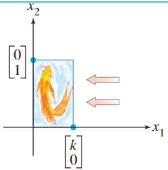
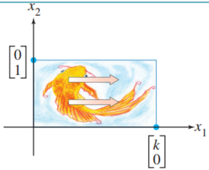
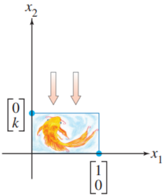
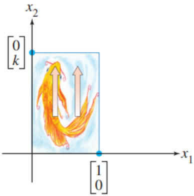
Projection onto X axis.

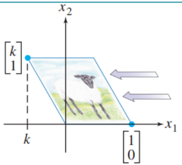
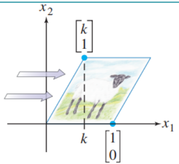
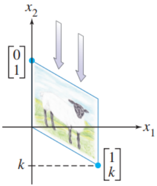
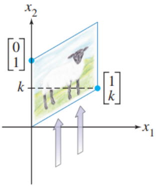
Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ scaling by 2.

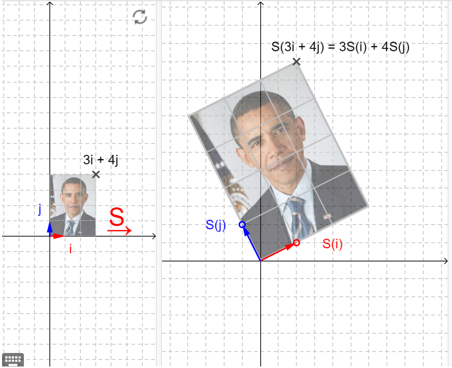
Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

rotation by 45 degree.

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k<0$  $k>0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k<0$  $k>0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$



$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

$$\begin{aligned} S\begin{pmatrix}3\\4\end{pmatrix} &= 3\,S\begin{pmatrix}1\\0\end{pmatrix} + 4\,S\begin{pmatrix}0\\1\end{pmatrix} \\ &= 3\begin{pmatrix}2\\1\end{pmatrix} + 4\begin{pmatrix}-1\\2\end{pmatrix} \\ &= \begin{pmatrix}2\\11\end{pmatrix} \end{aligned}$$

Represent S by $\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}$

$$\begin{aligned} S\begin{pmatrix}3\\4\end{pmatrix} &= \begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} \\ &:= 3\begin{pmatrix}2\\1\end{pmatrix} + 4\begin{pmatrix}-1\\2\end{pmatrix} \end{aligned}$$

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How does T look like?

$1, A$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

for some real numbers $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$.

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This matrix is called standard matrix representation of T.

Question: Given T How to find its matrix representation-$[T]$?

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Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

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Second column of $[T]$ is $T(e_2)$. where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Question: Given T How to find its matrix representation-$[T]$?

Answer: First column of $[T]$ is $T(e_1)$. where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Second column of $[T]$ is $T(e_2)$. where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

So on and last column of $[T]$ is $T(e_n)$. where $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if for each $b \in \mathbb{R}^m$ there exists $u \in \mathbb{R}^n$ such that $T(u) = b$

Properties of L. T.

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A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one to one if $T(u) = 0$ then u must be 0.

$T(u - u_2) = 0$

$T(u_1) = T(u_2)$

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$T \longrightarrow A \longrightarrow Ax = 0$

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$T \longrightarrow A$

$Col(A) = \mathbb{R}^m$

$\Updownarrow$

No. of pivot entries in $REF(A) = m$

$u \longrightarrow v$

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$Im(T) = \mathbb{R}^n$

Properties of Linear Transformation

$$\underbrace{\begin{bmatrix} 1 & -4 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}$$

$$X = B^{-1} b$$

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be bijective/invertible if T is one to one and onto.

Isomerphism.

Col(A):

Let T be the linear transformation whose standard matrix is $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}_{3\times 4}$ Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

$$\left[\begin{array}{cccc|c} 1 & -4 & 8 & 1 & b_1 \\ 0 & 2 & -1 & 3 & b_2 \\ 0 & 0 & 0 & 5 & b_3 \end{array}\right]$$

no. of pivot entries = 3
$\Rightarrow$ T is onto.

$x_4 = b_3/5$

$x_2 = \frac{1}{2}(b_2 - \frac{3}{5}b_3)$

$x_3 = 0$

$[A \mid 0]$

$$\left[\begin{array}{cccc|c} 1 & -4 & 8 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 \end{array}\right]$$

$AX = 0$

existence of free variable

gives $0 \neq x$ s.t. $Ax = 0$

$\Rightarrow T$ is not 1-1

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

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T is one-to-one if and only if the columns of A are linearly independent. if $Ax = 0$ has only trivial sol^n

does not have non-trivial sol^n

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$A: \begin{bmatrix} \quad \end{bmatrix}$ $m \times n$

onto $\Rightarrow$ no. of pivot entries $= n$

1-1 $\Rightarrow$ no. free variable

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T is one-to-one if and only if the columns of A are linearly independent.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. T is bijective iff $n = m$ and A is an invertible matrix.

Composite Transformation

$$\begin{array}{cc} T_1 & T_2 \\ \downarrow & \downarrow \\ A & B \end{array}$$

$$T_1 \circ T_2 (v) = T_1(T_2(v)) = (AB)\, v$$

If T_1, T_2 are two linear transformations then so is composite of T_1, T_2.

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