TUTORIAL_10

Problem 7.12 A long solenoid, of radius a, is driven by an alternating current, so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0 \cos(\omega t)\,\hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

Problem 7.12

$$\Phi = \pi\left(\frac{a}{2}\right)^2 B = \frac{\pi a^2}{4}B_0\cos(\omega t);\ \mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi a^2}{4}B_0\omega\sin(\omega t).\quad I(t) = \frac{\mathcal{E}}{R} = \boxed{\frac{\pi a^2\omega}{4R}B_0\sin(\omega t).}$$

Problem 7.22 A small loop of wire (radius a) is held a distance z above the center of a large loop (radius b), as shown in Fig. 7.37. The planes of the two loops are parallel, and perpendicular to the common axis.

(a) Suppose current I flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant.)

(b) Suppose current I flows in the little loop. Find the flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)

(c) Find the mutual inductances, and confirm that $M_{12} = M_{21}$.

Problem 7.22

(a) From Eq. 5.41, the field (on the axis) is $\mathbf{B} = \frac{\mu_0 I}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\hat{\mathbf{z}}$, so the flux through the little loop (area πa^2) is $\boxed{\Phi = \dfrac{\mu_0\pi I a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

(b) The field (Eq. 5.88) is $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$, where $m = I\pi a^2$. Integrating over the spherical "cap" (bounded by the big loop and centered at the little loop):

$$\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\int(2\cos\theta)(r^2\sin\theta\,d\theta\,d\phi) = \frac{\mu_0 I a^2}{2r}2\pi\int_0^{\bar{\theta}}\cos\theta\sin\theta\,d\theta$$

where $r = \sqrt{b^2+z^2}$ and $\sin\bar{\theta} = b/r$. Evidently $\Phi = \frac{\mu_0 I\pi a^2}{r}\left.\frac{\sin^2\theta}{2}\right|_0^{\bar{\theta}} = \boxed{\dfrac{\mu_0\pi I a^2 b^2}{2(b^2+z^2)^{3/2}},}$ the same as in (a)!!

(c) Dividing off I ($\Phi_1 = M_{12}I_2$, $\Phi_2 = M_{21}I_1$): $\boxed{M_{12} = M_{21} = \dfrac{\mu_0\pi a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

Problem 7.35 The preceding problem was an artificial model for the charging capacitor, designed to avoid complications associated with the current spreading out over the surface of the plates. For a more realistic model, imagine *thin* wires that connect to the centers of the plates (Fig. 7.46a). Again, the current I is constant, the radius of the capacitor is a, and the separation of the plates is $w \ll a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t = 0$.

(a) Find the electric field between the plates, as a function of t.

(b) Find the displacement current through a circle of radius s in the plane midway between the plates. Using this circle as your "Amperian loop," and the flat surface that spans it, find the magnetic field at a distance s from the axis.

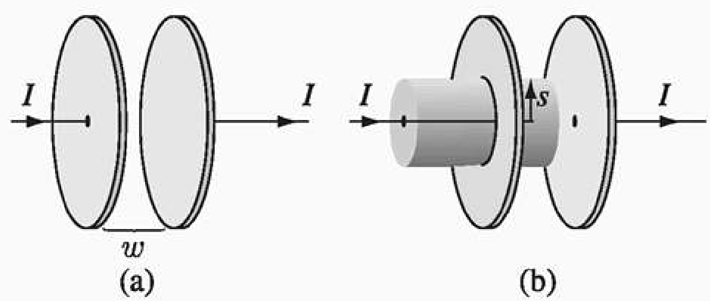

FIGURE 7.46

(c) Repeat part (b), but this time use the cylindrical surface in Fig. 7.46(b), which is open at the right end and extends to the left through the plate and terminates outside the capacitor. Notice that the displacement current through this surface is zero, and there are two contributions to I_{enc}.[22]

Problem 7.35

(a) $\mathbf{E} = \dfrac{\sigma(t)}{\epsilon_0}\,\hat{\mathbf{z}}; \quad \sigma(t) = \dfrac{Q(t)}{\pi a^2} = \dfrac{It}{\pi a^2}; \quad \boxed{\dfrac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$

(b) $I_{d_{\text{enc}}} = J_d \pi s^2 = \epsilon_0 \dfrac{dE}{dt}\pi s^2 = \boxed{I\dfrac{s^2}{a^2}.} \quad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{d_{\text{enc}}} \Rightarrow B\,2\pi s = \mu_0 I \dfrac{s^2}{a^2} \Rightarrow \mathbf{B} = \boxed{\dfrac{\mu_0 I}{2\pi a^2}\,s\,\hat{\boldsymbol{\phi}}.}$

(c) A surface current flows radially outward over the left plate; let $I(s)$ be the total current crossing a circle of radius s. The charge density (at time t) is

$$\sigma(t) = \frac{[I - I(s)]t}{\pi s^2}.$$

Since we are told this is independent of s, it must be that $I - I(s) = \beta s^2$, for some constant β. But $I(a) = 0$, so $\beta a^2 = I$, or $\beta = I/a^2$. Therefore $I(s) = I(1 - s^2/a^2)$.

$$B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0[I - I(s)] = \mu_0 I \frac{s^2}{a^2} \Rightarrow \boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi a^2}\,s\,\hat{\boldsymbol{\phi}}.} \checkmark$$

Problem 7.37 Suppose

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(r - vt)\hat{\mathbf{r}}; \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{0}$$

(The theta function is defined in Prob. 1.46b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and $\mathbf{J}$. Describe the physical situation that gives rise to these fields.

Problem 7.37

Physically, this is the field of a point charge q at the origin, out to an expanding spherical shell of radius vt; outside this shell the field is zero. Evidently the shell carries the opposite charge, $-q$. *Mathematically*, using product rule #5 and Eq. 1.99:

$$\nabla \cdot \mathbf{E} = \theta(vt - r)\nabla \cdot \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\right) + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \cdot \nabla[\theta(vt - r)] = \frac{q}{\epsilon_0}\delta^3(\mathbf{r})\theta(vt - r) + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\frac{\partial}{\partial r}\theta(vt - r).$$

But $\delta^3(\mathbf{r})\theta(vt - r) = \delta^3(\mathbf{r})\theta(t)$, and $\frac{\partial}{\partial r}\theta(vt - r) = -\delta(vt - r)$ (Prob. 1.46), so

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \boxed{q\delta^3(\mathbf{r})\theta(t) - \frac{q}{4\pi r^2}\delta(vt - r).}$$

(For $t < 0$ the field and the charge density are zero everywhere.)

Clearly $\nabla \cdot \mathbf{B} = 0$, and $\nabla \times \mathbf{E} = \mathbf{0}$ (since $\mathbf{E}$ has only an r component, and it is independent of θ and ϕ). There remains only the Ampére/Maxwell law, $\nabla \times \mathbf{B} = \mathbf{0} = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial\mathbf{E}/\partial t$. Evidently

$$\mathbf{J} = -\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\epsilon_0\left\{\frac{q}{4\pi\epsilon_0 r^2}\frac{\partial}{\partial t}[\theta(vt - r)]\right\}\hat{\mathbf{r}} = \boxed{-\frac{q}{4\pi r^2}v\delta(vt - r)\,\hat{\mathbf{r}}.}$$

(The stationary charge at the origin does not contribute to $\mathbf{J}$, of course; for the expanding shell we have $\mathbf{J} = \rho\mathbf{v}$, as expected—Eq. 5.26.)

Problem 7.40 Sea water at frequency $\nu = 4 \times 10^8$ Hz has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega \cdot$ m. What is the ratio of conduction current to displacement current? [*Hint:* Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi\nu t)$.]

Problem 7.40

$$E = \frac{V}{d} \Rightarrow J_c = \sigma E = \frac{1}{\rho}E = \frac{V}{\rho d}.\ J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = \epsilon\frac{\partial}{\partial t}\left[\frac{V_0\cos(2\pi\nu t)}{d}\right] = \frac{\epsilon V_0}{d}[-2\pi\nu\sin(2\pi\nu t)].$$

The ratio of the amplitudes is therefore:

$$\frac{J_c}{J_d} = \frac{V_0}{\rho d}\frac{d}{2\pi\nu\epsilon V_0} = \frac{1}{2\pi\nu\epsilon\rho} = \left[2\pi(4 \times 10^8)(81)(8.85 \times 10^{-12})(0.23)\right]^{-1} = \boxed{2.41.}$$

Problem 8.2 Consider the charging capacitor in Prob. 7.34.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that Eq. 8.12 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

Problem 8.2

(a) $\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}};\ \sigma = \frac{Q}{\pi a^2};\ Q(t) = It \ \Rightarrow\ \mathbf{E}(t) = \boxed{\frac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$

$$B\,2\pi s = \mu_0\epsilon_0\frac{\partial E}{\partial t}\pi s^2 = \mu_0\epsilon_0\frac{I\pi s^2}{\pi\epsilon_0 a^2} \ \Rightarrow\ \mathbf{B}(s,t) = \boxed{\frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

(b) $u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left[\epsilon_0\left(\frac{It}{\pi\epsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \boxed{\frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].}$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = \boxed{-\frac{I^2 t}{2\pi^2\epsilon_0 a^4}\,s\,\hat{\mathbf{s}}.}$$

$$\frac{\partial u_{em}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4}2c^2 t = \frac{I^2 t}{\pi^2\epsilon_0 a^4};\quad -\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\boldsymbol{\nabla}\cdot(s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2\epsilon_0 a^4} = \frac{\partial u_{em}}{\partial t}.\ \checkmark$$

(c) $U_{em} = \int u_{em} w 2\pi s\,ds = 2\pi w\frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b\left[(ct)^2 + (s/2)^2\right]s\,ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]\Big|_0^b$

$= \boxed{\frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right].}$ Over a surface at radius b: $P_{in} = -\int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\left[b\,\hat{\mathbf{s}}\cdot(2\pi b w\,\hat{\mathbf{s}})\right] = \boxed{\frac{I^2 w t b^2}{\pi\epsilon_0 a^4}.}$

$$\frac{dU_{em}}{dt} = \frac{\mu_0 w I^2 b^2}{2\pi a^4}2c^2 t = \frac{I^2 w t b^2}{\pi\epsilon_0 a^4} = P_{in}.\ \checkmark\ \text{(Set } b = a \text{ for total.)}$$