# Linear System

## Definition

*A linear System of m equations in n variables-* $X_1, X_2, \ldots, X_n$ *is*

$$\begin{gathered} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2 \\ . \\ . \\ . \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \end{gathered}$$

*where* $a_{ij}, b_j \in \mathbb{R}$

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*where* $a_{ij}, b_j \in \mathbb{R}$

Let's look at linear system of 2 equations in 2 variables:

Solve the system: $(1)\quad x + 2y = 3, \qquad (2)\quad 3x + y = 4.$

**Elimination of variables:**

Eliminate $x$ by $(2) - 3 \times (1)$ to get $y = 1$.

**Cramer's Rule (determinant):** $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{4-9}{1-6} = 1$

In either case, back substitution gives $x = 1$

We could also solve for $x$ first and use back substitution for $y$.

**Comparison:** For a large system, say $100$ equations in $100$ variables, elimination method is preferred, since computing the determinants of a $101$ matrices of size $100 \times 100$ is time-consuming.

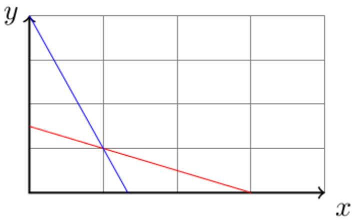
# Geometry of Linear Equations

$$\boxed{x+2y=3}$$

and

$$\boxed{3x+y=4}$$

represent lines in $\mathbb{R}^2$ passing through $\boxed{(0,3/2) \quad \text{and} \quad (3,0)}$ and through $\boxed{(0,4) \quad \text{and} \quad (4/3,0)}$ respectively.



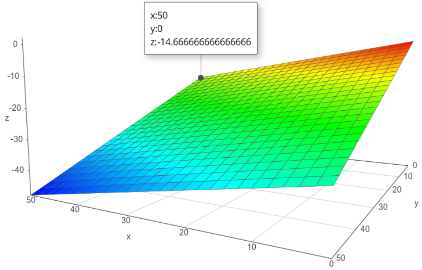

The intersection of the two lines is the unique point $(1,1)$. Hence $x=1$ and $y=1$ is the solution of above system of linear equations.

# 3 Equations in 3 Variables

A linear equation in 3 variables represents a plane in a 3-dimensional space $\mathbb{R}^3$.

$$x + 2y + 3z = 6$$

passes through $(0, 0, 2)$, $(0, 3, 0)$, $(6, 0, 0)$.




$x + 2y + 3z = 12$ passes through $(0, 0, 4)$, $(0, 6, 0)$, $(12, 0, 0)$. which is parallel to above plane.

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- If the line $L$ does intersects with the plane $P_3$ , then the linear system has unique solution.
- If the line $L$ does not intersects with the plane $P_3$ , then the linear system has no solution.
- If the line $L$ is contained in the plane $P_3$, then the system has infinitely many solutions.
  In this case, every point of $L$ is a solution.

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Question: Can we do the same when number of variables are $> 3$?

# Gaussian Elimination: Unique solution

**Example:** $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

**Algorithm:** Eliminate $x$ from last $2$ equations by $(2) - 2(1)$, and $(3) + (1)$ to get the *equivalent system:*

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 3z = 14$$

The first *pivot* is $2$, second pivot is $-8$. Eliminate $y$ from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad z = 2$$

Solve this triangular system by *back substitution*, we get

$$z = 2, \quad y = 1, \quad x = 1$$

**Observe:** This is the only possible solution!

# Gaussian Elimination: No solution

**Example:** $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 9.$

$\boxed{\text{Step 1}}$ Eliminate $x$ (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 14$

$\boxed{\text{Step 2}}$: Eliminate $y$ (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 2.$

The last equation shows that there is no solution,
i.e., the system is *inconsistent*.

**Geometric reasoning:** In Step 1, notice we get two distinct parallel planes $8y + 2z = 12$ and $8y + 2z = 14$.
They have no point in common.

**Note:** The planes in the original system were not parallel, but in an equivalent system, we get two distinct parallel planes!

# Gaussian Elimination: Infinitely solution

**Example:** $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 7.$

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$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 12$

$\boxed{\text{Step 2}}$: Eliminate $y$ (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 0.$

There are only two equations. For every value of $z$, values for $x$ and $y$ are obtained by back-substitution, e.g, $(1, 1, 2)$ or $\left(\frac{7}{4}, \frac{3}{2}, 0\right)$. Hence the system has infinitely many solutions.

**Geometric reasoning:** In Step 1, notice we get two parallel planes $-8y - 2z = 12$ and $8y + 2z = 12$.
They give the same plane. Hence we are looking at the intersection of the two planes, $2x + y + z = 5$ and $8x + 2z = 12$, which is a line.

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1. no solution; (inconsistent system)
2. exactly one solution; (consistent system)
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Solve

# Gaussian Elimination: Matrix form AX=b

**Example:** $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

Note that the last column is the RHS column vector $b$.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right) \rightarrow$$

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The last matrix corresponds to
$z = 2,$
$-8y - 2z = -12,$
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# Gaussian Algorithm

## For Row Echelon Form

1. Find the leftmost nonzero column.
2. Select a nonzero entry in that column (pivot). Bring it on top by interchanging rows
3. Use Elementary row operations to make all entries below that nonzero entry 0.
4. Ignore that row and columns before (including of leading entry) above found column. Apply steps 1-3 for remaining submatrix. Go on till all rows are exhausted.

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   For Reduced Row Echelon Form

5. Make all leading entries 1 by elementary row operation ($R_i \to cR_i$)
6. Make all entries in a column above leading 1 zero by elementary row operation ($R_i \to R_i - cR_j$)

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$$

Reduced Row Echelon Form (RREF) of A is

$$A' = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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## Theorem

*1. A linear system is inconsistent if and only if REF or RREF of its augmented matrix has a row of the form* $0\ 0\ \cdots\ 0\ b$*, for some nonzero* $b$*.*
*2. A linear system is has unique solution if and only if no. of nonzero rows and no. of columns of REF or RREF of its augmented matrix are same.*

Triangular factorization: $A = LU$ Given a square matrix $A$, we can find $L$ a lower Triangular matrix with 1s on the diagonal and $U$ upper Triangular matrix such that $A = LU$.

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

# Calculation of $A^{-1}$: Gauss-Jordan Method

Let $AA^{-1} = I$
If $X_1$ is a first column of $A^{-1}$ then $AX_1 = e_1$, where
$e_1 = [1\ 0\ \cdots\ 0]^T$.
We can use Gaussian Elimination to find $X_1$−first column of $A^{-1}$.

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We can use Gaussian Elimination to find $X_1-$first column of $A^{-1}$.
Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$.
Consider

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

Perform elementary row operations on $[A|I]$ to convert $A$ into $I$.
Then $I$ will change to $A^{-1}$.

## Vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ is a vector in } \mathbb{R}^2.$$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ is a vector in } \mathbb{R}^3.$$

Vector addition and scalar multiplication are done like matrices.

Alpropvectors.PNG

# Linear Combinations

Consider a linear system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

It can be written as

$$x_1 \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

LHS of $=$ is called linear combination of

$$u = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} \& \ w = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}$$

Above equation is called vector equation.

# Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1C_1 + x_2C_2 + \cdots + x_nC_n = b$ where $C_i$ is $i^{th}$ column of $A$.

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**Theorem**

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Hence we need to study linear combination of columns of $A$.
Column space of A= set of all linear combinations of columns of $A$.

$$Col(A) = \{x_1C_1 + x_2C_2 + \cdots + x_nC_n | x_i \in \mathbb{R} \text{ for all } i\}$$

$\therefore\ AX = b$ has a solution iff $b \in Col(A)$.

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.

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Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{Span } \{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$$

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Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$ Let $u \in \mathbb{R}^n$. Then
$span\{u\} = \{x.u | x \in \mathbb{R}\}$.
Geometrically it is a line passing through origin and point $u$.
Let $u, v \in \mathbb{R}^2$. Then $span\{u, v\} = \{x.u + y.v | x, y \in \mathbb{R}\}$.
Geometrically, what will it be?

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Geometrically, what will it be?
For a matrix $A$,
$Col(A) := span\{C_1, C_2, \ldots, C_n\} \subseteq \mathbb{R}^m$

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

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### Theorem

*Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent*

1. *For each $b \in \mathbb{R}^m$, the equation $AX = b$ has a solution.*
2. *Each $b \in \mathbb{R}^m$, is a linear combination of the columns of $A$.*
3. *$Col(A) = \mathbb{R}^m$.*
4. *$A$ has a pivot position in every row.*

# Homogeneous Linear System

The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the REF(A) has at least one free variable.

Suppose augmented form $[A|0]$ is

7_homo.PNG

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7_homo.PNG

7_ex3.PNG

# Non homogeneous linear system

**Theorem**

*Suppose the equation $Ax = b$ is consistent for some given $b$, and let $p$ be a solution (i.e., $Ap = b$). Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where $v_h$ is any solution of the homogeneous equation $Ax = 0$.*

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7_para.PNG

Write down the solutions of $AX = 0$ in parametric form where $A =$

7_para1.PNG

# Chemical Equations

Consider a chemical reaction of propane gas with oxygen to form carbon dioxide and water.
$x_1 C_3H_8 + x_2 O_2 = x_3 CO_2 + x_4 H_2O$,
where $x_i$ are the no. of molecules

# Network flow