# Vector Space

$$\mathbb{R}^n$$

$$\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_i \in \mathbb{R} \right\}$$

$$(V, +, \cdot)$$

$$M_2(\mathbb{R}), \quad M_n(\mathbb{R}), \quad M_{m,n}(\mathbb{R})$$

$$\mathbb{R}_4[X] = \{ P(X) = a_0 + \dots + a_4 X^4 \mid a_i \in \mathbb{R} \}$$

$$\mathbb{R}[X] = \{ a_0 + \dots + a_n X^n \mid a_i \in \mathbb{R}, n \geq 0 \}$$

# Basis of a vector space

$AX = b$

$\Rightarrow X = A^{-1} b$

std basis

$\mathbb{R}^3 \to B = \{e_1, e_2, e_3\}$

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

Take $A_{3\times 3}$ which is invertible.

Columns of $A = \{v_1, v_2, v_3\}$ will form a basis of $\mathbb{R}^3$.

A set $B$ is a basis for a vector space $V$ if $B$ is lin. indep. & $\text{Span}(B) = V$

$\mathbb{R}_3[X] = \{a_0 + a_1 X + a_2 X^2 + a_3 X^3 \mid a_i \in \mathbb{R}\}$

$B = \{1, X, X^2, X^3\}$

$a_0 \cdot 1 + a_1 \cdot X + a_2 X^2 + a_3 \cdot X^3 = 0$

$X = 0 \Rightarrow a_0 = 0 \Rightarrow a_1 X + a_2 X^2 + a_3 X^3 = 0$

$\underset{\neq 0}{X}\underline{(a_1 + a_2 X + a_3 X^2)} = 0 \Rightarrow a_1 = 0$ after $X = 0$

Is there any other basis of $\mathbb{R}_3[x]$

$1, x, x^2, x^3$

Take $\underline{A_{4\times4}}$ inv. matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+x \\ 1+x+x^2 \\ 1+x+x^2+x^3 \end{bmatrix}$$

<u>Check</u>: $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is again a basis of $\mathbb{R}_3[x]$.

$$\begin{aligned} B_1 &\to A_1 \\ B_2 &\to A_1 + A_2 \\ B_3 &\to A_1 + A_2 + A_3 \\ B_4 &\to A_1 + A_2 + A_3 + A_4 \end{aligned}$$

$$\| $$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

$$B_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big/ a, b, c, d \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \underset{A_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{A_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{A_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{A_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \text{Span}\{A_1, \ldots A_4\} = M_2(\mathbb{R})$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 = 0$$

$$\Rightarrow x_i = 0 \;\forall i \Rightarrow \text{L.I. of } \{A_i\}$$

$$S_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \Big/ a, b, d \in \mathbb{R} \right\}$$

$1 + x + \frac{x^2}{4} - - - - -$

power series.

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\sum_{i=0}^{\infty} a_i x^i = 0$ ✗

$a_{i_1} x^{i_1} + a_{i_2} x^{i_2} + \cdots + a_{i_n} x^{i_n}$

$e^x = 1 + x + \frac{x^2}{2} + \cdots$

$$\mathbb{R}[x] \longrightarrow \text{Basis} = \{1, x, x^2, x^3, x^4, \ldots\}$$

is an infinite dimensional vector space.

$A_{m\times n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T : V \longrightarrow W$ is a linear transformation

if 1) $T(v_1+v_2) = T(v_1)+T(v_2) \quad \forall\, v_1, v_2 \in V$

2) $T(c \cdot v_1) = c \cdot T(v_1) \quad \forall\, c \in \mathbb{R},\ v_1 \in V$

Suspace of $\mathbb{R}^n$

Subspace of $\mathbb{R}^m$

$\{x \mid Ax = 0\}$

$\shortparallel$

$\mathrm{Null}(A_{m\times n}) \longrightarrow$ kernel of $T = \{ v \in V \mid T(v) = 0 \} \leq$ subspace of $V$

$\mathrm{Col}(A)$

$\shortparallel$

$\{Ax \mid x \in \mathbb{R}^n\} \longrightarrow$ Image of $T$ / Range of $T = \underbrace{\{ w \in W \mid w = T(v) \text{ for some } v \in V \}}_{\text{Subspace of } W.}$

$\frac{4}{2} = \frac{\cancel{2} \cdot 2}{\cancel{2}}$

If $T : V \longrightarrow W$ is a L.T.

then $V / \ker(T) \cong$ Range of $T$ ~~~~ Y axis

Example.

$$\boxed{V = \mathbb{R}^2}$$

$\Longleftarrow$ $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$.

X axis

$\ker(T) = \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

X

$V / \ker(T) = \left\{ d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| \, d \in \mathbb{R} \right\}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ — not L.I.

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{y}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{y}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓

$\begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cancel{0\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix}} + (x-y)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

L.I

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\ 5\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}2\\ 5\end{bmatrix} = \begin{bmatrix}-3\\ 5\end{bmatrix}$$

$\mathbb{R}^2$

Std basis = $\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\end{bmatrix}$

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}2\\ 5\end{bmatrix} \qquad \begin{bmatrix}2\\ 5\end{bmatrix} = 2e_1 + 5e_2 \longrightarrow \begin{bmatrix}2\\ 5\end{bmatrix}$$

$$\begin{bmatrix}2\\ 5\end{bmatrix} = \alpha_1\begin{bmatrix}1\\ 0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\ 1\end{bmatrix} \Rightarrow \alpha_2 = 5, \ \alpha_1 = -3$$

$B' = \left\{\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}\right\}$ $\quad \begin{bmatrix}2\\ 5\end{bmatrix} = \begin{bmatrix}-3\\ 5\end{bmatrix}_{B'}$ $\longrightarrow$ co-ordinate vector w.r.t. basis $B'$

$[X]_{B'} = A^{-1} \cancel{B} X_B$ wh. columns of $A$ are new basis vectors.