Eigenvalues and Eigenvectors

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Hence for $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 2 and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue -1.

Question Is $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ also an eigenvector?

How to find eigenvalue/vector?

$$Ax = \lambda x$$

$Ax - \lambda.x = 0 \Rightarrow (A - \lambda.I)x = 0$
$\Rightarrow x \in Null(A - \lambda.I)$.
$Null(A - \lambda.I) \neq \{0\} \Leftrightarrow det(A - \lambda.I) = 0$
Given a matrix A, the polynomial $det(A - \lambda.I) = 0$ is called characteristic polynomial of A(here λ is treated as a variable). Its roots are the eigenvalues of A.

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Determinant and eigenvalues

Recall for A an $n \times n$ matrix, let U be any echelon form obtained from A by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges. Then the determinant of A, $det(A) = (-1)^r u_{11} u_{22} \cdots u_{nn}$ where u_{ii} are diagonal entries of U.

Theorem (Invertible matrix theorem)

Let A be an $n \times n$ matrix. Then A is invertible (i.e., det $(A) \neq 0$) iff 0 is not an eigenvalue of A.

Theorem

If matrices A and B are similar ($A = PBP^{-1}$), then they have the same characteristic polynomial and hence the same eigenvalues.

Theorem

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then det $(A) = \prod_i \lambda_i$ and trace of $(A) = \sum_i \lambda_i$

Diagonalization

Definition

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, i.e.,

$$A = PDP^{-1}$$

for some invertible matrix P and diagonal matrix D.

Theorem

An $n \times n$ matrix A is diagonalizable ($A = PDP^{-1}$) iff A has n linearly independent eigenvectors.

In this case eigenvectors will form a basis of $\mathbb{R}^n$.

Theorem

An $n \times n$ matrix with n **distinct eigenvalues** *is diagonalizable*

Proof: Let $v_1, \dots, v_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{v_1, \dots, v_n\}$ is linearly independent set, hence basis of $\mathbb{R}^n$.

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

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Eigenvectors and Difference and Differential Equations

Consider a dynamical system $x_{k+1} = Ax_k$.
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$$
\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{aligned}
$$