MA102: Linear Algebra and Matrices: Course Content

Matrices and Linear systems: The System of Linear Equations: Ax=b, Row Reduction and Echelon forms (Gaussian Elimination), Matrix operations(addition, multiplication), Block-Partitioned Matrices and Block Operations, Elementary Row and Column Operations, Determinant and its Properties, Cofactor Expansion, Rank of a Matrix, Gauss Jordan Method for marix inversion

Canonical Factorizations: Eigenvalues and Eigenvectors, Companion Matrices and Characteristic Polynomial, diagonalization of Matrices with a Full-Set of Eigenvectors, The Cayley-Hamilton Theorem, Triangulization and Unitary Diagonalization of a Matrix, Schur's Lemma and the Spectral Theorem, QR-Decomposition, QR-Algorithm, Singular Value Decomposition.

Vector Spaces: Vector Space over the set real numbers (Field), Linear Independence of Vectors, Bases in a Vector Space, Dimension of a Vector Space, Direct Sum Decomposition of a Vector Space, Linear Transformation (LT), Change of Bases, Canonical forms, Rank of a LT.

Numerical methods: Iterative methods (Jacobi, Gauss-Seidel, Relaxation) for linear systems, computing eigenvalues and eigenvectors.

Evaluation and Grading policy:

Assignments and Quizzes: 5%
Tutorial: 15%
Mini-Project: 5%
Mid-semester Exam.: 30%
End-semester Exam.: 45% **Text & Reference books:**

Linear Algebra and its Applications, David C. Lay, 4th Ed, Pearson, 2016.

Introduction to Linear Algebra, Gilbert Strang, 5th Ed, SIAM, 2016.

Linear Algebra, Kunze Ray, Hoffman Kenneth, 2nd Ed, Phi Learning, 2014.

Fundamentals of Matrix Computations, David S. Watkins, 3rd ed, Wiley.

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ChatGPT:

1. Foundational for Computer Graphics and Computer Vision: Linear algebra is fundamental in understanding and implementing algorithms for computer graphics, computer vision, and image processing. Concepts like transformations, rotations, and scaling can be efficiently represented and manipulated using linear algebra.
2. Machine Learning and Data Science: Many machine learning algorithms rely on linear algebra for operations such as matrix multiplication, eigenvalue decomposition, and singular value decomposition. Understanding linear algebra is crucial for designing and implementing algorithms in data science and machine learning.
3. Computer Networking and Distributed Systems: In computer networking and distributed systems, linear algebra is used to model and analyze various aspects, such as network flows, connectivity, and optimization problems. It is particularly important for understanding algorithms in graph theory.

4. Algorithm Analysis and Complexity: Understanding linear algebra helps in analyzing the time and space complexity of algorithms. Matrices and vectors often appear in algorithmic analysis, and knowledge of linear algebra enables students to analyze the efficiency of algorithms in terms of computational resources.
5. Computer Simulations and Numerical Methods: Linear algebra is used extensively in computer simulations and numerical methods. Solving systems of linear equations, eigenvalue problems, and numerical optimization are common tasks in scientific computing, physics simulations, and engineering applications.
6. Database Systems and Information Retrieval: In database systems and information retrieval, linear algebra is employed for tasks such as indexing, similarity calculations, and data representation. Understanding linear algebra is beneficial for developing efficient and scalable data storage and retrieval systems.

7. Coding and Software Development: Linear algebra provides a powerful and concise way to express mathematical and computational concepts in coding. Libraries and frameworks in various programming languages often leverage linear algebra operations to optimize code and enhance performance.
8. Signal Processing: Signal processing, used in areas like audio processing and image processing, heavily relies on linear algebra. Understanding concepts like Fourier transforms, convolution, and signal representation requires a solid foundation in linear algebra.
9. Robotics and Control Systems: In robotics and control systems, linear algebra is used for modeling the dynamics of robotic systems, designing control algorithms, and analyzing the stability of control systems.

10 Problem-solving and Critical Thinking: Learning linear algebra helps develop problem-solving skills and encourages abstract and critical thinking. It provides a structured and formalized way to represent and solve a wide range of problems, fostering a deeper understanding of mathematical concepts.

In summary, linear algebra is a versatile and foundational mathematical tool that has widespread applications in various areas of computer science. It forms the basis for understanding and solving complex problems across different domains within the field.

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Find all solutions to equation $(*)$ without non-negative constraints/conditions.

Covid Pandemic

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$$\begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 300 \\ 1 \end{bmatrix} = \begin{bmatrix} 60.9 \\ 240.1 \end{bmatrix}$$

$AX_0 = X_1$
$X_{10} = A^{10} X_0$

System of linear equations

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If $A = [a_{ij}]$ of order $m \times n$ & $B = [b_{ij}]$ of order $n \times p$ then AB is of size $m \times p$ with

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Question Is $AB = BA$ for any two matrices A, B?

Let A, B, C be matrices of same dimension and $r, s \in \mathbb{R}$. Then

1. A+B=B+A
2. r.(A+B)=r.A+r.B
3. (A+B)+C=A+(B+C)
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7. A.(B+C)=A.B+A.C
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Question: Does $AB = 0$ mean either $A = 0$ or $B = 0$?

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5. $(A^{-1})^{-1} = A$.
6. $(AB)^{-1} = B^{-1}A^{-1}$.
7.$(A^T)^{-1} = (A^{-1})^T$.
Question: If A, B, C are square matrices of same dimension such that $AB = CA = I$ then is $B = C$?

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Does $AB = I$ imply $BA = I$, i.e., A is invertible?

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$$M^2 = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}$$

Block Matrix Inversion

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} A^{-1}(I + B(D - CA^{-1}B)^{-1}CA^{-1}) & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

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A block diagonal matrix is a block matrix that is a square matrix such that the main diagonal blocks are square matrices and all off diagonal blocks are zero matrices. A block diagonal matrix and its inverse have the form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{bmatrix}$$

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Block partition is useful in many computer science applications, VLSI chip design, the Strassen algorithm for fast matrix multiplication, coding theory.

Linear System

Definition

A linear System of m equations in n variables- $X_1, X_2, \dots, X_n$ *is*

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\ &. \\ &. \\ &. \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m \end{aligned}$$

where $a_{ij}, b_j \in \mathbb{R}$

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Let's look at linear system of 2 equations in 2 variables:

Solve the system: $(1)\ \ x + 2y = 3, \qquad (2)\ \ 3x + y = 4.$

Elimination of variables:

Eliminate x by $(2) - 3 \times (1)$ to get $y = 1$.

Cramer's Rule (determinant): $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{4-9}{1-6} = 1$

In either case, back substitution gives $x = 1$

We could also solve for x first and use back substitution for y.

Comparison: For a large system, say 100 equations in 100 variables, elimination method is preferred, since computing the determinants of a 101 matrices of size 100×100 is time-consuming.

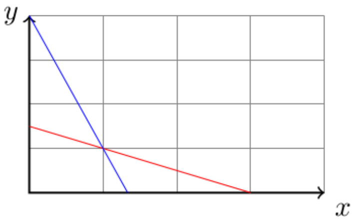
Geomtry of Linear Equations

$$\boxed{x+2y=3}$$

and

$$\boxed{3x+y=4}$$

represent lines in $\mathbb{R}^2$ passing through $\boxed{(0,3/2) \quad \text{and} \quad (3,0)}$ and through $\boxed{(0,4) \quad \text{and} \quad (4/3,0)}$ respectively.



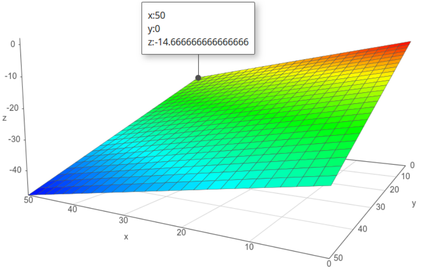
The intersection of the two lines is the unique point $(1,1)$. Hence $x = 1$ and $y = 1$ is the solution of above system of linear equations.

3 Equations in 3 Variables

A linear equation in 3 variables represents a plane in a 3-dimensional space $\mathbb{R}^3$.

$$x + 2y + 3z = 6$$

passes through $(0, 0, 2)$, $(0, 3, 0)$, $(6, 0, 0)$.



$x + 2y + 3z = 12$ passes through $(0, 0, 4)$, $(0, 6, 0)$, $(12, 0, 0)$. which is parallel to above plane.

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  In this case, every point of L is a solution.

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Question: Can we do the same when number of variables are > 3?

Gaussian Elimination: Unique solution

Example: $2x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + 2z = 9.$

Algorithm: Eliminate x from last 2 equations by $(2) - 2(1)$, and $(3) + (1)$ to get the *equivalent system:*

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 3z = 14$$

The first *pivot* is 2, second pivot is -8. Eliminate y from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad z = 2$$

Solve this triangular system by *back substitution*, we get

$$z = 2, \quad y = 1, \quad x = 1$$

Observe: This is the only possible solution!

Gaussian Elimination: No solution

Example: $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 9.$

$\boxed{\text{Step 1}}$ Eliminate x (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 14$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 2.$

The last equation shows that there is no solution,
i.e., the system is *inconsistent*.

Geometric reasoning: In Step 1, notice we get two distinct parallel planes $8y + 2z = 12$ and $8y + 2z = 14$.
They have no point in common.

Note: The planes in the original system were not parallel, but in an equivalent system, we get two distinct parallel planes!

Gaussian Elimination: Infinitely solution

Example: $\mathbf{2}x + y + z = 5, \quad 4x - 6y = -2, \quad -2x + 7y + z = 7.$

$\boxed{\text{Step 1}}$ Eliminate x (using the 1st pivot 2) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 8y + 2z = 12$

$\boxed{\text{Step 2}}$: Eliminate y (using the 2nd pivot -8) to get:

$\mathbf{2}x + y + z = 5, \quad -\mathbf{8}y - 2z = -12, \quad 0 = 0.$

There are only two equations. For every value of z, values for x and y are obtained by back-substitution, e.g, $(1, 1, 2)$ or $\left(\frac{7}{4}, \frac{3}{2}, 0\right)$. Hence the system has infinitely many solutions.

Geometric reasoning: In Step 1, notice we get two parallel planes $-8y - 2z = 12$ and $8y + 2z = 12$.
They give the same plane. Hence we are looking at the intersection of the two planes, $2x + y + z = 5$ and $8x + 2z = 12$, which is a line.

A system of linear equations has exactly one of following

1. no solution; (inconsistent system)
2. exactly one solution; (consistent system)
3. infinitely many solutions. (consistent system)

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Solve