PH110: Waves and Electromagnetics

Lecture 10

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Charge distribution in terms of electric potential:

Gauss's Law $\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$

But $\boldsymbol{\nabla} \times \mathbf{E} = 0 \quad \Rightarrow \mathbf{E} = -\boldsymbol{\nabla} V$

Therefore, $\boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$

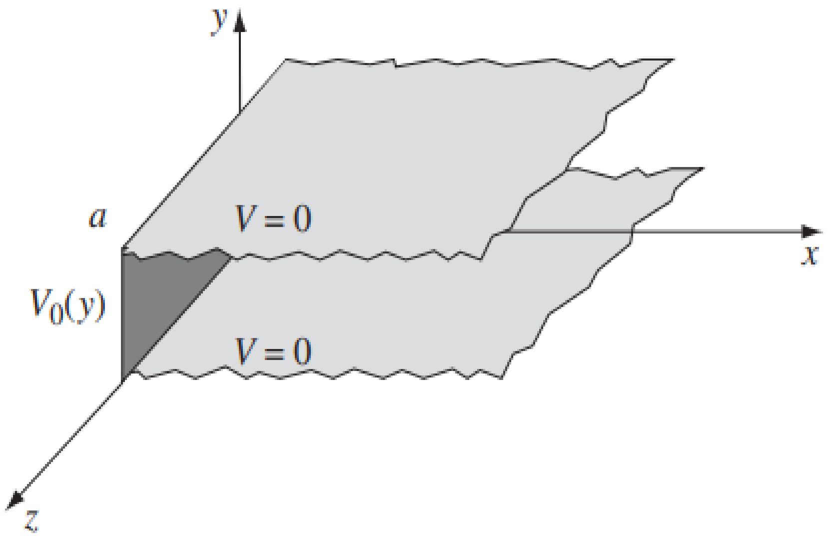
$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$ **Poisson's Equation**

In the region of space where there is no charge, $\rho=0$

$$\boxed{\nabla^2 V = 0}$$ **Laplace's Equation**

Solving Laplace Equation (Cartesian Coordinate)

Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

boundary conditions

(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = V_0(y)$ when $x = 0$,
(iv) $V \to 0$ as $x \to \infty$.

$$V(x, y) = X(x)Y(y).$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0.$$

$$\frac{d^2 X}{dx^2} = k^2 X, \qquad \frac{d^2 Y}{dy^2} = -k^2 Y.$$

$$X(x) = Ae^{kx} + Be^{-kx}, \qquad Y(y) = C \sin ky + D \cos ky,$$

$$\boxed{V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky).}$$

condition (iv) requires that A equal zero. $$V(x, y) = e^{-kx}(C \sin ky + D \cos ky).$$

Condition (i) now demands that D equal zero, so $$V(x, y) = Ce^{-kx} \sin ky.$$

(ii) yields $\sin ka = 0$, $$k = \frac{n\pi}{a}$$

the method: Separation of variables has given us an *infinite family* of solutions (one for each n), and whereas none of them *by itself* satisfies the final boundary condition, it is possible to combine them in a way that *does*. Laplace's equation is *linear*, in the sense that if $V_1, V_2, V_3, \ldots$ satisfy it, so does any **linear combination**, $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \ldots$, where $\alpha_1, \alpha_2, \ldots$ are arbitrary constants. For

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \ldots = 0\alpha_1 + 0\alpha_2 + \ldots = 0.$$

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a).$$

This still satisfies three of the boundary conditions; the question is, can we (by astute choice of the coefficients C_n) fit the final boundary condition (iii)?

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y).$$

Fourier's trick.

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \int_0^a V_0(y) \sin(n'\pi y/a)\, dy.$$

$$\int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \dfrac{a}{2}, & \text{if } n' = n. \end{cases}$$

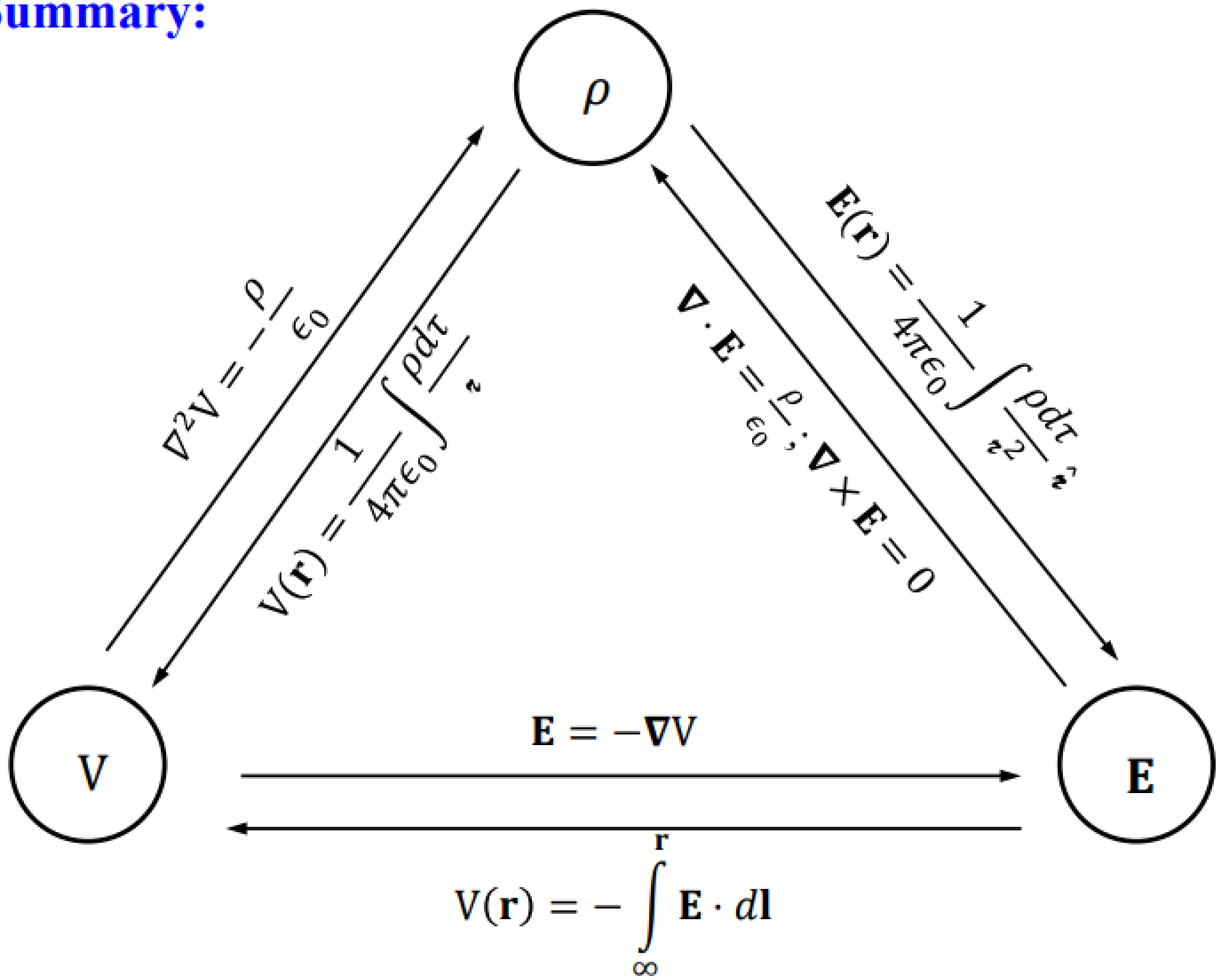
$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy.$$

$$C_n = \frac{2V_0}{a} \int_0^a \sin(n\pi y/a)\, dy = \frac{2V_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a).$$

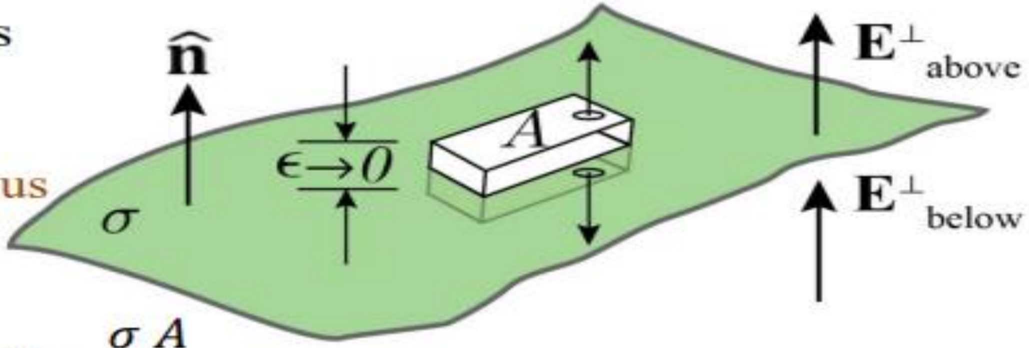
Summary:



Electrostatics

Electrostatic Boundary Conditions (Consequences of the fundamental laws):

How does electric field ($\mathbf{E}$) change across a boundary containing surface charge σ?



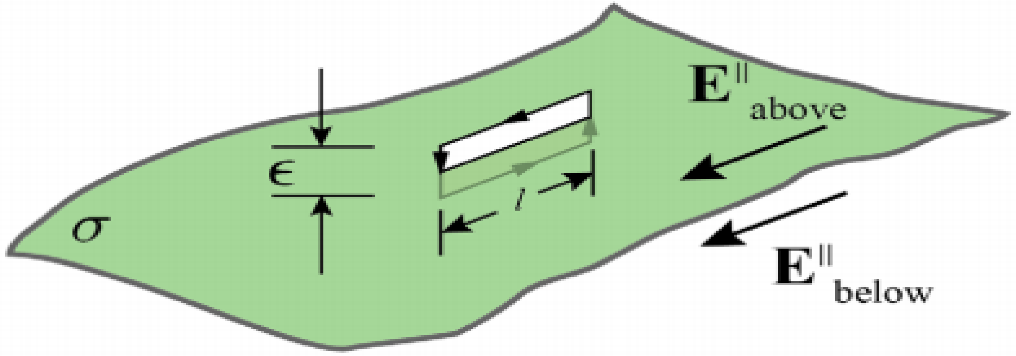
1. Normal component of E is Discontinuous

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \longleftrightarrow \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$\mathbf{E}^{\perp}{}_{\text{above}} A - \mathbf{E}^{\perp}{}_{\text{below}} A + 0 + 0 + 0 + 0 = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{\mathrm{E}^{\perp}{}_{\text{above}} - \mathrm{E}^{\perp}{}_{\text{below}} = \frac{\sigma}{\epsilon_0}}$$

2. Parallel component of E is Continuous



$$\nabla \times \mathbf{E} = 0 \longleftrightarrow \oint_{path} \mathbf{E} \cdot d\mathbf{l} = \mathbf{0}$$

$$\mathbf{E}^{\parallel}{}_{\text{above}}\, l - \mathbf{E}^{\parallel}{}_{\text{below}}\, l + 0 + 0 = 0$$

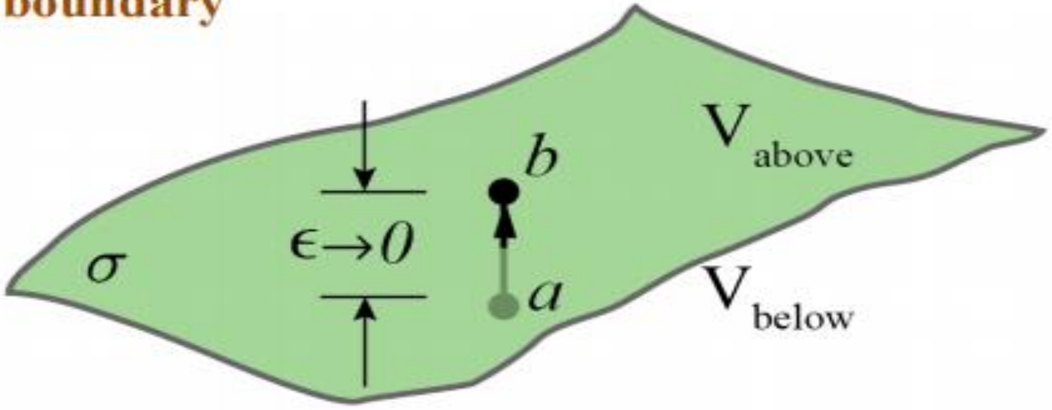
$$\boxed{\mathrm{E}^{\parallel}{}_{\text{above}} - \mathrm{E}^{\parallel}{}_{\text{below}} = 0}$$

The electrostatic boundary condition $\boxed{\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}}$

Electrostatic Boundary Conditions (Consequences of the fundamental laws):

How does electric potential (V) change across a boundary containing surface charge σ?

3. Potential V is continuous across a boundary



$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$V_{above} - V_{below} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$\text{For } \epsilon \to 0, \qquad -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l} = 0$$

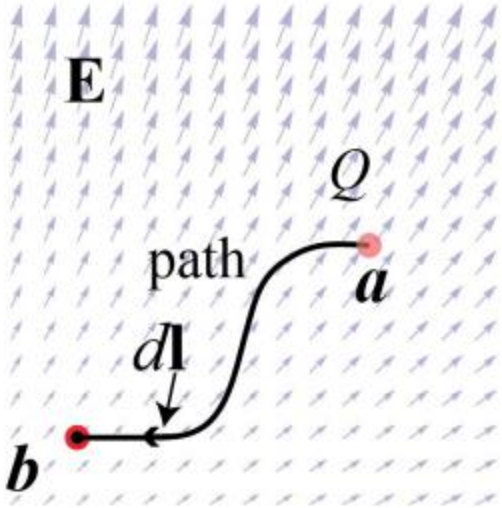
$$\boxed{V_{above} - V_{below} = 0}$$

Work and Energy in Electrostatics

There is a charge Q in an electrostatic field $\mathbf{E}$. How much work needs to be done in order to move the charge from point $\boldsymbol{a}$ to $\boldsymbol{b}$?

$$W = \int_{a}^{b} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

- $\mathbf{F} = -Q\mathbf{E}$ is the force one has to exert in order to counteract the electrostatic force $\mathbf{F} = Q\mathbf{E}$.
- Work done to move a unit charge from point $\boldsymbol{a}$ to $\boldsymbol{b}$ is the potential difference between points $\boldsymbol{b}$ and $\boldsymbol{a}$
- Work is independent of the path.



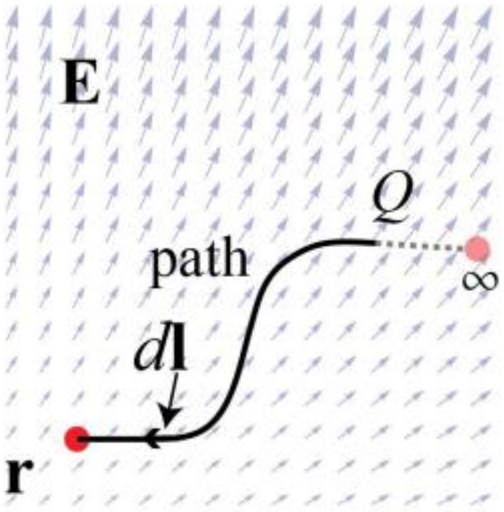
Take $V(\mathbf{a})=V(\infty)=0$ and $V(\mathbf{b}) = V(\mathbf{r})$

$$W = QV(\mathbf{r})$$

If $Q = 1$,

$$\boxed{W = V(\mathbf{r})}$$

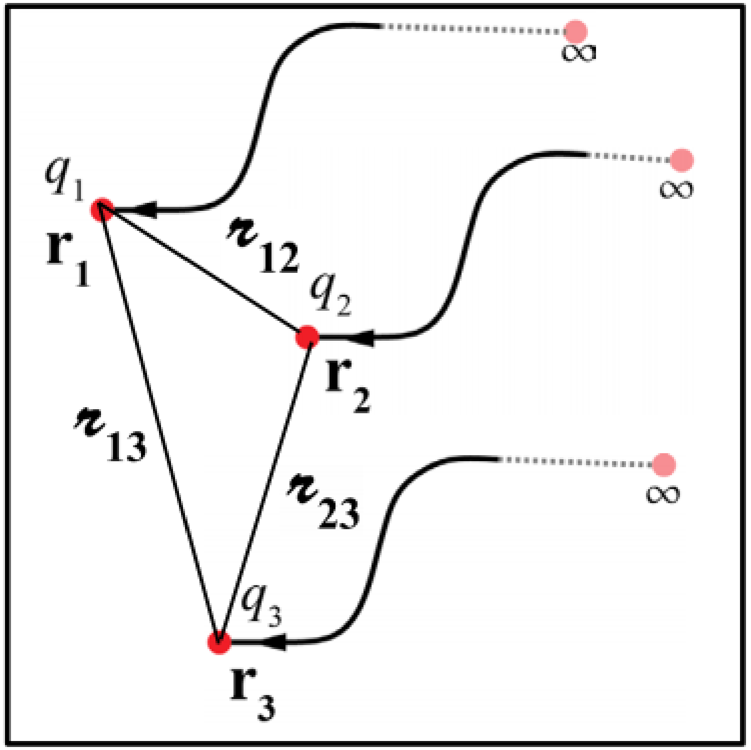
- Work done to construct a system of unit charge (to bring a unit charge from ∞ to $\mathbf{r}$ is the electric potential.
- Thus, electric potential is the potential energy per unit charge



Work required to assemble n point charges:

The work required to construct a system of one point charge Q is: $W = QV(\mathbf{r})$.

Work required to bring in the charge q_1 from ∞ to $\mathbf{r_1}$: $W_1 = q_1 V_0 = q_1 \times 0 = 0$



Work required to bring in the charge q_2 from ∞ to $\mathbf{r_2}$: $W_2 = q_2 V_1 = q_2 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_{12}}\right)$

Work required to bring in the charge q_3 from ∞ to $\mathbf{r_3}$: $W_3 = q_3 V_2 = q_3 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$

Total work required to bring in the first three charges:

$$W = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}}\right)$$

Total work required to bring in the first four charges:

$$W = W_1 + W_2 + W_3 + W_4 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right)$$

Work required to assemble n point charges:

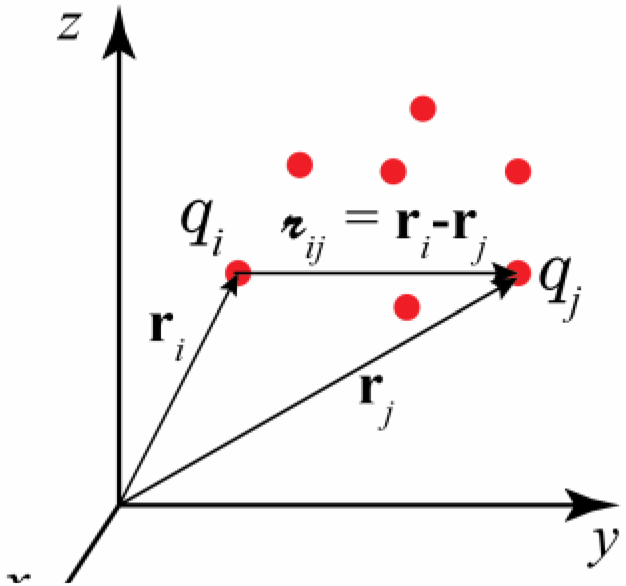
Total work required to bring in the first four charges:

$$W = W_1 + W_2 + W_3 + W_4 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right)$$

Total work required to bring in n point charges, with charge $q_1, q_2, q_3 \cdots q_n$, respectively:

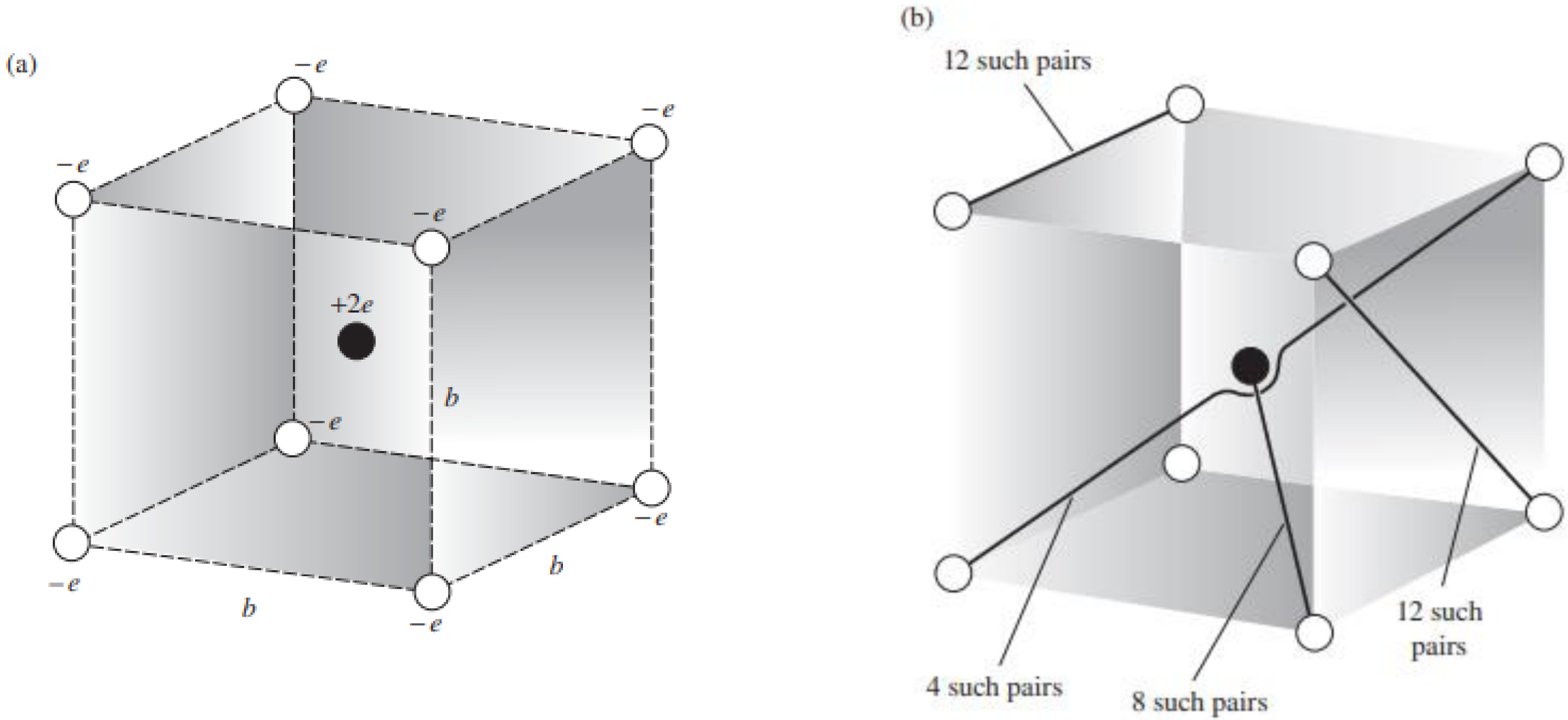
$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_j}{\mathscr{r}_{ij}}\right)$$

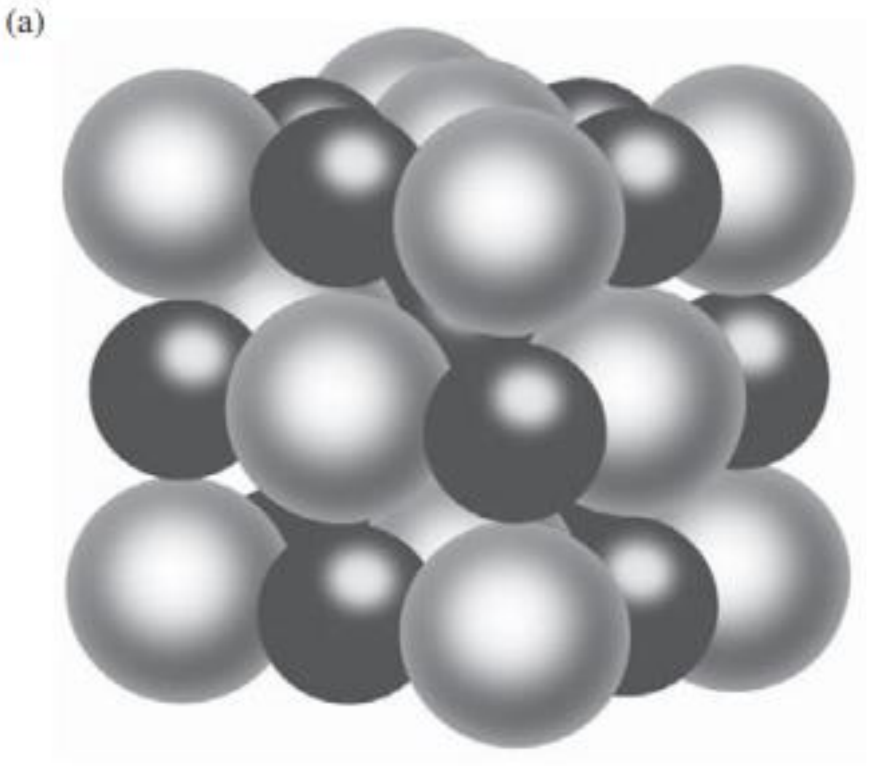


$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_\mathrm{i})}$$

- This is the total work required to assemble n point charges
- The potential $V(\mathbf{r}_\mathrm{i})$ is the potential at $\mathbf{r}_\mathrm{i}$ due to all charges, except the charge at $\mathbf{r}_\mathrm{i}$.



$$U = \frac{1}{4\pi\epsilon_0}\left(8\cdot\frac{(-2e^2)}{(\sqrt{3}/2)b} + 12\cdot\frac{e^2}{b} + 12\cdot\frac{e^2}{\sqrt{2}\,b} + 4\cdot\frac{e^2}{\sqrt{3}\,b}\right) \approx \frac{1}{4\pi\epsilon_0}\frac{4.32e^2}{b}.$$



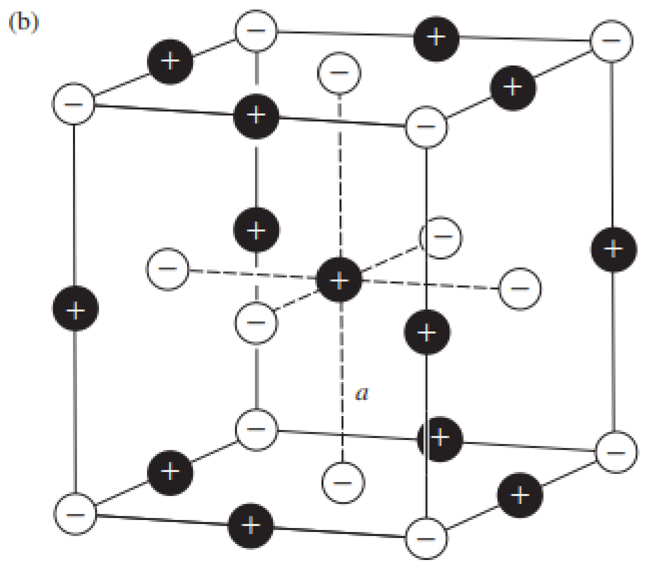


Figure 1.7.
A portion of a sodium chloride crystal, with the ions Na^+ and Cl^- shown in about the right relative proportions (a), and replaced by equivalent point charges (b).

$$U = \frac{1}{2} N \frac{1}{4\pi\epsilon_0} \left(-\frac{6e^2}{a} + \frac{12e^2}{\sqrt{2}\,a} - \frac{8e^2}{\sqrt{3}\,a} + \cdots \right).$$

The Work required to assemble a continuous charge Distribution:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\, V(\mathbf{r_i})$$

- This is the total worked required to assemble n **point** charges
- The potential $V(\mathbf{r_i})$ is the potential at $\mathbf{r_i}$ due to all the other charges, except the charge at $\mathbf{r_i}$.

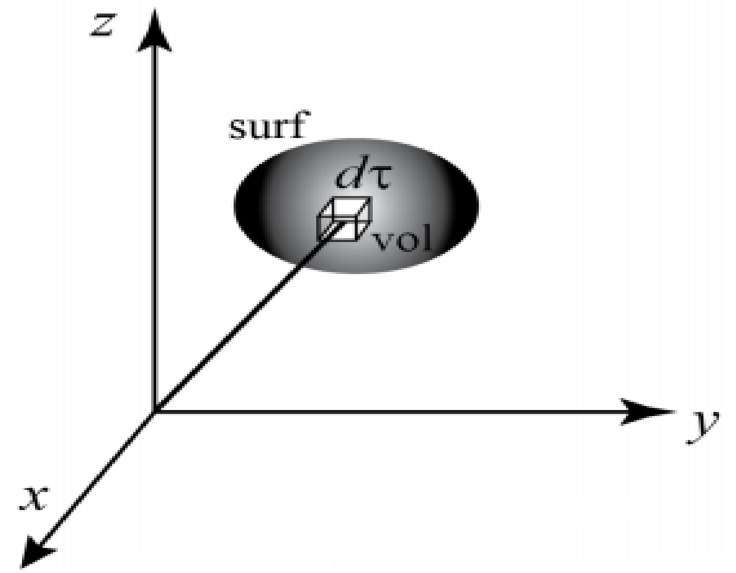
What would be the required work if it is a continuous distribution of charge ?

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\, V(\mathbf{r_i}) \;\rightarrow\; \frac{1}{2}\int_{vol} dq\, V(\mathbf{r})$$

$$\rightarrow\; \frac{1}{2}\int_{vol} \rho\, V(\mathbf{r})\,d\tau$$

Is this correct? Not really !

The potential $V(\mathbf{r})$ inside the integral is the potential at point $\mathbf{r}$. However, the potential $V(\mathbf{r_i})$ inside the summation in the potential at $\mathbf{r_i}$ due to all the charges except the charge at $\mathbf{r_i}$. Because of this difference in the definition of the potentials, the integral formula turns out to be different.



The Work required to assemble a continuous charge Distribution:

$$W = \frac{1}{2}\int_{vol} \rho\, V d\tau \quad = \frac{\epsilon_0}{2}\int_{vol} (\nabla \cdot \mathbf{E})\, V d\tau \qquad \left[\text{Using } \rho = \epsilon_0(\nabla \cdot \mathbf{E})\right]$$

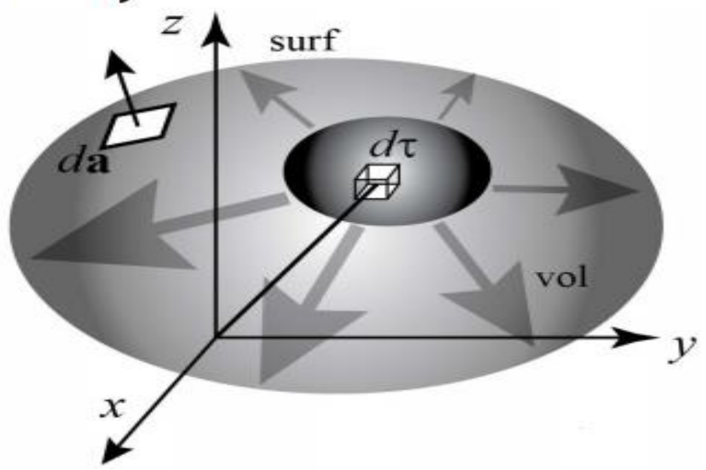
$$W = -\frac{\epsilon_0}{2}\int_{vol} \mathbf{E}\cdot\nabla V d\tau + \frac{\epsilon_0}{2}\int_{vol} \nabla\cdot V\mathbf{E} d\tau \qquad \left(\begin{array}{l}\text{Using the product rule} \\ \nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)\end{array}\right)$$

$$W = -\frac{\epsilon_0}{2}\int_{vol} \mathbf{E}\cdot\nabla V d\tau + \frac{\epsilon_0}{2}\oint_{surf} V\mathbf{E}\cdot d\mathbf{a} \qquad \left(\begin{array}{l}\text{Using the divergence theorem} \\ \int_{Vol}(\nabla\cdot\mathbf{A})d\tau = \oint_{Surf}\mathbf{A}\cdot d\mathbf{a}\end{array}\right)$$

$$W = \frac{\epsilon_0}{2}\int_{vol} E^2 d\tau + \frac{\epsilon_0}{2}\oint_{surf} V\mathbf{E}\cdot d\mathbf{a} \qquad \left[\text{Using } -\nabla V = \mathbf{E}\right]$$

$$\boxed{W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau}$$

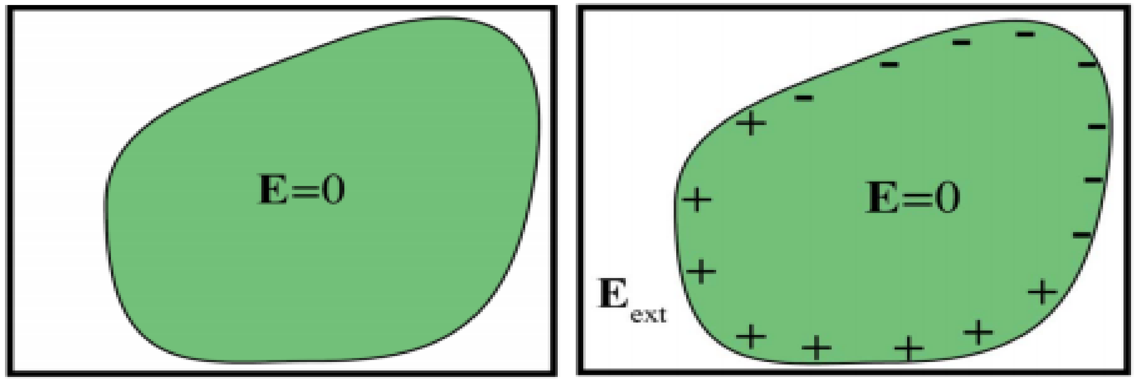
When the volume we are integrating over is very large, the contribution due to the surface integral is negligibly small.



Conductors (Materials containing unlimited supply of electrons)

(1) The electric field $\mathbf{E} = 0$ inside a conductor

This is true even when the conductor is placed in an external electric field $\mathbf{E}_{\mathbf{ext}}$.



(2) The charge density $\rho = 0$ inside a conductor.

This is because $\mathbf{E} = 0$ inside a conductor and therefore $\rho = \epsilon_0 \nabla \cdot \mathbf{E} = 0$.

(3) Any net charge resides on the surface.

Why?
To minimize the energy $\boxed{W_{\text{sphere}} = \frac{q^2}{4\pi\epsilon_0}\frac{3}{5R}} > \boxed{W_{\text{shell}} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2R}}$

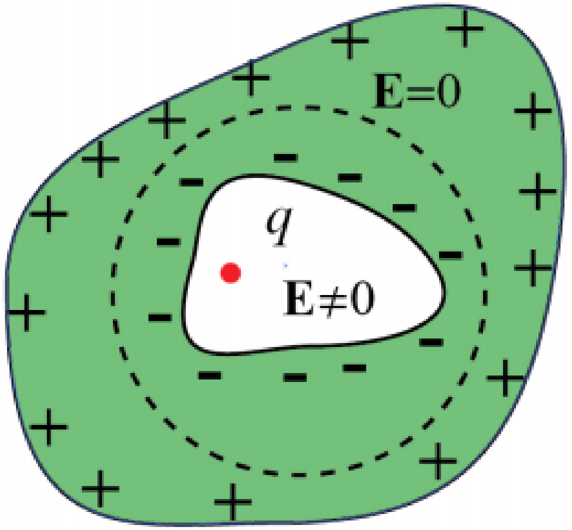
(4) A conductor is an equipotential.

This is because $\mathbf{E} = 0$. So, for any two points $\mathbf{a}$ and $\mathbf{b}$,

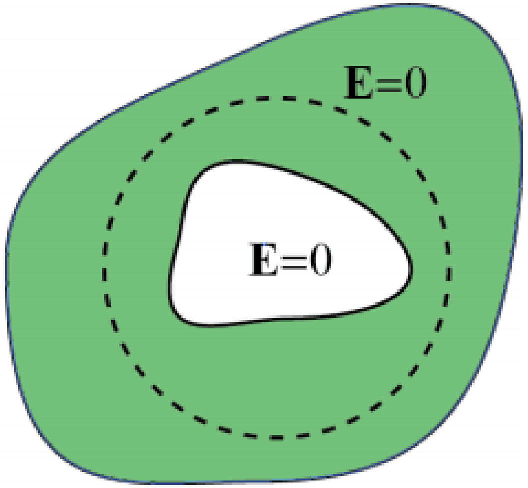
$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = 0.$$ This means $V(\mathbf{b}) = V(\mathbf{a})$.

(5) $\mathbf{E}$ is perpendicular to the surface, just outside the conductor.

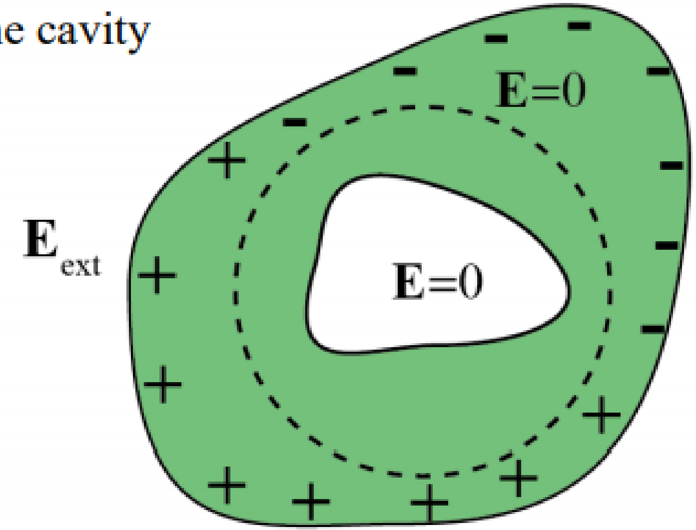
Induced Charges



Charge inside the cavity



No charge inside the cavity

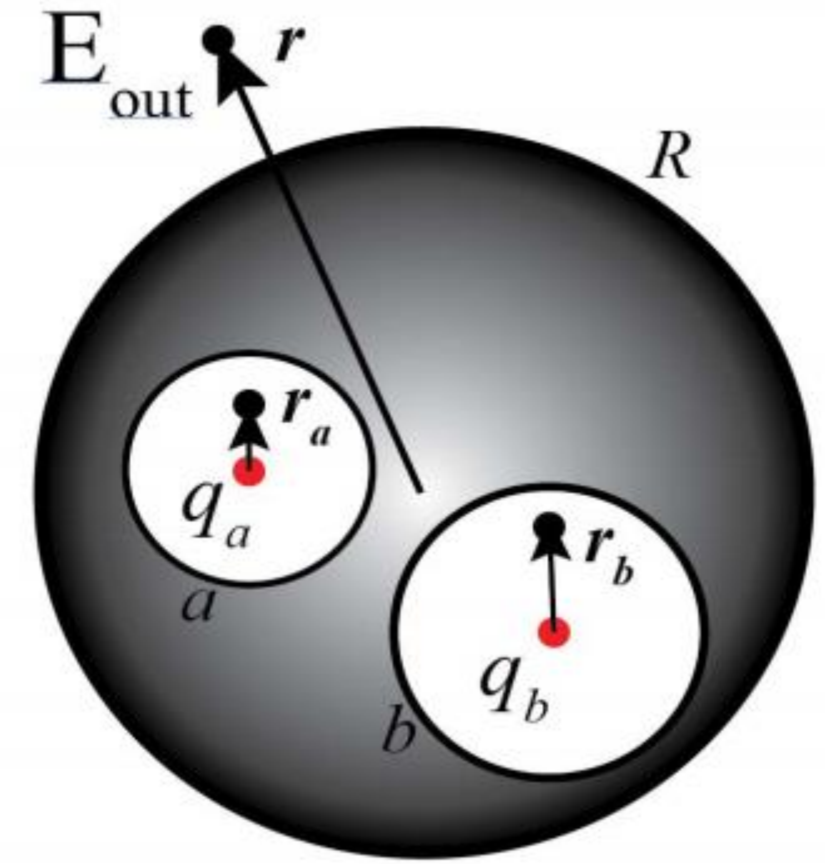


No charge inside the cavity,
Conductor in an external field

Induced Charges

Prob. 2.36 (Griffiths, 3rd Ed.):

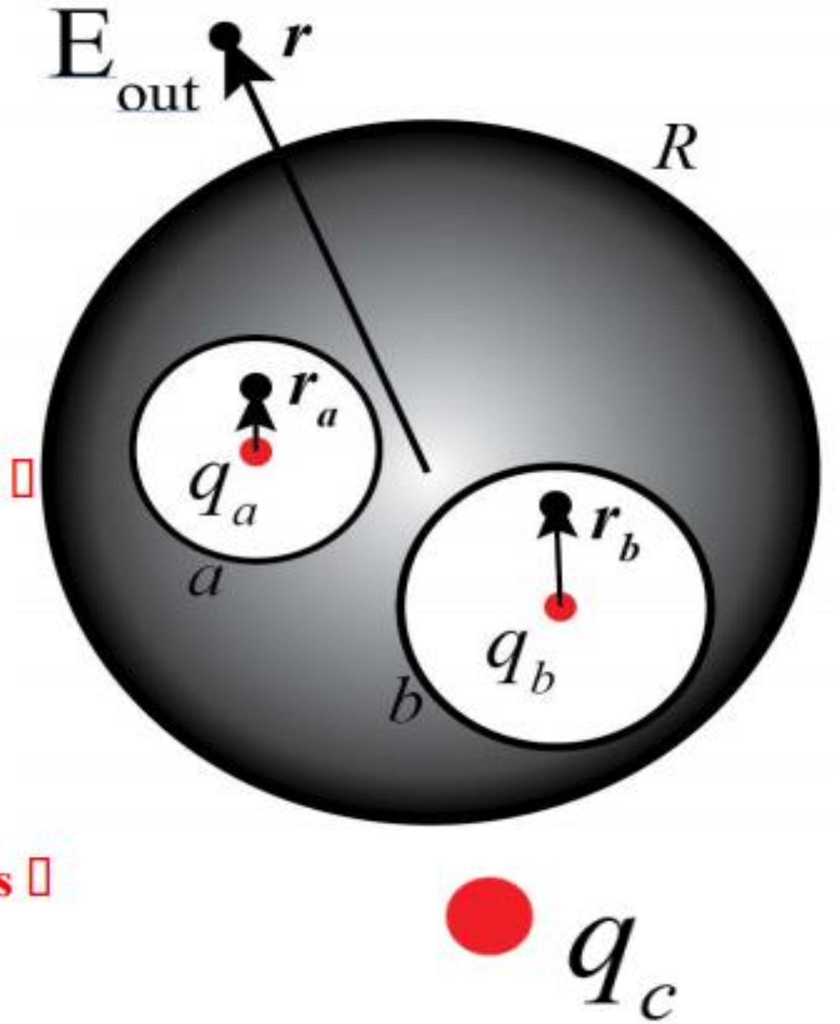
- Surface charge σ_a? $\sigma_a = -\dfrac{q_a}{4\pi a^2}$
- Surface charge σ_b? $\sigma_b = -\dfrac{q_b}{4\pi b^2}$
- Surface charge σ_R? $\sigma_R = \dfrac{q_a + q_b}{4\pi R^2}$
- $\mathbf{E}(\mathbf{r}_a)$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a}{{r_a}^2}\widehat{\mathbf{r}_a}$
- $\mathbf{E}(\mathbf{r}_b)$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_b}{{r_b}^2}\widehat{\mathbf{r}_b}$
- $\mathbf{E}_{\text{out}}(\mathbf{r})$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a + q_b}{r^2}\hat{\mathbf{r}}$
- Force on q_a ? 0
- Force on q_b ? 0



Induced Charges

Prob. 2.36 (Griffiths, 3rd Ed.):

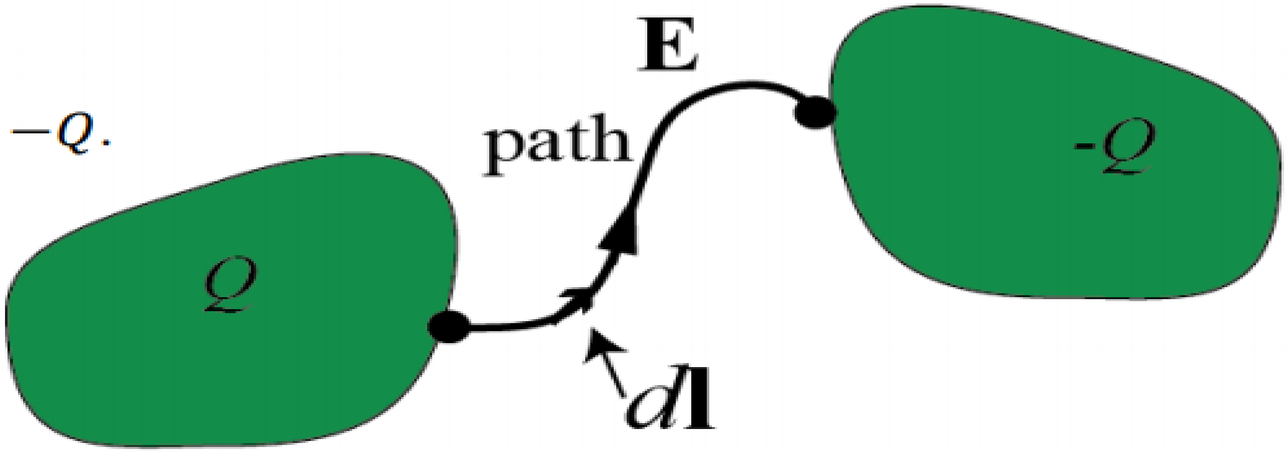
- Surface charge σ_a? $\sigma_a = -\frac{q_a}{4\pi a^2}$ **Same** ✓
- Surface charge σ_b? $\sigma_b = -\frac{q_b}{4\pi b^2}$ **Same** ✓
- Surface charge σ_R? $\sigma_R = \frac{q_a + q_b}{4\pi R^2}$ **Changes** ▯
- $\mathbf{E}(\mathbf{r}_a)$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a}{{r_a}^2}\hat{\mathbf{r}}_a$ **Same** ✓
- $\mathbf{E}(\mathbf{r}_b)$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_b}{{r_b}^2}\hat{\mathbf{r}}_b$ **Same** ✓
- $\mathbf{E}_{\text{out}}(\mathbf{r})$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$ **Changes** ▯
- Force on q_a ? 0 **Same** ✓
- Force on q_b ? 0 **Same** ✓



Bring in a third charge q_c

Capacitor:

Two conductors with charge Q and $-Q$.



What is the potential difference between them?

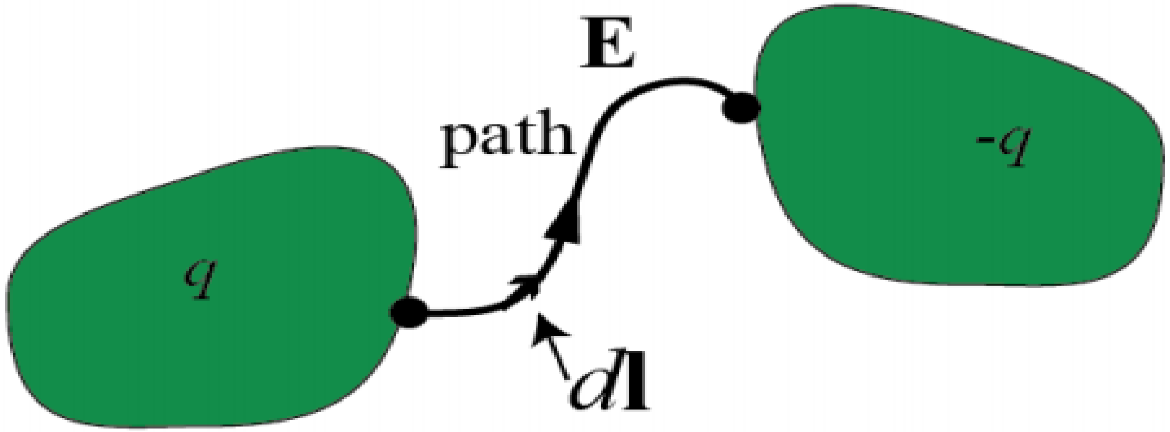
$$V = V_{+} - V_{-} = -\int_{(-)}^{(+)} \mathbf{E}.\,d\mathbf{l}$$

Capacitance C is defined as: $C \equiv \frac{Q}{V}$

- Capacitance is the ability of a system to store electric charge.
- It is purely a geometric quantity.
- C is measured in farads (F), Coulomb/Volt.
- Practical units are microfarad (10^{-6}) or picofarad (10^{-12}).

Work needed to charge a Capacitor:

Two conductors with charge q and $-q$.



How much work needs to be done to increase the charge by dq

Recall The work required to create a system of a point charge Q: $W = QV(\mathbf{r})$

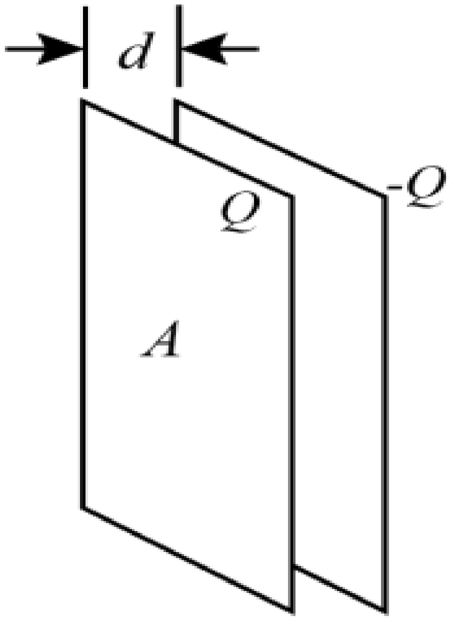
$$dW = V\,dq \;= \left(\frac{q}{C}\right) dq$$

The work necessary to go from $q = 0$ to $q = Q$ is

$$W = \int_0^Q \left(\frac{q}{C}\right) dq \;= \frac{1}{2}\frac{Q^2}{C} \;= \frac{1}{2} C V^2$$

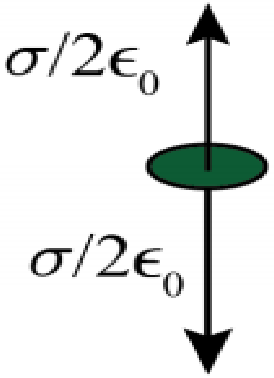
Capacitor:

Ex. 2.10 (Griffiths, 3rd Ed.): Find the capacitance of a parallel plate capacitor. Area = A, Separation = d



The electric field between the plates

$$\mathrm{E} = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$



The potential difference is therefore,

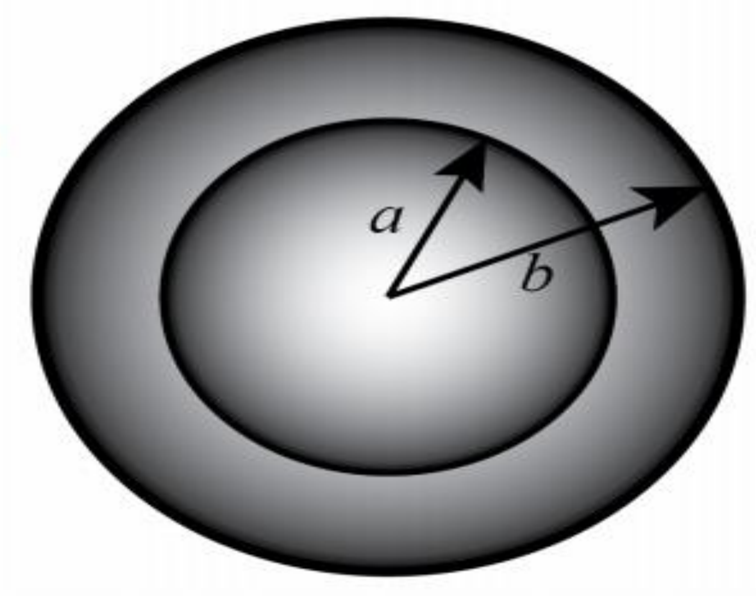
$$V = -\int \mathbf{E}.\,d\mathbf{l} = E\ d = \frac{Q}{A\epsilon_0} d$$

Capacitance C is:

$$C = \frac{Q}{V} = \frac{A\epsilon_0}{d}$$

Capacitor:

Ex. 2.11 (Griffiths, 3rd Ed.): Find the capacitance of two concentric spherical metal shells, with radii a and b.

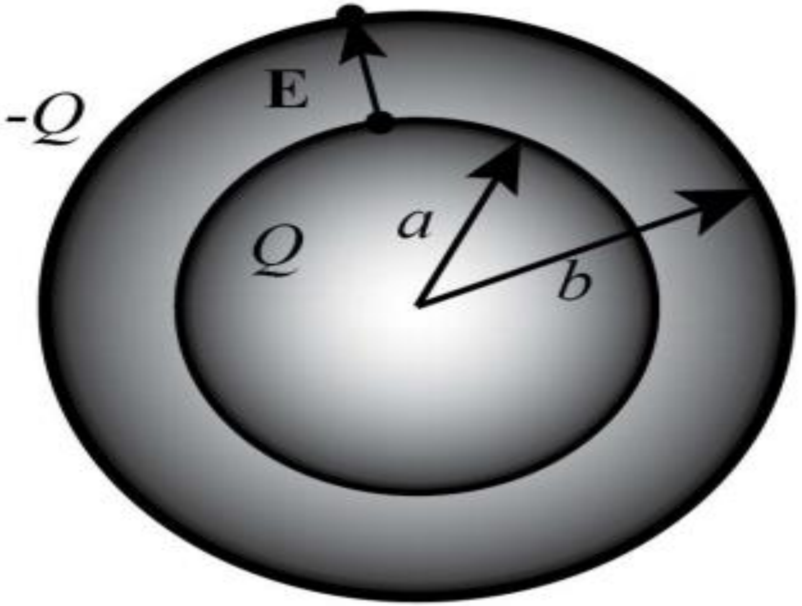


Suppose there is charge Q on the inner shell and $-Q$ on the outer shell.

The electric field between the two shells is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\boldsymbol{r}}$$

The potential difference is therefore,



$$V = V_b - V_a = -\int_a^b \mathbf{E}.\,d\mathbf{l} = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\,dr \quad = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Capacitance is: $C = \dfrac{Q}{V} \;= 4\pi\epsilon_0 \dfrac{ab}{(b-a)}$

Superposition principle for electrostatic energy:

We have seen several electrostatic systems, including conductors.

We know how to calculate electrostatic energy for different system.

We know that electric field ($\mathbf{E}$) and electric potential (V) follow the principle of superposition.

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} + \cdots \qquad V = V_1 + V_2 + \cdots$$

Does electrostatic energy also follow the principle of superposition? No

Why? Because W is quadratic in E?

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau = \frac{\epsilon_0}{2}\int (\mathbf{E_1} + \mathbf{E_2})^2\, d\tau = \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\mathbf{E_1}\cdot\mathbf{E_2})\, d\tau$$

$$= \frac{\epsilon_0}{2}\int E_1^2\, d\tau + \frac{\epsilon_0}{2}\int E_2^2\, d\tau + \epsilon_0 \int \mathbf{E_1}\cdot\mathbf{E_2}\, d\tau$$

$$= W_1 + W_2 + \epsilon_0 \int \mathbf{E_1}\cdot\mathbf{E_2}\, d\tau$$

Laplace's Equation

Q: How to find electric field $\mathbf{E}$?

Ans: $\mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$ (Coulomb's Law)

Very difficult to calculate the integral except for very simple situation

Alternative: First calculate the electric potential

$$\mathrm{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}}$$

This integral is relatively easier but in general still difficult to handle

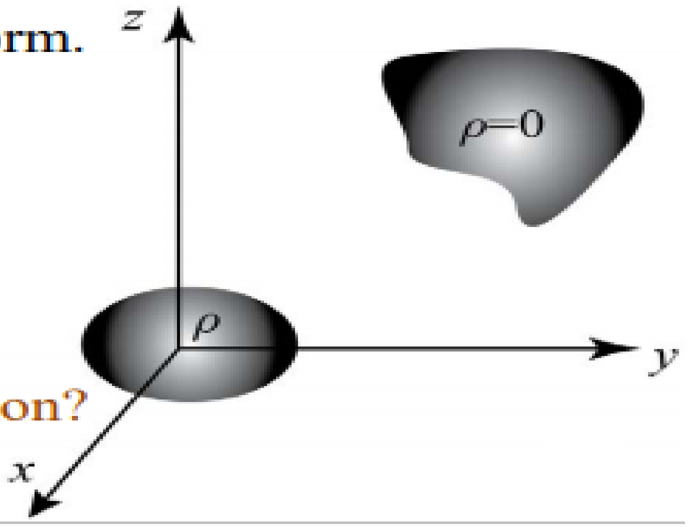
Alternative: Express the above equation in the different form.

$$\nabla^2 \mathrm{V} = -\frac{\rho}{\epsilon_0} \quad \text{(Poisson's Equation)}$$

When $\rho = 0$ $\quad \nabla^2 \mathrm{V} = 0$ (Laplace's Equation)

If $\rho = 0$ everywhere, $V = 0$ everywhere

If ρ is localized, what is V away from the charge distribution?



Laplace's Equation in One Dimension

$$\nabla^2 V = 0 \qquad \text{(Laplace's Equation)}$$

In Cartesian coordinates,

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

If $V(x, y, z)$ depends on only one variable, x, We have

$$\frac{d^2}{dx^2}V = 0$$

(One-dimensional Laplace's Equation, ordinary differential equation)

General Solution: $V(x) = mx + b$

How to calculate the constants m and b ?

Using boundary conditions

What decides the boundary condition?

The charge distribution

Laplace's Equation in one dimension

If the potential V(x) is a solution to the Laplace's equation then V(x) is the average of the potential at $x + a$ and $x - a$

$$V(x, y) = \frac{1}{2}[V(x + a) + V(x - a)]$$

As a result, V(x) cannot have local maxima or minima; the extreme values of V(x) must occur at the end points.

Laplace's Equation in two dimensions

If the potential V(x, y) is a solution to the Laplace's equation then V(x, y) is the average value of potential over a circle of radius R centered at (x, y).

$$V(x, y) = \frac{1}{2\pi R}\oint_{circle} V dl$$

As a result, V(x, y) cannot have local maxima or minima; the extreme values of V(x, y) must occur at the boundaries.

Thank You