PH110: Waves and Electromagnetics

Lecture 14

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Magnetostatics

Stationary charges $\longrightarrow$ Constant Electric field $\longrightarrow$ Electrostatics

Steady currents $\longrightarrow$ Constant Magnetic field $\longrightarrow$ Magnetostatics

Non-stationary charges and/or Non-steady currents $\longrightarrow$ Changing Electric and Magnetic field $\longrightarrow$ Electrodynamics

Force on a point charge *Q*:

Electric Force: $\mathbf{F}_{\text{elec}} = Q\mathbf{E}$

Magnetic Force: $\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$ **Lorentz Force Law**

Total Force: $\boxed{\mathbf{F} = \mathbf{F}_{\text{elec}} + \mathbf{F}_{\text{mag}} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$

Work done by magnetic forces

The work done by a magnetic force is zero !

Why?

$$\mathbf{W}_{\mathrm{mag}} = \int \mathbf{F}_{\mathrm{mag}} \cdot d\mathbf{l} = \int Q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= \int Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt$$

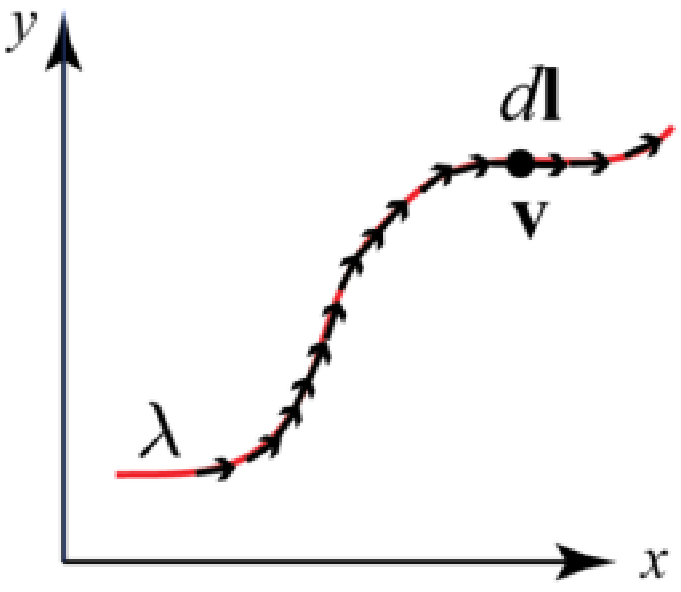
$$= 0$$

- Magnetic forces do no work
- Magnetic forces can change the direction in which a particle moves.
- Magnetic forces do not change the speed with which a particle moves.

Currents

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).

Charge flowing in a wire is described by **Current**



$$\mathbf{I} = \frac{dq}{dt} = \frac{\lambda\, d\mathbf{l}}{dt} = \lambda \mathbf{v}$$

- The direction of current is in the direction of charge-flow.
- Conventionally, this is the direction opposite to the flow of electrons.

Magnetic force on a current carrying wire:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\, \lambda dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

$$\boxed{\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\, dl = I \int (d\mathbf{l} \times \mathbf{B})}$$

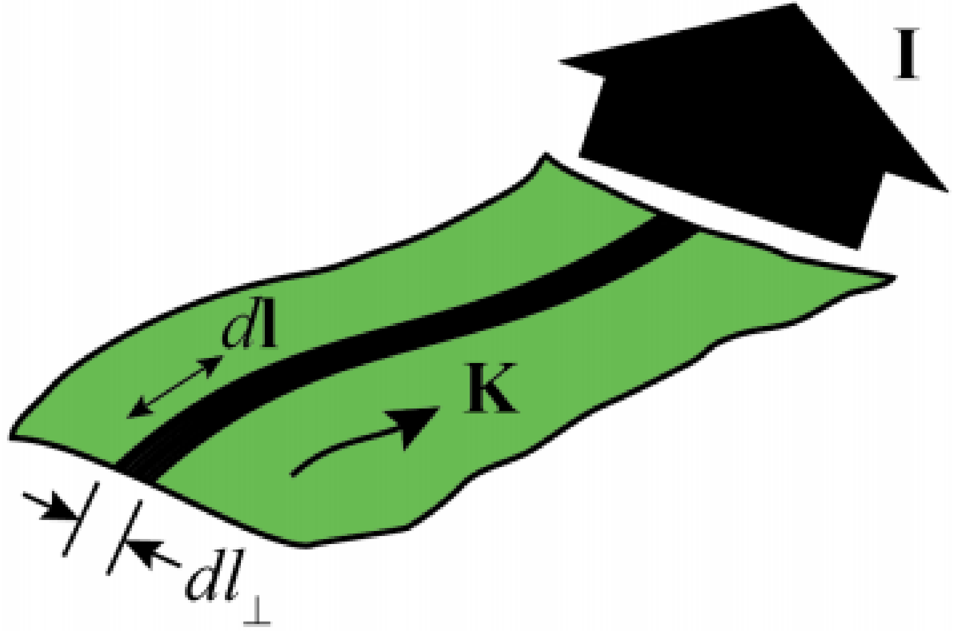
Currents

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).

Charge flowing on a surface is described by **surface current density**

$$\mathbf{K} = \frac{d\mathbf{I}}{dl_{\perp}} = \sigma\mathbf{v}$$

- Current density is a vector quantity.



Magnetic force on the surface current:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, \sigma da = \int (\mathbf{K} \times \mathbf{B})\, da$$

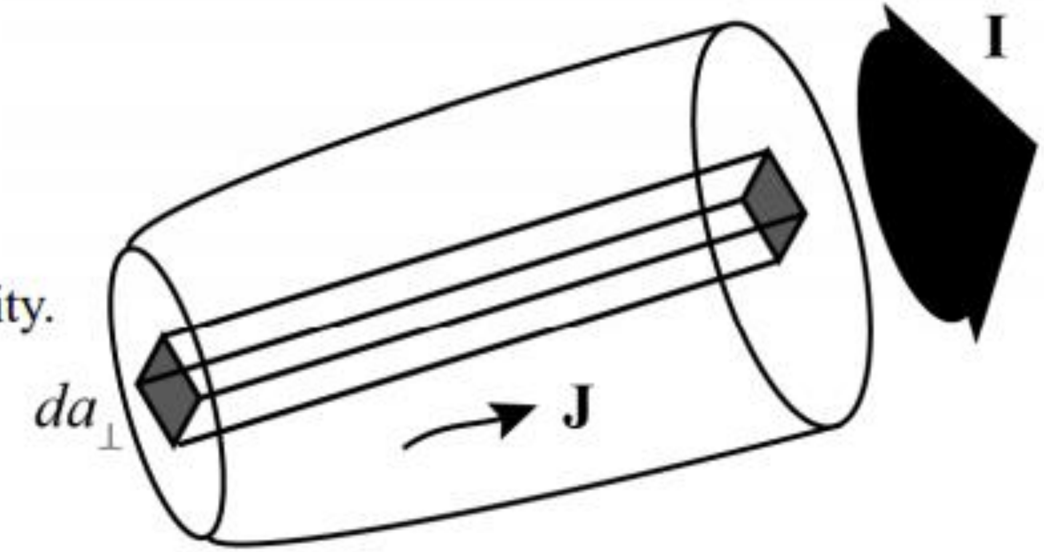
Currents

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).

Charge flowing in a volume is described by **volume current density**

$$\mathbf{J} = \frac{d\mathbf{I}}{da_{\perp}} = \rho\mathbf{v}$$

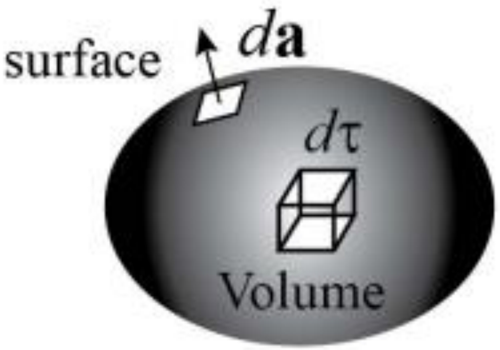
- Current density is a vector quantity.



Magnetic force on the volume current:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, \rho d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

The Continuity Equation (Conservation of Charge)



$$\mathbf{J} = \frac{d\mathbf{I}}{da_{\perp}} \qquad \Rightarrow \mathrm{I} = \int_S \mathrm{J}\, da_{\perp}$$

$$\Rightarrow \mathrm{I} = \int_S \mathbf{J} \cdot d\mathbf{a}$$

- Current crossing a surface
- Total charge per unit time crossing a surface

For a closed surface

$$\mathrm{I} = \oint_S \mathbf{J} \cdot d\mathbf{a}$$ Total charge per unit time crossing a closed surface

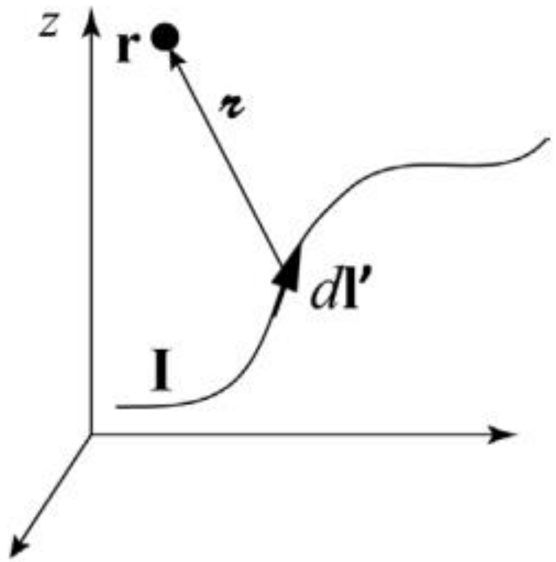
$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{J}) d\tau$$ Total charge per unit time leaving the volume V.

But, total charge per unit time leaving the volume V is $-\frac{d}{dt}\left(\int_V \rho d\tau\right) = -\int_V \frac{d\rho}{dt}\, d\tau$

So, $\int_V (\nabla \cdot \mathbf{J}) d\tau = -\int_V \frac{d\rho}{dt}\, d\tau \quad \Rightarrow \quad \boxed{\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}}$ **The Continuity Equation**

The Biot-Savart Law

The magnetic field produced by a steady line current



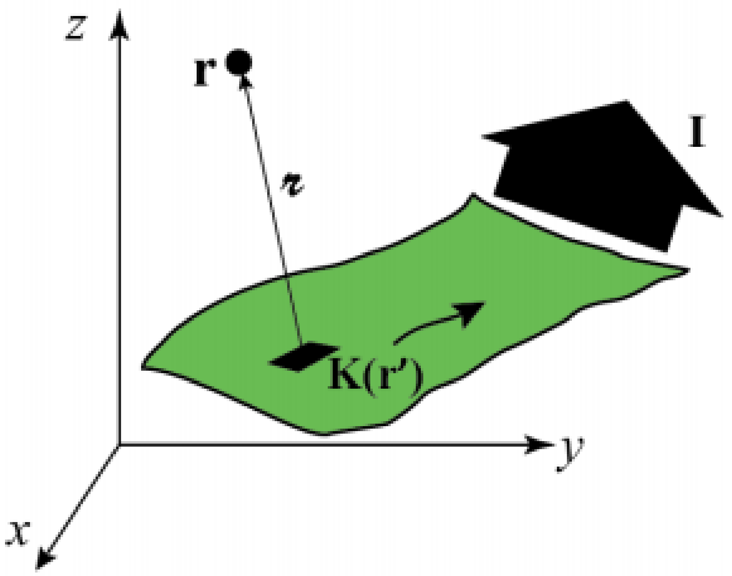
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \quad = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

- μ_0 is the permeability of free space
- $\mu_0 = 4\pi\times 10^{-7}\ \mathrm{N/A^2}$
- The unit of magnetic field is Newton per Ampere-meter, or Tesla
- 1 Tesla is a very strong magnetic field. Earth's magnetic field is about 10^{-4} times smaller
- Biot-Savart law for magnetic field is analogous to Coulomb's law for electric field

The Biot-Savart Law

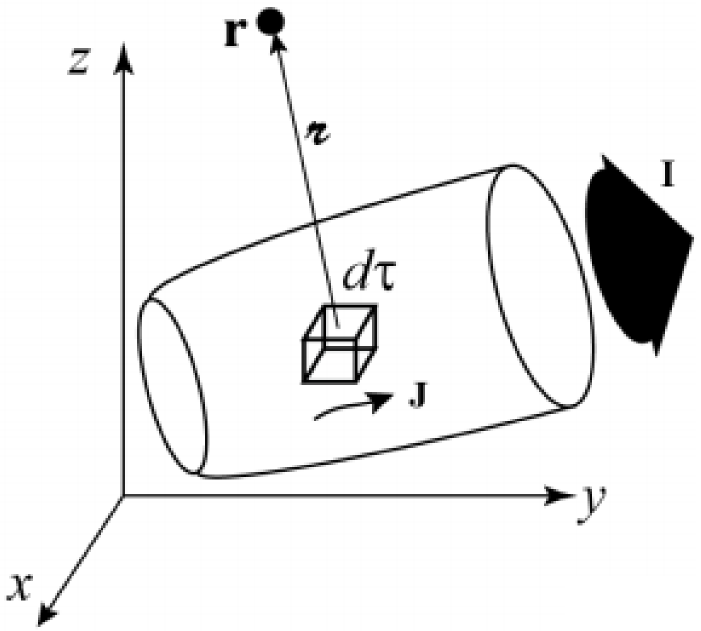
The magnetic field produced by a surface current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da'$$



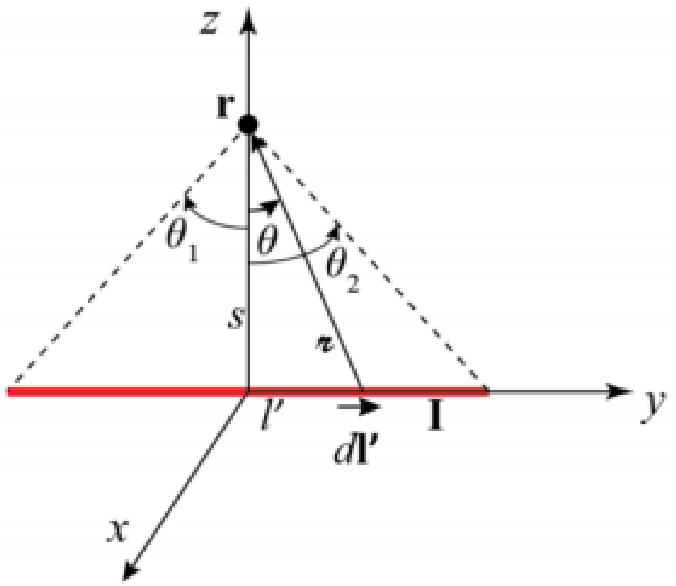
The magnetic field produced by a volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$



The Biot-Savart Law

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to a long straight wire carrying a steady current I .



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad l' = s\tan\theta \;\Rightarrow\; dl' = \frac{s}{\cos^2\theta} d\theta$$

$$\mathbf{B}(\mathbf{r}) = B\,\hat{\mathbf{x}}$$

$$|d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}| = dl'\cos\theta \qquad s = \mathscr{r}\cos\theta \;\Rightarrow\; \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta\,\hat{\mathbf{x}} = \frac{\mu_0 I}{4\pi s}\int_{-\theta_1}^{\theta_2}\cos\theta\, d\theta\,\hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{4\pi s}(\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

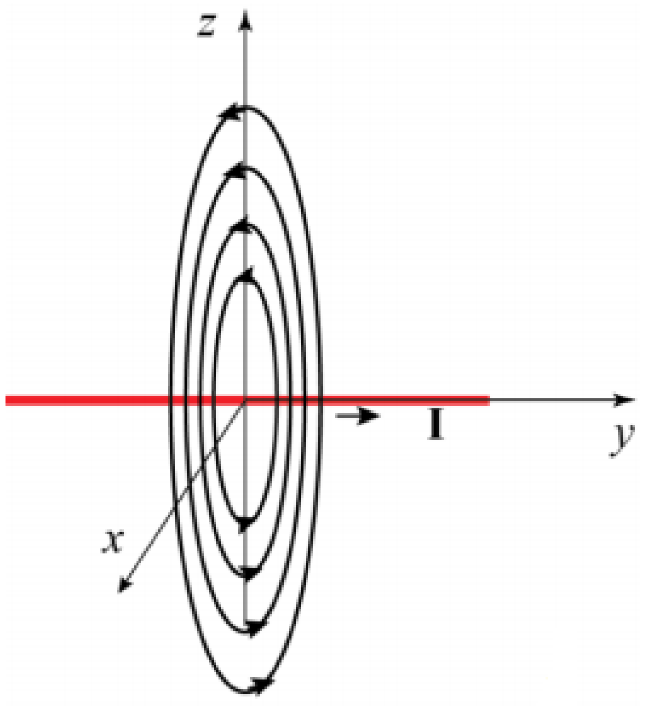
Field due to an infinite wire ?

$$\theta_1 = \frac{\pi}{2} \qquad \theta_2 = \frac{\pi}{2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{4\pi s}(1 + 1)\hat{\mathbf{x}}$$

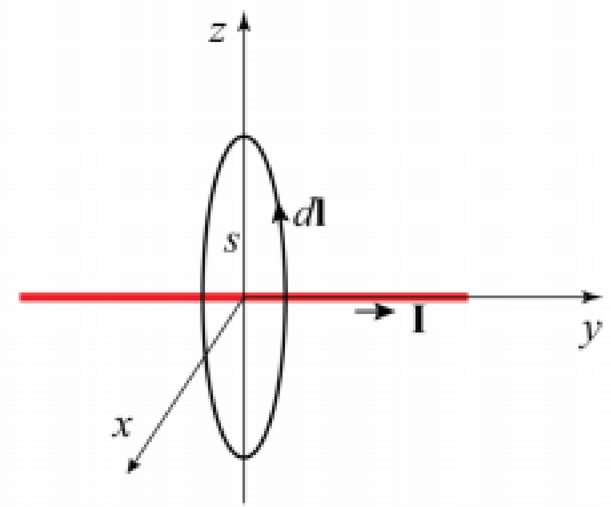
$$= \frac{\mu_0 I}{2\pi s}\hat{\mathbf{x}}$$



The Divergence and Curl of B

What is the divergence of **B** ? $\nabla \cdot \mathbf{B} = 0$

What is the curl of **B** ? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$



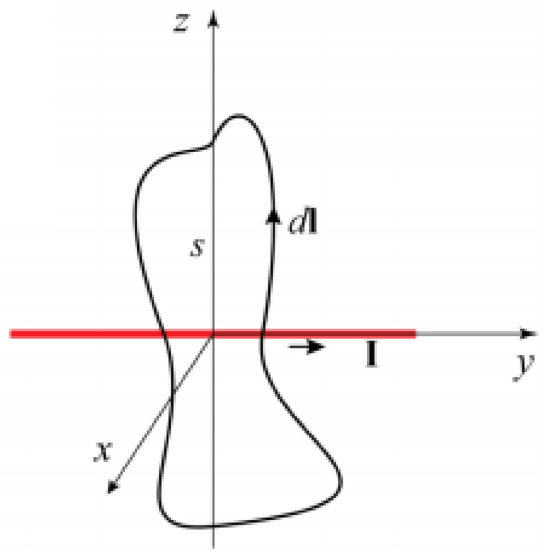
Check for the case of straight wire with current I

$$\mathrm{B}(s) = \frac{\mu_0 I}{2\pi s}$$

For circular path of radius s

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

The line integral is independent of s



For an arbitrary path enclosing the current carrying wire

The field is best represented in the cylindrical coordinate

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \qquad d\mathbf{l} = ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

So, $$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \cdot (ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}) = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

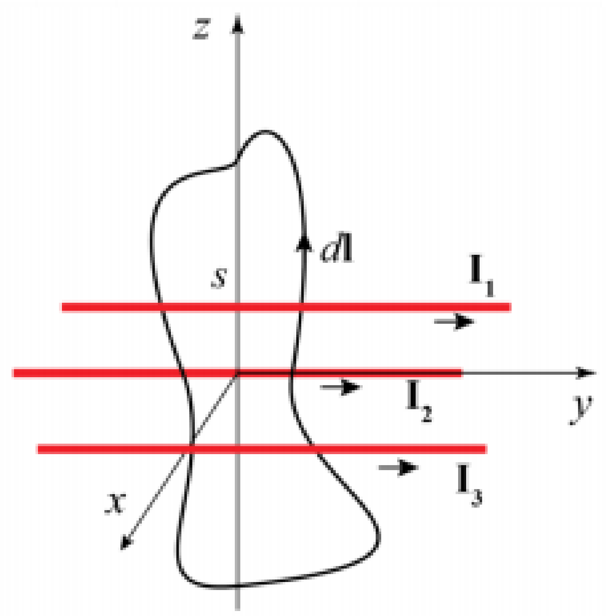
If the path encloses more than one current carrying wire

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3 = \mu_0 I_{\text{enc}}$$

But $\quad I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \quad \Rightarrow \quad \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$



- Is this valid only for straight wires? No
- It is valid in general

The Ampere's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ Ampere's law in differential form

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$ Ampere's law in integral form

- Ampere's law is analogous to Gauss's law
- Ampere's law makes the calculation of magnetic field very easy if there is symmetry.
- If there is no symmetry, one has to use Biot-Savart law to calculate the magnetic field.

Magnetostatics and Electrostatics

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')d\tau$$

Coulomb's Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'$$

Biot-Savart Law

$$\mathbf{F}_{\text{elec}} = Q\mathbf{E}$$

Electric Force

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

Magnetic Force

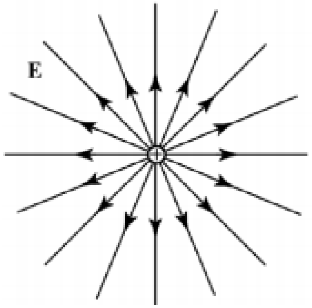
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ Gauss's Law

$\nabla \cdot \mathbf{B} = 0$ No Name

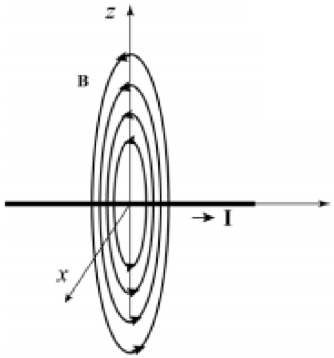
$\nabla \times \mathbf{E} = 0$ No Name

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Amperes's Law

Maxwell's equations (Electrostatics)



Electric field diverges away from a positive charge



Magnetic field curls around a current.

$\nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V$

Electric Potential (scalar)

$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$

Magnetic Vector Potential

Magnetostatic Boundary Conditions (Consequences of the fundamental laws):

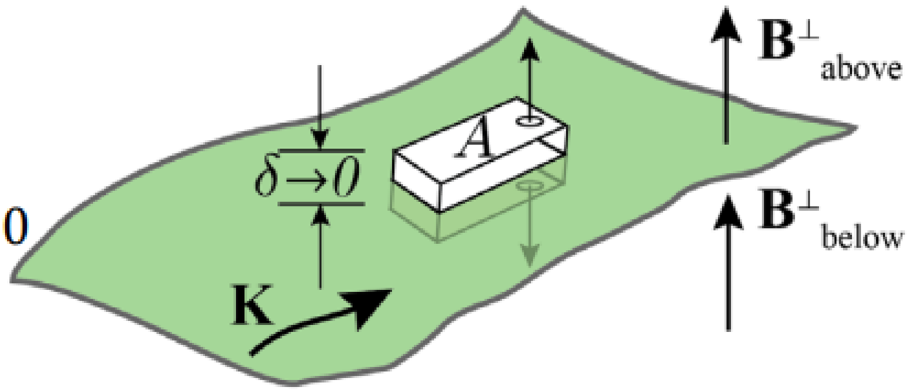
How does magnetic field (**B**) change across a boundary containing surface current **K**?

1. Normal component of B is continuous

$$\nabla \cdot \mathbf{B} = 0 \longleftrightarrow \oint_{surf} \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\mathbf{B}^{\perp}{}_{\text{above}} A - \mathbf{B}^{\perp}{}_{\text{below}} A + 0 + 0 + 0 + 0 = 0$$

$$\boxed{B^{\perp}{}_{\text{above}} = B^{\perp}{}_{\text{below}}}$$

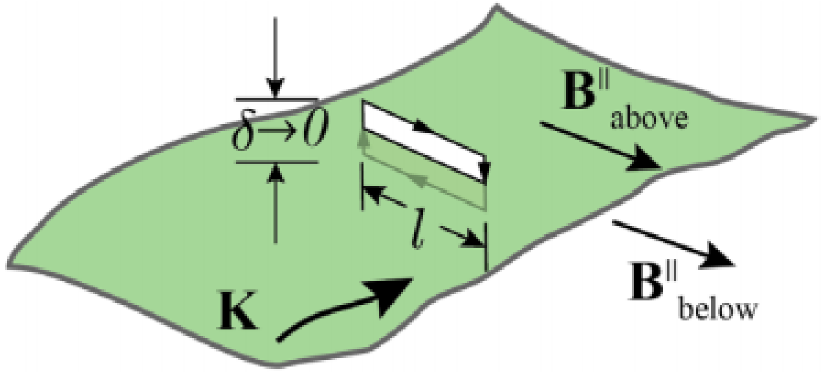


2. Parallel component of B is Discontinuous

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \longleftrightarrow \oint_{path} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{I}_{\text{enc}}$$

$$\mathbf{B}^{||}{}_{\text{above}}\, l - \mathbf{B}^{||}{}_{\text{below}}\, l + 0 + 0 = \mu_0 K l$$

$$\boxed{B^{||}{}_{\text{above}} - B^{||}{}_{\text{below}} = \mu_0 K}$$



$$(B^{\perp}{}_{\text{above}} - B^{\perp}{}_{\text{below}})\hat{\mathbf{n}} + (B^{||}{}_{\text{above}} - B^{||}{}_{\text{below}})\widehat{\mathbf{n}^{||}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$$

$$\boxed{\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}}$$

Magnetostatic Boundary Conditions (Consequences of the fundamental laws):

How does the magnetic potential (**A**) change across a boundary containing surface current **K**?

1. Normal component of **A** is continuous

$$\nabla \cdot \mathbf{A} = 0 \longleftrightarrow \oint_{surf} \mathbf{A} \cdot d\mathbf{a} = 0$$

$$\boxed{A^{\perp}{}_{\text{above}} = A^{\perp}{}_{\text{below}}}$$

2. Parallel component of **A** is continuous

$$\nabla \times \mathbf{A} = \mathbf{B} \longleftrightarrow \oint_{path} \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\boxed{A^{\parallel}{}_{\text{above}} = A^{\parallel}{}_{\text{below}}}$$

Magnetic Vector Potential

$\nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \mathbf{B} = \nabla \times \mathbf{A}$

- $\mathbf{A}$ is the Magnetic Vector Potential
- A gradient $\nabla\lambda$ of a scalar function can be added to $\mathbf{A}$ without affecting the magnetic field.

What happens to the Ampere's Law ?

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\Rightarrow\; \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\Rightarrow\; \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

- This is not in a very nice form.
- Ampere's law in terms of $\mathbf{B}$ seems better

- However, if we can ensure that $\nabla \cdot \mathbf{A} = 0$, we can have it in a nice form.
- This can be done since we know that a $\nabla\lambda$ can be added to $\mathbf{A}$ without changing $\mathbf{B}$

Suppose we start with $\mathbf{A}_0$, such that, $\mathbf{B} = \nabla \times \mathbf{A}_0$ but, $\nabla \cdot \mathbf{A_0} \neq \mathbf{0}$.

Then, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\Rightarrow\; \nabla(\nabla \cdot \mathbf{A_0}) - \nabla^2 \mathbf{A_0} = \mu_0 \mathbf{J}$

Re-define by adding $\nabla\lambda$: $\mathbf{A_0} + \nabla\lambda \equiv \mathbf{A}$ such that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2 \lambda = 0$

Then $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\Rightarrow\; \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \;\Rightarrow\; -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

Magnetic Vector Potential

What is the requirement on λ that $\nabla \cdot \mathbf{A} = 0$?? Or, $\nabla \cdot \mathbf{A_0} + \nabla^2 \lambda = 0$??

For a given $\mathbf{A_0}$ the gradient λ should be such that

$\nabla^2 \lambda = -\nabla \cdot \mathbf{A_0}$ (Poisson's Equation)

The solution is: $\lambda(\mathbf{r}) = \dfrac{1}{4\pi} \displaystyle\int \dfrac{\nabla \cdot \mathbf{A_0}}{\mathscr{r}} d\tau'$

If $\nabla \cdot \mathbf{A_0} \rightarrow \mathbf{0}$, when $\mathbf{r} \rightarrow \infty$.

Recall: $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ (Poisson's Equation)

The solution is: $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$

If the localized charge distribution $\rho(\mathbf{r}') \rightarrow \mathbf{0}$, when $\mathbf{r} \rightarrow \infty$.

Thus, one can always redefine the vector potential such that $\nabla \cdot \mathbf{A} = 0$

So, the Ampere's law can be written as $\boxed{-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}}$ It is three Poisson's Equations

$$\text{So,} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

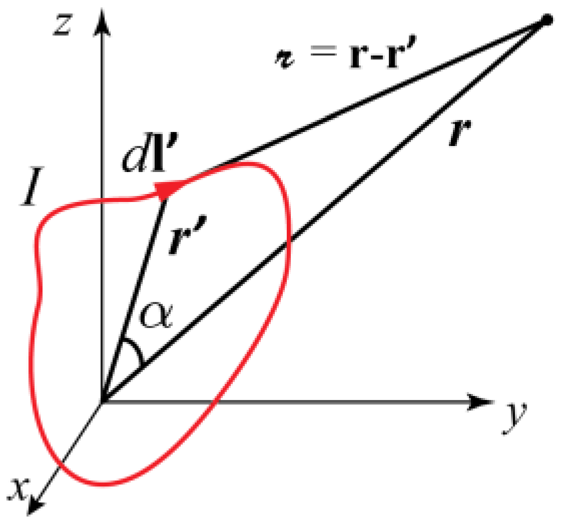
This is simpler than Biot-Savart Law.

For surface current: $$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} da'$$

For line current: $$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'}{\mathscr{r}}$$

Multipole Expansion of the Vector Potential

Using the cosine rule,



Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between **r** and **r**′: α

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\mathscr{r} = r\sqrt{1 + \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)}$$

$$\mathscr{r} = r\sqrt{1+\epsilon}$$

Define: $\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

(using binomial expansion)

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2(3\cos^2\alpha - 1)/2 - \left(\frac{r'}{r}\right)^3(5\cos^3\alpha - 3\cos\alpha)/2 + \cdots\right]$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\mathbf{l}'}{\mathscr{r}}$$

$$= \underbrace{\frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}'}_{\substack{\text{Monopole potential} \\ (\,1/r\text{ dependence})}} + \underbrace{\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}'}_{\substack{\text{Dipole potential} \\ (\,1/r^2\text{ dependence})}} + \underbrace{\frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos\alpha - \frac{1}{2}\right) d\mathbf{l}'}_{\substack{\text{Quadrupole potential} \\ (\,1/r^3\text{ dependence})}} + \cdots$$

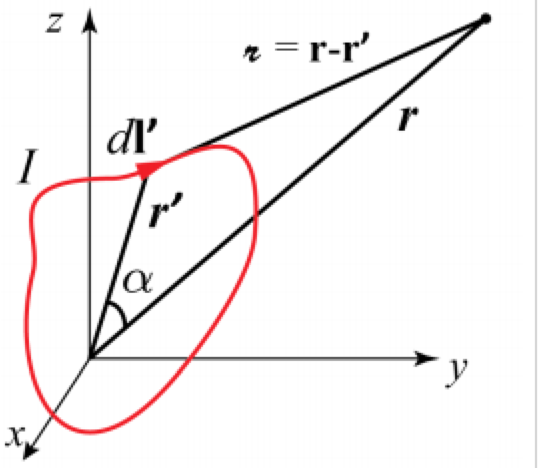
Multipole Expansion of the Vector Potential

Monopole potential

$$\mathbf{A}_{\text{mono}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' \quad = 0$$

Dipole potential

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' \;= \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$



Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between $\mathbf{r}$ and $\mathbf{r}'$: α

Magnetostatics in matter (magnetic field in matter)

What is Polarization? - dipole moment per unit volume

- The dipole moment is caused either by stretch of an atom/molecule or by rotation of polar molecules
- Polarization in the direction parallel to the applied electric field

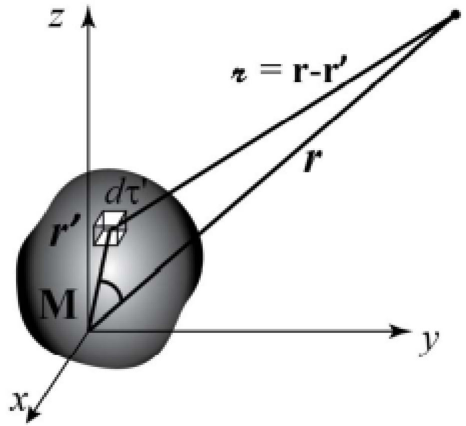
What is Magnetization? - magnetic dipole moment per unit volume

- The magnetic dipole moment is caused by electric charges in motion: (i) electrons orbiting around nuclei & (ii) electrons spinning about their own axes.
- In some material, magnetization is in the direction parallel to **B** (Paramagnets).
- In some other material, magnetization is opposite to **B** (Diamagnets).
- In other, there can be magnetization even in the absence of **B** (Ferromagnets).

➢ Magnetization in Ferromagnetic material is much higher.

➢ If a ferromagnetic material is exposed to strong-enough magnetic field, the magnetization in the material does not return to zero after the field is removed and this way the material becomes a permanent magnet.

The Field of a Magnetized Object:



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{vol}\left[\mathbf{M}(\mathbf{r}')\times\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

[Using $\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]d\tau' - \frac{\mu_0}{4\pi}\int_{vol}\boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'$$

[Using $\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f)$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}']$$

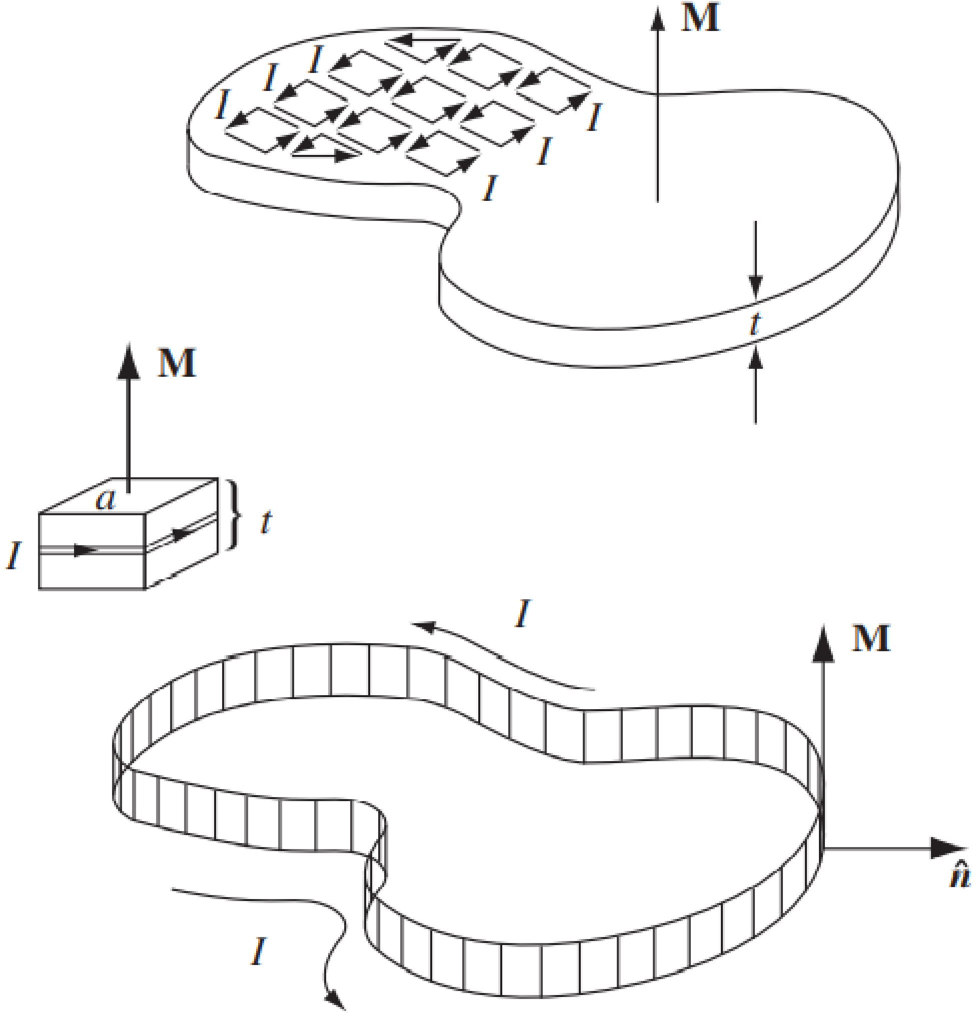
[Griffiths Prob. 1.60 (b)
$\int_{vol}(\boldsymbol{\nabla}\times\mathbf{V})d\tau = -\oint_{surf}\mathbf{V}\times d\mathbf{a}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}d\mathbf{a}'$$

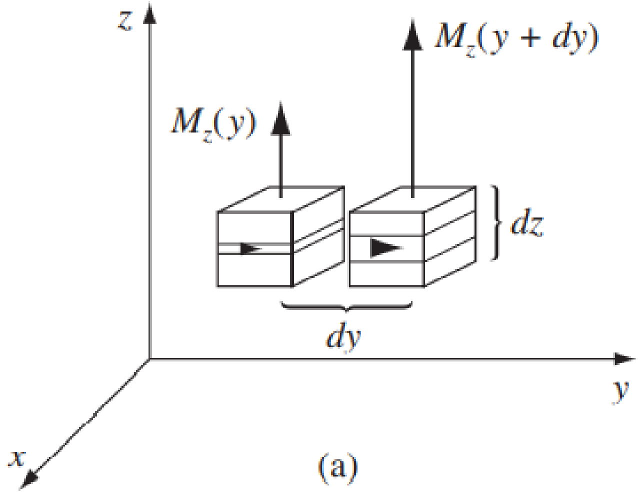
$\mathbf{J}_b(\mathbf{r}') = \boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')$ Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}')\times\hat{\mathbf{n}}$ Surface current

Uniform Polarization



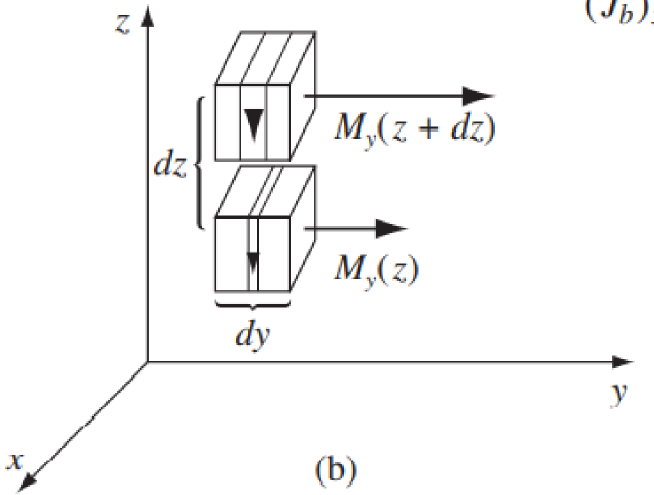
Non-Uniform Polarization



(a)

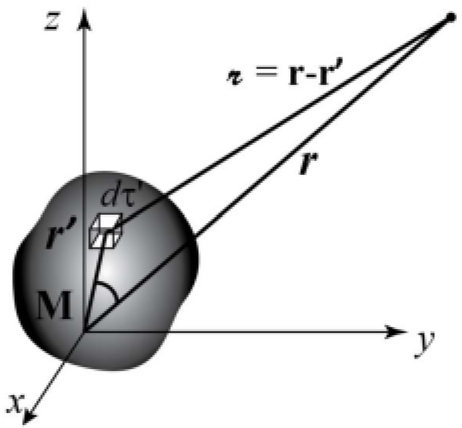
$$I_x = [M_z(y + dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz.$$

$$(J_b)_x = \frac{\partial M_z}{\partial y}.$$



(b)

Ampere's law in magnetized material:



$$\mathbf{J}_b(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}')$$

Volume current

$$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}$$

Surface current

Total volume current is

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

What happens to the Ampere's law?

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$$

Or, $\nabla \times \left(\dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f$

Define: $\mathbf{H} \equiv \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$ Which means $\mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M})$

So, $\nabla \times \mathbf{H} = \mathbf{J}_f$ Ampere's law in magnetized material (differential form)

And, $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\mathrm{enc}}}$ Ampere's law in magnetized material (integral form)

Thank You