Tut-11

Problem 9.1 By explicit differentiation, check that the functions f_1, f_2, and f_3 in the text satisfy the wave equation. Show that f_4 and f_5 do *not*.

Problem 9.1

$$\frac{\partial f_1}{\partial z} = -2Ab(z-vt)e^{-b(z-vt)^2}; \quad \frac{\partial^2 f_1}{\partial z^2} = -2Ab\left[e^{-b(z-vt)^2} - 2b(z-vt)^2 e^{-b(z-vt)^2}\right];$$

$$\frac{\partial f_1}{\partial t} = 2Abv(z-vt)e^{-b(z-vt)^2}; \quad \frac{\partial^2 f_1}{\partial t^2} = 2Abv\left[-ve^{-b(z-vt)^2} + 2bv(z-vt)^2 e^{-b(z-vt)^2}\right] = v^2\frac{\partial^2 f_1}{\partial z^2}. \checkmark$$

$$\frac{\partial f_2}{\partial z} = Ab\cos[b(z-vt)]; \quad \frac{\partial^2 f_2}{\partial z^2} = -Ab^2\sin[b(z-vt)];$$

$$\frac{\partial f_2}{\partial t} = -Abv\cos[b(z-vt)]; \quad \frac{\partial^2 f_2}{\partial t^2} = -Ab^2v^2\sin[b(z-vt)] = v^2\frac{\partial^2 f_2}{\partial z^2}. \checkmark$$

$$\frac{\partial f_3}{\partial z} = \frac{-2Ab(z-vt)}{[b(z-vt)^2+1]^2}; \quad \frac{\partial^2 f_3}{\partial z^2} = \frac{-2Ab}{[b(z-vt)^2+1]^2} + \frac{8Ab^2(z-vt)^2}{[b(z-vt)^2+1]^3};$$

$$\frac{\partial f_3}{\partial t} = \frac{2Abv(z-vt)}{[b(z-vt)^2+1]^2}; \quad \frac{\partial^2 f_3}{\partial t^2} = \frac{-2Abv^2}{[b(z-vt)^2+1]^2} + \frac{8Ab^2v^2(z-vt)^2}{[b(z-vt)^2+1]^3} = v^2\frac{\partial^2 f_3}{\partial z^2}. \checkmark$$

$$\frac{\partial f_4}{\partial z} = -2Ab^2ze^{-b(bz^2+vt)}; \quad \frac{\partial^2 f_4}{\partial z^2} = -2Ab^2\left[e^{-b(bz^2+vt)} - 2b^2z^2e^{-b(bz^2+vt)}\right];$$

$$\frac{\partial f_4}{\partial t} = -Abve^{-b(bz^2+vt)}; \quad \frac{\partial^2 f_4}{\partial t^2} = Ab^2v^2e^{-b(bz^2+vt)} \neq v^2\frac{\partial^2 f_4}{\partial z^2}.$$

$$\frac{\partial f_5}{\partial z} = Ab\cos(bz)\cos(bvt)^3; \quad \frac{\partial^2 f_5}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3; \quad \frac{\partial f_5}{\partial t} = -3Ab^3v^3t^2\sin(bz)\sin(bvt)^3;$$

$$\frac{\partial^2 f_5}{\partial t^2} = -6Ab^3v^3t\sin(bz)\sin(bvt)^3 - 9Ab^6v^6t^4\sin(bz)\cos(bvt)^3 \neq v^2\frac{\partial^2 f_5}{\partial z^2}.$$

$$f_1(z,t) = Ae^{-b(z-vt)^2}, \quad f_2(z,t) = A\sin[b(z-vt)], \quad f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

Problem 9.3 Use Eq. 9.19 to determine A_3 and δ_3 in terms of A_1, A_2, δ_1, and δ_2.

$$\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2, \quad \text{or} \quad A_3e^{i\delta_3} = A_1e^{i\delta_1} + A_2e^{i\delta_2}. \tag{9.19}$$

Problem 9.3

$$
\begin{aligned}
(A_3)^2 &= \left(A_3e^{i\delta_3}\right)\left(A_3e^{-i\delta_3}\right) = \left(A_1e^{i\delta_1}+A_2e^{i\delta_2}\right)\left(A_1e^{-i\delta_1}+A_2e^{-i\delta_2}\right) \\
&= (A_1)^2+(A_2)^2+A_1A_2\left(e^{i\delta_1}e^{-i\delta_2}+e^{-i\delta_1}e^{i\delta_2}\right) = (A_1)^2+(A_2)^2+A_1A_22\cos(\delta_1-\delta_2); \\
A_3 &= \boxed{\sqrt{(A_1)^2+(A_2)^2+2A_1A_2\cos(\delta_1-\delta_2)}.} \\
A_3e^{i\delta_3} &= A_3(\cos\delta_3+i\sin\delta_3) = A_1(\cos\delta_1+i\sin\delta_1)+A_2(\cos\delta_2+i\sin\delta_2) \\
&= (A_1\cos\delta_1+A_2\cos\delta_2)+i(A_1\sin\delta_1+A_2\sin\delta_2). \quad \tan\delta_3 = \frac{A_3\sin\delta_3}{A_3\cos\delta_3} = \frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}; \\
\delta_3 &= \boxed{\tan^{-1}\left(\frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}\right).}
\end{aligned}
$$

Problem 9.6

(a) Formulate an appropriate boundary condition, to replace Eq. 9.27, for the case of two strings under tension T joined by a knot of mass m.

(b) Find the amplitude and phase of the reflected and transmitted waves for the case where the knot has a mass m and the second string is massless.

Problem 9.6

(a) $T\sin\theta_+ - T\sin\theta_- = ma \Rightarrow \boxed{T\left(\left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-}\right) = m\left.\frac{\partial^2 f}{\partial t^2}\right|_0.}$

(b) $\tilde{A}_I+\tilde{A}_R=\tilde{A}_T$; $T[ik_2\tilde{A}_T - ik_1(\tilde{A}_I-\tilde{A}_R)] = m(-\omega^2\tilde{A}_T)$, or $k_1(\tilde{A}_I-\tilde{A}_R) = \left(k_2-\frac{im\omega^2}{T}\right)\tilde{A}_T$.

Multiply first equation by k_1 and add: $2k_1\tilde{A}_I = \left(k_1+k_2-i\frac{m\omega^2}{T}\right)\tilde{A}_T$, or $\tilde{A}_T = \left(\frac{2k_1}{k_1+k_2-im\omega^2/T}\right)\tilde{A}_I$.

$\tilde{A}_R = \tilde{A}_T-\tilde{A}_I = \frac{2k_1-(k_1+k_2-im\omega^2/T)}{k_1+k_2-im\omega^2/T}\tilde{A}_I = \left(\frac{k_1-k_2+im\omega^2/T}{k_1+k_2-im\omega^2/T}\right)\tilde{A}_I.$

If the second string is massless, so $v_2=\sqrt{T/\mu_2}=\infty$, then $k_2/k_1=0$, and we have $\tilde{A}_T = \left(\frac{2}{1-i\beta}\right)\tilde{A}_I$,

$\tilde{A}_R = \left(\frac{1+i\beta}{1-i\beta}\right)\tilde{A}_I$, where $\beta \equiv \frac{m\omega^2}{k_1T} = \frac{m(k_1v_1)^2}{k_1T} = \frac{mk_1}{T}\frac{T}{\mu_1}$, or $\boxed{\beta = m\frac{k_1}{\mu_1}.}$ Now $\left(\frac{1+i\beta}{1-i\beta}\right) = Ae^{i\phi}$, with

$A^2 = \left(\frac{1+i\beta}{1-i\beta}\right)\left(\frac{1-i\beta}{1+i\beta}\right) = 1 \Rightarrow A=1$, and $e^{i\phi} = \frac{(1+i\beta)^2}{(1-i\beta)(1+i\beta)} = \frac{1+2i\beta-\beta^2}{1+\beta^2} \Rightarrow$

$\tan\phi = \frac{2\beta}{1-\beta^2}$. Thus $A_Re^{i\delta_R} = e^{i\phi}A_Ie^{i\delta_I} \Rightarrow \boxed{A_R=A_I,}\ \boxed{\delta_R = \delta_I+\tan^{-1}\left(\frac{2\beta}{1-\beta^2}\right).}$

Similarly, $\left(\frac{2}{1-i\beta}\right) = Ae^{i\phi} \Rightarrow A^2 = \left(\frac{2}{1-i\beta}\right)\left(\frac{2}{1+i\beta}\right) = \frac{4}{1+\beta^2} \Rightarrow A = \frac{2}{\sqrt{1+\beta^2}}.$

Problem 9.7 Suppose string 2 is embedded in a viscous medium (such as molasses), which imposes a drag force that is proportional to its (transverse) speed:

$$\Delta F_{\text{drag}} = -\gamma \frac{\partial f}{\partial t} \Delta z.$$

(a) Derive the modified wave equation describing the motion of the string.

(b) Solve this equation, assuming the string vibrates at the incident frequency ω. That is, look for solutions of the form $\tilde{f}(z, t) = e^{i\omega t} \tilde{F}(z)$.

(c) Show that the waves are **attenuated** (that is, their amplitude decreases with increasing z). Find the characteristic penetration distance, at which the amplitude is $1/e$ of its original value, in terms of γ, T, μ, and ω.

(d) If a wave of amplitude A_I, phase $\delta_I = 0$, and frequency ω is incident from the left (string 1), find the reflected wave's amplitude and phase.

Problem 9.7

(a) $F = T\frac{\partial^2 f}{\partial z^2}\Delta z - \gamma\frac{\partial f}{\partial t}\Delta z = \mu\Delta z\frac{\partial^2 f}{\partial t^2}$, or $\boxed{T\frac{\partial^2 f}{\partial z^2} = \mu\frac{\partial^2 f}{\partial t^2} + \gamma\frac{\partial f}{\partial t}.}$

(b) Let $\tilde{f}(z,t) = \tilde{F}(z)e^{-i\omega t}$; then $Te^{-i\omega t}\frac{d^2\tilde{F}}{dz^2} = \mu(-\omega^2)\tilde{F}e^{-i\omega t} + \gamma(-i\omega)\tilde{F}e^{-i\omega t} \Rightarrow$

$T\frac{d^2\tilde{F}}{dz^2} = -\omega(\mu\omega + i\gamma)\tilde{F}$, $\frac{d^2\tilde{F}}{dz^2} = -\tilde{k}^2\tilde{F}$, where $\tilde{k}^2 \equiv \frac{\omega}{T}(\mu\omega + i\gamma)$. Solution : $\tilde{F}(z) = \tilde{A}e^{i\tilde{k}z} + \tilde{B}e^{-i\tilde{k}z}$.

Resolve $\tilde{k}$ into its real and imaginary parts: $\tilde{k} = k + i\kappa \Rightarrow \tilde{k}^2 = k^2 - \kappa^2 + 2ik\kappa = \frac{\omega}{T}(\mu\omega + i\gamma)$.

$2k\kappa = \frac{\omega\gamma}{T} \Rightarrow \kappa = \frac{\omega\gamma}{2kT}$; $k^2 - \kappa^2 = k^2 - \left(\frac{\omega\gamma}{2T}\right)^2\frac{1}{k^2} = \frac{\mu\omega^2}{T}$; or $k^4 - k^2(\mu\omega^2/T) - (\omega\gamma/2T)^2 = 0 \Rightarrow$

$k^2 = \frac{1}{2}\left[(\mu\omega^2/T) \pm \sqrt{(\mu\omega^2/T)^2 + 4(\omega\gamma/2T)^2}\right] = \frac{\mu\omega^2}{2T}\left[1 \pm \sqrt{1 + (\gamma/\mu\omega)^2}\right]$. But k is real, so k^2 is positive, so we need the plus sign: $k = \omega\sqrt{\frac{\mu}{2T}}\sqrt{1 + \sqrt{1 + (\gamma/\mu\omega)^2}}$. $\kappa = \frac{\omega\gamma}{2kT} = \frac{\gamma}{\sqrt{2T\mu}}\left[1 + \sqrt{1 + (\gamma/\mu\omega)^2}\right]^{-1/2}$.

Plugging this in, $\tilde{F} = \tilde{A}e^{i(k+i\kappa)z} + \tilde{B}e^{-i(k+i\kappa)z} = \tilde{A}e^{-\kappa z}e^{ikz} + \tilde{B}e^{\kappa z}e^{-ikz}$. But the B term gives an exponentially *increasing* function, which we don't want (I assume the waves are propagating in the $+z$ direction), so $B = 0$, and the solution is $\boxed{\tilde{f}(z,t) = \tilde{A}e^{-\kappa z}e^{i(kz - \omega t)}.}$ (The actual displacement of the string is the real part of this, of course.)

(c) The wave is attenuated by the factor $e^{-\kappa z}$, which becomes $1/e$ when

$z = \frac{1}{\kappa} = \boxed{\frac{\sqrt{2T\mu}}{\gamma}\sqrt{1 + \sqrt{1 + (\gamma/\mu\omega)^2}};}$ this is the characteristic penetration depth.

(d) This is the same as before, except that $k_2 \to k + i\kappa$. From Eq. 9.29, $\tilde{A}_R = \left(\frac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\tilde{A}_I$;

$\left(\frac{A_R}{A_I}\right)^2 = \left(\frac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\left(\frac{k_1 - k + i\kappa}{k_1 + k - i\kappa}\right) = \frac{(k_1 - k)^2 + \kappa^2}{(k_1 + k)^2 + \kappa^2}$. $\boxed{A_R = \sqrt{\frac{(k_1 - k)^2 + \kappa^2}{(k_1 + k)^2 + \kappa^2}}A_I}$

(where $k_1 = \omega/v_1 = \omega\sqrt{\mu_1/T}$, while k and κ are defined in part b). Meanwhile

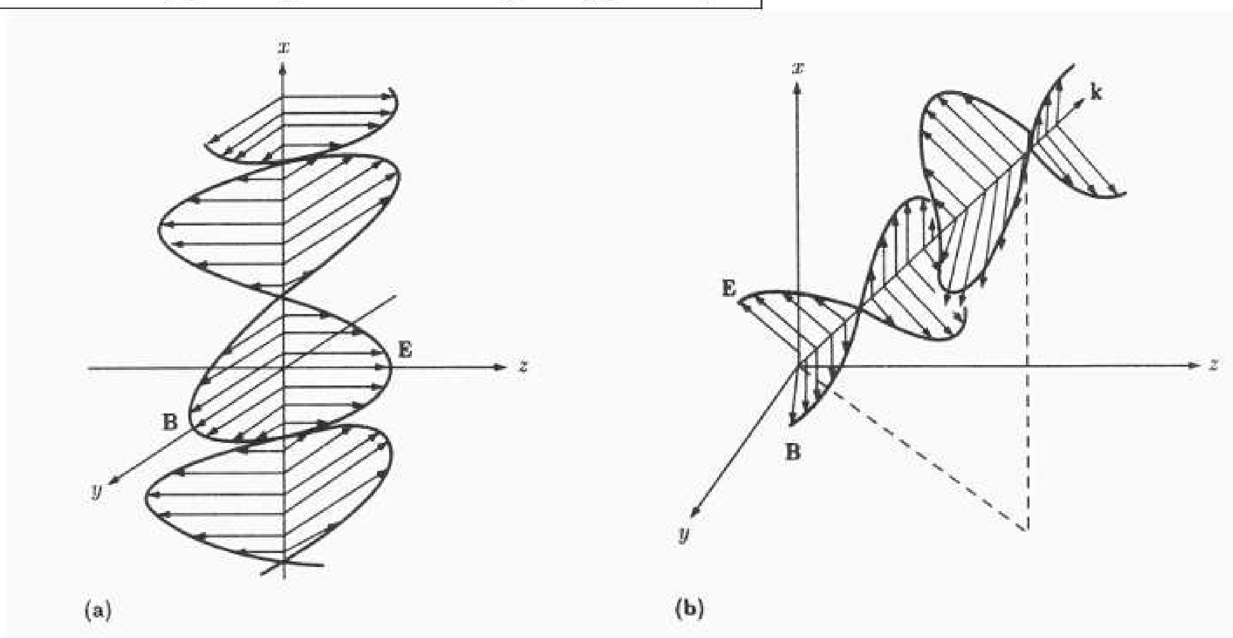
$\left(\frac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right) = \frac{(k_1 - k - i\kappa)(k_1 + k + i\kappa)}{(k_1 + k)^2 + \kappa^2} = \frac{(k_1)^2 - k^2 - \kappa^2 - 2i\kappa k_1}{(k_1 + k)^2 + \kappa^2} \Rightarrow \boxed{\delta_R = \tan^{-1}\left(\frac{-2k_1\kappa}{(k_1)^2 - k^2 - \kappa^2}\right).}$

Problem 9.9 Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to the point (1, 1, 1), with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$.

Problem 9.9

(a) $\boxed{\mathbf{k} = -\frac{\omega}{c}\,\hat{\mathbf{x}};\ \hat{\mathbf{n}} = \hat{\mathbf{z}}.}$ $\mathbf{k}\cdot\mathbf{r} = \left(-\frac{\omega}{c}\,\hat{\mathbf{x}}\right)\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = -\frac{\omega}{c}x;$ $\mathbf{k}\times\hat{\mathbf{n}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}.$

$$\boxed{\mathbf{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{z}};\quad \mathbf{B}(x,t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{y}}.}$$



(b) $\boxed{\mathbf{k} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}}{\sqrt{3}}\right);\ \hat{\mathbf{n}} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{2}}.}$ (Since $\hat{\mathbf{n}}$ is parallel to the $x\,z$ plane, it must have the form $\alpha\,\hat{\mathbf{x}} + \beta\,\hat{\mathbf{z}}$; since $\hat{\mathbf{n}}\cdot\mathbf{k} = 0$, $\beta = -\alpha$; and since it is a unit vector, $\alpha = 1/\sqrt{2}$.)

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{\sqrt{3}c}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = \frac{\omega}{\sqrt{3}c}(x + y + z);\ \hat{\mathbf{k}}\times\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} - \hat{\mathbf{z}}).$$

$$\boxed{\begin{aligned} \mathbf{E}(x,y,z,t) &= E_0\cos\left[\frac{\omega}{\sqrt{3}c}(x + y + z) - \omega t\right]\left(\frac{\hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{2}}\right); \\ \mathbf{B}(x,y,z,t) &= \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x + y + z) - \omega t\right]\left(\frac{-\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}}}{\sqrt{6}}\right). \end{aligned}}$$

Problem 9.11 Consider a particle of charge q and mass m, free to move in the xy plane in response to an electromagnetic wave propagating in the z direction (Eq. 9.48—might as well set $\delta = 0$).

(a) Ignoring the magnetic force, find the velocity of the particle, as a function of time. (Assume the average velocity is zero.)

(b) Now calculate the resulting magnetic force on the particle.

(c) Show that the (time) average magnetic force is *zero*.

The problem with this naive model for the pressure of light is that the velocity is 90° out of phase with the fields. For energy to be absorbed, there's got to be some *resistance* to the motion of the charges. Suppose we include a force of the form $-\gamma m\mathbf{v}$, for some damping constant γ.

(d) Repeat part (a) (ignore the exponentially damped transient). Repeat part (b), and find the average magnetic force on the particle.[9]

Problem 9.11 The fields are $\mathbf{E}(z,t) = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}}$, $\mathbf{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t)\,\hat{\mathbf{y}}$, with $\omega = ck$.

(a) The electric force is $\mathbf{F}_e = q\mathbf{E} = qE_0 \cos(kz - \omega t)\,\hat{\mathbf{x}} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$, so

$$\mathbf{v} = \frac{qE_0}{m}\,\hat{\mathbf{x}} \int \cos(kz - \omega t)\,dt = -\frac{qE_0}{m\omega}\sin(kz - \omega t)\,\hat{\mathbf{x}} + \mathbf{C}.$$

But $\mathbf{v}_{\text{ave}} = \mathbf{C} = \mathbf{0}$, so $\mathbf{v} = \boxed{-\frac{qE_0}{m\omega}\sin(kz - \omega t)\,\hat{\mathbf{x}}.}$

(b) The magnetic force is

$$\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B}) = q\left(-\frac{qE_0}{m\omega}\right)\left(\frac{E_0}{c}\right)\sin(kz - \omega t)\cos(kz - \omega t)(\hat{\mathbf{x}} \times \hat{\mathbf{y}}) = \boxed{-\frac{q^2E_0^2}{m\omega c}\sin(kz - \omega t)\cos(kz - \omega t)\,\hat{\mathbf{z}}.}$$

(c) The (time) average force is $(\mathbf{F}_m)_{\text{ave}} = -\frac{q^2E_0^2}{m\omega c}\,\hat{\mathbf{z}} \int_0^T \sin(kz - \omega t)\cos(kz - \omega t)\,dt$, where $T = 2\pi/\omega$ is the period. The integral is $-\frac{1}{2\omega}\sin^2(kz - \omega t)\Big|_0^T = -\frac{1}{2\omega}\left[\sin^2(kz - 2\pi) - \sin^2(kz)\right] = 0$, so $\boxed{(\mathbf{F}_m)_{\text{ave}} = \mathbf{0}.}$

(d) Adding in the damping term,

$$\mathbf{F} = q\mathbf{E} - \gamma m\mathbf{v} = qE_0\cos(kz - \omega t)\hat{\mathbf{x}} - \gamma m\mathbf{v} = m\frac{d\mathbf{v}}{dt} \;\Rightarrow\; \frac{d\mathbf{v}}{dt} + \gamma\mathbf{v} = \frac{qE_0}{m}\cos(kz - \omega t)\,\hat{\mathbf{x}}.$$

The steady state solution has the form $\mathbf{v} = A\cos(kz - \omega t + \theta)\,\hat{\mathbf{x}}$, $\frac{d\mathbf{v}}{dt} = A\omega\sin(kz - \omega t + \theta)\,\hat{\mathbf{x}}$. Putting this in, and using the trig identity $\cos u = \cos\theta\cos(u + \theta) + \sin\theta\sin(u + \theta)$,

$$A\omega\sin(kz - \omega t + \theta) + \gamma A\cos(kz - \omega t + \theta) = \frac{qE_0}{m}\left[\cos\theta\cos(kz - \omega t + \theta) + \sin\theta\sin(kz - \omega t + \theta)\right].$$

Equating like terms:

$$A\omega = \frac{qE_0}{m}\sin\theta,\; A\gamma = \frac{qE_0}{m}\cos\theta \Rightarrow \tan\theta = \frac{\omega}{\gamma},\; A^2(\omega^2 + \gamma^2) = \left(\frac{qE_0}{m}\right)^2 \Rightarrow A = \frac{qE_0}{m\sqrt{\omega^2 + \gamma^2}}.$$

So

$$\mathbf{v} = \boxed{\frac{qE_0}{m\sqrt{\omega^2 + \gamma^2}}\cos(kz - \omega t + \theta)\hat{\mathbf{x}},}\;\theta \equiv \tan^{-1}(\omega/\gamma);\, \mathbf{F}_m = \boxed{\frac{q^2E_0^2}{mc\sqrt{\omega^2 + \gamma^2}}\cos(kz - \omega t + \theta)\cos(kz - \omega t)\,\hat{\mathbf{z}}.}$$

To calculate the time average, write $\cos(kz - \omega t + \theta) = \cos\theta\cos(kz - \omega t) - \sin\theta\sin(kz - \omega t)$. We already know that the average of $\cos(kz - \omega t)\sin(kz - \omega t)$ is zero, so

$$(\mathbf{F}_m)_{\text{ave}} = \frac{q^2E_0^2}{mc\sqrt{\omega^2 + \gamma^2}}\,\hat{\mathbf{z}}\,\cos\theta \int_0^T \cos^2(kz - \omega t)\,dt.$$

The integral is $T/2 = \pi/\omega$, and $\cos\theta = \gamma/\sqrt{\omega^2 + \gamma^2}$ (see figure), so $(\mathbf{F}_m)_{\text{ave}} = \boxed{\frac{\pi\gamma q^2E_0^2}{m\omega c(\omega^2 + \gamma^2)}\,\hat{\mathbf{z}}.}$