Indian Institute of Information Technology Vadodara
MA 102: Linear Algebra and Matrices
Tutorial 7

1. Find the eigenvalues and eigenvectors of the following matrices. How many maximal linearly independent eigenvectors does a matrix have?
   a) $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$, b) $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.
   a) Find a basis of $\mathbb{R}^2$ consisting of v_1 and another eigenvector v_2 of A.
   b) Write x_0 as $v_1 + cv_2$ for some scalar c.
   c) Let $x_k = Ax_{k-1}$ for $k \geq 1$. Compute x_1, x_2 and write a formula for x_k. Show that x_k converges to v_1 as k increases.

3. Construct a 3×3 matrix which has only one eigenvalue but does not have three linearly independent eigenvectors over $\mathbb{R}$.

4. Compute A^{100}, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

5. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ be eigenvectors of $A_{3\times 3}$. Then find eigenvectors of A^T.

6. Let $B = \{e_1, e_2, e_3\}$ be standard basis of $\mathbb{R}^3$ and $\{v_1, v_2, v_3\}$ be another basis of $\mathbb{R}^3$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as $T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = (x_3 - x_2)v_1 - (x_1 + x_3)v_2 + (x_1 - x_2)v_3$. Find maximal linearly independent eigenvectors of T.

7. Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$, where $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

8. Let $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ with eigenvalue $4 - i$. Find $u, v \in \mathbb{R}^2$ such that $Au = 4u + v, Av = -u + 4v$. Compute $A(u + iv)$.

9. Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $Ax = \lambda x$ for some nonzero vector x in $\mathbb{C}^n$ then in fact, λ is real number and the real part of x is an eigenvector of A. (Show that $\overline{x}^T Ax \in \mathbb{R}$ and examine the real and imaginary parts of Ax.)