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$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a(d) - b(c)$$

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$$C_{ij} := (-1)^{i+j} det(A_{ij})$$

Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

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Corollary

$$det(A) = det(A^T)$$

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Note: If a computer performs one trillion multiplications per second, it would have to run for more than 500,000 years to compute determinant of a 25×25 matrix.
We need better "definition" or properties of determinant to calculate determinant easily.

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Definition (Determinant)

Let $\mathbb{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrix with entries from $\mathbb{R}$. Then det: $\mathbb{M}_n(\mathbb{R}) \to \mathbb{R}$ is a function which satisfies

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Verify your definition of determinant satisfies all 3 properties.

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Another definition of determinant: Let $A = [a_{ij}]$ be a $n \times n$ matrix.

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

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Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

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$$A^{-1} = \frac{1}{\det(A)}[c_{ij}]^T,$$

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(3) It also helps you to find solution to the linear system $AX = b$, when A is invertible matrix.

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Determinant and REF

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges only.
Then
$$det(A) = (-1)^r det(U),$$
where r is the no. of row interchanges required to get $U =$REF(A).

Determinant of Block diagonal matrices

Let $A = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_r \end{bmatrix}$ be a matrix in block diagonal diagonal form, where A_i are submatrices of A.

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Let $X = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. Then
$det(X) = det(A).det(D)$
$M = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} . \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} . \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$
Therefore, $det(X) = det(A).det(D)$.

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