Triangular factorization: $A = LU$ Given a square matrix A, we can find L a lower Triangular matrix with 1s on the diagonal and U upper Triangular matrix such that $A = LU$.

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

Calculation of A^{-1}: Gauss-Jordan Method

Let $AA^{-1} = I$

If X_1 is a first column of A^{-1} then $AX_1 = e_1$, where $e_1 = [1\ 0\ \cdots\ 0]^T$.

We can use Gaussian Elimination to find X_1−first column of A^{-1}.

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We can use Gaussian Elimination to find X_1−first column of A^{-1}.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$.

Consider

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

Perform elementary row operations on $[A|I]$ to convert A into I.

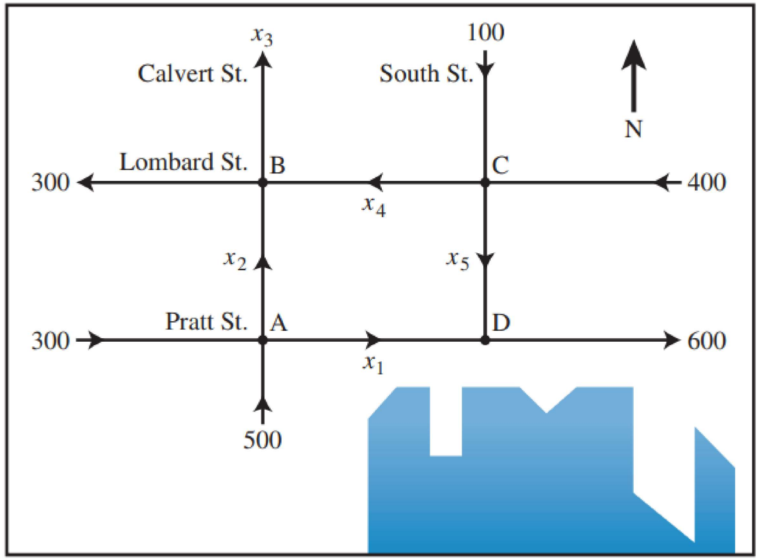
Then I will change to A^{-1}.

Write down the solutions of $AX = 0$ in parametric form where $A =$
$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chemical Equations

Consider a chemical reaction of propane gas with oxygen to form carbon dioxide and water.
$x_1 C_3H_8 + x_2 O_2 = x_3 CO_2 + x_4 H_2O$,
where x_i are the no. of molecules

Network flow



$A_{m\times n} b_{m\times 1} = \begin{bmatrix} \cdot \end{bmatrix}_{m\times 1}$

$Ax = b$

Vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ is a vector in } \mathbb{R}^2.$$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ is a vector in } \mathbb{R}^3.$$

Vector addition and scalar multiplication are done like matrices.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

Linear Combinations

Consider a linear system Q. Does this vector eq^n have a sol^n?

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

It can be written as

$$\underline{x_1 \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

LHS of $=$ is called linear combination of

$$u = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix} \;\&\; w = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}$$

Above equation is called vector equation.

Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$ where C_i is i^{th} column of A.

1st coln of A

last coln of A

Linear system and Vector Equation

A linear system $AX = b$ is same as $x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$ where C_i is i^{th} column of A.

Theorem

$AX = b$ has a solution if and only if b is a linear combination of columns of A.

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Theorem

$AX = b$ has a solution if and only if b is a linear combination of columns of A.

Hence we need to study linear combination of columns of A.
Column space of A $=$ set of all linear combinations of columns of A.

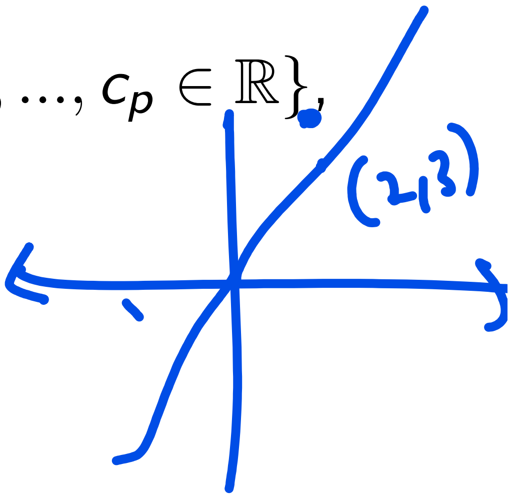
$$Col(A) = \{x_1 C_1 + x_2 C_2 + \cdots + x_n C_n | x_i \in \mathbb{R} \text{ for all } i\}$$

$\therefore\ AX = b$ has a solution iff $b \in Col(A)$.

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.

$$\left[\begin{array}{cc|c} 2 & 1 & r \\ 3 & 0 & s \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & r \\ 0 & -\frac{3}{2} & s-\frac{3}{2}r \end{array}\right] \leftarrow \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d \\ 3c \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

If $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$, then
$\text{Span}\,\{v_1, v_2, \ldots, v_p\} = \{c_1 v_1 + c_2 v_2 + \ldots + c_p v_p \mid c_1, c_2, \ldots, c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, \ldots, v_p$.



Let

$$d = -\frac{2}{3}\left(s - \frac{3}{2}r\right)$$

$$c = \frac{1}{2}\left(r + \frac{2}{3}\left(s - \frac{3}{2}r\right)\right)$$

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{Span}\,\{v\} = \{c.v \mid c \in \mathbb{R}\} = \left\{\begin{bmatrix} c \\ 0 \end{bmatrix} \mid c \in \mathbb{R}\right\}$$

$$v = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{Span}\{v\} = \left\{\begin{bmatrix} 2c \\ 0 \end{bmatrix} \,\middle|\, c \in \mathbb{R}\right\} = X \text{ axis}$$

$$\text{Span}\{u\} = \left\{\begin{bmatrix} 2c \\ 3c \end{bmatrix} \,\middle|\, c \in \mathbb{R}\right\} \quad 3x - 2y = 0$$

$$y = \frac{3}{2}x$$

$$\mathbb{R}^2 = \text{Span}\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 2 \\ 3 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \end{bmatrix} \,\middle|\, c, d \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} 2c+d \\ 3c \end{bmatrix} \,\middle|\, c, d \in \mathbb{R}\right\}$$

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.
Let

$$u = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$ Let $u \in \mathbb{R}^n$. Then

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ no sol^n

$span\{u\} = \{x.u | x \in \mathbb{R}\}$.
Geometrically it is a line passing through origin and point u.
Let $u, v \in \mathbb{R}^2$. Then $span\{u, v\} = \{x.u + y.v | x, y \in \mathbb{R}\}$.
Geometrically, what will it be?

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has a sol^n iff $\underline{c=0}$

If $v_1, v_2, ..., v_p \in \mathbb{R}^n$, then
Span $\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p | c_1, c_2, ..., c_p \in \mathbb{R}\}$,
the set of all linear combinations of $v_1, v_2, ..., v_p$.
Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Span $\{v\} = \{c.v | c \in \mathbb{R}\} = \{\begin{bmatrix} c \\ 0 \end{bmatrix} | c \in \mathbb{R}\}$ Let $u \in \mathbb{R}^n$. Then
$span\{u\} = \{x.u | x \in \mathbb{R}\}$.
Geometrically it is a line passing through origin and point u.
Let $u, v \in \mathbb{R}^2$. Then $span\{u, v\} = \{x.u + y.v | x, y \in \mathbb{R}\}$.
Geometrically, what will it be?
For a matrix A,
$Col(A) := span\{C_1, C_2, \ldots, C_n\} \subseteq \mathbb{R}^m$

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

$Col(A) = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right\} = \mathbb{R}^2$

$$\underbrace{\begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \qquad \left[\begin{array}{ccc|c}1 & 2 & 3 & b_1\\0 & 1 & 1 & b_2\end{array}\right]$$

$x_3 = c$
$x_2 = b_2 - c$
$x_1 = b_1 - c - 2b_2$

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent

1. *For each $b \in \mathbb{R}^m$, the equation $AX = b$ has a solution.* $= Col(A) = \mathbb{R}^m$
2. *Each $b \in \mathbb{R}^m$, is a linear combination of the columns of A.*
3. *$Col(A) = \mathbb{R}^m$.*
4. *A has a pivot position in every row.*

REF(A) has that row completely zero.

$$\underbrace{\begin{bmatrix}2 & 1\\3 & 0\\0 & 0\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \left[\begin{array}{cc|c}2 & 1 & 1\\3 & 0 & 1\\0 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|c}2 & 1 & 1\\0 & -3/2 & -1/2\\0 & 0 & 1\end{array}\right]$$

No pivot entry.

$\parallel$
$b=0$

$A(0) = 0$
trivial sol^n.

The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the REF(A) has at least one free variable.
Suppose augmented form $[A|0]$ is

$$\underbrace{\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]}_{A} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = c$
$x_2 = 0$
$x_1 = \frac{4}{3} c$

$c = 3$ $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ is a nontrivial sol^n to $AX=0$

Homogeneous Linear System

The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the REF(A) has at least one free variable.
Suppose augmented form $[A|0]$ is

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

Non homogeneous linear system

particular sol^n

Theorem

Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution (i.e., $Ap = b$). Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

sol^n of also H.L.S

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 7 \end{bmatrix} = b \qquad p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}c \\ 0 \\ c \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} 1+\frac{4}{3}c \\ 1 \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 8 \\ -5 \\ 7 \end{bmatrix}}_{b}$$

$$3\left(1+\frac{4}{3}c\right)+5.1-4c$$

Non homogeneous linear system

Theorem

Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution (i.e., $Ap = b$). Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Linear Independence

$AX = 0$

If $AX = 0$ has only one soln (i.e. $X = 0$)

Let A be a matrix of order $m \times n$. We say columns of A to be **linearly independent** if $AX = 0$ has **only trivial (0) solution**.

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 5 & 6 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -9 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{9}{2}R_2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Leftarrow$ No free variable $\Leftarrow$

$AX = 0$ has only trivial soln

Linear Independence

$$\text{Span}\left\{\begin{bmatrix}2\\3\end{bmatrix}\right\} = \left\{c\cdot\begin{bmatrix}2\\3\end{bmatrix} \,\middle|\, c\in\mathbb{R}\right\} = \left\{\begin{bmatrix}2c\\3c\end{bmatrix} \,\middle|\, c\in\mathbb{R}\right\} = \text{line } y=\tfrac{3}{2}x$$

Let A be a matrix of order $m \times n$. We say columns of A to be linearly independent if $AX = 0$ has only trivial (0) solution.

Definition

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ is said to be linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$$

has only trivial solution.