Indian Institute of Information Technology Vadodara
MA 102: Linear Algebra and Matrices
Tutorial 8

1. Consider a polynomial $p(t) = a_0 + a_1 t + \cdots + a_3 t^3 + t^4$ and a matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}$. Find a connection between characteristic polynomial of A and $p(t)$.

2. Can you find a set of orthonormal eigenvectors of $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$.

3. Find the distance between $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

4. Let $u = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, and let W be the set of all vectors of $\mathbb{R}^3$ which are orthogonal to u. Show that W is a subspace of $\mathbb{R}^3$ and find an orthogonal basis of W. Give an orthogonal basis of $\mathbb{R}^3$ containing u.

5. Verify the parallelogram law for vectors u and v:
$||u + v||^2 + ||u - v||^2 = 2.||u||^2 + 2.||v||^2$

6. Compute the orthogonal projection of $u = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ on to the line $L = \{\begin{bmatrix} -2t \\ t \end{bmatrix} | t \in \mathbb{R}\}$ and the shortest distance of u to the line, L.

7. Write $v = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$ as a linear combination of u_i, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

8. Let $W = \text{span}\{u_1, u_2\}$ and $U = [u_1 \;\; u_2]$. Compute $UU^T, U^TU, \text{Proj}_W y, UU^Ty$. What do you observe?

$$\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

9. Let $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 2 & -2 & 9 \\ 4 & -14 & -3 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $p, u \in \mathbb{R}^3$ such that $x = p + u, p \in \text{Row}(A), u \in \text{Null}(A)$.

10. Find an orthonormal basis-$\{u_1, u_2, u_3\}$ for the column space of the matrix $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.