Standard Normal Distribution

13 April 2024 09:31

Before Discussion letus start some properties of Normal Distribution:

(I) $X \sim N(\mu, \sigma^2)$

(i) Normal distⁿ has no shape parameter. It shape is fixed it is always bell shaped

(ii) It is symmetric which mean, median & mode are same.

(iii) coeff of skewness $\beta_1 = 0$

(iv) $E(X - E(X))^n = E(X-\mu)^n = 0$, $n = \text{odd}$

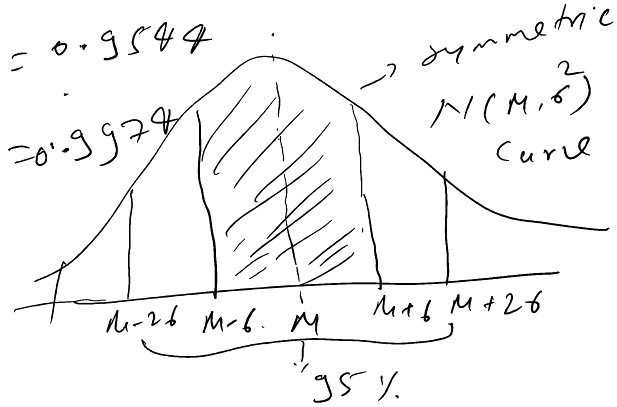
Area properties of any $N(\mu, \sigma^2)$ distⁿ :-

Let $X \sim N(\mu, \sigma^2)$

(i) $P(\mu - \sigma \le X \le \mu + \sigma) = 0.6826$

(ii) $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = 0.9544$

(iii) $P(\mu - 3\sigma \le X \le \mu + 3\sigma) = 0.9974$



Cumulative Distⁿ funcⁿ (CDF)

$X \sim N(\mu, \sigma^2)$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\frac{x-\mu}{\sigma} = t \quad \Rightarrow \quad \frac{dx}{\sigma} = dt$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= \int_{-\infty} \sigma \frac{1}{\sqrt{2\pi}}$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right) \rightarrow$$

Next we learn the some properties of $\Phi(z)$:

Considering $Z = \frac{X - E(X)}{\sqrt{V(X)}}$ ($E(X)$ → expectation, $V(X)$ → variance), then r.v.s Z is known as standardized version of random variable X.

$X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$, & Z is known as standard Normal Distribution.

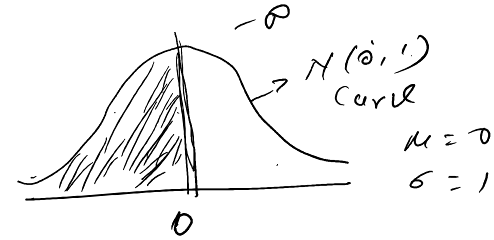
PDF: $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}$

CDF: $F(z) = \Phi(z)$

Properties of CDF:

(i) $\Phi(0) = \frac{1}{2}$, $\quad P(Z \leq 0) = \int_{-\infty}^{0} f(z)\,dz$

$\Phi(z) = P(Z \leq z)$



(ii)

$X \sim N(\mu, \sigma^2)$

CDF: $F(x) = P(X \leq x)$

$= P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$

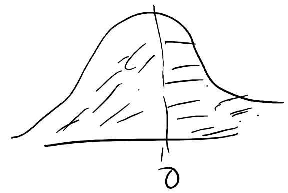
$= P\left(Z \leq \frac{x-\mu}{\sigma}\right)$

$= \Phi\left(\frac{x-\mu}{\sigma}\right)$

$\Phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}$

$= \underline{\underline{Y}}$

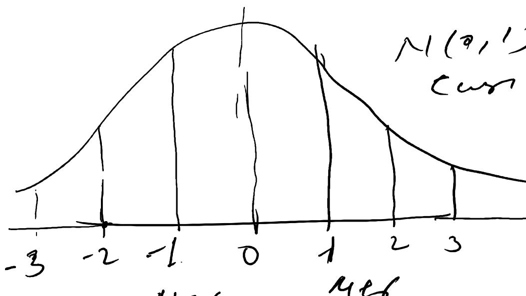
(ii) $\Phi(-z) = 1 - \Phi(z)$



$$\Phi(-z) = P(Z \le -z)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-t^2/2}\, dt \qquad \text{Let } -t = u$$
$$= \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-u^2/2}\, du \qquad t = -z \quad u = z, \quad t = -\infty \quad u = \infty, \quad -dt = du$$
$$= P(Z \ge z)$$
$$= 1 - P(Z \le z)$$
$$= \underline{1 - \Phi(z)}$$
proved.

Area properties of standard Normal $\underline{N(0,1)}$ Dist^n:-

$Z \sim N(0,1)$



$P(-1 < Z \le 1) = 0.6826$

$P(-2 < Z \le 2) = 0.9544$

$P(-3 < Z \le 3) = .9974$

$[-3, 3]$ cover 99.74 % of observation of a $N(0,1)$ dist^n

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what about any $N(\mu, \sigma^2)$ dist^n

(The main idea is "probability for any $N(\mu, \sigma^2)$ dist^n can be obtained using $N(0,1)$ CDF $\Phi(z)$

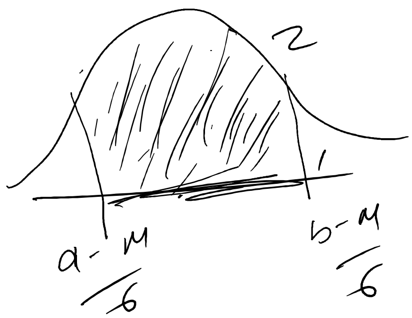
$X \sim N(\mu, \sigma^2)$

$\ldots -\mu, \ b-\mu \ )$

$X \sim N(\mu, \sigma^2)$

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$



$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

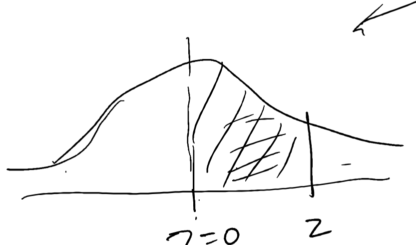
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

0.00 0.01 0.02 0.03

$Z \sim N(0,1)$

Exp: $P(0 \le Z < 1.5) = 0.4932$

z-table → left table
Right table
full table



mean = median = mode

$P(-3 < Z < 0)$ → symmetric

$= P(0 < Z < 3)$

$=$

$P(0 \le Z < 1.53) = 0.4370$

1.54

Ex: $P(Z \le 1.69) = P(-\infty < Z \le 1.69)$

$= P(-\infty < Z < 0) + P(0 \le Z \le 1.69)$

$= 0.5 + 0.4545$

$= 0.9545$

$P(-\infty < Z < \infty) = 1$

$= 0.9545$ Ans

$P(-\infty < z < \infty) = 1$

symmetric

$P(-\infty < z < 0) = \frac{1}{2}$

Exp: $X \sim N(9, 9) \Rightarrow \mu = 9, \quad \sigma = 3$ ($\sigma^2 = 9$)

(i) $P(X \geq 15)$ (ii) $P(X \leq 15)$ (iii) $P(0 < X \leq 9)$

$P(X \geq 15) = P\left(\frac{X-\mu}{\sigma} \geq \frac{15-\mu}{\sigma}\right)$

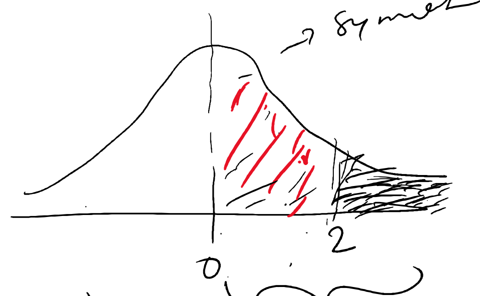
$= P\left(Z \geq \frac{15-9}{3}\right)$

$= P(Z \geq 2)$

$= 0.5 - P(0 < Z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228$ Ans



$P(0 < Z < \infty) = 0.5$

$\boxed{P(X < x) = 1 - P(X \geq x)}$

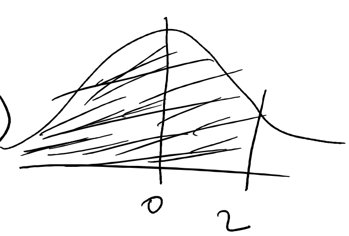
$P(X \leq 15) = 1 - P(X \geq 15) = 1 - 0.0228$

$= 0.9772$

M2

$P(X \leq 15) = P(Z \leq 2)$

$= 0.5 + P(0 \leq Z \leq 2)$



$= 0.5 + 0.4772$

$= 0.9772$

(iii) $P(0 \leq X \leq 9) = P\left(\frac{0-9}{3} \leq Z \leq 0\right)$

$X \sim N(9, 9)$ $= P(-3 \leq Z \leq 0) =$

(iii) P(

$X \sim N(9, 9)$ $\quad = P(-3 \le Z \le 0) =$

$\mu = 9,\ \sigma = 3$ $\quad = P(0 \le Z \le 3)$

$= 0.4987$

$= P\left(\frac{0-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{9-\mu}{\sigma}\right)$

$= P\left(\frac{0-9}{3} \le Z \le \frac{9-9}{\sigma}\right) = P(-3 < Z \le 0)$

P1) If the rv X is distributed as $N(\mu, \sigma^2)$ identify the constant C in terms of μ & σ for which

$$P(X \le C) = 2 - 9P(X > C)$$

P2) The distn of I.Qs of the people in a given group is approximated by the Normal distn with $\mu = 105$ & $\sigma = 20$. What is the proportion of the individuals in the group in questions has an IQ: (i) At least 50?

(ii) At most 80

(iii) Between 95 & 125.

P3)