INDIAN INSTITUTE OF TECHNOLOGY PATNA

MA501 ASSIGNMENT - II

SUBMITTED BY,

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M.Tech AI & DSE

1). consider probability mass function $p(X=x) = \frac{x}{15}$,

$x = 1,2,3,4,5$; $= 0$ otherwise. Find

(i) $P(X = 1$ or $2)$, and

(ii) $P(1/2 < 5/2 \mid x > 1)$

Addition Rule of probability is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(X = 1 \text{ or } 2) = P(X=1) \cup P(X=2) - \underbrace{P(X=1) \cdot P(X=2)}_{\text{Ignore}}$$

$$= P(X=1) + P(X=2) - 0$$

Total no. is 15.

if $x = 1$, and 2 in $\frac{x}{15}$.

$$= \frac{1}{15} + \frac{2}{15}$$

$$= \frac{3}{15} = \frac{1}{5}$$

$$\boxed{P(X=1 \text{ or } 2) = 1/5}$$

(ii) $P(1/2 < \frac{5}{2} \mid x > 1)$

Apply conditional probability

$$\boxed{P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A)}}$$

$$P(1/2 < \frac{5}{2} \mid x > 1) = \frac{P(1/2 < \frac{5}{2} \cap x > 1)}{P(x > 1)}$$

When we simplify upper section

$$= \frac{P(1/2 < x < \frac{5}{2} \cap x > 1)}{P(x > 1)}$$

$= P(x) =$ minimum is 1 and greater than 1 is 2

$$= \frac{P(x=2)}{[P(x=2) + P(x=3) + P(x=4) + P(x=5)}$$

$$= \frac{2/15}{\frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15}} = \frac{2/15}{\frac{14}{15}}$$

$$= \frac{\cancel{2}}{\cancel{15}} \times \frac{\cancel{15}}{\cancel{14}\ 7} = 1/7$$

$$\boxed{P(1/2 < x < 5/2 \mid x > 1) = 1/7 = 1/7}$$

(2) If A and B are independent events then show that (i) A and B^c (ii) A^c and B (iii) A^c and B^c are also independent

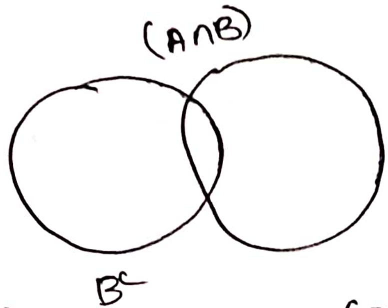
(i) A and B^c

Solution:-

Given A & B are independent events

disjoint events.

$$P(A \cap B) = P(A).P(B) \quad \text{———} \quad (1)$$



$$P(A \cap B^c) = P(A) - P(AB)$$

$$= P(A) - P(A).P(B) \quad \text{- - - - - -} \quad (2)$$

$$= P(A)(1 - P(B))$$

$$\boxed{P(A \cap B^c) = P(A) \cdot P(B^c)}$$

(ii) A^c & B are independent events. Lets prove

$$P(A^c \cap B) = P(A^c) \cdot P(B)$$

also

$$= P(B) - P(A \cap B)$$

apply eqn(1)

$$= P(B) - P(A).P(B)$$

$$= P(B)[1 - P(A)]$$

$$\boxed{P(A^c \cap B) = P(B) \cdot P(A^c)}$$

(iii) A^c and B^c are Independent events

Lets prove

$$P(A^c \cap B^c) = P(A^c) . P(B^c)$$

Apply De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$

$$P(A \cup B)^c = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A).P(B)]$$

$$= 1 - P(A) - P(B) + P(A).P(B)$$

apply eqn 1

$$= (1 - P(A)).(1 - P(B))$$

$$\boxed{P(A^c \cap B^c) = (1 - P(A)).(1 - P(B))}$$

(3) Let X be a random variable with PDF
$f(x) = kx, \quad 0 \leq x 1; = k, \quad 1 \leq x < 2;$
$-kx + 3k, \quad 2 \leq x < 3, = 0$; otherwise
determine (i) constant k (ii) CDF

The given PDF is

$$f(x) = \begin{cases} kx & 0 \le x < 1 \\ k & 1 \le x < 2 \\ -kx+3k & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$$

To find k, we integrate the PDF over each interval and set the result equal to 1.

(i.e) $$\int_{-\infty}^{\infty} f_x(x) = 1 \quad \text{——— (1)}$$

now apply for each interval

$$\int_0^1 Kx\,dx + \int_1^2 k\,dx + \int_2^3 (-Kx+3k)\,dx = 1$$

$$\frac{k}{2}x^2\Big|_0^1 + kx\Big|_1^2 + \left(-\frac{k}{2}x^2+3kx\right)\Big|_2^3 = 1$$

$$\frac{k}{2}(1)^2 + k(2-1) + \left(-\frac{k}{2}(3)^2+3k(3)\right) - \left(-\frac{k}{2}(2)^2+3k(2)\right) = 1$$

$$\frac{k}{2} + k + \left(-\frac{9k}{2} + 9k\right) - \left(-\frac{4k}{2} + 6k\right) = 1$$

$$\frac{k}{2} + k + -\frac{9k}{2} + 9k + \frac{4k}{2} - 6k = 1$$

$$= \frac{k}{2} + k - \frac{9k}{2} + 9k + \frac{41k}{2} - 6k$$

$$= \frac{k}{2} + k - \frac{5k}{2} + 3k$$

$$= \frac{k + 2k}{2} - \frac{5k + 6k}{2}$$

$$= \frac{k + 2k - 5k + 6k}{2}$$

$$= \frac{\cancel{4}k}{\cancel{2}} 2$$

$$= 2k$$

$$\boxed{\int_{-\infty}^{\infty} f_x(x) = 2k}$$

so,

$$2k = 1 \rightarrow (2)$$

$$\boxed{k = 1/2}$$

(ii) $0 \le x < 1$

For $-\infty < x < 0$

$$F(x) = \int_{-\infty}^{\infty} f_x \, dx = \int_{-\infty}^{\infty} 0 \, dx = 0.$$

$$F(x) = \int_{-\infty}^{0} f(x) dx + \int_{0}^{x} kx \, dx$$

$$= 0 + \left[\frac{kx^2}{2}\right]_0^x$$

$= 0 + \frac{kx^2}{2}$.

Apply $\boxed{k = 1/2}$

$= 0 + \frac{x^2 (1/2)}{2} = 0 + \frac{1/2 (x)^2}{2} = \frac{1}{4} x^2$ — (3)

For $0 \le x < 1$ $F(x) = \frac{1}{4} x^2$

For $1 \le x < 2$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} \frac{1}{2} k\,dt + \int_{1}^{x} \frac{1}{2}\,dt$$

$$= \frac{1}{2} \times \frac{1}{2} (1-0) + \frac{1}{2}(x-1)$$

$$= \frac{1}{4} + 1/2 (x-1)$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{1 + 2x - 2}{4} = \frac{2x-1}{4} \text{ (or)}$$

$$= \frac{1}{4} - \frac{2}{4} + \frac{x}{2}$$

$$= \frac{-1}{4} + \frac{x}{2}$$

for $2 \le x \le 3$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \left[\frac{k x^2}{2}\right]_0^1 + \left[kx\right]_1^2 + \left[-\frac{kx^2}{2} + 3kx\right]_2^x$$

$$= \frac{k}{2} + (2k - k) + \left[-\frac{k(x^2)}{2} + 3kx - \frac{2^2 \cdot 4k}{2} + 6k\right]$$

$$= \frac{k}{2} + k - \frac{kx^2}{2} - 4k + 3kx$$

$$= \frac{k}{2} + \frac{2k}{2} - \frac{kx^2}{2} - \frac{8k}{2} + 3kx$$

$$= \frac{-5k}{2} - \frac{kx^2}{2} + 3kx$$

apply $k = 1/2$ $$= \frac{-5}{4} - \frac{x^2}{4} + \frac{3x}{2}$$

$$= \frac{-5 - x^2 + 6x}{4}$$

$$\boxed{F(x) = \frac{6x - x^2 - 5}{4}}$$

or $3 \le x < \infty$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$= 0 + \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} kx\,dx + \int_{1}^{2} k\,dx + \int_{2}^{3} (-kx+3k)\,dx + \int_{3}^{x} 0\,dx$$

$$= 0 + \left[\frac{kx^2}{2}\right]_0^1 + \left[k(x)\right]_1^2 + \left[-\frac{kx^2}{2} + 3kx\right]_2^3 + 0$$

$$= \frac{k}{2} + (2k-1) + -\frac{9k}{2} + 9k + \frac{4k}{2} - 6k$$

$$= \frac{k}{2} + k + 3k + \left(-\frac{5k}{2}\right)$$

$$= \frac{k}{2} + k + 3k - \frac{5k}{2}$$

$$= -\frac{4k}{2} + 4k$$

apply = $k = 1/2$

$$= -\frac{4}{4} + \frac{4}{2}$$

$$= \frac{-4+8}{4} = \frac{4}{4} = 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ & 0 < x \le 1 \\ \frac{x}{2} - \frac{1}{4} & 1 < x < 2 \\ -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4}, & \\ & 3 < x < \infty \end{cases}$$

(4) The diameter say X_i of an electric [illegible] is assumed to be a continous random Variable with PDF $f(x) = 6x(1-x)$, $0 \le x \le 1$ (i) obtained CDF of x,
(ii) $P(X \le 1/2 \mid 1/3 \le X \le 2/3)$
(iii) Determine k such that $P(X < k) = P(X$ [illegible]

(i) CDF of x

$$f(x) = 6x(1-x), \quad 0 \le x \le 1$$

for,

$$-\infty < x < 0$$

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} 0 \,dx = 0$$

for $0 < x < 1$

$$f(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} 6x(1-x)\,dx$$

$$= 0 + \left[\frac{\overset{3}{\cancel{6}}x^2}{\cancel{2}}\right]_0^x - \left[\frac{\overset{2}{\cancel{6}}x^3}{\cancel{3}}\right]_0^x$$

$$= \left[3x^2\right]_0^x - \left[2x^3\right]_0^x$$

$$= 3(x^2 - 0) - 2(x^3 - 0)$$

$$= \underline{3x^2 - 2x^3}$$

for $x > 1$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^x f(x)\,dx$$

$$= 0 + \int_0^1 6x\,(1-x)\,dx + \int_1^x 0\,dx$$

$$= \left[3x^2 - 2x^3\right]_0^1$$

$$= 1 + 0$$

$$= \underline{1}$$

PDF function is

$$F(x) = \begin{cases} 0 & x > 0 \\ 3x^2 - 2x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

(ii) $P\left(x \le \frac{1}{2} \,\middle|\, \frac{1}{3} \le x \le \frac{2}{3}\right) = \dfrac{P(x \le 1/2 \wedge \frac{1}{3} \le \dots}{P(\frac{1}{3} \le x \le \frac{2}{3})}$

RHS:

$$= \frac{P\left(\frac{1}{3} \le x \le 1/2\right)}{P\left(1/3 \le x \le \frac{2}{3}\right)}$$

Let's find upper section.

$$P\left(\frac{1}{3} \le x \le 1/2\right) = \int_{1/3}^{1/2} f(x)\,dx$$

$$= \int_{1/3}^{1/2} 6x(1-x)\,dx$$

$$= \left[3x^2 - 2x^3\right]_{1/3}^{1/2}$$

$$= 3 \times 1/4 - 2 \times \frac{1}{8} - 3 \times \frac{1}{9} + 2 \times \frac{1}{27}$$

$$= \frac{3}{4} - \frac{1}{4} - \frac{3}{9} + \frac{2}{27}$$

$$= \frac{1}{2} - \frac{9}{27} + \frac{2}{27}$$

$$= \frac{1}{2} - \frac{7}{27} = \frac{27-14}{54} = \frac{13}{54}$$

Let's find Lower section

$$P\left(\frac{1}{3} \le x \le \frac{2}{3}\right)$$

$$= \int_{1/3}^{2/3} f(x)\,dx = \int_{1/3}^{2/3} 6x(1-x)\,dx$$

$$= \left(3x^2 - 2x^3\right)_{1/3}^{2/3}$$

$$= 3 \times \frac{4}{9} - 2 \times \frac{8}{27} - 3 \times \frac{1}{9} + 2 \times \frac{1}{27}$$

$$= \frac{12}{9} - \frac{16}{27} - \frac{3}{7} + \frac{2}{27}$$

$$= 1 - \frac{14}{27}$$

$$= \frac{13}{27}$$

$$\frac{\text{Upper section}}{\text{Lower section}}$$

$$= \frac{13/54}{13/27} = \frac{\cancel{13}}{\cancel{54}_{2}} \times \frac{\cancel{27}}{\cancel{13}} = \frac{1}{2}$$

(iii) $P(X < k) = P(X > k)$

$F(k) = 1 - F(k)$

$2F(k) = 1$

$\boxed{F(k) = 1/2}$

$$(3k^2 - 2k^3) = 1/2 \longrightarrow ①$$

$$2k^3 - 3k^2 + 1/2 = 0$$

$$4k^3 - 6k^2 + 1 = 0$$

$$(k - 1/2)(4k^2 - 4k - 2) = 0$$

Apply $k = 1/2$ $k = \frac{1 \pm \sqrt{3}}{2}$

It is an irrational

$$\boxed{k = \frac{1}{2}}$$

(5) Given that probability mass function of random variable X is,

x	0	1	2	3
$p(x)$	0.1	0.3	0.5	0.1

Let $Y = x^2 + 2x$

Find (i) the probability mass function of Y
(ii) means and variance of Y.

$$Y = x^2 + 2x$$

From eqn x^2+2x

From pmf

$y(x=0) = 0+0 = 0$ | $P(Y=0) = P(X=0) = 0.1$

$y(x=1) = 1+2 = 3$ | $P(Y=3) = P(X=1) = 0.3$

$y(x=2) = 4+4 = 8$ | $P(Y=8) = P(X=2) = 0.5$

$y(x=3) = 9+6 = 15$ | $P(Y=15) = P(X=3) = 0.1$

$$E(Y) = \sum y_i\, P(Y=x_i) = 0\times 0.1 + 3\times 0.3 + 8\times 0.5 + 15\times 0.1$$

$$= 0 + 0.9 + 4 + 1.5$$

$$= \underline{6.4}$$

$$E(y^2) = \sum y_i^2\, P(Y=x_i)$$

$$= 0^2\times 0.1 + 3^2\times 0.3 + 8^2\times 0.5 + 15^2\times 0.1$$

$$= 0 + 2.7 + 32 + 22.5$$

$$= 2.7 + 54.5$$

$$= 57.2$$

$$\text{Variance } (Y) = E(Y^2) - (E(Y))^2$$

$$= 57.2 - (6.4)^2$$

$$= 57.2 - 40.96$$

$$\boxed{V(Y) = 16.24}$$

(6) A random variable x has the following probability function

x	0	1	2	3	4	5	6	7	8
$p(x_i)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	1[illegible]

(i) Determine value of k.

(ii) Find $P(X \geq 2 \mid 0 < x < 5)$

(iii) What is the smallest value for x which $P(X \leq x) > 0.5$?

(iv) Find mean and median of X.

$$\sum_{i=0}^{8} P(X = x_i) = k + 3k + 5k + 7k + 9k + 11k + 13k + 15k + 17k$$

$$= \underline{81k}$$

Total probability must be equal to 1

hence,

$$K \times 81 = 1$$

$$\boxed{K = 1/81}$$

) $P(X \geq 2 | 0 < x < 5) = \dfrac{P(X \geq 2 \cap 0 < x < 5)}{P(0 < x < 5)}$

$= \dfrac{P(2 \leq x \leq 5)}{P(0 < x < 5)}$

$= \dfrac{P(x=2) + P(x=3) + P(x=4)}{[P(x=1) + P(x=2) + P(x=3) + P(x=4)]}$

$= \dfrac{(5k + 7k + 9k)}{3k + 5k + 7k + 9k}$

$= \dfrac{21k}{24k} = \boxed{\dfrac{21}{24}}$

ii) $P(X \leq x) > 0.5$

$P(x \leq 0) = P(x=0) = k = \dfrac{1}{81}$

$P(X \leq 1) = P(x=0) + P(x=1) = \dfrac{1}{81} + 3k$

$= \dfrac{1}{81} + \dfrac{3}{81} = \dfrac{4}{81}$

$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$= \dfrac{4}{81} + 5k = \dfrac{4}{81} + \dfrac{5}{81} = \dfrac{9}{81} = \boxed{\dfrac{1}{9}}$

$P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= \dfrac{1}{9} + 7k = \dfrac{1}{9} + 7 \times \dfrac{1}{9} =$

$= \dfrac{9}{81} + \dfrac{7}{81} = \boxed{\dfrac{16}{81}}$

$P(X \leq 4) = P(X=0) \cdots\cdots P(X=4)$

$= \frac{16}{81} + 9k = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$

$P(X \leq 5) = P(X=0) \cdots\cdots P(X=5)$

$= \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$

$P(X \leq 6) = P(X=0) \cdots\cdots P(X=6)$

$= \frac{36}{81} + 13k.$

$= \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$

therefore 6 is the smallest values

E

Mean of X is $= E(X) = \sum x f(x)$

$= 0 \times k + 1 \times 3k + 2 \times 5k + 3 \times 7k + 4 \times 9k + 5 \times 11k + 6 \times 13k + 7 \times 15k + 8 \times 17k$

$= 3k + 10k + 21k + 36k + 55k + 78k + 105k + 136k$

$= 444k = \frac{444}{81} = \frac{148}{27} = \boxed{5.48}$

Median of $x = \frac{x_1 + x_2}{2}$

sinle,

$P(x \leq 5) = 0.444 < 0.5$ and

$P(x \leq 6) = 0.6 > 0.5$

So,

$$\boxed{\text{median} = \frac{5+6}{2} = \frac{11}{2} = 5.5}$$