Indian Institute of Technology PATNA

probability, statistics & stochastic processes (MA501)

Assignment NO: 1

Date : 21-03-2024

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M.Tech (AI &DSE)

Jan 2024 Batch.

P1

1) Prove the general addition formula for n events $A_1, A_2 \ldots A_n$

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i<j \\ i,j=1}}^{n} P(A_i \cap A_j)$$

$$+ \sum_{\substack{i=j=k \\ i<j<k}}^{n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n+1} P(A_1 \cap A_2 \ldots A$$

<u>To prove</u>:- General addition formula of probability for 'n' events.

'n' events in sample space 's'

Events = $\{A_1\ A_2 \ldots An\}$

$(P(E))' \rightarrow$ probability of events occur in sample space 's'.

Formula used:- Axiom of probability

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$A = \underbrace{(A \cap B^c) \cup (A \cap B)}$$

↳ are disjoint events

so, $P(A) = P(A \cap B^c) + P(A \cap B)$

B^c = complement of 'B'

$$\underline{P(A \cap B^c)} = \underline{P(A)} - \underline{P(A \cap B)} \rightarrow (1)$$

proved (1)

P_2

Lets, apply mathematical Induction.

In equation (1)

if $\underline{n=1}$

$P(A_1) = P(A_1)$

Let's take $n=2$

$$P(A_1 \cup A_2) = \sum_{i=1}^{2} P(A_i) - \sum_{\substack{i=j=1 \\ (i<j)}}^{2} P(A_i \cap A_j)$$

Lets take LHS,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

equate RHS

$$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

LHS & RHS are proved.

Lets apply $\underline{n=m+1}$ in given equation

$P(A_1 \cup A_2 \cup \ldots \cup A_{m+1}) =$

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i\right) \cup P(A_{m+1})$$

now apply $P(A \cup B)$ eqn

$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) -$$

$$P\left(\bigcup_{i=1}^{m} A_i\right) \cdot \left(P(A_{m+1})\right)$$

now apply the above eqn.

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = \sum_{i=1}^{m} P(A_i) - \sum_{i=j=1}^{m} P(A_i \cap A_j)$$

$$+ (-1)^{m+1} P\left(\bigcap_{i=1}^{m} A_i\right) + P(A_{m+1})$$

$$- P\left[\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right]$$

joining $\sum_{i=1}^{m} P(A_i) + P(A_{m+1})$

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = \sum_{i=1}^{m+1} P(A_i) - \sum_{\substack{i=j=1 \\ i<j}}^{m} P(A_i \cap A_j) +$$

$$\sum_{\substack{i=j=k=1 \\ i<j<k}}^{m} P(A_i \cap A_j \cap A_k) \ldots$$

$$\ldots (-1)^{m+1} \cdot P\left(\bigcap_{i=1}^{m} A_i\right) - P\left[\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right]$$

now we join $(A_i \cap A_j)$ and $(a_i \cap A_{m+1})$

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = \sum_{i=1}^{m+1} P(A_i) - \sum_{\substack{i=j=1 \\ i<j}}^{m} P(A_i \cap A_j) +$$

$$\sum_{i=1}^{m} P(A_i \cap A_{m+1}) + \sum_{\substack{i=j=k=1 \\ i<j<k}}^{m} P(A_i \cap A_j \cap A_k)$$

$$+ \sum_{i=j=1}^{m} P(A_i \cap A_j \cap A_{m+1}) + \ldots (-1)^{m+1} \cdot P\left(\bigcap_{i=1}^{m} A_i\right)$$

continued

$$+(-1)^{m+1} \cdot (-1) P\left(\bigcap_{i=1}^{m+1} A_i\right)$$

At the end we get the value identical to the equation given in question.

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = \sum_{i=1}^{m+1} P(A_i) - \sum_{\substack{i=j=1 \\ i<j}}^{m+1} P(A_i \cap A_j) + \sum_{\substack{i=j=k=1 \\ i<j<k}}^{m+1} P(A_i \cap A_j \cap A_k) \cdots +$$

$$(-1)^{m+1} \cdot P\left(\bigcap_{i=1}^{m} A_i\right) \cdots -$$

This is proving the general addition formula.

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② Boole's Inequality: if $A_1 A_2 \ldots A_n$ are n event connected to a sample space. Then prove that.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad - (1)$$

proof: mathematical Induction method.

Lets apply $n=1$

$$P(A_1) \leq P(A_1) \quad \cdots \quad (2)$$

Lets apply $n=2$

$$P\left(\bigcup_{i=1}^{2} A_i\right) \leq P(A_1) \; P(A_1 \cup A_2)$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

General addition Rule of probability

According to Axiom 1

$$\boxed{P(A_1 \cap A_2) \geq 0}$$

so,

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2) \quad \dots (3)$$

Lets consider for m events

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i) \quad \text{---} \quad (4)$$

Lets consider $m+1 = n$, split $m+1$ from m.

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i \cup A_{m+1}\right)$$

$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1})$$

$$- P\left(\bigcup_{i=1}^{m} A_i\right) \cap (P(A_{m+1}))$$

By axiom 1

$$P\left(\bigcup_{i=1}^{m} A_i\right) \cap P(A_{m+1}) \geq 0$$

so,

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) \leq P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) \quad \dots (5)$$

Now lets join RHS, using Induction hypothesis.

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) \leq \sum_{i=1}^{m+1} P(A_i) \longrightarrow 6$$

hence proved

③ Bonferroni's Inequalities : If $A_1, A_2, \dots A_n$ events belonging to a sample space 'S' then prove that

(i) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$ $P(\bar{A}) \Rightarrow$ Complement of 'A'

(ii) $P\left(\bigcap_{i=1}^{m} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

proof :- (i) eqn.

let $n=1$

$$P(A_1) \geq 1 - P(A_1^c) \quad \text{----} \quad (1)$$

Let $n=2$

$$P\left(\bigcap_{i=1}^{2} A_i\right) = P(A_1 \cap A_2)$$

$$P(A_1 \cap A_2) = ?$$

Lets try to apply general addition rule,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$$

now we need complement of $A_1 \cup A_2$

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - [1 - P(A_1 \cup A_2)^c]$$

$$= P(A_1) + P(A_2) - 1 - P(A_1 \cup A_2)^c$$

According to the Axiom-1 [complement event's probability also ≥ 0]

$$P(A_1 \cup A_2) \geq 0$$

$$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$$

$$\geq [1 - P(A_1^c)] + [1 - P(A_2^c)] - 1$$

$$P(A_1 \cap A_2) \geq 1 - \sum_{i=1}^{2} P(A_i^c) \quad \text{----} \quad (2)$$

Lets try Induction method

apply $m, \in N$

$$P\left(\bigcap_{i=1}^{m} A_i\right) \geq 1 - \sum_{i=1}^{m} P(A_i^c) \rightarrow ③$$

now apply $m+1$

LHS

$$P\left(\bigcap_{i=1}^{m+1} A_i\right) = P\left(\bigcap_{i=1}^{m} A_i\right) \cap P(A_{m+1})$$

Apply general addition formula

$$= P\left(\bigcap_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\bigcap_{i=1}^{m} A_i\right] \cap P(A_{m+1})$$

By Axiom: 1

$$P\left(\bigcap_{i=1}^{m} A_i\right) \cap P(A_{m+1}) \geq 0$$

So, before that take complement

$$= P\left(\bigcap_{i=1}^{m} A_i\right) + P(A_{m+1}) - \left(1 - \left[P\bigcap_{i=1}^{m} A_i \cap P(A_{m+1})\right]\right]$$

$$= P\left(\bigcap_{i=1}^{m} A_i\right) + P(A_{m+1}) - 1 + \underline{P\left(\bigcap_{i=1}^{m} A_i \cap A_{m+1}\right]}$$

Axiom: 1

$$P\left(\bigcap_{i=1}^{m+1} A_i\right) \geq P\left(\bigcap_{i=1}^{m} A_i\right) + P(A_{m+1}) - 1$$

Apply eqn 3

$$\geq 1 - \sum_{i=1}^{m} P(A_i^c) + P(A_{m+1}) - 1$$

take complement for $P(A_{m+1})$ & $1 - P(A^c_{m+1})$

$$\geq 1 - \sum_{i=1}^{m} P(A_i^c) + P(A^c_{m+1}) - 1$$

$$\geq 1 - \sum_{i=1}^{m} P(A_i^c) + (1 - P(A^c_{m+1})) - 1$$

$$\geq 1 - \sum_{i=1}^{m} P(A_i^c) + \underline{1} - P(A_{m+1}^c) - \underline{1}$$

$$= 1 - \sum_{i=1}^{m+1} P(A_i^c) \rightarrow 4$$

It's identical, hence proved equation (i) given in equestion

Let's take eqn (ii)

we will use eqn (i) to solve eqn (ii)

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

Let's elebrate RHS.

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \left[P(A_1^c) + P(A_2^c) \ldots - P(A_n^c)\right]$$

$$\geq 1 - \left((1 - P(A_1) + (1 - P(A_2) \ldots (1 - P(A_n)\right]$$

$$\geq 1 - \left[n - \sum_{i=1}^{n} P(A_i)\right]$$

$$\geq \sum_{i=1}^{n} P(A_i) - n + 1$$

$$\geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

$$P\left(\bigcap_{i=1}^{n} P(A_i)\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1) \Rightarrow 3\,(ii)$$

hence proved the 3(ii)

(4) Let $A_1, A_2 \ldots A_n$ be n events connected to a sample space 'S' then prove that

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1).P(A_2|A_1) \cdots P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)$$

provided $P\left(\bigcap_{i=1}^{n-1} A_i\right) > 0$.

proof:-

Above is general multiplication formula.

Lets apply mathematical Induction

Apply $n=1$

$$P(A_1) = P(A_1) \quad \text{—} (1)$$

Apply $n=2$

LHS:-

$$P\left(\bigcap_{i=1}^{2} A_i\right) = P(A_1 \cap A_2)$$

now apply conditional probability

$$P(A_2|A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)}$$

$$P(A_1 \cap A_2) = P(A_2 \cap A_1) = P(A_1).P(A_2|A_1) \rightarrow (2)$$

Lets apply mathematical Induction

$m \in N$

$$P\left(\bigcap_{i=1}^{m} A_i\right) = P(A_1) \cdot P(A_2|A_1) \dots P\left(A_m \middle| \bigcap_{i=1}^{m-1} A_i\right) \rightarrow (3)$$

lets apply m+1

LHS:

$$P\left(\bigcap_{i=1}^{m+1} A_i\right) = P\left(\bigcap_{i=1}^{m} A_i\right) \cap (A_{m+1})$$

now equate with (2) conditional P

$$P\left(\bigcap_{i=1}^{m} A_i\right) \cap (A_{m+1}) = P\left(\bigcap_{i=1}^{m} A_i\right) \cdot \left(A_{m+1} \middle| P \bigcap_{i=1}^{m} A_i\right)$$

now,

$$P\left(\bigcap_{i=1}^{m+1} A_i\right) = P(A_1) \cdot (P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \dots$$
$$P\left(A_m \middle| P \bigcap_{i=1}^{m-1} A_i\right) \cdot P\left(A_{m+1} \middle| P \bigcap_{i=1}^{m} A_i\right) \rightarrow 4$$

hence proved

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(5) Let A&B be the two events such that

$P(A) = 1/4$, $P(B|A) = 1/2$, $P(A|B) = 1/4$

Find (i) $P(A^c|B^c)$ and (ii) $P(A|B) + P(A|B^c)$

proof:-

Lets use conditional probability

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

What we have is,

$P(A) = 1/4$, $P(B|A) = 1/2$ $P(A|B) = 1/4$

$$P(A \cap B) = P(A) . P(B|A)$$

$$= 1/4 \times 1/2 = \boxed{1/8}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) = \frac{P(A \cap B)}{P(A|B)} = \frac{1/8}{1/4} = \frac{1}{8} \times \frac{4}{1} = \boxed{1/2}$$

now, let's calculate complement

$$P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)}$$

According to D'morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$

Lets apply.

$$P(A^c|B^c) = \cancel{P(A^c A1} \frac{P(A \cup B)^c}{P(B^c)}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{1 - P(A \cup B)}{1 - \frac{1}{2}}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$$

$$= \frac{1 - \frac{5}{8}}{\frac{1}{2}} = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{6}{8} \boxed{\frac{3}{4}}$$

so (i) $P(A^c | B^c) = \frac{3}{4}$

Let's take (ii)

$$P(A|B) + P(A|B^c) = \frac{1}{4} + \frac{P(A \cap B^c)}{P(B^c)}$$

Apply addition rule of probability

$$P(A \cap B^c) = P(A) - P(B \cap A)$$

$$= 1/4 + \frac{P(A) - P(B \cap A)}{1 - P(B)}$$

$$= 1/4 + \frac{1/4 - \frac{1}{8}}{1 - 1/2} = \frac{\frac{1}{8}}{1/2} = \frac{1}{4} + 1/4$$

$$= \frac{2}{4} = \underline{1/2}$$

$$\boxed{P(A|B) + P(A|B^c) = 1/2}$$

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(6) A missile is fired at a plane on which there are two targets t_1, t_2. The probability of hitting T_1 is $= 0.7$ and that of hitting T_2 is $= 0.65$. It is known as T_2 was not hit. Find the probability that T_1 was hit.

<u>proof:</u>

Apply conditional probability

$$P(T_1 | T_2) = \frac{P(T_1 \cap T_2)}{P(T_2)}$$

To find:

$P(T_1) = 0.7$

$P(T_2) = 0.65$

It is known than T_2 never hit, so take complement.

$P(T_1 | T_2^c) = ?$

Solution:-

$$P(T_1 | T_2^c) = \frac{P(T_1 \cap T_2^c)}{P(T_2^c)} \quad -①$$

Apply addition rule of P.

$P(A \cap B^c) = P(A) - P(B \cap A)$

$P(T_1 \cap T_2^c) = P(T_1) - P(T_1 \cap T_2)$

Let's Apply in (1)

$$P(T_1 T_2^c) = \frac{P(T_1) - P(T_1 \cap T_2)}{P(T_2^c)}$$

$P(T_1 \cup T_2) = P(T_1) + P(T_2) - P(T_1 \cap T_2)$

$P(T_1 \cap T_2) = P(T_1) + P(T_2) - P(T_1 \cup T_2)$

$$= \frac{0.7 - (0.7 \times 0.65)}{(1 - 0.65)} = \frac{0.245}{0.35} = 0.7$$

The probability of T_1 hit $= 0.7$

(7) Each co-efficient in equation $ax^2+bx+c=0$ is determind by Throwing a fair dice. Find the probability that the equation will have real roots?

$ax^2+bx+c=0.$ —①

solution:..

given quadratic equation is

$f(x)= ax^2+bx+c=0$, in this case find that

$\sqrt{b^2-4ac}$ also 0, should be real.

$b^2-4ac \geq 0$

Lets throw fair dice.

$a,b,c \in \{1,2,3,4,5,6\} \leftarrow S.$

a,b,c are coefficients.

a,b,c are coefficients. Lets take b and apply value from 1-6. and see if a&c getting the correct value.

$b^2-4ac \geq 0$ $a=\{1,2,3,4,5,6\}$

$b^2-4ac \geq 0$ $b=\{1,2,3,4,5,6\}$

$c=\{1,2,3,4,5,6\}$

(e,) b=1 apply

$1^2-4ac \geq 0$, Lets check min values of a&c

$a,c=1$ $1^2-4(1)(1) \geq 0$

$1-4 \geq 0$ (wrong) (b=1, NO a, b compination)

Negative

(e2) Lets apply $b=2$

$b^2-4ac \geq 0$

$2^2-4ac \geq 0$ lets take min values of a,b,c

$a=1, c=1$

$4-4(1)(1) \geq 0$ = true $\{1,1\}$

Other possibilities of a & c wont be there since, the result will go negative

(e3) Lets apply $b=3$

$3^2-4ac \geq 0$ $a=1, c=1$

$9-4 \geq 0$ - true

$a=1, c=2$

$9-4(1)(2) \geq 0$ true

$9-(4)(2)(1) \geq 0$ true $a=2, c=1$

Other a & c possibilities will go negative with eqn.

So, the possibilities are,

a,c : $\{(1,1), (1,2), (2,1)\}$

(e4) Lets apply $b=4$

same way we calculate like above,

we get

$a,c = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (4,1), (3,1)\}$

total possibilities are $= \underline{\underline{8}}$

P6

(e5) Lets apply $b=5$

possiblities are.

$[(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1)$
$(2,2), (2,3), (3,1), (3,2), (4,1), (5,1), (6,1)]$

total possiblities are $= \underline{14}$

(e6) Lets apply $b=6$

The possiblities are.

$[(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (3,1)$
$(3,2), (3,3), (4,1), (4,2), (5,1), (6,1)]$

total possiblities are $=6$.

Lets evaluate the total probability of

$f(x)=0$

Let A is set of $\{a,b,c\} \mid f(x)$ has real roots.

Lets consider

$a_1 : \{b=1\}$ e_1 : set of $\{a,c\}$
value got in above eqn with respect to b

$a_2 : \{b=2\}$ e_2 : "

$a_3 : \{b=3\}$ e_3 : "

$a_4 : \{b=4\}$ e_4 : "

$a_5 : \{b=5\}$ e_5 : "

$a_6 : \{b=6\}$ e_6 : "

now findout for (a,b,c) from above (a,b) with respect to b.

Lets say.

$e_1^* = A/a_1$ $\quad e_3^k = A/a_3$

$e_2^k = A/a_2$ $\quad e_4^k = A/a_4$

$e_5^k = A/a_5$ $\quad e_6^k = A/a_6$

Lets apply total probability thearem. (for fair dice)

$$\underline{P(A)} = \underline{\sum_{i=1}^{6} P(a_i)} \cdot \underline{P(A/a_i)}$$

$$= P(a_1) \cdot P(A/a_1) + P(a_2) \cdot P(A/a_2)$$
$$P(a_3) \cdot P(A/a_3) + P(a_4) \cdot P(A/a_4)$$
$$+ P(a_5) \cdot P(A/a_5) + P(a_6) \cdot P(A/a_6)$$

$P(a_1) \rightarrow$ will be value we assigned to b in total number of available values.

for example,

$P(a_1) = \frac{1}{6}$ $\quad b=1$ total available values in b is 6.

$P(A/a_1) = e_1^k = \text{count}\{e_1\}$

$e_1 = \{0\}$

$e_1^k = \underline{0}$ $\quad e_2 = \{1\}$

$e_2^k = \text{count}\{1\} = \underline{1}$

and total possiblities are $\{a, c\} = 36$
6×6

So. lets apply the value.

$$P(A) = \left(\tfrac{1}{6} \cdot \tfrac{0}{36}\right) + \left(\tfrac{1}{6} \cdot \tfrac{1}{36}\right) + \left(\tfrac{1}{6} \cdot \tfrac{3}{36}\right) + \left(\tfrac{1}{6} \cdot \tfrac{8}{36}\right) + \left(\tfrac{1}{6} \cdot \tfrac{14}{36}\right) + \left(\tfrac{1}{6} \cdot \tfrac{17}{36}\right)$$

$$= \frac{0+1+3+8+14+17}{6 \times 36} = \boxed{\frac{43}{216}}$$

$$\boxed{P(A) = \frac{43}{216}}$$

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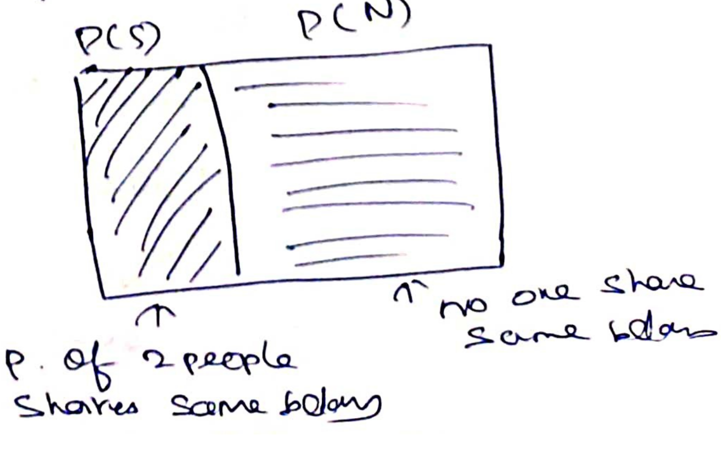
(8) Let there be n people in a room and $n \leq 365$. Let no person have birthday on 29th Feb. Find the probability that atleast two person share the same birthday.

To prove:
$= P\{$at least 2 people have the same bday$\}$

proof:
above will have lots of combination, Instead of find the complement. and subract it

$P(S) + P(N) = 100\%$



P(S) is the proboblity which we need two people shares same birthday.

$P(S) = 100\% - P(N)$

Lets find P(N)
↳ No one has same bday.

lets consider 1st guy for P(N)

P1 ⇒ first persion P2 → second person

P_1	P_2	P_3	
$\frac{365}{365}$	$\frac{364}{365}$	$\frac{363}{365}$	→ this is the Pattern

if P_1 has birthday he will have 365 possiblities.
whereas P2 will have only ~~368~~ 364.
and P_3 will have 363.

so, for 3 people.

$$P(N) = \frac{365 \cdot 364 \cdot 363}{(365)^3} \quad (\text{for } 3)$$

Lets simply ~~fo~~ this

$$= \frac{\frac{365!}{(365-3)!}}{(365)^3}$$

$$\frac{\frac{365 \cdot 364 \cdot 363 \cdot 362 \ldots}{\cancel{365} \cdot \cancel{364} \cdot \cancel{363} \ldots\ \cancel{362} \cdot 361 \cdot 362 \ldots}}{(365)^3}$$

$$= \frac{\frac{365!}{363!}}{(365)^3}$$

Lets apply for n people.

$$P(N) = \frac{365(n-1)!}{(365)^n}$$

lets find out

$$P(S) = 100\% - P(N)$$

$$= 1 - \frac{365(n-1)!}{(365)^n}.$$

so, to conside 2 people will not have same birthday.

$$P(N) = \left(\frac{\cancel{8}1!}{(36\cancel{5})^{2}} \cdot \frac{365 \cdot 364}{2!}\right)^{nC_2}$$

nC_2 = 2 people share same bday in room of n people.

$$PS = 1 - P(N)$$

$$= 1 - \left(\frac{36\cancel{8} \cdot 364}{\cancel{2}!} \cdot \frac{2\cancel{1}}{(365)^{\cancel{2}}}\right)^{nC_2}$$

$$\boxed{P(S) = 1 - \left(\frac{364}{365}\right)^{nC_2}}$$

(9) Suppose each of n men at Party thrown his hat in to the centre of the room. The hats are mixed up and then each man randomly selects a hat. Find the probability that none of the men selects his one hat.

→ It is a matching problem.

proof: (solution)

Lets consider following occurances

(a) none of the men select their own hat

(b) K men select thier own hat.

we need to find (a)

Lets apply Exclusion & Inclusion principle

Let's say A_i = event that the i^{th} person gets his own hat.

so, $\{A_1 \cup A_2 \cup \dots \cup A_n\}$ = the event one gets his hat.

from that, probability of none of them getting own hat is $= 1 - P(A_1 \cup A_2 \cup \dots \cup A_N)$

Lets find $P(A_1 \cup A_2 \cdots \cup A_N)$

Using Inclusion & Exclusion principle.

$$P(A_1 \cup A_2 \cdots A_N) = \underbrace{\sum_{i=1}^{N} P(A_i)}_{①} - \underbrace{\sum_{i<j} P(A_{ij})}_{②} + \cdots (-1)^{N+1} P(A_1 A_2 \cdots A_N)$$

Lets consider each,

$P(A_1)$ = probability of first man gets his hat.

Only 1 way for him to get his hat :

$$\frac{1 \times (N-1)!}{N!} = \frac{1 \times (N-1)!}{N!} = \frac{1}{N}$$

this is true for all the ith person gets his own hat. (1)

so, $\sum_{i=1}^{N} P(A_i) = N\left(\frac{1}{N}\right) = 1$

Now consider (2)

$$P(A_i A_j) = -\frac{1}{i} - \frac{1}{j}$$

arrange remaining $(N-2)!$ people.

$$= \frac{(N-2)!}{N!}$$

Since there are $\binom{N}{2}$ pairs with $i<j$

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$$\sum_{i<j} P(A_i A_j) = \binom{N}{2} \cdot \frac{1}{N(N-1)} =$$

$$= \frac{N!}{(N-2)!\,2!} \cdot \frac{(N-2)!}{N!}$$

$$= \frac{1}{2!}$$

Similarly $P(A_i A_j A_k) = \frac{(N-3)!}{N!}$
$1 < j < N$

So $\sum P(A_i A_j A_k) = \binom{N}{3} \frac{(N-3)!}{N!}$

$$= \frac{N!}{3!\,(N-3)!} \times \frac{(N-3)!}{N!}$$

$$= \frac{1}{3!}$$

and for n it will be $= 1/N!$

Lets apply all the found values in to formula.

$$\sum P(A_i) - \sum P(A_i A_j) + \dots (-1)^{N+1} P(A_1 A_2 \dots A_n)$$

$$1 - \frac{1}{2!} + 1/3! \dots + (-1)^{N+1} \frac{1}{N!}$$

And

$$1 - P(A_1 \cup A_2 \cdots A_N) = 1 - \left[1 - \frac{1}{2!} + \frac{1}{3!} \cdots + (-1)^{N+1}\frac{1}{N!}\right]$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^N \cdot \frac{1}{N!} \approx e^{-1}$$

for large N

$$\boxed{P(\text{Nobody gets thier hat back}) = \sum_{i=0}^{N} \frac{(-1)^i}{i!}}$$

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(10) Let population of a certain city is 40% males, 60% females. suppose also that 50% males & 30% females smoke. Find the probability of a randomly selected smoker is male.

Solution:-

Let P(A) - Randomly selected person is smoker.

so, we need to find P(A).

Let P_1 : Randomly selected person is male

P_2 : Randomly selected person is female.

By using Total probability theorem.

$$P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)$$

in out solution.

$$P(A) = P(P_1) \cdot P(A|P_1) + P(P_2) \cdot P(A|P_2)$$

$$= (0.4) \cdot (0.5) + (0.6) \cdot (0.3)$$

$$= 0.20 + 0.18 = 0.38$$

$$P(A) = 0.38$$

Now given A, (smoking)

Lets find P_1 which is the randomly Selected smoker is male.

$$P(P_1|A) = \frac{P(P_1) \cdot P(A|B_1)}{P(A)}$$

Came from below formula

$$P(P_1|A) = \frac{P(P_1 \cap A)}{P(A)}$$

$$P(P_1 \cap A) = P(A) \cdot P(P_1|A)$$

$P(P_1 \cap A)$ Apply general multiplication formula

$$P(P_1 \cap A) = P(P_1) \cdot P(A|P_1)$$

$$P(P_1 | A) = \frac{P(P_1) \cdot P(A|P_1)}{P(A)}$$

$$= \frac{(0.4)(0.5)}{0.38}$$

$$\boxed{P(P_1|A) = 0.526}$$

P_1 is randomly selected person is male

A is smoker who is randomly selected.