Boundary behavior in stochastic differential equations used in Finance

A Thesis by Báskay János

Supervisor: Fáth Gábor

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Abstract

Mathematical finance uses certain stochastic models, that possess rich phase diagrams in the space spanned by their parameters. The different phases can be separated based on the processes' behavior near its boundary: whether or not the boundary can be reached from the bulk, or the bulk can be reached from the boundary.

This thesis will be focused on investigating the boundary behavior of the Cox-Ingersoll-Ross model, which was introduced in 1985 by John C. Cox, Johnathan E. Ingersoll and Stephen A. Ross to describe the evolution of interest rates. The condition commonly referred to as Feller condition ensures the solution of this process is bounded below by zero. The attainability of the zero-boundary is dependent on the parameters of the model, and well known analytically.

My goal is to show that numerical simulation methods can also reproduce the analytical boundary behavior for both Reflective and Absorbing boundary conditions. To do this I will have to investigate multiple discretization schemes of the CIR model, introduce suitable order parameters for both the boundary conditions, and then approximate the continuum limit $(dt \to 0)$ of the phase diagram.

Feller's boundary classification scheme

(Karlin & Taylor, 1981)

Criteria				Terminology		
$S(l,x]$	$M(l,x]$	$\Sigma(l)$	$N(l)$			
$< \infty^*$	$< \infty^*$	$< \infty$	$< \infty$	Regular	Attracting	Attainable
$< \infty$	$= \infty^*$	$< \infty^*$	$= \infty$	Exit		
$< \infty^*$	$= \infty^*$	$= \infty^*$	$= \infty$	Natural	Nonattracting	Unattainable
$= \infty^*$	$< \infty^*$	$= \infty$	$= \infty^*$			
$= \infty^*$	$= \infty^*$	$= \infty$	$= \infty$			
$= \infty^*$	$< \infty$	$= \infty$	$< \infty^*$	Entrance		
* marks the minimal sufficient criteria to establish the nature of the boundary						

- $S(l,x]$: Scale measure, related to the probability $\Pr\{T_l \leq T_b | X(0) = x\}$
- $M(l,x]$: Speed measure, related to the expected value of the time to reach the boundary
- $\Sigma(l)$: Measures the time it takes to either reach the boundary l or the interior point b starting from an other interior point
- $N(l)$: Measures the time it takes to reach the interior point x starting from the boundary l

The Cox-Ingersoll-Ross model

(Cox, Ingersoll, & Ross, 1985)

- Created to describe the term structure of the short rate
- Defined by the SDE

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$$
$$dW_t = z\sqrt{dt} \quad and \quad z \in \mathcal{N}(0,1)$$

- The PDF of the process started from (X_0, t_0) at time t:

$$\Pr(X, t|X_0, t_0) = f_{\chi^2_\nu(\lambda)}\big(x\eta(t - t_0)\big)$$

  - Where $f_{\chi^2_\nu(\lambda)}(x)$ is the PDF of the non-central χ^2 distribution with the non-centrality parameter $\lambda = X_0 e^{-\kappa(t-t_0)}\eta(t - t_0)$ and degrees of freedom $\nu = \frac{4\kappa\theta}{\sigma^2} = 2\gamma$. Furthermore $\eta(\Delta t) = \frac{4\kappa}{\sigma^2}\left(1 - e^{-\kappa\Delta t}\right)$
  - We also defined the control parameter as $\gamma = \frac{2\kappa\theta}{\sigma^2}$

Analytical boundary properties of the CIR model

(Fáth, 2011)

- The probability to reach the $l = 0$ boundary from an arbitrary bulk point x:

$$u(x) = \begin{cases} 0 & if\ \ \gamma \geq 1 \\ finite & if\ \gamma < 1 \end{cases}$$

- Can the boundary be reached from the bulk in finite expected time?

$$\lim_{l \to 0} \Sigma(l) = \begin{cases} \infty & if\ \gamma \geq 1 \\ finite & if\ \gamma < 1 \end{cases}$$

- Can the balk reached from the boundary in finite expected time?

$$\lim_{l \to 0} N(l) = \begin{cases} \infty, & if\ \gamma < 0 \\ finite, & if\ 0 < \gamma < 1 \\ finite, & if\ \gamma \geq 1 \end{cases}$$

- Based on Feller's boundary classification scheme the

$$boundary(l = 0) = \begin{cases} Exit, & if\ \gamma < 0 \\ Regular, & if\ 0 < \gamma < 1 \\ Entrance, & if\ \gamma \geq 1 \end{cases}$$

Ornstein-Uhlenbeck process

(Uhlenbeck & Ornstein, 1930)

- The original model described a heavy Brownian particle in an environment with friction, now it's widely used in both Physics and Finance
- The SDE of the process

$$dY_t = -\kappa' Y_t dt + \sigma' dW_t$$

- The transition probability given the initial condition $\Pr(Y_0, t_0) = \delta(Y - Y_0)$ is known: *(Risken, 1984)*

$$\Pr(Y, t|Y_0, t_0) = \sqrt{\frac{\kappa'}{\pi\sigma'^2(1 - e^{-2\kappa'(t-t_0)})}} \exp\left[-\frac{\kappa'}{\sigma'^2}\frac{\left(Y - Y_0 e^{-\kappa'(t-t_0)}\right)^2}{1 - e^{-2\kappa'(t-t_0)}}\right]$$

  - The general solution for any initial distribution $W(Y', t')$ will be given by the convolution

$$W(Y, t) = \int P(Y, t|Y', t')W(Y', t')dY'$$

Connection of the OU and CIR processes

- Starting from the CIR Process and using Itô's Lemma for the function $Y_t(X_t) = \sqrt{X_t}$ we arrive to

$$dY_t = \left[\left(\frac{\kappa\theta}{2} - \frac{\sigma^2}{8}\right)\frac{1}{Y_t} - \frac{\kappa}{2}Y_t\right]dt + \frac{\sigma}{2}dW_t$$

- In the special case of $\frac{\kappa\theta}{2} - \frac{\sigma^2}{8} = 0$ this will be an OU-type SDE with $\kappa' = \frac{\kappa}{2}$ and $\sigma' = \frac{\sigma}{2}$.
- This has a known transition probability, which can be transformed back to obtain an alternate transition probability for the CIR model

$$\Pr(X, t|X_0, t_0) = \frac{1}{2}\frac{1}{\sqrt{X}}\sqrt{\frac{2\kappa}{\pi\sigma^2(1 - e^{-\kappa(t-t_0)})}}\exp\left[-\frac{2\kappa}{\sigma^2}\frac{\left(\sqrt{X} - \sqrt{X_0}e^{-\frac{\kappa}{2}(t-t_0)}\right)^2}{1 - e^{-\kappa(t-t_0)}}\right]$$

- Due to taking the square of an OU process, this will describe a CIR process with a **Regular** and **Reflective** $l = 0$ boundary.

CIR process with absorbing $l = 0$ boundary from OU process

- We can also construct this way an analytical formula for the Absorbing $l = 0$ boundary
- We take the sum of two Anti-symmetric OU processes. (One started from Y_0 and the other from $-Y_0$, with the second process having negative probabilities)
- Since the sum now integrates for < 1, it'll have to be normalized:

$$\mathcal{n} \coloneqq \int_0^\infty P(Y, t|Y_0, t_0) - P(Y, t|-Y_0, t_0)dY \ < 1$$

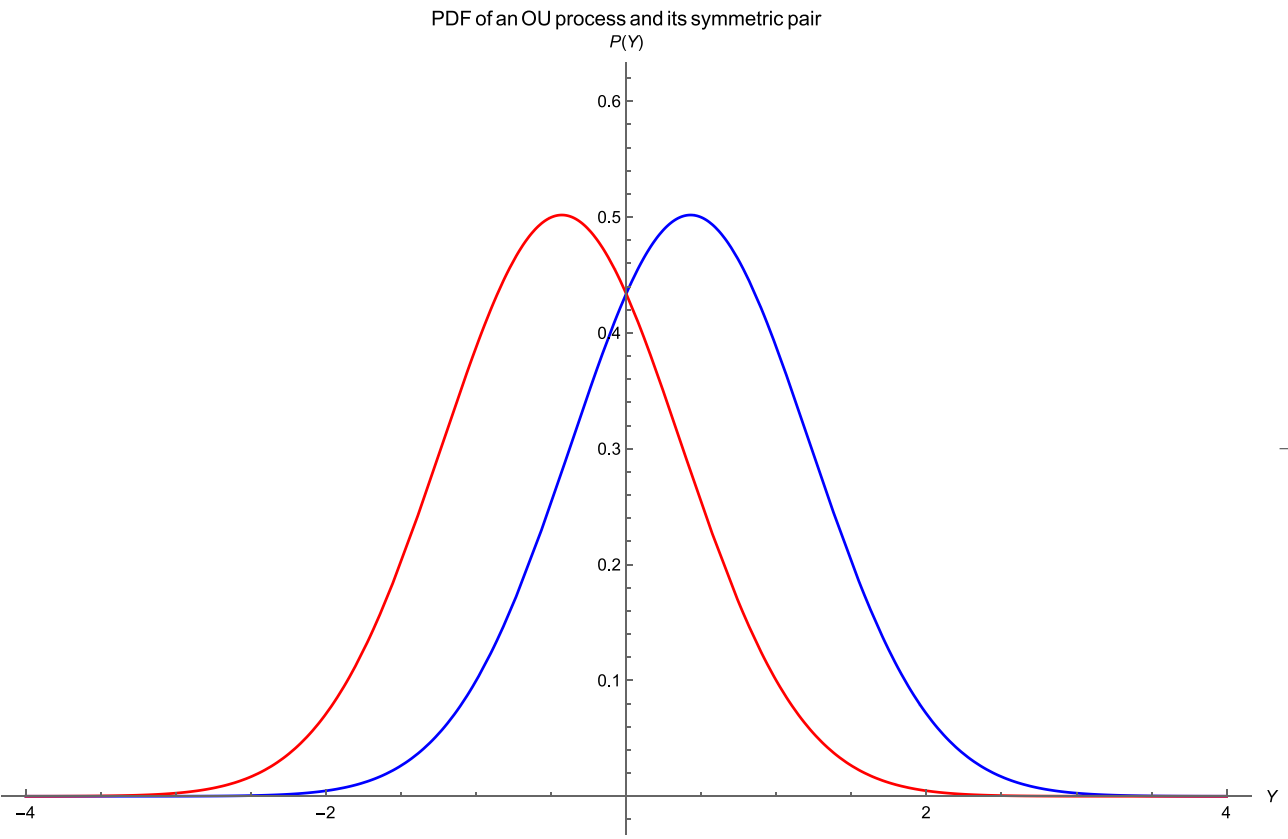
- We can also introduce the rate of absorption:

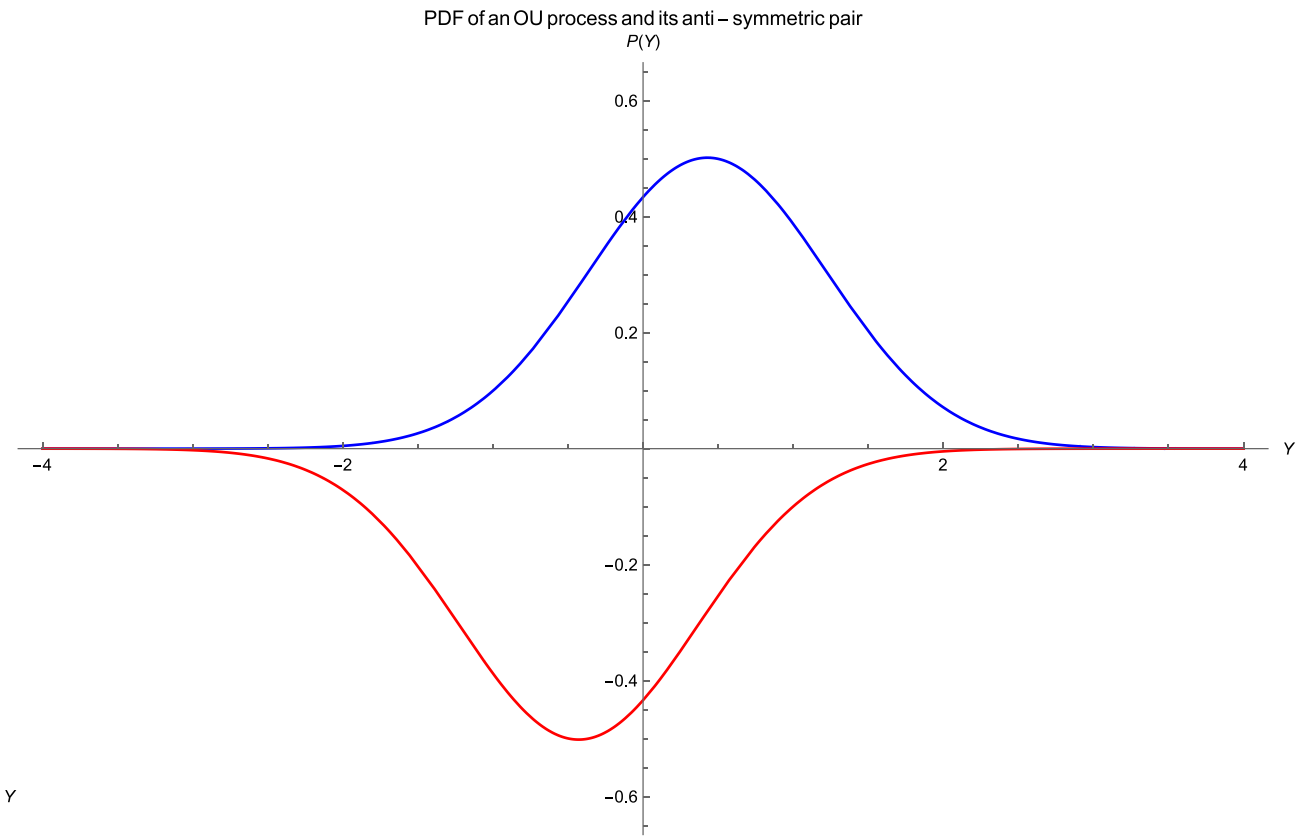
$$R_a = 1 - \mathcal{n}$$

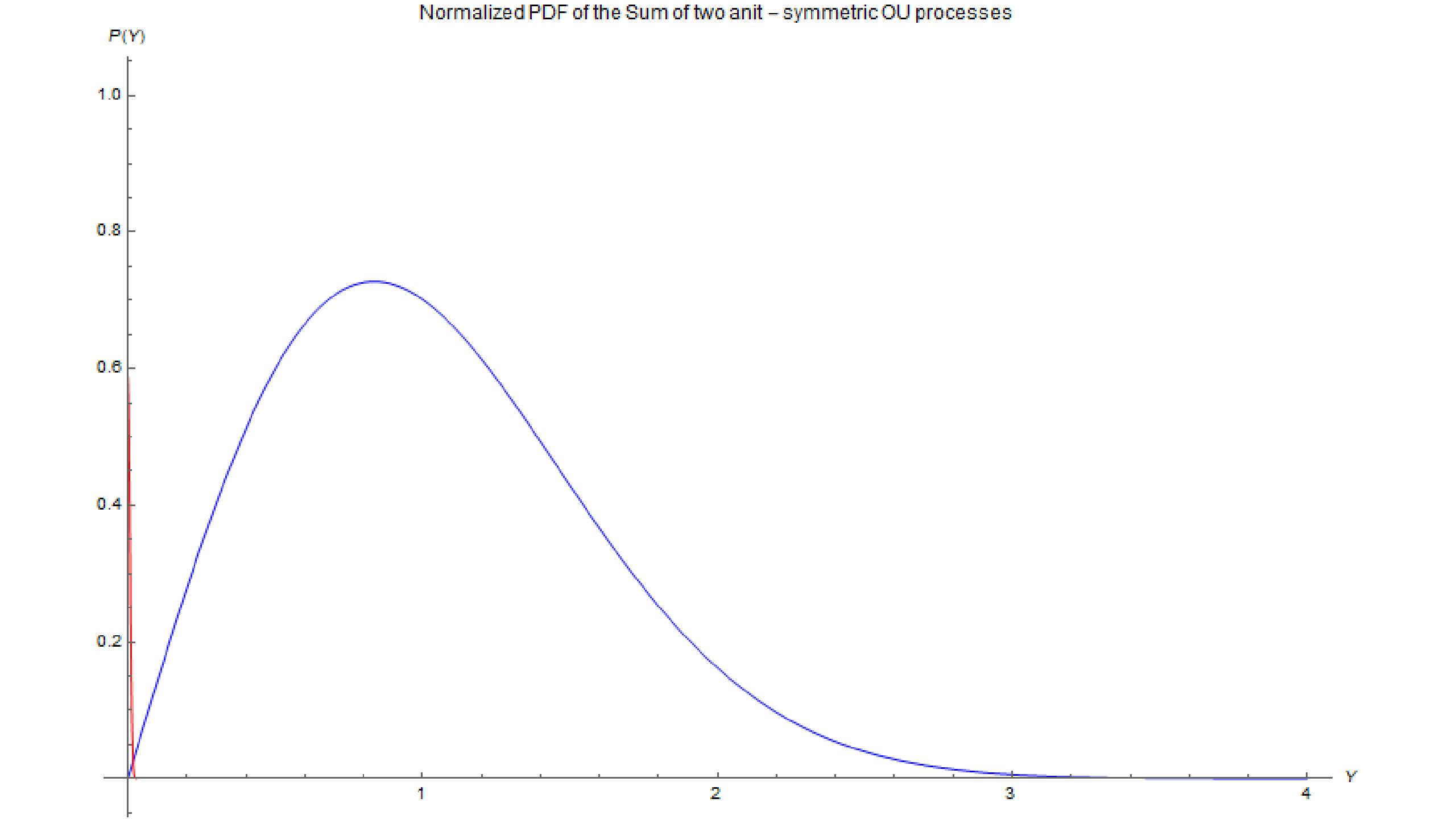
- The PDF then can be transformed back to describe a CIR process

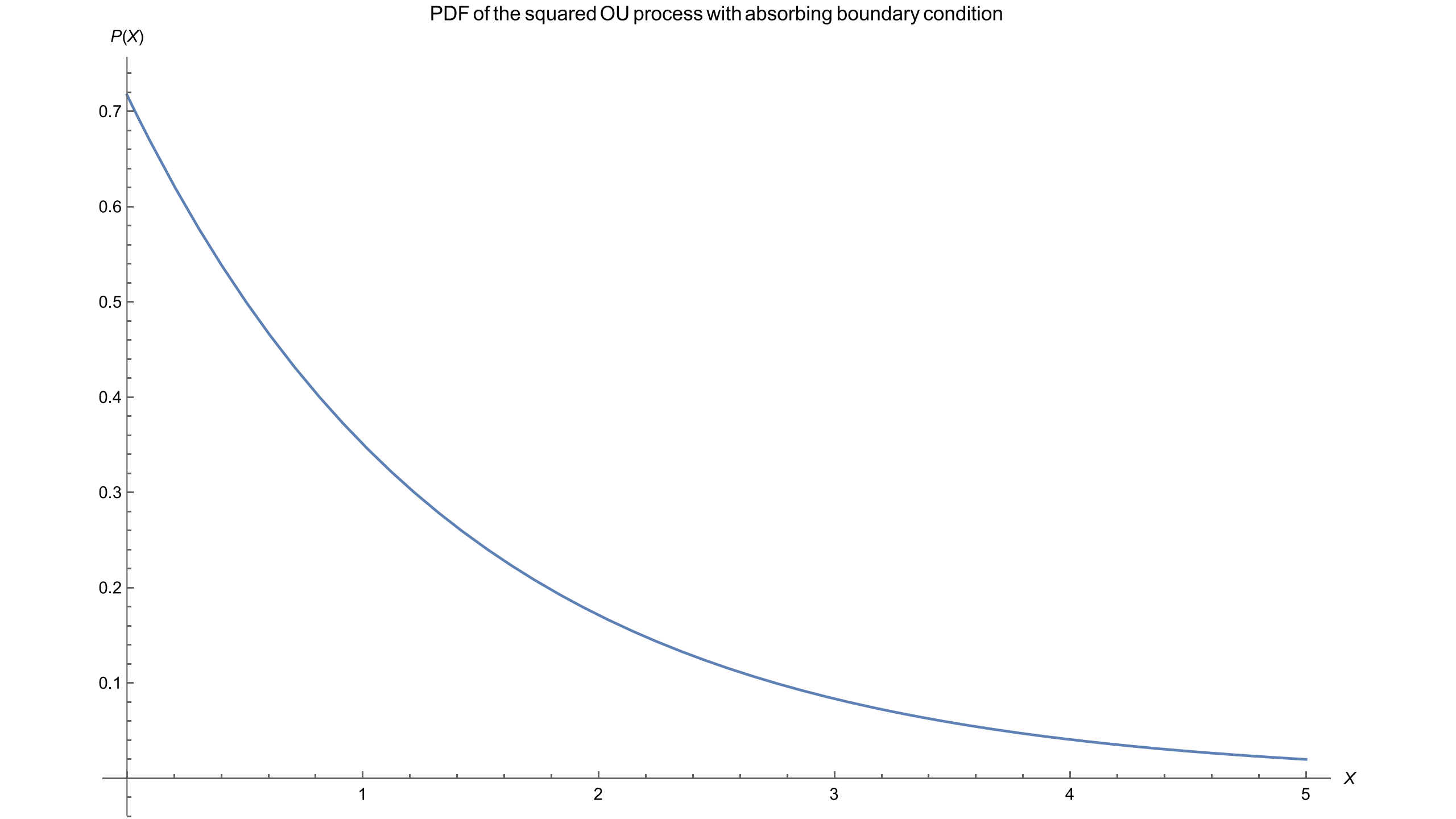
$$\Pr(X, t|X_0, t_0) =$$

$$= \frac{1}{2\mathcal{n}} \frac{1}{\sqrt{X}} \sqrt{\frac{2\kappa}{\pi\sigma^2(1 - e^{-\kappa(t-t_0)})}} \left\{ \exp\left[-\frac{2\kappa}{\sigma^2} \frac{\left(\sqrt{X} - \sqrt{X_0} e^{-\frac{\kappa}{2}(t-t_0)}\right)^2}{1 - e^{-\kappa(t-t_0)}} \right] - \exp\left[-\frac{2\kappa}{\sigma^2} \frac{\left(\sqrt{X} + \sqrt{X_0} e^{-\frac{\kappa}{2}(t-t_0)}\right)^2}{1 - e^{-\kappa(t-t_0)}} \right] \right\}$$


PDF of an OU process and its symmetric pair
P(Y)
0.6
0.5
0.4
0.3
0.2
0.1
−4
−2
2
4
Y


PDF of an OU process and its anti – symmetric pair
P(Y)
0.6
0.4
0.2
−0.2
−0.4
−0.6
−4
−2
2
4
Y


Normalized PDF of the Sum of two anit – symmetric OU processes
P(Y)
1.0
0.8
0.6
0.4
0.2
1
2
3
4
Y


PDF of the squared OU process with absorbing boundary condition
P(X)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1
2
3
4
5
X

Monte Carlo implementation of the CIR process

- Used discretization schemes
  - Euler-Maruyama scheme *(Haugh, 2017)*

$$X_{t+\Delta t} = X_t + \kappa(\theta - X_t)\Delta t + \sigma\sqrt{X_t}\Delta W_t$$

  - Weighted Milstein scheme, "WMS" *(Mil'shtein, 1974)*

$$X_{t+\Delta t} = \frac{X_t + \kappa(\theta - \alpha X_t)\Delta t + \sigma\sqrt{X_t}\Delta W_t + \frac{1}{4}\sigma^2\Delta t(z^2 - 1)}{1 + (1-\alpha)\kappa\Delta t}$$

  - Non-central χ^2 scheme, "Exact Monet Carlo" *(Malham & Wiese, 2012)*

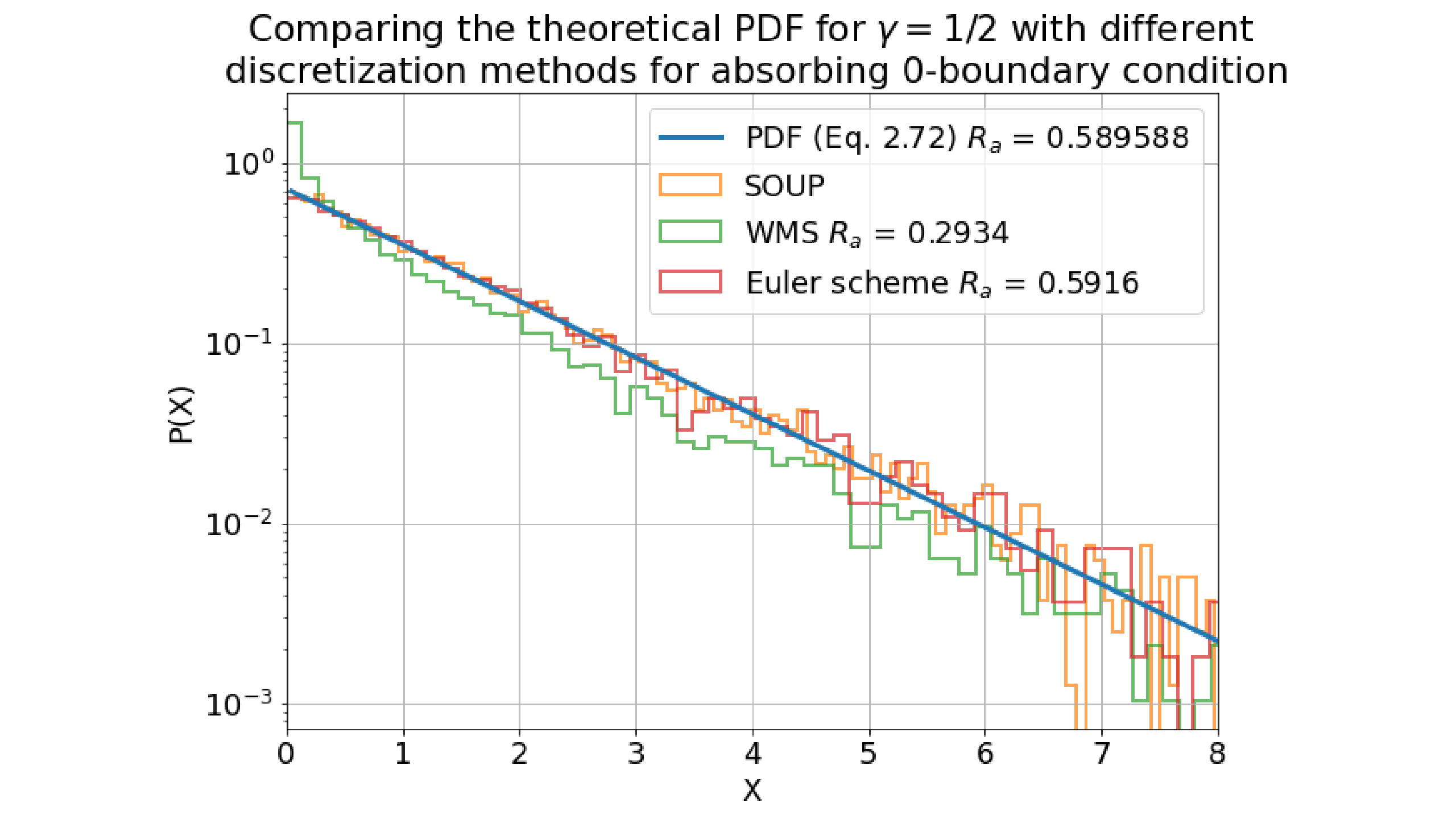
$$X_{t+\Delta t} = \chi^2_\nu(\lambda)\frac{e^{-\kappa\Delta t}}{\eta(\Delta t)}$$

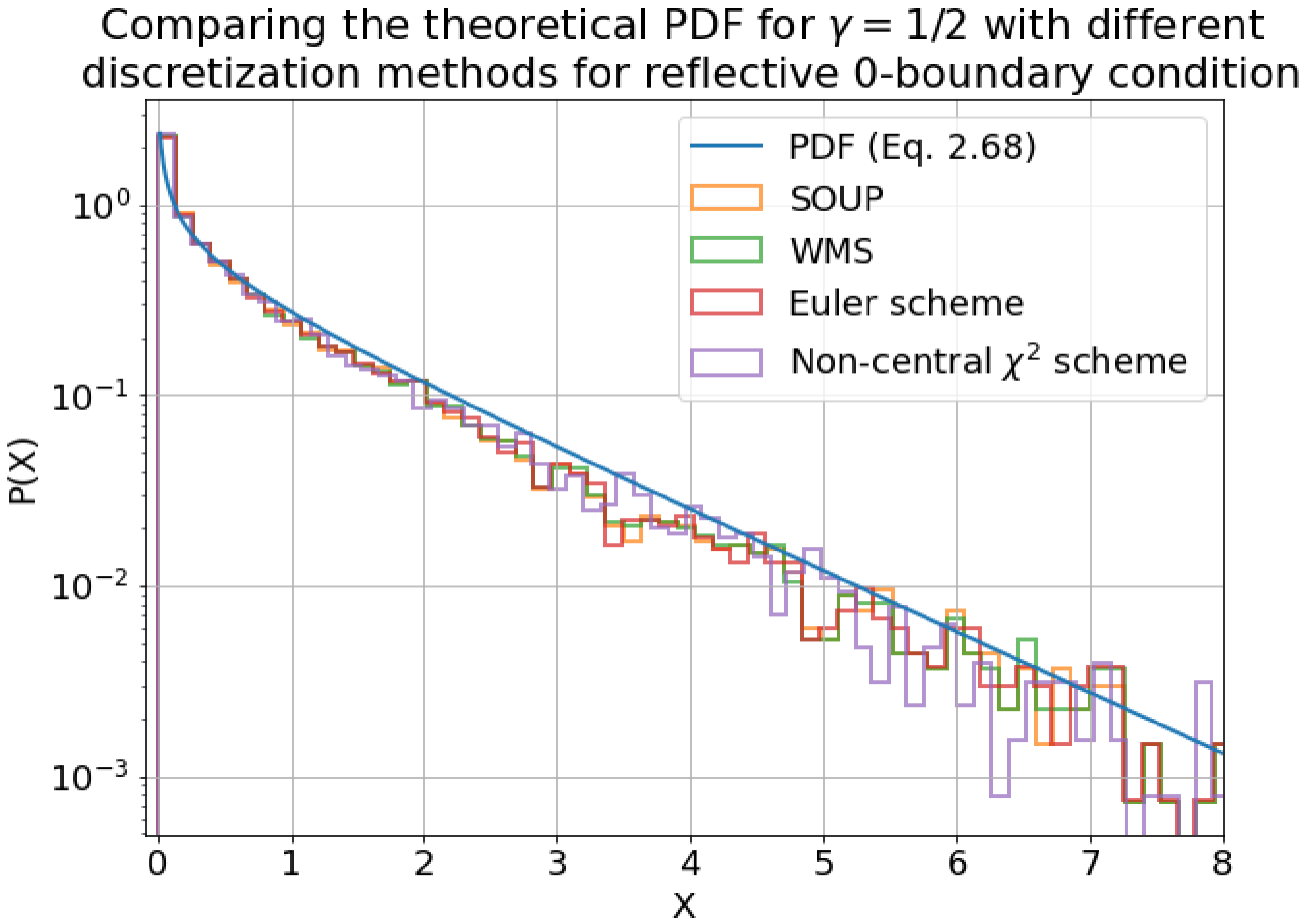
  - Some schemes require specific handling at the $l = 0$ boundary, to have **Reflective**/**Absorbing** boundary conditions
- The random normally distributed variable generation was handled by *numpy.random.normal()* and the non-central χ^2 distributed variables were created by *numpy.random.noncentral_chisquare()*
- Then the code can be optimized, for example the unnecessary cycles and memory allocations can be eliminated, and parallel processing introduced
- Once the simulated trajectories are ready, the calculation of order parameters, or other measurements can be carried out

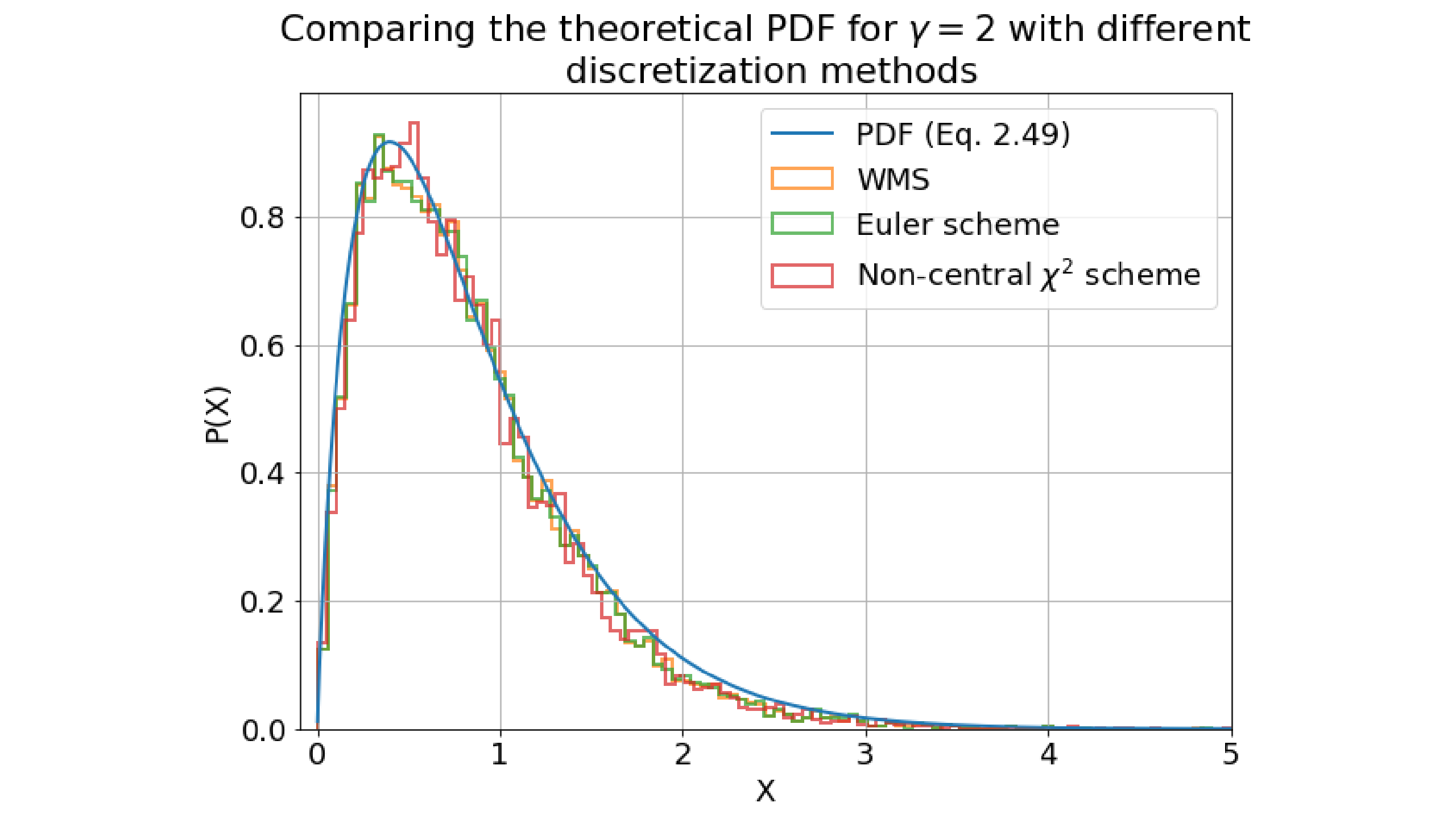
Results

Comparing the discretization schemes

- Runtime
  - $RT_{WMS} > RT_{\chi^2} > RT_{Euler} > RT_{OU^2}$ in all configurations
  - An order of magnitude change in $\Delta t = \frac{T}{N}$ results in approx. an order of magnitude change in the runtime
- Kolmogorov-Smirnov test, comparing the distributions
  - Accuracy is improved with $\Delta t \to 0$
  - Reflective WMS and Euler schemes performed poorly for $\gamma < 1$
    → Might be an implementation error…
- Monte Carlo error
  - Standard Deviation for a simulation of 1000 trajectories repeated 100 times: $\mathbf{2.717}$ %
  - SD for a simulation of 10000 trajectories repeated 10 times: $\mathbf{0.932}$ %


Comparing the theoretical PDF for $\gamma = 1/2$ with different discretization methods for absorbing 0-boundary condition
PDF (Eq. 2.72) $R_a = 0.589588$
SOUP
WMS $R_a = 0.2934$
Euler scheme $R_a = 0.5916$
P(X)
X
10^0
10^{-1}
10^{-2}
10^{-3}
0
1
2
3
4
5
6
7
8


Comparing the theoretical PDF for $\gamma = 1/2$ with different discretization methods for reflective 0-boundary condition
PDF (Eq. 2.68)
SOUP
WMS
Euler scheme
Non-central χ^2 scheme
10^0
10^{-1}
10^{-2}
10^{-3}
P(X)
0
1
2
3
4
5
6
7
8
X


Comparing the theoretical PDF for $\gamma = 2$ with different discretization methods
PDF (Eq. 2.49)
WMS
Euler scheme
Non-central χ^2 scheme
P(X)
X
0.0
0.2
0.4
0.6
0.8
0
1
2
3
4
5

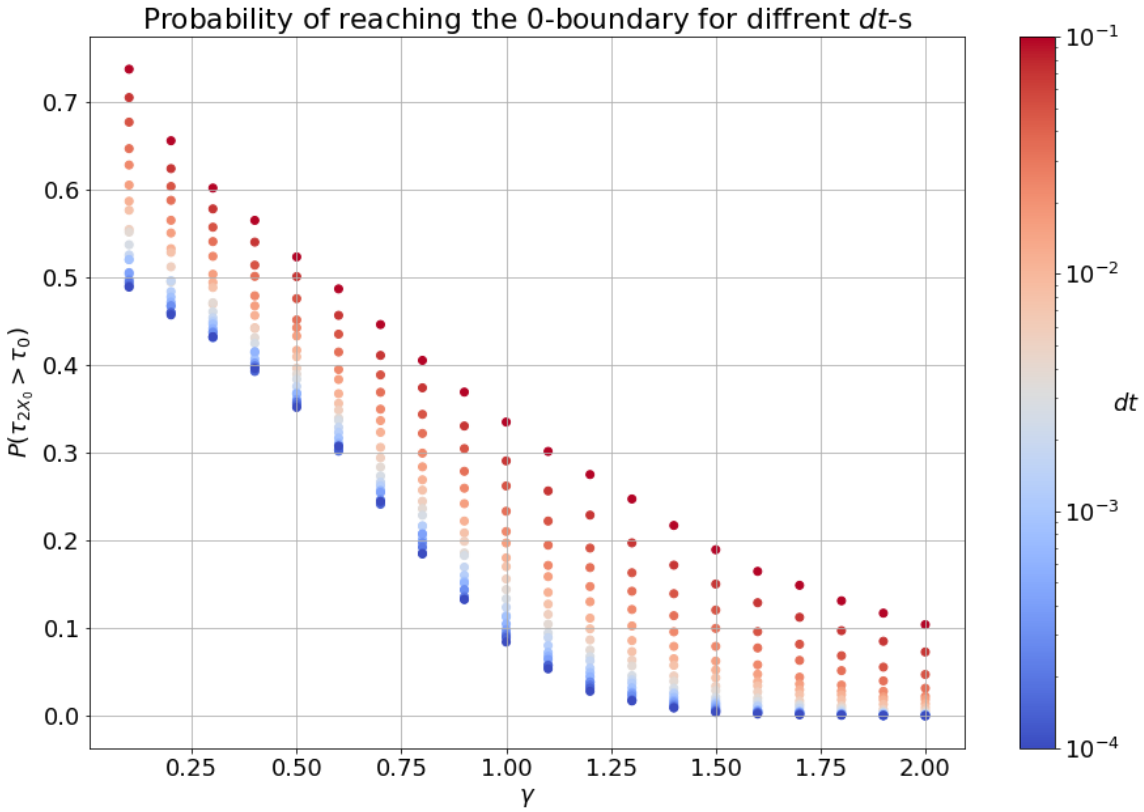
Results

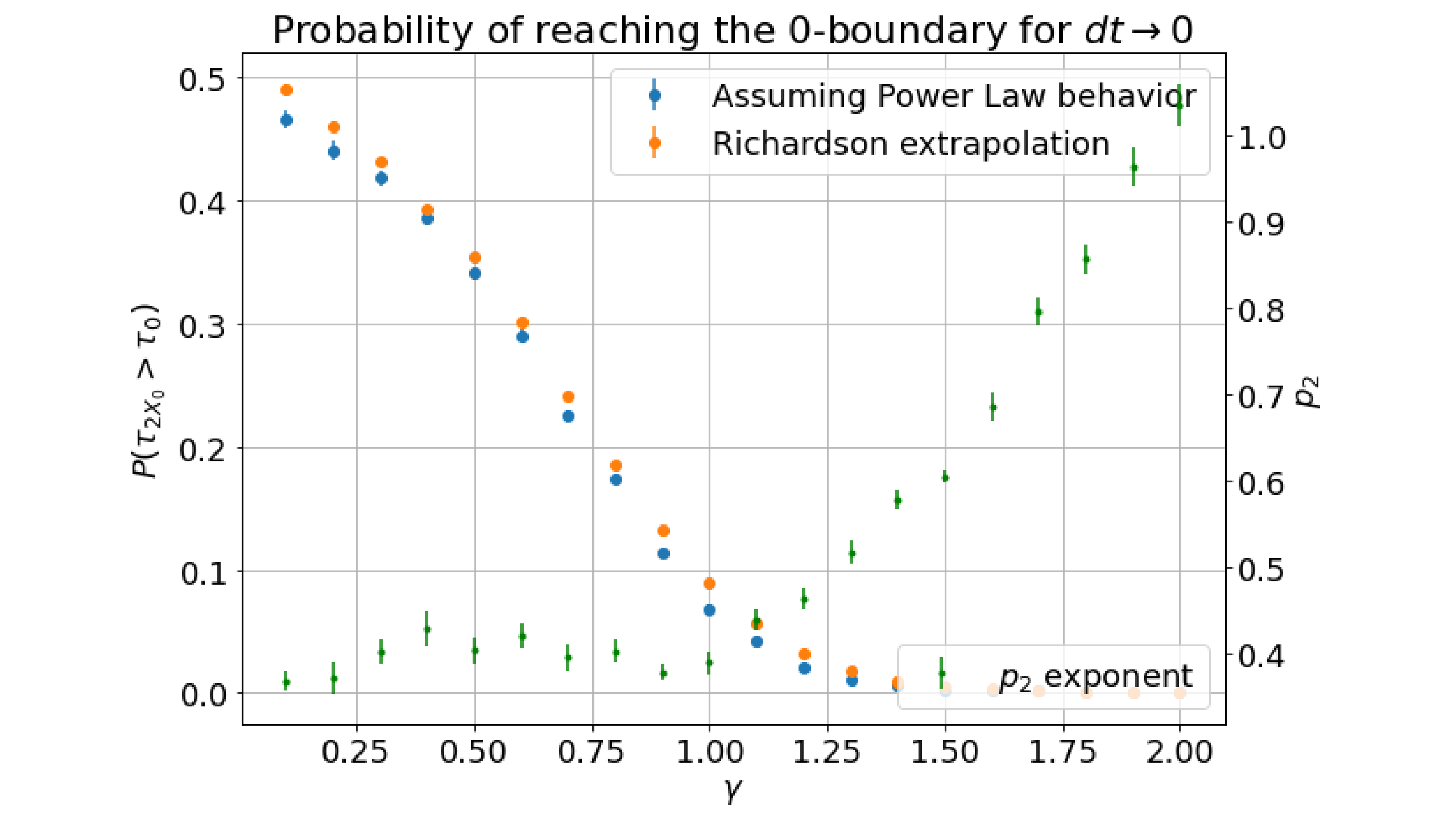
Finding γ_c numerically

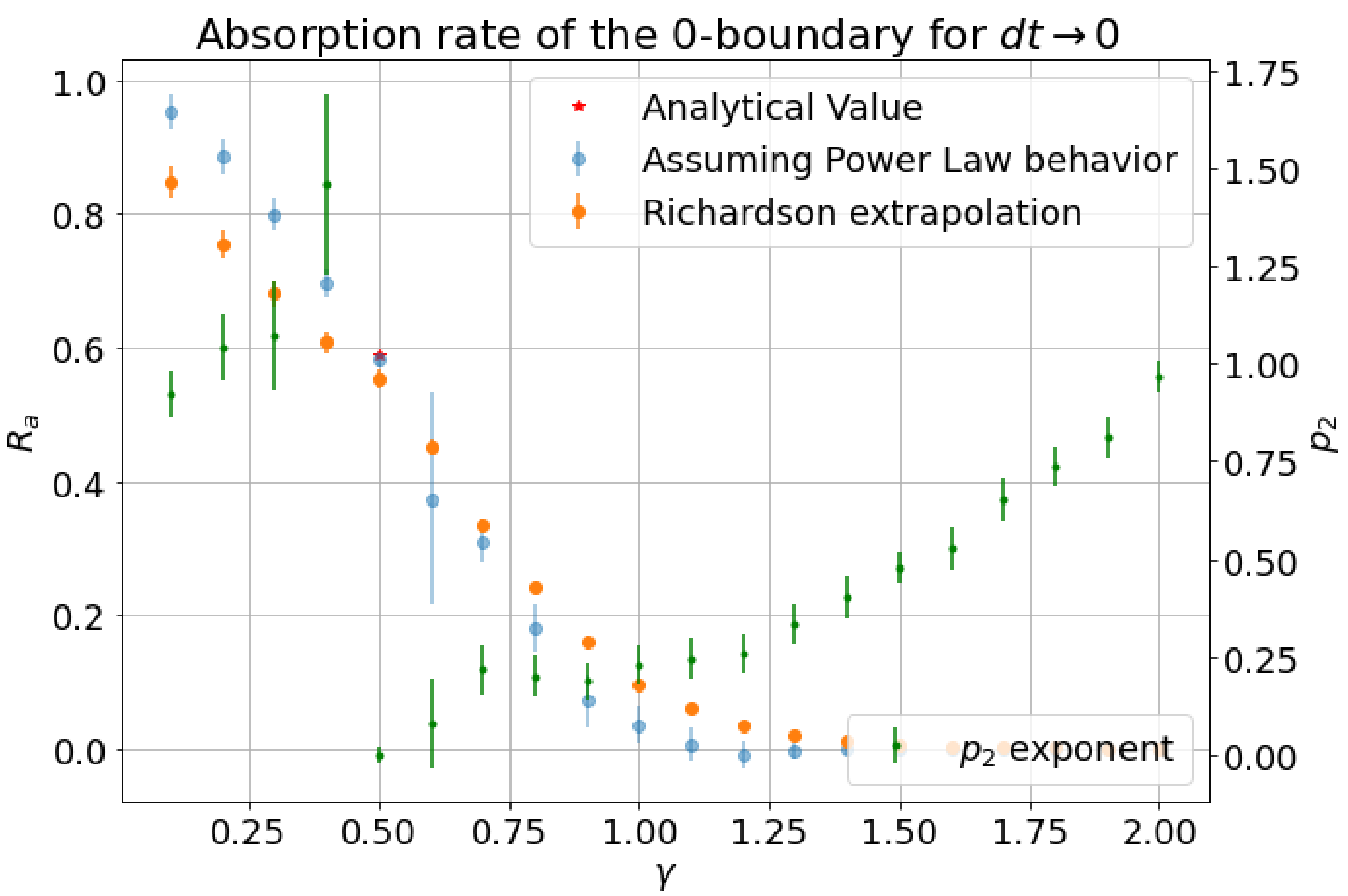
- Used parameters: $\kappa = 1, \theta = 1, X_0 = 0.5, T = 1$
- Order parameter
  - In the **Absorbing** case: R_a
  - In the **Reflective** case: $\Pr(\tau_{2X_0} > \tau_0)$
- Taking the $\Delta t \to 0$ limit
  - Richardson extrapolation
  - Power Law fitting
    - $P(\tau_{2X_0} > \tau_0|\gamma, \Delta t) = P(\tau_{2X_0} > \tau_0|\gamma) + p_1 \Delta t^{p_2}$*
    - $p_2 > 0 \; \forall \, \gamma \Rightarrow \lim_{\Delta t \to 0} (p_1 \Delta t^{p_2}) = 0$
    - Fitting error dominates over the Monte Carlo error
- For γ_c we'll get

γ_c	Richardson	Power Law
Absorbing	1.5	1.1
Reflective	1.5	1.3

*: For the ***Absorbing*** case $R_a(\gamma, \Delta t) = R_a(\gamma) + p_1 \Delta t^{p_2}$




Probability of reaching the 0-boundary for $dt \to 0$
Assuming Power Law behavior
Richardson extrapolation
p_2 exponent
$P(\tau_{2X_0} > \tau_0)$
p_2
γ
0.0
0.1
0.2
0.3
0.4
0.5
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


Absorption rate of the 0-boundary for $dt \to 0$
Analytical Value
Assuming Power Law behavior
Richardson extrapolation
p_2 exponent
R_a
p_2
γ
1.0
0.8
0.6
0.4
0.2
0.0
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Conclusion

- Two different order parameters were introduced $\left(R_a, \Pr\left(\tau_{2X_0} > \tau_0\right)\right)$
- The $\Delta t \to 0$ limit of the Phase Diagram was estimated using Power Law fitting and Richardson extrapolation
- Both methods gave a higher critical point, than the analytical expectation, this could be due to singularities near the critical point

- Further examinations should focus on fining a better extrapolation method, as well as increasing the number and resolution of the simulations

References

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