- Let's suppose u, v fullfills the Bogoliubov-eq.-s.

$$\left.\begin{aligned} H_{HF} u_i - g \Psi_0^2 v_i &= E_i u_i \\ -g \Psi_0^{*2} u_i + H_{HF} v_i &= -E_i v_i \end{aligned}\right\} \qquad E_i \in \mathbb{R}^+$$

$$K_2 = \sum_{ij} \int d^3\vec{r} \Big\{ \alpha_i^+ \alpha_j \left(E_j u_i^* u_j + E_i u_j u_i^* \right) + \alpha_i \alpha_j^+ \left(-E_j v_i v_j^* - E_i v_j^* v_i \right) +$$

$$+ \alpha_i \alpha_j \left(E_j v_i u_j - E_i u_j v_i \right) - \alpha_i^+ \alpha_j^+ \left(-E_j u_i^* v_j^* + E_i v_j^* u_i^* \right) \Big\}$$

$$= \sum_{ij} \int d^3\vec{r} \Big\{ \left(E_i + E_j \right) \alpha_i^+ \alpha_j u_i^* u_j - \left(E_i + E_j \right) \alpha_i \alpha_j^+ v_i v_j^* -$$

$$- \alpha_i \alpha_j \left(E_j \cancel{} - E_i \right) v_i u_j - \alpha_i^+ \alpha_j^+ \left(E_i - E_j \right) u_i^* v_j^* \Big\}$$

$\boxed{2019.11.07.}$

$$K_2 = \frac{1}{2} \sum_{ij} \int d^3\vec{r} \Big\{ \left(E_i + E_j \right) \left[\alpha_i^+ \alpha_j u_i^* u_j - \alpha_i \alpha_j^+ v_i v_j^* \right] +$$

$$+ \left(E_i - E_j \right) \Big[\alpha_i \alpha_j v_i u_j \underset{\left(E_i - E_j \right)}{\underbrace{-}} \alpha_i^+ \alpha_j^+ u_i^* v_j^* \Big] \Big\}$$

- Orthogonality properties (proven similarly as in 1st semester):

$$\delta_{ij} = \int d^3\vec{r} \left(u_i^* u_j - v_i^* v_j \right)$$

$$0 = \int d^3\vec{r} \left(u_i v_j - v_i u_j \right)$$

$$0 = \int d^3\vec{r} \left(u_i^* v_j^* - v_i^* u_j^* \right)$$

- $i \leftrightarrow j$ replacement

$$K_2 = \frac{1}{2}\sum_{ij}\int d^3\vec{r}\,\Big\{(E_i+E_j)\left[\alpha_i^+\alpha_j u_i^* u_j - \alpha_j\alpha_i^+ v_i^* v_j\right] +$$
$$+ E_i\alpha_i\alpha_j\left(v_i u_j \cancel{- v_i v_j}\right) -$$
$$- E_i\alpha_i^+\alpha_j^+\left(u_i^* v_j^* \cancel{- u_j^* v_i^*}\right)\Big\}$$

$\rightsquigarrow$ useful commutator:

$$[\alpha_j, \alpha_i^+] = \delta_{ij}$$

$$\alpha_j\alpha_i^+ = \delta_{ij} + \alpha_i^+\alpha_j$$

$$K_2 = \frac{1}{2}\sum_{ij}\int d^3\vec{r}\,\Big\{(E_i+E_j)\Big[\alpha_i^+\alpha_j\, u_i^* u_j - v_i^* v_j\left(\delta_{ij}+\alpha_i^+\alpha_j\right)\Big]\Big\} =$$

$$= \frac{1}{2}\sum_{ij}\int d^3\vec{r}\,\Big\{(E_i+E_j)\,\alpha_i^+\alpha_j\left(u_i^* u_j - v_i^* v_j\right) - (E_i+E_j)\,\delta_{ij}\, v_i^* v_j\Big\} =$$

$$= \frac{1}{2}\sum_{ij}(E_i+E_j)\alpha_i^+\alpha_j\delta_{ij} - \sum_i\int d^3\vec{r}\; E_i|v_i|^2 =$$

$$= \sum_i E_i\left(\alpha_i^+\alpha_i - \int|v_i|^2\right) \Rightarrow$$

this, too is a 0th order term!

$$\hat{K}_2 = \sum_i E_i\left(\hat{\alpha}_i^+\hat{\alpha}_i - \int d^3\vec{r}\,|v_i|^2\right)$$

$$K^{Popov} = \int d^3\vec{r}\;\Psi_0^*\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + \frac{g}{2}|\Psi_0|^2\right)\Psi_0 - \sum_i E_i\int d^3\vec{r}\,|v_i|^2 +$$
$$+ \sum_i E_i\,\hat{\alpha}_i^+\hat{\alpha}_i$$

$$\langle \alpha_i^+ \alpha_j \rangle = \frac{Tr\left[e^{-\beta \hat{\mathcal{H}}} \alpha_i^+ \alpha_j\right]}{Tr\, e^{-\beta \hat{\mathcal{H}}}} = \frac{Tr\left[e^{-\beta(\mathcal{H}_0 + \sum_\ell \epsilon_\ell \alpha_\ell^+ \alpha_\ell)} \alpha_i^+ \alpha_j\right]}{Tr\left[e^{-\beta(\cancel{\mathcal{H}_0} + \sum_\ell \epsilon_\ell \alpha_\ell^+ \alpha_\ell)}\right]} =$$

$$= \delta_{ij} \frac{\sum_{n_i=0}^{\infty} e^{-\beta \epsilon_i n_i} n_i}{\sum_{n_i=0}^{\infty} e^{-\beta \epsilon_i n_i}} = (*)$$

Fock-space:

$\alpha_\ell |0\rangle = 0 \rightsquigarrow$ grund state energy $= 0$

$N (\alpha_1^+)^{n_1} (\alpha_2^+)^{n_2} \ldots |0\rangle = |n_1 n_2 \ldots\rangle$

$$e^{-\beta \sum_\ell \epsilon_\ell \alpha_\ell^+ \alpha_\ell} |n_1 n_2 \ldots\rangle = \underbrace{e^{-\beta \sum_\ell \epsilon_\ell n_\ell} |n_1 n_2 \ldots\rangle}_{\prod_\ell e^{-\beta \epsilon_\ell n_\ell} |n_\ell\rangle}$$

$|n_1\rangle |n_2\rangle \ldots$

$$Z = \underbrace{\sum_{n_1=0}^{\infty} e^{-\beta \epsilon_1 n_1}}_{\text{geometric series}} \sum_{n_2=0}^{\infty} e^{-\beta \epsilon_2 n_2} \ldots = \prod_{\ell=1}^{\infty} \frac{1}{1 - e^{-\beta \epsilon_\ell}}$$

$$Tr\left[e^{-\beta \sum_\ell \epsilon_\ell \alpha_\ell^+ \alpha_\ell} \alpha_i^+ \alpha_j\right] = \sum \ldots \langle n_1 | \langle n_2 | \ldots | e^{-\beta \epsilon_1 n_1} e^{-\beta \epsilon_2 n_2} \delta_{ij} n_i | \ldots \rangle$$

$$= \left(\sum_{n_1=0}^{\infty} e^{-\beta \epsilon_1 n_1}\right) \ldots \left(\sum_{n_i=0}^{\infty} e^{-\beta \epsilon_i n_i} \delta_{ij} n_i\right) \ldots$$

$$(*) = -\delta_{ij} \frac{\partial}{\partial(\beta \epsilon_i)} \ln\left(\sum_{n_i=0}^{\infty} e^{-\beta \epsilon_i n_i}\right) = \frac{\delta_{ij}}{e^{\beta \epsilon_i} - 1} = \underline{N_B(\epsilon_i) \delta_{ij}}$$

$$\langle \alpha_j \alpha_i^+ \rangle = \delta_{ij} (1 + N_B(\epsilon_i)) = \delta_{ij} \frac{e^{\beta \epsilon_i}}{e^{\beta \epsilon_i} - 1}$$

$$\langle \alpha_i \alpha_j \rangle = 0$$

$$n_T(r) = \langle \hat{\varphi}^+(r)\varphi(r)\rangle = \sum_{ij} \langle (u_i^*\alpha_i^+ - v_i^*\alpha_i^*)(u_j\alpha_j - v_j^*\alpha_j^+)\rangle =$$

$$= \sum_{ij}\Big(u_i^* u_j \underbrace{\langle \alpha_i^+ \alpha_j\rangle}_{\delta_{ij} N_B(\epsilon_i)} + v_i v_j^* \underbrace{\langle \alpha_i \alpha_j^+\rangle}_{\delta_{ij}(1+N_B(\epsilon_i))}\Big) =$$

$$= \sum_i \left[|u_i|^2 N_B(\epsilon_i) + |v_i|^2 (1 + N_B(\epsilon_i))\right]$$

$$\lim_{T\to 0} N_B(\epsilon_i) = 0$$

$$\lim_{T\to 0} n_T(r) = \sum_i |v_i|^2$$ $\rightarrow$ non-vanishing contribution to the density.

"Thermal-depletion"

- anomalical avarage:

$$m_T = \langle \varphi(r)\varphi(r)\rangle = \sum_{ij} \langle (u_i\alpha_i - v_i^*\alpha_i^+)(u_j\alpha_j - v_j^*\alpha_j^+)\rangle =$$

$$= \sum_{ij}(-1)\Big[u_i v_j^* \underbrace{\langle \alpha_i \alpha_j^+\rangle}_{\delta_{ij}(1+N_B)} + v_i^* u_j \underbrace{\langle \alpha_i^+ \alpha_j\rangle}_{\delta_{ij} N_B}\Big] =$$

$$= -\sum_i u_i v_i^* (2N_B(\epsilon_i) + 1) \neq 0$$ but small

$$\left.\begin{array}{ll} m_T \ll n_T(r) & T \ll T_c \\ m_T \ll n(r) & T \approx T_c \end{array}\right\}$$

$\rightarrow m_T$ can be regarded as an other order parameter

- we have to solve the generalized GP:

$$\left[-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + g\left(n_c(r) + 2n_T(r)\right)\right]\Psi_0(r) = 0$$

⇝ non-linear! $n_c(r) = |\Psi_0|^2$

⇝ has to be solved with the correct normalization.

$$N_0 = \int d^3\vec{r}\, |\Psi_0(r)|^2$$

→ Ψ_0, μ is unknown.

→ $n_T(r)$ is also needed!

- for this we need the Bogoliubov-eq:

$$\left.\begin{aligned} H_{HF} U_i - g|\Psi_0|^2 V_i &= E_i U_i \\ -g\Psi_0^{*2} U_i + H_{HF} V_i &= -E_i V_i \end{aligned}\right\}$$

⇝ linear in U_i, V_i

⇝ they have to be normalized accordingly:

$$\left.\begin{aligned} \delta_{ij} &= \int d^3\vec{r}\left(U_i^* U_j - V_i^* V_j\right) \\ 0 &= \int d^3\vec{r}\left(U_i V_j - V_i U_j\right) \\ 0 &= \int d^3\vec{r}\left(U_i^* V_j^* - V_j^* U_j^*\right) \end{aligned}\right\}$$

$$H_{HF} = -\frac{\hbar^2}{2m}\Delta + V(r) - \mu + 2g\left(n_c(r) + n_T(r)\right)$$

$$N = N_T + N_0 \qquad n_T(r) = \sum_i \left(|U_i|^2 N_b(E_i) + |V_i|^2 (N_b + 1)\right)$$

$$N_T = \int d^3\vec{r}\, n_T(r)$$

- this is the closed set of eq.-s
  - they have to be solved <u>simultaneously</u>
- self-consistent iteration.