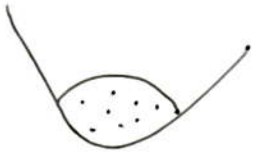
2019.02.18.

– N non-int bosons in HO potential, in 3D



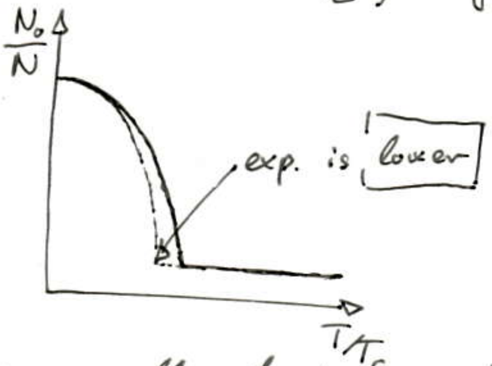
$$V = \frac{1}{2} m \omega_x^2 x^2 + \frac{1}{2} m \omega_y^2 y^2 + \frac{1}{2} m \omega_z^2 z^2$$

$$\rightsquigarrow \quad N = \left(\frac{k_B T_c}{\hbar \bar{\omega}}\right)^3 \zeta(3) \quad \text{in leading order.}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3 \quad \text{for this case.}$$

$$\mathcal{E}_{n_x, n_y, n_z} = \hbar\omega_x\left(n_x + \frac{1}{2}\right) + \hbar\omega_y\left(n_y + \frac{1}{2}\right) + \hbar\omega_z\left(n_z + \frac{1}{2}\right)$$

$$n_x, n_y, n_z = 0, 1, 2, \ldots$$



– there is small deviation from this curve in experiment
↳ finite-size limit.

$\boxed{T > T_c}$

$$N = \sum_{\substack{n_x\\n_y\\n_z}} \frac{1}{e^{\beta(\mathcal{E}_{n_x n_y n_z} - \mu)} - 1}$$

$\boxed{T < T_c}$

$$N = N_0 + \sum_{\substack{n_x\\n_y\\n_z}} \frac{1}{e^{\beta(\mathcal{E}_{n_x n_y n_z} - \mathcal{E}_{000})} - 1}$$

$\rightsquigarrow$ n_x, n_y, n_z cannot be simultaniously 0 in this sum!

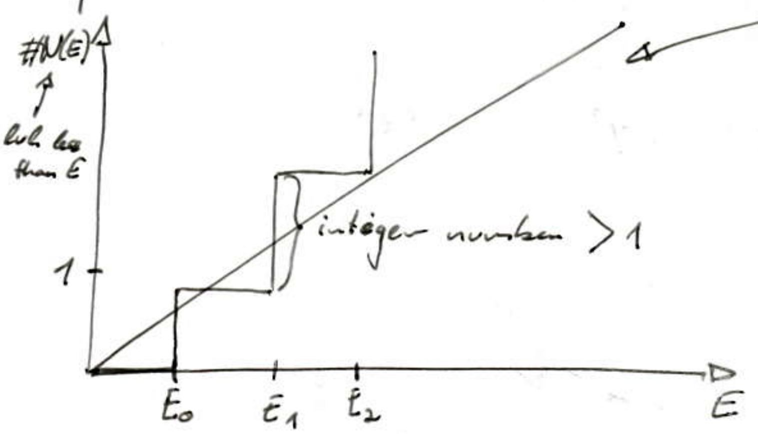
Chaotic behaviour of ~~ch~~ quantum systems

- no trajectory in phase-space
- useful quantity: "level spacing distribution"
  - classically: billiard problem

 circle → integrable

 stadium → chaotic

- Spectral staircase



smooth part

$\bar{N}(E)$

$$\frac{d\bar{N}}{dE} = \underset{\text{D.O.S.}}{\rho(E)}$$

for non-smooth we get Dirac-deltas at the jumps.

- for the biliard problem the $\bar{N}$ does not depend on wether it is chaotic or not.

$$H(p,r) = \frac{\vec{p}\,\vec{p}}{2m} + \text{boundary conditions}$$

$$N_{sc}(E) = \frac{1}{h^2} \int_{H<E} d^2p\, d^2r$$

N_{sc}: semiclassical smoothing

$H<E$: this does not know about energy lvl.

- for HO problem $N_{sc} = \frac{1}{6}\left(\frac{E}{\hbar_0 T}\right)^3 \to \rho_{cs}(E) = \frac{1}{2}\frac{E^2}{(\hbar_0 T)^3}$

  a diffrent smoothing might contain hints at finite size-effect.

Euler-Maclaurin-formula

$$\sum_{n=a}^{b} f(n) \approx \underbrace{\int_a^b dn\, f(n)}_{\text{leading term "L"}} + \underbrace{\frac{f(b)+f(a)}{2}}_{\text{1st correction "1C"}} + \underbrace{\frac{f'(b)-f'(a)}{12}}_{\text{"2C"}} \pm \dots$$

- generally only odd derivatives can be found in the series.

- problem $\to$ non-convergent series

- let $\underset{\in \mathbb{R}}{a, b} > 0$, $\underset{\in \mathbb{N}}{n \geq 0}$

$x\delta(x) = 0$ if $x = 0$

$$\int (a-bx)^n \Theta(a-bx)\,dx = \frac{(a-bx)^{n+1}}{(-b)(n+1)} \Theta(a-bx)$$

this will be the good primitive func.

$$\int_0^\infty (a-bx)^n \Theta(a-bx)\,dx = \left[\frac{(a-bx)^{n+1}}{(-b)(n+1)} \Theta(a-bx)\right]_0^\infty =$$

$$= -\frac{a^{n+1}\Theta(a)}{(-b)(n+1)} = \underline{\underline{\frac{a^{n+1}\Theta(a)}{b(n+1)}}}$$

- Spectral staircase for 3D HO problem:

$$N(E) = \sum_{\substack{n_x\\n_y\\n_z}} \Theta(E - \mathcal{E}_{n_x,n_y,n_z}) = \sum_{n_x n_y n_z = 0}^{\infty} \Theta(E - E_0 - \hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z))$$

$$E_0 = \frac{\hbar\omega_x + \hbar\omega_y + \hbar\omega_z}{2}$$

$$= \sum_{\substack{n_x\\n_y}} \Bigg[\Bigg(\underset{\overset{\uparrow}{L}}{\frac{(E - E_0 - \hbar(\omega_x n_x + \omega_y n_y)}{\hbar\omega_z}} + \underset{\overset{\uparrow}{C}}{\frac{1}{2}}\Bigg) \Theta(E - E_0 - \hbar(\omega_x n_x + \omega_y n_y)) =$$

$$= \sum_{n_x} \left[\frac{(E - E_0 - \hbar\omega_x n_x)^2}{(\hbar\omega_y)(\hbar\omega_z)\cdot 2} + \frac{(E - E_0 - \hbar\omega_x n_x)}{2\hbar\omega_z} + \frac{1}{2}\cdot\frac{(E - E_0 - \hbar\omega_x n_x)}{\hbar\omega_y}\right] \Theta(E - E_0 - \hbar\omega_x n_x)$$

$$= \left[\frac{(E - E_0)^3}{6(\hbar\omega_x)(\hbar\omega_y)(\hbar\omega_z)} + \frac{(E - E_0)^2}{4(\hbar\omega_y)(\hbar\omega_z)} + \frac{(E - E_0)^2}{4(\hbar\omega_x)(\hbar\omega_z)} + \frac{(E - E_0)^2}{4(\hbar\omega_x)(\hbar\omega_y)}\right] \cdot$$

$$\cdot\, \Theta(E - E_0)$$

$$\bar{N}(E) = \Theta(E-E_0)\left[\frac{1}{3}\gamma_2 \frac{(E-E_0)^3}{(\hbar\bar{\omega})^3} + \frac{1}{2}\gamma_1 \frac{(E-E_0)^2}{(\hbar\bar{\omega})^2}\right]$$

$$\bar{\omega} = (\omega_1 \omega_2 \omega_3)^{1/3}$$

$$\gamma_2 = \frac{1}{2} \; ; \; \gamma_1 = \frac{\omega_x + \omega_y + \omega_z}{\bar{\omega}}$$

$$\bar{\rho}(E) = \frac{d\bar{N}}{dE} = \Theta(E-E_0)\left[\gamma_2 \frac{(E-E_0)^2}{(\hbar\bar{\omega})^3} + \gamma_1 \frac{(E-E_0)}{(\hbar\bar{\omega})^2}\right]$$

appears naturally!
there is nothing below
grnd. state.

$T > T_c$

$$\langle N \rangle = \sum_{\substack{n_x=0\\n_y=0\\n_z=0}}^{\infty} \frac{1}{e^{\beta(\underbrace{\varepsilon_{n_x n_y n_z}}_{=\hbar(\underline{\omega}\,\underline{n})+\varepsilon_0}-\mu)}-1} = \int_{-\infty}^{\infty} \rho(E)\frac{dE}{e^{\beta(E-\mu)}-1} =$$

$$= \int_{E_0}^{\infty} dE \left[\frac{(E-E_0)^2}{(\hbar\bar{\omega})^3}\gamma_2 + \frac{(E-E_0)}{(\hbar\bar{\omega})^2}\gamma_1\right]\frac{1}{e^{\beta(E-\mu)}-1} \underset{z=E-E_0}{=}$$

$$= \int_0^{\infty} dz \left[\gamma_2 \frac{z^2}{(\hbar\bar{\omega})^3} + \gamma_1 \frac{z}{(\hbar\bar{\omega})^2}\right]\frac{1}{e^{\beta(z-(\mu-E_0))}-1} \underset{\text{both B-E integrals}}{=}$$

$$F_-(s,\alpha) = \frac{1}{\Gamma(s)}\int \frac{x^{s-1}\,dx}{e^{x+\alpha}-1}$$

$$\langle N \rangle = \frac{\gamma_2 \Gamma(3)}{(\beta\hbar\bar{\omega})^3} F_-\left(3, \frac{\varepsilon_0-\mu}{k_B T}\right) + \frac{\gamma_1 \Gamma(2)}{(\beta\hbar\bar{\omega})^2} F_-\left(2, \frac{\varepsilon_0-\mu}{k_B T}\right)$$

finite-size correction.

- if $T \to T_c$ then $\mu \to \varepsilon_0$ $\quad F_-(s,0) \to \zeta(s)$

$$\boxed{N = \left(\frac{k_B T_c}{\hbar\bar{\omega}}\right)^3 \zeta(3) + \gamma_1 \left(\frac{k_B T_c}{\hbar\bar{\omega}}\right)^2 \zeta(2)}$$

- we can calculate the correction to T_c

$\rightsquigarrow$ 3rd power order polynomial eq.

$\rightsquigarrow$ we calc. the corr. perturbatively.

$$N = \left(\frac{k_B T_0}{\hbar\bar{\omega}}\right)^3 \zeta(3) \qquad \text{then} \qquad T_c = T_0 + \delta T$$

leading order approx.

$\frac{\delta T}{T_0}$: small.

$$N = \left(\frac{k_B T_0}{\hbar\bar{\omega}}\right)^3 \zeta(3) \cdot \underbrace{\left(\frac{T_c}{T_0}\right)^3}_{\left(\frac{T_0+\delta T}{T_0}\right)^3 \approx 1 + 3\frac{\delta T}{T_0}} + \gamma_1 \left(\frac{k_B T_0}{\hbar\bar{\omega}}\right)^2 \zeta(2)$$

$$0 = 3\frac{\delta T}{T_0}\left(\frac{k_B T_0}{\hbar\bar{\omega}}\right)^3 \zeta(3) + \gamma_1 \left(\frac{k_B T_0}{\hbar\bar{\omega}}\right)^2 \zeta(2)$$

$\rightsquigarrow$ solve for $\frac{\delta T}{T_0}$

$$\frac{\delta T_0}{T} = -\frac{\gamma_1}{3}\left(\frac{\hbar\bar{\omega}}{k_B T_0}\right)\frac{\zeta(2)}{\zeta(3)} = -\frac{\gamma_1}{3}\frac{\zeta(2)}{\zeta^{2/3}(3)} N^{1/3}$$

this answers that δT is smaller than T_0 and can be considered as a correction

it lowers the critical Temp.