$$\omega^2 \delta n(\vec{r}) = -\frac{1}{m} \vec{\nabla}\left(\mu - \frac{1}{2} m \omega_0^2 r^2\right) \vec{\nabla} \delta n(\vec{r})$$

$$\omega^2 = \omega_0^2 (2n^2 + 2nl + 3n + l)$$

$$\text{with} \quad \left.\begin{array}{l} n \in 0, 1, 2, \dots \\ l \in 0, 1, 2, \dots \\ m \in -l, \dots, l \end{array}\right\}$$

this is called here hypergeometric polinomial.

$$\delta n_{n,l,m}(\vec{r}) = r^l Y_{lm}(\vartheta, \varphi)\, {}_2F_1\left(-n, n + l + \frac{3}{2}, l + \frac{3}{2}, \frac{r^2}{a^2}\right)$$

where $a = \sqrt{\frac{2\mu}{m\omega_0^2}}$ is the Thomas-Fermi-radius.

(apart from normalization.)

This is the general solution for isotopic system.

$\boxed{2019.09.26.}$

(Not for the exam...)

Standard map: $S_{t+1} = f(\underline{x}_t)$

- is the trajectory limited on a hyper-surface?
- components connected someway
- has a 1 step memory
- good way to test, wether a system is integrable or not.
  (or a subsytem)

  $\rightsquigarrow$ if "chatoic sea" is present (~~...~~) it isn't integrable..

The case of the Bogulibov-eq.:

$$\mathcal{E}\begin{pmatrix} U_i \\ V_i \end{pmatrix} = \begin{pmatrix} H_{HF} & -g\Psi_0^2 \\ g\Psi_0^{*2} & -H_{HF}\Psi_0 \end{pmatrix}\begin{pmatrix} U_i \\ V_i \end{pmatrix}$$

$$H_{HF} = \left(-\frac{\hbar^2}{2m}\Delta\right.$$

$$\hookrightarrow \frac{p^2}{2m}$$

$$0 = \begin{pmatrix} H_{HF} - E & -g\Psi_0^2 \\ g\Psi_0 & -H_{HF} - E \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$$\det \overset{!}{=} 0 \rightsquigarrow E = \sqrt{\left(\frac{p^2}{2m} \ldots\right.}$$

- then $E(p,q)$ will be used as the classical Hamiltonian.

$$\overset{\shortparallel}{H}(p,r) \qquad \dot{\underline{r}} = \frac{\partial H}{\partial \underline{p}} \qquad \underline{p} = -\frac{\partial H}{\partial \underline{r}}$$

- with this we can construct the 6D phase-space
- then choosing a nice hypersurface $\rightsquigarrow$ Poincaré-mapping
- for the Bogoliubov-eq. $\rightsquigarrow$ it is not integrable

$$H_{HF} = -\frac{\hbar^2}{2m}\Delta + \underset{\substack{\uparrow \\ \text{3d HO}}}{V(r)} + 2g|\Psi_0|^2 - \mu$$

$\rightsquigarrow$ E is conserved along the trajectory
$\rightsquigarrow$ it is suggested there is no other conserved quantity

$\rightsquigarrow$ other params: $\omega_x, \omega_y, \omega_z, m, \mu$
$\rightarrow$ if $\mu \sim E$ the system has large chaotic sea
$\left.\begin{array}{l}\rightarrow \text{if } E \rightarrow 0 \\ \rightarrow \text{if } E \gg \mu\end{array}\right\}$ the chaotic sea is gone

$\downarrow$

the system is integrable

- the small energy limit is interesting

↳ low-lying excitations

- if we go to $E \to 0$, & small / big λ

$\downarrow$

hydrodynamical approx:

$$\omega^2 \varphi = - \vec{\nabla}\left(\frac{\mu - V}{m}\right)\vec{\nabla}\varphi$$

- this is now an integrable system
- what is conserved?
  - $\to$ separating Hamilton - Jacobi eq. with some nice coordinates

Isotropic sys.: $(x_1, x_2, x_3) \to (r, \vartheta, \varphi)$ "spherical"

Axial sym.: $(x_1, x_2, x_3) \to (\rho, \varphi, z)$ "cylindrical"

Ellyptic sym.: $\to (\xi, \eta, \varphi)$

$\downarrow$ can be related to cylindrical

$\to$ with the good coordinates the QM - eq. becomes separable aswell.

- for $\forall$ separation constant $\exists$ a conserved quantity

ex.: $H\Psi = E\Psi$

(r, ϑ, φ)

$\Psi = R(r) \cdot \underbrace{Y_{lm}(\vartheta, \varphi)}_{P_{lm}(\cos\vartheta) \cdot e^{im\varphi}}$ $\to \Psi$ is a product.

separating ansatz: $\Psi = R(r) P(\vartheta) T(\varphi)$

$T(\varphi)$: quantization for m

$P(\vartheta)$: quantization for l

} separation const.-s

$R(r)$: with known const.-s this part can be solved...

• for the separation constant we can find the operators that have those eigenvalues.

$$V = \frac{1}{2} m\omega_1^2 x_1^2 + \frac{1}{2} m\omega_2^2 x_2^2 + \frac{1}{2} m\omega_3^2 x_3^2$$

$$\underset{\delta m}{\omega^2}\varphi = -\frac{1}{m}\vec{\nabla}(\mu - V)\vec{\nabla}\varphi \qquad \text{eigenvalue eq. for } \varphi$$

scalar product: $\langle \varphi_i | \varphi_j \rangle = \int_{V_{TF}} d^3r \, \varphi_i^*(x)\varphi_j(x) = \delta_{ij}$

if $x \in V_{TF}$ $\mu > V(r)$

$C_0 = \sqrt{\frac{\mu}{m}}$ has dimension of velocity "speed of sound"

$$\frac{1}{2}\frac{m\omega_1^2 x_1^2}{\mu} = \frac{x_1^2}{a^2} \rightsquigarrow a = \sqrt{\frac{2\mu}{m\omega_1^2}}$$

$$b = \sqrt{\frac{2\mu}{m\omega_2^2}}$$

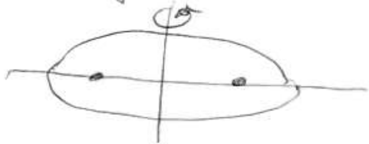
$$c = \sqrt{\frac{2\mu}{m\omega_3^2}}$$

semi-axis of the T-F ellipsoid.

$$\frac{\omega^2}{c_0^2}\varphi = -\vec{\nabla}\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}\right)\vec{\nabla}\varphi$$

• this is a general ellipsoid ~~for~~ with $a \neq b \neq c$

• there are 2 types of 3D elliptic coords



→ these cases are topologically different (prolate, oblate)

→ one of them has a "foci circle" the other keeps "2 foci"

$a > b > c > 0$

$$\frac{x_1^2}{a^2+\rho} + \frac{x_2^2}{b^2+\rho} + \frac{x_3^2}{c^2+\rho} = 1$$

→ eq. for ρ in any space point

↳ this is 3rd order!

$$x_1^2(b^2+\rho)(c^2+\rho) + x_2^2(a^2+\rho)(c^2+\rho) + x_3^2(a^2+\rho)(b^2+\rho) =$$

$$= (a^2+\rho)(b^2+\rho)(c^2+\rho)$$

<u>statement</u>: this has 3 <u>real</u> roots.

$$-a^2 \leq \nu \leq -b^2 \leq \mu \leq -c^2 \leq \lambda \leq 0$$

$$\rho_{1,2,3} = \nu, \mu, \lambda$$

- these ν, μ, λ are the good coordinates

$$x_1 = \pm\sqrt{\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}}$$ (others with cyclic permutation)

$$x_2 = \pm\sqrt{\frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)}}$$

$$x_3 = \pm\sqrt{\frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}}$$

$$\frac{\partial}{\partial\lambda} = \left(\frac{\partial x_1}{\partial\lambda}\right)\frac{\partial}{\partial\lambda} + \left(\frac{\partial x_2}{\partial\lambda}\right)\frac{\partial}{\partial\lambda} + \left(\frac{\partial x_3}{\partial\lambda}\right)\frac{\partial}{\partial\lambda}$$

$$\frac{\partial}{\partial\mu} = \ldots$$

$$\frac{\partial}{\partial\nu} = \ldots$$

$\rightsquigarrow$ the eq. in new coordinates:

$$a^2 b^2 c^2 \frac{\omega^2}{c_0^2}\varphi = -\frac{4\lambda\mu\nu}{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}\left\{(\mu-\nu)\left[F(\lambda)\frac{\partial^2}{\partial\lambda^2} + \left(\frac{F(\lambda)}{\lambda} + \frac{1}{2}F'(\lambda)\right)\frac{\partial}{\partial\lambda}\right] + \text{cyclic}\right\}\varphi$$

$$F(\rho) = (a^2+\rho^2)(b^2+\rho^2)(c^2+\rho^2)$$

- this can be separated:

$$\varphi = \varphi_\lambda(\lambda)\,\varphi_\mu(\mu)\,\varphi_\nu(\nu)$$

$$\rightarrow \quad 0 = (\mu-\nu)\,g_\lambda(\lambda) + (\nu-\lambda)\,g_\mu(\mu) + (\lambda-\mu)\,g_\nu(\nu)$$

$$g_\rho(\rho) = \frac{1}{\varphi_\rho(\rho)}\left[-F(\rho)\frac{d^2}{d\rho^2} - \left(\frac{F(\rho)}{\rho} + \frac{F'(\rho)}{2}\right)\frac{d}{d\rho} + \frac{a^2b^2c^2\omega^2}{4\rho\,c_0^2}\right]\varphi_\rho(\rho)$$

trivial solution: $\to g_\lambda = g_\mu = g_\nu = \text{const.}$

$\to g_\rho(\rho) = \rho$ linear func.

const + const. $\cdot\,\rho$

$\downarrow$

this is THE solution

there are no more.

$$g_\lambda(\lambda) - \underbrace{\frac{\mu g_\nu(\nu) - \nu g_\mu(\mu)}{\mu - \nu}}_{\substack{\text{this must} \\ \text{be a const.} \\ \text{in } \lambda!}} + \lambda \underbrace{\frac{g_\mu(\mu) - g_\nu(\nu)}{\mu - \nu}}_{-\frac{B}{4}}.$$

$$\left.\begin{aligned} g_\lambda(\lambda) &= -\frac{A}{4} - \frac{B}{4}\lambda \\ g_\mu(\mu) &= -\frac{A}{4} - \frac{B}{4}\mu \\ g_\nu(\nu) &= -\frac{A}{4} - \frac{B}{4}\nu \end{aligned}\right\}$$

A, B are non-trivial separation constant.

$\downarrow$

due to cyclic symmetry of the variables.

$$g_\rho(\rho) = \frac{1}{\varphi_\rho(\rho)} \left[\ldots \right] \varphi_\rho(\rho) = -\frac{A}{4} - \frac{B}{4}\rho$$

3 separational const.

$$\left.\begin{aligned} \hat{F}_\rho \, \varphi_\rho(\rho) &= \left(a^2 b^2 c^2 \frac{a^2}{c^2} + A\rho + B\rho^2 \right) \varphi_\rho(\rho) \\ \text{and } \hat{F}_\rho &= 4\left[\rho F(\rho) \frac{\partial^2}{\partial \rho^2} + \left(F(\rho) + \frac{1}{2}\rho F'(\rho) \right) \frac{\partial}{\partial \rho} \right] \end{aligned}\right\}$$

these are 3 eq.-s: λ, μ, ν