$$i\hbar \sum_i \left(-i\omega_i u_i e^{-i\omega_i t} - i\omega_i v_i^* e^{i\omega_i t}\right) =$$

$$- \sum_i \left[\left(e^{-i\omega_i t} \hat{H}_{HF} u_i - e^{i\omega_i t} \hat{H}_{HF} v_i^*\right) + g \Psi_0^2 \left(v_i^* e^{i\omega_i t} - v_i e^{-i\omega_i t}\right)\right]$$

- we gather all term $\sim e^{-i\omega_i t}$:

$$\hbar \omega_i u_i = \hat{H}_{HF} u_i - g \Psi_0^2 v_i$$

- terms with $\sim e^{i\omega_i t}$:

$$\hbar \omega_i v_i^* = - \hat{H}_{HF} v_i^* + g \Psi_0^2 u_i^* \quad / ()^*;\ (-1).$$

$$\left.\begin{aligned} \hbar \omega_i u_i &= \hat{H}_{HF} u_i - g \Psi_0^2 v_i \\ -\hbar \omega_i v_i &= \hat{H}_{HF} v_i - g \Psi_0^{*2} u_i \end{aligned}\right\}$$

- 2x2 mx structure:

$$\hbar \omega_i \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{H}_{HF} & -g\Psi_0^2 \\ g\Psi_0^{*2} & -\hat{H}_{HF} \end{pmatrix}}_{\underline{\underline{H}}} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$$\underline{u}_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$$\rightsquigarrow \boxed{\hbar \omega_i \underline{u}_i = \underline{\underline{H}}\, \underline{u}_i}$$

- delicate question: what is the scalar product for with $\underline{\underline{H}}$ is Hermitian?
  $\rightsquigarrow$ otherwise ω_i can be imaginary!!
- <u>Statement:</u> $\underline{\underline{H}} = \underline{\underline{H}}^+$ with the scalar product:

$$\langle \underline{u}_1 | \underline{u}_2 \rangle = \int d^3r \left(u_1^* u_2 - v_1^* v_2\right)$$

2019.04.08.

$\rightarrow$ usual "physics" way of scalar product: $\int_{-\infty}^{\infty} f^*(x) g(x) \cdot \rho(x) = \langle f | g \rangle$

ρ can be singular (weight function)

- finding the right ρ $\rightsquigarrow$ $\underline{\underline{H}}$ can be hermitian.
- other way: knowing the proper scalar product, and proving it's all right.

$$\underset{\substack{\uparrow\\ \text{spinor}}}{\underline{U}_i} = \begin{pmatrix} u_i(r) \\ v_i(r) \end{pmatrix} \rightsquigarrow \langle \underline{U}_1 | \underline{U}_2 \rangle = \int d^3r \left(u_1^*(r)\, u_2(r) - v_1^*(r) \cdot v_2(r) \right)$$

$$\underline{\underline{H}} = \begin{pmatrix} H_{HF} & -g\psi_0^2 \\ g\psi_0^{*2} & -H_{HF} \end{pmatrix} \quad \text{and} \quad H_{HF} = \left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + 2g|\psi_0(r)|^2 \right)$$

- we need to prove, that

$$\langle \underline{U}_1 | \underline{\underline{H}}\, \underline{U}_2 \rangle - \langle \underline{U}_2 | \underline{\underline{H}}\, \underline{U}_1 \rangle^* = 0 \quad \text{for } \forall\, \underline{U}_1, \underline{U}_2 \rightsquigarrow \underline{\underline{H}} = \underline{\underline{H}}^+$$

(since $\langle a|H|b\rangle$
$\langle a|H^+|b\rangle = \langle b|H|a\rangle^*$)

$$\underline{\underline{H}}\,\underline{U}_1 = \begin{pmatrix} H_{HF}u_1 - g\psi_0^2 v_1 \\ g\psi_0^{*2} u_1 - H_{HF} v_1 \end{pmatrix}$$

$$\langle \underline{U}_1 | \underline{\underline{H}}\underline{U}_2 \rangle - \langle \underline{U}_2 | \underline{\underline{H}}\underline{U}_1 \rangle = \int d^3r \left[u_1^* \left(H_{HF} u_2 - \cancel{g\psi_0^2 v_2} \right) - v_1^* \left(\cancel{g\psi_0^{*2} u_2} - \right.\right.$$
$$\left.\left. - H_{HF} v_2 \right) \right] - \int d^3r \left[u_2^* \left(H_{HF} u_1 - \cancel{g\psi_0^2 v_1} \right) - v_2^* \left(\cancel{g\psi_0^{*2}} u_1 - H_{HF} v_1 \right) \right]^* =$$

$H_{HF}^* = H_{HF}$ ✓

$$= \int d^3r \left[u_1^* \left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + 2g|\psi_0|^2 \right) u_2 - u_2 \left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + 2g|\psi_0|^2 \right) u_1^* \right] +$$
$$- \int d^3r \left[v_1^* (\quad - \text{''} - \quad) v_2 - v_2 (\quad - \text{''} - \quad) v_1^* \right] =$$

$\rightsquigarrow$ only the Δ is left:

$$= -\frac{\hbar^2}{2m}\int d^3r \left[u_1^*(\Delta u_2) - u_2(\Delta u_1^*) + v_2(\Delta v_1^*) - v_1^*(\Delta v_2)\right] =$$

identity: $a\Delta b - b\Delta a = \vec{\nabla}(a\vec{\nabla}b - b\vec{\nabla}a)$

$$= -\frac{\hbar^2}{2m}\int d^3r \; \mathrm{div}\left[u_1^*\vec{\nabla}u_2 - u_2\vec{\nabla}u_1^* + v_2\vec{\nabla}v_1^* - v_1^*\vec{\nabla}v_2\right] \overset{\oint dA \ldots}{=}$$

on infinite volumes the fields go $\to 0$

$= 0$

$\Longrightarrow \underline{\underline{H}} = \underline{\underline{H}}^+$ $\square$ ($\underline{\underline{H}}$ is hermitian.)

- for any other kind of scalar product $\underline{\underline{H}}$ won't be hermitian.

$\langle \underline{v}_i | \underline{v}_j \rangle = \delta_{ij} = \int d^3r \, (u_i^* u_j - v_i^* v_j)$ <u>Normalization</u> of the modes

problem: $\delta\Psi = \sum_i (u_i e^{-i\omega_i t} - v_i^* e^{i\omega_i t})$

$\rightsquigarrow$ has to be infinitesimally small!

$\rightarrow u_i = \alpha \, u_i$ $\leftarrow$ normalized u_i

small u_i in $\delta\Psi$; proportionality factor !!!

- there is an other scalar product. (with orthogonality)

$\hbar\omega_i = E_i$; $E_i \in \mathbb{R}^+$

$\rightsquigarrow$ for most of the confining $V(\vec{r})$ $\cancel{E_i = 0}$; $\boxed{E_i > 0}$

- the solution to the GP-eq. with stationary solution is <u>unique</u> for a given μ.

$$\left.\begin{aligned} H_{HF} v_i - g \Psi_0^2 v_i &= E_i v_i \\ -g \Psi_0^{*2} u_i + H_{HF} v_i &= -E_i v_i \end{aligned}\right\} \qquad \begin{aligned} &/ \textstyle\int v_j \cdot \\ &/ \textstyle\int u_j \cdot \end{aligned}$$

$$\int d^3r \left[v_j (H_{HF} u_i - g \Psi_0^2 v_i) + u_j (H_{HF} v_i - g \Psi_0^{*2} u_i) \right] = E_i \int d^3r \, (v_j u_i - u_j v_i)$$

$i \leftrightarrow j$ on both sides

$$\int d^3r \left[v_i (H_{HF} u_j - g \Psi_0^2 v_j) + u_i (H_{HF} v_j - g \Psi_0^{*2} u_j) \right] = E_j \int d^3r \, (u_j v_i - u_i v_j)$$

$-$

$$\int d^3r \left[v_j H_{HF} u_i - v_i H_{HF} u_j + u_j H_{HF} v_i - u_i H_{HF} v_j \right] = (E_i + E_j) \int d^3r \, (u_i v_j - v_i u_j)$$

$$(E_i + E_j) \int d^3r \, (u_i v_j - v_i u_j) = \int d^3r \Big[v_j \Big(-\frac{\hbar^2}{2m} \Delta + \cancel{V(r)} - \cancel{\mu} + \cancel{2g|\Psi_0|^2} \Big) u_i -$$

$$- v_i \Big(-\frac{\hbar^2}{2m} \Delta + \cancel{V(r)} - \cancel{\mu} + \cancel{2g|\Psi_0|^2} \Big) u_j + u_j \Big(-\frac{\hbar^2}{2m} \Delta + \ldots \Big) v_i - u_i \Big(-\frac{\hbar^2}{2m} \Delta + \ldots \Big) v_j \Big] =$$

$$= -\frac{\hbar^2}{2m} \int d^3r \left[v_j \Delta u_i - v_i \Delta u_j + u_j \Delta v_i - v_i \Delta u_j \right] \overset{\text{identity}}{=}$$

$$= -\frac{\hbar^2}{2m} \int d^3r \; \mathrm{div} \left[v_j \vec{\nabla} u_j - u_i \vec{\nabla} v_j + u_j \vec{\nabla} v_i - v_i \vec{\nabla} u_j \right] \longrightarrow 0$$

on the surface...

$$\rightarrow \quad (E_i + E_j) \int d^3r \, (u_i v_j - v_i u_j) = 0$$

- for true excitation $E_i > 0$, and non-degenerate ground state

$$\boxed{\left.\begin{aligned} &\int d^3r \, (u_i v_j - v_i u_j) = 0 \\ &\int d^3r \, (u_i^* v_j^* - v_i^* u_j^* = 0 \end{aligned}\right\}}$$

(also must be true)

- this kind of relation does not exist for scalar Hamiltonians

- $\underline{\underline{H}}$ can be diagonalized if all the <u>three</u> orthogonal properties are used.

- this model is good for weakly interacting bosons, like alkalie atoms. (for liquid He it is no good)

<u>Bogulibov excitations in a homogenious system</u>

- particles in a box
  - periodic boundary conditions
- $V = 0$
- G-P eq.:

$$\left(-\frac{\hbar^2}{2m}\Delta + V + g|\Psi_0|^2\right)\Psi_0 = \mu\Psi_0$$

$\rightarrow$ symmetry reasons: $\Psi_0 = \text{const.}$

$$\Rightarrow \boxed{\mu = g|\Psi_0|^2 = gn}$$

n: density of the condenced atoms

$$|\Psi_0|^2 = \frac{\mu}{g}$$

- problem: there is a $V \rightsquigarrow \Psi_0$ ain't const. $\rightsquigarrow \Delta$ does stuff..

- Bogulibov - eq:

$$\begin{pmatrix} H_{HF} & -g\Psi_0^2 \\ g\Psi_0^{*2} & -H_{HF} \end{pmatrix}\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \epsilon_i \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$E_i \longrightarrow E_{\vec{q}}$

- continuous spectra! (no confining potential...)

$$\begin{pmatrix} U_i \\ V_i \end{pmatrix} = e^{i\vec{q}\vec{r}} \begin{pmatrix} U_q \\ V_q \end{pmatrix}$$ $\longrightarrow$ same $\vec{r}$ dep. in both comp.

$$H_{HF}\, e^{i\vec{q}\vec{r}} = \Big(\frac{\hbar^2 q^2}{2m} + \cancel{V} \underbrace{-\mu + 2g|\psi_0|^2}_{\substack{-gn + 2gn \\ gn}}\Big) e^{i\vec{q}\vec{r}} = \Big(\frac{\hbar^2 q^2}{2m} + gn\Big) e^{i\vec{q}\vec{r}}$$

$$\begin{pmatrix} \frac{\hbar^2 q^2}{2m} + gn & -gn \\ gn & -\frac{\hbar^2 q^2}{2m} - gn \end{pmatrix} \cancel{e^{i\vec{q}\vec{r}}} \begin{pmatrix} U_q \\ V_q \end{pmatrix} = E_q \cancel{e^{i\vec{q}\vec{r}}} \begin{pmatrix} U_q \\ V_q \end{pmatrix}$$

$\longrightarrow$ linear problem $\longrightarrow$ homogenious linear algebraic eq. for U_q, V_q

on

eigenvalue problem

$$\begin{pmatrix} \frac{\hbar^2 q^2}{2m} + gn - E_q & -gn \\ gn & -\frac{\hbar^2 q^2}{2m} - gn - E_q \end{pmatrix} \begin{pmatrix} U_q \\ V_q \end{pmatrix} = 0$$

$$\Big(\frac{\hbar^2 q^2}{2m} + gn - E_q\Big)\Big(-\frac{\hbar^2 q^2}{2m} - gn - E_q\Big) - (gn)^2 \stackrel{!}{=} 0$$

$$-\Big[\Big(\frac{\hbar^2 q^2}{2m} + gn\Big)^2 - E_q^2\Big] - (gn)^2 \stackrel{!}{=} 0$$

$$E_q = \pm\sqrt{\Big(\frac{\hbar^2 q^2}{2m} + gn\Big)^2 - (gn)^2}$$

• cont. func. of k

$$\boxed{E_k = \sqrt{2\frac{\hbar^2 k^2}{2m} gn + \left(\frac{\hbar^2 k^2}{2m}\right)^2}}$$

Bogoliubov-spectrum

• dispersion relation for weakly interacting bosons

• for $k \to 0$ $(k \ll k_c)$ $E_k \sim |\vec{k}|$ phonon-like dependance

"cross-over" ↝ no critical behaviour!

$$E_k = \hbar\omega_k = \hbar c k$$

$$\parallel$$

$$\hbar\sqrt{\frac{gn}{m}}\,|\vec{k}| \longrightarrow c_B = \sqrt{\frac{gn}{m}} = \frac{\hbar}{m}\sqrt{4\pi a n}$$

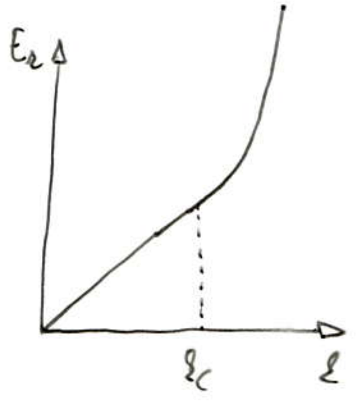
Bogoliubov sound speed

• for $k \to \infty$

$$\sqrt{1+x} = 1 + \frac{x}{2}$$

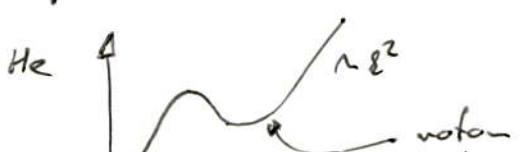
$$E_k = \left(\frac{\hbar^2 k^2}{2m}\right)\sqrt{1 + \underbrace{\frac{2gn}{\left(\frac{\hbar^2 k^2}{2m}\right)}}_{\text{small}}} \approx \left(\frac{\hbar^2 k^2}{2m}\right)\left(1 + \frac{gn}{\left(\frac{\hbar^2 k^2}{2m}\right)}\right) =$$

$$E_k = \frac{\hbar^2 k^2}{2m} + gn$$ ↝ shifted quadratic behavior in $\vec{k}$



• and k_c ↝ $1 = \frac{gn}{\left(\frac{\hbar^2 k_c^2}{2m}\right)}$

• for liquid He the excitation spectrum starts linearly:



→ it is wrong inbetween the limiting cases

can be measured by neutron scattering (not easy...)