const: $\omega^2 A = -\frac{\mu}{m}(4B + 2C)$ $\rightarrow$ solving the others for B, C, ω^2 we can get A

$$\omega^2 \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 4\omega_0^2 & \omega_0^2 \\ 2\omega_z^2 & 3\omega_z^2 \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}$$

$\rightarrow$ eigenvalue-eq for ω^2

$$(4\omega_0^2 - \omega^2)(3\omega_z^2 - \omega^2) - \omega_0^2\, 2\omega_z^2 = 0 \qquad (\det = 0)$$

$$\omega^2 = 2\omega_0^2 + \frac{3}{2}\omega_z^2 \pm \frac{1}{2}\sqrt{9\omega_z^2 - 16\omega_0^2\omega_z^2 + 16\omega_0^4}$$

- this was the easiest mode to prepare in BEC-experiments
  $\rightarrow$ the mode was exactly there
- μ does not play any role in ω freq.
  $\downarrow$
  ω is in the order of trap frequencies.

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3.

$$\omega^2 \delta n = -\frac{1}{m}\vec{\nabla}(\mu - V)\vec{\nabla}\delta n$$

$\omega_x = \omega_y = \omega_z = \omega_0$

with $V = \frac{1}{2} m \omega_0^2 r^2$ isotropic trap potential

$$\omega^2 \delta n = -\frac{1}{m}(\mu - V)\Delta\delta n + \underbrace{\frac{(\vec{\nabla} V)}{m}(\vec{\nabla}\delta n)}$$

$$\vec{\nabla} V = m\omega_0^2 (x, y, z) \longrightarrow \omega_0^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\delta n$$

$\omega_0^2\, r\frac{\partial}{\partial r}\delta n$ in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\Delta_{\theta,\varphi}}{r^2}$$

$$\Delta_{\vartheta,\varphi} Y_{lm}(\vartheta,\varphi) = -l(l+1)\,Y_{lm}(\vartheta,\varphi)$$

$$\omega^2 \delta n = -\frac{\mu}{m}\left(1-\frac{m\omega_0^2 r^2}{2\mu}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\Delta_{\vartheta,\varphi}}{r^2}\right)\delta n \,+$$

$$+\ \omega_0^2 r \frac{\partial}{\partial r}\delta n$$

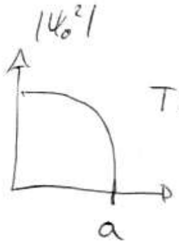
we are looking for a solution $\delta n = \overset{F(r)}{\cancel{P(r)}}\, \underbrace{r^l Y_{lm}(\vartheta,\varphi)}$

polinomial in x, y, z

polinomial of r^2
$\rightsquigarrow$ it knows all the rotational symmetries
$\Downarrow$
transforms like $\mathbb{1}$ of rotations

$$\omega^2 r^l \delta n = -\frac{\mu}{m}\left(1-\frac{m\omega_0^2 r^2}{2\mu}\right)\left(\frac{\partial}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}\right) r^l F(r) \,+$$

$$+\ \omega_0^2 r \frac{\partial}{\partial r} r^l F(r)$$



Thomas-Fermi-density profile

$$\mu = \frac{1}{2} m\omega_0^2 a^2 \rightsquigarrow a = \sqrt{\frac{2\mu}{m\omega_0^2}}$$

$$\omega^2 F(r) = r^{-l}\left[\frac{1}{2}\omega_0^2 a^2\left(1-\frac{r^2}{a^2}\right)\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right) +\right.$$

$$\left. +\ \omega_0^2 r \frac{\partial}{\partial r}\right] r^l F(r)$$

- we can pull through $r^{-l} \rightsquigarrow r^{-l}\frac{d}{dr} r^l$

similarity transformation of the derivative operator.

$$\left[\frac{d}{dx}, f(x)\right] = f'(x) \quad \text{we will use this}$$

$$r^{-\ell}\frac{d}{dr}r^{\ell} - r^{-\ell}\Big(r^{\ell}\frac{d}{dr} + \underbrace{\left[\frac{d}{dr}, r^{\ell}\right]}_{\ell \cdot r^{\ell-1}}\Big) = \left(\frac{d}{dr} + \frac{\ell}{r}\right)$$

$$r^{-\ell}\frac{d^2}{dr^2}r^{\ell} = r^{-\ell}\frac{d}{dr}\underbrace{r^{\ell}\cdot r^{-\ell}}_{1}\frac{d}{dr}r^{\ell} = \left(\frac{d}{dr} + \frac{\ell}{r}\right)^2 = \Big(\frac{d^2}{dr^2} + \frac{d}{dr}\frac{\ell}{r} +$$

$$+ \frac{\ell}{r}\frac{d}{dr} + \frac{\ell^2}{r^2}\Big) = \left(\frac{d^2}{dr^2} + 2\frac{\ell}{r}\frac{d}{dr} - \frac{\ell}{r^2} + \frac{\ell^2}{r^2}\right) \qquad \frac{\ell}{r}\frac{d}{dr} + (-1)\frac{\ell}{r^2}$$

$$\omega^2 F(r) = \Big[-\frac{\omega_0^2 a^2}{2}\left(1 - \frac{r^2}{a^2}\right)\Big(\frac{d^2}{dr^2} + \frac{2\ell}{r}\frac{d}{dr} - \cancel{\frac{\ell}{r^2}} + \cancel{\frac{\ell^2}{r^2}} + \frac{2}{r}\frac{d}{dr} + \cancel{\frac{2\ell}{r^2}}$$

$$\underbrace{- \frac{\ell(\ell+1)}{r^2}}_{-\frac{\ell^2}{r^2} - \frac{\ell}{r^2}}\Big) + \omega_0^2\left(r\frac{d}{dr} + \ell\right)\Big]F(r)$$

$\rightarrow$ now the $\frac{1}{r^2}$ potential dies!

- Now we introduce dimensionless quantities: $\tilde{r} = \frac{r}{a}$

$$\downarrow$$

$$r = \tilde{r}a$$

$$F(r) = \tilde{F}(\tilde{r})$$

$$\omega^2 \tilde{F}(\tilde{r}) = \Big[-\frac{\omega_0^2}{2}\left(1 - \tilde{r}^2\right)\left(\frac{d^2}{d\tilde{r}^2} + \frac{2\ell}{\tilde{r}}\frac{d}{d\tilde{r}} + \frac{2}{\tilde{r}}\frac{d}{d\tilde{r}}\right) + \omega_0^2\left(\tilde{r}\frac{d}{d\tilde{r}} + \ell\right)\Big]$$

$$\Big]\tilde{F}(\tilde{r})$$

$$\frac{\omega^2}{\omega_0^2}\tilde{F}(\tilde{r}) = \Big[-\frac{1}{2}\left(1 - \tilde{r}^2\right)\left(\frac{d^2}{d\tilde{r}^2} + \frac{2\ell+2}{\tilde{r}}\frac{d}{d\tilde{r}}\right) + \left(\tilde{r}\frac{d}{d\tilde{r}} + \ell\right)\Big]\tilde{F}(\tilde{r})$$

this operator either keeps the order, or lowers by 2.
which means even and odd orders do not mix!
we can introduce $\tilde{r}^2 = x$!

$r^2 = x$ $\qquad\qquad \tilde{F}(\tilde{r}) = G(x)$

$$\frac{d}{dr} = \underbrace{\frac{dx}{dr}}_{2r}\frac{d}{dx} = 2\sqrt{x}\frac{d}{dx}$$

$$\frac{d^2}{dr^2} = 2\sqrt{x}\frac{d}{dx}\,2\sqrt{x}\frac{d}{dx} = 2\sqrt{x}\left(2\sqrt{x}\frac{d}{dx} + \frac{1}{\sqrt{x}}\right)\frac{d}{dx} =$$

$$\frac{d^2}{dr^2} = 4x\frac{d^2}{dx^2} + 2\frac{d}{dx} = 2\left(2x\frac{d^2}{dx^2} + \frac{d}{dx}\right)$$

$$\frac{\omega^2}{\omega_0^2}G(x) = \Big[-\tfrac{1}{2}(1-x)\left(\cancel{4}^{\,2}x\frac{d^2}{dx^2} + \cancel{2}\frac{d}{dx} + \frac{2\ell+2}{\cancel{\sqrt{x}}}\cancel{2\sqrt{x}}\frac{d}{dx}\right) +$$

$$+\left(\sqrt{x}\,2\sqrt{x}\frac{d}{dx} + \ell\right)\Big]G(x)$$

$$0 = \left[(1-x)\left(2x\frac{d^2}{dx^2} + (2\ell+3)\frac{d}{dx}\right) - 2x\frac{d}{dx} - \ell + \frac{\omega^2}{\omega_0^2}\right]G(x)$$

$$0 = \left[x(1-x)\frac{d^2}{dx^2} + \underbrace{\left(\ell+\tfrac{3}{2}\right)(1-x)\frac{d}{dx} - x\frac{d}{dx}} - \frac{\ell}{2} + \frac{1}{2}\frac{\omega^2}{\omega_0^2}\right]G(x)$$

$$0 = \left[x(1-x)\frac{d^2}{dx^2} + \left(\ell+\tfrac{3}{2} - \left(\ell+\tfrac{5}{2}\right)x\right)\frac{d}{dx} + \frac{1}{2}\left(\frac{\omega^2}{\omega_0^2} - \ell\right)\right]G(x)$$

This is really similar to the eq. of hypergeometric func.

$${}_2F_1(a,b,c,z) = 1 + \frac{a\cdot b}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots$$

$\lim_{b\to\infty} F_1\left(a,b,c,\frac{z}{b}\right) \Rightarrow$ confluent hypergeometric func.

ex. Hermit-polinomials
generalized Legendre-pol. etc. $\Big\}$ solutions of the HO problem

$$\left[z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - a\cdot b\right]v(z) = 0$$

$$v(z) = {}_2F_1(a,b,c,z)$$ this func. fullfills the eq.

we need $G(x)$ on $[0,1]$ since $r \in [0,a]$.

How far is ${}_2F_1(a,b,c,z)$ regular after the origin?

⇝ ${}_2F_1$ diverges as $z \to \pm 1$!
⇝ except if it has a finite number of terms.
⇝ we need it to be regular even at the Thomas-Fermi surface

↓

if a or b = ⊖ intiger the function becomes finite

⇝ $a = -n$ $\quad n \in 0, 1, \dots$
⇝ this is the radial quantum number

$$\left.\begin{aligned} a &= -n \\ c &= l + \tfrac{3}{2} \\ a+b+1 &= l + \tfrac{5}{2} \\ -a\cdot b &= \tfrac{1}{2}\left(\tfrac{\omega^2}{\omega_0^2} - l\right) \end{aligned}\right\}$$ solve for a, b, c, ω^2

$$\left.\begin{aligned} a &= -n \\ c &= l + \tfrac{3}{2} \\ b &= n + l + \tfrac{3}{2} \end{aligned}\right\}$$

$$\omega^2 = \omega_0^2(l - 2ab) = \omega_0^2\left(l - 2(-n)\left(n + l + \tfrac{3}{2}\right)\right) =$$

$$\omega^2 = \omega_0^2\left(2n^2 + 2nl + 3n + l\right)$$

we have the excitation frequencies
and we also know the density oscillation modes.

$$\omega^2 \delta n(\vec{r}) = -\frac{1}{m} \vec{\nabla}\left(\mu - \frac{1}{2} m\omega_0^2 r^2\right) \vec{\nabla} \delta n(\vec{r})$$

$$\omega^2 = \omega_0^2 \left(2n^2 + 2nl + 3n + l\right)$$

$$\left.\begin{array}{ll} \text{with} & n \in 0, 1, 2, \dots \\ & l \in 0, 1, 2, \dots \\ & m \in -l, \dots, l \end{array}\right\}$$

this is called here hypergeometric polinomial.

$$\delta n_{n,l,m}(\vec{r}) = r^l Y_{lm}(\vartheta, \varphi)\, {}_2F_1\left(-n, n+l+\frac{3}{2}, l+\frac{3}{2}, \frac{r^2}{a^2}\right)$$

where $a = \sqrt{\frac{2\mu}{m\omega_0^2}}$ is the Thomas-Fermi-radius.

(apart from normalization.)

This is the general solution for isotopic system.