2019.03.18.

$$\left[-\frac{\hbar^2}{2m}\Delta + V_{ext}(r) + \frac{4\pi\hbar^2 a}{m}|\Psi_0|^2\right]\Psi_0 = \mu\Psi_0$$

with $\int |\Psi_0|^2 d^3r = N$

- in the T-F approx $\quad |\Psi_0|^2 = \frac{\mu - V_{ext}}{g}\,\Theta(\mu - V_{ext})$

$\mu \rightarrow \mu_{TF}$

- properties of μ_{TF}

$$V_{ext}(r) = \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

$$N = \int_{\mu_{TF} > \frac{1}{2}m\left(\sum_{i=1}^{3}\omega_i^2 x_i^2\right)} \frac{\mu_{TF} - \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)}{g}\,dx\,dy\,dz$$

integral is on a general ellipsoid
inside

~> the problem can be defined by scaling to an ~~ell~~ spherical integral.

$$N = \mu_{TF}\int_{\text{boundary}} \frac{1 - \frac{m\omega_x^2}{2\mu_{TF}}x^2 - \frac{m\omega_y^2}{2\mu_{TF}}y^2 - \frac{m\omega_z^2}{2\mu_{TF}}z^2}{g}\,dx\,dy\,dz$$

$$x'^2 = \frac{m\omega_x^2}{2\mu_{TF}}x^2 \qquad \text{rescaling}$$

$$dx = \sqrt{\frac{2\mu_{TF}}{m\omega_x^2}}\,dx'$$

$\vdots$

$$N = \frac{\mu_{TF}}{g} \frac{(2\mu_{TF})^{3/2}}{m^{3/2} \cdot \omega_x \omega_y \omega_z} \int dx' dy' dz' \, (1 - x'^2 - y'^2 - z'^2)$$

$$1 > x'^2 + y'^2 + z'^2$$

$\rightsquigarrow$ isotropic spherical problem $\rightsquigarrow$ spherical coords.

$$N = (\dots) \, 4\pi \underbrace{\int_0^1 r'^2 dr' \, (1 - r'^2)}_{\left[\frac{r^3}{3} - \frac{r^5}{5}\right]_0^1} = (\dots) \frac{8\pi}{15}$$

$$N = \frac{2\mu_{TF}}{\cancel{2}} \frac{m}{4\pi\hbar^2 a} (2\mu_{TF})^{3/2} \frac{1}{m^{3/2} \bar{\omega}^3} \frac{8\pi}{15} \qquad \bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$$

$$N = \frac{(2\mu_{TF})^{5/2}}{\underbrace{\hbar^2}_{\hbar^{5/2} \cdot \hbar^{-1/2}} m^{1/2} a} \frac{1}{\underbrace{\bar{\omega}^3}_{\bar{\omega}^{5/2} \bar{\omega}^{1/2}}} \frac{1}{15} = \frac{1}{15} \left(\frac{2\mu_{TF}}{\hbar \bar{\omega}} \right)^{5/2} \underbrace{\frac{\hbar^{1/2}}{m^{1/2} \bar{\omega}^{1/2}}}_{d} \frac{1}{a} =$$

$$N = \frac{1}{15} \left(\frac{2\mu_{TF}}{\hbar \bar{\omega}} \right)^{5/2} \frac{d}{a}$$

$$\boxed{\mu_{TF} = \frac{\hbar \bar{\omega}}{2} \left(15 \frac{Na}{d} \right)^{2/5}}$$ in the TF approx

- power law in the particle number

- for the condensate-density:

$$n_c(r) = \frac{\mu_{TF} - V_{ext}(r)}{g} \, \theta(\mu_{TF} - V_{ext}(r))$$

$$\sim \frac{\mu_{TF}}{g}\Big(1 - \underbrace{\frac{m\omega_x^2}{2\mu_{TF}}}_{\frac{1}{R_x^2}} x^2 - \dots\Big)\,\vartheta(\mu - V)$$

$\frac{1}{R_x^2}$ ⤳ length dim...

$$= \frac{\mu_{TF}}{g}\Big(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2}\Big)\,\vartheta(\mu - V)$$

and $R_x = \sqrt{\frac{2\mu_{TF}}{m\omega_x^2}} = \frac{\bar\omega}{\omega_x}\sqrt{\frac{2\mu_{TF}}{m\bar\omega^2}} =$

$$= \frac{\bar\omega}{\omega_x}\sqrt{\underbrace{\frac{\hbar}{m\bar\omega}}_{d^2}\Big(15\,\frac{Na}{d}\Big)^{2/5}} =$$

$$= \frac{\bar\omega}{\omega_x}\, d\,\Big(15\,\frac{Na}{d}\Big)^{1/5}$$

⤳ explains the width dependance on N!

- TF approx is applicable, if $\frac{Na}{d} \gg 1$!

- For homogenious system ⤳ no d, since we have $\emptyset$ potential

  $na^3 < 1$ ⤳ not too dence ensamble

$$\Big[-\frac{\hbar^2}{2m}\Delta + \frac{4\pi\hbar^2 a}{m}|\Psi_0|^2\Big]\Psi_0 = \mu\Psi_0$$

$\Psi_0 = \text{const.}$ is good

$n_c = \frac{\mu}{g}$ ⤳ well-known approx.

- there was no such material that fulfills the eq.

- μ is shifted by V_{ext} ⤳ <u>local density approx</u>

- ⇒ calc. everything in homogenious sys.
  ⤳ then switch on V_{ext}, which shifts μ like in TF-approx
  ⤳ this approx breaks down if there is a dependance on the derivatives of V_{ext} ⤳ V_{ext} changes <u>fast</u> → big derivatives

  ⇓

  in this case <u>Gradient-expension</u> should be used
  ⤳ somehow local density is the "leading order"

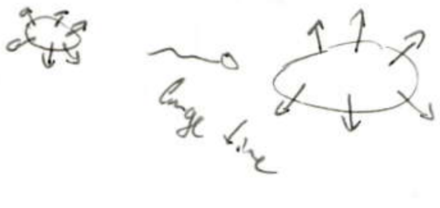
- <u>release-energy</u>

  - release the trap
  - cloud starts falling down and blowing up.

- just after releasing the trap:

$$E_{rel} = E_{kin} + \cancel{E_{pot}} + E_{int} \Big\} \text{ original energy content of the sys.}$$

$\omega \to 0$ ⤳



⇒ in the end everything is converted to E_{kin} ...

- prediction of the non-interacting model:

$E_{int} = \emptyset$ from the begining

$$E_{kin} = (\hbar\omega_x + \hbar\omega_y + \hbar\omega_z)\,\frac{1}{2}\cdot\underbrace{\frac{1}{2}}_{\text{Virial-theorem}}\cdot N$$

$\boxed{\dfrac{E_{rel}}{N} = \text{const.}}$ from the non int model.

- experiments show it is not const!

Release energy in TF-approx

$T = 0$

- general thermodynamics: $\frac{\partial E}{\partial N} = \mu$ /Gibbs-Duhem-relation/

$E = \cancel{E_{kin}} + E_{pot} + E_{int}$ from GP-functional.

$\mu N = \cancel{E_{kin}} + E_{pot} + 2 E_{int}$

due to TF approx.

$\mu \sim C \cdot N^{2/5}$ in this approx

$E = \int_0^N \mu(N')\, dN' = C' \cdot N^{7/5} = \frac{5}{7} \mu N$

inverse relation

$E = \frac{5}{7} \mu N = E_{pot} + E_{int}$
$\mu N = E_{pot} + 2 E_{int}$

$$\boxed{\begin{aligned} E_{int} &= \frac{2}{7} \mu N \\ E_{pot} &= \frac{3}{7} \mu N \\ E_{kin} &= \emptyset \end{aligned}}$$

- just after releasing the trap:

$E = E_{int} \rightsquigarrow \underbrace{E_{kin}}_{\text{can be measured.}}!$

$\rightsquigarrow E_{rel} \sim N^{7/5} \rightsquigarrow$ not const. even when looking at $\frac{E_{rel}}{N}$! $\rightsquigarrow$ non-trivial

• opposite limit

$\Downarrow$

negative scattering length situation.

ex.: Li experiment in Texas

Collapse with negative scattering length a

$a < 0$

$\frac{N|a|}{d} = 0{,}575$ — numerical exp. from GP-eq.

above this the eq. is numerically unstable

• let's use Gaussian ansatz, with variable widths!

$$\Psi_w(\vec{r}) = G\, e^{-\frac{r^2}{2 d_0^2 w^2}} \qquad V_{ext} = \frac{1}{2} m \omega_0^2 r^2, \text{ isotropic}$$

and w is the dimensionless width.

$$E = \int d^3r\, \Psi_0^*(\vec{r}) \left[-\frac{\hbar^2}{2m} \Delta + V_{ext} + \frac{4\pi \hbar^2 a}{2m} |\Psi_0|^2 \right] \Psi_0(\vec{r})$$

$$N = \int d^3r\, G^2 e^{-\frac{r^2}{d_0^2 w^2}} = 4\pi \int_0^\infty r^2 dr\, e^{-\frac{r^2}{d_0^2 w^2}}$$

$$t = \frac{r^2}{d_0^2 w^2}$$

$$dr = \frac{dt}{\sqrt{t}}\, d_0 w$$

$$N = G^2\, 4\pi \frac{d_0^3 w^3}{2} \underbrace{\int_0^\infty t^{1/2} e^{-t}\, dt}_{\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}}$$

$$N = G^2 \pi^{3/2} d_0^3 w^3 \rightsquigarrow G = \sqrt{\frac{N}{\pi^{3/2} d_0^3 w^3}}$$

and $d_0 = \sqrt{\frac{\hbar}{m\omega_0}}$ oscillator length.

$$F(w) = \frac{E}{N \hbar \omega_0}$$

$$F(w) = \frac{1}{\hbar\omega_0\, d_0^3\, w^3\, \pi^{3/2}} \int_0^\infty \Big[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{1}{2} m\omega_0^2 r^2 +$$

$$+ \frac{4\pi\hbar^2 a}{2m} \frac{N}{\pi d_0^3 w^3} \cdot e^{-\frac{r^2}{d_0^2 w^2}} \Big] e^{-\frac{r^2}{2d_0^2 w^2}} \; r^2 dr\, e^{-\frac{r^2}{2d_0^2 w^2}}$$

$$t^2 = \frac{r^2}{d_0^2 w^2} \rightarrow dr = d_0 w\, dt$$

$$= \frac{d_0^3 w^3}{\hbar\omega_0\, d_0^3 w^3\, \pi^{3/2}} \int_0^\infty t^2 dt\, e^{-\frac{t^2}{2}} \Big[-\frac{\hbar^2}{2m}\left(\frac{1}{d_0^2 w^2}\frac{d^2}{dt} + \frac{1}{d_0^2 w^2}\frac{2}{t}\frac{d}{dt}\right) +$$

$$+ \frac{1}{2} m\omega_0^2 d_0^2 w^2 t^2 + \frac{4\pi\hbar^2 a}{2m} \frac{N}{\pi d_0^3 w^3} e^{-t^2} \Big] e^{-t^2/2} =$$