- S_n is polinomial of x, y, z:

$$r^l Y_{ln}(\vartheta, \varphi) \cdot P_n(r^2) \sim r^l Y_{lm} r^{2n} \quad \text{(in highest order)}$$

$$\underset{x^2+y^2+z^2}{\uparrow}$$

- to calculate ω we only need what happens in <u>highest order.</u>

$$\omega^2 r^l \cancel{Y_{ln}}(\vartheta, \varphi) \cdot r^{2n} = \frac{1}{2}\omega_0^2 r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) r^{2n+l} \cancel{Y_{ln}} +$$

$$+ \omega_0^2 r \frac{\partial}{\partial r} r^{l+2n} \cancel{Y_{ln}}(\vartheta, \varphi)$$

after taking the scalar product, converted to spherical coords.

$$\frac{\omega^2}{\omega_0^2} r^{2n+l} = \frac{r^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) r^{2n+l} + r\frac{\partial}{\partial r} r^{l+2n}$$

2019.05.13.

- few examples: $r^1 Y_{11}(\vartheta, \varphi) = \underbrace{r \sin\vartheta}_{\rho} e^{i\varphi} = (x+iy)$

$$r Y_{10} = r\cos\vartheta = z$$

- we can take the derivatives and erase r^{2n+l}

$$\frac{\omega^2}{\omega_0^2} = \frac{1}{2}(2n+l)(2n+l-1) + (2n+l) - \frac{l}{2}(l+1) +$$

$$+ (2n+l) = (2n+l)\left(\frac{1}{2}(2n+l-1) + 1 + 1\right) - \frac{l}{2}(l+1)$$

$$\frac{\omega^2}{\omega_0^2} = \frac{1}{2}\left(4n^2 + 2nl + 2n(l+1)\right) + l + 2n$$

$$\boxed{\omega^2 = \omega_0^2\left(2n^2 + 2nl + 3n + l\right)}$$

Stringari excitation spectrum.

- $n \to$ radial quantum number $\quad n = 0, 1, 2, \ldots$
- $l \to$ ang. mom. $\quad$ — " — $\quad l = 0, 1, \ldots$
- $m \to \quad m = -l, \ldots, l$ /the usual stuff/

- here n and l are independent from each other.

- Kohn-modes: $\omega = \omega_0$

$$\left.\begin{array}{l} n=0 \\ l=1 \end{array}\right\} \rightsquigarrow m=-1,0,1$$

we have the three diffent directions.

- Characteristic energies: μ, $\hbar\omega_0$

the spectra is μ independent! (that's special)

The Hutchinson - Zaremba - Griffin method

- Boguliubov-eq.:

$$\left.\begin{array}{l} -H_{HF} u_i - g\Psi_0^2 v_i = \varepsilon_i u_i \\ -g\Psi_0^{*2} u_i + H_{HF} v_i = -\varepsilon_i v_i \end{array}\right\}$$

and

$$H_{HF} = \left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g|\Psi_0|^2\right)$$

- for the ground state (G-P-eq.): $\left(-\frac{\hbar^2}{2m}\Delta + V + g|\Psi_0|^2\right)\Psi_0 = \mu\Psi_0$

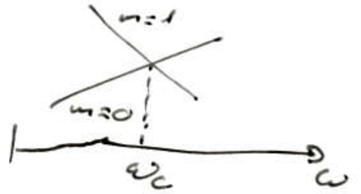
- if there is nothing to destroy the time-reversal sym. $\rightarrow \Psi_0 \in \mathbb{R}$ (like $\vec{B}$ field)

- Side remark: In a rotating frame

$$\Psi_0 = \Psi(\rho, z)\cdot e^{im\varphi} \quad \text{(vortex-like solution)}$$

$\rightsquigarrow$ cannot be transformed to real

$\rightsquigarrow$ only on a given φ



- let's suppose Ψ is real:

$$\left.\begin{array}{l} H_{HF} u_i - g\Psi_0^2 v_i = \varepsilon_i u_i \\ -g\Psi_0^2 u_i + H_{HF} v_i = -\varepsilon_i v_i \end{array}\right\}$$

- Let's introduce $\hat{h}$:

$$\hat{h} = H_{HF} - g\,\Psi_0^2$$

→ is a lin. op.
→ has a spectra

$$\hat{h}\,\phi_\alpha = \varepsilon_\alpha\,\phi_\alpha$$

there is ϕ_0 for $\varepsilon_0 = 0$!

$$\underset{\Psi_0}{\hat{h}\,\phi_0} = \underbrace{\left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g\,\Psi_0^2 - g\,\Psi_0^2\right)\Psi_0}_{\emptyset \text{ due to the G-P eq.}} = 0\cdot\Psi_0$$

→ $\phi_0 = \Psi_0$, $\varepsilon_0 = 0$ → this can allways be found numerically

$$h_{\alpha\beta} = \langle \varphi_\alpha | h | \varphi_\beta \rangle \qquad \varphi_\alpha = (\text{Gaussian})\cdot(\text{Hermit Polinomial})$$

⇝ harmonic oscillator basis.

$$h = \Big(\underbrace{-\frac{\hbar^2}{2m}\Delta + V - \mu}_{\text{this's part is diagonal}} + \underbrace{g\,\Psi_0^2}_{\text{non-diagonal part.}}\Big)$$

↳ can be obtained numerically

→ we can get ε_α, ϕ_α-s.

- u, v are coupled!

⇝ can we decouple the components (with somet lin comb.)

$$\left.\begin{aligned} H_{HF}(u_i + v_i) - g\,\Psi_0^2(u_i + v_i) &= \varepsilon_i(u_i - v_i) \\ H_{HF}(u_i - v_i) + g\,\Psi_0^2(u_i - v_i) &= \varepsilon_i(u_i + v_i) \end{aligned}\right\}$$

$$\left.\begin{aligned} \hat{h}(u_i + v_i) &= \varepsilon_i(u_i - v_i) \\ (\hat{h} + 2g\,\Psi_0^2)(u_i - v_i) &= \varepsilon_i(u_i + v_i) \end{aligned}\right\}$$

$$\left(\hat{h} + 2g\Psi_0^2\right)\left(\frac{\hat{h}}{E_i}(u_i + v_i)\right) = E_i(u_i + v_i)$$

$$\left(\hat{h} + 2g\Psi_0^2\right)\hat{h}(u_i + v_i) = E_i^2(u_i + v_i)$$

⇝ now the eq. are decoppled

⇝ but this is a 4th order eq. (Δ^2!)

$$\hat{h}\left(\hat{h} + 2g\Psi_0^2\right)(u_i - v_i) = E_i^2(u_i - v_i)$$

- the 2 operators are not the same!
- but they have the same spectra.

$$(u_i + v_i) = \sum_\alpha c_\alpha^i \phi_\alpha$$

$$\sum_\beta c_\beta^i \varepsilon_\beta^2 \phi_\beta + \sum_\beta 2g\Psi_0^2 \varepsilon_\beta c_\beta^i \phi_\beta = E_i^2 \sum_\beta c_\beta^i \phi_\beta \qquad \Big/ \int d^3r\, \phi_\alpha(r)$$

Normalization: $\int \phi_\alpha(r)\phi_\beta(r)\,d^3r = \delta_{\alpha\beta}$ for real ϕ-s.

$$c_\alpha^i \varepsilon_\alpha^2 + \sum_\beta 2g\,\varepsilon_\beta c_\beta^i \underbrace{\int \phi_\alpha(r)\,\Psi_0^2(r)\,\phi_\beta(r)\,d^3r}_{M_{\alpha\beta} \rightsquigarrow \text{ symmetric mx.}} = E_i^2 c_\alpha^i$$

$$\underbrace{(U_0 + V_0)}_{2\Psi_0} = c_0^0 \Psi_0$$

⇝ non-physical solution... but only one component, others are 0-s.

$$\boxed{\sum_\beta G_{\alpha\beta} c_\beta^i = E_i^2 c_\alpha^i} \qquad \text{with } \alpha \neq 0,\ \beta \neq 0$$

$$G_{\alpha\beta} = \varepsilon_\alpha^2 \cdot \delta_{\alpha\beta} + 2g M_{\alpha\beta} \underbrace{\varepsilon_\beta}_{}$$

this makes this mx.
non symmetric.

so we don't know if ϵ_i^2 is real.

- we can transform it to be symmetric:

$$\tilde{G} = D G D^{-1}$$

D is diagonal, $D_{\alpha\alpha} = \sqrt{\varepsilon_\alpha}$

$$D_{\alpha\beta} = \delta_{\alpha\beta} \sqrt{\varepsilon_\alpha}$$

$$(D^{-1})_{\alpha\beta} = \delta_{\alpha\beta} \frac{1}{\sqrt{\varepsilon_\alpha}}$$

- with this

$$\tilde{G}_{\alpha\beta} = \varepsilon_\alpha^2 \delta_{\alpha\beta} + 2g \sqrt{\varepsilon_\alpha} M_{\alpha\beta} \sqrt{\varepsilon_\beta}$$

$$\underline{\underline{G}}\, \underline{c}_i = \epsilon_i^2 \underline{c}_i$$

$$\underbrace{\underline{\underline{D}}\,\underline{\underline{G}}\,\underline{\underline{D}}^{-1}}_{\tilde{\underline{\underline{G}}}} \underbrace{\underline{\underline{D}}\,\underline{c}_i}_{\tilde{\underline{c}}_i} = \epsilon_i^2 \underbrace{\underline{\underline{D}}\,\underline{c}_i}_{\tilde{\underline{c}}_i}$$

$$\boxed{\tilde{\underline{\underline{G}}}\,\tilde{\underline{c}}_i = \epsilon_i^2 \tilde{\underline{c}}_i} \quad \text{with} \quad \tilde{\underline{\underline{G}}} = \underline{\underline{D}}\,\underline{\underline{G}}\,\underline{\underline{D}}^{-1}, \quad \tilde{\underline{c}}_i = \underline{\underline{D}}\,\underline{c}_i$$

- this can be solved by standard means.
- we then get real ϵ_i^2
- D^{-1} would be problematic with $\varepsilon_0 = 0$!
- Let's go back:

$$\underline{c}_i = \underline{\underline{D}}^{-1} \tilde{\underline{c}}_i$$

$$(u_i + v_i) = \sum_\alpha c_i^\alpha \phi_\alpha$$

- How to calc. u_i, v_i separetely?

$$(u_i - v_i) = \frac{\hat{h}\,(u_i + v_i)}{E_i} \qquad E_i = 0 \text{ is forbidden} \qquad E_i > 0$$

$$\left.\begin{aligned}(u_i - v_i) &= \sum_\alpha \frac{\varepsilon_\alpha}{E_i} c_i^\alpha \phi_\alpha \\ (u_i + v_i) &= \sum_\alpha c_i^\alpha \phi_\alpha\end{aligned}\right\} \quad \boxed{\begin{aligned} u_i &= \frac{1}{2}\sum_\alpha \left(1 + \frac{\varepsilon_\alpha}{E_i}\right) c_i^\alpha \phi_\alpha \\ v_i &= \frac{1}{2}\sum_\alpha \left(1 - \frac{\varepsilon_\alpha}{E_i}\right) c_i^\alpha \phi_\alpha\end{aligned}}$$

- normalization:

→ usually $\tilde{\underline{c}}_i \tilde{\underline{c}}_j = \delta_{ij}$ (by default)

→ $$\left.\begin{aligned}\delta_{ij} &= \int d^3r \,(u_i^* u_j - v_i^* v_j) \\ 0 &= \int d^3r \,(u_i v_j - u_j v_i) \\ 0 &= \int d^3r \,(u_i^* v_j^* - u_j^* v_i^*)\end{aligned}\right\} \text{ original normalizations}$$

$$\int (u_i + v_i)(u_j - v_j) = \int d^3r \,\big(\underbrace{u_i u_j - v_i v_j}_{\delta_{ij}} + \underbrace{v_i u_j - u_i v_j}_{0}\big) = \delta_{ij}$$

$\delta_{ij} = \int (u_i + v_i)(u_j - v_j)\, d^3r$ → is what we want.

$$\delta_{ij} = \int d^3r \sum_{\substack{\alpha \\ \beta \\ \alpha \neq 0 \\ \beta \neq 0}} c_i^\alpha \phi_\alpha(r) \frac{\varepsilon_\beta}{E_j} c_j^\beta \phi_\beta(r) =$$

$$= \sum_{\alpha,\beta}{}' \frac{\varepsilon_\beta}{E_j} c_i^\alpha c_j^\beta \underbrace{\int d^3r \,\phi_\alpha(r)\,\phi_\beta(r)}_{\delta_{\alpha\beta}} = \sum_\alpha \frac{\varepsilon_\alpha}{E_j} c_i^\alpha c_j^\alpha$$

$\boxed{E_i \delta_{ij} = \sum_\alpha \varepsilon_\alpha c_i^\alpha c_j^\alpha}$ is the correct normalization for $\underline{c}$-s.

- why is this any good?

  Diagonalization of $N \times N$ sym matrix $\sim N^3$

  Original Bogoliubov - problem $\sim (2N)^3$

  $\rightarrow$ we gain a factor of 4 $\rightarrow$ large matrices, this is huge!

  $\rightarrow$ less operation $\rightarrow$ more precise result.

- if the condensate has a spatial dependent phase, then this method cannot be used!