$$K = \sum_k \Big\{ \eta_k (u_k \alpha_k^+ + v_k \beta_{-k})(u_k \alpha_k + v_k \beta_{-k}^+) +$$
$$+ \eta_k (-v_k \alpha_k + u_k \beta_{-k}^+)(-v_k \alpha_k^+ + u_k \beta_{-k}) -$$
$$- \Delta (u_k \alpha_k^+ + v_k \beta_{-k})(-v_k \alpha_k + u_k \beta_{-k}^+) -$$
$$- \Delta (-v_k \alpha_k^+ + u_k \beta_{-k})(u_k \alpha_k + v_k \beta_{-k}^+) \Big\} =$$

$$= \sum_k \Big\{ \alpha_k^+ \alpha_k (\eta_k u_k^2 - \eta_k v_k^2 + \Delta u_k v_k + \Delta u_k v_k) +$$
$$+ \beta_{-k}^+ \beta_{-k} (-\eta_k v_k^2 + \eta_k u_k^2 + \Delta u_k v_k + \Delta u_k v_k) +$$
$$+ \alpha_k \beta_{-k} (-\eta_k u_k v_k - \eta_k u_k v_k - \Delta v_k^2 + \Delta u_k^2) +$$
$$+ \alpha_k^+ \beta_{-k}^+ (\eta_k u_k v_k + \eta_k u_k v_k - \Delta u_k^2 + \Delta v_k^2) +$$
$$+ (2\eta_k v_k^2 - 2\Delta u_k v_k) \Big\}$$

this last constant term does not matter in normal averages. (just shifts the grnd. state energy)

- we can choose u_k, v_k so the cross-terms die:

$$\Delta (u_k^2 - v_k^2) = 2\eta_k u_k v_k$$

Normalization: $u_k^2 + v_k^2 = 1$

Statement:

$$\begin{pmatrix} \eta_k & \Delta \\ \Delta & -\eta_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

2019. 12. 05.

$$\begin{pmatrix} \eta_k & \Delta \\ \Delta & -\eta_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} \quad \text{with } u_k^2 + v_k^2 = 1$$

$$\begin{pmatrix} \eta_\ell - \varepsilon_\ell & \Delta \\ \Delta & -\eta_\ell - \varepsilon_\ell \end{pmatrix} \begin{pmatrix} u_\ell \\ v_\ell \end{pmatrix} = 0$$

$$(\eta_\ell - \varepsilon_\ell)(-\eta_\ell - \varepsilon_\ell) - \Delta^2 \overset{!}{=} 0$$

$$\underline{\underline{\varepsilon_\ell = \sqrt{\eta_\ell^2 + \Delta^2}}} \quad \left(\text{with } \eta_\ell = \frac{\hbar^2 \ell^2}{2m} - \mu\right)$$

$\rightsquigarrow$ with having the eigenvalues we can solve for the eigenvector.

$$(\eta_\ell - \varepsilon_\ell) u_\ell + \Delta v_\ell = 0$$

$$-\frac{v_\ell}{u_\ell} = \frac{\eta_\ell - \varepsilon_\ell}{\Delta}$$

$$u_\ell^2 \left(1 + \left(\frac{\eta_\ell - \varepsilon_\ell}{\Delta}\right)^2\right) = 1$$

$$\underbrace{-(\eta_\ell - \varepsilon_\ell)}_{\varepsilon_\ell - \eta_\ell}(\eta_\ell + \varepsilon_\ell) - \Delta^2 \overset{!}{=} 0$$

$$\frac{\varepsilon_\ell - \eta_\ell}{\Delta} = \frac{\Delta}{\varepsilon_\ell + \eta_\ell}$$

$$\frac{(\eta_\ell - \varepsilon_\ell)^2}{\Delta^2} = \left(\frac{\varepsilon_\ell - \eta_\ell}{\Delta}\right)^2 = \left(\frac{\varepsilon_\ell - \eta_\ell}{\cancel{\Delta}}\right)\left(\frac{\cancel{\Delta}}{\varepsilon_\ell + \eta_\ell}\right)$$

$$u_\ell^2 \underbrace{\left(1 + \frac{\varepsilon_\ell - \eta_\ell}{\varepsilon_\ell + \eta_\ell}\right)}_{\frac{2\varepsilon_\ell}{\varepsilon_\ell + \eta_\ell}} = 1$$

$$u_\ell^2 = \frac{\varepsilon_\ell + \eta_\ell}{2\varepsilon_\ell} \rightsquigarrow \boxed{\begin{aligned} u_\ell &= \sqrt{\frac{1}{2}\left(1 + \frac{\eta_\ell}{\varepsilon_\ell}\right)} \\ v_\ell &= \sqrt{\frac{1}{2}\left(1 - \frac{\eta_\ell}{\varepsilon_\ell}\right)} \end{aligned}}$$

• we can use these in the diagonalization:

$$H = \sum_{\ell} \Big\{ \alpha_\ell^+ \alpha_\ell \left[\eta_\ell (u_\ell^2 - v_\ell^2) + 2\Delta u_\ell v_\ell \right] +$$

$$+ \beta_{-\ell}^+ \beta_{-\ell} \left[\eta_\ell (u_\ell^2 - v_\ell^2) + 2\Delta u_\ell v_\ell \right] +$$

$$+ \alpha_\ell \beta_{-\ell} \left[-2\eta_\ell u_\ell v_\ell + \Delta (u_\ell^2 - v_\ell^2) \right] +$$

$$+ \alpha_\ell^+ \beta_{-\ell}^+ \left[2\eta_\ell u_\ell v_\ell - \Delta (u_\ell^2 - v_\ell^2) \right] \Big\}$$

$$u_\ell v_\ell = \frac{1}{2} \sqrt{\left(1 + \frac{\eta_\ell}{\varepsilon_\ell}\right)\left(1 - \frac{\eta_\ell}{\varepsilon_\ell}\right)} = \frac{1}{2}\sqrt{1 - \frac{\eta_\ell^2}{\varepsilon_\ell^2}} = \frac{1}{2\varepsilon_\ell} \underbrace{\sqrt{\varepsilon_\ell^2 - \eta_\ell^2}}_{\Delta} = \frac{\Delta}{2\varepsilon_\ell}$$

$$u_\ell^2 - v_\ell^2 = \frac{1}{2}\left(\cancel{1} + \frac{\eta_\ell}{\varepsilon_\ell} - \cancel{1} + \frac{\eta_\ell}{\varepsilon_\ell}\right) = \frac{\eta_\ell}{\varepsilon_\ell}$$

$$\rightsquigarrow \quad -\cancel{2}\eta_\ell \cdot \frac{\Delta}{\cancel{2}\varepsilon_\ell} + \Delta \cdot \frac{\eta_\ell}{\varepsilon_\ell} = 0$$

$$\rightsquigarrow \quad \cancel{2}\eta_\ell \frac{\Delta}{\cancel{2}\varepsilon_\ell} - \Delta \cdot \frac{\eta_\ell}{\varepsilon_\ell} = 0$$

non-diagonal terms vanish!

$$H = \sum_{\ell} \Big\{ \underbrace{\left[\eta_\ell \cdot \frac{\eta_\ell}{\varepsilon_\ell} + \cancel{2}\Delta \cdot \frac{\Delta}{\cancel{2}\varepsilon_\ell} \right]}_{\frac{\eta_\ell^2}{\varepsilon_\ell} + \frac{\Delta^2}{\varepsilon_\ell} = \frac{\varepsilon_\ell^2}{\varepsilon_\ell} = \varepsilon_\ell} \left[\alpha_\ell^+ \alpha_\ell + \beta_{-\ell}^+ \beta_{-\ell} \right] \Big\}$$

$\rightarrow$ $\mathcal{E}_k$ is the excitation energies

$$H = \sum_k \mathcal{E}_k \left(\alpha_k^+ \alpha_k + \beta_k^+ \beta_k \right) + H_0$$

$\mathcal{E}_k$ is an even func. of k,
$-k \leftrightarrow k$ can be done.

$$\alpha_k |\Psi_{BCS}\rangle = 0 \; , \; \beta_k |\Psi_{BCS}\rangle = 0 \Big\} \text{ grnd state}$$

$$\sum_k \mathcal{E}_k \left(\alpha_k^+ \alpha_k + \beta_k^+ \beta_k \right) |\Psi_{BCS}\rangle = 0$$

- Statement:

$$|\Psi_{BCS}\rangle = \prod_k \left(u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow} \right) |\emptyset\rangle$$

$$\left.\begin{aligned} a^+_{k\uparrow} &= u_k \alpha_k^+ + v_k \beta_{-k} \\ a^+_{-k\downarrow} &= -v_k \alpha_k + u_k \beta^+_{-k} \end{aligned}\right\} \quad \begin{aligned} \alpha_k &= u_k a_{k\uparrow} - v_k a^+_{-k\downarrow} \\ \beta^+_{-k} &= v_k a_{k\uparrow} + u_k a^+_{-k\downarrow} \end{aligned}$$

$$\alpha_k |\Psi_{BCS}\rangle = \prod_{l \neq k} \left(u_l + v_l a^+_{l\uparrow} a^+_{-l\downarrow} \right) \left(u_k a_{k\uparrow} - v_k a^+_{-k\downarrow} \right) \left(u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow} \right) |\emptyset\rangle$$

these anticommute anyways..

$$\alpha_k |\Psi_{BCS}\rangle = \prod_{l \neq k} (u_l + v_l a^+_{l\uparrow} a^+_{-l\downarrow}) \Big(u_k^2 a_{k\uparrow} - v_k u_k a^+_{-k\downarrow} +$$

$$+ u_k v_k a_{k\uparrow} a^+_{k\uparrow} a^+_{-k\downarrow} - \underbrace{v_k^2 a^+_{-k\downarrow} a_{k\uparrow} a^+_{-k\downarrow}}_{-a^+_{k\uparrow} \underbrace{a^+_{-k\downarrow} a^+_{-k\downarrow}}_{0}} \Big) |\phi\rangle$$

0, anticommutation relations...

$$- v_k u_k a^+_{-k\downarrow} + u_k v_k (1 - a^+_{k\uparrow} a_{k\uparrow}) a^+_{-k\downarrow} =$$

$$= - \cancel{v_k u_k a^+_{-k\downarrow}} + \cancel{u_k v_k a^+_{-k\downarrow}} + u_k v_k a^+_{k\uparrow} a_{k\uparrow} a^+_{-k\downarrow} \rightarrow \text{also dies}$$

$\alpha_k |\Psi_{BCS}\rangle = 0 \rightsquigarrow |\Psi_{BCS}\rangle$ is the ground state for α.

$\rightsquigarrow \beta$ is completely similar...

$n_\uparrow = (n_\downarrow) \rightsquigarrow$ there is no imbalance between them.

$$n_\uparrow = \frac{1}{V} \sum_k \langle \Psi_{BCS} | a^+_{k\uparrow} a_{k\uparrow} | \Psi_{BCS} \rangle =$$

$$= \frac{1}{V} \sum_k \langle \Psi_{BCS} | (u_k \cancel{\alpha_k^+} + v_k \beta_{-k})(u_k \cancel{\alpha_k} + v_k \beta^+_{-k}) | \Psi_{BCS} \rangle$$

$$= \frac{1}{V} \sum_k \langle \Psi_{BCS} | v_k^2 \underbrace{\beta_{-k} \beta^+_{-k}}_{(1 - \beta^+_{-k} \cancel{\beta_{-k}})} | \Psi_{BCS} \rangle = \frac{1}{V} \sum_k v_k^2$$

$$n_\uparrow = \frac{1}{V}\sum_k \frac{1}{2}\left(1 - \frac{\eta_k}{\varepsilon_k}\right)$$

- normal state : $\Delta = 0 \rightsquigarrow \varepsilon_k = |\eta_k| = \left|\frac{\hbar^2 k^2}{2m} - \mu\right|$

  - this populates all the states inside the Fermi-sphere with $\uparrow\downarrow$

$$u_k = \begin{cases} 0 & k < k_f \\ 1 & k > k_f \end{cases}$$

$$v_k = \begin{cases} 1 & k < k_f \\ 0 & k > k_f \end{cases}$$

$\rightsquigarrow$ this describes a normal gas.

$$n_\uparrow^{\text{normal}} = \frac{1}{V}\sum_k v_k^2 = \frac{1}{V}\sum_k \frac{1}{2}\left(1 - \frac{\frac{\hbar^2 k^2}{2m} - \mu}{\left|\frac{\hbar^2 k^2}{2m} - \mu\right|}\right) = \frac{1}{V}\sum_k \frac{1}{2}\left(1 - \mathrm{sign}(k - k_F)\right) =$$

$$\mu = \frac{\hbar^2 k_F^2}{2m}$$

$$n_\uparrow^{\text{normal}} = \frac{1}{V}\sum_k \underbrace{\Theta(k - k_F)}_{\text{Fermi distribution at } T=0}$$

~~for supercon~~

- for the superconducting states:

$$F(R,s) = \left\langle \hat\Psi_\downarrow\left(\vec R + \frac{\vec s}{2}\right)\hat\Psi_\uparrow\left(\vec R - \frac{\vec s}{2}\right)\right\rangle_{BCS}$$

ground state average $\left(|\Psi_{BCS}\rangle\right)$ since $T = 0$

$$\left.\begin{aligned} \hat\Psi_\uparrow(r) &= \frac{1}{\sqrt V}\sum_k e^{ikr}\,\hat a_{k\uparrow} \\ \hat\Psi_\downarrow(r) &= \frac{1}{\sqrt V}\sum_k e^{-ikr}\,\hat a_{-k\downarrow} \end{aligned}\right\}$$

$$= \frac{1}{V} \sum_{q,q'} e^{-iq\left(R+\frac{s}{2}\right)} e^{iq'\left(R-\frac{s}{2}\right)} \underbrace{\langle a_{-q\downarrow} a_{q'\uparrow} \rangle_{BCS}}$$

$$\langle a_{-q\downarrow} a_{q'\uparrow} \rangle_{BCS} = \langle \Psi_{BCS} | (-v_q \cancel{\alpha_q^+} + u_q \beta_{-q})(u_{q'} \cancel{\alpha_{q'}} + v_{q'} \beta_{-q'}^+) | \Psi_{BCS} \rangle =$$

$$= u_q v_{q'} \langle \Psi_{BCS} | \underbrace{\beta_{-q} \beta_{-q'}^+}_{\delta_{qq'} - \cancel{\beta_{-q'}^+ \beta_{-q}}} | \Psi_{BCS} \rangle = \delta_{qq'} u_q v_{q'}$$

does not depend on the center of mass (in homogeneous sys. this is expected...)

$$= \frac{1}{V} \sum_q e^{-iqs} u_q v_q = F(s)$$

$$F(s) = \int \frac{d^3q}{(2\pi)^3} e^{-iqs} \frac{\Delta}{2E_q}$$

$$F(s=0) = \frac{\Delta}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{\left(\frac{\hbar^2 q^2}{2m} - \mu\right)^2 - \Delta^2}} \overset{q \gg \mu, \Delta}{\sim} \frac{\Delta}{2} \int \frac{q^2 dq}{\cancel{(2\pi)}} \frac{1}{\frac{\hbar^2 q^2}{2m}}$$

⇝ for very large q this is ~~(constant)~~ diverging

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{-iqs}}{q^2} = \frac{1}{4\pi s}$$ ⇝ also divergent at $s=0$

$$\lim_{s\to 0}\left[F(s) - \frac{m\Delta}{4\pi\hbar^2 s}\right]$$

$$\lim_{s\to 0} \Delta \int \frac{d^3 l}{(2\pi)^3} e^{-ils}\left(\frac{1}{2\varepsilon_l} - \frac{m}{\hbar^2 l^2}\right) = -\frac{m\Delta}{4\pi\hbar^2 a}$$

$$F(R, s) = \frac{m}{4\pi\hbar^2}\Delta(R)\left(\frac{1}{s} - \frac{1}{a}\right) + \mathcal{O}(s)$$ ← leading behavior of F.

- this is the regularized gap-eq.

$$\boxed{\Delta \int \frac{d^3 l}{(2\pi)^3}\left(\frac{1}{2\varepsilon_l} - \frac{m}{\hbar^2 l^2}\right) = -\frac{m\Delta}{4\pi\hbar^2 a}}$$