$$F(w) = \frac{1}{\hbar\omega_0 d_0^3 w^3 \pi^{3/2}} \int_0^\infty 4\pi r^2 dr\, e^{-\frac{r^2}{2d_0^2 w^2}} \left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{1}{2} m \omega_0^2 r^2 + \right.$$

$$\left. + \frac{4\pi\hbar^2 a}{2m} \frac{N}{\pi^{3/2} d_0^3 w^3} \cdot e^{-\frac{r^2}{d_0^2 w^2}} \right] e^{-\frac{r^2}{2d_0^2 w^2}}$$

Introducing a dimensionless quantity:

$$t^2 = \frac{r^2}{d_0^2 w^2} \rightsquigarrow dr = d_0 w\, dt$$

$$= \frac{d_0^3 w^3}{\hbar\omega_0 d_0^3 w^3 \pi^{3/2}} \int_0^\infty t^2 dt\, e^{-\frac{t^2}{2}} \left[-\frac{\hbar^2}{2m}\left(\frac{1}{d_0^2 w^2}\frac{d^2}{dt^2} + \frac{1}{d_0^2 w^2}\frac{2}{t}\frac{d}{dt}\right) + \right.$$

$$\left. + \frac{1}{2} m \omega_0^2 d_0^2 w^2 t^2 + \frac{4\pi\hbar^2 a}{2m}\frac{N}{\pi^{3/2} d_0^3 w^3} e^{-t^2} \right] e^{-t^2/2} = \ldots$$

$\boxed{2019.03.25.}$

$$\frac{d}{dt} e^{-t^2/2} = -t\, e^{-t^2/2}$$

$$\frac{d^2}{dt^2} e^{-t^2/2} = (t^2 - 1)\, e^{-t^2/2}$$

$$\left(\frac{d^2}{dt^2} + \frac{2}{t}\frac{d}{dt}\right) e^{-t^2/2} = $$

$$= (t^2 - 3)\, e^{-t^2/2}$$

/effect of the Laplacian.../

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

/Useful integrals.../

$$\underbrace{\frac{1}{\hbar\omega_0}\frac{\hbar^2}{2m}}_{\cancel{d_0^2}/2} \frac{1}{\cancel{d_0^2}} \frac{1}{w^2} \qquad \underbrace{\frac{1}{\hbar\omega_0}\frac{1}{2} m \omega_0^2}_{\frac{1}{2\cancel{d_0^2}}} w^2 \cancel{d_0^2} t^2 = \frac{w^2}{2} t^2$$

$$\frac{1}{\hbar\omega_0} \frac{{}^2\cancel{4}\pi\hbar^2 a}{\cancel{2}m} \frac{N}{w^3 d_0^3 \pi^{3/2}} = Na \frac{2\pi}{\pi^{3/2}} \underbrace{\frac{\hbar}{m\omega_0}}_{\cancel{d_0^2}} \frac{1}{w^3} \frac{1}{d_0^{\cancel{3}}}$$

$$\ldots = \frac{4\pi}{\pi^{3/2}} \int t^2 dt\, e^{-t^2/2} \left[-\frac{1}{2w^2}(t^2 - 3) + \frac{w^2}{2} t^2 + \frac{2}{\pi^{1/2}} \frac{1}{w^2} \frac{Na}{d_0} e^{-t^2} \right] e^{-t^2/2}$$

New variable:

$$t^2 = z \rightsquigarrow dt = \frac{1}{2\sqrt{z}}\, dz$$

$$= -\frac{\cancel{\chi}}{\sqrt{\pi}} \frac{1}{w^2} \int_0^\infty e^{-z} (z^2 - 3z) \frac{1}{\cancel{\chi}} \frac{1}{\sqrt{z}}\, dz + \frac{\cancel{\chi}}{\sqrt{\pi}} w^2 \int_0^\infty \frac{1}{\cancel{\chi}} z^{3/2} e^{-z}\, dz +$$

$$+ \frac{\cancel{8}^2}{\pi} \frac{1}{w^3} \frac{Na}{d_0} \int_0^\infty \frac{1}{\cancel{1}\sqrt{2}} \frac{\frac{1}{2} z}{\sqrt{z}} e^{-z}\, dz =$$

$2t^2 = z \ldots$

$$= -\frac{1}{\sqrt{\pi}} \frac{1}{w^2} \int_0^\infty (z^{3/2} - 3z^{1/2}) e^{-z}\, dz + \frac{w^2}{\sqrt{\pi}} \int_0^\infty z^{3/2} e^{-z}\, dz +$$

$$+ \frac{2}{\sqrt{2}\,\pi} \frac{1}{w^3} \frac{Na}{d_0} \int_0^\infty z^{1/2} e^{-z}\, dz =$$

$$= -\frac{1}{\sqrt{\cancel{\pi}}} \frac{1}{w^2} \cdot \frac{3\sqrt{\cancel{\pi}}}{4} + \frac{3}{\sqrt{\cancel{\pi}}} \frac{1}{w^2} \frac{\sqrt{\cancel{\pi}}}{2} + \frac{w^2}{\sqrt{\cancel{\pi}}} \frac{3\sqrt{\cancel{\pi}}}{4} +$$

$$+ \frac{\cancel{2}}{\sqrt{2}\,\cancel{\pi}\,\sqrt{\pi}} \frac{1}{w^3} \frac{Na}{d_0} \frac{\sqrt{\cancel{\pi}}}{\cancel{2}} = \frac{3}{4} \frac{1}{w^2} + \frac{3}{4} w^2 + \frac{1}{\sqrt{2\pi}} \left(\frac{Na}{d_0}\right) \frac{1}{w^3}$$

- Kinetic and potential part scale differently with w

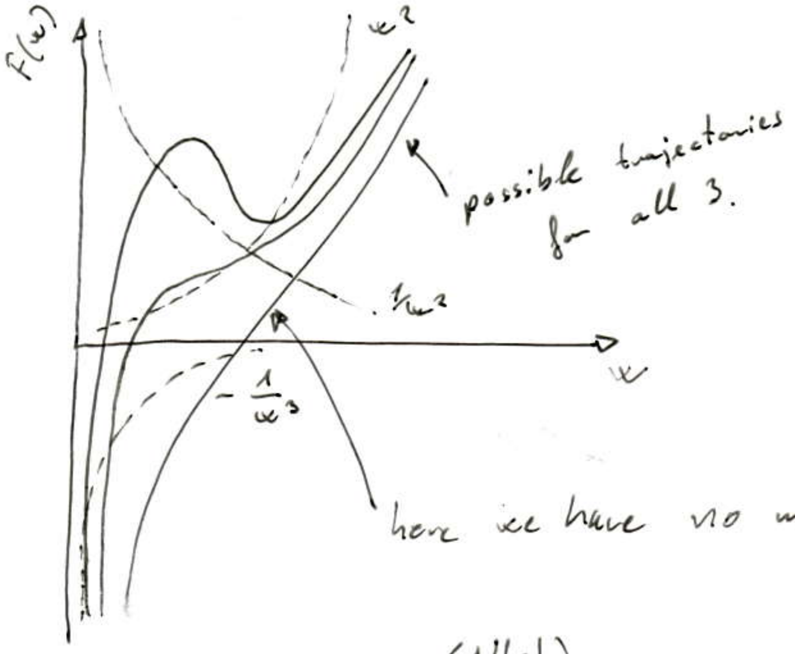
$$\min_w \left(\frac{1}{w^2} + w^2 \right) = 1$$

- then $\rightsquigarrow \frac{E}{N\hbar\omega_0} = \frac{3}{2} \rightsquigarrow$ that's good (neglecting non-linear interaction...)

- What happens when we add interaction, with $a < 0$?

$$F(w) = \frac{E(w)}{N\hbar\omega_0} = \frac{3}{4}\left(w^{-2} + w^2\right) - \frac{1}{\sqrt{2\pi}} \frac{N|a|}{d_0} w^{-3}$$

for $a < 0$



- by increasing $\left(\frac{N|a|}{d_0}\right)$ the minimum at $w \cong 1$ disapears.
- if $w \to 0$ the Gaussian $\to$ Dirac-delta
- there is a mechanical instability $\to$ collapse.
- but for lower $\left(\frac{N|a|}{d_0}\right)$ there is a local minima
  - $\to$ sys can be <u>metastable</u> there
  - $\to$ it can have long lifetime
  - $\to$ BEC for a while...

$\frac{\partial F(w)}{\partial w} = 0 \to$ where is the minima as a func. of w

$\frac{\partial^2 F(w)}{\partial w^2} = 0 \to$ criteria for marginal metastability.

$\to$ this fixes $\left(\frac{N|a|}{d_0}\right)$ to be the critical value.

$$0 = \frac{\partial F(w)}{\partial w} = -\frac{3}{2} w^{-3} + \frac{3}{2} w + \frac{3}{\sqrt{2\pi}} \left(\frac{N|a|}{d_0}\right)_{cr} w^{-4}$$

$$0 = \frac{\partial^2 F(w)}{\partial w} = +\frac{9}{2} w^{-4} + \frac{3}{2} - \frac{12}{\sqrt{2\pi}} \left(\frac{N|a|}{d_0}\right)_{cr} w^{-5}$$

- can be solved for $\left(\frac{N|a|}{d_0}\right)_{cr}$ and w_{cr}.

$$\frac{3}{\sqrt{2\pi}}\left(\frac{N|a|}{d_0}\right)_{cr} = \frac{3}{2}w - \frac{3}{2}w^5$$

$$-\frac{3}{\sqrt{2\pi}}\left(\frac{N|a|}{d_0}\right)_{cr} = -\frac{9}{8}w - \frac{3}{8}w^5$$

$\oplus$

$$\frac{3}{8}w - \frac{15}{8}w^5 = 0$$

$$w = 5w^5$$

$$\frac{1}{5} = w^4 \rightsquigarrow \underline{\underline{w_{crit} = \left(\frac{1}{5}\right)^{1/4}}}$$

$$\left(\frac{N|a|}{d_0}\right)_{cr} = \frac{\sqrt{2\pi}}{\cancel{3}}\left(\frac{\cancel{3}}{2}\left(\frac{1}{5}\right)^{1/4} - \frac{\cancel{3}}{2}\left(\frac{1}{5}\right)^{5/4}\right) = \frac{\sqrt{\pi}}{\sqrt{2}\,5^{1/4}}\underbrace{\left(1-\frac{1}{5}\right)}_{4/5} =$$

$$\underline{\underline{\left(\frac{N|a|}{d_0}\right)_{cr} = \frac{2}{5}\frac{\sqrt{2\pi}}{5^{1/4}} \approx 0.671}}$$

- the external potential helps keep the condensate up to $\left(\frac{N|a|}{d_0}\right)_{cr}$. $\rightsquigarrow$ w/o it the condensate disapears.

$\rightsquigarrow$ there is condensate for atoms with $\ominus$ scattering length.

$|a| \rightsquigarrow$ material property

$d_0 \rightsquigarrow$ dictated by trap

$N \rightsquigarrow$ only value experimentally to be set.

- in case of G-P, numerically $\left(\frac{N|a|}{d_0}\right)_{cr} \approx 0.575$

$\rightsquigarrow$ variational ansatz was not so bad

Imaginary time method for G-P eq.

$$\left[-\frac{\hbar^2}{2m}\Delta + V + g|\Psi|^2\right]\Psi = \mu\Psi$$

$$g = \frac{4\pi\hbar^2 a}{m} \;;\; \int |\Psi|^2 d^3r = N$$

- what is the grnd. state for a given V confing potential?

- First a linear system:

$$\left[-\frac{\hbar^2}{2m}\Delta + V\right]\Psi = E\Psi$$

$$\Psi(t=0) = \sum_n c_n \Psi_n(t=0) \;;\; \left[-\frac{\hbar^2}{2m}\Delta + V\right]\Psi_n = E_n\Psi_n$$

- for the dynamics of Ψ:

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = H\Psi(t)$$

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = \left[-\frac{\hbar^2}{2m}\Delta + V\right]\Psi(t)$$

$\rightsquigarrow$ $\Psi(t) = \sum_n e^{-\frac{iE_n t}{\hbar}} c_n \Psi_n(t=0)$ knowing the expansion.

$\rightsquigarrow$ starting from a mixed state.

- we introduce $t = -i\tau$

$$-\hbar\frac{\partial}{\partial\tau}\Psi(\tau,x) = \left[-\frac{\hbar^2}{2m}\Delta + V\right]\Psi(\tau,x)$$

$$\Psi(\tau,x) = \sum_n e^{-\frac{E_n\tau}{\hbar}} c_n \Psi_n(x)$$

$\rightsquigarrow$ all the states shrink to zero

$\rightsquigarrow$ the higher the n the faster it shrink

$\rightsquigarrow$ after a while only grnd. state remains.

→ exponential factor messes up the normalization!

$$e^{-\frac{\varepsilon_0 \tau}{\hbar}}\left(c_0 \Psi_0 + e^{-\frac{\varepsilon_1 - \varepsilon_0}{\hbar}\tau} c_1 \Psi_1 + \dots\right)$$

$$\to e^{-\frac{\varepsilon_0 \tau}{\hbar}} c_0 \Psi_0$$

taking it at τ_1, τ_2

ε_0 can be obtained...

$$\frac{\Psi(\tau_1)}{\Psi(\tau_2)} = e^{-\frac{\varepsilon_0}{\hbar}(\tau_1 - \tau_2)}$$

- repeating the same method (without reasoning) for the nonlinear case, it (for same reason) works and the grund. state can be found with imaginary time...

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\Psi|^2\right]\Psi(\vec{r}, t)$$

$$\Psi(\vec{r}, t) = e^{-i\frac{\mu t}{\hbar}} \Psi_0(\vec{r}, t)$$

$t = -i\tau$

$$-\hbar \frac{\partial}{\partial \tau} \Psi(\vec{r}, \tau) = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\Psi|^2\right]\Psi(\vec{r}, \tau)$$

$$\sim e^{-\frac{\mu \tau}{\hbar}} \Psi_0(\vec{r})$$