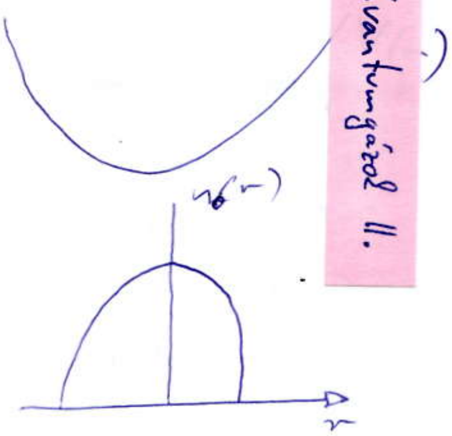
2019.09.12.

Kvantumgázok II.

1.

Quantum Gases II.

Stingari-hydro (continued)



$$\omega^2 \delta n(\vec{r}) = -\frac{1}{m}\vec{\nabla}\left[(\mu - V)\vec{\nabla}\delta n\right]$$

$$V(r) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2$$

- can solve for spectra
  - we expect discrete → we need some quantization

    ↓

    what is the function space in which δn lives?

  - we only require ℓ^2 in the support of the region $n_0(r) \neq 0$!

    $\delta n \in \ell^2$ (but not that one...)

$$\int_{\mu > V(r)} (\delta n)^2 \text{ must be finite}$$

⇓

It is fulfilled if δn is a polinomial in x, y, z

$$\omega^2 \delta n(\vec{r}) = -\frac{(\mu - V)}{m}\Delta\delta n - \frac{1}{m}\underbrace{\left(\vec{\nabla}(\mu - V)\right)}_{-\vec{\nabla}V = -\begin{pmatrix} m\omega_x^2 x \\ m\omega_y^2 y \\ m\omega_z^2 z\end{pmatrix}} \cdot \vec{\nabla}\delta n$$

$$+ \frac{1}{m}(\vec{\nabla}V)\cdot(\vec{\nabla}\delta n)$$

$$\boxed{\omega^2 \delta n(\vec{r}) = -\frac{(\mu - V)}{m}\Delta\delta n + \frac{1}{m}(\vec{\nabla}V)\cdot(\vec{\nabla}\delta n)}$$

- Laplacian decreases the total degree by two

but $V(\vec{r})$ raises the highest degree by two

→ the resulting polinomial will be the same order than S_n

- the same things happen on the second part.

- Altogether the highest order will be kept

• there is also no mixing between even and odd.

Some special solutions

$\boxed{1.}$ $x^\alpha y^\beta z^\gamma$ and $\alpha = 0, 1$, $\beta = 0, 1$, $\gamma = 0, 1$ independently

- this is 8 different monoms, all with different reflection properties. → corresponding 8 different symmetry classes

(0 → even, 1 → odd)

$$\Delta x^\alpha y^\beta z^\gamma = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) x^\alpha y^\beta z^\gamma = 0$$

$$\frac{\vec{\nabla} V}{m} \vec{\nabla} = \sum_{i=1}^{3} \omega_i^2 r_i \frac{\partial}{\partial r_i} \quad \text{(scalar operator)}$$

$$\frac{\vec{\nabla} V}{m} \vec{\nabla} x^\alpha y^\beta z^\gamma = \left(\alpha \omega_x^2 + \beta \omega_y^2 + \gamma \omega_z^2\right) x^\alpha y^\beta z^\gamma$$

$$\Longrightarrow \quad S_n = x^\alpha y^\beta z^\gamma, \quad \omega_x \neq \omega_y \neq \omega_z, \quad \alpha = 0,1,\ \beta = 0,1,\ \gamma = 0,1$$

$$\omega^2 = \left(\alpha \omega_x^2 + \beta \omega_y^2 + \gamma \omega_z^2\right)$$

1a) $\alpha = \beta = \gamma = 0$

$\omega^2 = 0 \longrightarrow \delta n = \text{const.} \rightarrow$ not number conserving!

$\rightsquigarrow$ this means the grnd. state is degenerate

$\rightsquigarrow$ this is a non-physical solution!

$$n = n_0 + \delta n(\vec{r}, t) \quad \Big/ \int_{\mu>V} d^3r$$

n_0: eq. density

$\delta n(\vec{r}, t)$: density profile

$$N = N + \int \delta n(\vec{r}, t)\, d^3r \rightarrow \text{extra condition for mode functions}$$

$$\boxed{\int_{\mu<V} \delta n(\vec{r}, t)\, d^3r \overset{!}{=} 0}$$

- the density must be number conserving

1b) $\alpha + \beta + \gamma = 1$

$$\left.\begin{array}{lll} \delta n = x & \alpha = 1, \beta = 0, \gamma = 0 & \omega = \omega_x \\ \delta n = y & \alpha = 0, \beta = 1, \gamma = 0 & \omega = \omega_y \\ \delta n = z & \alpha = 0, \beta = 0, \gamma = 1 & \omega = \omega_z \end{array}\right\} \text{these are the } \underline{\text{Kohn-modes}}$$

1c) $\alpha + \beta + \gamma = 2$

$$\left.\begin{array}{ll} \delta n = xy & \omega^2 = \omega_x^2 + \omega_y^2 \\ \delta n = yz & \omega^2 = \omega_y^2 + \omega_z^2 \\ \delta n = xz & \omega^2 = \omega_x^2 + \omega_z^2 \end{array}\right\}$$

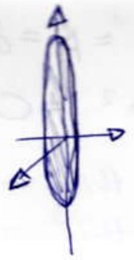
if $\omega_x, \omega_y << \omega_z$ at any ω_y, ω_x the Kohn modes corresponding to ω_z will show up!

1d) $\alpha + \beta + \gamma = 3$

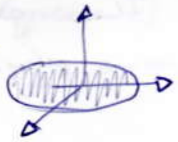
$\delta n = xyz \quad \omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$

$\boxed{2.}$ $\omega_x, \omega_y = \omega_0 \neq \omega_z$

- this is the case of cigar-shaped condensate



pancake - —"—



$$\delta n = \underbrace{z^{\alpha} \cdot \rho^{|m|} \cdot e^{\pm i m \varphi}}_{\text{axially symmetric}} \cdot T_n(\rho^2, z^2)$$

but still a polinomial

$$\left[\begin{array}{l} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ (x+iy)^m = \rho^m (\cos\varphi + i \sin\varphi) \end{array}\right]$$

$\alpha = 0, 1$

some general function with rotational symm.

~~and~~

forgás és tükr. szempontjából egységábrázolásként transzformálódik.

2a) $\delta n = 1 \rightsquigarrow \alpha = 0,\ m = 0 \rightsquigarrow \omega = 0 \to$ NO!

2b) $\alpha = 1,\ m = 0,\ n = 0$ • Kohn-modes

$\delta n = z^{\alpha} \rightsquigarrow \omega^2 = \omega_z^2$

rotational symmetry

$\alpha = 0,\ m = 1,\ n = 0$

$\delta n = (x \pm iy) = \rho e^{\pm i\varphi} \rightsquigarrow \omega^2 = \omega_0^2$

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\omega_0^2 \underbrace{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)}_{\rho\frac{\partial}{\partial\rho}} + \omega_z^2 z \frac{\partial}{\partial z} = \frac{\vec{\nabla} V}{m}\vec{\nabla} = \omega_0^2 \rho \frac{\partial}{\partial\rho} + \omega_z^2 z \frac{\partial}{\partial z}$$

2c) – the polinomial has a degree of 2.

$\alpha = 0, \quad m = 2, \quad n = 0$

- $\delta n = \rho^2 e^{\pm i 2\varphi} = (x \pm iy)^2$

$$\Delta \rho^2 e^{\pm 2i\varphi} = e^{\pm 2i\varphi} \cdot 4 - 4 e^{\pm 2i\varphi} = 0$$

$$\left(\omega_0^2 \rho \frac{\partial}{\partial \rho} + \cancel{\omega_z^2 z \frac{\partial}{\partial z}}\right) \rho^2 e^{\pm 2i\varphi} = \omega_0^2 e^{\pm 2i\varphi} \cdot 2\rho^2$$

$$\longrightarrow \boxed{\omega^2 = 2\omega_0^2}$$

these modes are called breath-modes

- $\alpha = 1, \quad m = 1, \quad n = 0$

$\delta n = z \rho e^{\pm i\varphi}$

$$\Delta\left(z \rho e^{\pm i\varphi}\right) = \frac{z}{\rho} e^{\pm i\varphi} - \frac{z}{\rho} e^{\pm i\varphi} = 0$$

$$\left(\omega_0^2 \rho \frac{\partial}{\partial \rho} + \omega_z^2 z \frac{\partial}{\partial z}\right) z \rho e^{\pm i\varphi} = \left(\omega_0^2 + \omega_z^2\right) z \rho e^{\pm i\varphi}$$

$$\longrightarrow \boxed{\omega^2 = \omega_0^2 + \omega_z^2}$$

- $\alpha = 0, \quad m = 0, \quad n = 1$

$\delta n = A + B\rho^2 + C z^2$

$$\omega^2 \left(A + B\rho^2 + C z^2\right) = - \frac{\left(\mu - \frac{m\omega_0^2 \rho^2 + m\omega_z^2 z^2}{2}\right)}{m} \cdot (4B + 2C) +$$

$$+ \omega_0^2 \, 2B\rho^2 + \omega_z^2 \, 2C z^2$$

– the two sides must be equal on all space points.

- $\rho^2 : \omega^2 B = \omega_0^2 (2B + C) + \omega_0^2 \, 2B = 4\omega_0^2 B + \omega_0^2 C$

$z^2 : \omega^2 C = \omega_z^2 (2B + C) + \omega_z^2 \, 2C = 2\omega_z^2 B + 3\omega_z^2 C$

const: $\omega^2 A = -\frac{\mu}{m}(4B+2C) \longrightarrow$ solving the others for B, C, ω^2 we can get A

$$\omega^2 \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 4\omega_0^2 & \omega_0^2 \\ 2\omega_z^2 & 3\omega_z^2 \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}$$

$\rightsquigarrow$ eigenvalue-eq for ω^2

$$(4\omega_0^2 - \omega^2)(3\omega_z^2 - \omega^2) - \omega_0^2 \, 2\omega_z^2 = 0 \qquad (\det = 0)$$

$$\omega^2 = 2\omega_0^2 + \frac{3}{2}\omega_z^2 \pm \frac{1}{2}\sqrt{9\omega_z^2 - 16\,\omega_0^2\omega_z^2 + 16\,\omega_0^4}$$

- this was the easiest mode to prepare in BEC-experiments
  - $\rightsquigarrow$ the mode was exactly there
- μ does not play any role in ω freq.

$\downarrow$

ω is in the order of trap frequencies.