⇝ exponential factor messes up the normalization!

$$e^{-\frac{\varepsilon_0 \tau}{\hbar}}\left(c_0 \psi_0 + e^{-\frac{\varepsilon_1-\varepsilon_0}{\hbar}\tau} c_1 \psi_1 + \dots\right)$$

$\longrightarrow e^{-\frac{\varepsilon_0 \tau}{\hbar}} c_0 \psi_0$ taking it at τ_1, τ_2

ε_0 can be obtained...

$$\frac{\psi(\tau_1)}{\psi(\tau_2)} = e^{-\frac{\varepsilon_0}{\hbar}(\tau_1-\tau_2)}$$

- repeating the same method (without reasoning) for the nonlinear case, it (for some reason) works and the grnd. state can be found with imaginary time...

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\psi|^2\right]\psi(\vec{r},t)$$

$$\psi(\vec{r},t) = e^{-i\frac{\mu t}{\hbar}}\psi_0(\vec{r},t)$$

$t = -i\tau$

$$-\hbar \frac{\partial}{\partial \tau}\psi(\vec{r},\tau) = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\psi|^2\right]\psi(\vec{r},\tau)$$

$$\sim e^{-\frac{\mu\tau}{\hbar}}\psi_0(\vec{r})$$

$\boxed{2019.04.01.}$

- two component description
  - ⟶ normal part $\longrightarrow \rho(\vec{r},\vec{p}) = \dfrac{1}{e^{-\beta\left(\frac{p^2}{2m}+V(r)-\mu\right)}-1}$
  - ⟶ condensate

$$n(\vec{r}) = \int \frac{d^3p}{(2\pi)^3}\rho(\vec{r},\vec{p}) \sim e^{-\frac{V(\vec{r})}{k_B T}}$$

at the border it behaves like a Boltzmann-distribution.

⇓

temperature can be extracted from the tail

⟵ at very low T there is no more normal atoms

⟶ we don't know the T anymore

- we need a different way to measure the T!
  - one trick ⇝ avg. over the angles to smooth out the fluctuation, and the tail can be fitted.

- attainable Temps ~ pK!

- BEC on a chip

- wires, they create a B field ("the chip")
- condensate (this way it can fall...)
- T.O.F. detector

<u>Excitations in BEC</u>, Bogoliubov - excitations

- with lasers the condensate can made to oscillate
  - After a while there will only be one excitation, and that can be measured. (Higher excitations die faster...)
  - both the condensate and the normal part oscillate
  - there is <u>characteristic length</u>
    ↳ good choice can be useful ⇝ ex.: diameter of the condensate...
  - the observed oscillation is damped: both damping, frequency can be obtained

- to describe the excitation <u>dynamics</u> have to be introduced
  - we will work at $\boxed{T=0}$ ⇝ no thermal atoms, only condensate } easier this way

- dispersion - relation of liquid He:

- for trapped gases
  - excitations live longer on lower temps..
  - static G-P eq.!

$$\mu \Psi_0(\vec{r}) = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\Psi_0|^2\right]\Psi_0(\vec{r})$$

- we go to time dep.:

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\Psi|^2\right]\Psi(\vec{r},t)}$$ Time-dependent G-P eq.

$\Psi(\vec{r},t) = e^{-\frac{i\mu}{\hbar}t}\Psi_0(\vec{r})$, where $\Psi_0(\vec{r})$ is the solution from the static eq.

ground state

- there are several methods on how to solve it.

- non-linear excitation: "drastic" effect.

  ⤳ not gonna happen..

- we will use linear approx. around the static condensate

$$\Psi(\vec{r},t) = e^{-\frac{i\mu}{\hbar}t}\left[\Psi_0(\vec{r}) + \delta\Psi(\vec{r},t)\right] \quad \text{and} \quad \delta\Psi(\vec{r},t) \ll \Psi_0(\vec{r})$$

- confining pot. ⤳ allways descrete excitations!

$$\delta\Psi(\vec{r},t) = \sum_i \left(u_i(\vec{r})e^{-i\omega_i t} - \underbrace{v_i^*(\vec{r})e^{i\omega_i t}}_{\text{we need this, too}}\right)$$

$\omega_i \in \mathbb{R}$, otherwise one of the terms is diverging.

$$\left.\begin{array}{l}\omega_i \leftrightarrow -\omega_i \\ u_i \leftrightarrow -v_i^*\end{array}\right\} \delta\Psi \text{ is invariant under this...}$$

⇓

for every ⊕ ω_i there is ⊖ ω_i

⤳ we can restrict $\boxed{\omega_i > 0}$

⤳ $\omega_i = 0$ ⤳ no unique ground. state ⤳ bad

$$i\hbar\left[-\frac{i\mu}{\hbar}\cancel{e^{-\frac{i\mu}{\hbar}t}}(\psi_0+\delta\psi)+\cancel{e^{-\frac{i\mu}{\hbar}t}}\frac{\partial\delta\psi}{\partial t}\right]=$$

$$=\cancel{e^{-\frac{i\mu}{\hbar}t}}\left[-\frac{\hbar^2}{2m}\Delta+V\right]\psi_0+\cancel{e^{-\frac{i\mu}{\hbar}t}}\left[-\frac{\hbar^2}{2m}\Delta+V\right]\delta\psi$$

$$+\cancel{e^{-\frac{i\mu}{\hbar}t}}g(\psi_0^*+\delta\psi^*)(\psi_0+\delta\psi)(\psi_0+\delta\psi)$$

$$\mu(\cancel{\psi_0}+\delta\psi)+i\hbar\frac{\partial}{\partial t}\delta\psi=\cancel{\left[-\frac{\hbar^2}{2m}\Delta+V\right]\psi_0}+\left[-\frac{\hbar^2}{2m}\Delta+V\right]\delta\psi+$$

$$+g\Big(\cancel{|\psi_0|^2\psi_0}+2(\delta\psi)|\psi_0|^2+\psi_0^2(\delta\psi^*)+$$

$$+(\delta\psi)(\delta\psi^*)\cdot 2\psi_0+\psi_0^*(\delta\psi)^2+$$

$$+(\delta\psi)^2(\delta\psi^*)\Big)$$

- we can get rid of the 0th order $\rightsquigarrow$ static GP-eq.
- we ignore 2nd, 3rd order terms.
- We only have the First Order (linearize...)

$$\mu\delta\psi+i\hbar\frac{\partial}{\partial t}\delta\psi=\left[-\frac{\hbar^2}{2m}\Delta+V+2g|\psi_0|^2\right](\delta\psi)+g\psi_0^2(\delta\psi^*)$$

$$i\hbar\frac{\partial}{\partial t}\delta\psi=\left[-\frac{\hbar^2}{2m}\Delta+V-\mu+2g|\psi_0|^2\right](\delta\psi)+\underbrace{g\psi_0^2(\delta\psi^*)}$$

- now we can insert

$$\delta\psi(\vec r,t)=\sum_i\left(v_i(\vec r)e^{-i\omega_i t}-v_i^*(\vec r)e^{i\omega_i t}\right)$$

this is why we need the second term in $\delta\psi$!

it does not allow a scalar Hamiltonian it becomes 2x2 mx.!

$$\hat H_{HF}=\left[-\frac{\hbar^2}{2m}\Delta+V-\mu+2g|\psi_0|^2\right]$$

notation

$$i\hbar \sum_i \left(-i\omega_i u_i e^{-i\omega_i t} - i\omega_i v_i^* e^{i\omega_i t}\right) =$$

$$- \sum_i \left[\left(e^{-i\omega_i t} \hat{H}_{HF} u_i - e^{i\omega_i t} \hat{H}_{HF} v_i^*\right) + g \Psi_0^2 \left(v_i^* e^{i\omega_i t} - v_i e^{-i\omega_i t}\right)\right]$$

- we gather all terms $\sim e^{-i\omega_i t}$:

$$\hbar \omega_i u_i = \hat{H}_{HF} u_i - g \Psi_0^2 v_i$$

- terms with $\sim e^{i\omega_i t}$:

$$\hbar \omega_i v_i^* = - \hat{H}_{HF} v_i^* + g \Psi_0^2 u_i^* \quad / ( )^* ; (-1) .$$

$$\left.\begin{aligned} \hbar \omega_i u_i &= \hat{H}_{HF} u_i - g \Psi_0^2 v_i \\ -\hbar \omega_i v_i &= \hat{H}_{HF} v_i - g \Psi_0^{*2} u_i \end{aligned}\right\}$$

- 2×2 mx structure:

$$\hbar \omega_i \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{H}_{HF} & -g \Psi_0^2 \\ g \Psi_0^{*2} & -\hat{H}_{HF} \end{pmatrix}}_{\underline{\underline{H}}} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$$\underline{v}_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

$$\rightsquigarrow \boxed{\hbar \omega_i \underline{v}_i = \underline{\underline{H}}\, \underline{v}_i}$$

- delicate question: what is the scalar product for with $\underline{\underline{H}}$ is Hermitian?

  $\rightsquigarrow$ otherwise ω_i can be imaginary!!

- <u>Statement:</u> $\underline{\underline{H}} = \underline{\underline{H}}^+$ with the scalar product:

$$\langle \underline{v}_1 | \underline{v}_2 \rangle = \int d^3r \left(u_1^* u_2 - v_1^* v_2\right)$$
