$$0 = \frac{4}{\vartheta_i} + \frac{4\ell + 6}{a^2 + \vartheta_i} + \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{8}{\vartheta_i - \vartheta_j} \qquad \text{force - eq.}$$

→ this fun gives a new evaluation of a hypergeometric polinomial (which is the known solution for this prob.)

→ it also gives the roots ...

$\boxed{2019.10.24.}$

Popov-approximation

$$\hat{\Psi}(\vec{r}) = \underbrace{\sqrt{\frac{N_0}{V}}}_{\text{goes to zero, when } T \to T_c} + \hat{\varphi}(\vec{r})$$

$$\langle \hat{\Psi}(\vec{r}) \rangle = \sqrt{\frac{N_0}{V}} \rightsquigarrow \langle \hat{\varphi}(\vec{r}) \rangle = 0$$

in homogenious systems $(V = 0)$

off diagonal long range order

$$\hat{\rho}(r, r', t) = \langle \Psi^+(r') \Psi(r) \rangle$$

→ if $r \neq r'$ the mx element goes to zero. (in normal systems.)

→ if we have BEC

$$\lim_{|r'-r| \to \infty} \hat{\rho}(r, r', t) = \lim_{|r'-r| \to \infty} \langle (\overset{\sqrt{N_0/V}}{\Psi_0^*} + \hat{\varphi}^+(r')) \cdot (\Psi_0 + \hat{\varphi}) \rangle = \underline{\underline{\frac{N_0}{V}}} \to \text{not zero!}$$

this is called off diagonal long range order

• Now $V \neq 0$

$$\hat{\Psi}(\vec{r}) = \Psi_0(\vec{r}) + \hat{\varphi}(\vec{r})$$

↳ condensate wf.

• $\langle \hat{\Psi}(\vec{r}) \rangle = \Psi_0$

$\langle \hat{\varphi}(\vec{r}) \rangle = 0$

→ in confined systems there is no off-diagonal long range order.

→ both $\Psi_0(\vec{r})$ and $\hat{\varphi}(\vec{r})$ are localized.

$$\hat{H} = \int d^3\vec{r}\, \Psi^+(\vec{r}) \left(-\frac{\hbar^2}{2m} \Delta + V(\vec{r}) - \mu \right) \Psi(\vec{r}) +$$

$$+ \frac{g}{2} \int d^3\vec{r}\, \Psi^+(\vec{r}) \Psi^+(\vec{r}^{\,*}) \Psi(\vec{r}^{\,*}) \Psi(\vec{r})$$

and $\quad V(r, r') = \frac{4\pi\hbar^2 a}{m} \delta(\vec{r} - \vec{r}') = g\,\delta(\vec{r} - \vec{r}')$

• mean-field approx to reduce to a quadratic problem.

$$\hat{\Psi}(\vec{r}) = \Psi_0 + \hat{\varphi}(\vec{r})$$
$$\hat{\Psi}^+(\vec{r}) = \Psi_0^* + \hat{\varphi}^+(\vec{r})$$

$$\hat{\Psi}^+(\vec{r})\,\hat{\Psi}^+(\vec{r})\,\Psi(\vec{r})\,\Psi(\vec{r}) = \left(\Psi_0^{*2} + 2\Psi_0^*\hat{\varphi}^+ + \hat{\varphi}^+\hat{\varphi}^+\right)\left(\Psi_0^2 + 2\Psi_0\hat{\varphi} + \hat{\varphi}\hat{\varphi}\right)$$

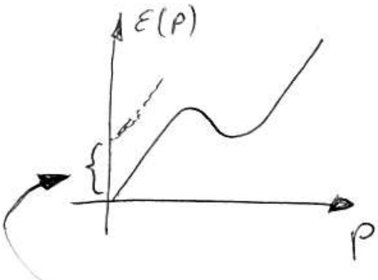
$$= |\Psi_0|^4 + 2|\Psi_0|^2\left(\Psi_0^*\hat{\varphi}^+ + \Psi_0^*\hat{\varphi}\right) +$$
$$+ \Psi_0^{*2}\hat{\varphi}\hat{\varphi} + \Psi_0^2\hat{\varphi}^+\hat{\varphi}^+ + 4|\Psi_0|^2\hat{\varphi}^+\hat{\varphi} +$$
$$+ 2\Psi_0^*\hat{\varphi}^+\hat{\varphi}\hat{\varphi} + 2\Psi_0\hat{\varphi}^+\hat{\varphi}^+\hat{\varphi} +$$
$$+ \hat{\varphi}^+\hat{\varphi}^+\hat{\varphi}\hat{\varphi}$$

$$\hat{\varphi}^+\hat{\varphi}\hat{\varphi} \approx 2\langle\hat{\varphi}^+(\vec{r})\hat{\varphi}(\vec{r})\rangle\hat{\varphi}(\vec{r}) + \hat{\varphi}^+\cancel{\langle\varphi(\vec{r})\varphi(\vec{r})\rangle}$$

possible contraction

⇝ 3rd, and 4th order replaced by mean field.

⇝ anomalous averages ~~[illegible]~~: $\langle\varphi(r)\varphi(r)\rangle = 0 = m_T(r)$
neglected $\langle\varphi^+(r)\varphi^+(r)\rangle = 0$



to have no gap in the dispersion relation.
(but actually it is non-zero)

$$n_T(\vec{r}) = \langle\hat{\varphi}^+(\vec{r})\,\hat{\varphi}(\vec{r})\rangle$$

close to T_c: $|m_T| \ll n_T$

$$n(\vec{r}) = \langle \Psi^+(\vec{r})\,\Psi(\vec{r})\rangle = \langle (\Psi_0^* + \varphi^+)(\Psi_0 + \varphi)\rangle =$$

$$= \underbrace{|\Psi_0(\vec{r})|^2}_{= n_C(\vec{r})} + \underbrace{\langle \varphi^+(r)\,\varphi(r)\rangle}_{= n_T(\vec{r})}$$

$\uparrow$ density of the system

$n_C(\vec{r})$: contribution of the condensed atom.

$n_T(\vec{r})$: contribution of the thermal atoms.

$$\hat{\varphi}^+(\vec{r})\,\hat{\varphi}(\vec{r})\,\hat{\varphi}(\vec{r}) = 2 n_T(\vec{r})\,\hat{\varphi}(\vec{r})$$

$$\hat{\varphi}^+(\vec{r})\,\hat{\varphi}^+(\vec{r})\,\hat{\varphi}(\vec{r}) = 2 n_T(\vec{r})\,\hat{\varphi}^+(\vec{r})$$

$$\hat{\varphi}^+(\vec{r})\,\hat{\varphi}^+(\vec{r})\,\hat{\varphi}(\vec{r})\,\hat{\varphi}(\vec{r}) = 4 n_T(\vec{r})\,\hat{\varphi}^+(\vec{r})\hat{\varphi}(\vec{r})$$

$$\hat{K} = \hat{K}_0 + \hat{K}_1 + \hat{K}_1^+ + \hat{K}_2$$

$\hat{K}_0$: 0th order; $\hat{K}_1$, $\hat{K}_1^+$: 1st order; $\hat{K}_2$: 2nd order in $\hat{\varphi}(\vec{r})$

$$\hat{K}_0 = \int d^3\vec{r}\;\Psi_0^*\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu\right)\Psi_0 + \frac{g}{2}|\Psi_0|^4 =$$

$$= \int d^3\vec{r}\;\Psi_0^*\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + \frac{g}{2}|\Psi_0|^2\right)\Psi_0$$

$$\hat{K}_1 = \int d^3\vec{r}\;\varphi^+(\vec{r})\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + g|\Psi_0|^2 + 2g\,n_T(r)\right)\Psi_0$$

$$\hat{K}_1^+ = \int d^3\vec{r}\;\Psi_0^*\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + g|\Psi_0|^2 + 2g\,n_T(r)\right)\hat{\varphi}(\vec{r})$$

$$\hat{K}_2 = \int d^3\vec{r}\ \hat{\varphi}^+(\vec{r})\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + 2g|\Psi_0|^2 + 2g\, n_T(r)\right)\hat{\varphi}(\vec{r}) +$$

$$+ \frac{g}{2}\int d^3\vec{r}\ \Psi_0^{*2}\,\hat{\varphi}(\vec{r})\,\hat{\varphi}(\vec{r}) + \frac{g}{2}\int d^3\vec{r}\ \Psi_0^2\,\hat{\varphi}^+(\vec{r})\,\hat{\varphi}^+(\vec{r})$$

$$\langle\hat{O}\rangle = \left\langle e^{-\beta(K_0 + K_1 + K_1^+ + K_2)}\,\hat{O}\right\rangle \cdot \frac{1}{\left\langle e^{-\beta(K_0 + K_1 + K_1^+ + K_2)}\right\rangle}$$

$\rightsquigarrow$ K_0 is just a number $\rightsquigarrow$ will not affect any thermal avg.-s

$$K_1 \stackrel{!}{=} 0$$

is true if $\left(-\frac{\hbar^2}{2m}\Delta + V(r) - \mu + g|\Psi_0|^2 + 2g\,n_T(r)\right)\Psi_0 = 0$

→ generalized G-P-eq. for Ψ_0

→ there is a contribution from the non-condensed atoms.

- only the second order terms should be diagonalized ($\hat{K}_2$)

$$\hat{H}_{HF} = -\frac{\hbar^2}{2m}\Delta + V(r) - \mu + g|\Psi_0|^2 + 2g\,n_T(r)$$

$$H_{HF}^+ = H_{HF} \quad (\text{self-adjoint})$$

$$\hat{K}_2 = \int d^3\vec{r}\ \hat{\varphi}^+(\vec{r})\,H_{HF}(\vec{r})\,\hat{\varphi}(\vec{r}) + \frac{g}{2}\int d^3\vec{r}\ \Psi_0^{*2}\,\hat{\varphi}(\vec{r})\,\hat{\varphi}(\vec{r}) + \frac{g}{2}\int d^3r\ \Psi_0^2\,\hat{\varphi}^+\hat{\varphi}^+$$

- above T_c: $\hat{\varphi} = \sum_i \varphi_i(\vec{r})\,\hat{a}_i$ (for normal system)

$$H_{HF}\,\varphi_i = \varepsilon_i\,\varphi_i$$

$$K_2 = \sum_{i,j}\int d^3r\ \varphi_i^*(r)\,a_i^+\,\varepsilon_j\,\varphi_j(r)\,a_j = \sum_{ij}\varepsilon_j\,\hat{a}_i^+\hat{a}_j\underbrace{\int d^3\vec{r}\ \varphi_i^*(r)\varphi_j(r)}_{\delta_{ij}} =$$

$$= \sum_i \varepsilon_i\, a_i^+ a_i$$

- under T_c, when there is BEC

Bogolibov - decomposition.

$$\hat{\varphi}(\vec{r}) = \sum_i \left(u_i(\vec{r}) \hat{\alpha}_i - v_i^*(\vec{r}) \hat{\alpha}_i^+ \right)$$

$$\hat{\varphi}^+(\vec{r}) = \sum_i \left(u_i^*(\vec{r}) \hat{\alpha}_i^+ - v_i(\vec{r}) \hat{\alpha}_i \right)$$

$$\hat{K}_2 = \int d^3\vec{r} \left\{ \sum_{ij} \left[(u_i^* \alpha_i^+ - v_i \alpha_i) H_{HF} (u_j \alpha_j - v_j^* \alpha_j^+) + \frac{g}{2} \psi_0^{*2} (u_i \alpha_i - v_i^* \alpha_i^+) \cdot (u_j \alpha_j - v_j^* \alpha_j^+) + \frac{g}{2} \psi_0^2 \left(u_i^* \alpha_i^+ - v_i \alpha_i \right) \left(u_j^* \alpha_j^+ - v_j \alpha_j \right) \right] \right\}$$

$$= \sum_{ij} \int d^3\vec{r} \left\{ \alpha_i^+ \alpha_j \left(u_i^* H_{HF} u_j - \frac{g}{2} \psi_0^{*2} v_i^* u_j - \frac{g}{2} \psi_0^2 u_i^* v_j \right) + \right.$$

$$+ \alpha_i \alpha_j^+ \left(v_i H_{HF} v_j^* - \frac{g}{2} \psi_0^{*2} u_i v_j^* - \frac{g}{2} \psi_0^2 v_i u_j^* \right) +$$

$$- \alpha_i \alpha_j \left(v_i H_{HF} u_j - \frac{g}{2} \psi_0^{*2} u_i u_j - \frac{g}{2} \psi_0^2 v_i v_j \right) +$$

$$\left. - \alpha_i^+ \alpha_j^+ \left(u_i^* H_{HF} v_j^* - \frac{g}{2} \psi_0^{*2} v_i^* v_j^* - \frac{g}{2} \psi_0^2 u_i^* u_j^* \right) \right\} =$$

$\rightarrow$ trick $\int a H_{HF} b = \int b H_{HF} a$
(partial integration...)

$$\longrightarrow \int a H_{HF} b = \frac{1}{2} \int \left(a H_{HF} b + b H_{HF} a \right)$$

$$= \frac{1}{2} \sum_{ij} \int d^3 r \left\{ \alpha_i^+ \alpha_j \left(u_i^* \left(H_{HF} u_j - g \psi_0^2 v_j \right) + u_j \left(H_{HF} u_i^* - g \psi_0^{*2} v_i^* \right) \right) + \right.$$

$$+ \alpha_i \alpha_j^+ \left(v_i \left(H_{HF} v_j^* - g \psi_0^2 u_j^* \right) + v_j^* \left(H_{HF} v_i - g \psi_0^{*2} u_i \right) \right) +$$

$$- \alpha_i \alpha_j \left(v_i \left(H_{HF} u_j - g \psi_0^2 v_j \right) + u_j \left(H_{HF} v_i - g \psi_0^{*2} u_i \right) \right) +$$

$$\left. - \alpha_i^+ \alpha_j^+ \left(u_i^* \left(H_{HF} v_j^* - g \psi_0^2 u_j^* \right) + v_j^* \left(H_{HF} u_i - g \psi_0^{*2} v_i^* \right) \right) \right\}$$

- Let's suppose u, v fullfills the Bogoliubov - eq-s.

$$\left.\begin{aligned} H_{HF} u_i - g\,\Psi_0^2 v_i &= E_i u_i \\ -g\,\Psi_0^{*2} u_i + H_{HF}\, v_i &= -E_i v_i \end{aligned}\right\} \qquad E_i \in \mathbb{R}^+$$

$$K_2 = \sum_{ij} \int d^3\vec{r} \Big\{ \alpha_i^+ \alpha_j \left(E_j u_i^* u_j + E_i u_j u_i^* \right) + \alpha_i \alpha_j^+ \left(-E_j v_i v_j^* - E_i v_j^* v_i \right) +$$

$$+ \alpha_i \alpha_j \left(E_j v_i u_j - E_i u_j v_i \right) - \alpha_i^+ \alpha_j^+ \left(-E_j u_i^* v_j^* + E_i v_j^* u_i^* \right) \Big\}$$

$$= \sum_{ij} \int d^3\vec{r} \Big\{ \left(E_i + E_j \right) \alpha_i^+ \alpha_j u_i^* u_j - \left(E_i + E_j \right) \alpha_i \alpha_j^+ v_i v_j^* -$$

$$- \alpha_i \alpha_j \left(E_j - E_i \right) v_i u_j - \alpha_i^+ \alpha_j^+ \left(E_i - E_j \right) u_i^* v_j^* \Big\}$$