2019.04.29.

Remarks

Static condensate [GP - eq.] $\xrightarrow{\text{large } N \text{ limit}}$ [Thomas-Fermi approx] (for static)

[GP - eq.] $\xrightarrow{\text{linear (perturbation)}}$ [Bogoliubov - eq. (Dynamics!)]

[Thomas-Fermi approx] $\xrightarrow{\text{linearization}}$ [Hydro]

[Bogoliubov - eq. (Dynamics!)] $\xrightarrow[\text{long wavelength limit}]{\text{low energy approx}}$ [Hydro]

- this works with GP
- incompatible with Thomas-Fermi

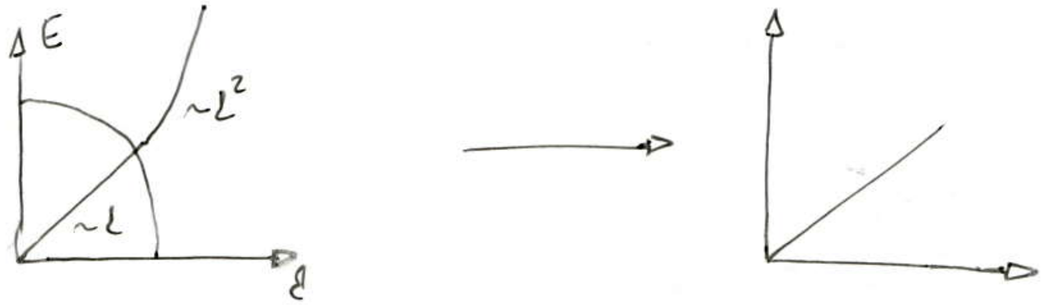
↓

Are there dynamics, compatible with T-F approx?

⇒ Hydrodynamical eq. (by Stringari)

↓

predicted where are the low lying excitations, 1 year before real condensates (!)



Stringari Hydro

- GP : $i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \Delta + V + g|\psi|^2 \right] \psi$

$$\Psi_0 = e^{-\frac{i\mu t}{\hbar}} \Psi_0(r)$$

$$\Psi(\vec{r},t) = e^{-\frac{i\mu t}{\hbar}} \left[\Psi_0(r) + \delta\Psi(\vec{r},t)\right]$$

→ this is how it went

- we will do smtg similar
- Ψ is $\mathbb{C}$ ⇝ EoM of Ψ can be transformed fm EoM of $\mathbb{R}$ fields!

$$\Psi(\vec{r}) = \sqrt{n(\vec{r})} \cdot e^{\frac{iS(\vec{r})}{\hbar}} \qquad n, S \in \mathbb{R}$$

$$\boxed{\vec{v}(\vec{r}) = \frac{\vec{\nabla} S}{m}}$$ ⇝ velocity field of the condensate atom

$\Psi^* \Psi = n$ ⇝ defines n

$$\Psi^*(\vec{\nabla}\Psi) - \Psi(\vec{\nabla}\Psi^*) = \sqrt{n}\, e^{-\frac{iS}{\hbar}} \left(e^{\frac{iS}{\hbar}}\left(\cancel{(\vec{\nabla}\sqrt{n})} + \frac{i}{\hbar}(\vec{\nabla} S)\right)\right) -$$

$$- \sqrt{n}\, e^{\frac{iS}{\hbar}} \left(e^{-\frac{iS}{\hbar}}\left(\cancel{(\vec{\nabla}\sqrt{n})} - \frac{i}{\hbar}(\vec{\nabla} S)\right)\right) =$$

$$= \frac{2i\,n(\vec{r})}{\hbar}(\vec{\nabla} S)$$

$$\downarrow$$

$$\left.\begin{aligned} \vec{\nabla} S &= \frac{\hbar}{2in}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) \\ \frac{\partial S}{\partial t} &= \frac{\hbar}{2in}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right) \end{aligned}\right\} \text{defines } S$$

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} =$$

$$= \psi^* \cdot \left(-\frac{i}{\hbar}\right)\left[-\frac{\hbar^2}{2m}\Delta + \cancel{V} + g|\psi|^2\right]\psi + \psi \cdot \left(\frac{i}{\hbar}\right)\left[-\frac{\hbar^2}{2m}\Delta + \cancel{V} + g|\psi|^2\right]\psi^* =$$

$$= i\frac{\hbar}{2m}\left(\psi^*\Delta\psi - \psi\Delta\psi^*\right) \underset{\text{identity}}{=} i\frac{\hbar}{2m}\,\mathrm{div}\underbrace{\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)}_{\frac{2in}{\hbar}\vec{\nabla}S} =$$

$$= -\mathrm{div}(n\cdot\vec{v})$$

$$\boxed{\frac{\partial n}{\partial t} + \mathrm{div}(n\cdot\vec{v}) = 0}$$

- sourceless continuity eq.
- $\vec{v}$ really is the velocity field of the condensate

$$\frac{\partial S}{\partial t} = \frac{\hbar}{2in}\Bigg[\psi^*\left(-\frac{i}{\hbar}\right)\left[-\frac{\hbar^2}{2m}\Delta + V + g\underbrace{|\psi|^2}_{n}\right]\psi -$$

$$- \psi\left(\frac{i}{\hbar}\right)\left[-\frac{\hbar^2}{2m}\Delta + V + g\underbrace{|\psi|^2}_{n}\right]\psi^*\Bigg] =$$

$$= -(V + gn) + \frac{1}{2n}\left[\psi^*\frac{\hbar^2}{2m}\Delta\psi + \psi\frac{\hbar^2}{2m}\Delta\psi^*\right]$$

identity: $\Delta(ab) = a\Delta b + 2(\vec{\nabla}a)(\vec{\nabla}b) + b\vec{\nabla}a$

$$= -(V + gn) + \frac{1}{2n}\frac{\hbar^2}{2m}\left[\sqrt{n}\,e^{-\frac{iS}{\hbar}}\Delta\left(\sqrt{n}\,e^{\frac{i}{\hbar}S}\right) + \sqrt{n}\,e^{\frac{i}{\hbar}S}\Delta\left(\sqrt{n}\,e^{-\frac{i}{\hbar}S}\right)\right] =$$

$$= -(V + gn) + \frac{1}{2n}\frac{\hbar^2}{2m}\Bigg[\sqrt{n}\,e^{-\frac{i}{\hbar}S}\left(\sqrt{n}\,\Delta e^{\frac{i}{\hbar}S} + 2(\vec{\nabla}\sqrt{n})(\vec{\nabla}e^{\frac{i}{\hbar}S}) + \right.$$

$$\left. + e^{\frac{i}{\hbar}S}\Delta\sqrt{n}\right) + \sqrt{n}\,e^{\frac{i}{\hbar}S}\left(\sqrt{n}\,\Delta e^{-\frac{i}{\hbar}S} + 2(\vec{\nabla}\sqrt{n})(\vec{\nabla}e^{-\frac{i}{\hbar}S}) + e^{-\frac{i}{\hbar}S}\Delta\sqrt{n}\right)\Bigg]$$

$$= -(V+gn) + \frac{\hbar^2}{2m}\frac{1}{\sqrt{n}}(\Delta\sqrt{n}) + \frac{i}{2}\,\frac{\hbar}{2m}\Big[e^{-\frac{i}{\hbar}s}\Big(e^{\frac{i}{\hbar}s}\Delta s + \frac{i}{\hbar}e^{\frac{i}{\hbar}s}(\vec{\nabla}s)(\vec{\nabla}s)\Big)$$

$$- e^{\frac{i}{\hbar}s}\Big(e^{-\frac{i}{\hbar}s}\Delta s - \frac{i}{\hbar}e^{-\frac{i}{\hbar}s}(\vec{\nabla}s)(\vec{\nabla}s)\Big)\Big]$$

$$\frac{\partial s}{\partial t} = -(V+gn) + \underbrace{\frac{\hbar^2}{2m}\frac{1}{\sqrt{n}}(\Delta\sqrt{n})}_{\text{quantum pressure}} - \underbrace{\frac{(\vec{\nabla}s)(\vec{\nabla}s)}{2m}}_{\frac{1}{2}mv^2}$$

- in a static condensat $n = \text{const.} \rightarrow \Delta n = 0$! (approx of Stringari)
- we can then neglect the quantum pressure
- usually we don't see the phase $(e^{\frac{i}{\hbar}s})$

$$\vec{\nabla}\left(\frac{\partial s}{\partial t}\right) = m\frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}\Big(V + gn - \frac{\hbar^2}{2m}\frac{1}{\sqrt{n}}\Delta\sqrt{n} + \frac{1}{2}mv^2\Big)$$

$\rightsquigarrow$ eom. of moving fluid $\rightsquigarrow$ Euler-eq. (without quantum-pressure)

- V: ext. force
- gn: effective internal force
- $\frac{\hbar^2}{2m}\frac{1}{\sqrt{n}}\Delta\sqrt{n}$: no classical analoge
- $\frac{1}{2}mv^2$: well known term.

- now after approx:

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\vec{v}) = 0$$

$$\frac{\partial S}{\partial t} = -\left(V + gn + \frac{1}{2} m v^2\right)$$

- in static condensate: $S_0 = -\mu t$

$$\left.\begin{array}{l} \dfrac{\partial n_0}{\partial t} = 0 \longrightarrow n_0(\vec{r}) \\ v_0 = 0 \end{array}\right\}$$

- linearization:

$$S(\vec{r},t) = -\mu t + \delta S(\vec{r},t)$$
$$n(\vec{r},t) = n_0(\vec{r}) + \delta n(\vec{r},t)$$
$$\vec{v}(\vec{r},t) = \frac{\vec{\nabla}(\delta S)}{m}$$

- calculating $n_0(\vec{r})$:

$$\frac{\partial S_0}{\partial t} = -\mu = -(V + g n_0) \rightsquigarrow n_0 = \frac{\mu - V}{g}$$

Thomas - Fermi condensate w.f.

- next step:

$$\frac{\partial}{\partial t}\left(\cancel{n_0(\vec{r})} + \delta n(\vec{r},t)\right) + \mathrm{div}\left[(n_0 + \underbrace{\delta n)\frac{\vec{\nabla}(\delta S)}{m}}_{\text{second order}}\right] = 0$$

$$\boxed{\frac{\partial \delta n}{\partial t} = -\mathrm{div}\left[\frac{n_0}{m}\vec{\nabla}(\delta S)\right]}$$

static - eq.

we forget 2nd order

$$-\cancel{\mu} + \frac{\partial(\delta S)}{\partial t} = -\left(\cancel{V} + g(\cancel{n_0} + \delta n) + \frac{1}{2} m \cancel{\left(\frac{\vec{\nabla}(\delta S)}{m}\right)^2}\right)$$

$$\boxed{\frac{\partial(\delta S)}{\partial t} = -g(\delta n)}$$

- from the coupled 1st order eq. we can eliminate one:

$$\frac{\partial^2 \delta n}{\partial t^2} = -\,div\left[\frac{n_0}{m}\vec{\nabla}\frac{\partial S}{\partial t}\right] = +\,div\left[\frac{n_0}{m}\vec{\nabla}(g\,\delta n)\right]$$

$$\frac{\partial^2 \delta n}{\partial t^2} = div\left(\frac{\mu - V}{m}\vec{\nabla}(\delta n)\right)$$

- spec. case: homogeneous system ($V=0$)

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{\mu}{m}\Delta(\delta n) = 0$$

$\rightarrow$ wave eq.: $\delta n = e^{i\vec{k}\vec{r} - i\omega t}$

$\rightarrow$ $\omega = c\cdot|\vec{k}|$ $\rightarrow$ low energy approx gets back the linear part.

$\longrightarrow$ phonons!

- trapped case:

$$\frac{\partial^2(\delta n)}{\partial t^2} = -\vec{\nabla}\left[\frac{\mu - V}{m}\vec{\nabla}(\delta n)\right]$$

$\downarrow$

some kind of wave operator

$\delta n = e^{\pm i\omega t}\cdot\delta n(\vec{r})$ $\rightarrow$ separation of variables

$$\boxed{\omega^2\,\delta n(\vec{r}) = -\vec{\nabla}\left[\frac{\mu - V}{m}\vec{\nabla}(\delta n(\vec{r}))\right]}$$

- we want to solve this if $V(r)$ in HO

$V(r) = \frac{1}{2}m\omega_0^2 r^2$ $\qquad$ ω_0 $\rightarrow$ trap freq.

$$\omega^2\,\delta n = -\frac{\mu - V}{m}\Delta\delta n - \frac{\vec{\nabla}V}{m}\cdot\vec{\nabla}\delta n$$

- S_n is polinomial of x, y, z :

$$r^l Y_{ln}(\vartheta, \varphi) \cdot P_n(r^2)$$

$x^2 + y^2 + z^2$

- to calculate ω we only need what happens in highest order.

$$\omega^2 r^l \cancel{Y_{ln}}(\vartheta, \varphi) \cdot r^{2n} = \frac{1}{2}\omega_0^2 r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) r^{2n+l} \cancel{Y_{ln}} +$$

$$+ \omega_0^2 r \frac{\partial}{\partial r} r^{l+2n} \cancel{Y_{ln}}(\vartheta, \varphi)$$

$$\frac{\omega^2}{\omega_0^2} r^{2n+l} = \frac{r^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) r^{2n+l} + r\frac{\partial}{\partial r} r^{l+2n}$$