$$g_\lambda(\lambda) = \underbrace{\frac{\mu g_\nu(\nu) - \nu g_\mu(\mu)}{\mu - \nu}}_{\substack{\text{this must} \\ \text{be a const.} \\ \text{in } \lambda!}} + \lambda \underbrace{\frac{g_\mu(\mu) - g_\nu(\nu)}{\mu - \nu}}_{-\frac{B}{4}}$$

$$\left.\begin{aligned} g_\lambda(\lambda) &= -\frac{A}{4} - \frac{B}{4}\lambda \\ g_\mu(\mu) &= -\frac{A}{4} - \frac{B}{4}\mu \\ g_\nu(\nu) &= -\frac{A}{4} - \frac{B}{4}\nu \end{aligned}\right\}$$

A, B are non-trivial separation constant.

↓

due to cyclic symmetry of the variables.

$$g_\varrho(\varrho) = \frac{1}{\varphi_\varrho(\varrho)} \left[\ldots \right] \varphi_\varrho(\varrho) = -\frac{A}{4} - \frac{B}{4}\varrho$$

3 separational const.

$$\left.\begin{aligned} \hat{F}_\varrho \varphi_\varrho(\varrho) &= \left(a^2 b^2 c^2 \frac{a^2}{c_0^2} + A\varrho + B\varrho^2 \right) \varphi_\varrho(\varrho) \\ \text{and} \quad \hat{F}_\varrho &= 4 \left[\varrho F(\varrho) \frac{\partial^2}{\partial \varrho^2} + \left(F(\varrho) + \frac{1}{2} \varrho F'(\varrho) \right) \frac{\partial}{\partial \varrho} \right] \end{aligned}\right\}$$

these are 3 eq.-s: λ, μ, ν

$\boxed{\text{2019. 10. 03.}}$

⤳ 1 vált. Schr. klet a sajátvált. szerint.

↓

már nem parc. diff.

⤳ A, B m, ℓ-nek megfelelő szeparációs konstansok.

⤳ megmaradó operátorok megkonstruálhatók belőlük.

$$\hat{G} = -\vec{\nabla}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\vec{\nabla} \rightsquigarrow \frac{\omega^2}{c_0^2}\varphi = \hat{G}\varphi$$

általánosított hullámegyenlet

$$\hat{F}_\lambda \varphi_\lambda(\lambda) = \left(a^2 b^2 c^2 \frac{\omega^2}{c_0^2} + A\lambda + B\lambda^2\right)\varphi_\lambda(\lambda) \qquad \Big/ \cdot \varphi_\mu \varphi_\nu$$

$$\left.\begin{aligned} \hat{F}_\lambda \varphi(\lambda,\mu,\nu) &= (\ldots)\,\varphi(\lambda,\mu,\nu) \\ \hat{F}_\mu \varphi(\mu,\nu,\lambda) &= (\ldots)\,\varphi(\mu,\nu,\lambda) \\ \hat{F}_\nu \varphi(\mu,\nu,\lambda) &= (\ldots)\,\varphi(\mu,\nu,\lambda) \end{aligned}\right\} \leftarrow \text{llet a teljes } \varphi\text{-re}$$

$$\begin{pmatrix} \hat{F}_\lambda \varphi \\ \hat{F}_\mu \varphi \\ \hat{F}_\nu \varphi \end{pmatrix} = \begin{pmatrix} 1 & \lambda & \lambda^2 \\ 1 & \mu & \mu^2 \\ 1 & \nu & \nu^2 \end{pmatrix} \begin{pmatrix} a^2 b^2 c^2 \frac{\omega^2}{c_0^2}\varphi \\ A\varphi \\ B\varphi \end{pmatrix}$$

$$\begin{pmatrix} a^2 b^2 c^2 \frac{\omega^2}{c_0^2}\varphi \\ A\varphi \\ B\varphi \end{pmatrix} = \begin{pmatrix} 1 & \lambda & \lambda^2 \\ 1 & \mu & \mu^2 \\ 1 & \nu & \nu^2 \end{pmatrix}^{-1} \begin{pmatrix} F_\lambda \varphi \\ F_\mu \varphi \\ F_\nu \varphi \end{pmatrix}$$

$\rightsquigarrow$ első llet a szorzás után: $\frac{\omega^2}{c_0^2}\varphi = \hat{G}\varphi$

$\rightsquigarrow$ második –"– : $A\varphi = \hat{A}\varphi$

$\rightsquigarrow$ harmadik –"– : $B\varphi = \hat{B}\varphi$

$\Rightarrow$ φ $\hat{A}, \hat{B}, \hat{G}$-nek szimultán sajátfv. $\rightsquigarrow$ kommutálnak (?)

$$\hat{B} = \frac{1}{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}\left((\mu-\nu)\hat{F}_\lambda + (\nu-\lambda)\hat{F}_\mu + (\lambda-\mu)\hat{F}_\nu\right)$$

$$\hat{A} = \frac{1}{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}\left((\mu^2-\nu^2)\hat{F}_\lambda + (\nu^2-\lambda^2)\hat{F}_\mu + (\lambda^2-\mu^2)\hat{F}_\nu\right)$$

⇝ visszatranszformálhatók Descartes-ba

$$\hat{B} = (\vec{r}\,\vec{\nabla})(\vec{r}\,\vec{\nabla}+3) - a^2\frac{\partial^2}{\partial x_1^2} - b^2\frac{\partial^2}{\partial x_2^2} - c^2\frac{\partial^2}{\partial x_3^2}$$

$$\hat{A} = \Big\{\Big[(b^2+c^2)(x_1-a^2) + a^2(x_2^2+x_3^2)\Big]\frac{\partial^2}{\partial x_1^2} +$$

$$+ 2a^2 x_2 x_3 \frac{\partial^2}{\partial x_2\,\partial x_3} +$$

$$+ 3(b^2+c^2)\,x_1\frac{\partial}{\partial x_1} +$$

+ ciklikus tagok

$(a \to b \to c\ ;\ x_1 \to x_2 \to x_3)$

⇝ ezek az op.-k nem a "normál" skalárszorzat mellett lesznek önadjungáltak (sajátért. valós.)

$$\langle \psi | \varphi \rangle = \int_{V_{T_F}} \psi^*(x)\,\varphi(x)\,d^3x$$

csak a T_F ellipszoid belsejében végezzük el!

→ itt az op. önadjungáltak lesznek.

$$\left.\begin{aligned}\hat{G} &= \hat{G}^* \\ \hat{A} &= \hat{A}^* \\ \hat{B} &= \hat{B}^+\end{aligned}\right\}$$

$$\langle \psi | \hat{G} | \varphi \rangle = \int_{\mathcal{V}_{T_F}} d^3x \, -\psi^* \vec{\nabla} \left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}\right) \vec{\nabla} \varphi =$$

$$= \int_{\mathcal{V}_{T_F}} d^3x \left\{ \underbrace{-\vec{\nabla}\left[\psi^* (1 - \dots)(\vec{\nabla}\varphi)\right]}_{\text{teljes divergencia} \to 0} + (\vec{\nabla}\psi^*)(1 - \dots)(\vec{\nabla}\varphi) \right\} =$$

$$= \int_{\mathcal{V}_{T_F}} d^3x \, (\vec{\nabla}\psi^*)(1 - \dots)(\vec{\nabla}\varphi)$$

$\rightsquigarrow$ láthatóan önadjungált

(A, B hasonlóan viselkednek)

$$[\hat{G}, \hat{A}] = [\hat{G}, \hat{B}] = [\hat{A}, \hat{B}] = 0$$

$\rightsquigarrow \hat{B}$ sajátértékei : $n \cdot (n+3)$

$\rightsquigarrow \hat{A}$ — " — : ???

$x_1^{\alpha} x_2^{\beta} x_3^{\gamma} (\dots) \Longrightarrow$ így is átírható szorzatra

$\downarrow$

polinom elliptikus koordinátákban.

$\left.\begin{array}{l} \alpha = 0, 1 \\ \beta = 0, 1 \\ \gamma = 0, 1 \end{array}\right\}$ szimmetria preferálódik különböző tükrözésekre

$$\varphi_s(s) = (s - \theta_1)(s - \theta_2) \dots (s - \theta_n)$$

n gyöke a polinomnak.

$$\varphi(\lambda,\mu,\nu) = \underbrace{(\lambda-\vartheta_1)(\mu-\vartheta_1)(\nu-\vartheta_1)}\ \ldots\ (\lambda-\vartheta_n)(\mu-\vartheta_n)(\nu-\vartheta_n)$$

$$\Downarrow$$

$$\left(1 - \frac{x_1^2}{a^2-\vartheta_1} - \frac{x_2^2}{b^2-\vartheta_1} - \frac{x_3^2}{c^2-\vartheta_1}\right)$$

ez teljesen szimmetrikus x_1, x_2, x_3-ban.

→ leszűkítette az n-ed fokú polinomok alakját.

- állítás:

$$\varphi(x_1,x_2,x_3) = x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \prod_{i=1}^{n} \left(1 - \frac{x_1^2}{a^2-\vartheta_i} - \frac{x_2^2}{b^2-\vartheta_i} - \frac{x_3^2}{c^2-\vartheta_i}\right)$$

a gerjesztési fv.-t ilyen alakban kell keresni.

→ ϑ_i-k és ω-k ismeretlenek

> innen kell vizsgára < - - - - - - - - - - - - - - -

$$\varphi(x_1,x_2,x_3) = x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \cdot \tilde{\varphi}$$

$$\frac{\omega^2}{c_0^2} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tilde{\varphi} = \hat{G}\, x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tilde{\varphi}$$

$$\frac{\omega^2}{c_0^2} \tilde{\varphi} = \underbrace{x_1^{-\alpha} x_2^{-\beta} x_3^{-\gamma}\, \hat{G}\, x_1^{\alpha} x_2^{\beta} x_3^{\gamma}}_{\tilde{G}\ \text{hasonlósági transzformált}} \tilde{\varphi}$$

$$\hat{G} = -\vec{\nabla} \underbrace{\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}\right)}_{=:\ \varphi_0} \vec{\nabla}$$

$$\varphi_i := \left(1 - \frac{x_1^2}{a^2-\vartheta_i} - \frac{x_2^2}{b^2-\vartheta_i} - \frac{x_3^2}{c^2-\vartheta_i}\right)$$

$$\hat{\varphi} = \prod_{i=1}^{n} \varphi_i$$

$$\hat{G} = -\varphi_0 \Delta + \left(\frac{2x_1}{a^2}\frac{\partial}{\partial x_1} + \frac{2x_2}{b^2}\frac{\partial}{\partial x_2} + \frac{2x_3}{c^2}\frac{\partial}{\partial x_3}\right)$$

- hasonlósági transzformáció:

$$x_1^{-\alpha} x_2^{-\beta} x_3^{-\gamma} \Delta x_1^{\alpha} x_2^{\beta} x_3^{\gamma} = x_1^{-\alpha}\frac{\partial^2}{\partial x_1^2} x_1^{\alpha} +$$

$$+ x_2^{-\beta}\frac{\partial^2}{\partial x_2^2} x_2^{\beta} +$$

$$+ x_3^{-\gamma}\frac{\partial^2}{\partial x_3^2} x_3^{\beta}$$

$$\rightsquigarrow \underbrace{\left(x_1^{-\alpha}\frac{\partial}{\partial x_1} x_1^{\alpha}\right)}\left(x_1^{-\alpha}\frac{\partial}{\partial x_1} x_1^{\alpha}\right) \qquad / ab = ba + [a,b] /$$

$$x_1^{-\alpha}\left(x_1^{\alpha}\frac{\partial}{\partial x_1} + \alpha x_1^{\alpha-1}\right) = \left(\frac{\partial}{\partial x_1} + \frac{\alpha}{x_1}\right)$$

$$\rightsquigarrow \left(\frac{\partial}{\partial x_1} + \frac{\alpha}{x_1}\right)^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\alpha}{x_1}\frac{\partial}{\partial x_1} + \left(\frac{\alpha}{x_1}\frac{\partial}{\partial x_1} - \frac{\alpha}{x_1^2}\right) + \frac{\alpha^2}{x_1^2}$$

$$= \frac{\partial^2}{\partial x_1^2} + \frac{2\alpha}{x_1}\frac{\partial}{\partial x_1} + \underbrace{\frac{\alpha(\alpha-1)}{x_1^2}}_{\alpha = 0,1}$$

$$\downarrow$$

$$\emptyset$$

$$x_1^{-\alpha} x_2^{-\beta} x_3^{-\gamma} \Delta x_1^{\alpha} x_2^{\beta} x_3^{\gamma} = \Delta + \left(\frac{2\alpha}{x_1}\frac{\partial}{\partial x_1} + \frac{2\beta}{x_2}\frac{\partial}{\partial x_2} + \frac{2\gamma}{x_3}\frac{\partial}{\partial x_3}\right)$$

$$x_1^{-\alpha} \cancel{x_2^{-\beta}} x_3^{-\gamma} \frac{2x_1}{a^2}\frac{\partial}{\partial x_1} x_1^{\alpha} \cancel{x_2^{\beta}} x_3^{\gamma} = \frac{2x_1}{a^2}\underbrace{\left(x_1^{-\alpha}\frac{\partial}{\partial x_1} x_1^{\alpha}\right)}_{\left(\frac{\partial}{\partial x_1} + \frac{\alpha}{x_1}\right)} =$$

$$= \frac{2x_1}{a^2}\frac{\partial}{\partial x_1} + \frac{2\alpha}{a^2}$$

$$\tilde{G} = G + \left(\frac{2\alpha}{x_1}\frac{\partial}{\partial x_1} + \frac{2\beta}{x_2}\frac{\partial}{\partial x_2} + \frac{2\gamma}{x_3}\frac{\partial}{\partial x_3}\right) + 2\left(\frac{\alpha}{a^2} + \frac{\beta}{b^2} + \frac{\gamma}{c^2}\right)$$

$$\tilde{G} = 2\left(\frac{\alpha}{a^2} + \frac{\beta}{b^2} + \frac{\gamma}{c^2}\right) - \varphi_0\left(\Delta + \left(\frac{2\alpha}{x_1}\frac{\partial}{\partial x_1} + \frac{2\beta}{x_2}\frac{\partial}{\partial x_2} + \frac{2\gamma}{x_3}\frac{\partial}{\partial x_3}\right)\right) +$$

$$+ \left(\frac{2x_1}{a^2}\frac{\partial}{\partial x_1} + \frac{2x_2}{b^2}\frac{\partial}{\partial x_2} + \frac{2x_3}{c^2}\frac{\partial}{\partial x_3}\right)$$

$$\left[\frac{\omega^2}{c_0^2}\tilde{\varphi} = \tilde{G}\tilde{\varphi}\right]$$

- Ha $\tilde{\varphi} = 1 \rightsquigarrow \frac{\omega^2}{c_0^2} = 2\left(\frac{\alpha}{a^2} + \frac{\beta}{b^2} + \frac{\gamma}{c^2}\right)$
- Ha $\tilde{\varphi} \neq 1$:

$$\frac{\partial\tilde{\varphi}}{\partial\varphi_i} := \tilde{\varphi}_{\varphi_i}$$

$$\Delta\tilde{\varphi} = \frac{\partial}{\partial x_1}\left(\sum_{i=1}^{n}\frac{\partial\tilde{\varphi}}{\partial\varphi_i}\frac{(-2)x_1}{a^2-\vartheta_i}\right) + \frac{\partial}{\partial x_2}\left(\sum_{i=1}^{n}\frac{\partial\tilde{\varphi}}{\partial\varphi_i}\frac{(-2)x_2}{b^2-\vartheta_i}\right) +$$

$$+ \frac{\partial}{\partial x_3}\left(\sum_{i=1}^{n}\frac{\partial\tilde{\varphi}}{\partial\varphi_i}\frac{(-2)x_3}{c^2-\vartheta_i}\right) =$$

$$= \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} \frac{\partial^2 \tilde{\varphi}}{\partial \varphi_i \partial \varphi_j} \left(\frac{4x_1^2}{(a^2-\vartheta_i)(a^2-\vartheta_j)} + \frac{4x_2^2}{(b^2-\vartheta_i)(b^2-\vartheta_i)} + \frac{4x_3^2}{(c^2-\vartheta_i)(c^2-\vartheta_i)} \right) +$$

$$+(-\cancel{2})\sum_{i=1}^{n} \frac{\partial \tilde{\varphi}}{\partial \varphi_i} \left(\frac{2}{a^2-\vartheta_i} + \frac{2}{b^2-\vartheta_i} + \frac{2}{c^2-\vartheta_i} \right) =$$

$$\frac{1}{a^2+\vartheta_i} - \frac{1}{a^2+\vartheta_j} = \frac{\vartheta_j - \vartheta_i}{(a^2+\vartheta_i)(a^2+\vartheta_j)} \quad \Big/ \frac{1}{\vartheta_j - \vartheta} x_1$$

$$\frac{x_1^2}{(a^2+\vartheta_i)(a^2+\vartheta_j)} = \frac{1}{\vartheta_j - \vartheta_i} \left(\frac{x_1^2}{a^2+\vartheta_i} - \frac{x_1^2}{a^2+\vartheta_j} \right)$$

---

azonos átaladítással

$$= 4 \sum_{i\neq j} \frac{\partial^2 \tilde{\varphi}}{\partial \varphi_i \partial \varphi_j} \frac{1}{\vartheta_j - \vartheta_i} \Bigg[\underbrace{\left(1 - \frac{x_1^2}{a^2+\vartheta_j} - \frac{x_2^2}{b^2+\vartheta_j} - \frac{x_3^2}{c^2+\vartheta_j} \right)}_{\varphi_j} - \underbrace{\left(1 - \frac{x_1^2}{a^2+\vartheta_i} - \frac{x_2^2}{b^2+\vartheta_i} - \frac{x_3^2}{c^2+\vartheta_i} \right)}_{\varphi_i} \Bigg] -$$

$$- \sum_{i=1}^{n} \frac{\partial \tilde{\varphi}}{\partial \varphi_i} \left(\frac{2}{a^2+\vartheta_i} + \frac{2}{b^2+\vartheta_i} + \frac{2}{c^2+\vartheta_i} \right) =$$

$\rightsquigarrow$ azonossággal: $\frac{\partial \tilde{\varphi}}{\partial \varphi_i} \varphi_i = \tilde{\varphi} \quad \forall i \qquad \frac{\partial^2 \tilde{\varphi}}{\partial \varphi_i \partial \varphi_j} \varphi_i = \frac{\partial \tilde{\varphi}}{\partial \varphi_j}$

$$= 4 \sum_{i\neq j} \frac{\partial \tilde{\varphi}}{\partial \varphi_i} \frac{1}{\vartheta_j - \vartheta_i} - 4 \sum_{i\neq j} \frac{\partial \tilde{\varphi}}{\partial \varphi_j} \frac{1}{\vartheta_j - \vartheta_i} - \sum_i \frac{\partial \tilde{\varphi}}{\partial \varphi_i} \left(\frac{2}{a^2+\vartheta_i} + \frac{2}{b^2+\vartheta_i} + \frac{2}{c^2+\vartheta_i} \right)$$

$$= -4 \sum_i \frac{\partial \tilde{\varphi}}{\partial \varphi_i} \sum_{\substack{j \\ j\neq 1}} \frac{1}{\vartheta_i - \vartheta_j} - 4 \sum_i \frac{\partial \tilde{\varphi}}{\partial \varphi_i} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{\vartheta_i - \vartheta_j} - (\ldots)$$

25.

$$\Delta\tilde{\varphi} = -\sum_{i=1}^{n} \frac{\partial\tilde{\varphi}}{\partial\varphi_i}\left(\frac{2}{a^2+\vartheta_i} + \frac{2}{b^2+\vartheta_i} + \frac{2}{c^2+\vartheta_i} + \sum_{j\neq i}\frac{8}{\vartheta_i-\vartheta_j}\right)$$