$na^3 << 1$ ⤳ the gas is not dense

- GP-eq. will be a good approx
- we don't need to know much about the real potential.

- we replaced the potential : $\boxed{\frac{4\pi\hbar^2 a}{m}\,\delta(r-r')}$
- for small energies ⤳ low T!

  ↓

  isotropic scattering ($L=0$) dominates the diff. x-section. ⤳ s-wave scattering

  ↳ "a" choosen appropriatly will result good thermodynamics

- normalization: $\int d^3x\, \underbrace{|\Psi_0(x)|^2}_{n_0\text{, condensate density}} = N$ ← all atoms are in the condensate

  at $T=0$

- for $T \neq T_0$ ⤳ $n(x) = |n_0| + n_T(x)$ ⟶ $N = N_0 + N_T$

  $|n_0| = |\Psi_0(x)|^2$, $n_T(x)$: thermal density

^{23}Na ⤳ $a = 2.75$ nm

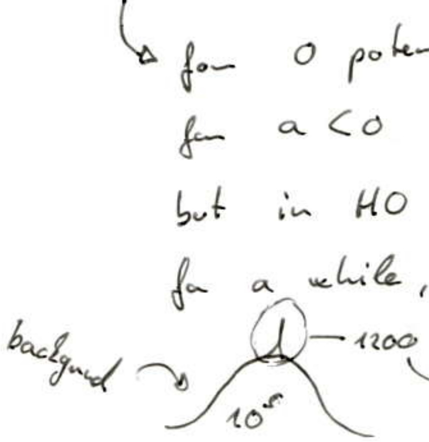
^{87}Rb ⤳ $a = 5.77$ nm

^{7}Li ⤳ $a = -1.45$ nm

↳ for 0 potential there can be no condensation for $a<0$

mechanically unstable

but in HO confining pot there can be con. for a while, for a given number of atoms



→ difficult to measure

- Let's try to estimate the different terms

$$\hat{H} = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\Delta_i + V(x_i)\right) + \frac{4\pi\hbar^2 a}{m}\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\delta(r_i - r_j)$$

$$V = \frac{1}{2}m\left(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2\right)$$

$$\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$$

- We can say $\Psi = \prod_{i=1}^{N}\phi_i(x_i)$

where ϕ is the solution to the 1 part

3D HO $\rightsquigarrow$ product of Gaussians. (grnd. state!)

$$\phi_0 = N e^{-\frac{1}{2}\left(\frac{x^2}{d_x^2} + \frac{y^2}{d_y^2} + \frac{z^2}{d_z^2}\right)} \qquad \text{and} \qquad d_i = \sqrt{\frac{\hbar}{m\omega_i}}$$

$$1 \overset{!}{=} \int |\phi_0|^2 \, dx\, dy\, dz = N^2 \underbrace{\left[\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{d_x^2}}\right]}_{\text{"}I_x\text{"}} I_y I_z$$

$$I_x = 2\int_0^{\infty} dx\, e^{-\frac{x^2}{d_x^2}} = d_x \int_0^{\infty} \overbrace{\frac{e^{-t}\,dt}{\sqrt{t}}}^{\Gamma(\frac{1}{2})} = \sqrt{\pi}\, d_x$$

$$t = \frac{x^2}{d_x^2} \rightsquigarrow x = d_x t^{1/2}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad dx = d_x \frac{dt}{\sqrt{t}}\cdot\frac{1}{2}$$

$$1 = N^2 \pi^{3/2} d_x d_y d_z = N^2\left(\frac{\pi\hbar}{m\bar{\omega}}\right)^{3/2}$$

$$\phi_0(\vec{x}) = \left(\frac{m\bar{\omega}}{\pi\hbar}\right)^{3/4} e^{-\frac{1}{2}\sum_i \omega_i^2 x_i^2}$$

$$E_{kin} = \langle \Psi | \hat{T} | \Psi \rangle = \frac{\hbar\omega_x + \hbar\omega_y + \hbar\omega_z}{4} \cdot N \sim N\hbar\bar{\omega}$$

using Virial-theorem. order...

$$E_{pot} = \langle \Psi | \hat{V} | \Psi \rangle = E_{kin} \sim N\hbar\bar{\omega}$$

$$\sum_{i \neq j} \int \phi_0^*(r_1) \dots \phi_0^*(r_N)\, v(r_i - r_j)\, \phi_0(r_1) \dots \phi_0(r_N) =$$

$r_i \to r$, $r_j \to r'$

$$= \frac{1}{2} \sum_{i \neq j} \int d^3r_i\, d^3r_j\, \phi_0^*(r_i) \phi_0^*(r_j)\, v(r_i - r_j)\, \phi_0(r_i) \phi_0(r_j) =$$

$$= \frac{N(N-1)}{2} \int d^3r\, d^3r'\, \phi_0^*(r) \phi_0^*(r') \frac{4\pi\hbar^2 a}{m} \delta(r - r')\, \phi_0(r) \phi_0(r') =$$

$$= \frac{1}{2} N(N-1) \frac{4\pi\hbar a}{m} \int d^3r\, |\phi_0|^4 = \frac{1}{2} N(N-1) \frac{4\pi\hbar a}{m} \left(\frac{m\bar{\omega}}{\pi\hbar}\right)^{3/2}.$$

$$\frac{1}{2^{3/2}} \cdot \underbrace{\int d^3r \left(\frac{2m\bar{\omega}}{\pi\hbar}\right)^{3/2} e^{-2\left(\frac{m\omega_x x^2}{\hbar} + \dots\right)}}_{1 \ /m \to 2m \dots/}$$

$$2 \langle \Psi | \hat{H}_{int} | \Psi \rangle = N(N-1) \sqrt{\frac{2}{\pi}}\, \hbar\bar{\omega} \underbrace{\left(\frac{\hbar}{m\bar{\omega}}\right)}_{d^2} \frac{a}{d^3} \sim \hbar\bar{\omega} N^2 \frac{a}{d}$$

- scales with N^2!

$$\frac{E_{int}}{E_{kin}} = \boxed{\frac{Na}{d}}$$

- JILA : $N = 10^5$; $\frac{a}{d} = 7 \cdot 10^{-3} \Rightarrow \frac{Na}{d} \sim 10$
- RICE : $N = 10^3$; $\frac{|a|}{d} = 0.5 \cdot 10^{-3} \Rightarrow \frac{N|a|}{d} < 1$
- MIT : $N = 10^6 - 10^7, \dots, \Rightarrow \frac{Na}{d} \sim 10^3 - 10^4$

experiment data

$\rightsquigarrow$ even tho $na^3 \ll 1$, $\frac{Na}{d} \gg 1$! $\rightsquigarrow$ interaction is not negligable!

- we will now use isotropic trapping potential:

$$\omega_x = \omega_y = \omega_z = \omega_0 \rightsquigarrow V_{ext}(r) = \frac{1}{2} m \omega_0^2 \underbrace{(x^2+y^2+z^2)}_{r^2}$$

$$\left[-\frac{\hbar^2}{2m}\Delta + V(x) + \frac{4\pi\hbar^2 a}{m}|\Psi_0|^2\right]\Psi_0 = \mu \Psi_0$$

and $\int |\Psi_0|^2 d^3x = N$

} dimensionless coordinate

$\rightarrow$ using oscillator - length $\rightsquigarrow$ char. prop. of the sys.

$$\int |\tilde{\Psi}_0|^2 d^3\tilde{x} = 1 \qquad \tilde{x} = \frac{x}{d_0} \qquad d_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$

- scaling the wf:

$$\Psi_0(x) = \sqrt{N}\,\frac{1}{d_0^{3/2}}\,\tilde{\Psi}_0(\tilde{x})$$

- scaling of μ:

$$\mu = \tilde{\mu}\,\hbar\bar{\omega}$$

$$\left[-\frac{1}{2}\Delta_{\tilde{x}} + \frac{1}{2}\tilde{x}^2 + 4\pi \left(N\frac{a}{d}\right)|\tilde{\Psi}_0(\tilde{x})|^2\right]\tilde{\Psi}_0(\tilde{x}) = \tilde{\mu}\,\tilde{\Psi}_0(\tilde{x})$$

and $\int |\tilde{\Psi}_0(\tilde{x})|^2 d^3\tilde{x} = 1$

only relevant parameter for the dimensionless GP - eq.

G-P functional

$$E[\Psi_0, \Psi_0^*]$$

$$\mu\,\Psi_0 = \frac{\delta E}{\delta \Psi_0^*(x)}$$

$$E = \int d^3r \left[\frac{\hbar^2}{2m}(\vec{\nabla}\Psi_0(\vec{r}))(\vec{\nabla}\Psi_0^*(\vec{r})) + \dots\right.$$

$$+ V(\vec{x})\,\psi_0^*(\vec{x})\,\psi(\vec{x}) + g\,\frac{\psi_0^*\psi_0^*\psi_0\psi_0}{2} \Big]$$

$$E_{GP} = \int d^3r \left[-\frac{\hbar^2}{2m}\psi_0^*\Delta\psi_0 + V|\psi_0|^2 + \frac{g}{2}|\psi_0|^4 \right] = E_{kin} + E_{pot} + E_{int}$$

$$\int d^3x\, \psi_0^*(\vec{x}) \Big/ \ \mu\,\psi_0(\vec{x}) = \frac{\delta E}{\delta\psi_0^*(\vec{x})}$$

$$\mu N = E_{kin} + E_{pot} + 2E_{int}$$

$$\Downarrow$$

$$\boxed{\mu = \frac{E_{kin} + E_{pot} + 2E_{int}}{N}}$$

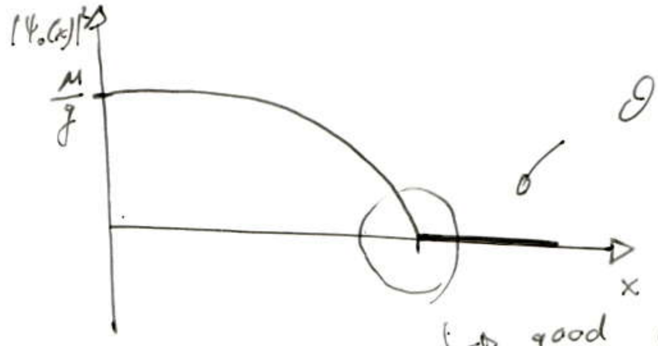
$\frac{Na}{d}$ ⇝ big param in several experiments

for $\frac{Na}{d} \gg 1$ then $\frac{E_{kin}}{E_{int}} = \frac{d}{Na} \ll 1$

⇝ we can "forget" E_{kin} } no weird

⇝ the confining pot. must be kept. } operators.

$$\boxed{|\psi_0(\vec{x})|^2 = \frac{\mu - V(\vec{x})}{g}\,\vartheta(\mu - V)}$$ Thomas-Fermi approximation



$\vartheta(\mu - V(x))$

↳ good approx until we are at the boundary.