2019. 04. 15.

- $\vec{k}$ in a homogenious system is a quasi-continuous quantity

$$E_{\vec{k}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m}\right)^2 + 2\left(\frac{\hbar^2 k^2}{2m}\right) g n}$$

- we can write this in a dimensionless form:
(to find the cross-over $\vec{k}_c$)

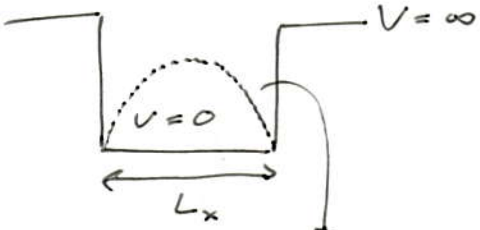
$$\frac{gn}{\frac{\hbar^2 k_c^2}{2m}} \stackrel{!}{=} 1$$

$$\underline{\underline{k_c^2}} = \frac{2m\, gn}{\hbar^2} = \frac{2\cancel{m}n}{\cancel{\hbar^2}} \frac{4\pi \cancel{\hbar^2} a}{\cancel{m}} = \underline{\underline{8\pi n \cdot a}}$$

$$\boxed{\xi = \frac{1}{k_c} = \frac{1}{\sqrt{8\pi n a}}}$$

healing length

- what happens, if we put the bose-gas in a container?



- what is wf?
- which mode is populated?

finite excitation for the ground state

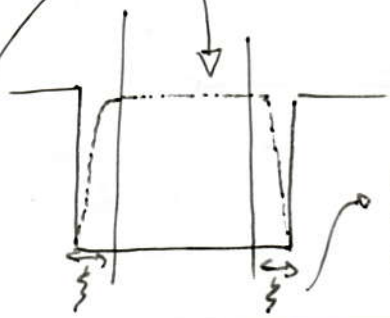
it is very different from the previous example!

- we can add small interaction:

$$V(\vec{r}_i - \vec{r}_j) = \frac{4\pi \hbar^2 a}{m} \delta(\vec{r}_i - \vec{r}_j)$$

$\rightsquigarrow$ $\Psi_0 \sim$ const for most size

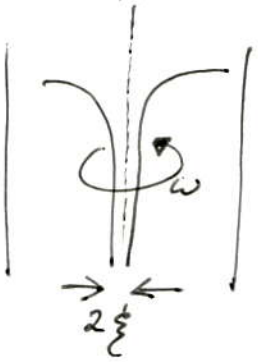
$\mu = g|\Psi_0|$



this is where the healing length appears

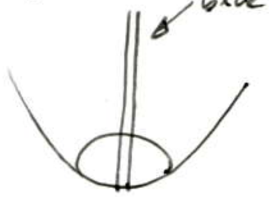
- $\xi << L_x, L_y, L_z$
- this is a similar argument than in solid state physics.
("we can use periodic boundary conditions in solids...")

- what happens in a rotating cilinder?

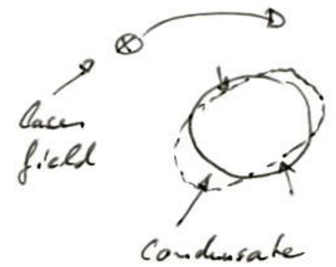


- it is doable in the rotating frame
↓
extra stuff in the G-P eq.

- vortex creation is possible in ultracold gases.

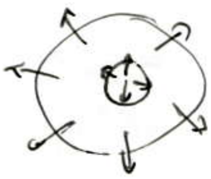


↓
move around the hole
⇝ didn't really work.



- this would give an additional dipole-force for the atoms
- rotating the laser rotates the modified profiles
↓
can add angular momentum

- vortexes are created on the surface, then go into the middle
↓
by that time the healing length was too small for CCD-s.
↓



TOF measurement.
as the condensate blows up, the size of the hole grows, too.

$$\frac{E_l}{gn} = \sqrt{\left(\frac{\frac{\hbar^2 l^2}{2m}}{gn}\right)^2 + 2\left(\frac{\frac{\hbar^2 l^2}{2m}}{gn}\right)} = \sqrt{\frac{l^4}{l_c^4}\underbrace{\left(\frac{\frac{\hbar^2 l_c^2}{2m}}{gn}\right)^2}_{1} + 2\frac{l^2}{l_c^2}\underbrace{\left(\frac{\frac{\hbar^2 l_c^2}{2m}}{gn}\right)}_{1}} =$$

$$= \sqrt{\xi^4 l^4 + 2 l^2 \xi^2} = \xi |l| \sqrt{\xi^2 l^2 + 2}$$

Special solutions of the B.-eq.:

$$H_{HF}\, u_i - g\,\psi_0^2 v_i = E_i u_i \qquad \text{and} \quad H_{HF} = \left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g|\psi_0|^2\right)$$

$$-g\,\psi_0^{*2} u_i + H_{HF}\, v_i = -E_i v_i$$

- the spectrum has a ⊕ and ⊖ part, but also a (degenaraite) 0 part.

- Normalization: $\delta_{ij} = \int d^3r\,(u_i^* u_j - v_i^* v_j)$

1.) $u_i = \psi_0$
$v_i = \psi_0^*$

$$\left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g|\psi_0|^2\right)\psi_0 - \underbrace{g\,\psi_0^2\psi_0^*}_{-g|\psi_0|^2\psi_0} = \underbrace{\left(-\frac{\hbar^2}{2m}\Delta + V - \mu + g|\psi_0|^2\right)\psi_0}_{\emptyset \text{ GP-eq.}}$$

$$\Downarrow$$

$$\underbrace{-g\,\psi_0^{*2}\psi_0}_{-g|\psi_0|^2\psi_0^*} + \left(-\frac{\hbar^2}{2m} + V - \mu + 2g|\psi_0|^2\right)\psi_0^* = 0 \xRightarrow[\text{GP}^*\text{-eq.}]{} \boxed{E = 0}$$

* denotes cc.

- let's look at the normalization:

$$\int d^3r \left(u_i^* u_i - v_i^* v_i\right) = \int d^3r \left(\varphi_0 \varphi_0^* - \varphi_0^* \varphi_0\right) = 0 \neq 1$$

it contradicts the normalization.

so $\left.\begin{array}{l} u_i = \varphi_0 \\ v_i = \varphi_0^* \end{array}\right\}$ is a formal soulution, which is not normalizable!

$\rightsquigarrow$ this sol. is allways found by numerical means, but it must be erased from the spectra.

<u>Kohn - theorem</u>

- For $V = \frac{1}{2} m \omega_1^2 x^2 + \frac{1}{2} m \omega_2^2 y^2 + \frac{1}{2} m \omega_3^2 z^2$

$\exists$ 3 modes of the N-particle problem, for which

$$E_1 = \hbar \omega_1$$
$$E_2 = \hbar \omega_2$$
$$E_3 = \hbar \omega_3$$

independently of the fact, what is the interaction between the particle.

$$\hat{H} = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \Delta_i + V(\vec{r}_i)\right) + \frac{1}{2} \sum_{\substack{i \neq j \\ i=1 \\ j=1}}^{N} v(\vec{r}_i - \vec{r}_j)$$

- the first part can be separated from the int. part in <u>Jakobian coordinates</u>

$N = 2$ $\qquad \vec{R} = \dfrac{\vec{r}_1 + \vec{r}_2}{2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$

$N = 3$ <there is a generalization ...>

$\vdots$

- int. part involves the $N-1$ Jacobian coordinates
- the first part $\rightsquigarrow$ Center of Mass part.

they separate

$\rightarrow$ then the whole system can be excited by the trap-freq. (only CM, relative not excited)

- independent of T
- does not dampen
- independent of the particles being bosons, fermions

$\rightarrow$ trapping potential can be <u>calibrated</u> with this

- these modes are exact even above T_c!

- After a lot of approx.-es, can we find these 3 modes in the B-eq.-s?
  - numerically yes, they are there.
  - analytically, too...

$\rightsquigarrow$ these are called <u>Kohn-modes</u>

2.) μ, Ψ_0 must be an exact solution of the GP-eq.

$$V(\vec{r}) = \sum_{i=1}^{3} \frac{1}{2} m \omega_i^2 r_i^2$$

$$b_i^+ = \frac{1}{\sqrt{2}} \left(\frac{x_i}{d_i} - d_i \frac{\partial}{\partial x_i} \right) \qquad i = 1, 2, 3$$

$$b_i = \frac{1}{\sqrt{2}} \left(\frac{x_i}{d_i} + d_i \frac{\partial}{\partial x_i} \right)$$

$d_i = \sqrt{\frac{\hbar}{m\omega_i}}$ oscillator - length

$b_i, b_i^+ \}$ creation, annihilation ops. for the 3D HO case.

$$\boxed{\begin{array}{ll} u_i = b_i^+ \varphi_0 & E_i = \hbar\omega_i \\ v_i = b_i \varphi_0^* & i = 1, 2, 3 \end{array}}$$

these are the Kohn - modes

$\rightsquigarrow$ CM motion in a HO. pot.

$$\left(-\frac{\hbar^2}{2m}\Delta + V_{HO}(\vec{r})\right) = \sum_{i=1}^{3} \hbar\omega_i \left(b_i^+ b_i + \frac{1}{2}\right)$$

$$\left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g|\varphi_0|^2\right) b_i^+ \varphi_0 - g\,\varphi_0^2 b_i \varphi_0^* =$$

- identities: $\hat{a}\hat{b} = \hat{b}\hat{a} + [\hat{a}, \hat{b}]$

$$= b_i^+ \left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g|\varphi_0|^2\right)\varphi_0 + \left[-\frac{\hbar^2}{2m}\Delta + V, b_i^+\right]\varphi_0 -$$

$$- b_i\, g\, \varphi_0^2 \varphi_0^* - g\left[\varphi_0^2, b_i\right]\varphi_0^* + 2g\left[|\varphi_0|^2, b_i^+\right]\varphi_0 =$$

- Commutators:

summation not needed, they commute (others)

$$\left[\hbar\omega_i\left(b_i^+ b_i + \cancel{\tfrac{1}{2}}\right), b_i^+\right] = \hbar\omega_i\left(b_i^+ \underbrace{[b_i, b_i^+]}_{\mathbb{1}} + \underbrace{[b_i^+, b_i^+]}_{\emptyset} b_i\right) = \underline{\underline{\hbar\omega_i b_i^+}}$$

$$= \hbar\omega_i \underbrace{b_i^+ \varphi_0}_{u_i} + b_i^+\left(-\frac{\hbar^2}{2m}\Delta + V - \mu + 2g|\varphi_0|^2\right)\varphi_0 - b_i^+ \varphi_0^2 \varphi_0^* + g(b_i^+ - b_i)\varphi_0^2\varphi_0^* -$$

$$- g\left[\varphi_0^2, \frac{1}{\sqrt{2}}\left(\frac{x_i}{d_i} + d_i\frac{\partial}{\partial x_i}\right)\right]\varphi_0^* + 2g\left[|\varphi_0|^2, \frac{1}{\sqrt{2}}\left(\frac{x_i}{d_i} - d_i\frac{\partial}{\partial x_i}\right)\right]\varphi_0$$

$$= \hbar\omega_i v_i + b_i^+ \underbrace{\left(-\frac{\hbar^2}{2m}\Delta + V - \mu + g|\Psi_0|^2\right)\Psi_0}_{\emptyset \text{ GP-eq.}} + g\underbrace{(b_i^+ - b_i)}_{-\sqrt{2}\, d_i \frac{\partial}{\partial x_i}}|\Psi_0|^2\Psi_0 -$$

⤳ that's why we need the exact solution of the eq.-s!

$$-\underbrace{g\left[\Psi_0^2, \frac{1}{\sqrt{2}}\left(\frac{x_i}{d_i} - d_i\frac{\partial}{\partial x_i}\right)\right]\Psi_0^*}_{\frac{d_i g}{\sqrt{2}}\left[\frac{\partial}{\partial x_i}, \Psi_0^2\right]\Psi_0^*} + \underbrace{2g\left[|\Psi_0|^2, \frac{1}{\sqrt{2}}\left(\frac{x_i}{d_i} - d_i\frac{\partial}{\partial x_i}\right)\right]\Psi_0}_{\sqrt{2}\, d_i g\left[\frac{\partial}{\partial x_i}, \Psi_0\Psi_0^*\right]\Psi_0} =$$

$$= (\ldots) - g\sqrt{2}\, d_i\left(2\Psi_0\cancel{\left(\frac{\partial\Psi_0}{\partial x_i}\right)}\Psi_0^* + \Psi_0^2\cancel{\left(\frac{\partial\Psi_0^*}{\partial x_i}\right)}\right) + \frac{g d_i}{\sqrt{2}}\left(2\Psi_0\cancel{\frac{\partial\Psi_0}{\partial x_i}}\right)\Psi_0^* +$$

$$+ \sqrt{2}\, g d_i\left(\Psi_0\cancel{\frac{\partial\Psi_0^*}{\partial x_i}} + \Psi_0^*\cancel{\frac{\partial\Psi_0}{\partial x_i}}\right)\Psi_0 = \hbar\omega_i v_i$$

$$\Longrightarrow \boxed{E_i = \hbar\omega_i}$$

- the second eq. works very similarly, and gives the same result for E_i, as here.
- So those 3 modes are exact modes.

→ one can excite selectively the Kohn-modes

- In other approximations (Green's func.) these 3 modes are not exact.
- numerical error can be estimated by calculating the Kohn-modes.