Problem Solving

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This class by Shreyash Sharma was focused on Problem Solving.

§1 Problem

Problem statement

Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

¶ **Solution.** **Answers :** $(0, \pm 2), (4, \pm 23)$

Notice that $x < 0$ doesn't work. Let us rewrite the equation as $2^x + 2^{2x+1} = (y-1)(y+1)$. Thus,

$$x = \nu_2(y-1) + \nu_2(y+1)$$

Thus, one of $\nu_2(y+1)$ or $\nu_2(y-1)$ has to be $x-1$ and the other has to be 1 for $x > 2$. One can manually check for $x \in \{1, 2\}$.

Case I : $y = 2^{x-1}m - 1$

One can plug in the value of y in the original solution, to get

$$m + 1 + 2^{x+1} = 2^{x-2}m^2$$
$$\implies 4m + 4 + 8 \cdot 2^x = 2^x m^2$$
$$\implies 4m + 4 = 2^x(m^2 - 8)$$

Thus, $m^2 - 8 \mid 4m + 4$ which implies $|m^2 - 8| < |4m + 4|$ which is false for $m \geq 7$ Thus, we now can get values of x and the values of y as well.

Case II : $y = 2^{x-1}m + 1$

One can plug in the value of y in the original solution, to get

$$1 + 2^{x+1} = 2^{x-2}m^2 + m$$
$$4 + 8 \cdot 2^x = 2^x m^2 + 4m$$
$$4 - 4m = 2^x(m^2 - 8)$$

Thus, $m^2 - 8 \mid 4 - 4m$ and we land in a similar situation to what we had in **Case I** and one can check it there are no solutions to this.
And we're done.