Number Theory

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This class by Shreyash Sharma was focused on Number Theory.

§1 Theory

Problem statement

Determine all pairs of (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1$$

.

Proof. From the equation we get that $\sqrt{ab}$ must be a integer, thus

$$ab = n^2 \implies b = \frac{n^2}{a}$$

Thus, our equation becomes

$$\frac{a + \frac{n^2}{a}}{2} - n = 1$$
$$\implies \frac{a^2 + n^2 - 2an}{2a} = 1$$
$$\implies a^2 - a(2n+2) + n^2 = 1$$

Thus, using quadratic formula we get that

$$a = \frac{2n + 2 \pm \sqrt{(2n+2)^2 - 4n^2}}{2} = n + 1 \pm \sqrt{2n+1}$$

Since, the equation is symmetric we get that

$$(a, b) = (n + 1 \pm \sqrt{2n+1}, n + 1 \mp \sqrt{2n+1})$$

□

Problem statement

A positive integer is called *perfect* if it is the sum of its proper divisors. Suppose n is an odd perfect number. Prove that $n = pk^2$ for some prime p and integer k.

Proof. Let σ be the sum of divisor function. Thus, n is *perfect* if and only if $\sigma(n) - n = n$. Let $n = p_1^{a_1} \cdots p_m^{a_m}$ and since n is odd we get $\nu_2(\sigma(n)) = 1$. Since,

$$\sigma(n) = (1 + p_1 + \cdots + p_1^{a_1}) \cdots (1 + p_m + \cdots + p_m^{a_m})$$

we get that exactly one of terms in the factorization must be even. Thus, exactly one of exponent is odd. Thus, we get that

$$n = p^{2a+1} c^2$$

for some prime p, and integers $a \geq 0$ and $c > 0$. Thus,

$$\implies n = p(p^a c)^2 = pk^2$$

as desired. $\square$