Number Theory

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This class by Shreyash Sharma was focused on Number Theory.

§1 Problem

> **Problem statement**
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> There were n positive integers. For each pair of those integers Boris wrote their arithmetic mean onto a blackboard and their geometric mean onto a whiteboard. It so happened that for each pair at least one of those means was integer. Prove that on at least one of the boards all the numbers are integer.

¶ **Solution.** Let $E = \{e_1, \ldots, e_k\}$ be the set of even numbers. And $O = \{o_1, \ldots, o_m\}$ be the set of odd numbers.

Since $\frac{e_i + o_j}{2}$ is not an integer, $e_i o_j$ must be a square. Thus, if we choose (o_i, e_1), (o_j, e_1). We must have, $o_i e_1$ and $o_j e_1$ be a square. Thus,

$$o_i e_1 = k^2$$

$$o_j e_1 = j^2$$

Thus $o_i o_j = (kj/e_1)^2$. Thus, $o_i o_j$ must be square for every $1 \leq i \leq k$ and $1 \leq j \leq m$. Similar calculation shows that $e_i e_j$ are square for every $1 \leq i \leq k$ and $1 \leq j \leq m$. This, shows that whiteboard must be full of integers.

§2 Exercise

Exercise 2.1. Alice and Bob play the following game: First, Alice chooses a polynomial $P(x)$ with integer coefficients. Then on each turn, Bob chooses an integer a, different from any integer he has chosen before, and Alice returns the number of integer solutions to the equation $P(x) = a$. Bob wins when Alice repeats a number she had already returned (not necessarily on the previous turn). Determine the minimum number of turns in which Bob can guarantee a win.