Inequalities

Basu Dev Karki

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This class by Sudivya Shah was focused on Inequalities.

§1 Problem

Problem statement

If $a+b+c=12$ such that $a,b,c \geq 0$ then find the maximum value of

$$(a+1)(b+2)(c+3)$$

¶ **Solution.** **Answers :** 216

Applying AM-GM inequality on $(a+1),(b+2),(c+3)$ we get

$$\frac{(a+1)+(b+2)+(c+3)}{3} \geq \sqrt[3]{(a+1)(b+2)(c+3)}$$

$$\implies 216 \geq (a+1)(b+2)(c+3)$$

Thus, the maximum is 216 which is attained at $(a,b,c)=(5,4,3)$.

> **Remark 1.1.** To achieve the maximum, set all the terms involved in the AM–GM inequality equal to each other.

Problem statement

If $a+b+c=216$ such that $a,b,c \geq 0$ then find the maximum value of

$$a^3b^2c$$

¶ **Solution.** **Answer :** $108^3 \cdot 72^2 \cdot 36$

Applying AM-GM inequality on $\frac{a}{3},\frac{a}{3},\frac{a}{3},\frac{b}{2},\frac{b}{2},c$ we get

$$\frac{\frac{a}{3}+\frac{a}{3}+\frac{a}{3}+\frac{b}{2}+\frac{b}{2}+c}{6} \geq \sqrt[6]{\frac{a^3b^2c}{27 \cdot 4}}$$

$$\implies 108^3 \cdot 72^2 \cdot 36 \geq a^3b^2c$$

Thus, the maximum is $108^3 \cdot 72^2 \cdot 36$ which is attained at $(a,b,c)=(108,72,36)$.

Problem statement

If $2a + 3b + c = 18$ such that $a, b, c \geq 0$ then find the maximum value of

$$a^2b^4c^3$$

¶ **Solution :** Apply AM-GM to $a, a, b, b, \frac{b}{2}, \frac{b}{2}, \frac{c}{3}, \frac{c}{3}, \frac{c}{3}$ and you should get the answer.

Problem statement

Let $0 \leq x \leq 1$. Find the maximum value of $x(1 - x^3)$.

¶ **Solution.** **Answer :** $\frac{3}{4 \cdot 4^{1/3}}$

Let $y = x(1 - x^3)$ then $y^3 = x^3(1 - x^3)^3$. Then, applying AM-GM

$$\frac{x^3 + \frac{1-x^3}{3} + \frac{1-x^3}{3} + \frac{1-x^3}{3}}{4} \geq \sqrt[4]{\frac{y^3}{3^3}}$$

$$\implies \frac{1}{4} \geq \sqrt[4]{\frac{y^3}{3^3}}$$

$$\implies \frac{3}{4 \cdot 4^{1/3}} \geq y$$

Thus, the maximum value of $x(1 - x^3)$ is $\frac{3}{4 \cdot 4^{1/3}}$ at $x = \frac{1}{4^{1/3}}$.

§2 Exercise

Exercise 2.1. Compare 101^{201} and $201!$.