Number Theory

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This class by Shreyash Sharma was focused on Number Theory.

§1 Problem

Problem statement

Find all primes p such that $5^p + 4p^4$ is a perfect square.

¶ Solution. **Answer :** $p = 5$

The equation can be rewritten as

$$5^p = (n - 2p^2)(n + 2p^2)$$

Then, $n - 2p^2 = 5^a$ and $5^{p-a} = n + 2p^2$ thus

$$4p^2 = 5^{p-a} - 5^a$$

Since, $n + 2p^2 > n - 2p^2 \implies 5^{p-a} > 5^a$. And if $a \geq 1$ then $5 \mid 4p^2 \implies p = 5$.
For $a = 0$ we get that $5^p - 1 = 4p^2$ which we know that there is no solution for $p \geq 2$.

Problem statement

Show that the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \ldots x_n} = 1$$

has at least one solution in integers for every positive integer n.

¶ Solution. We'll use induction. Our main claim will be that, if $(x_1, \ldots, x_k)$ is a solution for $n = k$ then $(x_1, \ldots, x_k, x_1 x_2 \cdots x_k + 1)$ is a solution for $n = k + 1$. One can check that our result is true for $n = 1$, and assume that for $n = k$ the result is true. Then,

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} + \frac{1}{x_1 x_2 \ldots x_k} = 1$$

Now, we have to show this is true, where $x_{k+1} = x_1 x_2 \cdots x_k + 1$

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} + \frac{1}{x_{k+1}} + \frac{1}{x_1 x_2 \ldots x_k x_{k+1}} = 1$$

Using our assumption we get

$$\begin{aligned} & 1 - \frac{1}{x_1 \cdots x_k} + \frac{1}{x_{k+1}} + \frac{1}{x_1 \cdots x_k x_{k+1}} \\ &= 1 + \frac{x_1 \cdots x_k - x_{k+1} + 1}{x_1 \cdots x_k x_{k+1}} \\ &= 1 \end{aligned}$$

and we're done.