Combinatorics

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This class by Shreyash Sharma was focused on Combinatorics.

§1 Problem

> **Problem statement**
>
> Starting with any 35 integers, you may select 23 of them and add 1 to each. By repeating this step, one can make all 35 integers equal. Prove this. Now replace 35 and 23 by m and n, respectively. What condition must m and n satisfy to make the equalization still possible?

¶ **Solution.** Let S be the list of 35 integers. Let $\max(S_t)$ and $\min(S_t)$ represent the maximum and minimum element at a turn t. Let A_t be the number of $\max(S_t)$ at a turn t. Now define $f(t)$ as follows

$$f(t) = \max(S_t) - \min(S_t)$$

Our goal is to show that $f(t) = 0$ at some turn t. That way we'd have $\max(S_t) = \min(S_t)$ and thus all the numbers in the list would be same. Our strategy would be

1. If $A_t \leq 12$ then we would try to pick the 23 numbers to be $\min(S_t)$ as much as possible and fill out the remaining with other numbers in the list other than the $\max(S_t)$ ones. This way the $f(t)$ would on decrease.

2. If $A_t > 12$ then we would select all the non $\max(S_t)$ (which is $35 - A_t$ of them) and then we'd select the remaining numbers to be the $\max(S_t)$(let it be z of them). Note that A_{t+1} would only come from increasing the $\max(S_t)$, thus $z = A_{t+1}$ and we get
$$(35 - A_t) + z = 23 \implies z = A_{t+1} = A_t - 12$$

Thus, the number of max integers decreases in each turn. And it keeps decreasing so we would eventually hit $A_k \leq 12$ and we'd be in our first case, where $f(t)$ decrease. This cycle keeps on happening and eventually we'd hit $f(t) = 0$ and we'd be done. The second part of this problem is left as an exercise.

Problem statement

Many handshakes are exchanged at a big international congress. We call a person an *odd person* if he has exchanged an odd number of handshakes. Otherwise he will be called an *even person*. Show that, at any moment, there is an even number of odd persons.

¶ Solution. Consider the people at the congress to be vertices and let a handshake between two individual be a edge between them. Let V be the set of all vertices, and let E be the set of edges. Then, using handshake lemma we get

$$\sum_{v \in V} \deg V = 2|E|$$

Now, since the sum of degrees of even person is always even we get that the odd number of odd person must be even as well.