Number Theory

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This class by Shreyash Sharma was focused on solving number theory related problems.

§1 Problems

Problem statement

Prove that $\frac{21n+4}{14n+3}$ is irreducible.

¶ Solution. Showing the fraction is irreducible is equivalent to showing $\gcd(21n+4, 14n+3) = 1$. We'll use the property of gcd that $\gcd(a, b) = \gcd(a - kb, b)$ for any $k \in \mathbb{Z}$. Then

$$\begin{aligned}
\gcd(21n+4, 14n+3) &= \gcd(7n+11, 14n+3) \\
&= \gcd(21n+4-14n-3, 14n+3) \\
&= \gcd(7n+1, 14n+3-2(7n+1)) \\
&= \gcd(7n+1, 1) \\
&= 1
\end{aligned}$$

Problem statement

For which integer n is $\frac{16(n^2-n-1)^2}{2n-1}$ an integer?

¶ Solution. Notice that $2n-1 \nmid 16$ thus $2n-1 \mid (n^2-n-1)^2$. Now, computing

$$\begin{aligned}
\gcd(n^2-n-1, 2n-1) &= \gcd(n^2-3n, 2n-1) \\
&= \gcd(n-3, 2n-1) \\
&= \gcd(n-3, 5) \\
&= 1, 5
\end{aligned}$$

The second equation is due to the fact that $\gcd(n, 2n-1) = 1$. Now, that means $\gcd((n^2-n-1)^2, 2n-1) = \pm1, \pm5, \pm25$. Now, we know that

$$a \mid b \iff \gcd(a, b) = a$$

Thus, $2n - 1 \mid (n^2 - n - 1)^2 \iff \gcd((n^2 - n - 1)^2, 2n - 1) = 2n - 1 = \pm 1, \pm 5, \pm 25$. Thus, we can now check the values of n.