Number Theory

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This class by Shreyash Sharma was focused on Number Theory.

§1 Problem

Problem statement

Find all positive integers n such that every decimal of $6^n + 1$ is the same.

¶ **Solution.** **Answer :** $n = 1, 5$
Notice that every digit of the number must be 7 because $6^n \equiv 6 \pmod{10}$ so $6^n + 1$ must have its last digit as 7. Now, $6^n + 1 = \overline{777\cdots7}$ thus, $6^n + 1 \equiv 777776 \pmod{2^6}$ for $n > 5$ as we can write $\overline{777\ldots7}$ in its digit form, which we can check gives a contradiction.

Problem statement

Determine, with proof, the smallest positive multiple of 99 all of whose digits are either 1 or 2.

¶ **Solution.** **Answer :** 1122222222
Since the number must be divisible by 9 and 11, we can use their divisibility rules to get that the sum of the digits must be a multiple of 9, and the alternating sum of the digits must be a multiple of 11. Suppose the smallest number was $\overline{a_1 \cdots a_k}$. Let us start with $a_1 + a_2 + \cdots + a_k = 9$ and $a_1 - a_2 + a_3 + \cdots + (-1)^{k+1} a_k \equiv 0 \pmod{11}$. Notice that the alternating sum can't be 0 because the parity of the sum of the positive terms and the negative terms in the alternating sum must be the same; that can't happen because the sum of all of them is odd. Since the alternating sum is always less than the sum of the digits, we get that there is no solution for this case.

Let us move to the case when $a_1 + \cdots + a_k = 18$ and $a_1 - a_2 + a_3 + \cdots + (-1)^{k+1} a_k = 0$. Notice that $9 \leq k \leq 18$, and $k = 9$ doesn't work, as it requires all the digits to be 2. Thus, $k \geq 10$. Let x be the number of 1s and y be the number of 2s. Thus, we get $x + 2y = 18$, which we can solve using $x + y = k$ and get $y = 8$ by setting $k = 10$. Thus, the number must have 10 digits, and 8 of them must be 2. Since the alternating sum is 0, we must have the digits at odd positions equal to the digits at even positions. Thus, they must sum to 9, and the only way that can happen is with $1 + 2 + 2 + 2 + 2 = 9$. This just leaves us with very few cases, which we can check easily. The cases can be reduced even further by saying we must have a 1 in order to minimize the number.