Inequalities

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This class by Sudivya Shah was focused on inequalities.

§1 Notation

The notation $\sum_{cyc}$ means to cycle over n variable. For example, if $n = 3$ then

$$\sum_{cyc} a^2 = a^2 + b^2 + c^2$$

$$\sum_{cyc} a^2b = a^2b + b^2c + c^2a$$

Moreover,

$$\sum_{cyc} f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b)$$

Then notation $\sum_{sym}$ means to go over all $n!$ permutation. For example, if $n = 3$ then

$$\sum_{sym} a^2b = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

Moreover,

$$\sum_{sym} f(a,b,c) = f(a,b,c) + f(a,c,b) + f(b,c,a) + f(b,a,c) + f(c,a,b) + f(c,b,a)$$

Similar convention also applies for $\prod$.

§2 Theory

Theorem 2.1 (AM-GM inequality)
For non-negative real numbers $a_1, \ldots, a_n$ the following holds true.

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdots a_n}$$

Equality holds when all the reals are equal.

Proof. Omitted. □

§3 Problems

Problem statement

Show that $x + \frac{1}{x} \geq 2$ for $x \in \mathbb{R}^+$.

¶ **Solution.** Using AM-GM inequality we get

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

$$\implies x + \frac{1}{x} \geq 2$$

as desired.

§4 Exercise

Exercise 4.1. Prove that $(a+b+c)^2 \geq 3(ab+bc+ac)$.

Exercise 4.2. Let $a_1, \ldots, a_n$ be n positive numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$\prod_{i=1}^{n} (1 + a_i) \geq 2^n$$

Exercise 4.3. Prove that for any $n \in \mathbb{N}$ such that $n > 1$,

$$(2n)! < (n(n+1))^n$$

holds.