Combinatorics(Class 8)

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Class 8 by Shreyash Sharma was mainly focused on solving combinatorial problem. The solution provided in this file is a detailed solution with motivation as well.

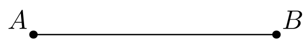
§1 Problems

Problem statement

There are $n \geq 2$ control points. We want to build k roads such that each road connects two different control points directly and no two roads connect the same pair of control points. Find the smallest k (in terms of n) such that no matter how we build the roads, we can always go from any control point to any other control point by traveling along one or two roads.

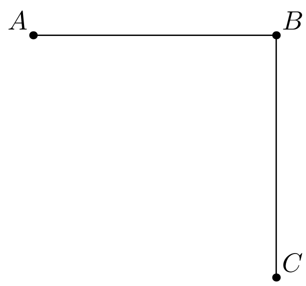
First, we test some small values of n to get some values of k and to get the fell for the problem.

For $\boxed{n = 2}$,



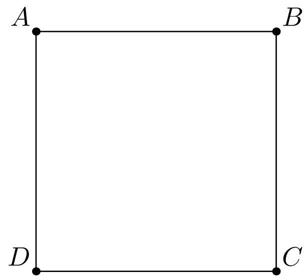
We see that $k = 1$ does the job.

For $\boxed{n = 3}$,



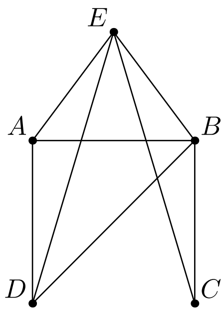
We see that $k = 2$ does the job.

For $\boxed{n=4}$,



We see that $k=4$ works.

For $\boxed{n=5}$,

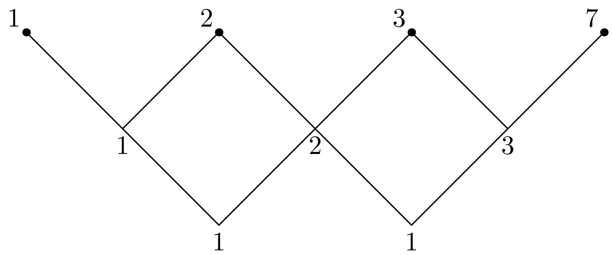


We see that $k=7$ works. Now, if we see our collected information and plot them in a table we get something like the table below.

n	k
2	1
3	2
4	4
5	7

Table 1: Values of n and k

If we look closely, we can see a pattern being formed.



Thus, we'd guess the k for $n=6$ to be 11 and for $n=7$ to be 16. These are known as quadratic progression(you can look it up on the internet how it works). Now, we can work out the constants for the general term of the quadratic progression by the following solving following equation.

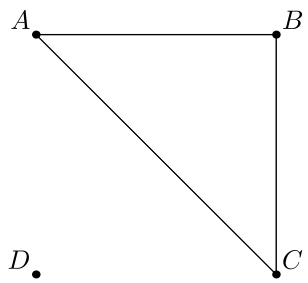
$$a+b+c=1$$

$$3a+b=1$$

$$2a=1$$

Solving these we get, $a = 1/2$, $b = -1/2$ and $c = 1$. Thus, for n we'd guess that our k would be equal to $\frac{n^2-n}{2} + 1$. This, however is not our answer for n but rather for $n+1$ as $n \geq 2$. Thus, for n we'd guess our k to be $\frac{(n-1)^2-(n-1)}{2} + 1$ which is just ${}^{n-1}C_2 + 1$.

Now, we need to prove that $k = \frac{(n-1)^2-(n-1)}{2} + 1$. First, we'll prove $k \geq \frac{(n-1)^2-(n-1)}{2} + 1$. Notice that if we have n control points then we can just build all possible roads between $n-1$ control points and leave one of the control points isolated.
Here is an example for $n = 4$,



The number of ways to build all possible roads between $n-1$ control points is ${}^{n-1}C_2$ since there is no way to reach the last control point $k \geq {}^{n-1}C_2 + 1$.

Now, we need to prove that if there are some control points that can't be reached within 2 or 1 roads then $k \leq {}^{n-1}C_2$. Suppose A and B are control points such that we can't reach each other within ≤ 2 roads. We also know that

$$k + (\text{missing roads}) = {}^{n}C_2$$

And missing roads $\geq (n-2) + 1$ as at least AX and BX can't be a road for any other point X other than A or B. Also, AB can't be a road either. Thus,

$$k \leq {}^{n}C_2 - n + 1 = {}^{n-1}C_2$$

And we're done.