Modular Arithemetic(Class 7)

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27 October 2025

The class by shreyash sharma was focused on modular arithmetic which was the continuation of the previous session on modular arithmetic. This class mainly focused on solving problems.

§1 Problems

Problem statement

Let $\sigma(a,b)$ be sum of all possible remainders of a^n when divided by b. For example, $\sigma(2,10) = 20$ since 2^n leaves $2,4,6,8$ as a remainder when divided by 10. Find

$$\sigma(3,20) + \sigma(5,52) + \sigma(7,100) + \sigma(9,164) + \sigma(11,244)$$

¶ Solution. If we look at first few remainders of 3^n when divided by 20 we see that $3^1 = 3 \pmod{20}, 3^2 \equiv 9 \pmod{20}, 3^3 \equiv 7 \pmod{20}$ and $3^4 \equiv 1 \pmod{20}$. Since, $3^4 \equiv 1 \pmod{20}$ we can multiply both sides by 3 and keep getting the same remainder. Thus, $1,3,9,7$ are all the remainders when 20.

Similarly, since $52 = 4 \times 13$ and $5^4 \equiv 1 \pmod 4$ and $5^4 \equiv 1 \pmod{13}$ we get that $5^4 \equiv 1^4 \pmod{52}$ as $\gcd(4,13) = 1$. Thus, we can get all the remainders. We can do similar stuff for other terms and the question just becomes a matter of calculation.

Problem statement

Find all positive integers m, n such that $\frac{2m-1}{n}$ and $\frac{2n-1}{m}$ are both integers.

¶ Solution. We can write $\frac{2m-1}{n} = k$ and $\frac{2n-1}{m} = \ell$ and since $m, n \geq 1$ the fractions are positive as well i.e $k, \ell \geq 1$. Simplifying a bit we get

$$2m - 1 = kn$$

$$2n - 1 = \ell m$$

Now, solving for m and substituting in the second equation gives

$$2n - 1 = \ell \left(\frac{kn+1}{2} \right)$$

$$\implies 4n - 2 = \ell kn + \ell$$
$$\implies 4n - \ell kn = \ell + 2$$

Now, since $\ell + 2 \geq 0$ we get $4n - \ell kn \geq 0 \implies 4 \geq \ell k$ since $n \geq 1$. Thus, we can list all the possible candidate for ℓ and k, which are $(k, \ell) = (1,1), (1,2), (2,1), (3,1), (1,3)$. Manually checking these answers will land us to the solutions which works.