

Real Analysis

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Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from [MIT OCW](#) and post the problems and the solutions here.

1 Problem Set 1

Problem 1.1

Let \mathbb{F} be a ordered field with $1 \neq 0$. Show that $1 > 0$.

Solution. First let us prove that $(-1) \cdot (-1) = 1$. We know that for all $x \in \mathbb{F}$, there is an inverse element $-x$ such that,

$$x + (-x) = 0$$

Thus, $1 + (-1) = 0$ which means that

$$\begin{aligned} 0 &= (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1) \\ &\implies 1 = (-1) \cdot (-1) \end{aligned}$$

Since \mathbb{F} is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that $1 \neq 0$ so the middle statement can't be true and if $1 < 0$ then $0 < (-1)$. But from **axiom of order and multiplication** $0 < (-1) \cdot (-1) = 1$. Thus a contradiction.

Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

Solution. Suppose $\frac{n}{m} = \frac{n_1}{m_1}$ and $\frac{p}{q} = \frac{p_1}{q_1}$ then using the definition of when two rational numbers are equal, we get $nm_1 = n_1m$ and $pq_1 = p_1q$. Thus,

$$\begin{aligned} m_1q_1(nq + pm) &= m_1q_1nq + m_1q_1pm \\ &= n_1m_1q_1q + p_1qm_1m \\ &= mq(n_1q_1 + m_1p_1) \end{aligned}$$

Thus, $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$.

Problem 1.3

Find the $\sup E$ and $\inf E$ for the following set E .

1. $E = \{n \in \mathbb{Z} \mid n < \sqrt{12}\}$
2. $E = \{r \in \mathbb{Q} \mid r < \sqrt{12}\}$
3. $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
4. $E = \left\{ \frac{n^2+n}{n+1} \mid n \in \mathbb{N} \right\}$

Solution.

1. $\sup E = 3$ but $\inf E$ doesn't exist.
2. $\sup E = \sqrt{12}$ but $\inf E$ doesn't exist.
3. $\sup E = \frac{1+\sqrt{5}}{2}$ and $\inf E = \frac{1-\sqrt{5}}{2}$.
4. $\inf E = 1$ but $\sup E$ doesn't exist.