Linear Algebra
Notes

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Content

1 Linear Maps

1.1 Vector Spaces of Linear Maps

1.1.1 Definiton and Examples of Linear Maps

Definition 1.1. A *linear map* from V to W is a function $T : V \to W$ with the following properties.

1. (**Additivity**) $T(u + w) = T(u) + T(w)$ for all $u, v \in V$
2. (**Homogeneity**) $T(\lambda u) = \lambda T(u)$ for all $\lambda \in \mathbf{F}$ and for all $u \in V$

Remark. Some mathematicians use the phrase *linear transformation*, which means the same as linear map.

Definition 1.2. (Notation)

1. The set of linear maps from $V \to W$ is denoted by $\mathcal{L}(V, W)$.
2. The set of linear maps from $V \to V$ is denoted bt $\mathcal{L}(V)$. In other words, $\mathcal{L}(V, V) = \mathcal{L}(V)$.

Examples:

zero
We will let the symbol 0 denote the liner map that takes every element of some vector space to additive identity of some another vector space. Thus, $0 \in \mathcal{L}(V, W)$ is defined by

$$0(v) = 0$$

identity operator
Let $I \in \mathcal{L}(V)$ be defined by

$$I(v) = v$$

differentiation
Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be defined by

$$D(p) = p'$$

integration
Let $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$ be defined by

$$T(p) = \int_0^1 p$$

composition
Fix a polynomial $q \in \mathcal{P}(\mathbf{R})$. Let $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be defined by

$$T(p) = (p \circ q)$$

Remark. We'll limit the Notation of $T(v)$ to just Tv for convenience.

Theorem 1.1. *Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then, there exists a unique linear map $T : V \to W$ such that*

$$Tv_k = w_k$$

for $i = 1, 2, 3, \ldots, n$.

Proof. First we show the existence of such map. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

where $c_i \in \mathbf{F}$. Since, $v_1, \ldots, v_n$ is a basis of V, it maps every element of V to W, thus it is a function.
Now, set $c_k = 1$ and all other c's to be 0 to show that $Tv_k = w_k$. From, here one can show that T is indeed a linear map. To show the uniqueness, suppose $T' \in \mathcal{L}(V, W)$ and $T'v_k = w_k$. Using the properties of linear map,

$$T'(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

Thus, T and T' agree on every input, thus $T = T'$.

□

1.1.2 Algebraic Operation on $\mathcal{L}(V, W)$

Definition 1.3. Suppose $T, S \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$ then the *sum* and the *product* of the linear maps from V to W is defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

Proposition 1.1. With the operations defined above, the set $\mathcal{L}(V, W)$ is a vector space.

Proof. The additive identity for $\mathcal{L}(V, W)$ is the zero linear map $0(v) = 0$. The inverse for T is $((-1)T)v = -(Tv)$. And the rest of the axioms are left for readers (future me) to verify. □

Definition 1.4. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Remark. Be careful about the domains of S and T. Here, the domain of S must be the co-domain of T.

Proposition 1.2. For the product of linear maps, the following holds

1. (**associativity**) $(T_1T_2)T_3 = T_1(T_2T_3)$ whenever the product makes sense(i.e T_3 must map to domain of T_2 and T_2 must map to the domain of T_1).
2. (**identity**) $TI_{W,V} = I_{V,W}T$ whenever $T \in \mathcal{L}(V, W)$. Here $I_{V,W}, I_{W,V}$ are the identity linear maps from V to W and W to V. We'll just limit he notation to $TI = IT$.
3. (**distributivity**) $(S_1+S_2)T = S_1T+S_2T$ and $S(T_1+T_2) = ST_1+ST_2$ for $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Proposition 1.3. Suppose T is a linear map from V to W. Then, $T(0) = 0$.

Proof. From the definition of linear map we have,

$$T(0) = T(0 + 0) = T(0) + T(0) \implies T(0) = 0$$

□

1.1.3 Exercises

Problem : Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z, 6x + cxyz)$$

Show that T is a linear map if and only if $b = c = 0$.

Solution : ($\Leftarrow$) is pretty simple as you just have to verify the two axioms. For ($\Rightarrow$), we know that if it is a linear map then $T(0) = 0 \implies b = 0$. Also, using the first axiom we get

$$T((x, y, z) + (1, 0, 0)) = T((x, y, z)) + T((1, 0, 0)) \implies c = 0$$

Problem : Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x) dx + c \sin(p(0))\right)$$

Show that T is a linear map if and only if $b = c = 0$.

Solution : ($\Leftarrow$) is pretty simple. For ($\Rightarrow$), we can use the first axiom

$$T(p + q) = Tp + Tq$$

we'll just look at the first component first,

$$\begin{aligned}\implies 3(p + q)(4) + 5(p + q)'(6) + b(p + q)(1)(p + q)(2) =& 3p(4) + 5p'(6) + bp(1)p(2) + \\ & 3q(4) + 5q'(6) + bq(1)q(2)\end{aligned}$$

Since, $(p + q)(4) = p(4) + q(4)$ and $(p + q)'(6) = p'(6) + q'(6)$, we can simplify is down to,

$$b(p(1)q(2) + p(2)q(1)) = 0$$

Now, if you choose polynomials $p, q > 0$ for $x > 0$ then $b = 0$. A similar argument works for $c = 0$.

Problem : Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Solution : Suppose on the contrary that $v_1, \ldots, v_m$ are not linearly independent in V, then there exists $\lambda_1, \ldots, \lambda_m$ not all zero such that

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0$$

But $T(\lambda_1 v_1 + \cdots + \lambda_m v_m) = \lambda_1 T(v_1) + \cdots + \lambda_m T(v_m)$ which implies

$$\lambda_1 T(v_1) + \cdots + \lambda_m T(v_m) = 0 \implies \lambda_i = 0$$

a contradiction.

Problem : Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution : Since $\dim V = 1$ there exists a $v \in V$ s.t every $v_i \in V$ can be written as $\lambda_i v$ for some $\lambda_i \in \mathbf{F}$. Thus, $v_i = \lambda_i v \implies T(v_i) = \lambda_i T(v)$. Since, $T(v) = v_j \in V$ for some j. Thus, $T(v_i) = \lambda_i \lambda_j v = \lambda_j v_i$.

Problem : Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and $v \in \mathbf{R}^2$ but φ is not linear.

Solution : $\varphi(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Problem : Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(v + w) = \varphi(v) + \varphi(w)$$

for all $v, w \in \mathbf{C}$ and but φ is not homogeneous.

Solution : $\varphi(x) = \mathrm{Re}(x)$.

Problem : Prove or give a counter example: Fix a polynomial $q \in \mathcal{P}(\mathbf{R})$.
Let $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ be defined by $Tp = q \circ p$ then T is a linear map.

Solution : Assume it was a linear map then $T(0) = q(0) = 0$. Just pick $q(0) \neq 0$. Example : $q(x) = x + 1$.

Problem : Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Solution : I really tried but it seems very hard to prove ($\Rightarrow$) but will come back later.

Problem : Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $Su \neq 0$ for some $u \in U$. Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that $T \notin \mathcal{L}(V, W)$.

Solution : Suppose it is a linear map, then $T(u + v) = Tu + Tv$ where $u \in U$ and $v \in V$ and $v \notin U$. One can check that $v + u \in V$ but $v + u \notin U$. Thus, $0 = Tu = Su$, but just take u s.t $Su \neq 0$.

Problem : Suppose V is finite-dimensional. Prove that every linear map on a subspace of U can be extended to a linear map on V. In other words, let U be a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists a $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution : Let $u_1, \ldots, u_m$ be the basis of U and let $u_1, \ldots, u_m, v_1, \ldots, v_k$ be the extended basis of V. Let $x = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_kv_k$ and define

$$T(x) = a_1Su_1 + \cdots a_mSu_m + b_1v_1 + \cdots + b_kv_k$$

From here its easy to see that $Tu = Su$ for all $u \in U$. We just have to prove this is a linear map on V. Let $y = x = c_1u_1 + \cdots + c_mu_m + d_1v_1 + \cdots + d_kv_k$ then

$$T(x+y) = (a_1 + c_1)Su_1 + \cdots + (a_m + c_m) + (b_1 + d_1)v_1 + \cdots + (b_k + d_k)v_k$$

$$\implies T(x+y) = a_1Su_1 + \cdots a_mSu_m + b_1v_1 + \cdots + b_kv_k + c_1Su_1 + \cdots c_mSu_m + d_1v_1 + \cdots + d_kv_k$$

$$\implies T(x+y) = Tx + Ty$$

Similarly one can prove $T(\lambda x) = \lambda T(x)$.

Problem : Suppose V is finite-dimensional with $\dim V > 0$ and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution : Let $v_1, \ldots, v_m$ be basis of V. Suppose $\mathcal{L}(V, W)$ is finite-dimensional then every $T \in \mathcal{L}(V, W)$ can be written as

$$T(x) = \lambda_1T_1(x) + \lambda_2T_2(x) + \cdots + \lambda_kT_k(x)$$

for some fixed $T_1, T_2, \cdots, T_k \in \mathcal{L}(V, W)$ and $\lambda_i \in \mathbf{F}$.
Since, W is infinite-dimensional, $\exists w \in W$ s.t $w \notin \operatorname{span}\{T_1(x), T_2(x), \ldots, T_k(x)\}$ for some fixed $x \in V$.
Now, set $x = v_i$ then $w \neq \lambda_1T_1(v_i) + \lambda_2T_2(v_i) + \cdots + \lambda_kT_k(v_i)$. One can find $T \in \mathcal{L}(V, W)$ such that $T(v_i) = w$ thus $T(v_i) \neq \lambda_1T_1(v_i) + \lambda_2T_2(v_i) + \cdots + \lambda_kT_k(v_i)$ contradicting our assumption.

Problem : Let V be finite-dimensional and let $\dim V > 1$. Prove that there exists $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

Solution : Let $v_1, \ldots, v_m$ be the basis of V and let $x = a_1v_1 + \cdots + a_mv_m$. Define $S(x) = a_1v_1$ and $T(x) = a_1v_2 + a_2v_3 + \cdots + a_{m-1}v_m + a_mv_1$. So, $T(S(x)) = a_1v_2$ and $S(T(x)) = a_mv_1$, thus

$$ST = TS \iff a_1 = a_m$$

but one can always choose x s.t $a_1 \neq a_m$.

Problem : Suppose V is finite-dimensional. Show that the only two-sided ideas of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a two-sided ideal if for every $E \in \mathcal{E}$ and $T \in \mathcal{L}(V)$, $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$.

Solution : Let $\mathcal{E}$ be a two-sided ideal of $\mathcal{L}(V)$. If $\mathcal{E} = \{0\}$ we are done. Otherwise pick a nonzero operator $A \in \mathcal{E}$. Choose $v \in V$ with $Av \neq 0$. Fix any $y, z \in V$. Choose $R \in \mathcal{L}(V)$ with $R(z) = v$ and choose $S \in \mathcal{L}(V)$ with $S(Av) = y$. Then

$$T := SAR \in \mathcal{E}$$

(since $\mathcal{E}$ is a two-sided ideal), and

$$T(z) = S(A(R(z))) = S(A(v)) = y.$$

Thus $\mathcal{E}$ contains, for every pair y, z, an operator that sends z to y. Taking a basis $u_1, \ldots, u_n$ of V and the operators E_{ij} defined by $E_{ij}(u_j) = u_i$ and $E_{ij}(u_k) = 0$ for $k \neq j$, we see each E_{ij} lies in $\mathcal{E}$. The set $\{E_{ij}\}$ spans $\mathcal{L}(V)$, so $\mathcal{E} = \mathcal{L}(V)$. Hence the only two-sided ideals are $\{0\}$ and $\mathcal{L}(V)$. $\square$

1.2 Null Spaces and Ranges

1.2.1 Null Space and Injectivity

Definition 1.5. let $T \in \mathcal{L}(V, W)$, the null space of T, written as $\operatorname{null} T$ is the following set

$$\operatorname{null} T = \{v \in V \mid Tv = 0\}$$

Examples

1. The zero map from V to W has a null space V as everything gets mapped to 0.

2. Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the differentiation map defined by $Dp = p'$. The only functions whose derivative is equal to 0 are the constant function. Thus, $\operatorname{null} D$ is the set of constant functions.

3. Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the multiplication by x^2 map i.e $Dp = x^2 p$. The only polynomial such that $x^2 p = 0$ is $p = 0$. Thus, $\operatorname{null} D = \{0\}$.

Proposition 1.4. Suppose $T \in \mathcal{L}(V, W)$. Then, $\operatorname{null} T$ is a subspace of V.

Proposition 1.5. Let $T \in \mathcal{T}(V, W)$. Then T is injective $\iff$ $\operatorname{null} T = \{0\}$.

Proof. Suppose T is injective. Since it is a linear map $T(0) = 0$, thus by injectivity the only thing that gets map to 0 is 0. Thus, $\operatorname{null} T = \{0\}$. Suppose T is such that $\operatorname{null} T = \{0\}$ then

$$T(v) = T(w) \implies T(v) + (-1)T(w) = 0$$
$$\implies T(v) + T(-w) = 0 \implies T(v - w) = 0 \implies v = w$$

Thus, the map is injective. □

1.2.2 Range and Surjectivity

Definition 1.6. Let $T \in \mathcal{L}(V, W)$, the *range* of T is the following set,

$$\operatorname{range} T = \{Tv \mid v \in V\}$$

Examples

1. If T is the zero map then the range of T is $\{0\}$.

2. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the differentiation map. Since, for every polynomial q there exists a polynomial p such that $p' = q$, the range is $\mathcal{P}(\mathbf{R})$.

Proposition 1.6. Let $T \in \mathcal{L}(V, W)$ then $\operatorname{range} T$ is a subspace of W.

Proof. Since, $0 \in V$ we know $T(0) = 0 \in \operatorname{range} T$. Now, suppose $x, y \in \operatorname{range} T$ then $x = Tv$ and $y = Tw$. Since, $v + w \in V$, $T(v + w) \in \operatorname{range} T \implies Tv + Tw \in \operatorname{range} T$ which mean $x + y \in \operatorname{range} T$. And $x \in \operatorname{range} T \implies Tv \in \operatorname{range} T$ which means $T(\lambda v) \in \operatorname{range} T$ as $\lambda v \in V$. Thus, $\lambda x = \lambda T(v) = T(\lambda v) \in \operatorname{range} T$. □

1.2.3 Fundamental Theorem of Linear Maps

Theorem 1.2. *Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then* $\operatorname{range} T$ *is finite-dimensional and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. We know that $\operatorname{null} T$ is a subspace of V then since V is finite-dimensional it has a basis. Let $u_1, \ldots, u_m$ be the basis of $\operatorname{null} T$. Then we can extend this basis to a basis of V. Let $u_1, \ldots, u_m, v_1, \ldots, v_n$ be the basis of V. Then,

$$x = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots b_n v_n$$

$$\implies Tx = b_1 T v_1 + \cdots + b_n T v_n$$

Thus, every Tv can be written as a linear combination of $Tv_1, \ldots, Tv_n$. Thus, $\operatorname{range} T$ is finite-dimensional. To prove our main result, we need to show that $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$. We already proved it spanned $\operatorname{range} T$, now suppose

$$b_1 T v_1 + \cdots + b_n T v_n = 0$$

$$\implies T(b_1 v_1 + \cdots + b_n v_n) = 0$$

Thus, $b_1 v_1 + \cdots + b_n v_n \in \operatorname{null} T$ and we can write it as $b_1 v_1 + \cdots + b_n v_n = a_1 u_1 + \cdots + a_m u_m$. Since, $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis vector we can say that $b_i = a_i = 0$. Thus, we have proved that $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$ and our theorem follows. □

Theorem 1.3. *Suppose V and W are both finite-dimensional vector spaces such that* $\dim V > \dim W$. *Then, there exists no* ***injective*** *linear map from V to W.*

Proof. We know that, for a $T \in \mathcal{L}(V, W)$,

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Since $\operatorname{range} T$ is a subspace of W, $\dim W \geq \dim \operatorname{range} T$. Thus,

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W \\ &> 0 \end{aligned}$$

Thus, $\operatorname{null} T$ has more than one vector, so T it's not injective by **Proposition 1.5**. □

Theorem 1.4. *Suppose V and W are both finite-dimensional vector spaces such that* $\dim V < \dim W$. *Then, there exists no* ***surjective*** *linear map from V to W.*

Proof. Similar to the proof above. □

Definition 1.7. Define $T : \mathbf{F}^n \to \mathbf{F}^m$ as

$$T(x_1, \cdots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right)$$

Definition 1.8. A homogeneous system of linear equations defined is as

$$\left(\sum_{k=1}^{n} A_{1,k}x_k, \ldots, \sum_{k=1}^{n} A_{m,k}x_k\right) = (0, \ldots, 0)$$

And a Inhomogeneous system of linear equation is defined as

$$\left(\sum_{k=1}^{n} A_{1,k}x_k, \ldots, \sum_{k=1}^{n} A_{m,k}x_k\right) = (c_1, \ldots, c_m)$$

where not all c_i are zero.

Proposition 1.7. A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof. Use **Theorem 1.3** and **Theorem 1.4**. □

Proposition 1.8. An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof. Use **Theorem 1.3** and **Theorem 1.4**. □

1.2.4 Excercise

Problem : Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution : $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2)$.

Problem : Suppose $S, T \in \mathcal{L}(V, W)$ are such that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

Solution : Since $\operatorname{range} S \subseteq \operatorname{null} T$, $T(Sv) = 0$. Thus,

$$(ST)^2 = (ST)(ST) = S(T(S(Tv))) = S(0) = 0$$

Problem : Suppose $v_1, \ldots, v_m$ is a list of vector in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1v_1 + \ldots + z_mv_m$$

a. What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

b. What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent on V?

Solution : If $v_1, \ldots, v_m$ spans V then $\operatorname{range} T = V$ thus T being surjective corresponds to $v_1, \ldots, v_m$ spanning V.
If $v_1, \ldots, v_m$ is linearly independent on V then $\operatorname{null} T = \{0\}$, thus T being injective corresponds to $v_1, \ldots, v_m$ being linearly independent on V.

Problem: Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Solution : Let $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$ and $T'(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, 0)$. Both of them are in $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$. Also $\dim \operatorname{null} T = 3$, $\dim \operatorname{null} T' = 4$ but $T + T' = (x_1, x_2, x_3, 0) \implies \dim \operatorname{null}(T + T') = 2 \not> 2$.

Problem : Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

Solution : $T(x_1, x_2, x_3, x_4) = (0, 0, x_1, x_2)$.

Problem : Prove that there doesn't exists a $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

Solution : Suppose there exists such T, then $\dim \operatorname{range} T = \dim \operatorname{null} T$ but from the fundamental theorem of linear maps we have

$$\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T$$

$$\implies \dim \operatorname{range} T = \dim \operatorname{null} T = \frac{5}{2}$$

which is impossible.

Problem : Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution : Let $v_1, \ldots, v_m$ be the basis of V and $w_1, \ldots, w_n$ be the basis of W. Then, define $T_i(a_1 v_1 + \cdots + a_m v_m) = a_i w_i$. One can check that this is not injective thus

$$T(a_1 v_1 + \cdots + a_m v_m) = \left(\sum_{i=1}^{m} T_i \right) (a_1 v_1 + \cdots + a_m v_m) = \sum_{i=1}^{m} a_i w_i$$

Now, suppose $T(v) = T(v')$ then

$$a_1 w_1 + \cdots + a_m w_m = a'_1 w_1 + \cdots + a'_m w_m$$

$$\implies b_1 = b'_1 \quad \textit{(because of linear independence)}$$

Thus, $v = v'$.

Problem : Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U and V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu \mid u \in U\}$$

Solution : We know that $\operatorname{null} T$ is the subspace of V. Thus, there exists a U such that $V = U \oplus \operatorname{null} T$ and since it is a direct sum $U \cap \operatorname{null} T = \{0\}$. Now, for the range of T

$$\operatorname{range} T = \{Tv \mid v \in V\}$$

$$\implies \{T(u + z) \mid u \in U, z \in \operatorname{null} T\} = \{Tu \mid u \in U\}$$

Problem : Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is a surjective linear map.

Solution : We can write T as

$$\operatorname{null} T = \{(5x_2, x_2, 7x_4, x_4) \mid x_2, x_4 \in \mathbf{F}\}$$

Now, since $(5x_2, x_2, 7x_4, x_4) = x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1) \implies \dim \operatorname{null} T = 2$. Thus,

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &\implies 4 = 2 + \dim \operatorname{range} T \\ &\implies \dim \operatorname{range} T = 2 \end{aligned}$$

Since, $\dim \mathbf{F}^2 = 2 = \dim \operatorname{range} T \implies \operatorname{range} T = \mathbf{F}^2$. Thus, T is surjective.

Problem : Suppose U is three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

Solution : Since, $\operatorname{null} T = U \implies \dim \operatorname{null} T = 3$. Thus,

$$\begin{aligned} \dim \mathbf{R}^8 &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &\implies \dim \operatorname{range} T = 5 \end{aligned}$$

Since, $\operatorname{range} T$ is a subspace of $\mathbf{R}^5$ and $\dim \operatorname{range} T = \dim \mathbf{R}^5$, $\mathbf{R}^5 = \operatorname{range} T$. Thus, T is surjective.

Problem : Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space doesn't equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

Solution : Suppose there does exist such a T. Then the null space can be written as

$$\operatorname{null} T = \{(3x_2, x_2, k, k, k) \mid x_2, k \in \mathbf{F}\}$$

One can check that $\dim \operatorname{null} T = 2$ but

$$\begin{aligned} \dim \mathbf{F}^5 &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &\implies \dim \operatorname{range} T = 3 \end{aligned}$$

But $2 = \dim \mathbf{F}^2 \geq \dim \operatorname{range} T = 3$ which is false.

Problem : Suppose there exists a linear map on V such that the null space and range of T is finite dimensional. Prove that V is finite-dimensional.

Solution : Since, the range of T is finite-dimensional it must have a basis. Suppose $Tv_1, \ldots, Tv_m$ is the basis then

$$\begin{aligned} T(x) &= \lambda_1 T v_1 + \cdots + \lambda_m T v_m \\ &\implies T(x - \lambda_1 v_1 - \cdots - \lambda_m v_m) = 0 \end{aligned}$$

Since, the null space is also finite-dimensional

$$x - \lambda_1 v_1 - \cdots - \lambda_m v_m = \lambda_1' v_1' + \cdots + \lambda_n' v_n'$$

where $v_1', \ldots, v_n'$ is the basis of the null space. Thus,

$$V = \operatorname{span}(v_1, \ldots, v_m, v_1' \ldots, v_n')$$

Problem : Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution : Suppose $T \in \mathcal{L}(V, W)$ is an injective map, then

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

$$\implies \dim V = \dim \operatorname{range} T \leq \dim W$$

Now suppose $\dim V \leq \dim W$ then we can construct a injective map from V to W.

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n$$

where $v_1, \ldots, v_n$ is the basis of V and $w_1, \ldots, w_m$ is the basis of W. One can check this is a linear map and suppose $T(x) = T(y)$ and let $x = a_1v_1 + \cdots + a_nv_n$ and $y = b_1v_1 + \cdots + b_nv_n$ then

$$T(x) = T(y)$$

$$\implies (a_1 - b_1)w_1 + \cdots + (a_n - b_n)w_n = 0$$

$$\implies a_i = b_i$$

Thus, $x = y$.

Problem : Suppose V and W are finite-dimensional vector spaces and U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution : Suppose $\operatorname{null} T = U$ then

$$\dim V - \dim \operatorname{range} T = \dim U$$

$$\implies \dim V - \dim W \leq \dim V - \dim \operatorname{range} T = \dim U$$

Now, suppose $\dim U \geq \dim V - \dim W$ then let $u_1, \ldots, u_k$ be the basis of U and

$$u_1, \ldots, u_k, v_1, \ldots, v_m$$

be the extended basis of V. Let $w_1, \ldots, w_j$ be the basis of W. From our condition, we know $k \geq k + m - j \implies j \geq m$. Thus we define

$$T(a_1u_1 + \cdots a_ku_k + b_1v_1 + \cdots b_mv_m) = b_1w_1 + \cdots b_mw_m$$

Here, $\operatorname{null} T = U$.

Problem : Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $X = \{v \in V \mid Tv \in U\}$ is a subspace of V and

$$\dim X = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$$

Solution : The subspace part is pretty simple. Let $S : X \to U$ be a map such that $S(v) = T(v)$. Here, $\operatorname{range} S = U \cap \operatorname{range} T$ and $\operatorname{null} S = \operatorname{null} T$.

Problem : Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} T + \dim \operatorname{null} S$$

Solution : One can find that $\operatorname{null} ST = \operatorname{null} T \cup \{x \in U \mid S(T(x)) = 0, T(x) \neq 0\}$.

Note : It seems that these exercises are taking way too long to do. I'll however come back to it and finish

1.3 Matrices

1.3.1 Representing a Linear Map by a Matrix

Definition 1.9. Suppose m and n are two non-negative integers. A $m \times n$ matrix is A is a rectangular array of elements in $\mathbf{F}$ with m rows and n columns

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

The $A_{i,j}$ represents the entry in i-th row and j-th column.

Definition 1.10. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is the basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ whose entries $A_{i,j}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots A_{m,k}w_m$$

Examples :

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$$

Then,

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

As $T(1, 0) = 1(1, 0, 0) + 2(0, 1, 0) + 7(0, 0, 1)$ and $T(0, 1) = 3(1, 0, 0) + 5(0, 1, 0) + 9(0, 0, 1)$.

1.3.2 Addition and Scalar Multiplication of Matrices

Definition 1.11. The sum of two matrices of same size is obtained by adding corresponding entries in the matrices i.e

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & & \vdots \\ B_{m,1} & \cdots & B_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} + B_{1,1} & \cdots & A_{1,n} + B_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + B_{1,1} & \cdots & A_{m,n} + B_{m,n} \end{pmatrix}$$

Proposition 1.9. Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Proof. Follows from the definition. $\square$

Definition 1.12. The product of a scalar and a matrix is obtained by multiplying each entry by the scalar i.e

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

Proposition 1.10. Suppose $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$ then $\lambda \mathcal{M}(T) = \mathcal{M}(\lambda T)$.

Proof. Again, just use the definitions. □

Theorem 1.5. *Suppose $\mathbf{F}^{m,n}$ be the set of all the matrices with entries in $\mathbf{F}$. Then, with addition and scalar multiplication defined above $\mathbf{F}^{m,n}$ is a vector space and $\dim \mathbf{F}^{m,n} = mn$.*

Proof. Proving it is a vector space is pretty easy. To verify $\dim \mathbf{F}^{m,n} = mn$ define

$$X_{i,j} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & 1 & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where every entry is 0 expect the $A_{i,j}$ entry which is equal to 1. Now, its easy to see that every $Z \in \mathbf{F}^{m,n}$ can be written as some linear combination of $X_{i,j}$'s. Thus, $F^{m,n} = \text{span}\{X_{i,j}\}$ where i, j vary with $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Also, every matrix with only 0 as its entry can only be written as linear combination of $X_{i,j}$ with all of its scalars equal to 0. Since, there are mn entries the dimension of $\mathbf{F}^{m,n}$ is equal to mn. □

1.3.3 Matrix Multiplication

Definition 1.13. Suppose A is a $m \times n$ matrix and B is $n \times p$ matrix. Then AB is defined as to be an $m \times p$ matrix whose entry in row j and column k is given by

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{k,r}$$

Remark. The motivation for us to define the product like this comes from questioning, does $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$? Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is the basis of W. Suppose $u_1, \cdots, u_k$ is the basis of U then consider the map $T : U \to V$ and $S : V \to W$. Suppose $\mathcal{M}(S) = A$ and $\mathcal{T} = B$. Then

$$\begin{aligned} (ST)u_k &= S\left(\sum_{r=1}^{n} B_{r,k} v_r\right) \\ &= \sum_{r=1}^{n} B_{r,k} S v_r \\ &= \sum_{r=1}^{n} B_{r,k} \sum_{j=1}^{m} A_{j,r} w_j \\ &= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} A_{j,r} B_{r,k}\right) w_j \end{aligned}$$

That is how we define $M(ST)$ and that is why **Definition 1.13.** makes sense.

Proposition 1.11. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then

$$\mathcal{M}(ST) = \mathcal{M}(S)\,\mathcal{M}(T)$$

Proof. It follows from our remark and how we defined the product of the matrix. ☐

Definition 1.14. Suppose A is a $m \times n$ matrix then

1. If $1 \leq j \leq m$ then $A_{j,\cdot}$ denotes the $1 \times n$ matrix consisting of row j of A.
2. If $1 \leq j \leq n$ then $A_{\cdot,j}$ denotes $m \times 1$ matrix consisting of column j of A.

Example :
Suppose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then

$$A_{1,\cdot} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$A_{\cdot,3} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Theorem 1.6. *Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. Then*

$$(AB)_{j,k} = A_{j,\cdot}.B_{\cdot,k}$$

where $1 \leq j \leq m$ and $1 \leq k \leq p$.

Proof. The definition of matrix multiplication states that

$$\begin{aligned}(AB)_{j,k} &= \sum_{r=1}^{n} A_{j,r}B_{r,k} \\ &= A_{j,1}B_{1,k} + \cdots + A_{j,n}B_{n,k}\end{aligned}$$

Now, if you take $A_{j,\cdot}$ and $B_{\cdot,k}$ and multiply it out you'll get the same thing. ☐

Theorem 1.7. *Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. Then*

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

for $1 \leq k \leq p$.

Proof. Both of the matrix have size $m \times 1$. The j-th row of $(AB)_{\cdot,k}$ has the element $(AB)_{j,k}$ and the j-th row of $AB_{\cdot,k}$ has element $A_{j,1}B_{1,1} + A_{j,2}B_{2,1} + \cdots A_{j,n}B_{n,1}$. Thus, from our previous theorem they're equal. ☐

Remark. The row version of this is

$$(AB)_{k,\cdot} = A_{k,\cdot}B$$

Theorem 1.8. *Suppose A is a $m \times n$ matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is a $n \times 1$ matrix. Then,*

$$Ab = b_1A_{\cdot,1} + \cdots + b_nA_{\cdot,n}$$

Proof. They both have same size and the entries of of Ab is the same as of the right side. ☐

Theorem 1.9. *Suppose C is an $m \times c$ matrix and R is a $c \times n$ matrix*

1. *The column k of CR is the linear combination of the columns of C, with coefficients of this linear combination coming from column k of R.*
2. *Then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.*

Proof. Use **Theorem 1.7.** and **Theorem 1.8.** for 1. and we'll prove 2. in the exercise section. □

1.3.4 Column-Row Factorization and Rank of a Matrix

Definition 1.15. Suppose A is a $m \times n$ matrix with entries in $\mathbf{F}$.

1. The **column rank** of A is the dimension of the span of columns of A in $\mathbf{F}^{1,m}$.
2. The **row rank** of A is the dimension of the span of rows of A in $\mathbf{F}^{n,1}$.

Example : Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then the column rank is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}\right)$$

and the row rank is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}\right)$$

Definition 1.16. The *transpose* of a $m \times n$ matrix A, denoted by A^t, is the $n \times m$ matrix whose entries are given by

$$(A^t)_{i,j} = A_{j,i}$$

Theorem 1.10. *Suppose A is an $m \times n$ matrix with entries in $\mathbf{F}$ and column rank $c \geq 1$. Then there exists a $m \times c$ matrix C and $c \times n$ matrix R, both with entries in F, such that $A = CR$.*

Proof. The list $A_{\cdot,1}, \ldots, A_{\cdot,n}$ of columns of A can be reduced to a basis of the span of the columns of A. This basis has length c by definition of column rank. The c columns can be put together to form $m \times c$.
Now, each column k of A is a linear combination of columns of C. Make the coefficients of this linear combination column k of R. This matrix R has size $c \times n$. Thus, $A = CR$ follows form **Theorem 1.9.**(a). □

Theorem 1.11. *Suppose $A \in \mathbf{F}^{m,n}$ then the column rank of A equals row rank of A.*

Proof. Let c be the column rank of A. Then $A = CR$ by the previous theorem where C and R are the matrix whose size are $m \times c$ and $c \times n$ respectively. Now, from textbfTheorem 1.9. (b) each row of A is a linear combination of rows of R. Since, R has c columns this implies that

$$\operatorname{rowrank} A \leq c = \operatorname{columnrank} A$$

Now applying the same thing to A^t we get

$$\begin{aligned} \operatorname{columnrank} A &= \operatorname{rowrank} A^t \\ &\leq \operatorname{columnrank} A^t \\ &= \operatorname{rowrank} A \end{aligned}$$

Thus, we're done. $\square$

Remark. From now on, we'll limit our use our terminology of "row rank" and "column rank" to just "rank".

1.3.5 Exercise

Problem : Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of basis of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Solution : Let $v_1, \ldots, v_n$ be the the basis of V and $w_1, \ldots, w_m$ be the basis of W then suppose k of those vectors are 0 under T and let those vectors be $v_1, \ldots, v_k$. Thus,

$$\operatorname{range} T = \operatorname{span}\{Tv_{k+1}, \ldots, Tv_n\}$$

Thus, $\dim \operatorname{range} T \leq n-k$. But since $T(v_j) \neq 0$ for each $k+1 \leq j \leq n$, there must be one entry thats not 0 for each $T(v_j)$. Since, the number of $T(v_j) \neq 0$ are exactly $n-k$ and $n-k \geq \dim \operatorname{range} T$ this means there is at least $\dim \operatorname{range} T$ nonzero entries in matrix of T.

Problem : Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all the entries of $\mathcal{M}(T)$ is 1.

Solution : Let us first prove ($\Leftarrow$). Suppose $A_{i,j} = 1$ for all i, j. That means, $T(v_i) = \sum w_j$ where $v_1, \ldots, v_n$ is the basis of V and $w_1, \ldots, w_m$ is the basis of W. Thus, $Tv_1 = Tv_2 = \cdots = Tv_n = k$ and $\operatorname{range} T = \operatorname{span}\{Tv_1, \cdots, Tv_n\} = \operatorname{span}\{k\} \implies \dim \operatorname{range} T = 1$.
Now, for the ($\Rightarrow$) we use the Fundamental theorem of linear maps,

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

$$\implies \dim V = \dim \operatorname{null} T + 1$$

Now, suppose $v_2, \ldots, v_n$ be that basis of $\operatorname{null} T$. Then extend this basis to V, suppose $v_2, \ldots, v_n, v$ then

$$T(v_2) = T(v_3) = \cdots = T(v_n) = 0$$
$$\implies T(v) = T(v_2 + v) = \cdots = T(v_n + v)$$

One can check that $v_2 + v, \ldots, v$ is a basis of V(as its linearly independent and has length n). Now, since $\dim \operatorname{range} T = 1$ we have $T(x) = \lambda T(v)$ and we choose $T(v), w_2, \ldots, w_m$ as our basis for W. Now, we use a clever trick and set $w_1 = T(v) - w_2 - w_3 - \cdots - w_m$ and notice that $w_1, \cdots, w_m$ is a basis of W. Thus,

$$T(v_i) = T(v) = \sum w_j$$

Thus, we're done.

Problem : Suppose that $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution : Take the the basis of $\mathcal{P}_3(\mathbf{R})$ to be $z, \frac{z^2}{2}, \frac{z^3}{3}, 1$ and $\mathcal{P}_2(\mathbf{R})$ to be $1, z, z^2$.

Problem : Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k, equal 1 if $1 \leq k \leq$ dim range T.

Solution : Let $\dim V = n$ and $\dim \operatorname{range} T = m$. Now, let $v_{m+1}, \cdots, v_n$ be the basis of $\dim \operatorname{null} T$. Now, extend these basis such in the following way

$$(v_1, \ldots, v_m, v_{m+1}, \ldots, v_n)$$

Here, $T(v_i) \neq 0$ for $1 \leq i \leq m$. Now, since $(v_1, \cdots, v_m, v_{m+1}, \cdots, v_n)$ spans V the list $(Tv_1, , \ldots, Tv_m)$ must span $\operatorname{range} T$ and in fact it is the basis of $\operatorname{range} T$ (one can check that its linearly independent). We can now extend this basis to the basis of W. Suppose $(Tv_1, \ldots, Tv_m, w_1, \ldots, w_k)$ is the basis of W. Then,

$$T(v_i) = 0 \cdot T(v_1) + \cdots + 1 \cdot T(v_i) + \cdots + 0 \cdot w_k$$

for $1 \leq i \leq m = \dim \operatorname{range} T$. But for $i > \dim \operatorname{range} T$ we have

$$0 = T(v_i) = 0 \cdot T(v_1) + 0 \cdot T(v_2) + \cdots + 0 \cdot w_k$$

Problem : Suppose $\sigma_1, ..., \sigma_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, ..., w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $\sigma_1, ..., \sigma_m$ and $w_1, ..., w_n$] are 0 except for possibly a 1 in the first row, first column.

Solution : We know that $\operatorname{range} T = \operatorname{span}\{T(\sigma_1), T(\sigma_2), \cdots, T(\sigma_m)\}$. Thus, we can make this span a basis. If $T(\sigma_1) = 0$ then we're done but if not then the basis of $\operatorname{range} T$ would be

$$(T(\sigma_1), z_2, \ldots, z_k)$$

Now, we can extend this basis to the basis of W, suppose its

$$(T(\sigma_1), z_2, \ldots, z_k, s_{k+1}, \ldots, s_m)$$

then

$$T(\sigma_1) = 1 \cdot T(\sigma_1) + 0 \cdot z_2 + \cdots + 0 \cdot s_m$$

Problem : Suppose $w_1, ..., w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $\sigma_1, ..., \sigma_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases $\sigma_1, ..., \sigma_m$ and $w_1, ..., w_n$] are 0 except for possibly a 1 in the first row, first column.

Solution : Take any basis $v_1, \ldots, v_m$ of V. Then, suppose

$$\begin{aligned} T(v_i) &= \sum_{j=1}^{n} {}_i\lambda_j w_j \\ &= {}_i\lambda_1 w_1 + \sum_{j=2}^{n} {}_i\lambda_j w_j \end{aligned}$$

Now, if all the $T(v_i)$ has 0 as the coefficient of w_1 then we're done. If not then take a v_k for which ${}_k\lambda_1 \neq 0$ then swap it with v_1. Then, define

$$\begin{aligned}\sigma_1 &= \frac{v_1}{{}_1\lambda_1}\\ \sigma_i &:= v_i - \frac{{}_i\lambda_1}{{}_1\lambda_1} v_1 \quad \text{for } i \geq 2\end{aligned}$$

Now, one can check that $(\sigma_1, \ldots, \sigma_m)$ is a basis and

$$\begin{aligned}T(\sigma_i) &= T(v_i) - \frac{{}_i\lambda_1}{{}_1\lambda_1} T(v_k)\\ &= 0 \cdot w_1 + \sum b_j w_j\end{aligned}$$

Problem : Give an example of 2×2 matrices A and B such that $AB \neq BA$.

Solution :

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Problem : Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, F are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that

$$A(B + C) = AB + AC \quad \text{and} \quad (D + E)F = DF + EF.$$

Solution : If $A(B + C)$ and $(D + E)F$ makes sense then B and C must be of the same size and D and E must be of the same size. Also, the number of columns in A must be same as the number of rows in B and C. Let A be a $n \times p$ matrix and $X = B + C$ then

$$\begin{aligned}(AX)_{i,j} &= \sum_{r=1}^{n} A_{i,r} X_{r,j}\\ &= \sum_{r=1}^{n} A_{i,r}(B_{r,j} + C_{r,j})\\ &= \sum_{r=1}^{n} A_{i,r} B_{r,j} + \sum_{r=1}^{n} A_{i,r} C_{r,j}\\ &= (AB)_{i,j} + (AC)_{i,j}\end{aligned}$$

Thus, $A(B + C) = AB + AC$. Similar proof works for $(D + E)F$.

Problem : Prove that matrix multiplication is associative. In other words, suppose A, B, C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $(AB)C$ makes sense and prove that

$$(AB)C = A(BC).$$

Solution : To make $(AB)C$ sense, we need A to have same number of columns as the number of rows in B. Also, we need B to have same number of columns as number of rows in C. To prove the associativity, you can just definition of matrix multiplication.

Problem : Suppose A is an $n \times n$ matrix and $1 \le j, k \le n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{r=1}^{n} \sum_{i=1}^{n} A_{j,r} A_{r,i} A_{i,k}.$$

Solution : It follows directly from the definition of matrix multiplication.

Problem : Suppose m and n are positive integers. Prove that the function $A \mapsto A^t$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Solution : Define $T : \mathbf{F}^{m,n} \to \mathbf{F}^{n,m}$ by $T(A) = A^t$. We must show that T is linear, i.e.

$$T(A+B) = T(A) + T(B) \quad \text{and} \quad T(\lambda A) = \lambda T(A),$$

for all $A, B \in \mathbf{F}^{m,n}$ and all scalars $\lambda \in \mathbf{F}$.
By definition of the transpose,

$$(A^t)_{ij} = A_{ji}, \qquad (1 \le i \le n,\ 1 \le j \le m).$$

Now let $A, B \in \mathbf{F}^{m,n}$. Then for each i, j,

$$\big((A+B)^t\big)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^t)_{ij} + (B^t)_{ij} = \big(A^t + B^t\big)_{ij}.$$

Hence $(A+B)^t = A^t + B^t$.
Similarly, for $\lambda \in \mathbf{F}$,

$$\big((\lambda A)^t\big)_{ij} = (\lambda A)_{ji} = \lambda A_{ji} = \lambda (A^t)_{ij} = \big(\lambda A^t\big)_{ij}.$$

So $(\lambda A)^t = \lambda A^t$.
Therefore T preserves both addition and scalar multiplication. Thus T is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Problem : Prove that if A is an $m \times n$ matrix and C is an $n \times p$ matrix, then

$$(AC)^t = C^t A^t.$$

Solution :

Solution : Let $A \in \mathbf{F}^{m,n}$ and $C \in \mathbf{F}^{n,p}$. By definition of matrix multiplication, the (i,j)-entry of AC is

$$(AC)_{ij} = \sum_{k=1}^{n} A_{ik} C_{kj}, \qquad (1 \le i \le m,\ 1 \le j \le p).$$

Taking the transpose, we get

$$\big((AC)^t\big)_{ij} = (AC)_{ji} = \sum_{k=1}^{n} A_{jk} C_{ki}.$$

On the other hand, consider the product C^tA^t. Here C^t is $p \times n$ and A^t is $n \times m$, so C^tA^t is $p \times m$. Its (i,j)-entry is

$$(C^tA^t)_{ij} = \sum_{k=1}^{n} (C^t)_{ik}(A^t)_{kj}.$$

By definition of the transpose,

$$(C^t)_{ik} = C_{ki}, \qquad (A^t)_{kj} = A_{jk}.$$

Hence

$$(C^tA^t)_{ij} = \sum_{k=1}^{n} C_{ki}A_{jk}.$$

But scalar multiplication in $\mathbf{F}$ is commutative, so

$$\sum_{k=1}^{n} C_{ki}A_{jk} = \sum_{k=1}^{n} A_{jk}C_{ki}.$$

Therefore,

$$(C^tA^t)_{ij} = \big((AC)^t\big)_{ij}, \qquad (1 \le i \le p,\ 1 \le j \le m).$$

Since all entries are equal, we conclude

$$(AC)^t = C^tA^t.$$

Problem : Suppose A is an $m \times n$ matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that

$$A_{j,k} = c_j d_k \quad \text{for every } j = 1, \ldots, m \text{ and every } k = 1, \ldots, n.$$

Solution : For ($\Leftarrow$), Use **Theorem 1.10.** then use the definition of matrix multiplication. For ($\Rightarrow$), the matrix

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

can produce any column thus the rank is 1.

Problem : Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent:

(a) T is injective.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

Solution : Will do later.

1.4 Invertibility and Isomorphism

1.4.1 Invertible Linear Maps

Definition 1.17. A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals identity operator on V and TS equals identity operator on W.

Definition 1.18. A linear map $S \in \mathcal{L}(V, W)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of T.

Proposition 1.12. An invertible map has an unique inverse.

Proof. Suppose $T \in \mathcal{L}(V, W)$ and let S_1 and S_2 be its inverses then

$$S_1 = S_1 I = S_1(TS_2) = (S_1T)S_2 = IS_2 = S_2$$

□

Remark. Since inverses are unique for a invertible map T, we will denote it by T^{-1}.

Proposition 1.13. A linear map is invertible if and only if it is injective and surjective.

Proof. Suppose $T \in \mathcal{L}(V, W)$ is an invertible map and suppose $T(v) = T(w)$ then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

Hence, T is injective. To prove surjectivity, notice that

$$w = T^{-1}(Tw)$$

which proves T is surjective.
Now, suppose T is injective and surjective. Then, there exists a unique element $S(w)$ such that

$$T(S(w)) = w$$

the uniqueness is due to the injectivity of T. Let us show that, $S \in \mathcal{L}(W, V)$

$$\begin{aligned} T(S(w_1) + S(w_2)) &= T(S(w_1)) + T(S(w_2)) \\ &= w_1 + w_2 \\ &= T(S(w_1 + w_2)) \end{aligned}$$

Thus, $S(w_1) + S(w_2) = S(w_1 + w_2)$. Also,

$$\begin{aligned} T(\lambda S(w)) &= \lambda T(S(w)) \\ &= \lambda w \\ &= T(S(\lambda w)) \end{aligned}$$

Thus, $\lambda S(w) = S(\lambda w)$. □

Now, by how we defined S, it implies that $TS = I$ on W. Also,

$$T(ST)v = (TS)(T)v = Tv$$

$$\implies (ST)v = v$$

Thus, ST is an identity operator on V.

Proposition 1.14. Suppose that V and W are finite-dimensional vector spaces, such that, $\dim W = \dim V$ and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible } \iff T \text{ is injective } \iff T \text{ is surjective}$$

Proof. From the Fundamental theorem of linear maps,

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

If T is injective then $\operatorname{null} T = \{0\}$. Thus

$$\dim V = \dim W = \dim \operatorname{range} T$$

$$\implies \operatorname{range} T = W$$

Now, if T is surjective then $\operatorname{range} T = W$. Thus

$$\dim V = \dim \operatorname{null} T + \dim W$$

$$\implies \dim \operatorname{range} T = 0$$
$$\implies \operatorname{range} T = \{0\}$$

Thus, T is injective $\iff$ T is surjective. From **Proposition 1.13.** we get our final result. ☐

Proposition 1.15. Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(V, W)$. Then, $ST = I \iff TS = I$.

Proof. First $ST = I$ then take $v \in \operatorname{null} T$. Thus,

$$v = STv = S(0) = 0$$

Thus, $\operatorname{null} T = \{0\}$ and T is injective. Since $\dim V = \dim W$, this implies T is invertible. Thus, there exists a T^{-1}. Now,

$$T^{-1} = (ST)(T^{-1}) = S$$

We can now apply the same idea for ($\Leftarrow$) of the proof. We just need to swap V with W, and T with S. ☐

1.4.2 Isomorphic Vector Spaces

Definition 1.19. An *isomorphism* is an invertible linear map and two vector spaces are isomorphic if there is an isomorphism between them.

Proposition 1.16. Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

Proof. Suppose V and W are isomorphic. Then there exists a injective and surjective map T from V to W. Thus, $\operatorname{null} T = \{0\}$. Then

$$\dim V = \dim \operatorname{range} T$$

Also, since T is surjective $\operatorname{range} T = W$. Then

$$\dim V = \dim W$$

Now, suppose $\dim W = \dim V$. Define $T : V \to W$ as

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

where v_i's and w_i's are the basis of V and W respectively. One can check this is a linear map. Now, this map is surjective as $\sum c_i w_i$ covers W. Also, $\operatorname{null} T = \{0\}$ as

$$\begin{aligned}\dim W = \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \operatorname{null} T + \dim W\end{aligned}$$

Thus, T is injective and surjective which means that V and W are isomorphic. □

Proposition 1.17. Suppose $v_1, \ldots, v_n$ be the basis of V and $w_1, \ldots, w_m$ be the basis of W. Then $\mathcal{M}(T)$ is a isomorphism between $\mathcal{L}(V, W)$ to $\mathbf{F}^{m,n}$

Proof. We know that $\mathcal{M}(T)$ is a linear map as

$$\mathcal{M}(T+S) = \mathcal{M}(T) + \mathcal{M}(S) \quad \text{and} \quad \mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$$

We need to prove that $\mathcal{M}$ is injective and surjective. We know that $\mathcal{M}(T)$ is injective $\iff$ $\operatorname{null} \mathcal{M}(T) = \{0\}$. And we know $\mathcal{M}(T) = 0 \iff T(x) = 0$ for all $x \in V$. Thus, $T = 0$.
To prove $\mathcal{M}(T)$ is surjective. We know that there exists a $T \in \mathcal{L}(V, W)$ such that

$$T(v_k) = \sum_{i=1}^{m} A_{i,j} w_j$$

which proves the surjectivity of $\mathcal{M}(T)$. □

Proposition 1.18. Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Proof. Use **Proposition 1.17.** and **Proposition 1.16.** and

$$\dim \mathcal{L}(V, W) = mn = (\dim V)(\dim W)$$

□

1.4.3 Linear Map Thought of as Matrix Multiplication

Definition 1.20. Suppose $v \in V$ and $v_1, \ldots, v_n$ is the basis of V. The matrix of v with respect to the basis is the n matrix

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

where $b_1, \ldots, b_n$ are scalar such that $v = b_1 v_1 + \cdots + b_n v_n$.

Proposition 1.19. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n$ is a basis of W. Let $1 \leq k \leq n$. Then,

$$\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$$

Proof. Immediate from the definition of $\mathcal{M}(Tv_k)$. □

Proposition 1.20. Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Let $v_1, \ldots, v_n$ be the basis of V and $w_1, \ldots, w_m$ be the basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\,\mathcal{M}(v)$$

Proof. Suppose $v = b_1 v_1 + \cdots + b_n v_n$. Then,

$$Tv = b_1 Tv_1 + \cdots + b_n Tv_n$$

Hence,

$$\begin{aligned}
\mathcal{M}(Tv) &= b_1\,\mathcal{M}(Tv_1) + \cdots + b_n\,\mathcal{M}(Tv_n) && \textit{(Linearity of } \mathcal{M}\textit{)}\\
&= b_1\,\mathcal{M}(T)_{\cdot,1} + \cdots + b_n\,\mathcal{M}(T)_{\cdot,n} && \textit{(Proposition 1.19.1)}\\
&= \mathcal{M}(T)\,\mathcal{M}(v) && \textit{(Theorem 1.8.)}
\end{aligned}$$

□

Proposition 1.21. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

Proof. Suppose $v_1, \ldots, v_n$ be the basis of V and $w_1, \ldots, w_m$ be the basis of W. Now, define $\varphi : W \to \mathbf{F}^{m,1}$ as $\varphi(w) = \mathcal{M}(w)$. One can prove that this is an isomorphism. If we restrict our domain to just $\operatorname{range} T$ we see that our co-domain is going to be $\mathcal{O} = \operatorname{span}\{\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_m)\}$. Also,

$$\varphi\,|_{\operatorname{range} T} : \operatorname{range} T \to \mathcal{O}$$

is a isomorphism and since isomorphism preserves dimension. We have

$$\dim \operatorname{range} T = \dim \mathcal{O} = \text{column rank of } T$$

□

1.4.4 Change of Basis

Definition 1.21. We define the $n \times n$ matrix, called *identity matrix* by

$$A_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The identity matrix is denoted by I.

Definition 1.22. A square matrix is called *invertible* if there is a square matrix B of the same size such that

$$AB = BA = I$$

we call the matrix B the *inverse* of A.

Remark. The inverse of a square matrix A is unique and therefore will be denoted by A^{-1}. Here, is a short proof of the uniqueness of the inverse. Suppose A has two inverses B_1 and B_2. Thus,

$$B_1 = IB_1 = (B_2A)B_1 = B_2(AB_1) = B_2I = B_2$$

Also, $(A^{-1})^{-1} = I$ and $(AC)^{-1} = C^{-1}A^{-1}$. You can verify these.

Definition 1.23. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. If $u_1, \ldots, u_m$ is a basis of U, $v_1, \ldots, v_n$ is a basis of V, and $w_1, \ldots, w_p$ is the basis of W then

$$\mathcal{M}(ST, (u_1, \ldots, u_m), (w_1, \ldots, w_p)) = \mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p)) \\ \mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_n))$$

This is just the matrix multiplication which we had defined earlier but with respect to the basis. See **Proposition 1.11.**

Proposition 1.22. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the basis of V. Then the matrices

$$\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) \quad \text{and} \quad \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$$

are inverses of each other. Here, I is the identity operator.

Proof. Use **Definition 1.23.** and replace w_k with u_k. And replace S, T with the identity operator. Then

$$I = \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))\, \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$$

Now interchange the roles of u's and v's to get

$$I = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))\, \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$$

□

Remark. For convenience, we'll write

$$\mathcal{M}(T, (u_1, \ldots, u_n), (u_1, \ldots, u_n)) = \mathcal{M}(T, (u_1, \ldots, u_n))$$

Proposition 1.23. Suppose $T \in \mathcal{L}(V)$. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the basis of V. Let

$$A = \mathcal{M}(T, (u_1, \ldots, u_n)) \quad \text{and} \quad B = \mathcal{M}(T, (v_1, \ldots, v_n))$$

and $C = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$. Then,

$$A = C^{-1}BC$$

Proof. Use **Definition 1.23.** and replace w_k with u_k and S with I. Then, use **Proposition 1.22.** to get

$$A = C^{-1}\, \mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) \tag{1}$$

Now, again use the definition and this time replace w_k with v_k. Then

$$\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) = BC$$

We can now substitute this equation in equation (1) to get

$$A = C^{-1}BC$$

□

Proposition 1.24. Suppose that $v_1, \ldots, v_n$ is the basis of V and $T \in \mathcal{L}(V)$ is invertible. Then, $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to basis $v_1, \ldots, v_n$.

Proof. Use **Definition 1.23.** □

1.4.5 Exercise

Problem : Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T$$

Solution : Since, T is invertible, we have

$$TT^{-1} = I \quad \text{and} \quad T^{-1}T = I$$

If we switch our perspective from T to T^{-1}, we get that T is invertible from **Definition 1.17.** and from **Proposition 1.12.** we have

$$(T^{-1})^{-1} = T$$

Problem : Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Solution : Since, S and T are both invertible then S^{-1} and T^{-1} both exist. Also, $T^{-1}S^{-1} \in \mathcal{L}(W, U)$. Thus,

$$\begin{aligned}(ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\ &= S(I)S^{-1} \\ &= SS^{-1} \\ &= I\end{aligned}$$

One can do the same thing for $(T^{-1}S^{-1})(ST)$. Thus, ST is invertible and from the above calculation so we can say $(ST)^{-1} = T^{-1}S^{-1}$.

Problem : Suppose V is finite-dimensional and $V \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) T is invertible

(b) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.

(c) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

Solution : Suppose T is invertible then it is also injective and surjective. Let $v_1, \ldots, v_n$ be a basis of V. Then, we know that $\text{span}\{Tv_1, \ldots, Tv_n\} = V$ because of the surjectivity. Also, if

$$\begin{aligned} a_1Tv_1 + \cdots + a_nTv_n &= 0 \\ \implies T(a_1v_1 + \cdots + a_nv_n) &= 0 \\ \implies a_1v_1 + \cdots + a_nv_n &= 0 \\ \implies a_1 = a_2 = \cdots = a_n &= 0 \end{aligned}$$

The last line is from injectivity of T. Thus, $Tv_1, \ldots, Tv_n$ is a basis of V for any basis of V.

Now, suppose $Tv_1, \dots, Tv_n$ is a basis of V for every basis $v_1, \dots, v_n$ of V. Then, (c) automatically holds. Also,

$$a_1Tv_1 + \ldots + a_nTv_n = 0$$
$$\implies a_1 = a_2 = \cdots = a_n = 0$$

Thus, $\operatorname{null} T = \{0\}$ which implies T is injective. Now, since $Tv_1, \dots, Tv_n$ is a basis, every element of V can be written as some combination of V. Thus,

$$a_1Tv_1 + \cdots + a_nTv_n = y$$
$$\implies T(a_1v_1 + \cdots + a_nv_n) = y$$

Thus, for every $y \in V$ there exists some element which gets mapped to y. Thus, T is surjective. Thus, from **Proposition 1.13.** we get that T is invertible.
Now, suppose $Tv_1, \dots, Tv_n$ is a basis of V for some basis $v_1, \dots, v_n$ of V. Then we can apply the same argument as we did for above to get to T is invertible. Since, T is invertible we get (b).

Problem : Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of non-invertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Solution : We can construct two non-invertible linear maps which form a invertible map when added. Suppose $v_1, \dots, v_n$ is a basis of V.

$$T(a_1v_1 + \cdots + a_nv_n) = a_2v_2 + \cdots + a_nv_n$$
$$S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$$

One can check that both of them are linear maps and both of them lack injectivity property so they're not invertible. But

$$(S+T)(a_1v_1 + \cdots + a_nv_n) = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$
$$(S+T)(x) = I(x)$$

which is a invertible linear map. Thus, set of non-invertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Remark. We used the $\dim V > 1$ when we defined T and S.

Problem : Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists a invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution : For ($\Rightarrow$), if $S(x) = S(y)$ then $T(x) = T(y)$ which implies $x = y$ because T is invertible. Now, for ($\Leftarrow$) choose a basis of U and extend it to the basis of V say $\mathcal{B} = (u_1, \dots, u_k, v_{k+1}, \dots, v_n)$, here $n = \dim V$. Now since S is injective, the list $(Su_1, \dots, Su_k)$ is linearly independent and can be extended to a basis of V. Let

$$\mathcal{C} = \{Su_1, \dots, Su_k, w_{k+1}, \dots, w_n\}$$

be the basis of V. Define $T : V \to V$ as following

$$T(u_i) = S(u_i) \text{ for } 1 \le i \le k \quad \text{and} \quad T(v_j) = w_j \text{ for } k+1 \le j \le n$$

One can check this is a invertible linear map.

Problem : Suppose W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S = \operatorname{null} T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

Solution : For $(\Leftarrow)$, take $x \in \operatorname{null} T$ then $S(x) = E(T(x)) = 0$ thus $x \in \operatorname{null} S$. Now, if $x \in \operatorname{null} S$ then $0 = S(x) = ET(x) \implies T(x) = 0$ thus $x \in \operatorname{null} T$. Thus, $\operatorname{null} T = \operatorname{null} S$. I'll do the $\Leftarrow$ later.

1.5 Product and Quotients of Vector Spaces

1.5.1 Products of Vector Spaces

Definition 1.24. Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$.

- The product $V_1 \times \cdots \times V_m$ is defined by
$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V, \dots, v_m \in V\}$$
- Addition on $V_1 \times \cdots \times V_m$ is defined by
$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$$
- Scalar Multiplication on $V_1 \times \cdots \times V_m$ is defined by
$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$$

Proposition 1.25. Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbf{F}$.

Proof. Just check the vector axioms. □

Proposition 1.26. Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and
$$\dim(V_1 \times \cdots V_m) = \dim V_1 + \cdots + \dim V_m$$

Proof. Choose a basis of V_k and consider every element of $V_1 \times \cdots \times V_k$ that equals a element from the basis of the vector V_k in the k-th sloth and 0 in others. The list of vector spans $V_1 \times \cdots \times V_m$ and is linearly independent. Thus, it is the basis of $V_1 \times \cdots \times V_m$. The length of the basis is $\dim V_1 + \cdots + \dim V_m$. □

Proposition 1.27. Suppose that $V_1, \dots, V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by
$$\Gamma(v_1, \cdots, v_m) = v_1 + \cdots + v_m$$
Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

Proof. If $V_1 + \cdots + V_m$ is a direct sum then the only way we can write 0 is by choosing 0 from each V_i. Thus,
$$\Gamma(v_1, \dots, v_m) = 0 \iff v_1 = v_2 = \cdots = v_m = 0$$
Thus, $\operatorname{null}\Gamma = \{0\}$ which implies Γ is injective.
Now, suppose Γ is injective then $\operatorname{null}\Gamma = \{0\}$, which means that the only element that gets mapped to 0 is $(0, \dots, 0)$. Thus, the only way to write 0 is by choosing 0 from each V_i. Thus, $V_1 + \cdots + V_m$ is a direct sum. □

Proposition 1.28. Suppose V is finite-dimensional and $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is direct sum if and only if
$$\dim(V_1 + \cdots V_m) = \dim V_1 + \cdots \dim V_m$$

Proof. The map Γ is surjective. And $V_1 + \cdots + V_m$ is a direct sum
$$\begin{aligned} &\iff \Gamma \text{ is injective} \\ &\iff \operatorname{null}\Gamma = \{0\} \\ &\iff \dim(V_1 \times \cdots \times V_m) = \dim(V_1 + \cdots + V_m) \end{aligned}$$
Combining the result of **Proposition 1.26.** we get our desired result. □

1.5.2 Quotients Spaces

Definition 1.25. Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is a subset of V defined by

$$v + U = \{v + u \mid u \in U\}$$

Definition 1.26. For $v \in V$ and U a subset of V, the set $v + U$ is said to be a *translate* of U.

Definition 1.27. Suppose U is a subspace of V. Then the *quotient space* V/U is the set of all translate of U,

$$V/U = \{v + U \mid v \in V\}$$

Proposition 1.29. Suppose U is a subspace of V and $v, w \in V$ then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (u + V) \neq \emptyset$$

Proof. Suppose $v - w \in U$ then $v = w + u'$ for some $u \in U$ thus, $v + u = w + (u' + u) \in w + U$ which implies $v + U \subseteq w + U$. Thus, similarly $w + U \subseteq v + U \implies v + U = w + U \implies (v + U) \cap (w + U) \neq \emptyset$.
Now, suppose $(v + U) \cap (w + U) \neq \emptyset$ then $v + u_1 = w + u_2$ for some $u_1, u_2 \in U$ which implies $v - w \in U$, which implies $v + U = w + U$. And $v + U = w + U \implies v - w \in U$. Thus, we proved every direction of the proof. □

Definition 1.28. Suppose U is a subspace of V. Then *addition* and *scalar multiplication* are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

for all $v, w \in V$ and all $\lambda \in F$.

Proposition 1.30. Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

Proof. Just use previous definitions and check the vector axioms. □

Proposition 1.31. Suppose U is a subspace of V. The *quotient map* $\pi : V \to V/U$ is a linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

Proof. Note that $\pi(a + b) = (a + b) + U = (a + U) + (b + U) = \pi(a) + \pi(b)$ and $\pi(\lambda a) = (\lambda a) + U = \lambda(a + U) = \lambda\pi(a)$. □

Proposition 1.32. Suppose V is finite-dimensional vector space and U is a subspace of V then

$$\dim V/U = \dim V - \dim U$$

Proof. We use the map quotient map π introduced before. We know that $a + U = 0 + U \iff a \in U$ this $\operatorname{null} \pi = U$ and $\operatorname{range} \pi = V/U$ as the map is surjective. Thus, using the fundamental theorem of linear map we get our desired result. □

Definition 1.29. Suppose $T \in \mathcal{L}(V,W)$. Define $\widetilde{T} : V/(\operatorname{null} T) \to W$ by

$$\widetilde{T}(v + \operatorname{null} T) = Tv$$

This map indeed is well-defined as $u + \operatorname{null} T = v + \operatorname{null} T \implies v - u \in \operatorname{null} T$ and thus $T(v-u) = 0 \implies Tv = Tu$. Also, the map is a linear map.

Proposition 1.33. Suppose $T \in \mathcal{L}(V,W)$ then

(a) $\widetilde{T} \circ \pi = T$ where π is the quotient map with $U = \operatorname{null} T$

(b) $\widetilde{T}$ is injective

(c) $\operatorname{range} T = \operatorname{range} \widetilde{T}$

(d) $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are isomorphic vector spaces.

Proof. We prove each point individually

(a) If $v \in V$ then $\widetilde{T} \circ \pi(v) = \widetilde{T}(v + \operatorname{null} T) = Tv$ as desired.

(b) If $\widetilde{T}(v + \operatorname{null} T) = 0$ then $Tv = 0$ thus $v \in \operatorname{null} T$. Thus, $v + \operatorname{null} T = 0 + \operatorname{null} T \implies \operatorname{null} \widetilde{T} = \{0 + \operatorname{null} T\}$.

(c) By definition of $\widetilde{T}$.

(d) From (b) and (c).

□

1.5.3 Exercise

Problem : Suppose T is a function from V to W. The graph of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}$$

Prove that T is a linear map if and only if graph of T is a subspace of $V \times W$.

Solution : For ($\Rightarrow$), you just need to check the vector axioms and just the definition of a linear map. For the ($\Leftarrow$), consider

$$(v, Tv) + (w, Tw) = (v + w, Tv + Tw)$$

Since, graph of T is a subspace and all of its element is of the form (v, Tv) for any v, it must be that $(v + w, T(v + w)) = (v + w, Tv + Tw)$. Similarly, $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$. And we're done.

Problem : Suppose $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \cdots, m$.

Solution : Let $\dim(V_1 \times \cdots \times V_m) = k$ and suppose $e_1, \ldots, e_k$ are the basis of $V_1 \times \cdots \times V_m$. Let $e_i = (e_{1i}, e_{2i}, \ldots, e_{mi})$ for $1 \leq i \leq k$. Notice that to cover elements of V_j, only the j-th component of e_i's play a role. Thus,

$$\operatorname{span}\{e_{j1}, e_{j2}, \ldots, e_{jm}\} = V_k$$

Since, every span can be reduced to a basis we get that each V_k are finite-dimensional.

Problem : Suppose $V_1, V_2, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Solution :

Solution. Let $U = V_1 \times \cdots \times V_m$. We want to prove that

$$\mathcal{L}(U, W) \cong \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W).$$

Define a map

$$\Phi : \mathcal{L}(U, W) \to \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$$

by

$$T \mapsto (T \circ i_1,\, T \circ i_2,\, \ldots,\, T \circ i_m),$$

where for each $j = 1, \ldots, m$, the map $i_j : V_j \to U$ is defined by

$$i_j(v) = (0, \ldots, 0, v, 0, \ldots, 0),$$

with v in the j-th position.
Now, for $T, T' \in \mathcal{L}(U, W)$ and scalars $\alpha, \beta \in \mathbf{F}$,

$$\Phi(\alpha T + \beta T') = ((\alpha T + \beta T') \circ i_1, \ldots, (\alpha T + \beta T') \circ i_m) = \alpha\Phi(T) + \beta\Phi(T').$$

Hence, Φ is linear.
For invertibility of the map, define

$$\Psi : U \to W$$

by

$$\Psi(S_1, \ldots, S_m)(v_1, \ldots, v_m) = S_1(v_1) + \cdots + S_m(v_m).$$

where $S_j \in \mathcal{L}(V_j, W)$. Now, one can check that $\Psi \circ \Phi(T) = T$ and $\Phi \circ \Psi(S_1, \cdots, S_2) = (S_1, \cdots, S_2)$. Thus, the map is invertible.

Problem : For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m}$$

Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic vector spaces.

Solution : Define a map from $\Phi : V^m \to \mathcal{L}(\mathbf{F}^m, V)$ by

$$(v_1, \cdots, v_m) \mapsto (a_1, \ldots, a_m) \mapsto a_1 v_1 + \cdots + a_m v_m$$

One can check that this is linear as well as invertible.

Problem : Suppose x, v are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Solution : Suppose u is a arbitrary vector in U, then $v + u = x + w$ for some $w \in W$. Thus, setting $u = 0$ gives $x - v = -w \in W$. Thus, for all $u \in U$, $u = x - v + w$ for some $w \in W$ since W is a subspace $x - v + w \in W$. Thus, $U \subseteq W$ and repeating a similar argument for U gives $W \subseteq W$.

Problem : Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$. Prove that A is a translate of U if and only if there exists $k \in \mathbf{R}$ such that

$$A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = k\}$$

Solution : Suppose A is a translate of U, then $(a, b, c) + U = A$ for some $(a, b, c) \in \mathbf{R}^3$. Thus

$$A = \{(x + a, y + b, z + c) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$$

Now let $x + a = x_1, y + b = y_1, z + c = z_1$ thus,

$$A = \{(x_1, y_1, z_1) \in \mathbb{R}^3 : 2x_1 + 3y_1 + 5z_1 = 2a + 3b + 5c\}$$

$$\implies A = \{(x_1, y_1, z_1) \in \mathbb{R}^3 : 2x_1 + 3y_1 + 5z_1 = k\}$$

Now, suppose $A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = k\}$ then $\left(\frac{k}{6}, \frac{k}{9}, \frac{k}{15}\right) + U = A$.

1.6 Duality

1.6.1 Dual Space and Dual Map

Definition 1.30. A *linear functional* on V is a linear map from V to $\mathbf{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

Definition 1.31. The *dual space* of V, denoted V', is the vector space of all linear functional on V. In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

Proposition 1.34. Suppose V is a finite-dimensional vector space. Then V' is also finite-dimensional and

$$\dim V' = \dim V$$

Proof. From **Proposition 1.18.** we have

$$\dim V' = \dim \mathcal{L}(V, \mathbf{F}) = (\dim V)(\dim \mathbf{F}) = \dim V$$

as desired. □

Definition 1.32. If $v_1, \ldots, v_n$ is a basis of V, then the *dual space* of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements in V' such that

$$\varphi_j(k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Proposition 1.35. Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

for each $v \in V$.

Proof. Let

$$v = c_1v_1 + \cdots + c_nv_n$$

Then, $\varphi_n(v) = c_n$ thus

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

as desired. □

Proposition 1.36. Suppose V is a finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

Proof. Suppose $v_1, \ldots, v_n$ is a basis of V and let $\varphi_1, \ldots, \varphi_n$ be the dual basis. To show $\varphi_1, \ldots, \varphi_n$ is linearly independent, suppose there exists $a_1, \ldots, a_n \in \mathbf{F}$ such that

$$a_1\varphi_1 + \cdots + a_n\varphi_n = 0$$

Then, $(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_k) = a_k$ for $1 \leq k \leq n$. Thus, $a_1 = a_2 = \cdots = a_n = 0$. And since the list is of the length $\dim V'$, we can conclude that the list is the basis of V'. □

Definition 1.33. Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T$$

Remark. Since T' is a composition of linear maps φ and T, it is a linear map as well. Also, $T'(\varphi) \in V'$ as T' as it takes an element from V to $\mathbf{F}$. Also, one can verify $T' \in \mathcal{L}(W', V')$.

Proposition 1.37. Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S+T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$,

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$,

(c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Proof. The proofs of (a) and (b) directly follow from the definitions. For (c),

$$(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = T'S'$$

The fourth equation is due to $\varphi \circ S \in W'$. □

1.6.2 Null Space and Range of Linear Map

Definition 1.34. For $U \subseteq V$, the annihilator of U, denoted by U^0, is defined by

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \text{ for all } u \in U\}$$

Proposition 1.38. Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

Proof. One can just check the vector axioms. □

Proposition 1.39. Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^0 = \dim V - \dim U$$

Proof. Let $i \in \mathcal{L}(U, V)$ be the linear map such that $i(u) = u$ for each $u \in U$. Thus, $i' \in \mathcal{L}(V', U')$ and from fundamental theorem of linear maps we have,

$$\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V' = \dim V$$

Also, $\operatorname{null} i' = \{\varphi \in V' \mid i'(\varphi) = 0\} = \{\varphi \in V' \mid \varphi \circ i = 0\} = \{\varphi \in V' \mid \varphi(x) = 0\} = U^0$. Thus, $\dim \operatorname{null} i' = \dim U^0$ and the equation above becomes

$$\dim \operatorname{range} i' + \dim U^0 = \dim V$$

If $\varphi \in U'$, then φ can be extended to a linear functional ϕ on V (**Exercise 10** of the first section). Thus, $i'(\phi) = \varphi$ and $\operatorname{range} i' = U'$. Hence

$$\dim \operatorname{range} i' = \dim U' = \dim U$$

And thus $\dim U + \dim U^0 = \dim V$ as desired. □

Proposition 1.40. Suppose V is finite-dimensional and U is a subspace of V. Then

(a) $U^0 = \{0\} \iff U = V$,

(b) $U^0 = V' \iff U = \{0\}$

Proof. For (a) we have,

$$\begin{aligned} U^0 = \{0\} &\iff \dim U^0 = 0 \\ &\iff \dim U = \dim V \\ &\iff U = V \end{aligned}$$

Similarly, to prove (b) we have

$$\begin{aligned} U^0 = V' &\iff \dim U^0 = \dim V' \\ &\iff \dim U^0 = \dim V \\ &\iff \dim U = 0 \\ &\iff U = \{0\} \end{aligned}$$

And we're done. □

Proposition 1.41. Let V and W be vector spaces and let $T \in \mathcal{L}(V, W)$. Then

$$\operatorname{null} T' = (\operatorname{range} T)^0$$

Proof. First suppose $\varphi \in \operatorname{null} T'$, then $T'(\varphi) = (\varphi \circ T)(x) = 0$ for every $x \in V$. Since, $\varphi(Tx) = 0$ we have $\varphi \in (\operatorname{range} T)^0$ and thus $\operatorname{null} T' \subseteq (\operatorname{range} T)^0$.
Now, suppose $\varphi \in (\operatorname{range} T)^0$ then $\varphi(Tv) = 0$ for all $v \in V$ which implies $T'(\varphi) = 0$. Thus, $\varphi \in \operatorname{null} T'$ and $(\operatorname{range} T)^0 \subseteq \operatorname{null} T'$ as desired. □

Proposition 1.42. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ then

$$\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$$

Proof. We have

$$\begin{aligned} \dim \operatorname{null} T' &= \dim (\operatorname{range} T)^0 \\ &= \dim W - \dim \operatorname{range} T \\ &= \dim W - (\dim V - \dim \operatorname{null} T) \\ &= \dim \operatorname{null} T + \dim W - \dim V \end{aligned}$$

□

Proposition 1.43. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is surjective} \iff T' \text{ is injective}$$

Proof. To prove this, we have

$$\begin{aligned} T \text{ is surjective} &\iff \operatorname{range} T = W \\ &\iff (\operatorname{range} T)^0 = \{0\} \\ &\iff \operatorname{null} T' = \{0\} \\ &\iff T' \text{ is injective} \end{aligned}$$

as desired. □

Proposition 1.44. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$,

(b) $\operatorname{range} T' = (\operatorname{null} T)^0$

Proof. For (a) we have,

$$\begin{aligned}\dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T' \\ &= \dim W - (\dim \operatorname{null} T + \dim W - \dim V) \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim \operatorname{range} T\end{aligned}$$

For (b), suppose $\varphi \in \operatorname{range} T'$ then there exists a $\phi \in W'$ such that $T'(\phi) = \varphi$. Thus, for all $v \in \operatorname{null} T$ we have

$$\varphi(v) = T'(\phi)v = (\phi \circ T)(v) = \phi(0) = 0$$

Thus, $\varphi \in (\operatorname{null} T)^0$. Thus, $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. Now, we'll complete the proof by showing $\dim \operatorname{range} T' = \dim(\operatorname{null} T)^0$. Note, that

$$\begin{aligned}\dim \operatorname{range} T' &= \dim \operatorname{range} T \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim(\operatorname{null} T)^0\end{aligned}$$

where the last equation is from **Proposition 1.39**. □

Proposition 1.45. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then,

$$T \text{ is injective} \iff T' \text{ is surjective}$$

Proof. We have

$$\begin{aligned}T \text{ is injective} &\iff \operatorname{null} T = \{0\} \\ &\iff (\operatorname{null} T)^0 = V' \\ &\iff \operatorname{range} T' = V'\end{aligned}$$

as desired. □

1.6.3 Matrix of Dual Linear Map

Proposition 1.46. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T', (\psi_1, \ldots, \psi_m), (\varphi_1, \ldots, \varphi_n)) = (\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_n)))^t$$

where $(\psi_1, \ldots, \psi_m)$ and $(\varphi_1, \ldots, \varphi_n)$ are the dual basis of W' and V' respectively.

Proof. Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Suppose $1 \leq j \leq m$ and $1 \leq k \leq n$. From the definition of $\mathcal{M}(T')$ we have

$$T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r.$$

The left side of the equation above equals $\psi_j \circ T$. Thus applying both sides of the equation above to v_k gives

$$(\psi_j \circ T)(v_k) = \sum_{r=1}^{n} C_{r,j}\varphi_r(v_k) = C_{k,j}.$$

We also have

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k) = \psi_j\left(\sum_{r=1}^{m} A_{r,k}w_r\right) = \sum_{r=1}^{m} A_{r,k}\psi_j(w_r) = A_{j,k}.$$

Comparing the last line of the last two sets of equations, we have $C_{k,j} = A_{j,k}$. Thus $C = A^{\mathrm{t}}$. In other words, $\mathcal{M}(T') = \mathcal{M}(T)^{\mathrm{t}}$, as desired. $\square$

2 Polynomials

2.1 Zeros of Polynomials

Definition 2.1. A number $\lambda \in \mathbf{F}$ is called a *zero*(or *root*) of a polynomial $p \in \mathcal{P}(\mathbf{F})$ if

$$p(\lambda) = 0$$

Proposition 2.1. Suppose m is a positive integer and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial of degree m. Suppose $\lambda \in \mathbf{F}$. Then $p(\lambda) = 0$ if and only if there exists a polynomial $q \in \mathcal{P}(\mathbf{F})$ of degree $m-1$ such that

$$p(z) = (z - \lambda)q(z)$$

for every $z \in \mathbf{F}$.

Proof. Not so hard. □

Proposition 2.2. Suppose m is a positive integer and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial of degree m. Then p has at most m roots in $\mathbf{F}$.

Proof. We'll use induction. For $m = 1$, it is quite straight forward, as the polynomial $a_0 + a_1 z$ only has one zero which is $-a_0/a_1$. Now, suppose the assumption holds for all polynomial with degree $m-1$. Let p be a polynomial of degree m, then if p has no zeros then we're done. Suppose $\lambda \in \mathbb{F}$ such that $p(\lambda) = 0$, then using our previous proposition we get

$$p(z) = (z - \lambda)q(z)$$

where $q(z)$ has degree $m-1$. This shows that zeros of p are exactly the zeros of $q(z)$ and λ, which is at most m. □

Remark. The result above implies that, coefficients of a polynomial are uniquely determined

2.2 Division Algorithm for Polynomials

Proposition 2.3. Suppose that $p, s \in \mathcal{P}(\mathbf{F})$ with $s \neq 0$. Then there exists a unique polynomials $q, r \in \mathcal{P}(\mathbf{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

Proof. Suppose $\deg p = n$ and $\deg s = m$. If $n < m$ then, $q = 0$ and $r = p$. Thus assume that $n \geq m$. Then, take the list

$$1, z, z^2, \ldots, z^{m-1}, s, sz, \ldots, sz^{n-m}$$

this list is linearly independent in $\mathcal{P}_n(\mathbf{F})$ as every element has a different degree. Also, the length of the list is $n+1$, thus this list is a basis of $\mathcal{P}_n(\mathbf{F})$. But since $p \in \mathcal{P}_n(\mathbf{F})$, we can write p as

$$\begin{aligned} p &= a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} + b_0 s + b_1 zs + \cdots + b_{n-m}sz^{n-m} \\ &= \underbrace{a_0 + a_1 z + \cdots + a_{m-1}z^{m-1}}_{r} + s\underbrace{(b_0 + b_1 z + \cdots + b_{n-m}z^{n-m})}_{q} \end{aligned}$$

as desired. □

3 Eigenvalues & Eigenvectors

3.1 Invariant Subspaces

3.1.1 Eigenvalues

Definition 3.1. A linear map from vector space to itself is called an *operator*.

Definition 3.2. Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called a *invariant* under T if $Tu \in U$ for every $u \in U$.

From our definition, U is invariant under T if $T|_U$ is an operator on U.

Definition 3.3. Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called a *eigenvalue* of T if there exists a $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Remark. Thus, V has one-dimensional subspace invariant under T if and only if T has an eigenvalue. If U is an one-dimensional subspace then $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$. Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$ then $\operatorname{span} v$ is one-dimensional subspace V invariant under T.

Proposition 3.1. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then the following are equivalent.

1. λ is an eigenvalue of T.
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

where I is the identity operator on V.

Proof. Condition 1. and 2. are equivalent because of $Tv = \lambda v \iff (T - \lambda I)v = 0$ and if it was injective then $v = 0$. And 2., 3. and 4. are equivalent from **Proposition 1.14.** □

Definition 3.4. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called and *eigenvector* corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Proposition 3.2. Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Proof. For the sake of contradiction, suppose the result is false. Then there exists a smallest positive integer $m > 1$ such that there exists a list $v_1, \ldots, v_m$ of linearly dependent eigenvectors $\lambda_1, \ldots, \lambda_m$ of T, corresponding to distinct eigenvalues. Due to the minimality of m, there exists $a_1, \ldots, a_m \in \mathbf{F}$ such that

$$a_1 v_1 + \ldots + a_m v_m = 0$$

Applying $T - \lambda I$ we get

$$a_1(\lambda_1 - \lambda_m)v_1 + \cdots + a_m(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$$

Since, all the eigenvalues are different, none of the coefficient above is 0. Thus, we get a new list of linearly dependent vector with length $m-1$, which contradicts the minimality of m. □

Proposition 3.3. Suppose V is a finite-dimensional. Then each operator of V has at most $\dim V$ distinct eigenvalue.

Proof. Since, every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent by above proposition, we get the bound immediately. $\square$

3.1.2 Polynomials Applied to Operatos

Definition 3.5. Suppose $T \in \mathcal{L}(V)$ and m is a positive integer

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m}$
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverses T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by
$$T^{-m} = (T^{-1})^m$$

Definition 3.6. Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by
$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$
for all $z \in \mathbf{F}$. Then $p(T)$ is the operator on V defined by
$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$$

Proposition 3.4. Let $T \in \mathcal{L}(V)$, then the function $f : \mathcal{P}(\mathbf{F}) \to \mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear.

Proof. Left for future me as a exercise. $\square$

Proposition 3.5. Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$ Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

Proof. Just define the polynomials and plug T. $\square$

Definition 3.7. Let $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is defined by
$$(pq)(z) = p(z)q(z)$$

Proposition 3.6. Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Then $\operatorname{null} p(T)$ and $\operatorname{range} p(T)$ are invariant under T.

Proof. Suppose $u \in \operatorname{null} p(T)$ then $p(T)u = 0$
$$p(T)(Tu) = T(p(T)u) = T(0) = 0$$
Hence, $Tu \in \operatorname{null} p(T)$. Thus, $\operatorname{null} p(T)$ is invariant under T
Suppose, $u \in \operatorname{range} p(T)$. Then,
$$\begin{aligned} p(T)v &= u \\ \implies T(p(T)v) &= Tu \\ \implies p(T)(Tv) &= Tu \end{aligned}$$
Hence, $Tu \in \operatorname{range} p(T)$. Thus, $\operatorname{range} p(T)$ is invariant under T, as desired. $\square$

3.1.3 Exercise

Problem : Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

1. Prove that $U \subseteq \operatorname{null} T$, then U is invariant under T.
2. Prove that $\operatorname{range} T \subseteq U$, then U is invariant under T.

Solution : For 1. just notice that $Tu = 0 \in U$ for all $u \in U$. Thus, it is invariant under T. For 2. notice that $Tu = k \in \operatorname{range} T \subseteq U$, thus $k \in U$. Therefore it is invariant under T.

Problem : Suppose that $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is also invariant under T.

Solution : Let $u \in V_1 + \cdots + V_m$ then $Tu = Tv_1 + \cdots + Tv_m$. Since $Tv_i \in V_i$ we have that

$$Tv_1 + \cdots + Tv_m \in V_1 + \cdots + V_m$$

Thus $V_1 + \cdots + V_m$ is invariant under T.

Problem : Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Solution : Let $\bigcap_{\alpha} V_\alpha = K$ where V_α is a subspace of V invariant under T. If $v \in K$ then $Tv \in V_\alpha$ for every α, hence $Tv \in K$.

Problem : If V is a finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Solution : Suppose $U \neq \{0\}$ and $U \neq V$. Let $u_1, \ldots, u_n$ be a basis of U and extend it to basis of V, $u_1, \ldots, u_n, v_1, \ldots, v_{m-n}$. Define

$$T(a_1u_1 + \cdots + a_nu_n + \cdots a_mv_{m-n}) = a_1v_1$$

It is easy to see that $T \in \mathcal{L}(V)$ but U is not invariant under T as $T(u_1) = v_1$.

Problem : Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution : If λ is a eigenvalues then $(-3y, x) = T(x, y) = \lambda(x, y) = (\lambda x, \lambda y)$ then $-3y = \lambda^2 y$. Notice that $y \neq 0$ because if $y = 0$ then $x = 0$ but $(x, y) \neq (0, 0)$. Thus, $\lambda = \pm\sqrt{3}i$.

Problem : Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or 1.

Solution : If λ is a eigenvalue then $\exists v \neq 0$ such that $P(v) = \lambda v$. Therefore,

$$\lambda v = P(P(v)) = P(\lambda v) = \lambda \cdot \lambda v$$
$$\implies \lambda^2 v = \lambda v \implies v(\lambda^2 - \lambda) = 0$$
$$\implies \lambda^2 = \lambda \implies \lambda = 1, 0$$

Problem : Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

1. Prove that T and $S^{-1}TS$ have the same eigenvalues.
2. What is the relationship between eigenvectors of T and eigenvectors of $S^{-1}TS$?

Solution : If λ is a eigenvalue of T then $T(v) = \lambda v$. Since S is invertible $\exists w \in V$ such that $S(w) = v$ thus $(S^{-1}(T(S(w))) = S^{-1}(\lambda v) = \lambda w$. If λ is a eigenvalue of $S^{-1}TS$ then $S^{-1}(T(Sw)) = \lambda w$ thus $T(S(w)) = S(\lambda w) = \lambda S(w)$. The relationship between the eigenvectors is that, eigenvectors of $S^{-1}TS$ maps to eigenvectors of T.

Problem : Give an example of a operator in $\mathbf{R}^4$ that has no eigenvalues.

Solution : $T(w, x, y, z) = (-x, w, -z, y)$

Problem : Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Solution : From exercise 17 of $3F$ we know that T is invertible if and only if T' is invertible. Thus, $T - \lambda I$ is not invertible if and only if $(T - \lambda I) = T' - \lambda I$ is not invertible. Thus we are done.

Problem : Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbf{C}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

Solution : If λ is an eigenvalue then $T(v + iw) = \lambda(v + iw)$ thus we have

$$\overline{T(v + iw)} = T(\overline{v + iw}) = \overline{\lambda}(\overline{v + iw})$$

Since $v + iw \neq 0$ so their conjugate isn't zero as well.

Problem : Suppose $T \in \mathcal{L}(V)$ is invertible.

1. Suppose $\lambda \in F$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
2. Prove that T and T^{-1} have same eigenvectors.

Solution : If λ is an eigenvalue of T then

$$T(v) = \lambda v \iff v = \lambda T^{-1}(v) \iff \frac{1}{\lambda}v = T^{-1}(v)$$

This proves both of the statement.

Problem : Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Solution : If ST has eigenvalue λ then

$$ST(v) = \lambda v$$

$$\implies T(STv) = \lambda T(v)$$

If $Tv \neq 0$ then we are done. If $Tv = 0$ then

$$S(Tv) = S(0) = 0 = \lambda v \implies \lambda = 0$$

Thus, $ST - \lambda I = ST$ is no invertible thus TS is also not invertible. Therefore 0 is also an eigenvalue of TS.

Left to do exercises. $23, 24, 25, 26, 27, 30, 31, 34, 35, 39$

3.2 The Minimal Polynomial

3.2.1 Existence of Eigenvalue on Complex Vector Spaces

Theorem 3.1. *Every operator on a finite-dimensional non-zero complex vector space has an eigenvalue.*

Proof. Let V be a finite-dimensional non-zero complex vector space of dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v \in V$ such that $v \neq 0$. Then

$$v, Tv, T^2v, \ldots, T^nv$$

This has length $n + 1 > n$ thus it is linearly dependent list. Therefore there exists $a_0, \ldots, a_n \in \mathbf{C}$ not all zero, such that

$$a_0v + a_1Tv + \cdots + a_nT^nv = 0$$

Let $p(z) = a_0 + a_1z + \cdots + a_nz^n$ be a polynomial with minimal degree such that $p(T) = 0$. Since $a_i \in \mathbf{C}$ there exist complex roots λ and a polynomial $q(z)$ such that

$$p(z) = (z - \lambda)q(z)$$

Apply T to this polynomial gives us,

$$p(T) = (T - \lambda)(q(T))$$

and applying v gives us,

$$(T - \lambda)(q(T)v) = 0 \implies T(q(T)v) = \lambda q(T)v$$

Thus λ is an eigenvalue as $Tv \neq 0$ because $q(T)v \neq 0$ deu to the minimality of p. $\square$

Theorem 3.2. *Suppose V is a finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$. Furthermore,* $\deg p \leq \dim V$.

Proof. We proceed by induction on $\dim V$. If $\dim V = 0$, then I is the zero operator, and we take $p(z) = 1$, which is monic and has $\deg p = 0 \leq 0$.
Suppose $\dim V > 0$ and the result holds for all operators on spaces of smaller dimension. Pick a non-zero vector $v \in V$. The list $v, Tv, \ldots, T^{\dim V}v$ has length $1 + \dim V$, so it is linearly dependent. By linear dependence, lemma there exists a smallest positive integer m such that T^mv is a linear combination of $v, Tv, \ldots, T^{m-1}v$. Thus, there exist scalars $c_0, \ldots, c_{m-1}$ such that:

$$c_0v + c_1Tv + \cdots + c_{m-1}T^{m-1}v + T^mv = 0$$

Define the monic polynomial $q(z) = z^m + c_{m-1}z^{m-1} + \cdots + c_0$. Then $q(T)v = 0$.
By the choice of m, the list $v, Tv, \ldots, T^{m-1}v$ is linearly independent. Since $q(T)(T^kv) = T^k(q(T)v) = 0$, all m vectors in this list are in null $q(T)$. Thus, $\dim \text{null } q(T) \geq m$ (by Independent list $>$ span list inequality). By the Fundamental theorem of linear maps,

$$\dim \text{range } q(T) = \dim V - \dim \text{null } q(T) \leq \dim V - m$$

Since range $q(T)$ is invariant under T, we apply the induction hypothesis to $T|_{\text{range } q(T)}$. There exists a monic polynomial s with $\deg s \leq \dim V - m$ such that $s(T|_{\text{range } q(T)}) = 0$. Let $p = sq$. Then p is monic and $\deg p = \deg s + \deg q \leq (\dim V - m) + m = \dim V$. For any $v \in V$, $q(T)v \in \text{range } q(T)$, so $s(T)(q(T)v) = 0$. Thus $p(T) = 0$. $\square$

Here is another way to prove the existence of minimal polynomial and its uniqueness. The proof below doesn't prove the inequality $\dim V \geq \deg p$.

Proof. let $n = \dim V$ then $\dim \mathcal{L}(V) = n^2$. Choose the list

$$I, T, \dots, T^{n^2}$$

This list is linearly dependent because $n^2+1 > n^2$. Let m be the smallest positive number such that $I, T, \dots, T^m$ is linearly dependent. By the linear dependence lemma we have,

$$T^m + a_{m-1}T^{m-1} + \cdots + a_0 I = 0$$

Construct a monic polynomial $p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$. By the choice of m this is the smallest degree monic polynomial which satisfies $p(T) = 0$.
For the uniqueness, suppose $p \neq q \in \mathcal{P}(\mathbf{F})$ are monic polynomial such that $p(T) = 0$ and $q(T) = 0$. Let $h(z) = p(z) - q(z)$. Notice that $\deg h < m$ because both are monic and have same degree and since $p \neq q$ we have $h \not\equiv 0$. Thus $h(T) = p(T) - q(T) = 0$ which contradicts our previous statement. □

Definition 3.8. Suppose V is a finite-dimensional and $T \in \mathcal{L}(V)$. Then the *minimal polynomial* of T is the unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$.

Proposition 3.7. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

1. λ is a root of the minimal polynomial $\iff$ λ is an eigenvalues of T.

2. If V is a complex vector space, then minimal polynomial has the form

$$(z - \lambda_1) \cdots (z - \lambda_m)$$

where $\lambda_1, \cdots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetition.

Proof. For (1.), Let p be a minimal polynomial and suppose λ is a root of the minimal polynomial. Then, we know by **FTA** we can write p as $p(z) = (z - \lambda)q(z)$ for some $q(z)$. Applying this Polynomial to T we get

$$p(T)v = (T - \lambda I)(q(T)v)$$

for all $v \in V$. We know that $q(T)v \neq 0$ for some $v \in V$ because $\deg q < \deg p$ and $q(T)v = k$, thus $T(k) = \lambda k$. Hence λ is an eigenvalue.

Now, let $\lambda \in \mathbf{F}$ be a eigenvalue of T. Then, $T^k(v) = \lambda^k v$. Thus,

$$p(T)v = p(\lambda)v \implies p(\lambda)v = 0 \implies p(\lambda) = 0$$

Thus λ is a root of the polynomial.

For (2.), Use (1.) and **FTA**. □

Proposition 3.8. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$. Then $q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial.

Proof. Let p be the minimal polynomial. Then, $\exists s, r$ such that $\deg r < \deg p$ and

$$q = ps + r$$

We have

$$0 = q(T) = p(T)s(T) + r(T) = r(T)$$

Thus $r \equiv 0$ as if not then would violets p being the minimal polynomial. Thus $q = ps$. $\square$

Proposition 3.9. Suppose V is a finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of T is a multiple of minimal polynomial of $T|_U$.

Proof. Let p be the minimal polynomial of T. Then, $p(T)v = 0$ for every $v \in V$. Therefore, $p(T)u = 0$ for every $u \in U$. Since T is invariant under U we have $p(T|_U) = 0$. Thus using previous results we get that p is a multiple of the minimal polynomial of $T|_U$. $\square$

Proposition 3.10. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if constant term of minimal polynomial p of T is 0.

Proof.

$$\begin{aligned} T \text{ is not invertible } &\iff 0 \text{ is an eigenvalue of } T \\ &\iff 0 \text{ is a root of } p \\ &\iff \text{constant term of } p \text{ is zero.} \end{aligned}$$

$\square$

3.2.2 Eigenvalue on Odd-Dimensional Real Vector Space

Proposition 3.11. Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Then $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

Proof. We know that $\dim \operatorname{null}(T^2 + bT + cI)$ is invariant under T. Thus, we will work on $V' = \operatorname{null}(T^2 + bT + cI)$ and $T' = T|_{\operatorname{null}(T^2+bT+cI)}$. Let $v \in \operatorname{null}(T^2 + bT + cI)$ such that $Tv = \lambda v$.

$$0 = (T^2 + bT + cI)v = (\lambda^2 + b\lambda + c)v = \left(\left(\lambda + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}\right)v$$

Thus $v = 0$ as the first term is positive. This proves that T' has no eigenvalue.
Let U be a subspace of V' that is invariant under T' and has the largest dimension among all subspaces of V' that are invariant under T' and have even dimension. If $U = V'$ then we are done, if not then suppose $w \in V'$ and $w \notin U$. Let $W = \operatorname{span}(w, Tw)$. Then W is invariant under T' and $\dim W = 2$. Also,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = \dim U + 2$$

where $\dim(U \cap W) = \{0\}$ as $\dim(U \cap W) \leq 2$ and if $\dim(U \cap W) = 2 = \dim W$ then $U \cap W = W$ but that is false as $w \notin U$ but $w \in W$. Also, $\dim(U \cap W) = 1$ implies that T' has an eigenvalue. Because $U + W$ is invariant under T' , the equation above shows that there exists a subspace of V' invariant under T' of even dimension larger than $\dim U$. Thus the assumption that $U \neq V'$ was incorrect. Hence V' has even dimension. $\square$