Real Analysis

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Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from MIT OCW and write the solutions to the problem set here.

1 Problem Set 1

Problem 1.1

Let $\mathbb{F}$ be a ordered field with $1 \neq 0$. Show that $1 > 0$.

Solution.
First let us prove that $(-1) \cdot (-1) = 1$. We know that for all $x \in \mathbb{F}$, there is an inverse element $-x$ such that,

$$x + (-x) = 0$$

Thus, $1 + (-1) = 0$ which means that

$$0 = (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1)$$

$$\implies 1 = (-1) \cdot (-1)$$

Since $\mathbb{F}$ is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that $1 \neq 0$ so the middle statement can't be true and if $1 < 0$ then $0 < (-1)$. But from **axiom of order and multiplication** $0 < (-1) \cdot (-1) = 1$. Thus a contradiction.

Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

Solution.
Suppose $\frac{n}{m} = \frac{n_1}{m_1}$ and $\frac{p}{q} = \frac{p_1}{q_1}$ then using the definition of when two rational numbers are equal, we get $nm_1 = n_1 m$ and $pq_1 = p_1 q$. Thus,

$$\begin{aligned} m_1 q_1 (nq + pm) &= m_1 q_1 nq + m_1 q_1 pm \\ &= n_1 m q_1 q + p_1 q m_1 m \\ &= mq(n_1 q_1 + m_1 p_1) \end{aligned}$$

Thus, $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$.

Problem 1.3

Find the $\sup E$ and $\inf E$ for the following set E.

1. $E = \left\{n \in \mathbb{Z} \mid n < \sqrt{12}\right\}$
2. $E = \left\{r \in \mathbb{Q} \mid r < \sqrt{12}\right\}$
3. $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
4. $E = \left\{\frac{n^2+n}{n+1} \mid n \in \mathbb{N}\right\}$

Solution.

1. $\sup E = 3$ but $\inf E$ doesn't exist.
2. $\sup E = \sqrt{12}$ but $\inf E$ doesn't exist.
3. $\sup E = \frac{1+\sqrt{5}}{2}$ and $\inf E = \frac{1-\sqrt{5}}{2}$.
4. $\inf E = 1$ but $\sup E$ doesn't exist.

Problem 1.4

Let $\mathbb{M}$ be the set of polynomials with integer coefficients i.e,

$$\mathbb{M} := \{f(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Z}\}$$

Define the relation $0 \prec f$ if $0 < f(x)$ for x large enough. More precisely, we say

$$0 \prec f \quad \text{if there exists } M > 0 \text{ such that } f(x) > 0 \text{ for all } x > M.$$

Then define

$$f \prec g \quad \text{if } 0 \prec (g - f).$$

Show that $(M, \prec)$ is an ordered set.

Solution.
We'll use the fact that for large enough x, $f(x) > 0$ for $a_n > 0$. Let $f, g \in \mathbb{M}$ such that $f \neq g$ and

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

Define $h(x) = g(x) - f(x) = (b_0 - a_0) + (b_1 - a_1)x + \cdots + (b_n - a_n)x^n$ and let $c_i = b_i - a_i$. Suppose c_k is the highest degree non zero coefficient, then if

1. $c_k > 0$ then $f \prec g$
2. $c_k < 0$ then $g \prec f$

Suppose $f \prec g$ and $g \prec h$. Then, $f(x) < g(x)$ and $g(x) < h(x)$ for large x and hence $f(x) < h(x)$ for large x. Thus, $0 \prec (h - f)$ which means $f \prec h$.

Problem 1.5

Prove that $(\mathbb{M}, \prec)$ **doesn't** satisfy the archimedean property.

Solution.
To show that it doesn't satisfy the archimedean property, we need to show that $\exists f, g \in \mathbb{M}$ such that

$$\forall n \in \mathbb{N}, g \not\prec nf$$

If we choose $g(x) = x^2$ and $f(x) = x$ and assume $g \prec nf$ then , $0 \prec x(n - x)$ which means that $0 < x(n - x)$ which we know is false for large x. Thus, $g \not\prec nf$.

Problem 1.6

Show that for any non-empty set $E \subset R$ which is bounded from below, E has the greatest lower bound.

Solution.
To show that E has greatest lower bound define

$$-E = \{-x \mid x \in E\}$$

If α is any lower bound of E then $x \geq \alpha \Rightarrow -x \leq -\alpha$. That means that $-E$ is bounded above by $-\alpha$. Since $-E$ is bounded above by $-\alpha$, it must have the least upper bound property. Let

$$\sup -E = \beta$$

Thus, $\beta \geq -x \Rightarrow x \geq -\beta$ and $\beta \leq -\alpha$ for any lower bound α of E. Thus, $\alpha \leq -\beta$. Hence, $-\beta = \inf E$.

Problem 1.7

Show that for any real number $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that $y^3 = x$.

Solution.
If $x = 0$ then $y = 0^3 = 0$. Let us define

$$A = \left\{a \mid a > 0 \text{ and } a^3 \leq x\right\}$$

for $x > 0$. Since A is non-empty and bounded, let $y = \sup A$. We will show that $y^3 = x$. Suppose $x > y^3$. Let $h = \min\{\frac{1}{2}, \frac{x-y^3}{3y^2+3y+1}\}$ then,

$$\begin{aligned}
(y+h)^3 &= y^3 + h^3 + 3yh^2 + 3y^2h \\
&< y^3 + h(1 + 3y + 3y^2) \\
&\leq y^3 + \frac{x - y^3}{3y^2 + 3y + 1} \cdot (1 + 3y + 3y^2) \\
&\leq x
\end{aligned}$$

Thus, $y + h \in A$ but since y is the least upper bound of A we have, $y + h \leq y \Rightarrow h \leq 0$. This is a contradiction.

Suppose $x < y^3$. Let $h = \frac{y^3 - x}{3y^2}$ then

$$\begin{aligned}(y-h)^3 &= y^3 - h^3 - 3y^2h + 3yh^2 \\ &= y^3 - h^3 - 3y^2 \cdot \frac{(y^3 - x)}{3y^2} + 3yh^2 \\ &= x - h^3 + 3yh^2\end{aligned}$$

Since $3yh^2 - h^3 > 0$ we have $(y-h)^3 > x$. Let $q \in A$ then

$$q^3 \leq x < (y-h)^3$$

Thus, $(y-h)^3 - q^3 > 0 \Rightarrow (y-h-q)\left((y-h)^2 + (y-h)q + q^3\right) > 0$. Since, $(y-h)^2 + (y-h)q + q^3 > 0$ we have $y - h > q$. Thus $(y-h)$ is an upper bound for A. Thus $y \leq y - h \Rightarrow h \leq 0$. Contradiction!!

The proof also works for $x < 0$ as $-x > 0$. Thus,

$$\exists k \in \mathbb{R} \text{ s.t } k^3 = -x$$

Let $z := -k$ then $z^3 = (-k)^3 = -k^3 = -(-x) = x$.

2 Problem Set 2

Problem 2.1

Let a_n ad b_n be a sequence of real numbers.

1. Assume that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists. Show that $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} a_n\right)$ exists.
2. Give an example in which $\lim_{n\to\infty}(a_n b_n)$ exists but neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty} b_n$ exits.

Solutions.
To show that $\lim_{n\to\infty} a_n b_n = ab$ we need to show $\exists N \in \mathbb{N}$ such that for all $\varepsilon > 0$ we have

$$|a_n b_n - ab| < \varepsilon$$

for $n \geq N$. Now,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

Since $\{a_n\}$ is convergent, we know that it is bounded above. Suppose $|a_n| < C$ then

$$|a_n||b_n - b| + |b||a_n - a| < |C||b_n - b| + |b||a_n - a|$$

Also let $N_1 \in \mathbb{N}$ such that $|b_n - b| < \dfrac{\varepsilon}{2(|C|+1)}$ for all $n \geq N_1$ and let $N_2 \in \mathbb{N}$ such that $|a_n - a| < \dfrac{\varepsilon}{2(|b|+1)}$ for all $n \geq N_2$. Thus for $n \geq \max\{N_1, N_2\}$ we have,

$$\begin{aligned} 2|a_n||b_n - b| + |b||a_n - a| &< |C||b_n - b| + |b||a_n - a| \\ &< |C| \cdot \frac{\varepsilon}{2(|C|+1)} + |b| \cdot \frac{\varepsilon}{2(|b|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

For the second part just take $a_n = (-1)^n$ and $b_n = (-1)^n$. We know that $\lim_{n\to\infty} a_n b_n = 1$ but $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ doesn't exist.

Problem 2.2

Find the limit of the following sequence if it exists or show that the limit doesn't exists.

1. $a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$
2. $a_n = \frac{\sin(n)}{n}$
3. $a_n = \displaystyle\sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6 + i}}$

Solution.
For the first part we know that $a_n = \frac{1}{n+1} - \frac{1}{n} - 1$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n} - 1 &= \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 1 \\ &= 0 - 0 - 1 \\ &= -1\end{aligned}$$

The first equation is due to the algebraic property of of limit.

For the second part, we need to show that

$$\left|\frac{\sin(n)}{n}\right| < \varepsilon$$

We know that $0 \le |\sin(n)| \le 1 \Rightarrow 0 \le \frac{|\sin(n)|}{|n|} \le \frac{1}{|n|}$. Thus, for any $N > \frac{1}{\varepsilon}$ we have

$$\left|\frac{\sin(n)}{n}\right| \le \frac{1}{|n|} < \frac{1}{N} < \varepsilon$$