

Linear Algebra

Notes

Basu Dev Karki

Content

1	Linear Maps	2
1.1	Vector Spaces of Linear Maps	2
1.1.1	Definiton and Examples of Linear Maps	2
1.1.2	Algebraic Operation on $\mathcal{L}(V, W)$	3
1.1.3	Exercises	4
1.2	Null Spaces and Ranges	8
1.2.1	Null Space and Injectivity	8
1.2.2	Range and Surjectivity	8
1.2.3	Fundamental Theorem of Linear Maps	9
1.2.4	Excercise	10
1.3	Matrices	15
1.3.1	Representing a Linear Map by a Matrix	15
1.3.2	Addition and Scalar Multiplication of Matrices	15
1.3.3	Matrix Multiplication	16
1.3.4	Column-Row Factorization and Rank of a Matrix	18
1.3.5	Exercise	20
1.4	Invertibility and Isomorphism	25
1.4.1	Invertible Linear Maps	25
1.4.2	Isomorphic Vector Spaces	26
1.4.3	Linear Map Thought of as Matrix Multiplication	27
1.4.4	Change of Basis	28
1.4.5	Exercise	30
1.5	Product and Quotients of Vector Spaces	33
1.5.1	Products of Vector Spaces	33
1.5.2	Quotients Spaces	34
1.5.3	Exercise	35
1.6	Duality	38
1.6.1	Dual Space and Dual Map	38
1.6.2	Null Space and Range of Linear Map	39
1.6.3	Matrix of Dual Linear Map	41
2	Polynomials	43
2.1	Zeros of Polynomials	43
2.2	Division Algorithm for Polynomials	43
3	Eigenvalues & Eigenvectors	44
3.1	Invariant Subspaces	44
3.1.1	Eigenvalues	44
3.1.2	Polynomials Applied to Operatos	45
3.1.3	Exercise	46
3.2	The Minimal Polynomial	49

1 Linear Maps

1.1 Vector Spaces of Linear Maps

1.1.1 Definiton and Examples of Linear Maps

Definition 1.1. A *linear map* from V to W is a function $T : V \rightarrow W$ with the following properties.

1. **(Additivity)** $T(u + w) = T(u) + T(w)$ for all $u, v \in V$
2. **(Homogeneity)** $T(\lambda u) = \lambda T(u)$ for all $\lambda \in \mathbf{F}$ and for all $u \in V$

Remark. Some mathematicians use the phrase *linear transformation*, which means the same as linear map.

Definition 1.2. (Notation)

1. The set of linear maps from $V \rightarrow W$ is denoted by $\mathcal{L}(V, W)$.
2. The set of linear maps from $V \rightarrow V$ is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V, V) = \mathcal{L}(V)$.

Examples:

zero

We will let the symbol 0 denote the liner map that takes every element of some vector space to additive identity of some another vector space. Thus, $0 \in \mathcal{L}(V, W)$ is defined by

$$0(v) = 0$$

identity operator

Let $I \in \mathcal{L}(V)$ be defined by

$$I(v) = v$$

differentiation

Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be defined by

$$D(p) = p'$$

integration

Let $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$ be defined by

$$T(p) = \int_0^1 p$$

composition

Fix a polynomial $q \in \mathcal{P}(\mathbf{R})$. Let $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be defined by

$$T(p) = (p \circ q)$$

Remark. We'll limit the Notation of $T(v)$ to just Tv for convenience.

Theorem 1.1. Suppose v_1, \dots, v_n is a basis of V and $w_1, \dots, w_n \in W$. Then, there exists a unique linear map $T : V \rightarrow W$ such that

$$Tv_k = w_k$$

for $i = 1, 2, 3, \dots, n$.

Proof. First we show the existence of such map. Define $T : V \rightarrow W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

where $c_i \in \mathbf{F}$. Since, v_1, \dots, v_n is a basis of V , it maps every element of V to W , thus it is a function.

Now, set $c_k = 1$ and all other c 's to be 0 to show that $Tv_k = w_k$. From, here one can show that T is indeed a linear map. To show the uniqueness, suppose $T' \in \mathcal{L}(V, W)$ and $T'v_k = w_k$. Using the properties of linear map,

$$T'(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

Thus, T and T' agree on every input, thus $T = T'$. □

1.1.2 Algebraic Operation on $\mathcal{L}(V, W)$

Definition 1.3. Suppose $T, S \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$ then the *sum* and the *product* of the linear maps from V to W is defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

Proposition 1.1. With the operations defined above, the set $\mathcal{L}(V, W)$ is a vector space.

Proof. The additive identity for $\mathcal{L}(V, W)$ is the zero linear map $0(v) = 0$. The inverse for T is $((-1)T)v = -(Tv)$. And the rest of the axioms are left for readers (future me) to verify. □

Definition 1.4. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Remark. Be careful about the domains of S and T . Here, the domain of S must be the co-domain of T .

Proposition 1.2. For the product of linear maps, the following holds

1. (**associativity**) $(T_1T_2)T_3 = T_1(T_2T_3)$ whenever the product makes sense (i.e T_3 must map to domain of T_2 and T_2 must map to the domain of T_1).
2. (**identity**) $TI_{W,V} = I_{V,W}T$ whenever $T \in \mathcal{L}(V, W)$. Here $I_{V,W}, I_{W,V}$ are the identity linear maps from V to W and W to V . We'll just limit the notation to $TI = IT$.
3. (**distributivity**) $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$ for $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Proposition 1.3. Suppose T is a linear map from V to W . Then, $T(0) = 0$.

Proof. From the definition of linear map we have,

$$T(0) = T(0 + 0) = T(0) + T(0) \implies T(0) = 0$$

□

1.1.3 Exercises

Problem : Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z, 6x + cxyz)$$

Show that T is a linear map if and only if $b = c = 0$.

Solution : (\Leftarrow) is pretty simple as you just have to verify the two axioms. For (\Rightarrow), we know that if it is a linear map then $T(0) = 0 \Rightarrow b = 0$. Also, using the first axiom we get

$$T((x, y, z) + (1, 0, 0)) = T((x, y, z)) + T((1, 0, 0)) \Rightarrow c = 0$$

Problem : Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^2 x^3 p(x) dx + c \sin(p(0)) \right)$$

Show that T is a linear map if and only if $b = c = 0$.

Solution : (\Leftarrow) is pretty simple. For (\Rightarrow), we can use the first axiom

$$T(p + q) = Tp + Tq$$

we'll just look at the first component first,

$$\begin{aligned} \Rightarrow 3(p + q)(4) + 5(p + q)'(6) + b(p + q)(1)(p + q)(2) &= 3p(4) + 5p'(6) + bp(1)p(2) + \\ &3q(4) + 5q'(6) + bq(1)q(2) \end{aligned}$$

Since, $(p + q)(4) = p(4) + q(4)$ and $(p + q)'(6) = p'(6) + q'(6)$, we can simplify is down to,

$$b(p(1)q(2) + p(2)q(1)) = 0$$

Now, if you choose polynomials $p, q > 0$ for $x > 0$ then $b = 0$. A similar argument works for $c = 0$.

Problem : Suppose $T \in \mathcal{L}(V, W)$ and v_1, \dots, v_m is a list of vectors in V such that Tv_1, \dots, Tv_m is a linearly independent list in W . Prove that v_1, \dots, v_m is linearly independent.

Solution : Suppose on the contrary that v_1, \dots, v_m are not linearly independent in V , then there exists $\lambda_1, \dots, \lambda_m$ not all zero such that

$$\lambda_1 v_1 + \dots + \lambda_m v_m = 0$$

But $T(\lambda_1 v_1 + \dots + \lambda_m v_m) = \lambda_1 T(v_1) + \dots + \lambda_m T(v_m)$ which implies

$$\lambda_1 T(v_1) + \dots + \lambda_m T(v_m) = 0 \Rightarrow \lambda_i = 0$$

a contradiction.

Problem : Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution : Since $\dim V = 1$ there exists a $v \in V$ s.t every $v_i \in V$ can be written as $\lambda_i v$ for some $\lambda_i \in \mathbf{F}$. Thus, $v_i = \lambda_i v \implies T(v_i) = \lambda_i T(v)$. Since, $T(v) = v_j \in V$ for some j . Thus, $T(v_i) = \lambda_i \lambda_j v = \lambda_j v_i$.

Problem : Give an example of a function $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and $v \in \mathbf{R}^2$ but φ is not linear.

$$\text{Solution : } \varphi(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Problem : Give an example of a function $\varphi : \mathbf{C} \rightarrow \mathbf{C}$ such that

$$\varphi(v + w) = \varphi(v) + \varphi(w)$$

for all $v, w \in \mathbf{C}$ and but φ is not homogeneous.

$$\text{Solution : } \varphi(x) = \operatorname{Re}(x).$$

Problem : Prove or give a counter example: Fix a polynomial $q \in \mathcal{P}(\mathbf{R})$.

Let $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ be defined by $Tp = q \circ p$ then T is a linear map.

Solution : Assume it was a linear map then $T(0) = q(0) = 0$. Just pick $q(0) \neq 0$.
Example : $q(x) = x + 1$.

Problem : Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Solution : I really tried but it seems very hard to prove (\Rightarrow) but will come back later.

Problem : Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $Su \neq 0$ for some $u \in U$. Define $T : V \rightarrow W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that $T \notin \mathcal{L}(V, W)$.

Solution : Suppose it is a linear map, then $T(u + v) = Tu + Tv$ where $u \in U$ and $v \in V$ and $v \notin U$. One can check that $v + u \in V$ but $v + u \notin U$. Thus, $0 = Tu = Su$, but just take u s.t $Su \neq 0$.

Problem : Suppose V is finite-dimensional. Prove that every linear map on a subspace of U can be extended to a linear map on V . In other words, let U be a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists a $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution : Let u_1, \dots, u_m be the basis of U and let $u_1, \dots, u_m, v_1, \dots, v_k$ be the extended basis of V . Let $x = a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_kv_k$ and define

$$T(x) = a_1Su_1 + \dots + a_mSu_m + b_1v_1 + \dots + b_kv_k$$

From here its easy to see that $Tu = Su$ for all $u \in U$. We just have to prove this is a linear map on V . Let $y = x = c_1u_1 + \dots + c_mu_m + d_1v_1 + \dots + d_kv_k$ then

$$\begin{aligned} T(x+y) &= (a_1+c_1)Su_1 + \dots + (a_m+c_m)Su_m + (b_1+d_1)v_1 + \dots + (b_k+d_k)v_k \\ \implies T(x+y) &= a_1Su_1 + \dots + a_mSu_m + b_1v_1 + \dots + b_kv_k + c_1Su_1 + \dots + c_mSu_m + d_1v_1 + \dots + d_kv_k \\ &\implies T(x+y) = Tx + Ty \end{aligned}$$

Similarly one can prove $T(\lambda x) = \lambda T(x)$.

Problem : Suppose V is finite-dimensional with $\dim V > 0$ and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution : Let v_1, \dots, v_m be basis of V . Suppose $\mathcal{L}(V, W)$ is finite-dimensional then every $T \in \mathcal{L}(V, W)$ can be written as

$$T(x) = \lambda_1 T_1(x) + \lambda_2 T_2(x) + \dots + \lambda_k T_k(x)$$

for some fixed $T_1, T_2, \dots, T_k \in \mathcal{L}(V, W)$ and $\lambda_i \in \mathbf{F}$.

Since, W is infinite-dimensional, $\exists w \in W$ s.t $w \notin \text{span}\{T_1(x), T_2(x), \dots, T_k(x)\}$ for some fixed $x \in V$.

Now, set $x = v_i$ then $w \neq \lambda_1 T_1(v_i) + \lambda_2 T_2(v_i) + \dots + \lambda_k T_k(v_i)$. One can find $T \in \mathcal{L}(V, W)$ such that $T(v_i) = w$ thus $T(v_i) \neq \lambda_1 T_1(v_i) + \lambda_2 T_2(v_i) + \dots + \lambda_k T_k(v_i)$ contradicting our assumption.

Problem : Let V be finite-dimensional and let $\dim V > 1$. Prove that there exists $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

Solution : Let v_1, \dots, v_m be the basis of V and let $x = a_1v_1 + \dots + a_mv_m$. Define $S(x) = a_1v_1$ and $T(x) = a_1v_2 + a_2v_3 + \dots + a_{m-1}v_m + a_mv_1$. So, $T(S(x)) = a_1v_2$ and $S(T(x)) = a_mv_1$, thus

$$ST = TS \iff a_1 = a_m$$

but one can always choose x s.t $a_1 \neq a_m$.

Problem : Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$. A subspace \mathcal{E} of $\mathcal{L}(V)$ is called a two-sided ideal if for every $E \in \mathcal{E}$ and $T \in \mathcal{L}(V)$, $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$.

Solution : Let \mathcal{E} be a two-sided ideal of $\mathcal{L}(V)$. If $\mathcal{E} = \{0\}$ we are done. Otherwise pick a nonzero operator $A \in \mathcal{E}$. Choose $v \in V$ with $Av \neq 0$. Fix any $y, z \in V$. Choose $R \in \mathcal{L}(V)$ with $R(z) = v$ and choose $S \in \mathcal{L}(V)$ with $S(Av) = y$. Then

$$T := SAR \in \mathcal{E}$$

(since \mathcal{E} is a two-sided ideal), and

$$T(z) = S(A(R(z))) = S(A(v)) = y.$$

Thus \mathcal{E} contains, for every pair y, z , an operator that sends z to y . Taking a basis u_1, \dots, u_n of V and the operators E_{ij} defined by $E_{ij}(u_j) = u_i$ and $E_{ij}(u_k) = 0$ for $k \neq j$, we see each E_{ij} lies in \mathcal{E} . The set $\{E_{ij}\}$ spans $\mathcal{L}(V)$, so $\mathcal{E} = \mathcal{L}(V)$. Hence the only two-sided ideals are $\{0\}$ and $\mathcal{L}(V)$. \square

1.2 Null Spaces and Ranges

1.2.1 Null Space and Injectivity

Definition 1.5. let $T \in \mathcal{L}(V, W)$, the null space of T , written as $\text{null } T$ is the following set

$$\text{null } T = \{v \in V \mid Tv = 0\}$$

Examples

1. The zero map from V to W has a null space V as everything gets mapped to 0.
2. Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the differentiation map defined by $Dp = p'$. The only functions whose derivative is equal to 0 are the constant function. Thus, $\text{null } D$ is the set of constant functions.
3. Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the multiplication by x^2 map i.e $Dp = x^2p$. The only polynomial such that $x^2p = 0$ is $p = 0$. Thus, $\text{null } D = \{0\}$.

Proposition 1.4. Suppose $T \in \mathcal{L}(V, W)$. Then, $\text{null } T$ is a subspace of V .

Proposition 1.5. Let $T \in \mathcal{T}(V, W)$. Then T is injective $\iff \text{null } T = \{0\}$.

Proof. Suppose T is injective. Since it is a linear map $T(0) = 0$, thus by injectivity the only thing that gets map to 0 is 0. Thus, $\text{null } T = \{0\}$. Suppose T is such that $\text{null } T = \{0\}$ then

$$\begin{aligned} T(v) = T(w) &\implies T(v) + (-1)T(w) = 0 \\ \implies T(v) + T(-w) &= 0 \implies T(v - w) = 0 \implies v = w \end{aligned}$$

Thus, the map is injective. □

1.2.2 Range and Surjectivity

Definition 1.6. Let $T \in \mathcal{L}(V, W)$, the *range* of T is the following set,

$$\text{range } T = \{Tv \mid v \in V\}$$

Examples

1. If T is the zero map then the range of T is $\{0\}$.
2. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the differentiation map. Since, for every polynomial q there exists a polynomial p such that $p' = q$, the range is $\mathcal{P}(\mathbf{R})$.

Proposition 1.6. Let $T \in \mathcal{L}(V, W)$ then $\text{range } T$ is a subspace of W .

Proof. Since, $0 \in V$ we know $T(0) = 0 \in \text{range } T$. Now, suppose $x, y \in \text{range } T$ then $x = Tv$ and $y = Tw$. Since, $v + w \in V$, $T(v + w) \in \text{range } T \implies Tv + Tw \in \text{range } T$ which mean $x + y \in \text{range } T$. And $x \in \text{range } T \implies Tv \in \text{range } T$ which means $T(\lambda v) \in \text{range } T$ as $\lambda v \in V$. Thus, $\lambda x = \lambda T(v) = T(\lambda v) \in \text{range } T$. □

1.2.3 Fundamental Theorem of Linear Maps

Theorem 1.2. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\text{range } T$ is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

Proof. We know that $\text{null } T$ is a subspace of V then since V is finite-dimensional it has a basis. Let u_1, \dots, u_m be the basis of $\text{null } T$. Then we can extend this basis to a basis of V . Let $u_1, \dots, u_m, v_1, \dots, v_n$ be the basis of V . Then,

$$x = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n$$

$$\implies Tx = b_1 T v_1 + \dots + b_n T v_n$$

Thus, every Tv can be written as a linear combination of Tv_1, \dots, Tv_n . Thus, $\text{range } T$ is finite-dimensional. To prove our main result, we need to show that Tv_1, \dots, Tv_n is a basis of $\text{range } T$. We already proved it spanned $\text{range } T$, now suppose

$$b_1 T v_1 + \dots + b_n T v_n = 0$$

$$\implies T(b_1 v_1 + \dots + b_n v_n) = 0$$

Thus, $b_1 v_1 + \dots + b_n v_n \in \text{null } T$ and we can write it as $b_1 v_1 + \dots + b_n v_n = a_1 u_1 + \dots + a_m u_m$. Since, $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis vector we can say that $b_i = a_i = 0$. Thus, we have proved that Tv_1, \dots, Tv_n is a basis of $\text{range } T$ and our theorem follows. \square

Theorem 1.3. Suppose V and W are both finite-dimensional vector spaces such that $\dim V > \dim W$. Then, there exists no **injective** linear map from V to W .

Proof. We know that, for a $T \in \mathcal{L}(V, W)$,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

Since $\text{range } T$ is a subspace of W , $\dim W \geq \dim \text{range } T$. Thus,

$$\begin{aligned} \dim \text{null } T &= \dim V - \dim \text{range } T \\ &\geq \dim V - \dim W \\ &> 0 \end{aligned}$$

Thus, $\text{null } T$ has more than one vector, so T is not injective by **Proposition 1.5**. \square

Theorem 1.4. Suppose V and W are both finite-dimensional vector spaces such that $\dim V < \dim W$. Then, there exists no **surjective** linear map from V to W .

Proof. Similar to the proof above. \square

Definition 1.7. Define $T : \mathbf{F}^n \rightarrow \mathbf{F}^m$ as

$$T(x_1, \dots, x_n) = \left(\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k \right)$$

Definition 1.8. A homogeneous system of linear equations defined is as

$$\left(\sum_{k=1}^n A_{1,k}x_k, \dots, \sum_{k=1}^n A_{m,k}x_k \right) = (0, \dots, 0)$$

And a Inhomogeneous system of linear equation is defined as

$$\left(\sum_{k=1}^n A_{1,k}x_k, \dots, \sum_{k=1}^n A_{m,k}x_k \right) = (c_1, \dots, c_m)$$

where not all c_i are zero.

Proposition 1.7. A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof. Use **Theorem 1.3** and **Theorem 1.4**. □

Proposition 1.8. An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof. Use **Theorem 1.3** and **Theorem 1.4**. □

1.2.4 Exercise

Problem : Give an example of a linear map T such that $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$.

Solution : $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2)$.

Problem : Suppose $S, T \in \mathcal{L}(V, W)$ are such that $\text{range } S \subseteq \text{null } T$. Prove that $(ST)^2 = 0$.

Solution : Since $\text{range } S \subseteq \text{null } T$, $T(Sv) = 0$. Thus,

$$(ST)^2 = (ST)(ST) = S(T(S(Tv))) = S(0) = 0$$

Problem : Suppose v_1, \dots, v_m is a list of vector in V . Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \dots, z_m) = z_1v_1 + \dots + z_mv_m$$

- What property of T corresponds to v_1, \dots, v_m spanning V ?
- What property of T corresponds to v_1, \dots, v_m being linearly independent on V ?

Solution : If v_1, \dots, v_m spans V then $\text{range } T = V$ thus T being surjective corresponds to v_1, \dots, v_m spanning V .

If v_1, \dots, v_m is linearly independent on V then $\text{null } T = \{0\}$, thus T being injective corresponds to v_1, \dots, v_m being linearly independent on V .

Problem: Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null } T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Solution : Let $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$ and $T'(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, 0)$. Both of them are in $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$. Also $\dim \text{null } T = 3$, $\dim \text{null } T' = 4$ but $T + T' = (x_1, x_2, x_3, 0) \implies \dim \text{null}(T + T') = 2 \not> 2$.

Problem : Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\text{range } T = \text{null } T$.

Solution : $T(x_1, x_2, x_3, x_4) = (0, 0, x_1, x_2)$.

Problem : Prove that there doesn't exist a $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\text{range } T = \text{null } T$.

Solution : Suppose there exists such T , then $\dim \text{range } T = \dim \text{null } T$ but from the fundamental theorem of linear maps we have

$$\begin{aligned} \dim V &= \dim \text{range } T + \dim \text{null } T \\ \implies \dim \text{range } T &= \dim \text{null } T = \frac{5}{2} \end{aligned}$$

which is impossible.

Problem : Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution : Let v_1, \dots, v_m be the basis of V and w_1, \dots, w_n be the basis of W . Then, define $T_i(a_1v_1 + \dots + a_mv_m) = a_iw_i$. One can check that this is not injective thus

$$T(a_1v_1 + \dots + a_mv_m) = \left(\sum_{i=1}^m T_i \right) (a_1v_1 + \dots + a_mv_m) = \sum_{i=1}^m a_iw_i$$

Now, suppose $T(v) = T(v')$ then

$$\begin{aligned} a_1w_1 + \dots + a_mw_m &= a'_1w_1 + \dots + a'_mw_m \\ \implies b_1 &= b'_1 \quad (\text{because of linear independence}) \end{aligned}$$

Thus, $v = v'$.

Problem : Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U and V such that

$$U \cap \text{null } T = \{0\} \quad \text{and} \quad \text{range } T = \{Tu \mid u \in U\}$$

Solution : We know that $\text{null } T$ is the subspace of V . Thus, there exists a U such that $V = U \oplus \text{null } T$ and since it is a direct sum $U \cap \text{null } T = \{0\}$. Now, for the range of T

$$\begin{aligned} \text{range } T &= \{Tv \mid v \in V\} \\ \implies \{T(u + z) \mid u \in U, z \in \text{null } T\} &= \{Tu \mid u \in U\} \end{aligned}$$

Problem : Suppose T is a linear map from \mathbf{F}^4 to \mathbf{F}^2 such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is a surjective linear map.

Solution : We can write T as

$$\text{null } T = \{(5x_2, x_2, 7x_4, x_4) \mid x_2, x_4 \in \mathbf{F}\}$$

Now, since $(5x_2, x_2, 7x_4, x_4) = x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1) \implies \dim \text{null } T = 2$. Thus,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$\implies 4 = 2 + \dim \text{range } T$$

$$\implies \dim \text{range } T = 2$$

Since, $\dim \mathbf{F}^2 = 2 = \dim \text{range } T \implies \text{range } T = \mathbf{F}^2$. Thus, T is surjective.

Problem : Suppose U is three-dimensional subspace of \mathbf{R}^8 and that T is a linear map from \mathbf{R}^8 to \mathbf{R}^5 such that $\text{null } T = U$. Prove that T is surjective.

Solution : Since, $\text{null } T = U \implies \dim \text{null } T = 3$. Thus,

$$\dim \mathbf{R}^8 = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim \text{range } T = 5$$

Since, $\text{range } T$ is a subspace of \mathbf{R}^5 and $\dim \text{range } T = \dim \mathbf{R}^5$, $\mathbf{R}^5 = \text{range } T$. Thus, T is surjective.

Problem : Prove that there does not exist a linear map from \mathbf{F}^5 to \mathbf{F}^2 whose null space doesn't equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

Solution : Suppose there does exist such a T . Then the null space can be written as

$$\text{null } T = \{(3x_2, x_2, k, k, k) \mid x_2, k \in \mathbf{F}\}$$

One can check that $\dim \text{null } T = 2$ but

$$\dim \mathbf{F}^5 = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim \text{range } T = 3$$

But $2 = \dim \mathbf{F}^2 \geq \dim \text{range } T = 3$ which is false.

Problem : Suppose there exists a linear map on V such that the null space and range of T is finite dimensional. Prove that V is finite-dimensional.

Solution : Since, the range of T is finite-dimensional it must have a basis. Suppose Tv_1, \dots, Tv_m is the basis then

$$T(x) = \lambda_1 Tv_1 + \dots + \lambda_m Tv_m$$

$$\implies T(x - \lambda_1 v_1 - \dots - \lambda_m v_m) = 0$$

Since, the null space is also finite-dimensional

$$x - \lambda_1 v_1 - \dots - \lambda_m v_m = \lambda'_1 v'_1 + \dots + \lambda'_n v'_n$$

where v'_1, \dots, v'_n is the basis of the null space. Thus,

$$V = \text{span}(v_1, \dots, v_m, v'_1, \dots, v'_n)$$

Problem : Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution : Suppose $T \in \mathcal{L}(V, W)$ is an injective map, then

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim V = \dim \text{range } T \leq \dim W$$

Now suppose $\dim V \leq \dim W$ then we can construct an injective map from V to W .

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n$$

where v_1, \dots, v_n is the basis of V and w_1, \dots, w_m is the basis of W . One can check this is a linear map and suppose $T(x) = T(y)$ and let $x = a_1v_1 + \cdots + a_nv_n$ and $y = b_1v_1 + \cdots + b_nv_n$ then

$$T(x) = T(y)$$

$$\implies (a_1 - b_1)w_1 + \cdots + (a_n - b_n)w_n = 0$$

$$\implies a_i = b_i$$

Thus, $x = y$.

Problem : Suppose V and W are finite-dimensional vector spaces and U is a subspace of V . Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\text{null } T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution : Suppose $\text{null } T = U$ then

$$\dim V - \dim \text{range } T = \dim U$$

$$\implies \dim V - \dim W \leq \dim V - \dim \text{range } T = \dim U$$

Now, suppose $\dim U \geq \dim V - \dim W$ then let u_1, \dots, u_k be the basis of U and

$$u_1, \dots, u_k, v_1, \dots, v_m$$

be the extended basis of V . Let w_1, \dots, w_j be the basis of W . From our condition, we know $k \geq k + m - j \implies j \geq m$. Thus we define

$$T(a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m) = b_1w_1 + \cdots + b_mw_m$$

Here, $\text{null } T = U$.

Problem : Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W . Prove that $X = \{v \in V \mid Tv \in U\}$ is a subspace of V and

$$\dim X = \dim \text{null } T + \dim(U \cap \text{range } T)$$

Solution : The subspace part is pretty simple. Let $S : X \rightarrow U$ be a map such that $S(v) = T(v)$. Here, $\text{range } S = U \cap \text{range } T$ and $\text{null } S = \text{null } T$.

Problem : Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{null } ST \leq \dim \text{null } T + \dim \text{null } S$$

Solution : One can find that $\text{null } ST = \text{null } T \cup \{x \in U \mid S(T(x)) = 0, T(x) \neq 0\}$.

Note : It seems that these exercises are taking way too long to do. I'll however come back to it and finish

1.3 Matrices

1.3.1 Representing a Linear Map by a Matrix

Definition 1.9. Suppose m and n are two non-negative integers. A $m \times n$ matrix is A is a rectangular array of elements in \mathbf{F} with m rows and n columns

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

The $A_{i,j}$ represents the entry in i -th row and j -th column.

Definition 1.10. Suppose $T \in \mathcal{L}(V, W)$ and v_1, \dots, v_n is the basis of V and w_1, \dots, w_m is a basis of W . The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ whose entries $A_{i,j}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

Examples :

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$$

Then,

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

As $T(1, 0) = 1(1, 0, 0) + 2(0, 1, 0) + 7(0, 0, 1)$ and $T(0, 1) = 3(1, 0, 0) + 5(0, 1, 0) + 9(0, 0, 1)$.

1.3.2 Addition and Scalar Multiplication of Matrices

Definition 1.11. The sum of two matrices of same size is obtained by adding corresponding entries in the matrices i.e

$$\begin{aligned} & \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & & \vdots \\ B_{m,1} & \cdots & B_{m,n} \end{pmatrix} \\ &= \begin{pmatrix} A_{1,1} + B_{1,1} & \cdots & A_{1,n} + B_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + B_{m,1} & \cdots & A_{m,n} + B_{m,n} \end{pmatrix} \end{aligned}$$

Proposition 1.9. Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Proof. Follows from the definition. □

Definition 1.12. The product of a scalar and a matrix is obtained by multiplying each entry by the scalar i.e

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

Proposition 1.10. Suppose $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$ then $\lambda \mathcal{M}(T) = \mathcal{M}(\lambda T)$.

Proof. Again, just use the definitions. \square

Theorem 1.5. Suppose $\mathbf{F}^{m,n}$ be the set of all the matrices with entries in \mathbf{F} . Then, with addition and scalar multiplication defined above $\mathbf{F}^{m,n}$ is a vector space and $\dim \mathbf{F}^{m,n} = mn$.

Proof. Proving it is a vector space is pretty easy. To verify $\dim \mathbf{F}^{m,n} = mn$ define

$$X_{i,j} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & 1 & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where every entry is 0 except the $A_{i,j}$ entry which is equal to 1. Now, its easy to see that every $Z \in \mathbf{F}^{m,n}$ can be written as some linear combination of $X_{i,j}$'s. Thus, $\mathbf{F}^{m,n} = \text{span}\{X_{i,j}\}$ where i, j vary with $i = 1, \dots, m$ and $j = 1, \dots, n$. Also, every matrix with only 0 as its entry can only be written as linear combination of $X_{i,j}$ with all of its scalars equal to 0. Since, there are mn entries the dimension of $\mathbf{F}^{m,n}$ is equal to mn . \square

1.3.3 Matrix Multiplication

Definition 1.13. Suppose A is a $m \times n$ matrix and B is $n \times p$ matrix. Then AB is defined as to be an $m \times p$ matrix whose entry in row j and column k is given by

$$(AB)_{j,k} = \sum_{r=1}^n A_{j,r} B_{r,k}$$

Remark. The motivation for us to define the product like this comes from questioning, does $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$? Suppose v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is the basis of W . Suppose u_1, \dots, u_k is the basis of U then consider the map $T : U \rightarrow V$ and $S : V \rightarrow W$. Suppose $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = B$. Then

$$\begin{aligned} (ST)u_k &= S \left(\sum_{r=1}^n B_{r,k} v_r \right) \\ &= \sum_{r=1}^n B_{r,k} S v_r \\ &= \sum_{r=1}^n B_{r,k} \sum_{j=1}^m A_{j,r} w_j \\ &= \sum_{j=1}^m \left(\sum_{r=1}^n A_{j,r} B_{r,k} \right) w_j \end{aligned}$$

That is how we define $\mathcal{M}(ST)$ and that is why **Definition 1.13.** makes sense.

Proposition 1.11. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then

$$\mathcal{M}(ST) = \mathcal{M}(S) \mathcal{M}(T)$$

Proof. It follows from our remark and how we defined the product of the matrix. \square

Definition 1.14. Suppose A is a $m \times n$ matrix then

1. If $1 \leq j \leq m$ then $A_{j,\cdot}$ denotes the $1 \times n$ matrix consisting of row j of A .
2. If $1 \leq j \leq n$ then $A_{\cdot,j}$ denotes $m \times 1$ matrix consisting of column j of A .

Example :

Suppose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then

$$A_{1,\cdot} = (1 \quad 2 \quad 3)$$

$$A_{\cdot,3} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Theorem 1.6. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

where $1 \leq j \leq m$ and $1 \leq k \leq p$.

Proof. The definition of matrix multiplication states that

$$\begin{aligned} (AB)_{j,k} &= \sum_{r=1}^n A_{j,r} B_{r,k} \\ &= A_{j,1} B_{1,k} + \cdots + A_{j,n} B_{n,k} \end{aligned}$$

Now, if you take $A_{j,\cdot}$ and $B_{\cdot,k}$ and multiply it out you'll get the same thing. \square

Theorem 1.7. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. Then

$$(AB)_{\cdot,k} = A B_{\cdot,k}$$

for $1 \leq k \leq p$.

Proof. Both of the matrix have size $m \times 1$. The j -th row of $(AB)_{\cdot,k}$ has the element $(AB)_{j,k}$ and the j -th row of $A B_{\cdot,k}$ has element $A_{j,1} B_{1,k} + A_{j,2} B_{2,k} + \cdots + A_{j,n} B_{n,k}$. Thus, from our previous theorem they're equal. \square

Remark. The row version of this is

$$(AB)_{k,\cdot} = A_{k,\cdot} B$$

Theorem 1.8. Suppose A is a $m \times n$ matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is a $n \times 1$ matrix. Then,

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$$

Proof. They both have same size and the entries of Ab is the same as of the right side. \square

Theorem 1.9. Suppose C is an $m \times c$ matrix and R is a $c \times n$ matrix

1. The column k of CR is the linear combination of the columns of C , with coefficients of this linear combination coming from column k of R .
2. Then row j of CR is a linear combination of the rows of R , with the coefficients of this linear combination coming from row j of C .

Proof. Use **Theorem 1.7.** and **Theorem 1.8.** for 1. and we'll prove 2. in the exercise section. \square

1.3.4 Column-Row Factorization and Rank of a Matrix

Definition 1.15. Suppose A is a $m \times n$ matrix with entries in \mathbf{F} .

1. The **column rank** of A is the dimension of the span of columns of A in $\mathbf{F}^{1,m}$.
2. The **row rank** of A is the dimension of the span of rows of A in $\mathbf{F}^{n,1}$.

Example : Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then the column rank is the dimension of

$$\text{span} \left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right)$$

and the row rank is the dimension of

$$\text{span} \left((1 \ 2 \ 3), (4 \ 5 \ 6) \right)$$

Definition 1.16. The *transpose* of a $m \times n$ matrix A , denoted by A^t , is the $n \times m$ matrix whose entries are given by

$$(A^t)_{i,j} = A_{j,i}$$

Theorem 1.10. Suppose A is an $m \times n$ matrix with entries in \mathbf{F} and column rank $c \geq 1$. Then there exists a $m \times c$ matrix C and $c \times n$ matrix R , both with entries in \mathbf{F} , such that $A = CR$.

Proof. The list $A_{\cdot,1}, \dots, A_{\cdot,n}$ of columns of A can be reduced to a basis of the span of the columns of A . This basis has length c by definition of column rank. The c columns can be put together to form $m \times c$.

Now, each column k of A is a linear combination of columns of C . Make the coefficients of this linear combination column k of R . This matrix R has size $c \times n$. Thus, $A = CR$ follows from **Theorem 1.9.(a)**. \square

Theorem 1.11. Suppose $A \in \mathbf{F}^{m,n}$ then the column rank of A equals row rank of A .

Proof. Let c be the column rank of A . Then $A = CR$ by the previous theorem where C and R are the matrix whose size are $m \times c$ and $c \times n$ respectively. Now, from **Theorem 1.9. (b)** each row of A is a linear combination of rows of R . Since, R has c columns this implies that

$$\text{rowrank } A \leq c = \text{columnrank } A$$

Now applying the same thing to A^t we get

$$\begin{aligned}\text{columnrank } A &= \text{rowrank } A^t \\ &\leq \text{columnrank } A^t \\ &= \text{rowrank } A\end{aligned}$$

Thus, we're done. □

Remark. From now on, we'll limit our use of terminology of “row rank” and “column rank” to just “rank”.

1.3.5 Exercise

Problem : Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of basis of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

Solution : Let v_1, \dots, v_n be the basis of V and w_1, \dots, w_m be the basis of W then suppose k of those vectors are 0 under T and let those vectors be v_1, \dots, v_k . Thus,

$$\text{range } T = \text{span}\{Tv_{k+1}, \dots, Tv_n\}$$

Thus, $\dim \text{range } T \leq n - k$. But since $T(v_j) \neq 0$ for each $k+1 \leq j \leq n$, there must be one entry that is not 0 for each $T(v_j)$. Since, the number of $T(v_j) \neq 0$ are exactly $n - k$ and $n - k \geq \dim \text{range } T$ this means there is at least $\dim \text{range } T$ nonzero entries in matrix of T .

Problem : Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \text{range } T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all the entries of $\mathcal{M}(T)$ is 1.

Solution : Let us first prove (\Leftarrow). Suppose $A_{i,j} = 1$ for all i, j . That means, $T(v_i) = \sum w_j$ where v_1, \dots, v_n is the basis of V and w_1, \dots, w_m is the basis of W . Thus, $Tv_1 = Tv_2 = \dots = Tv_n = \sum w_j$ and $\text{range } T = \text{span}\{Tv_1, \dots, Tv_n\} = \text{span}\{\sum w_j\} \implies \dim \text{range } T = 1$. Now, for the (\Rightarrow) we use the Fundamental theorem of linear maps,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim V = \dim \text{null } T + 1$$

Now, suppose v_2, \dots, v_n be that basis of $\text{null } T$. Then extend this basis to V , suppose v_2, \dots, v_n, v then

$$\begin{aligned} T(v_2) &= T(v_3) = \dots = T(v_n) = 0 \\ \implies T(v) &= T(v_2 + v) = \dots = T(v_n + v) \end{aligned}$$

One can check that $v_2 + v, \dots, v$ is a basis of V (as it is linearly independent and has length n). Now, since $\dim \text{range } T = 1$ we have $T(x) = \lambda T(v)$ and we choose $T(v), w_2, \dots, w_m$ as our basis for W . Now, we use a clever trick and set $w_1 = T(v) - w_2 - w_3 - \dots - w_m$ and notice that w_1, \dots, w_m is a basis of W . Thus,

$$T(v_i) = T(v) = \sum w_j$$

Thus, we're done.

Problem : Suppose that $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution : Take the basis of $\mathcal{P}_3(\mathbf{R})$ to be $z, \frac{z^2}{2}, \frac{z^3}{3}, 1$ and $\mathcal{P}_2(\mathbf{R})$ to be $1, z, z^2$.

Problem : Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k , column k , equal 1 if $1 \leq k \leq \dim \text{range } T$.

Solution : Let $\dim V = n$ and $\dim \text{range } T = m$. Now, let v_{m+1}, \dots, v_n be the basis of $\dim \text{null } T$. Now, extend these basis such in the following way

$$(v_1, \dots, v_m, v_{m+1}, \dots, v_n)$$

Here, $T(v_i) \neq 0$ for $1 \leq i \leq m$. Now, since $(v_1, \dots, v_m, v_{m+1}, \dots, v_n)$ spans V the list (Tv_1, \dots, Tv_m) must span $\text{range } T$ and in fact it is the basis of $\text{range } T$ (one can check that its linearly independent). We can now extend this basis to the basis of W . Suppose $(Tv_1, \dots, Tv_m, w_1, \dots, w_k)$ is the basis of W . Then,

$$T(v_i) = 0 \cdot T(v_1) + \dots + 1 \cdot T(v_i) + \dots + 0 \cdot w_k$$

for $1 \leq i \leq m = \dim \text{range } T$. But for $i > \dim \text{range } T$ we have

$$0 = T(v_i) = 0 \cdot T(v_1) + 0 \cdot T(v_2) + \dots + 0 \cdot w_k$$

Problem : Suppose $\sigma_1, \dots, \sigma_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis w_1, \dots, w_n of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $\sigma_1, \dots, \sigma_m$ and w_1, \dots, w_n] are 0 except for possibly a 1 in the first row, first column.

Solution : We know that $\text{range } T = \text{span}\{T(\sigma_1), T(\sigma_2), \dots, T(\sigma_m)\}$. Thus, we can make this span a basis. If $T(\sigma_1) = 0$ then we're done but if not then the basis of $\text{range } T$ would be

$$(T(\sigma_1), z_2, \dots, z_k)$$

Now, we can extend this basis to the basis of W , suppose its

$$(T(\sigma_1), z_2, \dots, z_k, s_{k+1}, \dots, s_m)$$

then

$$T(\sigma_1) = 1 \cdot T(\sigma_1) + 0 \cdot z_2 + \dots + 0 \cdot s_m$$

Problem : Suppose w_1, \dots, w_n is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $\sigma_1, \dots, \sigma_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases $\sigma_1, \dots, \sigma_m$ and w_1, \dots, w_n] are 0 except for possibly a 1 in the first row, first column.

Solution : Take any basis v_1, \dots, v_m of V . Then, suppose

$$\begin{aligned} T(v_i) &= \sum_{j=1}^n i\lambda_j w_j \\ &= i\lambda_1 w_1 + \sum_{j=2}^n i\lambda_j w_j \end{aligned}$$

Now, if all the $T(v_i)$ has 0 as the coefficient of w_1 then we're done. If not then take a v_k for which $k\lambda_1 \neq 0$ then swap it with v_1 . Then, define

$$\sigma_1 = \frac{v_1}{1\lambda_1}$$

$$\sigma_i := v_i - \frac{i\lambda_1}{1\lambda_1}v_1 \quad \text{for } i \geq 2$$

Now, one can check that $(\sigma_1, \dots, \sigma_m)$ is a basis and

$$\begin{aligned} T(\sigma_i) &= T(v_i) - \frac{i\lambda_1}{1\lambda_1}T(v_k) \\ &= 0 \cdot w_1 + \sum b_j w_j \end{aligned}$$

Problem : Give an example of 2×2 matrices A and B such that $AB \neq BA$.

Solution :

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Problem : Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, F are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that

$$A(B + C) = AB + AC \quad \text{and} \quad (D + E)F = DF + EF.$$

Solution : If $A(B + C)$ and $(D + E)F$ makes sense then B and C must be of the same size and D and E must be of the same size. Also, the number of columns in A must be same as the number of rows in B and C . Let A be a $n \times p$ matrix and $X = B + C$ then

$$\begin{aligned} (AX)_{i,j} &= \sum_{r=1}^n A_{i,r}X_{r,j} \\ &= \sum_{r=1}^n A_{i,r}(B_{r,j} + C_{r,j}) \\ &= \sum_{r=1}^n A_{i,r}B_{r,j} + \sum_{r=1}^n A_{i,r}C_{r,j} \\ &= (AB)_{i,j} + (AC)_{i,j} \end{aligned}$$

Thus, $A(B + C) = AB + AC$. Similar proof works for $(D + E)F$.

Problem : Prove that matrix multiplication is associative. In other words, suppose A, B, C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $(AB)C$ makes sense and prove that

$$(AB)C = A(BC).$$

Solution : To make $(AB)C$ sense, we need A to have same number of columns as the number of rows in B . Also, we need B to have same number of columns as number of rows in C . To prove the associativity, you can just definition of matrix multiplication.

Problem : Suppose A is an $n \times n$ matrix and $1 \leq j, k \leq n$. Show that the entry in row j , column k , of A^3 (which is defined to mean AAA) is

$$\sum_{r=1}^n \sum_{i=1}^n A_{j,r} A_{r,i} A_{i,k}.$$

Solution : It follows directly from the definition of matrix multiplication.

Problem : Suppose m and n are positive integers. Prove that the function $A \mapsto A^t$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Solution : Define $T : \mathbf{F}^{m,n} \rightarrow \mathbf{F}^{n,m}$ by $T(A) = A^t$. We must show that T is linear, i.e.

$$T(A + B) = T(A) + T(B) \quad \text{and} \quad T(\lambda A) = \lambda T(A),$$

for all $A, B \in \mathbf{F}^{m,n}$ and all scalars $\lambda \in \mathbf{F}$.

By definition of the transpose,

$$(A^t)_{ij} = A_{ji}, \quad (1 \leq i \leq n, 1 \leq j \leq m).$$

Now let $A, B \in \mathbf{F}^{m,n}$. Then for each i, j ,

$$((A + B)^t)_{ij} = (A + B)_{ji} = A_{ji} + B_{ji} = (A^t)_{ij} + (B^t)_{ij} = (A^t + B^t)_{ij}.$$

Hence $(A + B)^t = A^t + B^t$.

Similarly, for $\lambda \in \mathbf{F}$,

$$((\lambda A)^t)_{ij} = (\lambda A)_{ji} = \lambda A_{ji} = \lambda (A^t)_{ij} = (\lambda A^t)_{ij}.$$

So $(\lambda A)^t = \lambda A^t$.

Therefore T preserves both addition and scalar multiplication. Thus T is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Problem : Prove that if A is an $m \times n$ matrix and C is an $n \times p$ matrix, then

$$(AC)^t = C^t A^t.$$

Solution :

Solution : Let $A \in \mathbf{F}^{m,n}$ and $C \in \mathbf{F}^{n,p}$. By definition of matrix multiplication, the (i, j) -entry of AC is

$$(AC)_{ij} = \sum_{k=1}^n A_{ik} C_{kj}, \quad (1 \leq i \leq m, 1 \leq j \leq p).$$

Taking the transpose, we get

$$((AC)^t)_{ij} = (AC)_{ji} = \sum_{k=1}^n A_{jk} C_{ki}.$$

On the other hand, consider the product $C^t A^t$. Here C^t is $p \times n$ and A^t is $n \times m$, so $C^t A^t$ is $p \times m$. Its (i, j) -entry is

$$(C^t A^t)_{ij} = \sum_{k=1}^n (C^t)_{ik} (A^t)_{kj}.$$

By definition of the transpose,

$$(C^t)_{ik} = C_{ki}, \quad (A^t)_{kj} = A_{jk}.$$

Hence

$$(C^t A^t)_{ij} = \sum_{k=1}^n C_{ki} A_{jk}.$$

But scalar multiplication in \mathbf{F} is commutative, so

$$\sum_{k=1}^n C_{ki} A_{jk} = \sum_{k=1}^n A_{jk} C_{ki}.$$

Therefore,

$$(C^t A^t)_{ij} = ((AC)^t)_{ij}, \quad (1 \leq i \leq p, 1 \leq j \leq m).$$

Since all entries are equal, we conclude

$$(AC)^t = C^t A^t.$$

Problem : Suppose A is an $m \times n$ matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that

$$A_{j,k} = c_j d_k \quad \text{for every } j = 1, \dots, m \text{ and every } k = 1, \dots, n.$$

Solution : For (\Leftarrow) , Use **Theorem 1.10**. then use the definition of matrix multiplication. For (\Rightarrow) , the matrix

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

can produce any column thus the rank is 1.

Problem : Suppose $T \in \mathcal{L}(V)$, and u_1, \dots, u_n and v_1, \dots, v_n are bases of V . Prove that the following are equivalent:

- (a) T is injective.
- (b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.
- (c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.
- (d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.
- (e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

Solution : Will do later.

1.4 Invertibility and Isomorphism

1.4.1 Invertible Linear Maps

Definition 1.17. A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals identity operator on V and TS equals identity operator on W .

Definition 1.18. A linear map $S \in \mathcal{L}(V, W)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of T .

Proposition 1.12. An invertible map has an unique inverse.

Proof. Suppose $T \in \mathcal{L}(V, W)$ and let S_1 and S_2 be its inverses then

$$S_1 = S_1 I = S_1 (TS_2) = (S_1 T) S_2 = I S_2 = S_2$$

□

Remark. Since inverses are unique for a invertible map T , we will denote it by T^{-1} .

Proposition 1.13. A linear map is invertible if and only if it is injective and surjective.

Proof. Suppose $T \in \mathcal{L}(V, W)$ is an invertible map and suppose $T(v) = T(w)$ then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

Hence, T is injective. To prove surjectivity, notice that

$$w = T^{-1}(Tw)$$

which proves T is surjective.

Now, suppose T is injective and surjective. Then, there exists a unique element $S(w)$ such that

$$T(S(w)) = w$$

the uniqueness is due to the injectivity of T . Let us show that, $S \in \mathcal{L}(W, V)$

$$\begin{aligned} T(S(w_1) + S(w_2)) &= T(S(w_1)) + T(S(w_2)) \\ &= w_1 + w_2 \\ &= T(S(w_1 + w_2)) \end{aligned}$$

Thus, $S(w_1) + S(w_2) = S(w_1 + w_2)$. Also,

$$\begin{aligned} T(\lambda S(w)) &= \lambda T(S(w)) \\ &= \lambda w \\ &= T(S(\lambda w)) \end{aligned}$$

Thus, $\lambda S(w) = S(\lambda w)$. □

Now, by how we defined S , it implies that $TS = I$ on W . Also,

$$\begin{aligned} T(ST)v &= (TS)(T)v = Tv \\ \implies (ST)v &= v \end{aligned}$$

Thus, ST is an identity operator on V .

Proposition 1.14. Suppose that V and W are finite-dimensional vector spaces, such that, $\dim W = \dim V$ and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective}$$

Proof. From the Fundamental theorem of linear maps,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

If T is injective then $\text{null } T = \{0\}$. Thus

$$\dim V = \dim W = \dim \text{range } T$$

$$\implies \text{range } T = W$$

Now, if T is surjective then $\text{range } T = W$. Thus

$$\dim V = \dim \text{null } T + \dim W$$

$$\implies \dim \text{range } T = 0$$

$$\implies \text{range } T = \{0\}$$

Thus, T is injective $\iff T$ is surjective. From **Proposition 1.13.** we get our final result. \square

Proposition 1.15. Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(V, W)$. Then, $ST = I \iff TS = I$.

Proof. First $ST = I$ then take $v \in \text{null } T$. Thus,

$$v = STv = S(0) = 0$$

Thus, $\text{null } T = \{0\}$ and T is injective. Since $\dim V = \dim W$, this implies T is invertible. Thus, there exists a T^{-1} . Now,

$$T^{-1} = (ST)(T^{-1}) = S$$

We can now apply the same idea for (\Leftarrow) of the proof. We just need to swap V with W , and T with S . \square

1.4.2 Isomorphic Vector Spaces

Definition 1.19. An *isomorphism* is an invertible linear map and two vector spaces are isomorphic if there is an isomorphism between them.

Proposition 1.16. Two finite-dimensional vector spaces over \mathbf{F} are isomorphic if and only if they have the same dimension.

Proof. Suppose V and W are isomorphic. Then there exists a injective and surjective map T from V to W . Thus, $\text{null } T = \{0\}$. Then

$$\dim V = \dim \text{range } T$$

Also, since T is surjective $\text{range } T = W$. Then

$$\dim V = \dim W$$

Now, suppose $\dim W = \dim V$. Define $T : V \rightarrow W$ as

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

where v_i 's and w_i 's are the basis of V and W respectively. One can check this is a linear map. Now, this map is surjective as $\sum c_iw_i$ covers W . Also, $\text{null } T = \{0\}$ as

$$\begin{aligned} \dim W &= \dim V = \dim \text{null } T + \dim \text{range } T \\ &= \dim \text{null } T + \dim \text{range } T \\ &= \dim \text{null } T + \dim W \end{aligned}$$

Thus, T is injective and surjective which means that V and W are isomorphic. □

Proposition 1.17. Suppose v_1, \dots, v_n be the basis of V and w_1, \dots, w_m be the basis of W . Then $\mathcal{M}(T)$ is a isomorphism between $\mathcal{L}(V, W)$ to $\mathbf{F}^{m,n}$

Proof. We know that $\mathcal{M}(T)$ is a linear map as

$$\mathcal{M}(T + S) = \mathcal{M}(T) + \mathcal{M}(S) \quad \text{and} \quad \mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$$

We need to prove that \mathcal{M} is injective and surjective. We know that $\mathcal{M}(T)$ is injective $\iff \text{null } \mathcal{M}(T) = \{0\}$. And we know $\mathcal{M}(T) = 0 \iff T(x) = 0$ for all $x \in V$. Thus, $T = 0$.

To prove $\mathcal{M}(T)$ is surjective. We know that there exists a $T \in \mathcal{L}(V, W)$ such that

$$T(v_k) = \sum_{j=1}^m A_{i,j} w_j$$

which proves the surjectivity of $\mathcal{M}(T)$. □

Proposition 1.18. Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Proof. Use **Proposition 1.17.** and **Proposition 1.16.** and

$$\dim \mathcal{L}(V, W) = mn = (\dim V)(\dim W)$$

□

1.4.3 Linear Map Thought of as Matrix Multiplication

Definition 1.20. Suppose $v \in V$ and v_1, \dots, v_n is the basis of V . The matrix of v with respect to the basis is the n matrix

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

where b_1, \dots, b_n are scalar such that $v = b_1v_1 + \cdots + b_nv_n$.

Proposition 1.19. Suppose $T \in \mathcal{L}(V, W)$ and v_1, \dots, v_n is a basis of V and w_1, \dots, w_n is a basis of W . Let $1 \leq k \leq n$. Then,

$$\mathcal{M}(T)_{\cdot, k} = \mathcal{M}(Tv_k)$$

Proof. Immediate from the definition of $\mathcal{M}(Tv_k)$. \square

Proposition 1.20. Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Let v_1, \dots, v_n be the basis of V and w_1, \dots, w_m be the basis of W . Then

$$\mathcal{M}(Tv) = \mathcal{M}(T) \mathcal{M}(v)$$

Proof. Suppose $v = b_1 v_1 + \dots + b_n v_n$. Then,

$$Tv = b_1 Tv_1 + \dots + b_n Tv_n$$

Hence,

$$\begin{aligned} \mathcal{M}(Tv) &= b_1 \mathcal{M}(Tv_1) + \dots + b_n \mathcal{M}(Tv_n) && (\text{Linearity of } \mathcal{M}) \\ &= b_1 \mathcal{M}(T)_{\cdot, 1} + \dots + b_n \mathcal{M}(T)_{\cdot, n} && (\text{Proposition 1.19.1}) \\ &= \mathcal{M}(T) \mathcal{M}(v) && (\text{Theorem 1.8.}) \end{aligned}$$

\square

Proposition 1.21. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \text{range } T$ equals the column rank of $\mathcal{M}(T)$.

Proof. Suppose v_1, \dots, v_n be the basis of V and w_1, \dots, w_m be the basis of W . Now, define $\varphi : W \rightarrow \mathbf{F}^{m,1}$ as $\varphi(w) = \mathcal{M}(w)$. One can prove that this is an isomorphism. If we restrict our domain to just $\text{range } T$ we see that our co-domain is going to be $\mathcal{O} = \text{span}\{\mathcal{M}(Tv_1), \dots, \mathcal{M}(Tv_m)\}$. Also,

$$\varphi|_{\text{range } T} : \text{range } T \rightarrow \mathcal{O}$$

is a isomorphism and since isomorphism preserves dimension. We have

$$\dim \text{range } T = \dim \mathcal{O} = \text{column rank of } T$$

\square

1.4.4 Change of Basis

Definition 1.21. We define the $n \times n$ matrix, called *identity matrix* by

$$A_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The identity matrix is denoted by I .

Definition 1.22. A square matrix is called *invertible* if there is a square matrix B of the same size such that

$$AB = BA = I$$

we call the matrix B the *inverse* of A .

Remark. The inverse of a square matrix A is unique and therefore will be denoted by A^{-1} . Here, is a short proof of the uniqueness of the inverse. Suppose A has two inverses B_1 and B_2 . Thus,

$$B_1 = IB_1 = (B_2A)B_1 = B_2(AB_1) = B_2I = B_2$$

Also, $(A^{-1})^{-1} = I$ and $(AC)^{-1} = C^{-1}A^{-1}$. You can verify these.

Definition 1.23. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. If u_1, \dots, u_m is a basis of U , v_1, \dots, v_n is a basis of V , and w_1, \dots, w_p is the basis of W then

$$\mathcal{M}(ST, (u_1, \dots, u_m), (w_1, \dots, w_p)) = \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_p)) \mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_n))$$

This is just the matrix multiplication which we had defined earlier but with respect to the basis. See **Proposition 1.11**.

Proposition 1.22. Suppose u_1, \dots, u_n and v_1, \dots, v_n are the basis of V . Then the matrices

$$\mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) \quad \text{and} \quad \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n))$$

are inverses of each other. Here, I is the identity operator.

Proof. Use **Definition 1.23**. and replace w_k with u_k . And replace S, T with the identity operator. Then

$$I = \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n)) \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$$

Now interchange the roles of u 's and v 's to get

$$I = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n))$$

□

Remark. For convenience, we'll write

$$\mathcal{M}(T, (u_1, \dots, u_n), (u_1, \dots, u_n)) = \mathcal{M}(T, (u_1, \dots, u_n))$$

Proposition 1.23. Suppose $T \in \mathcal{L}(V)$. Let u_1, \dots, u_n and v_1, \dots, v_n be the basis of V . Let

$$A = \mathcal{M}(T, (u_1, \dots, u_n)) \quad \text{and} \quad B = \mathcal{M}(T, (v_1, \dots, v_n))$$

and $C = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$. Then,

$$A = C^{-1}BC$$

Proof. Use **Definition 1.23**. and replace w_k with u_k and S with I . Then, use

Proposition 1.22. to get

$$A = C^{-1} \mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n)) \tag{1}$$

Now, again use the definition and this time replace w_k with v_k . Then

$$\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n)) = BC$$

We can now substitute this equation in equation (1) to get

$$A = C^{-1}BC$$

□

Proposition 1.24. Suppose that v_1, \dots, v_n is the basis of V and $T \in \mathcal{L}(V)$ is invertible. Then, $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to basis v_1, \dots, v_n .

Proof. Use **Definition 1.23**. □

1.4.5 Exercise

Problem : Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T$$

Solution : Since, T is invertible, we have

$$TT^{-1} = I \quad \text{and} \quad T^{-1}T = I$$

If we switch our perspective from T to T^{-1} , we get that T is invertible from **Definition 1.17.** and from **Proposition 1.12.** we have

$$(T^{-1})^{-1} = T$$

Problem : Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Solution : Since, S and T are both invertible then S^{-1} and T^{-1} both exist. Also, $T^{-1}S^{-1} \in \mathcal{L}(W, U)$. Thus,

$$\begin{aligned} (ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\ &= S(I)S^{-1} \\ &= SS^{-1} \\ &= I \end{aligned}$$

One can do the same thing for $(T^{-1}S^{-1})(ST)$. Thus, ST is invertible and from the above calculation so we can say $(ST)^{-1} = T^{-1}S^{-1}$.

Problem : Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

- (a) T is invertible
- (b) Tv_1, \dots, Tv_n is a basis of V for every basis v_1, \dots, v_n of V .
- (c) Tv_1, \dots, Tv_n is a basis of V for some basis v_1, \dots, v_n of V .

Solution : Suppose T is invertible then it is also injective and surjective. Let v_1, \dots, v_n be a basis of V . Then, we know that $\text{span}\{Tv_1, \dots, Tv_n\} = V$ because of the surjectivity. Also, if

$$\begin{aligned} a_1Tv_1 + \dots + a_nTv_n &= 0 \\ \implies T(a_1v_1 + \dots + a_nv_n) &= 0 \\ \implies a_1v_1 + \dots + a_nv_n &= 0 \\ \implies a_1 = a_2 = \dots = a_n &= 0 \end{aligned}$$

The last line is from injectivity of T . Thus, Tv_1, \dots, Tv_n is a basis of V for any basis of V .

Now, suppose Tv_1, \dots, Tv_n is a basis of V for every basis v_1, \dots, v_n of V . Then, (c) automatically holds. Also,

$$\begin{aligned} a_1Tv_1 + \dots + a_nTv_n &= 0 \\ \implies a_1 &= a_2 = \dots = a_n = 0 \end{aligned}$$

Thus, $\text{null } T = \{0\}$ which implies T is injective. Now, since Tv_1, \dots, Tv_n is a basis, every element of V can be written as some combination of V . Thus,

$$\begin{aligned} a_1Tv_1 + \dots + a_nTv_n &= y \\ \implies T(a_1v_1 + \dots + a_nv_n) &= y \end{aligned}$$

Thus, for every $y \in V$ there exists some element which gets mapped to y . Thus, T is surjective. Thus, from **Proposition 1.13**, we get that T is invertible.

Now, suppose Tv_1, \dots, Tv_n is a basis of V for some basis v_1, \dots, v_n of V . Then we can apply the same argument as we did for above to get to T is invertible. Since, T is invertible we get (b).

Problem : Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of non-invertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Solution : We can construct two non-invertible linear maps which form an invertible map when added. Suppose v_1, \dots, v_n is a basis of V .

$$T(a_1v_1 + \dots + a_nv_n) = a_2v_2 + \dots + a_nv_n$$

$$S(a_1v_1 + \dots + a_nv_n) = a_1v_1$$

One can check that both of them are linear maps and both of them lack injectivity property so they're not invertible. But

$$(S + T)(a_1v_1 + \dots + a_nv_n) = a_1v_1 + a_2v_2 + \dots + a_nv_n$$

$$(S + T)(x) = I(x)$$

which is an invertible linear map. Thus, set of non-invertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Remark. We used the $\dim V > 1$ when we defined T and S .

Problem : Suppose V is finite-dimensional, U is a subspace of V , and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution : For (\Rightarrow) , if $S(x) = S(y)$ then $T(x) = T(y)$ which implies $x = y$ because T is invertible. Now, for (\Leftarrow) choose a basis of U and extend it to the basis of V say $\mathcal{B} = (u_1, \dots, u_k, v_{k+1}, \dots, v_n)$, here $n = \dim V$. Now since S is injective, the list (Su_1, \dots, Su_k) is linearly independent and can be extended to a basis of V . Let

$$\mathcal{C} = \{Su_1, \dots, Su_k, w_{k+1}, \dots, w_n\}$$

be the basis of V . Define $T : V \rightarrow V$ as following

$$T(u_i) = S(u_i) \text{ for } 1 \leq i \leq k \quad \text{and} \quad T(v_j) = w_j \text{ for } k+1 \leq j \leq n$$

One can check this is an invertible linear map.

Problem : Suppose W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null } S = \text{null } T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

Solution : For (\Leftarrow) , take $x \in \text{null } T$ then $S(x) = E(T(x)) = 0$ thus $x \in \text{null } S$. Now, if $x \in \text{null } S$ then $0 = S(x) = ET(x) \implies T(x) = 0$ thus $x \in \text{null } T$. Thus, $\text{null } T = \text{null } S$. I'll do the \Leftarrow later.

1.5 Product and Quotients of Vector Spaces

1.5.1 Products of Vector Spaces

Definition 1.24. Suppose V_1, \dots, V_m are vector spaces over \mathbf{F} .

- The product $V_1 \times \dots \times V_m$ is defined by

$$V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$$

- Addition on $V_1 \times \dots \times V_m$ is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$$

- Scalar Multiplication on $V_1 \times \dots \times V_m$ is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$$

Proposition 1.25. Suppose V_1, \dots, V_m are vector spaces over \mathbf{F} . Then $V_1 \times \dots \times V_m$ is a vector space over \mathbf{F} .

Proof. Just check the vector axioms. □

Proposition 1.26. Suppose V_1, \dots, V_m are finite-dimensional vector spaces. Then $V_1 \times \dots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$$

Proof. Choose a basis of V_k and consider every element of $V_1 \times \dots \times V_k$ that equals a element from the basis of the vector V_k in the k -th slot and 0 in others. The list of vector spans $V_1 \times \dots \times V_m$ and is linearly independent. Thus, it is the basis of $V_1 \times \dots \times V_m$. The length of the basis is $\dim V_1 + \dots + \dim V_m$. □

Proposition 1.27. Suppose that V_1, \dots, V_m are subspaces of V . Define a linear map $\Gamma : V_1 \times \dots \times V_m \rightarrow V_1 + \dots + V_m$ by

$$\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$$

Then $V_1 + \dots + V_m$ is a direct sum if and only if Γ is injective.

Proof. If $V_1 + \dots + V_m$ is a direct sum then the only way we can write 0 is by choosing 0 from each V_i . Thus,

$$\Gamma(v_1, \dots, v_m) = 0 \iff v_1 = v_2 = \dots = v_m = 0$$

Thus, $\text{null } \Gamma = \{0\}$ which implies Γ is injective.

Now, suppose Γ is injective then $\text{null } \Gamma = \{0\}$, which means that the only element that gets mapped to 0 is $(0, \dots, 0)$. Thus, the only way to write 0 is by choosing 0 from each V_i . Thus, $V_1 + \dots + V_m$ is a direct sum. □

Proposition 1.28. Suppose V is finite-dimensional and V_1, \dots, V_m are subspaces of V . Then $V_1 + \dots + V_m$ is direct sum if and only if

$$\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$$

Proof. The map Γ is surjective. And $V_1 + \dots + V_m$ is a direct sum

$$\iff \Gamma \text{ is injective}$$

$$\iff \text{null } \Gamma = \{0\}$$

$$\iff \dim(V_1 \times \dots \times V_m) = \dim(V_1 + \dots + V_m)$$

Combining the result of **Proposition 1.26.** we get our desired result. □

1.5.2 Quotients Spaces

Definition 1.25. Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is a subset of V defined by

$$v + U = \{v + u \mid u \in U\}$$

Definition 1.26. For $v \in V$ and U a subset of V , the set $v + U$ is said to be a *translate* of U .

Definition 1.27. Suppose U is a subspace of V . Then the *quotient space* V/U is the set of all translate of U ,

$$V/U = \{v + U \mid v \in V\}$$

Proposition 1.29. Suppose U is a subspace of V and $v, w \in V$ then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset$$

Proof. Suppose $v - w \in U$ then $v = w + u'$ for some $u' \in U$ thus, $v + u = w + (u' + u) \in w + U$ which implies $v + U \subseteq w + U$. Thus, similarly $w + U \subseteq v + U \implies v + U = w + U \implies (v + U) \cap (w + U) \neq \emptyset$.

Now, suppose $(v + U) \cap (w + U) \neq \emptyset$ then $v + u_1 = w + u_2$ for some $u_1, u_2 \in U$ which implies $v - w \in U$, which implies $v + U = w + U$. And $v + U = w + U \implies v - w \in U$. Thus, we proved every direction of the proof. \square

Definition 1.28. Suppose U is a subspace of V . Then *addition* and *scalar multiplication* are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U$$

$$\lambda(v + U) = (\lambda v) + U$$

for all $v, w \in V$ and all $\lambda \in F$.

Proposition 1.30. Suppose U is a subspace of V . Then V/U , with the operations of addition and scalar multiplication as defined above, is a vector space.

Proof. Just use previous definitions and check the vector axioms. \square

Proposition 1.31. Suppose U is a subspace of V . The *quotient map* $\pi : V \rightarrow V/U$ is a linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

Proof. Note that $\pi(a + b) = (a + b) + U = (a + U) + (b + U) = \pi(a) + \pi(b)$ and $\pi(\lambda a) = (\lambda a) + U = \lambda(a + U) = \lambda\pi(a)$. \square

Proposition 1.32. Suppose V is finite-dimensional vector space and U is a subspace of V then

$$\dim V/U = \dim V - \dim U$$

Proof. We use the map quotient map π introduced before. We know that $a + U = 0 + U \iff a \in U$ this null $\pi = U$ and $\text{range } \pi = V/U$ as the map is surjective. Thus, using the fundamental theorem of linear map we get our desired result. \square

Definition 1.29. Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : V/(\text{null } T) \rightarrow W$ by

$$\tilde{T}(v + \text{null } T) = Tv$$

This map indeed is well-defined as $u + \text{null } T = v + \text{null } T \implies v - u \in \text{null } T$ and thus $T(v - u) = 0 \implies Tv = Tu$. Also, the map is a linear map.

Proposition 1.33. Suppose $T \in \mathcal{L}(V, W)$ then

- (a) $\tilde{T} \circ \pi = T$ where π is the quotient map with $U = \text{null } T$
- (b) \tilde{T} is injective
- (c) $\text{range } T = \text{range } \tilde{T}$
- (d) $V/(\text{null } T)$ and $\text{range } T$ are isomorphic vector spaces.

Proof. We prove each point individually

- (a) If $v \in V$ then $\tilde{T} \circ \pi(v) = \tilde{T}(v + \text{null } T) = Tv$ as desired.
- (b) If $\tilde{T}(v + \text{null } T) = 0$ then $Tv = 0$ thus $v \in \text{null } T$. Thus, $v + \text{null } T = 0 + \text{null } T \implies \text{null } \tilde{T} = \{0 + \text{null } T\}$.
- (c) By definition of \tilde{T} .
- (d) From (b) and (c).

□

1.5.3 Exercise

Problem : Suppose T is a function from V to W . The graph of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}$$

Prove that T is a linear map if and only if graph of T is a subspace of $V \times W$.

Solution : For (\implies) , you just need to check the vector axioms and just the definition of a linear map. For the (\impliedby) , consider

$$(v, Tv) + (w, Tw) = (v + w, Tv + Tw)$$

Since, graph of T is a subspace and all of its element is of the form (v, Tv) for any v , it must be that $(v + w, T(v + w)) = (v + w, Tv + Tw)$. Similarly, $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$. And we're done.

Problem : Suppose V_1, \dots, V_m are vector spaces such that $V_1 \times \dots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \dots, m$.

Solution : Let $\dim(V_1 \times \dots \times V_m) = k$ and suppose e_1, \dots, e_k are the basis of $V_1 \times \dots \times V_m$. Let $e_i = (e_{1i}, e_{2i}, \dots, e_{mi})$ for $1 \leq i \leq k$. Notice that to cover elements of V_j , only the j -th component of e_i 's play a role. Thus,

$$\text{span}\{e_{j1}, e_{j2}, \dots, e_{jm}\} = V_k$$

Since, every span can be reduced to a basis we get that each V_k are finite-dimensional.

Problem : Suppose V_1, V_2, \dots, V_m are vector spaces. Prove that $\mathcal{L}(V_1 \times \dots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Solution :

Solution. Let $U = V_1 \times \dots \times V_m$. We want to prove that

$$\mathcal{L}(U, W) \cong \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W).$$

Define a map

$$\Phi : \mathcal{L}(U, W) \rightarrow \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$$

by

$$T \mapsto (T \circ i_1, T \circ i_2, \dots, T \circ i_m),$$

where for each $j = 1, \dots, m$, the map $i_j : V_j \rightarrow U$ is defined by

$$i_j(v) = (0, \dots, 0, v, 0, \dots, 0),$$

with v in the j -th position.

Now, for $T, T' \in \mathcal{L}(U, W)$ and scalars $\alpha, \beta \in \mathbf{F}$,

$$\Phi(\alpha T + \beta T') = ((\alpha T + \beta T') \circ i_1, \dots, (\alpha T + \beta T') \circ i_m) = \alpha \Phi(T) + \beta \Phi(T').$$

Hence, Φ is linear.

For invertibility of the map, define

$$\Psi : U \rightarrow W$$

by

$$\Psi(S_1, \dots, S_m)(v_1, \dots, v_m) = S_1(v_1) + \dots + S_m(v_m).$$

where $S_j \in \mathcal{L}(V_j, W)$. Now, one can check that $\Psi \circ \Phi(T) = T$ and $\Phi \circ \Psi(S_1, \dots, S_m) = (S_1, \dots, S_m)$. Thus, the map is invertible.

Problem : For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \dots \times V}_m$$

Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic vector spaces.

Solution : Define a map from $\Phi : V^m \rightarrow \mathcal{L}(\mathbf{F}^m, V)$ by

$$(v_1, \dots, v_m) \mapsto (a_1, \dots, a_m) \mapsto a_1 v_1 + \dots + a_m v_m$$

One can check that this is linear as well as invertible.

Problem : Suppose x, v are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Solution : Suppose u is an arbitrary vector in U , then $v + u = x + w$ for some $w \in W$. Thus, setting $u = 0$ gives $x - v = -w \in W$. Thus, for all $u \in U$, $u = x - v + w$ for some $w \in W$ since W is a subspace $x - v + w \in W$. Thus, $U \subseteq W$ and repeating a similar argument for U gives $W \subseteq U$.

Problem : Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$. Prove that A is a translate of U if and only if there exists $k \in \mathbf{R}$ such that

$$A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = k\}$$

Solution : Suppose A is a translate of U , then $(a, b, c) + U = A$ for some $(a, b, c) \in \mathbf{R}^3$. Thus

$$A = \{(x + a, y + b, z + c) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$$

Now let $x + a = x_1, y + b = y_1, z + c = z_1$ thus,

$$A = \{(x_1, y_1, z_1) \in \mathbf{R}^3 : 2x_1 + 3y_1 + 5z_1 = 2a + 3b + 5c\}$$

$$\implies A = \{(x_1, y_1, z_1) \in \mathbf{R}^3 : 2x_1 + 3y_1 + 5z_1 = k\}$$

Now, suppose $A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = k\}$ then $(\frac{k}{6}, \frac{k}{9}, \frac{k}{15}) + U = A$.

1.6 Duality

1.6.1 Dual Space and Dual Map

Definition 1.30. A *linear functional* on V is a linear map from V to \mathbf{F} . In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

Definition 1.31. The *dual space* of V , denoted V' , is the vector space of all linear functional on V . In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

Proposition 1.34. Suppose V is a finite-dimensional vector space. Then V' is also finite-dimensional and

$$\dim V' = \dim V$$

Proof. From **Proposition 1.18.** we have

$$\dim V' = \dim \mathcal{L}(V, \mathbf{F}) = (\dim V)(\dim \mathbf{F}) = \dim V$$

as desired. □

Definition 1.32. If v_1, \dots, v_n is a basis of V , then the *dual space* of v_1, \dots, v_n is the list $\varphi_1, \dots, \varphi_n$ of elements in V' such that

$$\varphi_j(k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Proposition 1.35. Suppose v_1, \dots, v_n is a basis of V and $\varphi_1, \dots, \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \dots + \varphi_n(v)v_n$$

for each $v \in V$.

Proof. Let

$$v = c_1v_1 + \dots + c_nv_n$$

Then, $\varphi_n(v) = c_n$ thus

$$v = \varphi_1(v)v_1 + \dots + \varphi_n(v)v_n$$

as desired. □

Proposition 1.36. Suppose V is a finite-dimensional. Then the dual basis of a basis of V is a basis of V' .

Proof. Suppose v_1, \dots, v_n is a basis of V and let $\varphi_1, \dots, \varphi_n$ be the dual basis. To show $\varphi_1, \dots, \varphi_n$ is linearly independent, suppose there exists $a_1, \dots, a_n \in \mathbf{F}$ such that

$$a_1\varphi_1 + \dots + a_n\varphi_n = 0$$

Then, $(a_1\varphi_1 + \dots + a_n\varphi_n)(v_k) = a_k$ for $1 \leq k \leq n$. Thus, $a_1 = a_2 = \dots = a_n = 0$. And since the list is of the length $\dim V'$, we can conclude that the list is the basis of V' . □

Definition 1.33. Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T$$

Remark. Since T' is a composition of linear maps φ and T , it is a linear map as well. Also, $T'(\varphi) \in V'$ as T' as it takes an element from V to \mathbf{F} . Also, one can verify $T' \in \mathcal{L}(W', V')$.

Proposition 1.37. Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$,
- (b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$,
- (c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Proof. The proofs of (a) and (b) directly follow from the definitions. For (c),

$$(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = T'S'$$

The fourth equation is due to $\varphi \circ S \in W'$. □

1.6.2 Null Space and Range of Linear Map

Definition 1.34. For $U \subseteq V$, the annihilator of U , denoted by U^0 , is defined by

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \text{ for all } u \in U\}$$

Proposition 1.38. Suppose $U \subseteq V$. Then U^0 is a subspace of V' .

Proof. One can just check the vector axioms. □

Proposition 1.39. Suppose V is finite-dimensional and U is a subspace of V . Then

$$\dim U^0 = \dim V - \dim U$$

Proof. Let $i \in \mathcal{L}(U, V)$ be the linear map such that $i(u) = u$ for each $u \in U$. Thus, $i' \in \mathcal{L}(V', U')$ and from fundamental theorem of linear maps we have,

$$\dim \text{range } i' + \dim \text{null } i' = \dim V' = \dim V$$

Also, $\text{null } i' = \{\varphi \in V' \mid i'(\varphi) = 0\} = \{\varphi \in V' \mid \varphi \circ i = 0\} = \{\varphi \in V' \mid \varphi(x) = 0\} = U^0$. Thus, $\dim \text{null } i' = \dim U^0$ and the equation above becomes

$$\dim \text{range } i' + \dim U^0 = \dim V$$

If $\varphi \in U'$, then φ can be extended to a linear functional ϕ on V (**Exercise 10** of the first section). Thus, $i'(\phi) = \varphi$ and $\text{range } i' = U'$. Hence

$$\dim \text{range } i' = \dim U' = \dim U$$

And thus $\dim U + \dim U^0 = \dim V$ as desired. □

Proposition 1.40. Suppose V is finite-dimensional and U is a subspace of V . Then

- (a) $U^0 = \{0\} \iff U = V$,
 - (b) $U^0 = V' \iff U = \{0\}$
-

Proof. For (a) we have,

$$\begin{aligned} U^0 = \{0\} &\iff \dim U^0 = 0 \\ &\iff \dim U = \dim V \\ &\iff U = V \end{aligned}$$

Similarly, to prove (b) we have

$$\begin{aligned} U^0 = V' &\iff \dim U^0 = \dim V' \\ &\iff \dim U^0 = \dim V \\ &\iff \dim U = 0 \\ &\iff U = \{0\} \end{aligned}$$

And we're done. □

Proposition 1.41. Let V and W be vector spaces and let $T \in \mathcal{L}(V, W)$. Then

$$\text{null } T' = (\text{range } T)^0$$

Proof. First suppose $\varphi \in \text{null } T'$, then $T'(\varphi) = (\varphi \circ T)(x) = 0$ for every $x \in V$. Since, $\varphi(Tx) = 0$ we have $\varphi \in (\text{range } T)^0$ and thus $\text{null } T' \subseteq (\text{range } T)^0$. Now, suppose $\varphi \in (\text{range } T)^0$ then $\varphi(Tv) = 0$ for all $v \in V$ which implies $T'(\varphi) = 0$. Thus, $\varphi \in \text{null } T'$ and $(\text{range } T)^0 \subseteq \text{null } T'$ as desired. □

Proposition 1.42. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ then

$$\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$$

Proof. We have

$$\begin{aligned} \dim \text{null } T' &= \dim(\text{range } T)^0 \\ &= \dim W - \dim \text{range } T \\ &= \dim W - (\dim V - \dim \text{null } T) \\ &= \dim \text{null } T + \dim W - \dim V \end{aligned}$$

□

Proposition 1.43. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is surjective} \iff T' \text{ is injective}$$

Proof. To prove this, we have

$$\begin{aligned} T \text{ is surjective} &\iff \text{range } T = W \\ &\iff (\text{range } T)^0 = \{0\} \\ &\iff \text{null } T' = \{0\} \\ &\iff T' \text{ is injective} \end{aligned}$$

as desired. □

Proposition 1.44. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

- (a) $\dim \text{range } T' = \dim \text{range } T$,
- (b) $\text{range } T' = (\text{null } T)^0$

Proof. For (a) we have,

$$\begin{aligned}
 \dim \text{range } T' &= \dim W' - \dim \text{null } T' \\
 &= \dim W - (\dim \text{null } T + \dim W - \dim V) \\
 &= \dim V - \dim \text{null } T \\
 &= \dim \text{range } T
 \end{aligned}$$

For (b), suppose $\varphi \in \text{range } T'$ then there exists a $\phi \in W'$ such that $T'(\phi) = \varphi$. Thus, for all $v \in \text{null } T$ we have

$$\varphi(v) = T'(\phi)v = (\phi \circ T)(v) = \phi(0) = 0$$

Thus, $\varphi \in (\text{null } T)^0$. Thus, $\text{range } T' \subseteq (\text{null } T)^0$. Now, we'll complete the proof by showing $\dim \text{range } T' = \dim(\text{null } T)^0$. Note, that

$$\begin{aligned}
 \dim \text{range } T' &= \dim \text{range } T \\
 &= \dim V - \dim \text{null } T \\
 &= \dim(\text{null } T)^0
 \end{aligned}$$

where the last equation is from **Proposition 1.39**. □

Proposition 1.45. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then,

$$T \text{ is injective} \iff T' \text{ is surjective}$$

Proof. We have

$$\begin{aligned}
 T \text{ is injective} &\iff \text{null } T = \{0\} \\
 &\iff (\text{null } T)^0 = V' \\
 &\iff \text{range } T' = V'
 \end{aligned}$$

as desired. □

1.6.3 Matrix of Dual Linear Map

Proposition 1.46. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T', (\psi_1, \dots, \psi_m), (\varphi_1, \dots, \varphi_n)) = (\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m)))^t$$

where (ψ_1, \dots, ψ_m) and $(\varphi_1, \dots, \varphi_n)$ are the dual basis of W' and V' respectively.

Proof. Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Suppose $1 \leq j \leq m$ and $1 \leq k \leq n$. From the definition of $\mathcal{M}(T')$ we have

$$T'(\psi_j) = \sum_{r=1}^n C_{r,j} \varphi_r.$$

The left side of the equation above equals $\psi_j \circ T$. Thus applying both sides of the equation above to v_k gives

$$(\psi_j \circ T)(v_k) = \sum_{r=1}^n C_{r,j} \varphi_r(v_k) = C_{k,j}.$$

We also have

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k) = \psi_j \left(\sum_{r=1}^m A_{r,k} w_r \right) = \sum_{r=1}^m A_{r,k} \psi_j(w_r) = A_{j,k}.$$

Comparing the last line of the last two sets of equations, we have $C_{k,j} = A_{j,k}$. Thus $C = A^t$. In other words, $\mathcal{M}(T') = \mathcal{M}(T)^t$, as desired. \square

2 Polynomials

2.1 Zeros of Polynomials

Definition 2.1. A number $\lambda \in \mathbf{F}$ is called a *zero*(or *root*) of a polynomial $p \in \mathcal{P}(\mathbf{F})$ if

$$p(\lambda) = 0$$

Proposition 2.1. Suppose m is a positive integer and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial of degree m . Suppose $\lambda \in \mathbf{F}$. Then $p(\lambda) = 0$ if and only if there exists a polynomial $q \in \mathcal{P}(\mathbf{F})$ of degree $m - 1$ such that

$$p(z) = (z - \lambda)q(z)$$

for every $z \in \mathbf{F}$.

Proof. Not so hard. □

Proposition 2.2. Suppose m is a positive integer and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial of degree m . Then p has at most m roots in \mathbf{F} .

Proof. We'll use induction. For $m = 1$, it is quite straight forward, as the polynomial $a_0 + a_1z$ only has one zero which is $-a_0/a_1$. Now, suppose the assumption holds for all polynomial with degree $m - 1$. Let p be a polynomial of degree m , then if p has no zeros then we're done. Suppose $\lambda \in \mathbf{F}$ such that $p(\lambda) = 0$, then using our previous proposition we get

$$p(z) = (z - \lambda)q(z)$$

where $q(z)$ has degree $m - 1$. This shows that zeros of p are exactly the zeros of $q(z)$ and λ , which is at most m . □

Remark. The result above implies that, coefficients of a polynomial are uniquely determined

2.2 Division Algorithm for Polynomials

Proposition 2.3. Suppose that $p, s \in \mathcal{P}(\mathbf{F})$ with $s \neq 0$. Then there exists a unique polynomials $q, r \in \mathcal{P}(\mathbf{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

Proof. Suppose $\deg p = n$ and $\deg s = m$. If $n < m$ then, $q = 0$ and $r = p$. Thus assume that $n \geq m$. Then, take the list

$$1, z, z^2, \dots, z^{m-1}, s, sz, \dots, sz^{n-m}$$

this list is linearly independent in $\mathcal{P}_n(\mathbf{F})$ as every element has a different degree. Also, the length of the list is $n + 1$, thus this list is a basis of $\mathcal{P}_n(\mathbf{F})$. But since $p \in \mathcal{P}_n(\mathbf{F})$, we can write p as

$$\begin{aligned} p &= a_0 + a_1z + \dots + a_{m-1}z^{m-1} + b_0s + b_1zs + \dots + b_{n-m}sz^{n-m} \\ &= \underbrace{a_0 + a_1z + \dots + a_{m-1}z^{m-1}}_r + s \underbrace{(b_0 + b_1z + \dots + b_{n-m}z^{n-m})}_q \end{aligned}$$

as desired. □

3 Eigenvalues & Eigenvectors

3.1 Invariant Subspaces

3.1.1 Eigenvalues

Definition 3.1. A linear map from vector space to itself is called an *operator*.

Definition 3.2. Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called a *invariant* under T if $Tu \in U$ for every $u \in U$.

From our definition, U is invariant under T if $T|_U$ is an operator on U .

Definition 3.3. Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called a *eigenvalue* of T if there exists a $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Remark. Thus, V has one-dimensional subspace invariant under T if and only if T has an eigenvalue. If U is an one-dimensional subspace then $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$. Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$ then $\text{span } v$ is one-dimensional subspace V invariant under T .

Proposition 3.1. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then the following are equivalent.

1. λ is an eigenvalue of T .
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

where I is the identity operator on V .

Proof. Condition 1. and 2. are equivalent because of $Tv = \lambda v \iff (T - \lambda I)v = 0$ and if it was injective then $v = 0$. And 2., 3. and 4. are equivalent from **Proposition 1.14**. \square

Definition 3.4. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T . A vector $v \in V$ is called an *eigenvector* corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Proposition 3.2. Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Proof. For the sake of contradiction, suppose the result is false. Then there exists a smallest positive integer $m > 1$ such that there exists a list v_1, \dots, v_m of linearly dependent eigenvectors $\lambda_1, \dots, \lambda_m$ of T , corresponding to distinct eigenvalues. Due to the minimality of m , there exists $a_1, \dots, a_m \in \mathbf{F}$ such that

$$a_1 v_1 + \dots + a_m v_m = 0$$

Applying $T - \lambda I$ we get

$$a_1(\lambda_1 - \lambda_m)v_1 + \dots + a_m(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$$

Since, all the eigenvalues are different, none of the coefficient above is 0. Thus, we get a new list of linearly dependent vector with length $m - 1$, which contradicts the minimality of m . \square

Proposition 3.3. Suppose V is a finite-dimensional. Then each operator of V has at most $\dim V$ distinct eigenvalue.

Proof. Since, every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent by above proposition, we get the bound immediately. \square

3.1.2 Polynomials Applied to Operatos

Definition 3.5. Suppose $T \in \mathcal{L}(V)$ and m is a positive integer

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_m$
- T^0 is defined to be the identity operator I on V .
- If T is invertible with inverses T^{-1} , then $T^{-m} \in \mathcal{L}(V)$ is defined by

$$T^{-m} = (T^{-1})^m$$

Definition 3.6. Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1z + \cdots + a_mz^m$$

for all $z \in \mathbf{F}$. Then $p(T)$ is the operator on V defined by

$$p(T) = a_0I + a_1T + a_2T^2 + \cdots + a_mT^m$$

Proposition 3.4. Let $T \in \mathcal{L}(V)$, then the function $f : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear.

Proof. Left for future me as a exercise. \square

Proposition 3.5. Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$ Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

Proof. Just define the polynomials and plug T . \square

Definition 3.7. Let $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is defined by

$$(pq)(z) = p(z)q(z)$$

Proposition 3.6. Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Then $\text{null } p(T)$ and $\text{range } p(T)$ are invariant under T .

Proof. Suppose $u \in \text{null } p(T)$ then $p(T)u = 0$

$$p(T)(Tu) = T(p(T)u) = T(0) = 0$$

Hence, $Tu \in \text{null } p(T)$. Thus, $\text{null } p(T)$ is invariant under T

Suppose, $u \in \text{range } p(T)$. Then,

$$\begin{aligned} p(T)v &= u \\ \implies T(p(T)v) &= Tu \\ \implies p(T)(Tv) &= Tu \end{aligned}$$

Hence, $Tu \in \text{range } p(T)$. Thus, $\text{range } p(T)$ is invariant under T , as desired. \square

3.1.3 Exercise

Problem : Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V .

1. Prove that $U \subseteq \text{null } T$, then U is invariant under T .
2. Prove that $\text{range } T \subseteq U$, then U is invariant under T .

Solution : For 1. just notice that $Tu = 0 \in U$ for all $u \in U$. Thus, it is invariant under T . For 2. notice that $Tu = k \in \text{range } T \subseteq U$, thus $k \in U$. Therefore it is invariant under T .

Problem : Suppose that $T \in \mathcal{L}(V)$ and V_1, \dots, V_m are subspaces of V invariant under T . Prove that $V_1 + \dots + V_m$ is also invariant under T .

Solution : Let $u \in V_1 + \dots + V_m$ then $Tu = Tv_1 + \dots + Tv_m$. Since $Tv_i \in V_i$ we have that

$$Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$$

Thus $V_1 + \dots + V_m$ is invariant under T .

Problem : Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T .

Solution : Let $\bigcap_{\alpha} V_{\alpha} = K$ where V_{α} is a subspace of V invariant under T . If $v \in K$ then $Tv \in V_{\alpha}$ for every α , hence $Tv \in K$.

Problem : If V is a finite-dimensional and U is a subspace of V that is invariant under every operator on V , then $U = \{0\}$ or $U = V$.

Solution : Suppose $U \neq \{0\}$ and $U \neq V$. Let u_1, \dots, u_n be a basis of U and extend it to basis of V , $u_1, \dots, u_n, v_1, \dots, v_{m-n}$. Define

$$T(a_1u_1 + \dots + a_nu_n + \dots + a_mv_{m-n}) = a_1v_1$$

It is easy to see that $T \in \mathcal{L}(V)$ but U is not invariant under T as $T(u_1) = v_1$.

Problem : Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T .

Solution : If λ is a eigenvalues then $(-3y, x) = T(x, y) = \lambda(x, y) = (\lambda x, \lambda y)$ then $-3y = \lambda^2 y$. Notice that $y \neq 0$ because if $y = 0$ then $x = 0$ but $(x, y) \neq (0, 0)$. Thus, $\lambda = \pm\sqrt{3}i$.

Problem : Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P , then $\lambda = 0$ or 1 .

Solution : If λ is a eigenvalue then $\exists v \neq 0$ such that $P(v) = \lambda v$. Therefore,

$$\begin{aligned} \lambda v &= P(P(v)) = P(\lambda v) = \lambda \cdot \lambda v \\ \implies \lambda^2 v &= \lambda v \implies v(\lambda^2 - \lambda) = 0 \\ \implies \lambda^2 &= \lambda \implies \lambda = 1, 0 \end{aligned}$$

Problem : Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

1. Prove that T and $S^{-1}TS$ have the same eigenvalues.
2. What is the relationship between eigenvectors of T and eigenvectors of $S^{-1}TS$?

Solution : If λ is a eigenvalue of T then $T(v) = \lambda v$. Since S is invertible $\exists w \in V$ such that $S(w) = v$ thus $(S^{-1}(T(S(w)))) = S^{-1}(\lambda v) = \lambda w$. If λ is a eigenvalue of $S^{-1}TS$ then $S^{-1}(T(Sw)) = \lambda w$ thus $T(S(w)) = S(\lambda w) = \lambda S(w)$. The relationship between the eigenvectors is that, eigenvectors of $S^{-1}TS$ maps to eigenvectors of T .

Problem : Give an example of a operator in \mathbf{R}^4 that has no eigenvalues.

Solution : $T(w, x, y, z) = (-x, w, -z, y)$

Problem : Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Solution : From exercise 17 of 3F we know that T is invertible if and only if T' is invertible. Thus, $T - \lambda I$ is not invertible if and only if $(T - \lambda I)' = T' - \lambda I$ is not invertible. Thus we are done.

Problem : Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbf{C}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

Solution : If λ is an eigenvalue then $T(v + iw) = \lambda(v + iw)$ thus we have

$$\overline{T(v + iw)} = T(\overline{v + iw}) = \bar{\lambda}(\overline{v + iw})$$

Since $v + iw \neq 0$ so their conjugate isn't zero as well.

Problem : Suppose $T \in \mathcal{L}(V)$ is invertible.

1. Suppose $\lambda \in F$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} .
2. Prove that T and T^{-1} have same eigenvectors.

Solution : If λ is an eigenvalue of T then

$$T(v) = \lambda v \iff v = \lambda T^{-1}(v) \iff \frac{1}{\lambda}v = T^{-1}(v)$$

This proves both of the statement.

Problem : Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Solution : If ST has eigenvalue λ then

$$ST(v) = \lambda v$$

$$\implies T(STv) = \lambda T(v)$$

If $Tv \neq 0$ then we are done. If $Tv = 0$ then

$$S(Tv) = S(0) = 0 = \lambda v \implies \lambda = 0$$

Thus, $ST - \lambda I = ST$ is not invertible thus TS is also not invertible. Therefore 0 is also an eigenvalue of TS .

Left to do exercises. 23, 24, 25, 26, 27, 30, 31, 34, 35, 39

3.2 The Minimal Polynomial