

# Real Analysis

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## Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from [MIT OCW](#) and post the problems and the solutions here.

# 1 Problem Set 1

## Problem 1.1

Let  $\mathbb{F}$  be an ordered field with  $1 \neq 0$ . Show that  $1 > 0$ .

*Solution.* First let us prove that  $(-1) \cdot (-1) = 1$ . We know that for all  $x \in \mathbb{F}$ , there is an inverse element  $-x$  such that,

$$x + (-x) = 0$$

Thus,  $1 + (-1) = 0$  which means that

$$\begin{aligned} 0 &= (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1) \\ &\implies 1 = (-1) \cdot (-1) \end{aligned}$$

Since  $\mathbb{F}$  is an ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that  $1 \neq 0$  so the middle statement can't be true and if  $1 < 0$  then  $0 < (-1)$ . But from **axiom of order and multiplication**  $0 < (-1) \cdot (-1) = 1$ . Thus a contradiction.

## Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

*Solution.* Suppose  $\frac{n}{m} = \frac{n_1}{m_1}$  and  $\frac{p}{q} = \frac{p_1}{q_1}$  then using the definition of when two rational numbers are equal, we get  $nm_1 = n_1m$  and  $pq_1 = p_1q$ . Thus,

$$\begin{aligned} m_1q_1(nq + pm) &= m_1q_1nq + m_1q_1pm \\ &= n_1mq_1q + p_1qm_1m \\ &= mq(n_1q_1 + m_1p_1) \end{aligned}$$

Thus,  $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$ .

## Problem 1.3

Find the  $\sup E$  and  $\inf E$  for the following set  $E$ .

1.  $E = \{n \in \mathbb{Z} \mid n < \sqrt{12}\}$
2.  $E = \{r \in \mathbb{Q} \mid r < \sqrt{12}\}$
3.  $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
4.  $E = \left\{ \frac{n^2+n}{n+1} \mid n \in \mathbb{N} \right\}$

*Solution.*

1.  $\sup E = 3$  but  $\inf E$  doesn't exist.
  2.  $\sup E = \sqrt{12}$  but  $\inf E$  doesn't exist.
  3.  $\sup E = \frac{1+\sqrt{5}}{2}$  and  $\inf E = \frac{1-\sqrt{5}}{2}$ .
  4.  $\inf E = 1$  but  $\sup E$  doesn't exist.
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