Real Analysis

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Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from MIT OCW and post the problems and the solutions here.

1 Problem Set 1

Problem 1.1

Let $\mathbf{F}$ be a ordered field with $1 \neq 0$. Show that $1 > 0$.

Solution. First let us prove that $(-1) \cdot (-1) = 1$. We know that for all $x \in \mathbf{F}$, there is an inverse element $-x$ such that,

$$x + (-x) = 0$$

Thus, $1 + (-1) = 0$ which means that

$$0 = (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1)$$
$$\implies 1 = (-1) \cdot (-1)$$

Since $\mathbf{F}$ is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that $1 \neq 0$ so the middle statement can't be true and if $1 < 0$ then $0 < (-1)$. But from **axiom of order and multiplication** $0 < (-1) \cdot (-1) = 1$. Thus a contradiction.

Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

Solution. Suppose $\frac{n}{m} = \frac{n_1}{m_1}$ and $\frac{p}{q} = \frac{p_1}{q_1}$ then using the definition of when two rational numbers are equal, we get $nm_1 = n_1 m$ and $pq_1 = p_1 q$. Thus,

$$\begin{aligned} m_1 q_1 (nq + pm) &= m_1 q_1 nq + m_1 q_1 pm \\ &= n_1 m q_1 q + p_1 q m_1 m \\ &= mq(n_1 q_1 + m_1 p_1) \end{aligned}$$

Thus, $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$.

Problem 1.3

Find the $\sup E$ and $\inf E$ for the following set E.

1. $E = \left\{n \in \mathbf{Z} \mid n \leq \sqrt{12}\right\}$
2. $E = \left\{r \in \mathbf{Q} \mid r \leq \sqrt{12}\right\}$
3. $E = \{x \in \mathbf{R} \mid x^2 - x - 1 < 0\}$
4. $E = \left\{\frac{n^2+n}{n+1} \mid n \in \mathbf{N}\right\}$