

# Linear Algebra

## Notes

Basu Dev Karki

# Content

<b>1</b>	<b>Linear Maps</b>	<b>2</b>
1.1	Vector Spaces of Linear Maps . . . . .	2
1.1.1	Definiton and Examples of Linear Maps . . . . .	2
1.1.2	Algebraic Operation on $\mathcal{L}(V, W)$ . . . . .	3
1.1.3	Exercises . . . . .	4
1.2	Null Spaces and Ranges . . . . .	8
1.2.1	Null Space and Injectivity . . . . .	8
1.2.2	Range and Surjectivity . . . . .	8
1.2.3	Fundamental Theorem of Linear Maps . . . . .	9
1.2.4	Excercise . . . . .	10
1.3	Matrices . . . . .	15
1.3.1	Representing a Linear Map by a Matrix . . . . .	15
1.3.2	Addition and Scalar Multiplication of Matrices . . . . .	15
1.3.3	Matrix Multiplication . . . . .	16
1.3.4	Column-Row Factorization and Rank of a Matrix . . . . .	18
1.3.5	Exercise . . . . .	20
1.4	Invertibility and Isomorphism . . . . .	25
1.4.1	Invertible Linear Maps . . . . .	25
1.4.2	Isomorphic Vector Spaces . . . . .	26
1.4.3	Linear Map Thought of as Matrix Multiplication . . . . .	27
1.4.4	Change of Basis . . . . .	28
1.4.5	Exercise . . . . .	30
1.5	Product and Quotients of Vector Spaces . . . . .	33
1.5.1	Products of Vector Spaces . . . . .	33
1.5.2	Quotients Spaces . . . . .	34
1.5.3	Exercise . . . . .	35
1.6	Duality . . . . .	38
1.6.1	Dual Space and Dual Map . . . . .	38
1.6.2	Null Space and Range of Linear Map . . . . .	39
1.6.3	Matrix of Dual Linear Map . . . . .	41
<b>2</b>	<b>Polynomials</b>	<b>43</b>
2.1	Zeros of Polynomials . . . . .	43
2.2	Division Algorithm for Polynomials . . . . .	43
<b>3</b>	<b>Eigenvalues &amp; Eigenvectors</b>	<b>44</b>
3.1	Invariant Subspaces . . . . .	44
3.1.1	Eigenvalues . . . . .	44
3.1.2	Polynomials Applied to Operatos . . . . .	45
3.1.3	Exercise . . . . .	46
3.2	The Minimal Polynomial . . . . .	49
3.2.1	Existence of Eigenvalue on Complex Vector Spaces . . . . .	49

# 1 Linear Maps

## 1.1 Vector Spaces of Linear Maps

### 1.1.1 Definiton and Examples of Linear Maps

**Definition 1.1.** A *linear map* from  $V$  to  $W$  is a function  $T : V \rightarrow W$  with the following properties.

1. **(Additivity)**  $T(u + w) = T(u) + T(w)$  for all  $u, v \in V$
2. **(Homogeneity)**  $T(\lambda u) = \lambda T(u)$  for all  $\lambda \in \mathbf{F}$  and for all  $u \in V$

**Remark.** Some mathematicians use the phrase *linear transformation*, which means the same as linear map.

**Definition 1.2.** (Notation)

1. The set of linear maps from  $V \rightarrow W$  is denoted by  $\mathcal{L}(V, W)$ .
2. The set of linear maps from  $V \rightarrow V$  is denoted by  $\mathcal{L}(V)$ . In other words,  $\mathcal{L}(V, V) = \mathcal{L}(V)$ .

**Examples:**

**zero**

We will let the symbol  $0$  denote the liner map that takes every element of some vector space to additive identity of some another vector space. Thus,  $0 \in \mathcal{L}(V, W)$  is defined by

$$0(v) = 0$$

**identity operator**

Let  $I \in \mathcal{L}(V)$  be defined by

$$I(v) = v$$

**differentiation**

Let  $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  be defined by

$$D(p) = p'$$

**integration**

Let  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$  be defined by

$$T(p) = \int_0^1 p$$

**composition**

Fix a polynomial  $q \in \mathcal{P}(\mathbf{R})$ . Let  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  be defined by

$$T(p) = (p \circ q)$$

**Remark.** We'll limit the Notation of  $T(v)$  to just  $Tv$  for convenience.

**Theorem 1.1.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n \in W$ . Then, there exists a unique linear map  $T : V \rightarrow W$  such that

$$Tv_k = w_k$$

for  $i = 1, 2, 3, \dots, n$ .

*Proof.* First we show the existence of such map. Define  $T : V \rightarrow W$  by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

where  $c_i \in \mathbf{F}$ . Since,  $v_1, \dots, v_n$  is a basis of  $V$ , it maps every element of  $V$  to  $W$ , thus it is a function.

Now, set  $c_k = 1$  and all other  $c$ 's to be 0 to show that  $Tv_k = w_k$ . From, here one can show that  $T$  is indeed a linear map. To show the uniqueness, suppose  $T' \in \mathcal{L}(V, W)$  and  $T'v_k = w_k$ . Using the properties of linear map,

$$T'(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

Thus,  $T$  and  $T'$  agree on every input, thus  $T = T'$ . □

### 1.1.2 Algebraic Operation on $\mathcal{L}(V, W)$

**Definition 1.3.** Suppose  $T, S \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbf{F}$  then the *sum* and the *product* of the linear maps from  $V$  to  $W$  is defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all  $v \in V$ .

**Proposition 1.1.** With the operations defined above, the set  $\mathcal{L}(V, W)$  is a vector space.

*Proof.* The additive identity for  $\mathcal{L}(V, W)$  is the zero linear map  $0(v) = 0$ . The inverse for  $T$  is  $((-1)T)v = -(Tv)$ . And the rest of the axioms are left for readers (future me) to verify. □

**Definition 1.4.** Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$  then the *product*  $ST \in \mathcal{L}(U, W)$  is defined by

$$(ST)(u) = S(Tu)$$

for all  $u \in U$ .

**Remark.** Be careful about the domains of  $S$  and  $T$ . Here, the domain of  $S$  must be the co-domain of  $T$ .

**Proposition 1.2.** For the product of linear maps, the following holds

1. (**associativity**)  $(T_1T_2)T_3 = T_1(T_2T_3)$  whenever the product makes sense (i.e  $T_3$  must map to domain of  $T_2$  and  $T_2$  must map to the domain of  $T_1$ ).
2. (**identity**)  $TI_{W,V} = I_{V,W}T$  whenever  $T \in \mathcal{L}(V, W)$ . Here  $I_{V,W}, I_{W,V}$  are the identity linear maps from  $V$  to  $W$  and  $W$  to  $V$ . We'll just limit the notation to  $TI = IT$ .
3. (**distributivity**)  $(S_1 + S_2)T = S_1T + S_2T$  and  $S(T_1 + T_2) = ST_1 + ST_2$  for  $T, T_1, T_2 \in \mathcal{L}(U, V)$  and  $S, S_1, S_2 \in \mathcal{L}(V, W)$ .

**Proposition 1.3.** Suppose  $T$  is a linear map from  $V$  to  $W$ . Then,  $T(0) = 0$ .

*Proof.* From the definition of linear map we have,

$$T(0) = T(0 + 0) = T(0) + T(0) \implies T(0) = 0$$

□

### 1.1.3 Exercises

**Problem :** Suppose  $b, c \in \mathbf{R}$ . Define  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  by

$$T(x, y, z) = (2x - 4y + 3z, 6x + cxyz)$$

Show that  $T$  is a linear map if and only if  $b = c = 0$ .

*Solution :* ( $\Leftarrow$ ) is pretty simple as you just have to verify the two axioms. For ( $\Rightarrow$ ), we know that if it is a linear map then  $T(0) = 0 \Rightarrow b = 0$ . Also, using the first axiom we get

$$T((x, y, z) + (1, 0, 0)) = T((x, y, z)) + T((1, 0, 0)) \Rightarrow c = 0$$

**Problem :** Suppose  $b, c \in \mathbf{R}$ . Define  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathbf{R}^2$  by

$$Tp = \left( 3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^2 x^3 p(x) dx + c \sin(p(0)) \right)$$

Show that  $T$  is a linear map if and only if  $b = c = 0$ .

*Solution :* ( $\Leftarrow$ ) is pretty simple. For ( $\Rightarrow$ ), we can use the first axiom

$$T(p + q) = Tp + Tq$$

we'll just look at the first component first,

$$\begin{aligned} \Rightarrow 3(p + q)(4) + 5(p + q)'(6) + b(p + q)(1)(p + q)(2) &= 3p(4) + 5p'(6) + bp(1)p(2) + \\ &3q(4) + 5q'(6) + bq(1)q(2) \end{aligned}$$

Since,  $(p + q)(4) = p(4) + q(4)$  and  $(p + q)'(6) = p'(6) + q'(6)$ , we can simplify is down to,

$$b(p(1)q(2) + p(2)q(1)) = 0$$

Now, if you choose polynomials  $p, q > 0$  for  $x > 0$  then  $b = 0$ . A similar argument works for  $c = 0$ .

**Problem :** Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_m$  is a list of vectors in  $V$  such that  $Tv_1, \dots, Tv_m$  is a linearly independent list in  $W$ . Prove that  $v_1, \dots, v_m$  is linearly independent.

*Solution :* Suppose on the contrary that  $v_1, \dots, v_m$  are not linearly independent in  $V$ , then there exists  $\lambda_1, \dots, \lambda_m$  not all zero such that

$$\lambda_1 v_1 + \dots + \lambda_m v_m = 0$$

But  $T(\lambda_1 v_1 + \dots + \lambda_m v_m) = \lambda_1 T(v_1) + \dots + \lambda_m T(v_m)$  which implies

$$\lambda_1 T(v_1) + \dots + \lambda_m T(v_m) = 0 \Rightarrow \lambda_i = 0$$

a contradiction.

**Problem :** Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if  $\dim V = 1$  and  $T \in \mathcal{L}(V)$ , then there exists  $\lambda \in \mathbf{F}$  such that  $Tv = \lambda v$  for all  $v \in V$ .

*Solution :* Since  $\dim V = 1$  there exists a  $v \in V$  s.t every  $v_i \in V$  can be written as  $\lambda_i v$  for some  $\lambda_i \in \mathbf{F}$ . Thus,  $v_i = \lambda_i v \implies T(v_i) = \lambda_i T(v)$ . Since,  $T(v) = v_j \in V$  for some  $j$ . Thus,  $T(v_i) = \lambda_i \lambda_j v = \lambda_j v_i$ .

**Problem :** Give an example of a function  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$  such that

$$\varphi(av) = a\varphi(v)$$

for all  $a \in \mathbf{R}$  and  $v \in \mathbf{R}^2$  but  $\varphi$  is not linear.

$$\text{Solution : } \varphi(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

**Problem :** Give an example of a function  $\varphi : \mathbf{C} \rightarrow \mathbf{C}$  such that

$$\varphi(v + w) = \varphi(v) + \varphi(w)$$

for all  $v, w \in \mathbf{C}$  and but  $\varphi$  is not homogeneous.

$$\text{Solution : } \varphi(x) = \operatorname{Re}(x).$$

**Problem :** Prove or give a counter example: Fix a polynomial  $q \in \mathcal{P}(\mathbf{R})$ .

Let  $T : \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$  be defined by  $Tp = q \circ p$  then  $T$  is a linear map.

*Solution :* Assume it was a linear map then  $T(0) = q(0) = 0$ . Just pick  $q(0) \neq 0$ .  
Example :  $q(x) = x + 1$ .

**Problem :** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that  $T$  is a scalar multiple of identity if and only if  $ST = TS$  for every  $S \in \mathcal{L}(V)$ .

*Solution :* I really tried but it seems very hard to prove ( $\Rightarrow$ ) but will come back later.

**Problem :** Suppose  $U$  is a subspace of  $V$  with  $U \neq V$ . Suppose  $S \in \mathcal{L}(U, W)$  and  $Su \neq 0$  for some  $u \in U$ . Define  $T : V \rightarrow W$  by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that  $T \notin \mathcal{L}(V, W)$ .

*Solution :* Suppose it is a linear map, then  $T(u + v) = Tu + Tv$  where  $u \in U$  and  $v \in V$  and  $v \notin U$ . One can check that  $v + u \in V$  but  $v + u \notin U$ . Thus,  $0 = Tu = Su$ , but just take  $u$  s.t  $Su \neq 0$ .

**Problem :** Suppose  $V$  is finite-dimensional. Prove that every linear map on a subspace of  $U$  can be extended to a linear map on  $V$ . In other words, let  $U$  be a subspace of  $V$  and  $S \in \mathcal{L}(U, W)$ , then there exists a  $T \in \mathcal{L}(V, W)$  such that  $Tu = Su$  for all  $u \in U$ .

*Solution :* Let  $u_1, \dots, u_m$  be the basis of  $U$  and let  $u_1, \dots, u_m, v_1, \dots, v_k$  be the extended basis of  $V$ . Let  $x = a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_kv_k$  and define

$$T(x) = a_1Su_1 + \dots + a_mSu_m + b_1v_1 + \dots + b_kv_k$$

From here its easy to see that  $Tu = Su$  for all  $u \in U$ . We just have to prove this is a linear map on  $V$ . Let  $y = x = c_1u_1 + \dots + c_mu_m + d_1v_1 + \dots + d_kv_k$  then

$$\begin{aligned} T(x+y) &= (a_1+c_1)Su_1 + \dots + (a_m+c_m)Su_m + (b_1+d_1)v_1 + \dots + (b_k+d_k)v_k \\ \implies T(x+y) &= a_1Su_1 + \dots + a_mSu_m + b_1v_1 + \dots + b_kv_k + c_1Su_1 + \dots + c_mSu_m + d_1v_1 + \dots + d_kv_k \\ &\implies T(x+y) = Tx + Ty \end{aligned}$$

Similarly one can prove  $T(\lambda x) = \lambda T(x)$ .

**Problem :** Suppose  $V$  is finite-dimensional with  $\dim V > 0$  and suppose  $W$  is infinite-dimensional. Prove that  $\mathcal{L}(V, W)$  is infinite-dimensional.

*Solution :* Let  $v_1, \dots, v_m$  be basis of  $V$ . Suppose  $\mathcal{L}(V, W)$  is finite-dimensional then every  $T \in \mathcal{L}(V, W)$  can be written as

$$T(x) = \lambda_1 T_1(x) + \lambda_2 T_2(x) + \dots + \lambda_k T_k(x)$$

for some fixed  $T_1, T_2, \dots, T_k \in \mathcal{L}(V, W)$  and  $\lambda_i \in \mathbf{F}$ .

Since,  $W$  is infinite-dimensional,  $\exists w \in W$  s.t  $w \notin \text{span}\{T_1(x), T_2(x), \dots, T_k(x)\}$  for some fixed  $x \in V$ .

Now, set  $x = v_i$  then  $w \neq \lambda_1 T_1(v_i) + \lambda_2 T_2(v_i) + \dots + \lambda_k T_k(v_i)$ . One can find  $T \in \mathcal{L}(V, W)$  such that  $T(v_i) = w$  thus  $T(v_i) \neq \lambda_1 T_1(v_i) + \lambda_2 T_2(v_i) + \dots + \lambda_k T_k(v_i)$  contradicting our assumption.

**Problem :** Let  $V$  be finite-dimensional and let  $\dim V > 1$ . Prove that there exists  $S, T \in \mathcal{L}(V)$  such that  $ST \neq TS$ .

*Solution :* Let  $v_1, \dots, v_m$  be the basis of  $V$  and let  $x = a_1v_1 + \dots + a_mv_m$ . Define  $S(x) = a_1v_1$  and  $T(x) = a_1v_2 + a_2v_3 + \dots + a_{m-1}v_m + a_mv_1$ . So,  $T(S(x)) = a_1v_2$  and  $S(T(x)) = a_mv_1$ , thus

$$ST = TS \iff a_1 = a_m$$

but one can always choose  $x$  s.t  $a_1 \neq a_m$ .

**Problem :** Suppose  $V$  is finite-dimensional. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ . A subspace  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called a two-sided ideal if for every  $E \in \mathcal{E}$  and  $T \in \mathcal{L}(V)$ ,  $TE \in \mathcal{E}$  and  $ET \in \mathcal{E}$ .

*Solution :* Let  $\mathcal{E}$  be a two-sided ideal of  $\mathcal{L}(V)$ . If  $\mathcal{E} = \{0\}$  we are done. Otherwise pick a nonzero operator  $A \in \mathcal{E}$ . Choose  $v \in V$  with  $Av \neq 0$ . Fix any  $y, z \in V$ . Choose  $R \in \mathcal{L}(V)$  with  $R(z) = v$  and choose  $S \in \mathcal{L}(V)$  with  $S(Av) = y$ . Then

$$T := SAR \in \mathcal{E}$$

(since  $\mathcal{E}$  is a two-sided ideal), and

$$T(z) = S(A(R(z))) = S(A(v)) = y.$$

Thus  $\mathcal{E}$  contains, for every pair  $y, z$ , an operator that sends  $z$  to  $y$ . Taking a basis  $u_1, \dots, u_n$  of  $V$  and the operators  $E_{ij}$  defined by  $E_{ij}(u_j) = u_i$  and  $E_{ij}(u_k) = 0$  for  $k \neq j$ , we see each  $E_{ij}$  lies in  $\mathcal{E}$ . The set  $\{E_{ij}\}$  spans  $\mathcal{L}(V)$ , so  $\mathcal{E} = \mathcal{L}(V)$ . Hence the only two-sided ideals are  $\{0\}$  and  $\mathcal{L}(V)$ .  $\square$



## 1.2 Null Spaces and Ranges

### 1.2.1 Null Space and Injectivity

**Definition 1.5.** let  $T \in \mathcal{L}(V, W)$ , the null space of  $T$ , written as  $\text{null } T$  is the following set

$$\text{null } T = \{v \in V \mid Tv = 0\}$$

#### Examples

1. The zero map from  $V$  to  $W$  has a null space  $V$  as everything gets mapped to 0.
2. Let  $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  be the differentiation map defined by  $Dp = p'$ . The only functions whose derivative is equal to 0 are the constant function. Thus,  $\text{null } D$  is the set of constant functions.
3. Let  $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  be the multiplication by  $x^2$  map i.e  $Dp = x^2p$ . The only polynomial such that  $x^2p = 0$  is  $p = 0$ . Thus,  $\text{null } D = \{0\}$ .

**Proposition 1.4.** Suppose  $T \in \mathcal{L}(V, W)$ . Then,  $\text{null } T$  is a subspace of  $V$ .

**Proposition 1.5.** Let  $T \in \mathcal{T}(V, W)$ . Then  $T$  is injective  $\iff \text{null } T = \{0\}$ .

*Proof.* Suppose  $T$  is injective. Since it is a linear map  $T(0) = 0$ , thus by injectivity the only thing that gets map to 0 is 0. Thus,  $\text{null } T = \{0\}$ . Suppose  $T$  is such that  $\text{null } T = \{0\}$  then

$$\begin{aligned} T(v) = T(w) &\implies T(v) + (-1)T(w) = 0 \\ \implies T(v) + T(-w) &= 0 \implies T(v - w) = 0 \implies v = w \end{aligned}$$

Thus, the map is injective. □

### 1.2.2 Range and Surjectivity

**Definition 1.6.** Let  $T \in \mathcal{L}(V, W)$ , the *range* of  $T$  is the following set,

$$\text{range } T = \{Tv \mid v \in V\}$$

#### Examples

1. If  $T$  is the zero map then the range of  $T$  is  $\{0\}$ .
2. Suppose  $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$  be the differentiation map. Since, for every polynomial  $q$  there exists a polynomial  $p$  such that  $p' = q$ , the range is  $\mathcal{P}(\mathbf{R})$ .

**Proposition 1.6.** Let  $T \in \mathcal{L}(V, W)$  then  $\text{range } T$  is a subspace of  $W$ .

*Proof.* Since,  $0 \in V$  we know  $T(0) = 0 \in \text{range } T$ . Now, suppose  $x, y \in \text{range } T$  then  $x = Tv$  and  $y = Tw$ . Since,  $v + w \in V$ ,  $T(v + w) \in \text{range } T \implies Tv + Tw \in \text{range } T$  which mean  $x + y \in \text{range } T$ . And  $x \in \text{range } T \implies Tv \in \text{range } T$  which means  $T(\lambda v) \in \text{range } T$  as  $\lambda v \in V$ . Thus,  $\lambda x = \lambda T(v) = T(\lambda v) \in \text{range } T$ . □

### 1.2.3 Fundamental Theorem of Linear Maps

**Theorem 1.2.** *Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then  $\text{range } T$  is finite-dimensional and*

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

*Proof.* We know that  $\text{null } T$  is a subspace of  $V$  then since  $V$  is finite-dimensional it has a basis. Let  $u_1, \dots, u_m$  be the basis of  $\text{null } T$ . Then we can extend this basis to a basis of  $V$ . Let  $u_1, \dots, u_m, v_1, \dots, v_n$  be the basis of  $V$ . Then,

$$x = a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_nv_n$$

$$\implies Tx = b_1Tv_1 + \dots + b_nTv_n$$

Thus, every  $Tv$  can be written as a linear combination of  $Tv_1, \dots, Tv_n$ . Thus,  $\text{range } T$  is finite-dimensional. To prove our main result, we need to show that  $Tv_1, \dots, Tv_n$  is a basis of  $\text{range } T$ . We already proved it spanned  $\text{range } T$ , now suppose

$$b_1Tv_1 + \dots + b_nTv_n = 0$$

$$\implies T(b_1v_1 + \dots + b_nv_n) = 0$$

Thus,  $b_1v_1 + \dots + b_nv_n \in \text{null } T$  and we can write it as  $b_1v_1 + \dots + b_nv_n = a_1u_1 + \dots + a_mu_m$ . Since,  $u_1, \dots, u_m, v_1, \dots, v_n$  is a basis vector we can say that  $b_i = a_i = 0$ . Thus, we have proved that  $Tv_1, \dots, Tv_n$  is a basis of  $\text{range } T$  and our theorem follows.  $\square$

**Theorem 1.3.** *Suppose  $V$  and  $W$  are both finite-dimensional vector spaces such that  $\dim V > \dim W$ . Then, there exists no **injective** linear map from  $V$  to  $W$ .*

*Proof.* We know that, for a  $T \in \mathcal{L}(V, W)$ ,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

Since  $\text{range } T$  is a subspace of  $W$ ,  $\dim W \geq \dim \text{range } T$ . Thus,

$$\begin{aligned} \dim \text{null } T &= \dim V - \dim \text{range } T \\ &\geq \dim V - \dim W \\ &> 0 \end{aligned}$$

Thus,  $\text{null } T$  has more than one vector, so  $T$  it's not injective by **Proposition 1.5**.  $\square$

**Theorem 1.4.** *Suppose  $V$  and  $W$  are both finite-dimensional vector spaces such that  $\dim V < \dim W$ . Then, there exists no **surjective** linear map from  $V$  to  $W$ .*

*Proof.* Similar to the proof above.  $\square$

**Definition 1.7.** Define  $T : \mathbf{F}^n \rightarrow \mathbf{F}^m$  as

$$T(x_1, \dots, x_n) = \left( \sum_{k=1}^n A_{1,k}x_k, \dots, \sum_{k=1}^n A_{m,k}x_k \right)$$

**Definition 1.8.** A homogeneous system of linear equations defined is as

$$\left( \sum_{k=1}^n A_{1,k}x_k, \dots, \sum_{k=1}^n A_{m,k}x_k \right) = (0, \dots, 0)$$

And a Inhomogeneous system of linear equation is defined as

$$\left( \sum_{k=1}^n A_{1,k}x_k, \dots, \sum_{k=1}^n A_{m,k}x_k \right) = (c_1, \dots, c_m)$$

where not all  $c_i$  are zero.

**Proposition 1.7.** A homogeneous system of linear equations with more variables than equations has nonzero solutions.

*Proof.* Use **Theorem 1.3** and **Theorem 1.4**. □

**Proposition 1.8.** An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

*Proof.* Use **Theorem 1.3** and **Theorem 1.4**. □

#### 1.2.4 Exercise

**Problem :** Give an example of a linear map  $T$  such that  $\dim \text{null } T = 3$  and  $\dim \text{range } T = 2$ .

*Solution :*  $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2)$ .

**Problem :** Suppose  $S, T \in \mathcal{L}(V, W)$  are such that  $\text{range } S \subseteq \text{null } T$ . Prove that  $(ST)^2 = 0$ .

*Solution :* Since  $\text{range } S \subseteq \text{null } T$ ,  $T(Sv) = 0$ . Thus,

$$(ST)^2 = (ST)(ST) = S(T(S(Tv))) = S(0) = 0$$

**Problem :** Suppose  $v_1, \dots, v_m$  is a list of vector in  $V$ . Define  $T \in \mathcal{L}(\mathbf{F}^m, V)$  by

$$T(z_1, \dots, z_m) = z_1v_1 + \dots + z_mv_m$$

- What property of  $T$  corresponds to  $v_1, \dots, v_m$  spanning  $V$ ?
- What property of  $T$  corresponds to  $v_1, \dots, v_m$  being linearly independent on  $V$ ?

*Solution :* If  $v_1, \dots, v_m$  spans  $V$  then  $\text{range } T = V$  thus  $T$  being surjective corresponds to  $v_1, \dots, v_m$  spanning  $V$ .

If  $v_1, \dots, v_m$  is linearly independent on  $V$  then  $\text{null } T = \{0\}$ , thus  $T$  being injective corresponds to  $v_1, \dots, v_m$  being linearly independent on  $V$ .

---

**Problem:** Show that  $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \text{null } T > 2\}$  is not a subspace of  $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ .

*Solution :* Let  $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$  and  $T'(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, 0)$ . Both of them are in  $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ . Also  $\dim \text{null } T = 3$ ,  $\dim \text{null } T' = 4$  but  $T + T' = (x_1, x_2, x_3, 0) \implies \dim \text{null}(T + T') = 2 \not> 2$ .

**Problem :** Give an example of  $T \in \mathcal{L}(\mathbf{R}^4)$  such that  $\text{range } T = \text{null } T$ .

*Solution :*  $T(x_1, x_2, x_3, x_4) = (0, 0, x_1, x_2)$ .

**Problem :** Prove that there doesn't exist a  $T \in \mathcal{L}(\mathbf{R}^5)$  such that  $\text{range } T = \text{null } T$ .

*Solution :* Suppose there exists such  $T$ , then  $\dim \text{range } T = \dim \text{null } T$  but from the fundamental theorem of linear maps we have

$$\begin{aligned} \dim V &= \dim \text{range } T + \dim \text{null } T \\ \implies \dim \text{range } T &= \dim \text{null } T = \frac{5}{2} \end{aligned}$$

which is impossible.

**Problem :** Suppose  $V$  and  $W$  are finite-dimensional with  $2 \leq \dim V \leq \dim W$ . Show that  $\{T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\}$  is not a subspace of  $\mathcal{L}(V, W)$ .

*Solution :* Let  $v_1, \dots, v_m$  be the basis of  $V$  and  $w_1, \dots, w_n$  be the basis of  $W$ . Then, define  $T_i(a_1v_1 + \dots + a_mv_m) = a_iw_i$ . One can check that this is not injective thus

$$T(a_1v_1 + \dots + a_mv_m) = \left( \sum_{i=1}^m T_i \right) (a_1v_1 + \dots + a_mv_m) = \sum_{i=1}^m a_iw_i$$

Now, suppose  $T(v) = T(v')$  then

$$\begin{aligned} a_1w_1 + \dots + a_mw_m &= a'_1w_1 + \dots + a'_mw_m \\ \implies b_1 &= b'_1 \quad (\text{because of linear independence}) \end{aligned}$$

Thus,  $v = v'$ .

**Problem :** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that there exists a subspace  $U$  and  $V$  such that

$$U \cap \text{null } T = \{0\} \quad \text{and} \quad \text{range } T = \{Tu \mid u \in U\}$$

*Solution :* We know that  $\text{null } T$  is the subspace of  $V$ . Thus, there exists a  $U$  such that  $V = U \oplus \text{null } T$  and since it is a direct sum  $U \cap \text{null } T = \{0\}$ . Now, for the range of  $T$

$$\begin{aligned} \text{range } T &= \{Tv \mid v \in V\} \\ \implies \{T(u + z) \mid u \in U, z \in \text{null } T\} &= \{Tu \mid u \in U\} \end{aligned}$$

**Problem :** Suppose  $T$  is a linear map from  $\mathbf{F}^4$  to  $\mathbf{F}^2$  such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that  $T$  is a surjective linear map.

*Solution :* We can write  $T$  as

$$\text{null } T = \{(5x_2, x_2, 7x_4, x_4) \mid x_2, x_4 \in \mathbf{F}\}$$

Now, since  $(5x_2, x_2, 7x_4, x_4) = x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1) \implies \dim \text{null } T = 2$ . Thus,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$\implies 4 = 2 + \dim \text{range } T$$

$$\implies \dim \text{range } T = 2$$

Since,  $\dim \mathbf{F}^2 = 2 = \dim \text{range } T \implies \text{range } T = \mathbf{F}^2$ . Thus,  $T$  is surjective.

**Problem :** Suppose  $U$  is three-dimensional subspace of  $\mathbf{R}^8$  and that  $T$  is a linear map from  $\mathbf{R}^8$  to  $\mathbf{R}^5$  such that  $\text{null } T = U$ . Prove that  $T$  is surjective.

*Solution :* Since,  $\text{null } T = U \implies \dim \text{null } T = 3$ . Thus,

$$\dim \mathbf{R}^8 = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim \text{range } T = 5$$

Since,  $\text{range } T$  is a subspace of  $\mathbf{R}^5$  and  $\dim \text{range } T = \dim \mathbf{R}^5$ ,  $\mathbf{R}^5 = \text{range } T$ . Thus,  $T$  is surjective.

**Problem :** Prove that there does not exist a linear map from  $\mathbf{F}^5$  to  $\mathbf{F}^2$  whose null space doesn't equals  $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$ .

*Solution :* Suppose there does exist such a  $T$ . Then the null space can be written as

$$\text{null } T = \{(3x_2, x_2, k, k, k) \mid x_2, k \in \mathbf{F}\}$$

One can check that  $\dim \text{null } T = 2$  but

$$\dim \mathbf{F}^5 = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim \text{range } T = 3$$

But  $2 = \dim \mathbf{F}^2 \geq \dim \text{range } T = 3$  which is false.

**Problem :** Suppose there exists a linear map on  $V$  such that the null space and range of  $T$  is finite dimensional. Prove that  $V$  is finite-dimensional.

*Solution :* Since, the range of  $T$  is finite-dimensional it must have a basis. Suppose  $Tv_1, \dots, Tv_m$  is the basis then

$$T(x) = \lambda_1 Tv_1 + \dots + \lambda_m Tv_m$$

$$\implies T(x - \lambda_1 v_1 - \dots - \lambda_m v_m) = 0$$

Since, the null space is also finite-dimensional

$$x - \lambda_1 v_1 - \dots - \lambda_m v_m = \lambda'_1 v'_1 + \dots + \lambda'_n v'_n$$

where  $v'_1, \dots, v'_n$  is the basis of the null space. Thus,

$$V = \text{span}(v_1, \dots, v_m, v'_1, \dots, v'_n)$$

**Problem :** Suppose  $V$  and  $W$  are both finite-dimensional. Prove that there exists an injective linear map from  $V$  to  $W$  if and only if  $\dim V \leq \dim W$ .

*Solution :* Suppose  $T \in \mathcal{L}(V, W)$  is an injective map, then

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim V = \dim \text{range } T \leq \dim W$$

Now suppose  $\dim V \leq \dim W$  then we can construct an injective map from  $V$  to  $W$ .

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n$$

where  $v_1, \dots, v_n$  is the basis of  $V$  and  $w_1, \dots, w_m$  is the basis of  $W$ . One can check this is a linear map and suppose  $T(x) = T(y)$  and let  $x = a_1v_1 + \cdots + a_nv_n$  and  $y = b_1v_1 + \cdots + b_nv_n$  then

$$T(x) = T(y)$$

$$\implies (a_1 - b_1)w_1 + \cdots + (a_n - b_n)w_n = 0$$

$$\implies a_i = b_i$$

Thus,  $x = y$ .

**Problem :** Suppose  $V$  and  $W$  are finite-dimensional vector spaces and  $U$  is a subspace of  $V$ . Prove that there exists  $T \in \mathcal{L}(V, W)$  such that  $\text{null } T = U$  if and only if  $\dim U \geq \dim V - \dim W$ .

*Solution :* Suppose  $\text{null } T = U$  then

$$\dim V - \dim \text{range } T = \dim U$$

$$\implies \dim V - \dim W \leq \dim V - \dim \text{range } T = \dim U$$

Now, suppose  $\dim U \geq \dim V - \dim W$  then let  $u_1, \dots, u_k$  be the basis of  $U$  and

$$u_1, \dots, u_k, v_1, \dots, v_m$$

be the extended basis of  $V$ . Let  $w_1, \dots, w_j$  be the basis of  $W$ . From our condition, we know  $k \geq k + m - j \implies j \geq m$ . Thus we define

$$T(a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m) = b_1w_1 + \cdots + b_mv_m$$

Here,  $\text{null } T = U$ .

**Problem :** Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V, W)$ , and  $U$  is a subspace of  $W$ . Prove that  $X = \{v \in V \mid Tv \in U\}$  is a subspace of  $V$  and

$$\dim X = \dim \text{null } T + \dim(U \cap \text{range } T)$$

*Solution :* The subspace part is pretty simple. Let  $S : X \rightarrow U$  be a map such that  $S(v) = T(v)$ . Here,  $\text{range } S = U \cap \text{range } T$  and  $\text{null } S = \text{null } T$ .

**Problem :** Suppose  $U$  and  $V$  are finite-dimensional vector spaces and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(U, V)$ . Prove that

$$\dim \text{null } ST \leq \dim \text{null } T + \dim \text{null } S$$

*Solution :* One can find that  $\text{null } ST = \text{null } T \cup \{x \in U \mid S(T(x)) = 0, T(x) \neq 0\}$ .

**Note :** It seems that these exercises are taking way too long to do. I'll however come back to it and finish

---

## 1.3 Matrices

### 1.3.1 Representing a Linear Map by a Matrix

**Definition 1.9.** Suppose  $m$  and  $n$  are two non-negative integers. A  $m \times n$  matrix is  $A$  is a rectangular array of elements in  $\mathbf{F}$  with  $m$  rows and  $n$  columns

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

The  $A_{i,j}$  represents the entry in  $i$ -th row and  $j$ -th column.

**Definition 1.10.** Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_n$  is the basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . The matrix of  $T$  with respect to these bases is the  $m \times n$  matrix  $\mathcal{M}(T)$  whose entries  $A_{i,j}$  are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

**Examples :**

Suppose  $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$  is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$$

Then,

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

As  $T(1, 0) = 1(1, 0, 0) + 2(0, 1, 0) + 7(0, 0, 1)$  and  $T(0, 1) = 3(1, 0, 0) + 5(0, 1, 0) + 9(0, 0, 1)$ .

### 1.3.2 Addition and Scalar Multiplication of Matrices

**Definition 1.11.** The sum of two matrices of same size is obtained by adding corresponding entries in the matrices i.e

$$\begin{aligned} & \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & & \vdots \\ B_{m,1} & \cdots & B_{m,n} \end{pmatrix} \\ &= \begin{pmatrix} A_{1,1} + B_{1,1} & \cdots & A_{1,n} + B_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + B_{m,1} & \cdots & A_{m,n} + B_{m,n} \end{pmatrix} \end{aligned}$$

**Proposition 1.9.** Suppose  $S, T \in \mathcal{L}(V, W)$ . Then  $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$ .

*Proof.* Follows from the definition. □

**Definition 1.12.** The product of a scalar and a matrix is obtained by multiplying each entry by the scalar i.e

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$



**Proposition 1.10.** Suppose  $T \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbf{F}$  then  $\lambda \mathcal{M}(T) = \mathcal{M}(\lambda T)$ .

*Proof.* Again, just use the definitions.  $\square$

**Theorem 1.5.** Suppose  $\mathbf{F}^{m,n}$  be the set of all the matrices with entries in  $\mathbf{F}$ . Then, with addition and scalar multiplication defined above  $\mathbf{F}^{m,n}$  is a vector space and  $\dim \mathbf{F}^{m,n} = mn$ .

*Proof.* Proving it is a vector space is pretty easy. To verify  $\dim \mathbf{F}^{m,n} = mn$  define

$$X_{i,j} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & 1 & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where every entry is 0 except the  $A_{i,j}$  entry which is equal to 1. Now, its easy to see that every  $Z \in \mathbf{F}^{m,n}$  can be written as some linear combination of  $X_{i,j}$ 's. Thus,  $\mathbf{F}^{m,n} = \text{span}\{X_{i,j}\}$  where  $i, j$  vary with  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . Also, every matrix with only 0 as its entry can only be written as linear combination of  $X_{i,j}$  with all of its scalars equal to 0. Since, there are  $mn$  entries the dimension of  $\mathbf{F}^{m,n}$  is equal to  $mn$ .  $\square$

### 1.3.3 Matrix Multiplication

**Definition 1.13.** Suppose  $A$  is a  $m \times n$  matrix and  $B$  is  $n \times p$  matrix. Then  $AB$  is defined as to be an  $m \times p$  matrix whose entry in row  $j$  and column  $k$  is given by

$$(AB)_{j,k} = \sum_{r=1}^n A_{j,r} B_{r,k}$$

**Remark.** The motivation for us to define the product like this comes from questioning, does  $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$ ? Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is the basis of  $W$ . Suppose  $u_1, \dots, u_k$  is the basis of  $U$  then consider the map  $T : U \rightarrow V$  and  $S : V \rightarrow W$ . Suppose  $\mathcal{M}(S) = A$  and  $\mathcal{M}(T) = B$ . Then

$$\begin{aligned} (ST)u_k &= S \left( \sum_{r=1}^n B_{r,k} v_r \right) \\ &= \sum_{r=1}^n B_{r,k} S v_r \\ &= \sum_{r=1}^n B_{r,k} \sum_{j=1}^m A_{j,r} w_j \\ &= \sum_{j=1}^m \left( \sum_{r=1}^n A_{j,r} B_{r,k} \right) w_j \end{aligned}$$

That is how we define  $\mathcal{M}(ST)$  and that is why **Definition 1.13.** makes sense.

**Proposition 1.11.** Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$ , then

$$\mathcal{M}(ST) = \mathcal{M}(S) \mathcal{M}(T)$$

*Proof.* It follows from our remark and how we defined the product of the matrix.  $\square$

**Definition 1.14.** Suppose  $A$  is a  $m \times n$  matrix then

1. If  $1 \leq j \leq m$  then  $A_{j,\cdot}$  denotes the  $1 \times n$  matrix consisting of row  $j$  of  $A$ .
2. If  $1 \leq j \leq n$  then  $A_{\cdot,j}$  denotes  $m \times 1$  matrix consisting of column  $j$  of  $A$ .

**Example :**

Suppose  $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$  then

$$A_{1,\cdot} = (1 \quad 2 \quad 3)$$

$$A_{\cdot,3} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

**Theorem 1.6.** Suppose  $A$  is a  $m \times n$  matrix and  $B$  is a  $n \times p$  matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

where  $1 \leq j \leq m$  and  $1 \leq k \leq p$ .

*Proof.* The definition of matrix multiplication states that

$$\begin{aligned} (AB)_{j,k} &= \sum_{r=1}^n A_{j,r} B_{r,k} \\ &= A_{j,1} B_{1,k} + \cdots + A_{j,n} B_{n,k} \end{aligned}$$

Now, if you take  $A_{j,\cdot}$  and  $B_{\cdot,k}$  and multiply it out you'll get the same thing.  $\square$

**Theorem 1.7.** Suppose  $A$  is a  $m \times n$  matrix and  $B$  is a  $n \times p$  matrix. Then

$$(AB)_{\cdot,k} = A B_{\cdot,k}$$

for  $1 \leq k \leq p$ .

*Proof.* Both of the matrix have size  $m \times 1$ . The  $j$ -th row of  $(AB)_{\cdot,k}$  has the element  $(AB)_{j,k}$  and the  $j$ -th row of  $A B_{\cdot,k}$  has element  $A_{j,1} B_{1,k} + A_{j,2} B_{2,k} + \cdots + A_{j,n} B_{n,k}$ . Thus, from our previous theorem they're equal.  $\square$

**Remark.** The row version of this is

$$(AB)_{k,\cdot} = A_{k,\cdot} B$$

**Theorem 1.8.** Suppose  $A$  is a  $m \times n$  matrix and  $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$  is a  $n \times 1$  matrix. Then,

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$$

*Proof.* They both have same size and the entries of  $Ab$  is the same as of the right side.  $\square$

**Theorem 1.9.** Suppose  $C$  is an  $m \times c$  matrix and  $R$  is a  $c \times n$  matrix

1. The column  $k$  of  $CR$  is the linear combination of the columns of  $C$ , with coefficients of this linear combination coming from column  $k$  of  $R$ .
2. Then row  $j$  of  $CR$  is a linear combination of the rows of  $R$ , with the coefficients of this linear combination coming from row  $j$  of  $C$ .

*Proof.* Use **Theorem 1.7.** and **Theorem 1.8.** for 1. and we'll prove 2. in the exercise section.  $\square$

### 1.3.4 Column-Row Factorization and Rank of a Matrix

**Definition 1.15.** Suppose  $A$  is a  $m \times n$  matrix with entries in  $\mathbf{F}$ .

1. The **column rank** of  $A$  is the dimension of the span of columns of  $A$  in  $\mathbf{F}^{1,m}$ .
2. The **row rank** of  $A$  is the dimension of the span of rows of  $A$  in  $\mathbf{F}^{n,1}$ .

**Example :** Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then the column rank is the dimension of

$$\text{span} \left( \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right)$$

and the row rank is the dimension of

$$\text{span} \left( (1 \quad 2 \quad 3), (4 \quad 5 \quad 6) \right)$$

**Definition 1.16.** The *transpose* of a  $m \times n$  matrix  $A$ , denoted by  $A^t$ , is the  $n \times m$  matrix whose entries are given by

$$(A^t)_{i,j} = A_{j,i}$$

**Theorem 1.10.** Suppose  $A$  is an  $m \times n$  matrix with entries in  $\mathbf{F}$  and column rank  $c \geq 1$ . Then there exists a  $m \times c$  matrix  $C$  and  $c \times n$  matrix  $R$ , both with entries in  $\mathbf{F}$ , such that  $A = CR$ .

*Proof.* The list  $A_{\cdot,1}, \dots, A_{\cdot,n}$  of columns of  $A$  can be reduced to a basis of the span of the columns of  $A$ . This basis has length  $c$  by definition of column rank. The  $c$  columns can be put together to form  $m \times c$ .

Now, each column  $k$  of  $A$  is a linear combination of columns of  $C$ . Make the coefficients of this linear combination column  $k$  of  $R$ . This matrix  $R$  has size  $c \times n$ . Thus,  $A = CR$  follows from **Theorem 1.9.(a)**.  $\square$

**Theorem 1.11.** Suppose  $A \in \mathbf{F}^{m,n}$  then the column rank of  $A$  equals row rank of  $A$ .

*Proof.* Let  $c$  be the column rank of  $A$ . Then  $A = CR$  by the previous theorem where  $C$  and  $R$  are the matrix whose size are  $m \times c$  and  $c \times n$  respectively. Now, from **Theorem 1.9. (b)** each row of  $A$  is a linear combination of rows of  $R$ . Since,  $R$  has  $c$  columns this implies that

$$\text{rowrank } A \leq c = \text{columnrank } A$$

Now applying the same thing to  $A^t$  we get

$$\begin{aligned}\text{columnrank } A &= \text{rowrank } A^t \\ &\leq \text{columnrank } A^t \\ &= \text{rowrank } A\end{aligned}$$

Thus, we're done. □

**Remark.** From now on, we'll limit our use of terminology of “row rank” and “column rank” to just “rank”.

### 1.3.5 Exercise

**Problem :** Suppose  $T \in \mathcal{L}(V, W)$ . Show that with respect to each choice of basis of  $V$  and  $W$ , the matrix of  $T$  has at least  $\dim \text{range } T$  nonzero entries.

*Solution :* Let  $v_1, \dots, v_n$  be the basis of  $V$  and  $w_1, \dots, w_m$  be the basis of  $W$  then suppose  $k$  of those vectors are 0 under  $T$  and let those vectors be  $v_1, \dots, v_k$ . Thus,

$$\text{range } T = \text{span}\{Tv_{k+1}, \dots, Tv_n\}$$

Thus,  $\dim \text{range } T \leq n - k$ . But since  $T(v_j) \neq 0$  for each  $k+1 \leq j \leq n$ , there must be one entry that is not 0 for each  $T(v_j)$ . Since, the number of  $T(v_j) \neq 0$  are exactly  $n - k$  and  $n - k \geq \dim \text{range } T$  this means there is at least  $\dim \text{range } T$  nonzero entries in matrix of  $T$ .

**Problem :** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that  $\dim \text{range } T = 1$  if and only if there exist a basis of  $V$  and a basis of  $W$  such that with respect to these bases, all the entries of  $\mathcal{M}(T)$  is 1.

*Solution :* Let us first prove  $(\Leftarrow)$ . Suppose  $A_{i,j} = 1$  for all  $i, j$ . That means,  $T(v_i) = \sum w_j$  where  $v_1, \dots, v_n$  is the basis of  $V$  and  $w_1, \dots, w_m$  is the basis of  $W$ . Thus,  $Tv_1 = Tv_2 = \dots = Tv_n = \sum w_j$  and  $\text{range } T = \text{span}\{Tv_1, \dots, Tv_n\} = \text{span}\{\sum w_j\} \implies \dim \text{range } T = 1$ . Now, for the  $(\Rightarrow)$  we use the Fundamental theorem of linear maps,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$\implies \dim V = \dim \text{null } T + 1$$

Now, suppose  $v_2, \dots, v_n$  be that basis of  $\text{null } T$ . Then extend this basis to  $V$ , suppose  $v_2, \dots, v_n, v$  then

$$\begin{aligned} T(v_2) &= T(v_3) = \dots = T(v_n) = 0 \\ \implies T(v) &= T(v_2 + v) = \dots = T(v_n + v) \end{aligned}$$

One can check that  $v_2 + v, \dots, v$  is a basis of  $V$  (as it is linearly independent and has length  $n$ ). Now, since  $\dim \text{range } T = 1$  we have  $T(x) = \lambda T(v)$  and we choose  $T(v), w_2, \dots, w_m$  as our basis for  $W$ . Now, we use a clever trick and set  $w_1 = T(v) - w_2 - w_3 - \dots - w_m$  and notice that  $w_1, \dots, w_m$  is a basis of  $W$ . Thus,

$$T(v_i) = T(v) = \sum w_j$$

Thus, we're done.

**Problem :** Suppose that  $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$  is the differentiation map defined by  $Dp = p'$ . Find a basis of  $\mathcal{P}_3(\mathbf{R})$  and a basis of  $\mathcal{P}_2(\mathbf{R})$  such that the matrix of  $D$  with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

*Solution :* Take the basis of  $\mathcal{P}_3(\mathbf{R})$  to be  $z, \frac{z^2}{2}, \frac{z^3}{3}, 1$  and  $\mathcal{P}_2(\mathbf{R})$  to be  $1, z, z^2$ .

**Problem :** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that there exist a basis of  $V$  and a basis of  $W$  such that with respect to these bases, all entries of  $\mathcal{M}(T)$  are 0 except that the entries in row  $k$ , column  $k$ , equal 1 if  $1 \leq k \leq \dim \text{range } T$ .

*Solution :* Let  $\dim V = n$  and  $\dim \text{range } T = m$ . Now, let  $v_{m+1}, \dots, v_n$  be the basis of  $\dim \text{null } T$ . Now, extend these basis such in the following way

$$(v_1, \dots, v_m, v_{m+1}, \dots, v_n)$$

Here,  $T(v_i) \neq 0$  for  $1 \leq i \leq m$ . Now, since  $(v_1, \dots, v_m, v_{m+1}, \dots, v_n)$  spans  $V$  the list  $(Tv_1, \dots, Tv_m)$  must span  $\text{range } T$  and in fact it is the basis of  $\text{range } T$  (one can check that its linearly independent). We can now extend this basis to the basis of  $W$ . Suppose  $(Tv_1, \dots, Tv_m, w_1, \dots, w_k)$  is the basis of  $W$ . Then,

$$T(v_i) = 0 \cdot T(v_1) + \dots + 1 \cdot T(v_i) + \dots + 0 \cdot w_k$$

for  $1 \leq i \leq m = \dim \text{range } T$ . But for  $i > \dim \text{range } T$  we have

$$0 = T(v_i) = 0 \cdot T(v_1) + 0 \cdot T(v_2) + \dots + 0 \cdot w_k$$

**Problem :** Suppose  $\sigma_1, \dots, \sigma_m$  is a basis of  $V$  and  $W$  is finite-dimensional. Suppose  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis  $w_1, \dots, w_n$  of  $W$  such that all entries in the first column of  $\mathcal{M}(T)$  [with respect to the bases  $\sigma_1, \dots, \sigma_m$  and  $w_1, \dots, w_n$ ] are 0 except for possibly a 1 in the first row, first column.

*Solution :* We know that  $\text{range } T = \text{span}\{T(\sigma_1), T(\sigma_2), \dots, T(\sigma_m)\}$ . Thus, we can make this span a basis. If  $T(\sigma_1) = 0$  then we're done but if not then the basis of  $\text{range } T$  would be

$$(T(\sigma_1), z_2, \dots, z_k)$$

Now, we can extend this basis to the basis of  $W$ , suppose its

$$(T(\sigma_1), z_2, \dots, z_k, s_{k+1}, \dots, s_m)$$

then

$$T(\sigma_1) = 1 \cdot T(\sigma_1) + 0 \cdot z_2 + \dots + 0 \cdot s_m$$

**Problem :** Suppose  $w_1, \dots, w_n$  is a basis of  $W$  and  $V$  is finite-dimensional. Suppose  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis  $\sigma_1, \dots, \sigma_m$  of  $V$  such that all entries in the first row of  $\mathcal{M}(T)$  [with respect to the bases  $\sigma_1, \dots, \sigma_m$  and  $w_1, \dots, w_n$ ] are 0 except for possibly a 1 in the first row, first column.

*Solution :* Take any basis  $v_1, \dots, v_m$  of  $V$ . Then, suppose

$$\begin{aligned} T(v_i) &= \sum_{j=1}^n i\lambda_j w_j \\ &= i\lambda_1 w_1 + \sum_{j=2}^n i\lambda_j w_j \end{aligned}$$

Now, if all the  $T(v_i)$  has 0 as the coefficient of  $w_1$  then we're done. If not then take a  $v_k$  for which  $k\lambda_1 \neq 0$  then swap it with  $v_1$ . Then, define

$$\sigma_1 = \frac{v_1}{1\lambda_1}$$

$$\sigma_i := v_i - \frac{i\lambda_1}{1\lambda_1}v_1 \quad \text{for } i \geq 2$$

Now, one can check that  $(\sigma_1, \dots, \sigma_m)$  is a basis and

$$T(\sigma_i) = T(v_i) - \frac{i\lambda_1}{1\lambda_1}T(v_k)$$

$$= 0 \cdot w_1 + \sum b_j w_j$$

**Problem :** Give an example of  $2 \times 2$  matrices  $A$  and  $B$  such that  $AB \neq BA$ .

*Solution :*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

**Problem :** Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose  $A, B, C, D, E, F$  are matrices whose sizes are such that  $A(B + C)$  and  $(D + E)F$  make sense. Explain why  $AB + AC$  and  $DF + EF$  both make sense and prove that

$$A(B + C) = AB + AC \quad \text{and} \quad (D + E)F = DF + EF.$$

*Solution :* If  $A(B + C)$  and  $(D + E)F$  makes sense then  $B$  and  $C$  must be of the same size and  $D$  and  $E$  must be of the same size. Also, the number of columns in  $A$  must be same as the number of rows in  $B$  and  $C$ . Let  $A$  be a  $n \times p$  matrix and  $X = B + C$  then

$$\begin{aligned} (AX)_{i,j} &= \sum_{r=1}^n A_{i,r}X_{r,j} \\ &= \sum_{r=1}^n A_{i,r}(B_{r,j} + C_{r,j}) \\ &= \sum_{r=1}^n A_{i,r}B_{r,j} + \sum_{r=1}^n A_{i,r}C_{r,j} \\ &= (AB)_{i,j} + (AC)_{i,j} \end{aligned}$$

Thus,  $A(B + C) = AB + AC$ . Similar proof works for  $(D + E)F$ .

**Problem :** Prove that matrix multiplication is associative. In other words, suppose  $A, B, C$  are matrices whose sizes are such that  $(AB)C$  makes sense. Explain why  $(AB)C$  makes sense and prove that

$$(AB)C = A(BC).$$

*Solution :* To make  $(AB)C$  sense, we need  $A$  to have same number of columns as the number of rows in  $B$ . Also, we need  $B$  to have same number of columns as number of rows in  $C$ . To prove the associativity, you can just definition of matrix multiplication.

**Problem :** Suppose  $A$  is an  $n \times n$  matrix and  $1 \leq j, k \leq n$ . Show that the entry in row  $j$ , column  $k$ , of  $A^3$  (which is defined to mean  $AAA$ ) is

$$\sum_{r=1}^n \sum_{i=1}^n A_{j,r} A_{r,i} A_{i,k}.$$

*Solution :* It follows directly from the definition of matrix multiplication.

**Problem :** Suppose  $m$  and  $n$  are positive integers. Prove that the function  $A \mapsto A^t$  is a linear map from  $\mathbf{F}^{m,n}$  to  $\mathbf{F}^{n,m}$ .

*Solution :* Define  $T : \mathbf{F}^{m,n} \rightarrow \mathbf{F}^{n,m}$  by  $T(A) = A^t$ . We must show that  $T$  is linear, i.e.

$$T(A + B) = T(A) + T(B) \quad \text{and} \quad T(\lambda A) = \lambda T(A),$$

for all  $A, B \in \mathbf{F}^{m,n}$  and all scalars  $\lambda \in \mathbf{F}$ .

By definition of the transpose,

$$(A^t)_{ij} = A_{ji}, \quad (1 \leq i \leq n, 1 \leq j \leq m).$$

Now let  $A, B \in \mathbf{F}^{m,n}$ . Then for each  $i, j$ ,

$$((A + B)^t)_{ij} = (A + B)_{ji} = A_{ji} + B_{ji} = (A^t)_{ij} + (B^t)_{ij} = (A^t + B^t)_{ij}.$$

Hence  $(A + B)^t = A^t + B^t$ .

Similarly, for  $\lambda \in \mathbf{F}$ ,

$$((\lambda A)^t)_{ij} = (\lambda A)_{ji} = \lambda A_{ji} = \lambda (A^t)_{ij} = (\lambda A^t)_{ij}.$$

So  $(\lambda A)^t = \lambda A^t$ .

Therefore  $T$  preserves both addition and scalar multiplication. Thus  $T$  is a linear map from  $\mathbf{F}^{m,n}$  to  $\mathbf{F}^{n,m}$ .

**Problem :** Prove that if  $A$  is an  $m \times n$  matrix and  $C$  is an  $n \times p$  matrix, then

$$(AC)^t = C^t A^t.$$

*Solution :*

**Solution :** Let  $A \in \mathbf{F}^{m,n}$  and  $C \in \mathbf{F}^{n,p}$ . By definition of matrix multiplication, the  $(i, j)$ -entry of  $AC$  is

$$(AC)_{ij} = \sum_{k=1}^n A_{ik} C_{kj}, \quad (1 \leq i \leq m, 1 \leq j \leq p).$$

Taking the transpose, we get

$$((AC)^t)_{ij} = (AC)_{ji} = \sum_{k=1}^n A_{jk} C_{ki}.$$



On the other hand, consider the product  $C^t A^t$ . Here  $C^t$  is  $p \times n$  and  $A^t$  is  $n \times m$ , so  $C^t A^t$  is  $p \times m$ . Its  $(i, j)$ -entry is

$$(C^t A^t)_{ij} = \sum_{k=1}^n (C^t)_{ik} (A^t)_{kj}.$$

By definition of the transpose,

$$(C^t)_{ik} = C_{ki}, \quad (A^t)_{kj} = A_{jk}.$$

Hence

$$(C^t A^t)_{ij} = \sum_{k=1}^n C_{ki} A_{jk}.$$

But scalar multiplication in  $\mathbf{F}$  is commutative, so

$$\sum_{k=1}^n C_{ki} A_{jk} = \sum_{k=1}^n A_{jk} C_{ki}.$$

Therefore,

$$(C^t A^t)_{ij} = ((AC)^t)_{ij}, \quad (1 \leq i \leq p, 1 \leq j \leq m).$$

Since all entries are equal, we conclude

$$(AC)^t = C^t A^t.$$

**Problem :** Suppose  $A$  is an  $m \times n$  matrix with  $A \neq 0$ . Prove that the rank of  $A$  is 1 if and only if there exist  $(c_1, \dots, c_m) \in \mathbf{F}^m$  and  $(d_1, \dots, d_n) \in \mathbf{F}^n$  such that

$$A_{j,k} = c_j d_k \quad \text{for every } j = 1, \dots, m \text{ and every } k = 1, \dots, n.$$

*Solution :* For  $(\Leftarrow)$ , Use **Theorem 1.10**. then use the definition of matrix multiplication. For  $(\Rightarrow)$ , the matrix

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

can produce any column thus the rank is 1.

**Problem :** Suppose  $T \in \mathcal{L}(V)$ , and  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are bases of  $V$ . Prove that the following are equivalent:

- (a)  $T$  is injective.
- (b) The columns of  $\mathcal{M}(T)$  are linearly independent in  $\mathbf{F}^{n,1}$ .
- (c) The columns of  $\mathcal{M}(T)$  span  $\mathbf{F}^{n,1}$ .
- (d) The rows of  $\mathcal{M}(T)$  span  $\mathbf{F}^{1,n}$ .
- (e) The rows of  $\mathcal{M}(T)$  are linearly independent in  $\mathbf{F}^{1,n}$ .

Here  $\mathcal{M}(T)$  means  $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$ .

*Solution :* Will do later.

## 1.4 Invertibility and Isomorphism

### 1.4.1 Invertible Linear Maps

**Definition 1.17.** A linear map  $T \in \mathcal{L}(V, W)$  is called *invertible* if there exists a linear map  $S \in \mathcal{L}(W, V)$  such that  $ST$  equals identity operator on  $V$  and  $TS$  equals identity operator on  $W$ .

**Definition 1.18.** A linear map  $S \in \mathcal{L}(V, W)$  satisfying  $ST = I$  and  $TS = I$  is called an *inverse* of  $T$ .

**Proposition 1.12.** An invertible map has an unique inverse.

*Proof.* Suppose  $T \in \mathcal{L}(V, W)$  and let  $S_1$  and  $S_2$  be its inverses then

$$S_1 = S_1 I = S_1 (TS_2) = (S_1 T) S_2 = I S_2 = S_2$$

□

**Remark.** Since inverses are unique for a invertible map  $T$ , we will denote it by  $T^{-1}$ .

**Proposition 1.13.** A linear map is invertible if and only if it is injective and surjective.

*Proof.* Suppose  $T \in \mathcal{L}(V, W)$  is an invertible map and suppose  $T(v) = T(w)$  then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

Hence,  $T$  is injective. To prove surjectivity, notice that

$$w = T^{-1}(Tw)$$

which proves  $T$  is surjective.

Now, suppose  $T$  is injective and surjective. Then, there exists a unique element  $S(w)$  such that

$$T(S(w)) = w$$

the uniqueness is due to the injectivity of  $T$ . Let us show that,  $S \in \mathcal{L}(W, V)$

$$\begin{aligned} T(S(w_1) + S(w_2)) &= T(S(w_1)) + T(S(w_2)) \\ &= w_1 + w_2 \\ &= T(S(w_1 + w_2)) \end{aligned}$$

Thus,  $S(w_1) + S(w_2) = S(w_1 + w_2)$ . Also,

$$\begin{aligned} T(\lambda S(w)) &= \lambda T(S(w)) \\ &= \lambda w \\ &= T(S(\lambda w)) \end{aligned}$$

Thus,  $\lambda S(w) = S(\lambda w)$ . □

Now, by how we defined  $S$ , it implies that  $TS = I$  on  $W$ . Also,

$$\begin{aligned} T(ST)v &= (TS)(T)v = Tv \\ \implies (ST)v &= v \end{aligned}$$

Thus,  $ST$  is an identity operator on  $V$ .

**Proposition 1.14.** Suppose that  $V$  and  $W$  are finite-dimensional vector spaces, such that,  $\dim W = \dim V$  and  $T \in \mathcal{L}(V, W)$ . Then

$$T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective}$$

*Proof.* From the Fundamental theorem of linear maps,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

If  $T$  is injective then  $\text{null } T = \{0\}$ . Thus

$$\dim V = \dim W = \dim \text{range } T$$

$$\implies \text{range } T = W$$

Now, if  $T$  is surjective then  $\text{range } T = W$ . Thus

$$\dim V = \dim \text{null } T + \dim W$$

$$\implies \dim \text{range } T = 0$$

$$\implies \text{range } T = \{0\}$$

Thus,  $T$  is injective  $\iff T$  is surjective. From **Proposition 1.13.** we get our final result.  $\square$

**Proposition 1.15.** Suppose  $V$  and  $W$  are finite-dimensional vector spaces of the same dimension,  $S \in \mathcal{L}(V, W)$ , and  $T \in \mathcal{L}(V, W)$ . Then,  $ST = I \iff TS = I$ .

*Proof.* First  $ST = I$  then take  $v \in \text{null } T$ . Thus,

$$v = STv = S(0) = 0$$

Thus,  $\text{null } T = \{0\}$  and  $T$  is injective. Since  $\dim V = \dim W$ , this implies  $T$  is invertible. Thus, there exists a  $T^{-1}$ . Now,

$$T^{-1} = (ST)(T^{-1}) = S$$

We can now apply the same idea for  $(\Leftarrow)$  of the proof. We just need to swap  $V$  with  $W$ , and  $T$  with  $S$ .  $\square$

### 1.4.2 Isomorphic Vector Spaces

**Definition 1.19.** An *isomorphism* is an invertible linear map and two vector spaces are isomorphic if there is an isomorphism between them.

**Proposition 1.16.** Two finite-dimensional vector spaces over  $\mathbf{F}$  are isomorphic if and only if they have the same dimension.

*Proof.* Suppose  $V$  and  $W$  are isomorphic. Then there exists a injective and surjective map  $T$  from  $V$  to  $W$ . Thus,  $\text{null } T = \{0\}$ . Then

$$\dim V = \dim \text{range } T$$

Also, since  $T$  is surjective  $\text{range } T = W$ . Then

$$\dim V = \dim W$$

Now, suppose  $\dim W = \dim V$ . Define  $T : V \rightarrow W$  as

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

where  $v_i$ 's and  $w_i$ 's are the basis of  $V$  and  $W$  respectively. One can check this is a linear map. Now, this map is surjective as  $\sum c_iw_i$  covers  $W$ . Also,  $\text{null } T = \{0\}$  as

$$\begin{aligned} \dim W &= \dim V = \dim \text{null } T + \dim \text{range } T \\ &= \dim \text{null } T + \dim \text{range } T \\ &= \dim \text{null } T + \dim W \end{aligned}$$

Thus,  $T$  is injective and surjective which means that  $V$  and  $W$  are isomorphic. □

**Proposition 1.17.** Suppose  $v_1, \dots, v_n$  be the basis of  $V$  and  $w_1, \dots, w_m$  be the basis of  $W$ . Then  $\mathcal{M}(T)$  is a isomorphism between  $\mathcal{L}(V, W)$  to  $\mathbf{F}^{m,n}$

*Proof.* We know that  $\mathcal{M}(T)$  is a linear map as

$$\mathcal{M}(T + S) = \mathcal{M}(T) + \mathcal{M}(S) \quad \text{and} \quad \mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$$

We need to prove that  $\mathcal{M}$  is injective and surjective. We know that  $\mathcal{M}(T)$  is injective  $\iff \text{null } \mathcal{M}(T) = \{0\}$ . And we know  $\mathcal{M}(T) = 0 \iff T(x) = 0$  for all  $x \in V$ . Thus,  $T = 0$ .

To prove  $\mathcal{M}(T)$  is surjective. We know that there exists a  $T \in \mathcal{L}(V, W)$  such that

$$T(v_k) = \sum_{j=1}^m A_{i,j} w_j$$

which proves the surjectivity of  $\mathcal{M}(T)$ . □

**Proposition 1.18.** Suppose  $V$  and  $W$  are finite-dimensional. Then  $\mathcal{L}(V, W)$  is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

*Proof.* Use **Proposition 1.17.** and **Proposition 1.16.** and

$$\dim \mathcal{L}(V, W) = mn = (\dim V)(\dim W)$$

□

### 1.4.3 Linear Map Thought of as Matrix Multiplication

**Definition 1.20.** Suppose  $v \in V$  and  $v_1, \dots, v_n$  is the basis of  $V$ . The matrix of  $v$  with respect to the basis is the  $n$  matrix

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

where  $b_1, \dots, b_n$  are scalar such that  $v = b_1v_1 + \cdots + b_nv_n$ .

**Proposition 1.19.** Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n$  is a basis of  $W$ . Let  $1 \leq k \leq n$ . Then,

$$\mathcal{M}(T)_{\cdot, k} = \mathcal{M}(Tv_k)$$

*Proof.* Immediate from the definition of  $\mathcal{M}(Tv_k)$ .  $\square$

**Proposition 1.20.** Suppose  $T \in \mathcal{L}(V, W)$  and  $v \in V$ . Let  $v_1, \dots, v_n$  be the basis of  $V$  and  $w_1, \dots, w_m$  be the basis of  $W$ . Then

$$\mathcal{M}(Tv) = \mathcal{M}(T) \mathcal{M}(v)$$

*Proof.* Suppose  $v = b_1 v_1 + \dots + b_n v_n$ . Then,

$$Tv = b_1 Tv_1 + \dots + b_n Tv_n$$

Hence,

$$\begin{aligned} \mathcal{M}(Tv) &= b_1 \mathcal{M}(Tv_1) + \dots + b_n \mathcal{M}(Tv_n) && (\text{Linearity of } \mathcal{M}) \\ &= b_1 \mathcal{M}(T)_{\cdot, 1} + \dots + b_n \mathcal{M}(T)_{\cdot, n} && (\text{Proposition 1.19.1}) \\ &= \mathcal{M}(T) \mathcal{M}(v) && (\text{Theorem 1.8.}) \end{aligned}$$

$\square$

**Proposition 1.21.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then  $\dim \text{range } T$  equals the column rank of  $\mathcal{M}(T)$ .

*Proof.* Suppose  $v_1, \dots, v_n$  be the basis of  $V$  and  $w_1, \dots, w_m$  be the basis of  $W$ . Now, define  $\varphi : W \rightarrow \mathbf{F}^{m,1}$  as  $\varphi(w) = \mathcal{M}(w)$ . One can prove that this is an isomorphism. If we restrict our domain to just  $\text{range } T$  we see that our co-domain is going to be  $\mathcal{O} = \text{span}\{\mathcal{M}(Tv_1), \dots, \mathcal{M}(Tv_m)\}$ . Also,

$$\varphi|_{\text{range } T} : \text{range } T \rightarrow \mathcal{O}$$

is a isomorphism and since isomorphism preserves dimension. We have

$$\dim \text{range } T = \dim \mathcal{O} = \text{column rank of } T$$

$\square$

#### 1.4.4 Change of Basis

**Definition 1.21.** We define the  $n \times n$  matrix, called *identity matrix* by

$$A_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The identity matrix is denoted by  $I$ .

**Definition 1.22.** A square matrix is called *invertible* if there is a square matrix  $B$  of the same size such that

$$AB = BA = I$$

we call the matrix  $B$  the *inverse* of  $A$ .

**Remark.** The inverse of a square matrix  $A$  is unique and therefore will be denoted by  $A^{-1}$ . Here, is a short proof of the uniqueness of the inverse. Suppose  $A$  has two inverses  $B_1$  and  $B_2$ . Thus,

$$B_1 = IB_1 = (B_2A)B_1 = B_2(AB_1) = B_2I = B_2$$

Also,  $(A^{-1})^{-1} = I$  and  $(AC)^{-1} = C^{-1}A^{-1}$ . You can verify these.

**Definition 1.23.** Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$ . If  $u_1, \dots, u_m$  is a basis of  $U$ ,  $v_1, \dots, v_n$  is a basis of  $V$ , and  $w_1, \dots, w_p$  is the basis of  $W$  then

$$\mathcal{M}(ST, (u_1, \dots, u_m), (w_1, \dots, w_p)) = \mathcal{M}(S, (v_1, \dots, v_n), (w_1, \dots, w_p)) \mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_n))$$

This is just the matrix multiplication which we had defined earlier but with respect to the basis. See **Proposition 1.11**.

**Proposition 1.22.** Suppose  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are the basis of  $V$ . Then the matrices

$$\mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) \quad \text{and} \quad \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n))$$

are inverses of each other. Here,  $I$  is the identity operator.

*Proof.* Use **Definition 1.23**. and replace  $w_k$  with  $u_k$ . And replace  $S, T$  with the identity operator. Then

$$I = \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n)) \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$$

Now interchange the roles of  $u$ 's and  $v$ 's to get

$$I = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)) \mathcal{M}(I, (v_1, \dots, v_n), (u_1, \dots, u_n))$$

□

**Remark.** For convenience, we'll write

$$\mathcal{M}(T, (u_1, \dots, u_n), (u_1, \dots, u_n)) = \mathcal{M}(T, (u_1, \dots, u_n))$$

**Proposition 1.23.** Suppose  $T \in \mathcal{L}(V)$ . Let  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  be the basis of  $V$ . Let

$$A = \mathcal{M}(T, (u_1, \dots, u_n)) \quad \text{and} \quad B = \mathcal{M}(T, (v_1, \dots, v_n))$$

and  $C = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$ . Then,

$$A = C^{-1}BC$$

*Proof.* Use **Definition 1.23**. and replace  $w_k$  with  $u_k$  and  $S$  with  $I$ . Then, use

**Proposition 1.22.** to get

$$A = C^{-1} \mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n)) \tag{1}$$

Now, again use the definition and this time replace  $w_k$  with  $v_k$ . Then

$$\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n)) = BC$$

We can now substitute this equation in equation (1) to get

$$A = C^{-1}BC$$

□

**Proposition 1.24.** Suppose that  $v_1, \dots, v_n$  is the basis of  $V$  and  $T \in \mathcal{L}(V)$  is invertible. Then,  $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$ , where both matrices are with respect to basis  $v_1, \dots, v_n$ .

*Proof.* Use **Definition 1.23**. □

### 1.4.5 Exercise

**Problem :** Suppose  $T \in \mathcal{L}(V, W)$  is invertible. Show that  $T^{-1}$  is invertible and

$$(T^{-1})^{-1} = T$$

*Solution :* Since,  $T$  is invertible, we have

$$TT^{-1} = I \quad \text{and} \quad T^{-1}T = I$$

If we switch our perspective from  $T$  to  $T^{-1}$ , we get that  $T$  is invertible from **Definition 1.17.** and from **Proposition 1.12.** we have

$$(T^{-1})^{-1} = T$$

**Problem :** Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$  are both invertible linear maps. Prove that  $ST \in \mathcal{L}(U, W)$  is invertible and that  $(ST)^{-1} = T^{-1}S^{-1}$ .

*Solution :* Since,  $S$  and  $T$  are both invertible then  $S^{-1}$  and  $T^{-1}$  both exist. Also,  $T^{-1}S^{-1} \in \mathcal{L}(W, U)$ . Thus,

$$\begin{aligned} (ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\ &= S(I)S^{-1} \\ &= SS^{-1} \\ &= I \end{aligned}$$

One can do the same thing for  $(T^{-1}S^{-1})(ST)$ . Thus,  $ST$  is invertible and from the above calculation so we can say  $(ST)^{-1} = T^{-1}S^{-1}$ .

**Problem :** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that the following are equivalent.

- (a)  $T$  is invertible
- (b)  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for every basis  $v_1, \dots, v_n$  of  $V$ .
- (c)  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for some basis  $v_1, \dots, v_n$  of  $V$ .

*Solution :* Suppose  $T$  is invertible then it is also injective and surjective. Let  $v_1, \dots, v_n$  be a basis of  $V$ . Then, we know that  $\text{span}\{Tv_1, \dots, Tv_n\} = V$  because of the surjectivity. Also, if

$$\begin{aligned} a_1Tv_1 + \dots + a_nTv_n &= 0 \\ \implies T(a_1v_1 + \dots + a_nv_n) &= 0 \\ \implies a_1v_1 + \dots + a_nv_n &= 0 \\ \implies a_1 = a_2 = \dots = a_n &= 0 \end{aligned}$$

The last line is from injectivity of  $T$ . Thus,  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for any basis of  $V$ .

Now, suppose  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for every basis  $v_1, \dots, v_n$  of  $V$ . Then, (c) automatically holds. Also,

$$\begin{aligned} a_1Tv_1 + \dots + a_nTv_n &= 0 \\ \implies a_1 &= a_2 = \dots = a_n = 0 \end{aligned}$$

Thus,  $\text{null } T = \{0\}$  which implies  $T$  is injective. Now, since  $Tv_1, \dots, Tv_n$  is a basis, every element of  $V$  can be written as some combination of  $V$ . Thus,

$$\begin{aligned} a_1Tv_1 + \dots + a_nTv_n &= y \\ \implies T(a_1v_1 + \dots + a_nv_n) &= y \end{aligned}$$

Thus, for every  $y \in V$  there exists some element which gets mapped to  $y$ . Thus,  $T$  is surjective. Thus, from **Proposition 1.13**, we get that  $T$  is invertible.

Now, suppose  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for some basis  $v_1, \dots, v_n$  of  $V$ . Then we can apply the same argument as we did for above to get to  $T$  is invertible. Since,  $T$  is invertible we get (b).

**Problem :** Suppose  $V$  is finite-dimensional and  $\dim V > 1$ . Prove that the set of non-invertible linear maps from  $V$  to itself is not a subspace of  $\mathcal{L}(V)$ .

*Solution :* We can construct two non-invertible linear maps which form an invertible map when added. Suppose  $v_1, \dots, v_n$  is a basis of  $V$ .

$$T(a_1v_1 + \dots + a_nv_n) = a_2v_2 + \dots + a_nv_n$$

$$S(a_1v_1 + \dots + a_nv_n) = a_1v_1$$

One can check that both of them are linear maps and both of them lack injectivity property so they're not invertible. But

$$(S + T)(a_1v_1 + \dots + a_nv_n) = a_1v_1 + a_2v_2 + \dots + a_nv_n$$

$$(S + T)(x) = I(x)$$

which is an invertible linear map. Thus, set of non-invertible linear maps from  $V$  to itself is not a subspace of  $\mathcal{L}(V)$ .

**Remark.** We used the  $\dim V > 1$  when we defined  $T$  and  $S$ .

**Problem :** Suppose  $V$  is finite-dimensional,  $U$  is a subspace of  $V$ , and  $S \in \mathcal{L}(U, V)$ . Prove that there exists an invertible linear map  $T$  from  $V$  to itself such that  $Tu = Su$  for every  $u \in U$  if and only if  $S$  is injective.

*Solution :* For  $(\Rightarrow)$ , if  $S(x) = S(y)$  then  $T(x) = T(y)$  which implies  $x = y$  because  $T$  is invertible. Now, for  $(\Leftarrow)$  choose a basis of  $U$  and extend it to the basis of  $V$  say  $\mathcal{B} = (u_1, \dots, u_k, v_{k+1}, \dots, v_n)$ , here  $n = \dim V$ . Now since  $S$  is injective, the list  $(Su_1, \dots, Su_k)$  is linearly independent and can be extended to a basis of  $V$ . Let

$$\mathcal{C} = \{Su_1, \dots, Su_k, w_{k+1}, \dots, w_n\}$$

be the basis of  $V$ . Define  $T : V \rightarrow V$  as following

$$T(u_i) = S(u_i) \text{ for } 1 \leq i \leq k \quad \text{and} \quad T(v_j) = w_j \text{ for } k+1 \leq j \leq n$$

One can check this is an invertible linear map.



**Problem :** Suppose  $W$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{null } S = \text{null } T$  if and only if there exists an invertible  $E \in \mathcal{L}(W)$  such that  $S = ET$ .

*Solution :* For  $(\Leftarrow)$ , take  $x \in \text{null } T$  then  $S(x) = E(T(x)) = 0$  thus  $x \in \text{null } S$ . Now, if  $x \in \text{null } S$  then  $0 = S(x) = ET(x) \implies T(x) = 0$  thus  $x \in \text{null } T$ . Thus,  $\text{null } T = \text{null } S$ . I'll do the  $\Leftarrow$  later.

## 1.5 Product and Quotients of Vector Spaces

### 1.5.1 Products of Vector Spaces

**Definition 1.24.** Suppose  $V_1, \dots, V_m$  are vector spaces over  $\mathbf{F}$ .

- The product  $V_1 \times \dots \times V_m$  is defined by

$$V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$$

- Addition on  $V_1 \times \dots \times V_m$  is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$$

- Scalar Multiplication on  $V_1 \times \dots \times V_m$  is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$$

**Proposition 1.25.** Suppose  $V_1, \dots, V_m$  are vector spaces over  $\mathbf{F}$ . Then  $V_1 \times \dots \times V_m$  is a vector space over  $\mathbf{F}$ .

*Proof.* Just check the vector axioms. □

**Proposition 1.26.** Suppose  $V_1, \dots, V_m$  are finite-dimensional vector spaces. Then  $V_1 \times \dots \times V_m$  is finite-dimensional and

$$\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$$

*Proof.* Choose a basis of  $V_k$  and consider every element of  $V_1 \times \dots \times V_k$  that equals a element from the basis of the vector  $V_k$  in the  $k$ -th slot and 0 in others. The list of vector spans  $V_1 \times \dots \times V_m$  and is linearly independent. Thus, it is the basis of  $V_1 \times \dots \times V_m$ . The length of the basis is  $\dim V_1 + \dots + \dim V_m$ . □

**Proposition 1.27.** Suppose that  $V_1, \dots, V_m$  are subspaces of  $V$ . Define a linear map  $\Gamma : V_1 \times \dots \times V_m \rightarrow V_1 + \dots + V_m$  by

$$\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m$$

Then  $V_1 + \dots + V_m$  is a direct sum if and only if  $\Gamma$  is injective.

*Proof.* If  $V_1 + \dots + V_m$  is a direct sum then the only way we can write 0 is by choosing 0 from each  $V_i$ . Thus,

$$\Gamma(v_1, \dots, v_m) = 0 \iff v_1 = v_2 = \dots = v_m = 0$$

Thus,  $\text{null } \Gamma = \{0\}$  which implies  $\Gamma$  is injective.

Now, suppose  $\Gamma$  is injective then  $\text{null } \Gamma = \{0\}$ , which means that the only element that gets mapped to 0 is  $(0, \dots, 0)$ . Thus, the only way to write 0 is by choosing 0 from each  $V_i$ . Thus,  $V_1 + \dots + V_m$  is a direct sum. □

**Proposition 1.28.** Suppose  $V$  is finite-dimensional and  $V_1, \dots, V_m$  are subspaces of  $V$ . Then  $V_1 + \dots + V_m$  is direct sum if and only if

$$\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$$

*Proof.* The map  $\Gamma$  is surjective. And  $V_1 + \dots + V_m$  is a direct sum

$$\iff \Gamma \text{ is injective}$$

$$\iff \text{null } \Gamma = \{0\}$$

$$\iff \dim(V_1 \times \dots \times V_m) = \dim(V_1 + \dots + V_m)$$

Combining the result of **Proposition 1.26.** we get our desired result. □

### 1.5.2 Quotients Spaces

**Definition 1.25.** Suppose  $v \in V$  and  $U \subseteq V$ . Then  $v + U$  is a subset of  $V$  defined by

$$v + U = \{v + u \mid u \in U\}$$

**Definition 1.26.** For  $v \in V$  and  $U$  a subset of  $V$ , the set  $v + U$  is said to be a *translate* of  $U$ .

**Definition 1.27.** Suppose  $U$  is a subspace of  $V$ . Then the *quotient space*  $V/U$  is the set of all translate of  $U$ ,

$$V/U = \{v + U \mid v \in V\}$$

**Proposition 1.29.** Suppose  $U$  is a subspace of  $V$  and  $v, w \in V$  then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset$$

*Proof.* Suppose  $v - w \in U$  then  $v = w + u'$  for some  $u' \in U$  thus,  $v + u = w + (u' + u) \in w + U$  which implies  $v + U \subseteq w + U$ . Thus, similarly  $w + U \subseteq v + U \implies v + U = w + U \implies (v + U) \cap (w + U) \neq \emptyset$ .

Now, suppose  $(v + U) \cap (w + U) \neq \emptyset$  then  $v + u_1 = w + u_2$  for some  $u_1, u_2 \in U$  which implies  $v - w \in U$ , which implies  $v + U = w + U$ . And  $v + U = w + U \implies v - w \in U$ . Thus, we proved every direction of the proof.  $\square$

**Definition 1.28.** Suppose  $U$  is a subspace of  $V$ . Then *addition* and *scalar multiplication* are defined on  $V/U$  by

$$(v + U) + (w + U) = (v + w) + U$$

$$\lambda(v + U) = (\lambda v) + U$$

for all  $v, w \in V$  and all  $\lambda \in F$ .

**Proposition 1.30.** Suppose  $U$  is a subspace of  $V$ . Then  $V/U$ , with the operations of addition and scalar multiplication as defined above, is a vector space.

*Proof.* Just use previous definitions and check the vector axioms.  $\square$

**Proposition 1.31.** Suppose  $U$  is a subspace of  $V$ . The *quotient map*  $\pi : V \rightarrow V/U$  is a linear map defined by

$$\pi(v) = v + U$$

for each  $v \in V$ .

*Proof.* Note that  $\pi(a + b) = (a + b) + U = (a + U) + (b + U) = \pi(a) + \pi(b)$  and  $\pi(\lambda a) = (\lambda a) + U = \lambda(a + U) = \lambda\pi(a)$ .  $\square$

**Proposition 1.32.** Suppose  $V$  is finite-dimensional vector space and  $U$  is a subspace of  $V$  then

$$\dim V/U = \dim V - \dim U$$

*Proof.* We use the map quotient map  $\pi$  introduced before. We know that  $a + U = 0 + U \iff a \in U$  this null  $\pi = U$  and  $\text{range } \pi = V/U$  as the map is surjective. Thus, using the fundamental theorem of linear map we get our desired result.  $\square$

**Definition 1.29.** Suppose  $T \in \mathcal{L}(V, W)$ . Define  $\tilde{T} : V/(\text{null } T) \rightarrow W$  by

$$\tilde{T}(v + \text{null } T) = Tv$$

This map indeed is well-defined as  $u + \text{null } T = v + \text{null } T \implies v - u \in \text{null } T$  and thus  $T(v - u) = 0 \implies Tv = Tu$ . Also, the map is a linear map.

**Proposition 1.33.** Suppose  $T \in \mathcal{L}(V, W)$  then

- (a)  $\tilde{T} \circ \pi = T$  where  $\pi$  is the quotient map with  $U = \text{null } T$
- (b)  $\tilde{T}$  is injective
- (c)  $\text{range } T = \text{range } \tilde{T}$
- (d)  $V/(\text{null } T)$  and  $\text{range } T$  are isomorphic vector spaces.

*Proof.* We prove each point individually

- (a) If  $v \in V$  then  $\tilde{T} \circ \pi(v) = \tilde{T}(v + \text{null } T) = Tv$  as desired.
- (b) If  $\tilde{T}(v + \text{null } T) = 0$  then  $Tv = 0$  thus  $v \in \text{null } T$ . Thus,  $v + \text{null } T = 0 + \text{null } T \implies \text{null } \tilde{T} = \{0 + \text{null } T\}$ .
- (c) By definition of  $\tilde{T}$ .
- (d) From (b) and (c).

□

### 1.5.3 Exercise

**Problem :** Suppose  $T$  is a function from  $V$  to  $W$ . The graph of  $T$  is the subset of  $V \times W$  defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W \mid v \in V\}$$

Prove that  $T$  is a linear map if and only if graph of  $T$  is a subspace of  $V \times W$ .

*Solution :* For  $(\implies)$ , you just need to check the vector axioms and just the definition of a linear map. For the  $(\impliedby)$ , consider

$$(v, Tv) + (w, Tw) = (v + w, Tv + Tw)$$

Since, graph of  $T$  is a subspace and all of its element is of the form  $(v, Tv)$  for any  $v$ , it must be that  $(v + w, T(v + w)) = (v + w, Tv + Tw)$ . Similarly,  $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$ . And we're done.

**Problem :** Suppose  $V_1, \dots, V_m$  are vector spaces such that  $V_1 \times \dots \times V_m$  is finite-dimensional. Prove that  $V_k$  is finite-dimensional for each  $k = 1, \dots, m$ .

*Solution :* Let  $\dim(V_1 \times \dots \times V_m) = k$  and suppose  $e_1, \dots, e_k$  are the basis of  $V_1 \times \dots \times V_m$ . Let  $e_i = (e_{1i}, e_{2i}, \dots, e_{mi})$  for  $1 \leq i \leq k$ . Notice that to cover elements of  $V_j$ , only the  $j$ -th component of  $e_i$ 's play a role. Thus,

$$\text{span}\{e_{j1}, e_{j2}, \dots, e_{jm}\} = V_k$$

Since, every span can be reduced to a basis we get that each  $V_k$  are finite-dimensional.

**Problem :** Suppose  $V_1, V_2, \dots, V_m$  are vector spaces. Prove that  $\mathcal{L}(V_1 \times \dots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$  are isomorphic vector spaces.

*Solution :*

**Solution.** Let  $U = V_1 \times \dots \times V_m$ . We want to prove that

$$\mathcal{L}(U, W) \cong \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W).$$

Define a map

$$\Phi : \mathcal{L}(U, W) \rightarrow \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$$

by

$$T \mapsto (T \circ i_1, T \circ i_2, \dots, T \circ i_m),$$

where for each  $j = 1, \dots, m$ , the map  $i_j : V_j \rightarrow U$  is defined by

$$i_j(v) = (0, \dots, 0, v, 0, \dots, 0),$$

with  $v$  in the  $j$ -th position.

Now, for  $T, T' \in \mathcal{L}(U, W)$  and scalars  $\alpha, \beta \in \mathbf{F}$ ,

$$\Phi(\alpha T + \beta T') = ((\alpha T + \beta T') \circ i_1, \dots, (\alpha T + \beta T') \circ i_m) = \alpha \Phi(T) + \beta \Phi(T').$$

Hence,  $\Phi$  is linear.

For invertibility of the map, define

$$\Psi : U \rightarrow W$$

by

$$\Psi(S_1, \dots, S_m)(v_1, \dots, v_m) = S_1(v_1) + \dots + S_m(v_m).$$

where  $S_j \in \mathcal{L}(V_j, W)$ . Now, one can check that  $\Psi \circ \Phi(T) = T$  and  $\Phi \circ \Psi(S_1, \dots, S_m) = (S_1, \dots, S_m)$ . Thus, the map is invertible.

**Problem :** For  $m$  a positive integer, define  $V^m$  by

$$V^m = \underbrace{V \times \dots \times V}_m$$

Prove that  $V^m$  and  $\mathcal{L}(\mathbf{F}^m, V)$  are isomorphic vector spaces.

*Solution :* Define a map from  $\Phi : V^m \rightarrow \mathcal{L}(\mathbf{F}^m, V)$  by

$$(v_1, \dots, v_m) \mapsto (a_1, \dots, a_m) \mapsto a_1 v_1 + \dots + a_m v_m$$

One can check that this is linear as well as invertible.

**Problem :** Suppose  $x, v$  are vectors in  $V$  and  $U, W$  are subspaces of  $V$  such that  $v + U = x + W$ . Prove that  $U = W$ .

*Solution :* Suppose  $u$  is an arbitrary vector in  $U$ , then  $v + u = x + w$  for some  $w \in W$ . Thus, setting  $u = 0$  gives  $x - v = -w \in W$ . Thus, for all  $u \in U$ ,  $u = x - v + w$  for some  $w \in W$  since  $W$  is a subspace  $x - v + w \in W$ . Thus,  $U \subseteq W$  and repeating a similar argument for  $U$  gives  $W \subseteq U$ .

**Problem :** Let  $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$ . Suppose  $A \subseteq \mathbf{R}^3$ . Prove that  $A$  is a translate of  $U$  if and only if there exists  $k \in \mathbf{R}$  such that

$$A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = k\}$$

*Solution :* Suppose  $A$  is a translate of  $U$ , then  $(a, b, c) + U = A$  for some  $(a, b, c) \in \mathbf{R}^3$ . Thus

$$A = \{(x + a, y + b, z + c) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$$

Now let  $x + a = x_1, y + b = y_1, z + c = z_1$  thus,

$$A = \{(x_1, y_1, z_1) \in \mathbf{R}^3 : 2x_1 + 3y_1 + 5z_1 = 2a + 3b + 5c\}$$

$$\implies A = \{(x_1, y_1, z_1) \in \mathbf{R}^3 : 2x_1 + 3y_1 + 5z_1 = k\}$$

Now, suppose  $A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = k\}$  then  $(\frac{k}{6}, \frac{k}{9}, \frac{k}{15}) + U = A$ .

---

## 1.6 Duality

### 1.6.1 Dual Space and Dual Map

**Definition 1.30.** A *linear functional* on  $V$  is a linear map from  $V$  to  $\mathbf{F}$ . In other words, a linear functional is an element of  $\mathcal{L}(V, \mathbf{F})$ .

**Definition 1.31.** The *dual space* of  $V$ , denoted  $V'$ , is the vector space of all linear functional on  $V$ . In other words,  $V' = \mathcal{L}(V, \mathbf{F})$ .

**Proposition 1.34.** Suppose  $V$  is a finite-dimensional vector space. Then  $V'$  is also finite-dimensional and

$$\dim V' = \dim V$$

*Proof.* From **Proposition 1.18.** we have

$$\dim V' = \dim \mathcal{L}(V, \mathbf{F}) = (\dim V)(\dim \mathbf{F}) = \dim V$$

as desired.  $\square$

**Definition 1.32.** If  $v_1, \dots, v_n$  is a basis of  $V$ , then the *dual space* of  $v_1, \dots, v_n$  is the list  $\varphi_1, \dots, \varphi_n$  of elements in  $V'$  such that

$$\varphi_j(k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

**Proposition 1.35.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $\varphi_1, \dots, \varphi_n$  is the dual basis. Then

$$v = \varphi_1(v)v_1 + \dots + \varphi_n(v)v_n$$

for each  $v \in V$ .

*Proof.* Let

$$v = c_1v_1 + \dots + c_nv_n$$

Then,  $\varphi_n(v) = c_n$  thus

$$v = \varphi_1(v)v_1 + \dots + \varphi_n(v)v_n$$

as desired.  $\square$

**Proposition 1.36.** Suppose  $V$  is a finite-dimensional. Then the dual basis of a basis of  $V$  is a basis of  $V'$ .

*Proof.* Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and let  $\varphi_1, \dots, \varphi_n$  be the dual basis. To show  $\varphi_1, \dots, \varphi_n$  is linearly independent, suppose there exists  $a_1, \dots, a_n \in \mathbf{F}$  such that

$$a_1\varphi_1 + \dots + a_n\varphi_n = 0$$

Then,  $(a_1\varphi_1 + \dots + a_n\varphi_n)(v_k) = a_k$  for  $1 \leq k \leq n$ . Thus,  $a_1 = a_2 = \dots = a_n = 0$ . And since the list is of the length  $\dim V'$ , we can conclude that the list is the basis of  $V'$ .  $\square$

**Definition 1.33.** Suppose  $T \in \mathcal{L}(V, W)$ . The *dual map* of  $T$  is the linear map  $T' \in \mathcal{L}(W', V')$  defined for each  $\varphi \in W'$  by

$$T'(\varphi) = \varphi \circ T$$

**Remark.** Since  $T'$  is a composition of linear maps  $\varphi$  and  $T$ , it is a linear map as well. Also,  $T'(\varphi) \in V'$  as  $T'$  as it takes an element from  $V$  to  $\mathbf{F}$ . Also, one can verify  $T' \in \mathcal{L}(W', V')$ .

**Proposition 1.37.** Suppose  $T \in \mathcal{L}(V, W)$ . Then

- (a)  $(S + T)' = S' + T'$  for all  $S \in \mathcal{L}(V, W)$ ,
- (b)  $(\lambda T)' = \lambda T'$  for all  $\lambda \in \mathbf{F}$ ,
- (c)  $(ST)' = T'S'$  for all  $S \in \mathcal{L}(W, U)$ .

*Proof.* The proofs of (a) and (b) directly follow from the definitions. For (c),

$$(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = T'S'$$

The fourth equation is due to  $\varphi \circ S \in W'$ . □

### 1.6.2 Null Space and Range of Linear Map

**Definition 1.34.** For  $U \subseteq V$ , the annihilator of  $U$ , denoted by  $U^0$ , is defined by

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \text{ for all } u \in U\}$$

**Proposition 1.38.** Suppose  $U \subseteq V$ . Then  $U^0$  is a subspace of  $V'$ .

*Proof.* One can just check the vector axioms. □

**Proposition 1.39.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then

$$\dim U^0 = \dim V - \dim U$$

*Proof.* Let  $i \in \mathcal{L}(U, V)$  be the linear map such that  $i(u) = u$  for each  $u \in U$ . Thus,  $i' \in \mathcal{L}(V', U')$  and from fundamental theorem of linear maps we have,

$$\dim \text{range } i' + \dim \text{null } i' = \dim V' = \dim V$$

Also,  $\text{null } i' = \{\varphi \in V' \mid i'(\varphi) = 0\} = \{\varphi \in V' \mid \varphi \circ i = 0\} = \{\varphi \in V' \mid \varphi(x) = 0\} = U^0$ . Thus,  $\dim \text{null } i' = \dim U^0$  and the equation above becomes

$$\dim \text{range } i' + \dim U^0 = \dim V$$

If  $\varphi \in U'$ , then  $\varphi$  can be extended to a linear functional  $\phi$  on  $V$  (**Exercise 10** of the first section). Thus,  $i'(\phi) = \varphi$  and  $\text{range } i' = U'$ . Hence

$$\dim \text{range } i' = \dim U' = \dim U$$

And thus  $\dim U + \dim U^0 = \dim V$  as desired. □

**Proposition 1.40.** Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then

- (a)  $U^0 = \{0\} \iff U = V$ ,
- (b)  $U^0 = V' \iff U = \{0\}$



*Proof.* For (a) we have,

$$\begin{aligned} U^0 = \{0\} &\iff \dim U^0 = 0 \\ &\iff \dim U = \dim V \\ &\iff U = V \end{aligned}$$

Similarly, to prove (b) we have

$$\begin{aligned} U^0 = V' &\iff \dim U^0 = \dim V' \\ &\iff \dim U^0 = \dim V \\ &\iff \dim U = 0 \\ &\iff U = \{0\} \end{aligned}$$

And we're done. □

**Proposition 1.41.** Let  $V$  and  $W$  be vector spaces and let  $T \in \mathcal{L}(V, W)$ . Then

$$\text{null } T' = (\text{range } T)^0$$

*Proof.* First suppose  $\varphi \in \text{null } T'$ , then  $T'(\varphi) = (\varphi \circ T)(x) = 0$  for every  $x \in V$ . Since,  $\varphi(Tx) = 0$  we have  $\varphi \in (\text{range } T)^0$  and thus  $\text{null } T' \subseteq (\text{range } T)^0$ .

Now, suppose  $\varphi \in (\text{range } T)^0$  then  $\varphi(Tv) = 0$  for all  $v \in V$  which implies  $T'(\varphi) = 0$ . Thus,  $\varphi \in \text{null } T'$  and  $(\text{range } T)^0 \subseteq \text{null } T'$  as desired. □

**Proposition 1.42.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$  then

$$\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$$

*Proof.* We have

$$\begin{aligned} \dim \text{null } T' &= \dim(\text{range } T)^0 \\ &= \dim W - \dim \text{range } T \\ &= \dim W - (\dim V - \dim \text{null } T) \\ &= \dim \text{null } T + \dim W - \dim V \end{aligned}$$

□

**Proposition 1.43.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then

$$T \text{ is surjective} \iff T' \text{ is injective}$$

*Proof.* To prove this, we have

$$\begin{aligned} T \text{ is surjective} &\iff \text{range } T = W \\ &\iff (\text{range } T)^0 = \{0\} \\ &\iff \text{null } T' = \{0\} \\ &\iff T' \text{ is injective} \end{aligned}$$

as desired. □

**Proposition 1.44.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then

- (a)  $\dim \text{range } T' = \dim \text{range } T$ ,
- (b)  $\text{range } T' = (\text{null } T)^0$

*Proof.* For (a) we have,

$$\begin{aligned}
 \dim \text{range } T' &= \dim W' - \dim \text{null } T' \\
 &= \dim W - (\dim \text{null } T + \dim W - \dim V) \\
 &= \dim V - \dim \text{null } T \\
 &= \dim \text{range } T
 \end{aligned}$$

For (b), suppose  $\varphi \in \text{range } T'$  then there exists a  $\phi \in W'$  such that  $T'(\phi) = \varphi$ . Thus, for all  $v \in \text{null } T$  we have

$$\varphi(v) = T'(\phi)v = (\phi \circ T)(v) = \phi(0) = 0$$

Thus,  $\varphi \in (\text{null } T)^0$ . Thus,  $\text{range } T' \subseteq (\text{null } T)^0$ . Now, we'll complete the proof by showing  $\dim \text{range } T' = \dim(\text{null } T)^0$ . Note, that

$$\begin{aligned}
 \dim \text{range } T' &= \dim \text{range } T \\
 &= \dim V - \dim \text{null } T \\
 &= \dim(\text{null } T)^0
 \end{aligned}$$

where the last equation is from **Proposition 1.39**. □

**Proposition 1.45.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then,

$$T \text{ is injective} \iff T' \text{ is surjective}$$

*Proof.* We have

$$\begin{aligned}
 T \text{ is injective} &\iff \text{null } T = \{0\} \\
 &\iff (\text{null } T)^0 = V' \\
 &\iff \text{range } T' = V'
 \end{aligned}$$

as desired. □

### 1.6.3 Matrix of Dual Linear Map

**Proposition 1.46.** Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then

$$\mathcal{M}(T', (\psi_1, \dots, \psi_m), (\varphi_1, \dots, \varphi_n)) = (\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m)))^t$$

where  $(\psi_1, \dots, \psi_m)$  and  $(\varphi_1, \dots, \varphi_n)$  are the dual basis of  $W'$  and  $V'$  respectively.

*Proof.* Let  $A = \mathcal{M}(T)$  and  $C = \mathcal{M}(T')$ . Suppose  $1 \leq j \leq m$  and  $1 \leq k \leq n$ . From the definition of  $\mathcal{M}(T')$  we have

$$T'(\psi_j) = \sum_{r=1}^n C_{r,j} \varphi_r.$$

The left side of the equation above equals  $\psi_j \circ T$ . Thus applying both sides of the equation above to  $v_k$  gives

$$(\psi_j \circ T)(v_k) = \sum_{r=1}^n C_{r,j} \varphi_r(v_k) = C_{k,j}.$$

We also have

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k) = \psi_j \left( \sum_{r=1}^m A_{r,k} w_r \right) = \sum_{r=1}^m A_{r,k} \psi_j(w_r) = A_{j,k}.$$

Comparing the last line of the last two sets of equations, we have  $C_{k,j} = A_{j,k}$ . Thus  $C = A^t$ . In other words,  $\mathcal{M}(T') = \mathcal{M}(T)^t$ , as desired.  $\square$

## 2 Polynomials

### 2.1 Zeros of Polynomials

**Definition 2.1.** A number  $\lambda \in \mathbf{F}$  is called a *zero*(or *root*) of a polynomial  $p \in \mathcal{P}(\mathbf{F})$  if

$$p(\lambda) = 0$$

**Proposition 2.1.** Suppose  $m$  is a positive integer and  $p \in \mathcal{P}(\mathbf{F})$  is a polynomial of degree  $m$ . Suppose  $\lambda \in \mathbf{F}$ . Then  $p(\lambda) = 0$  if and only if there exists a polynomial  $q \in \mathcal{P}(\mathbf{F})$  of degree  $m - 1$  such that

$$p(z) = (z - \lambda)q(z)$$

for every  $z \in \mathbf{F}$ .

*Proof.* Not so hard. □

**Proposition 2.2.** Suppose  $m$  is a positive integer and  $p \in \mathcal{P}(\mathbf{F})$  is a polynomial of degree  $m$ . Then  $p$  has at most  $m$  roots in  $\mathbf{F}$ .

*Proof.* We'll use induction. For  $m = 1$ , it is quite straight forward, as the polynomial  $a_0 + a_1z$  only has one zero which is  $-a_0/a_1$ . Now, suppose the assumption holds for all polynomial with degree  $m - 1$ . Let  $p$  be a polynomial of degree  $m$ , then if  $p$  has no zeros then we're done. Suppose  $\lambda \in \mathbf{F}$  such that  $p(\lambda) = 0$ , then using our previous proposition we get

$$p(z) = (z - \lambda)q(z)$$

where  $q(z)$  has degree  $m - 1$ . This shows that zeros of  $p$  are exactly the zeros of  $q(z)$  and  $\lambda$ , which is at most  $m$ . □

**Remark.** The result above implies that, coefficients of a polynomial are uniquely determined

### 2.2 Division Algorithm for Polynomials

**Proposition 2.3.** Suppose that  $p, s \in \mathcal{P}(\mathbf{F})$  with  $s \neq 0$ . Then there exists a unique polynomials  $q, r \in \mathcal{P}(\mathbf{F})$  such that

$$p = sq + r$$

and  $\deg r < \deg s$ .

*Proof.* Suppose  $\deg p = n$  and  $\deg s = m$ . If  $n < m$  then,  $q = 0$  and  $r = p$ . Thus assume that  $n \geq m$ . Then, take the list

$$1, z, z^2, \dots, z^{m-1}, s, sz, \dots, sz^{n-m}$$

this list is linearly independent in  $\mathcal{P}_n(\mathbf{F})$  as every element has a different degree. Also, the length of the list is  $n + 1$ , thus this list is a basis of  $\mathcal{P}_n(\mathbf{F})$ . But since  $p \in \mathcal{P}_n(\mathbf{F})$ , we can write  $p$  as

$$\begin{aligned} p &= a_0 + a_1z + \dots + a_{m-1}z^{m-1} + b_0s + b_1zs + \dots + b_{n-m}sz^{n-m} \\ &= \underbrace{a_0 + a_1z + \dots + a_{m-1}z^{m-1}}_r + s \underbrace{(b_0 + b_1z + \dots + b_{n-m}z^{n-m})}_q \end{aligned}$$

as desired. □

### 3 Eigenvalues & Eigenvectors

#### 3.1 Invariant Subspaces

##### 3.1.1 Eigenvalues

**Definition 3.1.** A linear map from vector space to itself is called an *operator*.

**Definition 3.2.** Suppose  $T \in \mathcal{L}(V)$ . A subspace  $U$  of  $V$  is called a *invariant* under  $T$  if  $Tu \in U$  for every  $u \in U$ .

From our definition,  $U$  is invariant under  $T$  if  $T|_U$  is an operator on  $U$ .

**Definition 3.3.** Suppose  $T \in \mathcal{L}(V)$ . A number  $\lambda \in \mathbf{F}$  is called a *eigenvalue* of  $T$  if there exists a  $v \in V$  such that  $v \neq 0$  and  $Tv = \lambda v$ .

**Remark.** Thus,  $V$  has one-dimensional subspace invariant under  $T$  if and only if  $T$  has an eigenvalue. If  $U$  is an one-dimensional subspace then  $Tv = \lambda v$  for some  $\lambda \in \mathbf{F}$ . Conversely, if  $Tv = \lambda v$  for some  $\lambda \in \mathbf{F}$  then  $\text{span } v$  is one-dimensional subspace  $V$  invariant under  $T$ .

**Proposition 3.1.** Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V)$ , and  $\lambda \in F$ . Then the following are equivalent.

1.  $\lambda$  is an eigenvalue of  $T$ .
2.  $T - \lambda I$  is not injective.
3.  $T - \lambda I$  is not surjective.
4.  $T - \lambda I$  is not invertible.

where  $I$  is the identity operator on  $V$ .

*Proof.* Condition 1. and 2. are equivalent because of  $Tv = \lambda v \iff (T - \lambda I)v = 0$  and if it was injective then  $v = 0$ . And 2., 3. and 4. are equivalent from **Proposition 1.14**.  $\square$

**Definition 3.4.** Suppose  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbf{F}$  is an eigenvalue of  $T$ . A vector  $v \in V$  is called an *eigenvector* corresponding to  $\lambda$  if  $v \neq 0$  and  $Tv = \lambda v$ .

**Proposition 3.2.** Suppose  $T \in \mathcal{L}(V)$ . Then every list of eigenvectors of  $T$  corresponding to distinct eigenvalues of  $T$  is linearly independent.

*Proof.* For the sake of contradiction, suppose the result is false. Then there exists a smallest positive integer  $m > 1$  such that there exists a list  $v_1, \dots, v_m$  of linearly dependent eigenvectors  $\lambda_1, \dots, \lambda_m$  of  $T$ , corresponding to distinct eigenvalues. Due to the minimality of  $m$ , there exists  $a_1, \dots, a_m \in \mathbf{F}$  such that

$$a_1 v_1 + \dots + a_m v_m = 0$$

Applying  $T - \lambda I$  we get

$$a_1(\lambda_1 - \lambda_m)v_1 + \dots + a_m(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$$

Since, all the eigenvalues are different, none of the coefficient above is 0. Thus, we get a new list of linearly dependent vector with length  $m - 1$ , which contradicts the minimality of  $m$ .  $\square$

**Proposition 3.3.** Suppose  $V$  is a finite-dimensional. Then each operator of  $V$  has at most  $\dim V$  distinct eigenvalue.

*Proof.* Since, every list of eigenvectors of  $T$  corresponding to distinct eigenvalues is linearly independent by above proposition, we get the bound immediately.  $\square$

### 3.1.2 Polynomials Applied to Operatos

**Definition 3.5.** Suppose  $T \in \mathcal{L}(V)$  and  $m$  is a positive integer

- $T^m \in \mathcal{L}(V)$  is defined by  $T^m = \underbrace{T \cdots T}_m$
- $T^0$  is defined to be the identity operator  $I$  on  $V$ .
- If  $T$  is invertible with inverses  $T^{-1}$ , then  $T^{-m} \in \mathcal{L}(V)$  is defined by

$$T^{-m} = (T^{-1})^m$$

**Definition 3.6.** Suppose  $T \in \mathcal{L}(V)$  and  $p \in \mathcal{P}(\mathbf{F})$  is a polynomial given by

$$p(z) = a_0 + a_1z + \cdots + a_mz^m$$

for all  $z \in \mathbf{F}$ . Then  $p(T)$  is the operator on  $V$  defined by

$$p(T) = a_0I + a_1T + a_2T^2 + \cdots + a_mT^m$$

**Proposition 3.4.** Let  $T \in \mathcal{L}(V)$ , then the function  $f : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{L}(V)$  given by  $p \mapsto p(T)$  is linear.

*Proof.* Left for future me as a exercise.  $\square$

**Proposition 3.5.** Suppose  $p, q \in \mathcal{P}(\mathbf{F})$  and  $T \in \mathcal{L}(V)$  Then

1.  $(pq)(T) = p(T)q(T)$
2.  $p(T)q(T) = q(T)p(T)$

*Proof.* Just define the polynomials and plug  $T$ .  $\square$

**Definition 3.7.** Let  $p, q \in \mathcal{P}(\mathbf{F})$ , then  $pq \in \mathcal{P}(\mathbf{F})$  is defined by

$$(pq)(z) = p(z)q(z)$$

**Proposition 3.6.** Suppose  $T \in \mathcal{L}(V)$  and  $p \in \mathcal{P}(\mathbf{F})$ . Then  $\text{null } p(T)$  and  $\text{range } p(T)$  are invariant under  $T$ .

*Proof.* Suppose  $u \in \text{null } p(T)$  then  $p(T)u = 0$

$$p(T)(Tu) = T(p(T)u) = T(0) = 0$$

Hence,  $Tu \in \text{null } p(T)$ . Thus,  $\text{null } p(T)$  is invariant under  $T$

Suppose,  $u \in \text{range } p(T)$ . Then,

$$\begin{aligned} p(T)v &= u \\ \implies T(p(T)v) &= Tu \\ \implies p(T)(Tv) &= Tu \end{aligned}$$

Hence,  $Tu \in \text{range } p(T)$ . Thus,  $\text{range } p(T)$  is invariant under  $T$ , as desired.  $\square$

### 3.1.3 Exercise

**Problem :** Suppose  $T \in \mathcal{L}(V)$  and  $U$  is a subspace of  $V$ .

1. Prove that  $U \subseteq \text{null } T$ , then  $U$  is invariant under  $T$ .
2. Prove that  $\text{range } T \subseteq U$ , then  $U$  is invariant under  $T$ .

*Solution :* For 1. just notice that  $Tu = 0 \in U$  for all  $u \in U$ . Thus, it is invariant under  $T$ . For 2. notice that  $Tu = k \in \text{range } T \subseteq U$ , thus  $k \in U$ . Therefore it is invariant under  $T$ .

**Problem :** Suppose that  $T \in \mathcal{L}(V)$  and  $V_1, \dots, V_m$  are subspaces of  $V$  invariant under  $T$ . Prove that  $V_1 + \dots + V_m$  is also invariant under  $T$ .

*Solution :* Let  $u \in V_1 + \dots + V_m$  then  $Tu = Tv_1 + \dots + Tv_m$ . Since  $Tv_i \in V_i$  we have that

$$Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$$

Thus  $V_1 + \dots + V_m$  is invariant under  $T$ .

**Problem :** Suppose  $T \in \mathcal{L}(V)$ . Prove that the intersection of every collection of subspaces of  $V$  invariant under  $T$  is invariant under  $T$ .

*Solution :* Let  $\bigcap_{\alpha} V_{\alpha} = K$  where  $V_{\alpha}$  is a subspace of  $V$  invariant under  $T$ . If  $v \in K$  then  $Tv \in V_{\alpha}$  for every  $\alpha$ , hence  $Tv \in K$ .

**Problem :** If  $V$  is a finite-dimensional and  $U$  is a subspace of  $V$  that is invariant under every operator on  $V$ , then  $U = \{0\}$  or  $U = V$ .

*Solution :* Suppose  $U \neq \{0\}$  and  $U \neq V$ . Let  $u_1, \dots, u_n$  be a basis of  $U$  and extend it to basis of  $V$ ,  $u_1, \dots, u_n, v_1, \dots, v_{m-n}$ . Define

$$T(a_1u_1 + \dots + a_nu_n + \dots + a_mv_{m-n}) = a_1v_1$$

It is easy to see that  $T \in \mathcal{L}(V)$  but  $U$  is not invariant under  $T$  as  $T(u_1) = v_1$ .

**Problem :** Suppose  $T \in \mathcal{L}(\mathbf{R}^2)$  is defined by  $T(x, y) = (-3y, x)$ . Find the eigenvalues of  $T$ .

*Solution :* If  $\lambda$  is a eigenvalues then  $(-3y, x) = T(x, y) = \lambda(x, y) = (\lambda x, \lambda y)$  then  $-3y = \lambda^2 y$ . Notice that  $y \neq 0$  because if  $y = 0$  then  $x = 0$  but  $(x, y) \neq (0, 0)$ . Thus,  $\lambda = \pm\sqrt{3}i$ .

**Problem :** Suppose  $P \in \mathcal{L}(V)$  is such that  $P^2 = P$ . Prove that if  $\lambda$  is an eigenvalue of  $P$ , then  $\lambda = 0$  or  $1$ .

*Solution :* If  $\lambda$  is a eigenvalue then  $\exists v \neq 0$  such that  $P(v) = \lambda v$ . Therefore,

$$\begin{aligned} \lambda v &= P(P(v)) = P(\lambda v) = \lambda \cdot \lambda v \\ \implies \lambda^2 v &= \lambda v \implies v(\lambda^2 - \lambda) = 0 \\ \implies \lambda^2 &= \lambda \implies \lambda = 1, 0 \end{aligned}$$

**Problem :** Suppose  $T \in \mathcal{L}(V)$ . Suppose  $S \in \mathcal{L}(V)$  is invertible.

1. Prove that  $T$  and  $S^{-1}TS$  have the same eigenvalues.
2. What is the relationship between eigenvectors of  $T$  and eigenvectors of  $S^{-1}TS$ ?

*Solution :* If  $\lambda$  is a eigenvalue of  $T$  then  $T(v) = \lambda v$ . Since  $S$  is invertible  $\exists w \in V$  such that  $S(w) = v$  thus  $(S^{-1}(T(S(w)))) = S^{-1}(\lambda v) = \lambda w$ . If  $\lambda$  is a eigenvalue of  $S^{-1}TS$  then  $S^{-1}(T(Sw)) = \lambda w$  thus  $T(S(w)) = S(\lambda w) = \lambda S(w)$ . The relationship between the eigenvectors is that, eigenvectors of  $S^{-1}TS$  maps to eigenvectors of  $T$ .

**Problem :** Give an example of a operator in  $\mathbf{R}^4$  that has no eigenvalues.

*Solution :*  $T(w, x, y, z) = (-x, w, -z, y)$

**Problem :** Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbf{F}$ . Show that  $\lambda$  is an eigenvalue of  $T$  if and only if  $\lambda$  is an eigenvalue of the dual operator  $T' \in \mathcal{L}(V')$ .

*Solution :* From exercise 17 of 3F we know that  $T$  is invertible if and only if  $T'$  is invertible. Thus,  $T - \lambda I$  is not invertible if and only if  $(T - \lambda I)' = T' - \lambda I$  is not invertible. Thus we are done.

**Problem :** Suppose  $\mathbf{F} = \mathbf{R}$ ,  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbf{C}$ . Prove that  $\lambda$  is an eigenvalue of the complexification  $T_{\mathbf{C}}$  if and only if  $\bar{\lambda}$  is an eigenvalue of  $T_{\mathbf{C}}$ .

*Solution :* If  $\lambda$  is an eigenvalue then  $T(v + iw) = \lambda(v + iw)$  thus we have

$$\overline{T(v + iw)} = T(\overline{v + iw}) = \bar{\lambda}(\overline{v + iw})$$

Since  $v + iw \neq 0$  so their conjugate isn't zero as well.

**Problem :** Suppose  $T \in \mathcal{L}(V)$  is invertible.

1. Suppose  $\lambda \in F$  with  $\lambda \neq 0$ . Prove that  $\lambda$  is an eigenvalue of  $T$  if and only if  $\frac{1}{\lambda}$  is an eigenvalue of  $T^{-1}$ .
2. Prove that  $T$  and  $T^{-1}$  have same eigenvectors.

*Solution :* If  $\lambda$  is an eigenvalue of  $T$  then

$$T(v) = \lambda v \iff v = \lambda T^{-1}(v) \iff \frac{1}{\lambda}v = T^{-1}(v)$$

This proves both of the statement.



**Problem :** Suppose  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V)$ . Prove that  $ST$  and  $TS$  have the same eigenvalues.

*Solution :* If  $ST$  has eigenvalue  $\lambda$  then

$$ST(v) = \lambda v$$

$$\implies T(STv) = \lambda T(v)$$

If  $Tv \neq 0$  then we are done. If  $Tv = 0$  then

$$S(Tv) = S(0) = 0 = \lambda v \implies \lambda = 0$$

Thus,  $ST - \lambda I = ST$  is not invertible thus  $TS$  is also not invertible. Therefore 0 is also an eigenvalue of  $TS$ .

Left to do exercises. 23, 24, 25, 26, 27, 30, 31, 34, 35, 39

## 3.2 The Minimal Polynomial

### 3.2.1 Existence of Eigenvalue on Complex Vector Spaces

**Theorem 3.1.** *Every operator on a finite-dimensional non-zero complex vector space has an eigenvalue.*

*Proof.* Let  $V$  be a finite-dimensional non-zero complex vector space of dimension  $n > 0$  and  $T \in \mathcal{L}(V)$ . Choose  $v \in V$  such that  $v \neq 0$ . Then

$$v, Tv, T^2v, \dots, T^nv$$

This has length  $n + 1 > n$  thus it is linearly dependent list. Therefore there exists  $a_0, \dots, a_n \in \mathbf{C}$  not all zero, such that

$$a_0v + a_1Tv + \dots + a_nT^nv = 0$$

Let  $p(z) = a_0 + a_1z + \dots + a_nz^n$  be a polynomial with minimal degree such that  $p(T) = 0$ . Since  $a_i \in \mathbf{C}$  there exist complex roots  $\lambda$  and a polynomial  $q(z)$  such that

$$p(z) = (z - \lambda)q(z)$$

Apply  $T$  to this polynomial gives us,

$$p(T) = (T - \lambda)(q(T))$$

and applying  $v$  gives us,

$$(T - \lambda)(q(T)v) = 0 \implies T(q(T)v) = \lambda q(T)v$$

Thus  $\lambda$  is an eigenvalue as  $Tv \neq 0$  because  $q(T)v \neq 0$  due to the minimality of  $p$ .  $\square$

**Theorem 3.2.** *Suppose  $V$  is a finite-dimensional and  $T \in \mathcal{L}(V)$ . Then there is a unique monic polynomial  $p \in \mathcal{P}(\mathbf{F})$  of smallest degree such that  $p(T) = 0$ . Furthermore,  $\deg p \leq \dim V$ .*

*Proof.* We proceed by induction on  $\dim V$ . If  $\dim V = 0$ , then  $I$  is the zero operator, and we take  $p(z) = 1$ , which is monic and has  $\deg p = 0 \leq 0$ .

Suppose  $\dim V > 0$  and the result holds for all operators on spaces of smaller dimension. Pick a non-zero vector  $v \in V$ . The list  $v, Tv, \dots, T^{\dim V}v$  has length  $1 + \dim V$ , so it is linearly dependent. By linear dependence, lemma there exists a smallest positive integer  $m$  such that  $T^mv$  is a linear combination of  $v, Tv, \dots, T^{m-1}v$ . Thus, there exist scalars  $c_0, \dots, c_{m-1}$  such that:

$$c_0v + c_1Tv + \dots + c_{m-1}T^{m-1}v + T^mv = 0$$

Define the monic polynomial  $q(z) = z^m + c_{m-1}z^{m-1} + \dots + c_0$ . Then  $q(T)v = 0$ .

By the choice of  $m$ , the list  $v, Tv, \dots, T^{m-1}v$  is linearly independent. Since  $q(T)(T^kv) = T^k(q(T)v) = 0$ , all  $m$  vectors in this list are in null  $q(T)$ . Thus,  $\dim \text{null } q(T) \geq m$  (by Independent list  $>$  span list inequality). By the Fundamental theorem of linear maps,

$$\dim \text{range } q(T) = \dim V - \dim \text{null } q(T) \leq \dim V - m$$

Since  $\text{range } q(T)$  is invariant under  $T$ , we apply the induction hypothesis to  $T|_{\text{range } q(T)}$ . There exists a monic polynomial  $s$  with  $\deg s \leq \dim V - m$  such that  $s(T|_{\text{range } q(T)}) = 0$ . Let  $p = sq$ . Then  $p$  is monic and  $\deg p = \deg s + \deg q \leq (\dim V - m) + m = \dim V$ . For any  $v \in V$ ,  $q(T)v \in \text{range } q(T)$ , so  $s(T)(q(T)v) = 0$ . Thus  $p(T) = 0$ .  $\square$

Here is another way to prove the existence of minimal polynomial and its uniqueness. The proof below doesn't prove the inequality  $\dim V \geq \deg p$ .

*Proof.* let  $n = \dim V$  then  $\dim \mathcal{L}(V) = n^2$ . Choose the list

$$I, T, \dots, T^{n^2}$$

This list is linearly dependent because  $n^2 + 1 > n^2$ . Let  $m$  be the smallest positive number such that  $I, T, \dots, T^m$  is linearly dependent. By the linear dependence lemma we have,

$$T^m + a_{m-1}T^{m-1} + \dots + a_0I = 0$$

Construct a monic polynomial  $p(z) = z^m + a_{m-1}z^{m-1} + \dots + a_0$ . By the choice of  $m$  this is the smallest degree monic polynomial which satisfies  $p(T) = 0$ .

For the uniqueness, suppose  $p \neq q \in \mathcal{P}(\mathbf{F})$  are monic polynomial such that  $p(T) = 0$  and  $q(T) = 0$ . Let  $h(z) = p(z) - q(z)$ . Notice that  $\deg h < m$  because both are monic and have same degree and since  $p \neq q$  we have  $h \neq 0$ . Thus  $h(T) = p(T) - q(T) = 0$  which contradicts our previous statement.  $\square$

**Definition 3.8.** Suppose  $V$  is a finite-dimensional and  $T \in \mathcal{L}(V)$ . Then the *minimal polynomial* of  $T$  is the unique monic polynomial  $p \in \mathcal{P}(\mathbf{F})$  of smallest degree such that  $p(T) = 0$ .

**Proposition 3.7.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ .

1.  $\lambda$  is a root of the minimal polynomial  $\iff \lambda$  is an eigenvalues of  $T$ .
2. If  $V$  is a complex vector space, then minimal polynomial has the form

$$(z - \lambda_1) \cdots (z - \lambda_m)$$

where  $\lambda_1, \dots, \lambda_m$  is a list of all eigenvalues of  $T$ , possibly with repetition.

*Proof.* For (1.), Let  $p$  be a minimal polynomial and suppose  $\lambda$  is a root of the minimal polynomial. Then, we know by **FTA** we can write  $p$  as  $p(z) = (z - \lambda)q(z)$  for some  $q(z)$ . Applying this Polynomial to  $T$  we get

$$p(T)v = (T - \lambda I)(q(T)v)$$

for all  $v \in V$ . We know that  $q(T)v \neq 0$  for some  $v \in V$  because  $\deg q < \deg p$  and  $q(T)v = k$ , thus  $T(k) = \lambda k$ . Hence  $\lambda$  is an eigenvalue.

Now, let  $\lambda \in \mathbf{F}$  be a eigenvalue of  $T$ . Then,  $T^k(v) = \lambda^k v$ . Thus,

$$p(T)v = p(\lambda)v \implies p(\lambda)v = 0 \implies p(\lambda) = 0$$

Thus  $\lambda$  is a root of the polynomial.

For (2.), Use (1.) and **FTA**.  $\square$

**Proposition 3.8.** Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V)$  and  $q \in \mathcal{P}(\mathbf{F})$ . Then  $q(T) = 0 \iff q$  is a polynomial multiple of the minimal polynomial.

*Proof.* Let  $p$  be the minimal polynomial. Then,  $\exists s, r$  such that  $\deg r < \deg p$  and

$$q = ps + r$$

We have

$$0 = q(T) = p(T)s(T) + r(T) = r(T)$$

Thus  $r \equiv 0$  as if not then would violate  $p$  being the minimal polynomial. Thus  $q = ps$ .  $\square$

**Proposition 3.9.** Suppose  $V$  is a finite-dimensional,  $T \in \mathcal{L}(V)$ , and  $U$  is a subspace of  $V$  that is invariant under  $T$ . Then the minimal polynomial of  $T$  is a multiple of minimal polynomial of  $T|_U$ .

*Proof.* Let  $p$  be the minimal polynomial of  $T$ . Then,  $p(T)v = 0$  for every  $v \in V$ . Therefore,  $p(T)u = 0$  for every  $u \in U$ . Since  $T$  is invariant under  $U$  we have  $p(T|_U) = 0$ . Thus using previous results we get that  $p$  is a multiple of the minimal polynomial of  $T|_U$ .  $\square$

**Proposition 3.10.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Then  $T$  is not invertible if and only if constant term of minimal polynomial  $p$  of  $T$  is 0.

*Proof.*

$$\begin{aligned} T \text{ is not invertible} &\iff 0 \text{ is an eigenvalue of } T \\ &\iff 0 \text{ is a root of } p \\ &\iff \text{constant term of } p \text{ is zero.} \end{aligned}$$

$\square$