Real Analysis

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Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll type out problems and their solutions here. I'm reading **Principle of Mathematical Analysis** by *Walter Rudin.*

Content

1 Basic Topology

> **Problem 1.1**
>
> A complex number z is said to be algebraic if there exists integers $a_0, \ldots, a_n$ not all zero such that
> $$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$
> Prove that all the algebraic numbers are countable.

Solution.
For a given positive integer N, we know that there are finitely many tuples $(n, a_0, \ldots, a_n)$ that satisfy,
$$n + |a_0| + |a_1| + \cdots + |a_n| = N$$

Let Q_N be the set of polynomials with degree n and coefficient $a_0, \ldots, a_n$ such that they satisfy the above condition, i.end

$$Q_N = \{P \in \mathbf{Z}[x] \mid \deg(P) = n \text{ and } n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

Then define

$$A_N = \{\#\text{all the algebraic solutions to the polynomials in } Q_N\}$$

Since Q_N is finite and each polynomial has at most n roots, A_N is also finite. Since A_N is finite we know its countable as well. Define

$$A = \bigcup_{i=1}^{\infty} A_i$$

Thus A is the set of all algebraic numbers. Since A is the union of countable sets, A is also countable.

> **Problem 1.2**
>
> Prove that there are real numbers which are not algebraic.

Solution. Immediate from above problem.

> **Problem 1.3**
>
> Is the set of irrational real numbers countable?

Solution. If set of irrational real numbers were countable then

$$\mathbf{R} = \mathbf{Q} \cup (\mathbf{R}/\mathbf{Q})$$

But we know that union of countable sets are countable, and since $\mathbf{Q}$ is countable we have $\mathbf{R}$ is countable which is a contradiction.

Problem 1.4

Construct a bounded set of real numbers with exactly three limit points.

Solution. Let A_0 , A_2 and A_4 be the sets such that

$$A_0 = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, \quad A_2 = \left\{ 2 + \frac{1}{n} : n \in \mathbb{N} \right\}, \quad A_4 = \left\{ 4 + \frac{1}{n} : n \in \mathbb{N} \right\}$$

One can check that the only limit points of A_0, A_2 and A_4 are 0, 2 and 4 respectively. Thus,

$$A = A_0 \cup A_2 \cup A_4$$

has limit points $0, 2$ and 4 and is clearly bounded within $(0, 5)$.