

Real Analysis

Basu Dev Karki

Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll type out problems and their solutions here. I'm reading **Principle of Mathematical Analysis** by *Walter Rudin*.

Content

| | |
|---|----------------|
| 1 | Basic Topology |
|---|----------------|

| |
|---|
| 3 |
|---|

1 Basic Topology

Problem 1.1

A complex number z is said to be algebraic if there exists integers a_0, \dots, a_n not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$$

Prove that all the algebraic numbers are countable.

Solution.

For a given positive integer N , we know that there are finitely many tuples (n, a_0, \dots, a_n) that satisfy,

$$n + |a_0| + |a_1| + \dots + |a_n| = N$$

Let Q_N be the set of polynomials with degree n and coefficient a_0, \dots, a_n such that they satisfy the above condition, i.e.

$$Q_N = \{P \in \mathbf{Z}[x] \mid \deg(P) = n \text{ and } n + |a_0| + |a_1| + \dots + |a_n| = N\}$$

Then define

$$A_N = \{\text{all the algebraic solutions to the polynomials in } Q_N\}$$

Since Q_N is finite and each polynomial has at most n roots, A_N is also finite. Since A_N is finite we know it's countable as well. Define

$$A = \bigcup_{i=1}^{\infty} A_i$$

Thus A is the set of all algebraic numbers. Since A is the union of countable sets, A is also countable.

Problem 1.2

Prove that there are real numbers which are not algebraic.

Solution. Immediate from above problem.

Problem 1.3

Is the set of irrational real numbers countable?

Solution. If set of irrational real numbers were countable then

$$\mathbf{R} = \mathbf{Q} \cup (\mathbf{R}/\mathbf{Q})$$

But we know that union of countable sets are countable, and since \mathbf{Q} is countable we have \mathbf{R} is countable which is a contradiction.

Problem 1.4

Construct a bounded set of real numbers with exactly three limit points.

Solution. Let A_0 , A_2 and A_4 be the sets such that

$$A_0 = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, \quad A_2 = \left\{ 2 + \frac{1}{n} : n \in \mathbb{N} \right\}, \quad A_4 = \left\{ 4 + \frac{1}{n} : n \in \mathbb{N} \right\}$$

One can check that the only limit points of A_0 , A_2 and A_4 are 0, 2 and 4 respectively. Thus,

$$A = A_0 \cup A_2 \cup A_4$$

has limit points 0, 2 and 4 and is clearly bounded within $(0, 5)$.