Real Analysis

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Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from MIT OCW and write the solutions to the problem set here.

1 Problem Set 1

Problem 1.1

Let $\mathbb{F}$ be a ordered field with $1 \neq 0$. Show that $1 > 0$.

Solution.
First let us prove that $(-1) \cdot (-1) = 1$. We know that for all $x \in \mathbb{F}$, there is an inverse element $-x$ such that,

$$x + (-x) = 0$$

Thus, $1 + (-1) = 0$ which means that

$$0 = (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1)$$

$$\implies 1 = (-1) \cdot (-1)$$

Since $\mathbb{F}$ is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that $1 \neq 0$ so the middle statement can't be true and if $1 < 0$ then $0 < (-1)$. But from **axiom of order and multiplication** $0 < (-1) \cdot (-1) = 1$. Thus a contradiction.

Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

Solution.
Suppose $\frac{n}{m} = \frac{n_1}{m_1}$ and $\frac{p}{q} = \frac{p_1}{q_1}$ then using the definition of when two rational numbers are equal, we get $nm_1 = n_1 m$ and $pq_1 = p_1 q$. Thus,

$$\begin{aligned} m_1 q_1 (nq + pm) &= m_1 q_1 nq + m_1 q_1 pm \\ &= n_1 m q_1 q + p_1 q m_1 m \\ &= mq(n_1 q_1 + m_1 p_1) \end{aligned}$$

Thus, $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$.

Problem 1.3

Find the $\sup E$ and $\inf E$ for the following set E.

1. $E = \left\{n \in \mathbb{Z} \mid n < \sqrt{12}\right\}$
2. $E = \left\{r \in \mathbb{Q} \mid r < \sqrt{12}\right\}$
3. $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
4. $E = \left\{\frac{n^2+n}{n+1} \mid n \in \mathbb{N}\right\}$

Solution.

1. $\sup E = 3$ but $\inf E$ doesn't exist.
2. $\sup E = \sqrt{12}$ but $\inf E$ doesn't exist.
3. $\sup E = \frac{1+\sqrt{5}}{2}$ and $\inf E = \frac{1-\sqrt{5}}{2}$.
4. $\inf E = 1$ but $\sup E$ doesn't exist.

Problem 1.4

Let $\mathbb{M}$ be the set of polynomials with integer coefficients i.e,

$$\mathbb{M} := \{f(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Z}\}$$

Define the relation $0 \prec f$ if $0 < f(x)$ for x large enough. More precisely, we say

$$0 \prec f \quad \text{if there exists } M > 0 \text{ such that } f(x) > 0 \text{ for all } x > M.$$

Then define

$$f \prec g \quad \text{if } 0 \prec (g - f).$$

Show that $(M, \prec)$ is an ordered set.

Solution.
We'll use the fact that for large enough x, $f(x) > 0$ for $a_n > 0$. Let $f, g \in \mathbb{M}$ such that $f \neq g$ and

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

Define $h(x) = g(x) - f(x) = (b_0 - a_0) + (b_1 - a_1)x + \cdots + (b_n - a_n)x^n$ and let $c_i = b_i - a_i$. Suppose c_k is the highest degree non zero coefficient, then if

1. $c_k > 0$ then $f \prec g$
2. $c_k < 0$ then $g \prec f$

Suppose $f \prec g$ and $g \prec h$. Then, $f(x) < g(x)$ and $g(x) < h(x)$ for large x and hence $f(x) < h(x)$ for large x. Thus, $0 \prec (h - f)$ which means $f \prec h$.

Problem 1.5

Prove that $(\mathbb{M}, \prec)$ **doesn't** satisfy the archimedean property.

Solution.
To show that it doesn't satisfy the archimedean property, we need to show that $\exists f, g \in \mathbb{M}$ such that

$$\forall n \in \mathbb{N}, g \not\prec nf$$

If we choose $g(x) = x^2$ and $f(x) = x$ and assume $g \prec nf$ then , $0 \prec x(n-x)$ which means that $0 < x(n-x)$ which we know is false for large x. Thus, $g \not\prec nf$.

Problem 1.6

Show that for any non-empty set $E \subset R$ which is bounded from below, E has the greatest lower bound.

Solution.
To show that E has greatest lower bound define

$$-E = \{-x \mid x \in E\}$$

If α is any lower bound of E then $x \geq \alpha \Rightarrow -x \leq -\alpha$. That means that $-E$ is bounded above by $-\alpha$. Since $-E$ is bounded above by $-\alpha$, it must have the least upper bound property. Let

$$\sup -E = \beta$$

Thus, $\beta \geq -x \Rightarrow x \geq -\beta$ and $\beta \leq -\alpha$ for any lower bound α of E. Thus, $\alpha \leq -\beta$. Hence, $-\beta = \inf E$.

Problem 1.7

Show that for any real number $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that $y^3 = x$.

Solution.
If $x = 0$ then $y = 0^3 = 0$. Let us define

$$A = \left\{a \mid a > 0 \text{ and } a^3 \leq x\right\}$$

for $x > 0$. Since A is non-empty and bounded, let $y = \sup A$. We will show that $y^3 = x$. Suppose $x > y^3$. Let $h = \min\{\frac{1}{2}, \frac{x-y^3}{3y^2+3y+1}\}$ then,

$$\begin{aligned}(y+h)^3 &= y^3 + h^3 + 3yh^2 + 3y^2h \\ &< y^3 + h(1 + 3y + 3y^2) \\ &\leq y^3 + \frac{x - y^3}{3y^2 + 3y + 1} \cdot (1 + 3y + 3y^2) \\ &\leq x\end{aligned}$$

Thus, $y + h \in A$ but since y is the least upper bound of A we have, $y + h \leq y \Rightarrow h \leq 0$. This is a contradiction.

Suppose $x < y^3$. Let $h = \frac{y^3-x}{3y^2}$ then

$$\begin{aligned}(y-h)^3 &= y^3 - h^3 - 3y^2h + 3yh^2 \\ &= y^3 - h^3 - 3y^2 \cdot \frac{(y^3-x)}{3y^2} + 3yh^2 \\ &= x - h^3 + 3yh^2\end{aligned}$$

Since $3yh^2 - h^3 > 0$ we have $(y-h)^3 > x$. Let $q \in A$ then

$$q^3 \leq x < (y-h)^3$$

Thus, $(y-h)^3 - q^3 > 0 \Rightarrow (y-h-q)\left((y-h)^2 + (y-h)q + q^3\right) > 0$. Since, $(y-h)^2 + (y-h)q + q^3 > 0$ we have $y - h > q$. Thus $(y-h)$ is an upper bound for A. Thus $y \leq y - h \Rightarrow h \leq 0$. Contradiction!!

The proof also works for $x < 0$ as $-x > 0$. Thus,

$$\exists k \in \mathbb{R} \text{ s.t } k^3 = -x$$

Let $z := -k$ then $z^3 = (-k)^3 = -k^3 = -(-x) = x$.

2 Problem Set 2

Problem 2.1

Let a_n ad b_n be a sequence of real numbers.

1. Assume that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists. Show that $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} a_n\right)$ exists.

2. Give an example in which $\lim_{n\to\infty}(a_n b_n)$ exists but neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty} b_n$ exits.

Solutions.

To show that $\lim_{n\to\infty} a_n b_n = ab$ we need to show $\exists N \in \mathbb{N}$ such that for all $\varepsilon > 0$ we have

$$|a_n b_n - ab| < \varepsilon$$

for $n \geq N$. Now,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

Since $\{a_n\}$ is convergent, we know that it is bounded above. Suppose $|a_n| < C$ then

$$|a_n||b_n - b| + |b||a_n - a| < |C||b_n - b| + |b||a_n - a|$$

Also let $N_1 \in \mathbb{N}$ such that $|b_n - b| < \dfrac{\varepsilon}{2(|C|+1)}$ for all $n \geq N_1$ and let $N_2 \in \mathbb{N}$ such that $|a_n - a| < \dfrac{\varepsilon}{2(|b|+1)}$ for all $n \geq N_2$. Thus for $n \geq \max\{N_1, N_2\}$ we have,

$$\begin{aligned} 2|a_n||b_n - b| + |b||a_n - a| &< |C||b_n - b| + |b||a_n - a| \\ &< |C| \cdot \frac{\varepsilon}{2(|C|+1)} + |b| \cdot \frac{\varepsilon}{2(|b|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

For the second part just take $a_n = (-1)^n$ and $b_n = (-1)^n$. We know that $\lim_{n\to\infty} a_n b_n = 1$ but $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ doesn't exist.

Problem 2.2

Find the limit of the following sequence if it exists or show that the limit doesn't exists.

1. $a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$

2. $a_n = \frac{\sin(n)}{n}$

3. $a_n = \displaystyle\sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6 + i}}$

Solution.
For the first part we know that $a_n = \frac{1}{n+1} - \frac{1}{n} - 1$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n} - 1 &= \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 1 \\ &= 0 - 0 - 1 \\ &= -1\end{aligned}$$

The first equation is due to the algebraic property of of limit.

For the second part, we need to show that

$$\left|\frac{\sin(n)}{n}\right| < \varepsilon$$

We know that $0 \leq |\sin(n)| \leq 1 \Rightarrow 0 \leq \frac{|\sin(n)|}{|n|} \leq \frac{1}{|n|}$. Thus, for any $N > \frac{1}{\varepsilon}$ we have

$$\left|\frac{\sin(n)}{n}\right| \leq \frac{1}{|n|} < \frac{1}{N} < \varepsilon$$

For the third part, we will use **squeeze theorem** to show that the the limit is 1. Notice that,

$$\frac{n^2}{\sqrt{n^6+n}} < \frac{n^2}{\sqrt{n^6+i}} < \frac{n^2}{\sqrt{n^6}}$$

for $i = 1, 2, \ldots, n$. Thus,

$$\begin{aligned} n \cdot \frac{n^2}{\sqrt{n^6+n}} &\leq \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq n \cdot \frac{n^2}{\sqrt{n^6}} \\ \Longrightarrow \frac{1}{\sqrt{1+\frac{1}{n^5}}} &\leq \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq 1 \\ \Longrightarrow \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^5}}} &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq \lim_{n\to\infty} 1 \\ \Longrightarrow 1 &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq 1 \end{aligned}$$

Thus, by **squeeze theorem** we have $\lim\limits_{n\to\infty} a_n = 1$.

Problem 2.3

Let a_n be a sequence of real numbers and L be a real number. Show that the following two statements are equivalent. One holds if and only if the other does.

1. There exists a subsequence a_{n_k} converging to L.
2. For any $\epsilon > 0$, there exist infinitely many a_n in $(L - \epsilon, L + \epsilon)$.

Solution.

Suppose (1.) holds then there exists a $N \in \mathbb{N}$ such that for all $k \geq N$ we have,

$$|a_{n_k} - L| < \varepsilon \implies L - \varepsilon < a_{n_k} < L + \varepsilon$$

Thus, there are infinitely many a_i in the interval $(L - \varepsilon, L + \varepsilon)$.

Suppose (2.) holds then for any $\varepsilon > 0$ there are infinitely many a_n such that

$$L - \varepsilon < a_n < L + \varepsilon$$

Since there are infinitely many terms in $(L - 1, L + 1)$ choose n_1 such that

$$|a_{n_1} - L| < 1$$

Since there are infinitely many terms in $(L - \frac{1}{2}, L + \frac{1}{2})$ choose n_2 such that

$$n_2 > n_1 \text{ and } |a_{n_2} - L| < \frac{1}{2}$$

Continuing inductively we can produce a sequence such that $\{a_{n_i}\}$

$$|a_{n_k} - L| < \frac{1}{k}$$

Hence, $a_{n_k} \to L$.

Problem 2.4

Where possible, find a subsequence that is monotone and a subsequence that is convergent for the following sequences.

1. $a_n = \sin(n\pi/8)$
2. $a_n = (-1)^n n$

Solution.

Notice that

$$\sin\left(\frac{n\pi}{8}\right) = \begin{cases} 0 & \text{if } n = 8k \\ 1 & \text{if } n = 4(4k+1) \\ -1 & \text{if } n = 4(4k-1) \end{cases}$$

So a monotone subsequence would be $a_{n_k} = 1$. And the same example works for convergent subsequence.

For the second part, notice that

$$a_n = \begin{cases} 2k & \text{if } n = 2k \\ -2k-1 & \text{if } n = 2k+1 \end{cases}$$

So a monotone subsequence would be $a_{n_k} = 2k$. But there is no convergent subsequence, because we know that every subsequence would have a infinitely many odd numbers or infinitely many even numbers but we know that those aren't bounded.

3 Problem Set 3

Problem 3.1

Prove that for reals $x < y$, there exists a $r \in \mathbb{Q}$ such that $x < r < y$.

Solution. Using the archimedean property of $\mathbb{R}$

$$\exists n \in \mathbb{N}, \text{ s.t } \frac{1}{n} < y - x$$

$$\implies n(y - x) > 1$$

Since the difference is greater than 1, there exists a integer m such that

$$nx < m < ny \implies x < \frac{m}{n} < y$$

Hence we are done.

Problem 3.2

Let E be an non-empty subset of $\mathbb{R}$ which is bounded. Define

$$F := \left\{x^2 \mid x \in E\right\}$$

Show that $\sup F$ exists and that $\sup F = \max\{(\sup E)^2, (\inf E)^2\}$.

Solution.
Since E is bounded F is also bounded. Then since $\inf E \le x \le \sup E$ we have

$$x^2 \le \max\{(\sup E)^2, (\inf E)^2\}$$

If $\max\{(\sup E)^2, (\inf E)^2\} = (\sup E)^2$ then $\sup E \ge 0$ and $(\sup E)^2$ is an upper bound for F. Let C be any upper bound for F. Then $C \ge x^2 \Rightarrow \sqrt{C} \ge |x| \ge x$ for all $x \in E$. Hence $\sqrt{C}$ is an upper bound for E. Thus, $\sqrt{C} \ge \sup E \Rightarrow C \ge (\sup E)^2$. It follows that $(\sup E)^2 = \sup F$.

Similarly if $\max\{(\sup E)^2, (\inf E)^2\} = (\inf E)^2$ then $\sup F = (\inf E)^2$.

Problem 3.3

Let E be an non-empty subset of $\mathbb{R}$ which is bounded from above. Show that there exists a sequence $\{a_n\}$ such that $a_n \in E$ and $\lim\limits_{n\to\infty} a_n = \sup E$.

Solution.
We know that for each $\varepsilon > 0$ there exists at least one $a \in E$ such that $a > \sup E - \varepsilon$. Let $a_k \in E$ such that $a_k > \sup E - \dfrac{1}{k}$. Thus, $\dfrac{1}{k} > \sup E - a_k \ge 0$. Thus the sequence $\{a_n\}$ converges to $\sup E$ and $\lim\limits_{n\to\infty} a_n = \sup E$.

Problem 3.4

Let $a_1 = 4$ and define a_n inductively by

$$a_n = 4 - \frac{4}{a_{n-1}} \text{ for } n \geq 2$$

Show that $\lim_{n\to\infty} a_n = 2$.

Solution. Using induction one can prove that

$$a_n = 2 + \frac{2}{n}, \text{ for } n \geq 1$$

Thus, $\lim_{n\to\infty} a_n = 2$.

Problem 3.5

Let $T : \mathbb{R} \to \mathbb{R}$ be a contraction map and $x \in \mathbb{R}$ be a number. Define a sequence a_n by requiring $a_1 = x$ and $a_{n+1} = T(a_n)$.

1. Show that for any $m \in \mathbb{N}$, $|a_1 - a_m| \leq \frac{1}{1-\lambda}|a_1 - a_2|$
2. Show that a_n is a Cauchy sequence.

Solution.
Notice that,

$$\begin{aligned}
|a_n - a_{n+1}| &\leq \lambda|a_{n-1} - a_n| \\
&\leq \lambda^2|a_{n-2} - a_{n-1}| \\
&\leq \quad \vdots \\
&\leq \lambda^{n-1}|a_1 - a_2|
\end{aligned}$$

Therefore,

$$\begin{aligned}
|a_1 - a_m| &\leq |a_1 - a_{m-1}| + |a_{m-1} - a_m| \\
&\leq |a_1 - a_{m-2}| + |a_{m-2} - a_{m-1}| + |a_{m-1} - a_m| \\
&\leq \quad \vdots \\
&\leq |a_1 - a_2| + \sum_{i=1}^{m-1} |a_i - a_{i+1}| \\
&\leq |a_1 - a_2| + \sum_{i=1}^{m-1} \lambda^i |a_1 - a_2| \\
&\leq |a_1 - a_2| \left(\sum_{i=0}^{m-1} \lambda^i \right) \\
&\leq |a_1 - a_2| \left(\frac{1 - \lambda^m}{1 - \lambda} \right) \leq \left(\frac{1}{1 - \lambda} \right) |a_1 - a_2|
\end{aligned}$$

For the second part, suppose $m > n$ then,

$$\begin{aligned}
|a_n - a_m| &\leq |a_n - a_{m-1}| + |a_{m-1} - a_m| \\
&\leq \qquad \vdots \\
&\leq \sum_{i=n}^{m-1} |a_i - a_{i+1}| \\
&\leq \sum_{i=n}^{m-1} \lambda^{i-1} |a_1 - a_2| \\
&\leq |a_1 - a_2| \cdot \left(\lambda^{n-1} \cdot \frac{1 - \lambda^{m-n}}{1 - \lambda} \right) \\
&\leq \frac{\lambda^{n-1}}{1 - \lambda} |a_1 - a_2|
\end{aligned}$$

So for any $\varepsilon > 0$ we can just choose a large n such that $\varepsilon > \dfrac{\lambda^{n-1}}{1 - \lambda}|a_1 - a_2|$.

4 Problem Set 4

Problem 4.1

Give an example of a sequence a_n that satisfies the following two conditions.

- a_n is divergent.
- For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_{n+1} - a_n| < \varepsilon \quad \text{for all } n \geq N.$$

Solution.
Take $a_n = \sum_{i=1}^{n} \frac{1}{i}$.

Problem 4.2

Let $p > 0$ be a positive number. Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

In the lecture we proved that the series diverges for $p = 1$ and converges for $p = 2$. Show that the series converges for $p > 1$ and diverges for $0 < p \leq 1$.

Solution.
For $0 < p \leq 1$ it is easy to see that $\frac{1}{n^p} \geq \frac{1}{n}$. From the comparison test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

For $p > 1$ we can use the same argument like we did for $p = 2$. We can group the terms and use the comparison test. I am skipping the details.

Problem 4.3

Let $r > 0$ be a positive number. Determine whether the series

$$\sum_{n=1}^{\infty} a_n$$

converges or diverges for the following cases.

1. $a_n = \sqrt{n + r} - \sqrt{n}$
2. $a_n = n^3 r^n$
3. $a_n = \frac{1}{n!} r^n$

The answer may depend on the value of r.

Solution.

For (1.), Let $r > 0$. Consider the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n+r}-\sqrt{n}\right).$$

Rationalizing the terms, we have

$$\sqrt{n+r}-\sqrt{n}=\frac{r}{\sqrt{n+r}+\sqrt{n}}.$$

Since $\sqrt{n+r} \leq \sqrt{n}+\sqrt{r}$, it follows that

$$\sqrt{n+r}+\sqrt{n} \leq 2\sqrt{n}+\sqrt{r}.$$

Hence,

$$\sqrt{n+r}-\sqrt{n}=\frac{r}{\sqrt{n+r}+\sqrt{n}} \geq \frac{r}{2\sqrt{n}+\sqrt{r}}.$$

Choose N such that $2\sqrt{n} \geq \sqrt{r}$ for all $n \geq N$. Then for $n \geq N$,

$$2\sqrt{n}+\sqrt{r} \leq 3\sqrt{n},$$

and therefore

$$\sqrt{n+r}-\sqrt{n} \geq \frac{r}{3\sqrt{n}}.$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

diverges, the comparison test implies that

$$\sum_{n=1}^{\infty}\left(\sqrt{n+r}-\sqrt{n}\right)$$

also diverges.

For (2.) we have and (3.) just do the ratio test.

Problem 4.4

Let b_n be a sequence of non-negative numbers which decreases to zero. That is,

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Let $a_n = (-1)^{n-1}b_n$. The purpose of this problem is to show that

$$\sum_{n=1}^{\infty} a_n$$

converges. This is called the *alternating series test*. Let

$$s_n = \sum_{k=1}^{n} a_k.$$

1. Show that s_{2k+1} is decreasing and that s_{2k} is increasing.
2. Show that s_{2k+1} is bounded from below and that s_{2k} is bounded from above.
3. Show that both $\lim_{k\to\infty} s_{2k+1}$ and $\lim_{k\to\infty} s_{2k}$ exist and are identical.
4. Show that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Solution.
For the first part notice that

$$a_{2k} + a_{2k+1} = b_{2k+1} - b_{2k} \leq 0$$

Thus $s_{2k+1} = s_{2k-1} + a_{2k} + a_{2k+1} \leq s_{2k-1}$.

Similarly,

$$a_{2k+1} + a_{2k+2} = b_{2k+1} - b_{2k+2} \geq 0$$

Thus, $s_{2k+2} = a_{2k+1} + a_{2k+2} + s_{2k} \geq s_{2k}$.

To show that $\{s_{2k+1}\}_{k=0}^{\infty}$ is bounded from below, we will show that $s_{2k+1} \geq 0$.

$$\begin{aligned} s_{2k+1} &= \underbrace{b_1 - b_2}_{\geq 0} + \underbrace{b_3 - b_4}_{\geq 0} + \cdots + \underbrace{b_{2k-1} - b_{2k}}_{\geq 0} + b_{2k+1} \\ &\geq b_{2k+1} \\ &\geq 0 \end{aligned}$$

Also,

$$\begin{aligned} s_{2k} &= b_1 - b_2 + b_3 - b_4 + \cdots + b_{2k-1} - b_{2k} \\ &\leq b_1 - b_2 + b_2 - b_3 + \cdots + b_{2k-2} - b_{2k-1} \\ &\leq b_1 - b_{2k-1} \\ &\leq b_1 \end{aligned}$$

Thus we're done with (2.).

For (3.), by **monotone convergence theorem**,

$$\lim_{n\to\infty} s_{2k+1} = \inf\{s_{2k+1}\} \quad \text{and} \quad \lim_{n\to\infty} s_{2k} = \sup\{s_{2k}\}$$

To show they are identical,

$$\begin{gathered} s_{2k+1} = s_{2k} + a_{2k+1} = s_{2k} + b_{2k-1} \\ \implies \lim_{n\to\infty} s_{2k+1} = \lim_{n\to\infty} s_{2k} = L \end{gathered}$$

For the (4.) part,

$$|s_n - L| < \varepsilon$$

for large enough even and odd n.

5 Problem Set 5

Problem 5.1

Let a_n, b_n be two sequence of real numbers and x be a real number. The fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Show that if $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$ converges, then so does the fourier series.

Solution.
We can ignore the $\frac{a_0}{2}$ term and just focus on the sum. Let

$$S_N(x) = \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

We will show that the sequence $\{S_N(x)\}$ is cauchy. Let $N, M \in \mathbb{N}$ such that $M > N$.

$$|S_M(x) - S_N(x)| = \sum_{n=N+1}^{M} a_n \cos(nx) + b_n \sin(nx) \leq \sum_{n=N+1}^{M} |a_n| + |b_n|$$

As $N \to \infty$ the $\sum_{n=N+1}^{M} |a_n| + |b_n| \to 0$. Thus the sequence is cauchy and from the **cauchy convergence theorem** we know the series converges.

Problem 5.2

Give an example of a function $f(x)$ defined on $[-1, 1]$ with the following property: $(f(x))^2$ is continuous on $[-1, 1]$ but $f(x)$ is not continuous on $[-1, 1]$.

Solution.
An example of such a function would be

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Problem 5.3

Determine at which points the function $f(x) = \lfloor x \rfloor$ is continuous or discontinuous.

Solution.
Let $n \in \mathbb{Z}$. Suppose, for contradiction, that $\lfloor x \rfloor$ is continuous at $x = n$. Then, by definition, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - n| < \delta \implies |\lfloor x \rfloor - \lfloor n \rfloor| < \varepsilon.$$

Take $\varepsilon = \frac{1}{2}$. Consider $x = n - \frac{\delta}{2}$. Clearly, $|x - n| = \frac{\delta}{2} < \delta$, so it should satisfy the continuity condition.
However,

$$\lfloor x \rfloor = \lfloor n - \tfrac{\delta}{2} \rfloor = n - 1$$

and

$$\lfloor n \rfloor = n.$$

Thus,

$$|\lfloor x \rfloor - \lfloor n \rfloor| = |(n-1) - n| = 1 > \varepsilon = \frac{1}{2},$$

which is a contradiction. Hence, $\lfloor x \rfloor$ is **discontinuous at every integer**.

Let $x_0 \notin \mathbb{Z}$ and set $n = \lfloor x_0 \rfloor$. Then $n < x_0 < n+1$. Define

$$\delta = \min\{x_0 - n,\, n + 1 - x_0\} > 0.$$

For any x with $|x - x_0| < \delta$, we have

$$n < x < n + 1 \implies \lfloor x \rfloor = n = \lfloor x_0 \rfloor.$$

Hence,

$$|\lfloor x \rfloor - \lfloor x_0 \rfloor| = 0,$$

proving that $\lfloor x \rfloor$ is **continuous at every non-integer**.

Problem 5.4

Let $f(x)$ and $g(x)$ be two continuous functions defined on $\mathbb{R}$ with $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Solution.
Let x_n be a rational number such that

$$r - \frac{1}{n} < x_n < r + \frac{1}{n}$$

where r is any irrational number. From our definition of x_n, we have that $x_n \to r$. Thus,

$$\begin{aligned} g(r) = g\left(\lim_{n\to\infty} x_n\right) &= \lim_{n\to\infty} g(x_n) \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\left(\lim_{n\to\infty} x_n\right) \\ &= f(r) \end{aligned}$$

Thus, $f(r) = g(r)$ for any $r \in \mathbb{R} \setminus \mathbb{Q}$ and therefore $f(x) = g(x)$ for any $x \in \mathbb{R}$.

Problem 5.5

Recall that

$$E(x) := 1 + x + \frac{x^2}{2!} + \cdots + = \sum_{n=0}^{infty} \frac{x^n}{n!}$$

1. For $k \in \mathbb{N}$, define
$$E_k(x) := 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$$
Show that $E_k(x)$ is continuous on $\mathbb{R}$ for any $k \in \mathbb{N}$.

2. Let $M > 0$ be a fixed number. Show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that
$$|E_k(x) - E(x)| < \varepsilon$$
for all $k \geq N$ and for all $x \in [-M, M]$.

   Remark: We require that a single number N that works for all $x \in [-M, M]$.

3. Show that $E(x)$ is continuous on $\mathbb{R}$.

Solution.
For (1.) just use the fact that if f is continuous then $c \cdot f$ is continuous for some fixed constant c.

For (2.), let $M > 0$ and fix $x \in [-M, M]$. Then $|x| \leq M$. We have

$$E(x) - E_k(x) = \sum_{n=k+1}^{\infty} \frac{x^n}{n!}.$$

By the triangle inequality,

$$|E(x) - E_k(x)| \leq \sum_{n=k+1}^{\infty} \frac{|x|^n}{n!}.$$

Since $|x| \leq M$, we have

$$|E(x) - E_k(x)| \leq \sum_{n=k+1}^{\infty} \frac{M^n}{n!}$$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{M^n}{n!}$ converges, hence its tail $\displaystyle\sum_{n=k+1}^{\infty} \frac{M^n}{n!} \to 0$ as $k \to \infty$. Therefore,

$$\sup_{x \in [-M,M]} |E(x) - E_k(x)| \leq \sum_{n=k+1}^{\infty} \frac{M^n}{n!} < \varepsilon$$

for some $k \geq N$. Therefore,

$$|E(x) - E_k(x)| < \varepsilon$$

for $k \geq N$ and for all $x \in [-M, M]$.

For (3.), notice that

$$|E(x) - E(x_0)| \leq |E(x) - E_k(x)| + |E_k(x) - E_k(x_0)| + |E_k(x_0) - E(x_0)|$$

Now, choose a $M > |x_0|$ such that $x \in [-M, M]$. Then, from (2.),

$$|E(x) - E_k(x)| < \frac{\varepsilon}{3} \quad \text{and} \quad |E(x_0) - E_k(x_0)| < \frac{\varepsilon}{3}$$

for large enough k. Also,

$$|E_k(x) - E_k(x_0)| < \frac{\varepsilon}{3}$$

by (1.) and thus,

$$|E(x) - E_k(x)| + |E_k(x) - E_k(x_0)| + |E_k(x_0) - E(x_0)| < 3 \times \frac{\varepsilon}{3}$$

$$\implies |E(x) - E(x_0)| < \varepsilon$$

6 Problem Set 6

Problem 6.1

Let $X = C([0,1])$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$$

Prove that d satisfies

$$d(f,g) \le d(f,h) + d(h,g)$$

for all $f, g, h \in X$.

Solution. Notice that

$$\begin{aligned} d(f,h) + d(h,g) &= \max_{x\in[0,1]} |f(x) - h(x)| + \max_{x\in[0,1]} |h(x) - g(x)| \\ &\ge |f(x) - h(x)| + |h(x) - g(x)| \quad (\text{for all } x \in [0,1]) \\ &\ge |f(x) - g(x)| \\ &= d(f,g) \end{aligned}$$

Problem 6.2

Let $X = \mathbb{N}_{>0}$ and $d : X \times X \to \mathbb{R}$ defined by

$$d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$$

1. Show that the sequence $x_n = 5n$ is a cauchy sequence.
2. Show that the sequence $x_n = 5n$ doesn't converge.

Thus the Metric space is not Cauchy complete.

Solution. Notice that,

$$\begin{aligned} d(x_n, x_m) &= \frac{1}{5}\left|\frac{1}{n} - \frac{1}{m}\right| \\ &\le \left|\frac{1}{n} - \frac{1}{m}\right| \\ &\le \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

the second last inequality is due to the archimedean property of $\mathbb{R}$.
Let $\varepsilon = \frac{1}{2} > 0$. For any candidate $x \in X$, we have

$$d(x_n, x) = \left|\frac{1}{5n} - \frac{1}{x}\right|.$$

Since $x \geq 1$, we have $\frac{1}{x} \geq 0$, so for sufficiently large n,

$$\frac{1}{5n} < \frac{1}{2x} \implies d(x_n, x) = \left| \frac{1}{5n} - \frac{1}{x} \right| > \frac{1}{2x} \geq \frac{1}{2} = \varepsilon.$$

Hence, for all $N \in \mathbb{N}$, we can find $n \geq N$ such that $d(x_n, x) \geq \varepsilon$. Thus the sequence is not convergent.

Problem 6.3

Let $X = \mathbb{R}^2$ and consider the metric

$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1. Let (x_n, y_n) be a sequence in $\mathbb{R}^2$. Show that (x_n, y_n) converges if and only if both x_n and y_n converge as sequence in $\mathbb{R}$.
2. Let (x_n, y_n) be a sequence in $\mathbb{R}^2$. Show that (x_n, y_n) is a Cauchy sequence if and only if both x_n and y_n are Cauchy sequence in $\mathbb{R}$.
3. Show that $\mathbb{R}^2$ is Cauchy Complete.

Solution.
For (1.), suppose the sequence $(x_n, y_n) \to (x, y)$ in $\mathbb{R}^2$ then

$$\begin{aligned}
|y_n - y| &= \sqrt{(y_n - y)^2} \\
&\leq \sqrt{(x_n - x)^2 + (y_n - y)^2} \\
&= d((x_n, y_n), (x, y)) \\
&< \varepsilon
\end{aligned}$$

Similarly for $|x_n - x|$. Now, suppose $x_n \to x$ and $y_n \to y$ then we claim that $(x_n, y_n) \to (x, y)$. Notice, that from the triangle inequality

$$\begin{aligned}
d((x_n, y_n), (x, y)) &\leq d((x_n, y_n), (x_n, y)) + d((x_n, y), (x, y)) \\
&= \sqrt{(y_n - y)^2} + \sqrt{(x_n - x)^2} \\
&= |y_n - y| + |x_n - x| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

Thus, $(x_n, y_n) \to (x, y)$.

For (2.), suppose (x_n, y_n) is cauchy sequence then

$$\begin{aligned}
|y_n - y_m| &= \sqrt{(y_n - y_m)^2} \\
&\leq \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} \\
&= d((x_n, y_n), (x_m, y_m)) \\
&< \varepsilon
\end{aligned}$$

Similarly for $|x_n - x_m|$. Now suppose x_n and y_n are cauchy sequence in $\mathbb{R}$. Then,

$$\begin{aligned} d((x_n, y_n), (x_m, y_m)) &\leq d((x_n, y_n), (x_n, y_m)) + d((x_n, y_m), (x_m, y_m)) \\ &= \sqrt{(y_n - y_m)^2} + \sqrt{(x_n - x_m)^2} \\ &= |y_n - y_m| + |x_n - x_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Thus, (x_n, y_n) is cauchy in $\mathbb{R}^2$.

For (3.), the sequence (x_n, y_n) is cauchy sequence if and only if x_n and y_n are cauchy sequence in $\mathbb{R}$ from (2.), and x_n and y_n are cauchy if and only if x_n and y_n are convergent and x_n and y_n are convergent if and only if (x_n, y_n) convergent from (1.).

Problem 6.4

Let A_j, $j \in \mathbb{N}$ be open sets in metric space (X, d).

1. Show that $\bigcup_{j=1}^{\infty} A_j$ is open.
2. Show that $\bigcap_{j=1}^{\infty} A_j$ maybe not be open.

Solution.

For (1.), take a arbitrary element $x \in \bigcup_{j=1}^{\infty} A_j$. That means $x \in A_k$ for some $k \in \mathbb{N}$. Thus, $\exists r > 0$ such that $B_r(x) \subseteq A_k \subseteq \bigcup_{j=1}^{\infty} A_j$.

For (2.) consider $A_j = \left(0, 1 + \frac{1}{j}\right)$ the intersection is just $(0, 1]$ and $B_\varepsilon(1)$ covers element greater than 1 as $1 - \varepsilon < y < 1 + \varepsilon$ for some $\varepsilon > 0$.

Problem 6.5

Let (X, d) be a metric space, K be a compact set in X and A_j, $j \in \mathbb{N}$ be closed set in X. Suppose for all $k \in \mathbb{N}$

$$K \cap \left(\bigcap_{j=1}^{k} A_j \right) \text{ is non empty}$$

Show that

$$K \cap \left(\bigcap_{j=1}^{\infty} A_j \right) \text{ is non empty}$$

Solution.

Suppose

$$K \cap \left(\bigcap_{j=1}^{\infty} A_j \right) \text{ is empty}$$

Then,

$$K \subseteq \left(\bigcap_{j=1}^{\infty} A_j \right)^c = \bigcup_{j=1}^{\infty} A_j^c$$

Using the compactness of K, we would have

$$K \subseteq \bigcup_{j=1}^{k} A_j^c = \left(\bigcap_{j=1}^{k} A_j \right)^c$$

for some $k \in \mathbb{N}$. But that means, $K \cap \left(\bigcap_{j=1}^{k} A_j \right) = \emptyset$.

7 Problem Set 7

Problem 7.1

Consider the set

$$X := \left\{ (a_1, a_2, a_3, \ldots) \;\middle|\; \sum_{j=1}^{\infty} a_j^2 \text{ converges} \right\}.$$

Define the function $d : X \times X \to \mathbb{R}$ as follows. For

$$x = (a_1, a_2, a_3, \ldots), \quad y = (b_1, b_2, b_3, \ldots) \in X,$$

define

$$d(x, y) := \sqrt{\sum_{j=1}^{\infty} (a_j - b_j)^2}.$$

1. Show that the function d is well-defined. Equivalently, show that for

   $$x = (a_1, a_2, a_3, \ldots) \quad \text{and} \quad y = (b_1, b_2, b_3, \ldots) \in X,$$

   the series

   $$\sum_{j=1}^{\infty} (a_j - b_j)^2$$

   converges.

2. Show that the function d satisfies the triangle inequality. You may use the following triangle inequality in $\mathbb{R}^n$ without proof. For all $n \geq 1$,

   $$\sqrt{\sum_{j=1}^{n} (a_j - c_j)^2} \leq \sqrt{\sum_{j=1}^{n} (a_j - b_j)^2} + \sqrt{\sum_{j=1}^{n} (b_j - c_j)^2}.$$

3. Consider a sequence in X defined as

   $$x_1 = (1, 0, 0, 0, \ldots), \quad x_2 = (0, 1, 0, 0, \ldots), \quad \text{and so on.}$$

   In general,

   $$x_n = (0, \ldots, 0, \underbrace{1}_{n\text{th position}}, 0, \ldots).$$

   Show that (x_n) has no convergent subsequence.

Solution.

For (1.), Let $S_N^a = \sum_{j=1}^{N} a_j^2$ and $S_N^b = \sum_{j=1}^{N} b_j^2$. We know that $\lim_{n\to\infty} S_n^a$ and $\lim_{n\to\infty} S_n^b$ exists, therefore $\lim_{n\to\infty} S_n^a + S_n^b$ exists. And since,

$$0 \leq (a_j - b_j)^2 = a_j^2 + b_j^2 - 2a_j b_j \leq a_j^2 + b_j^2 + 2|a_j b_j| \leq 2(a_j^2 + b_j^2)$$

The last inequality is due to AM-GM inequality. Therefore,

$$0 \leq (a_j - b_j)^2 \leq 2(a_j^2 + b_j^2)$$

and we have that $\sum_{j=1}^{\infty}(a_j - b_j)^2$ converges from comparison test.

For (2.), just takes the limit as $n \to \infty$.

For (3.), suppose there is a convergent subsequence and let $x_{n_k} \to x$

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, x) + d(x, x_{n_\ell})$$

Now, notice that for $n_k \neq n_\ell$ we have $d(x_{n_k}, x_{n_\ell}) = \sqrt{2}$. Thus, for large n_k and m_ℓ we have

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, x) + d(x, x_{n_\ell}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$\implies \sqrt{2} < \varepsilon$$

But for $\varepsilon < \sqrt{2}$ thats false.

Problem 7.2

Let (X, d) be a metric space and (x_n) a sequence in X. Denote

$$E = \{x_1, x_2, x_3, \ldots\}.$$

Suppose (x_n) has no convergent subsequence. Show that for all $k \in \mathbb{N}$, there exists $r_k > 0$ such that

$$B(x_k, r_k) \cap E = \{x_k\}.$$

You may use the following fact without proof. Fix $x \in X$. Suppose that for all $r > 0$, there are infinitely many elements in $E \cap B(x, r)$. Then (x_n) has a subsequence which converges to x.

Solution. Using the fact, since no sequence of (x_n) converges to x there exists a $r > 0$ such that $E \cap B(x, r)$ is finite. Since no subsequence converges to x_k for $k \in \mathbb{N}$, we have $r > 0$ s.t $E \cap B(x_k, r)$ is finite. If $|E \cap B(x_k, r)| = 1$ then we have the desired condition but suppose $|E \cap B(x_k, r)| > 1$ then

$$E \cap B(x_k, r) = \{x_k, x_{m_1}, \ldots, x_{m_\ell}\}$$

Then, since $d(x_k, x_{m_i}) > 0$ for all $1 \leq i \leq \ell$ we have

$$E \cap B\left(x_k, \min_{1 \leq i \leq \ell}\{d(x_k, x_{m_i})\}\right) = \{x_k\}$$

Problem 7.3

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $f(x)$ is differentiable on $(-\infty, 0) \cup (0, \infty)$ and that the limit $\lim\limits_{x \to 0} f'(x)$ exists. Show that $f(x)$ is differentiable at $x = 0$ and that $f'(0) = \lim_{x \to 0} f'(x)$.

Solution.
Let $h \neq 0$. Since f is continuous on the closed interval with endpoints 0 and h and differentiable on the open interval with end points 0 and h,we can use MVT. Thus, there exists a point c_h between 0 and h such that

$$\frac{f(h) - f(0)}{h} = f'(c_h).$$

As $h \to 0$, the point c_h lies between 0 and h, and therefore $c_h \to 0$. Since $\lim_{x\to 0} f'(x) = L$, it follows that

$$f'(c_h) \to L \quad \text{as } h \to 0.$$

Hence,

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} f'(c_h) = L.$$

Therefore, the derivative of f at 0 exists and satisfies

$$f'(0) = L = \lim_{x\to 0} f'(x).$$

Problem 7.4

Use $\frac{d}{dx}e^x = e^x$ to show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

Solution.
Notice that $f(x) = e^x$ is continuous everywhere and differentiable everywhere. For $x = 0$ it is obvious $e^0 = 1$. Assume that $x > 0$ then f is continuous on $[0, x]$ and differentiable on $(0, x)$ thus applying MVT we get,

$$\frac{f(x) - f(0)}{x} = f(c)$$

$$\implies \frac{e^x - 1}{x} = e^c$$

some for $0 < c < x$, we have $e^c \geq 1$ we have

$$\begin{aligned} e^x &= e^c \cdot x + 1 \\ &\geq x + 1 \end{aligned}$$

Now assume $x < 0$, we have $x = -a$ thus using MVT again we get,

$$\frac{e^{-a} - 1}{-a} = e^c$$

Since $0 > c > -a$ we get $e^c \leq 1 \implies e^{-a} = e^c(-a) + 1 \geq -a + 1$.

Problem 7.5

Let $X = C([0,1])$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

Suppose $(f_n) \subset X$ is a Cauchy sequence.

1. Fix an arbitrary $x_0 \in [0,1]$. Show that $\lim_{n\to\infty} f_n(x_0)$ exists.
2. Define the function $f : [0,1] \to \mathbb{R}$ by

   $$f(x) := \lim_{n\to\infty} f_n(x).$$

   Show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

   $$|f_n(x) - f(x)| \leq \varepsilon$$

   for all $x \in [0,1]$ and for all $n \geq N$.
3. Show that $f(x)$ is continuous on $[0,1]$. Equivalently, show that $f \in X$.
4. Show that

   $$\lim_{n\to\infty} f_n = f$$

   as a sequence in X.

Solution.
For (1.), since the f_n is Cauchy that means $f_n(x_0)$ is also Cauchy. But since its a real valued function and $\mathbb{R}$ is complete we have that $f_n(x_0)$ is convergent. Thus the $\lim_{n\to\infty} f_n(x_0)$ exists.

For (2.) we will use the fact that, $\lim_{n\to\infty} |a_n \pm b_n| = \left|\lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n\right|$ if both the limits exist. Since f_n is cauchy, we have

$$|f_n(x) - f_m(x)| \leq \max_{x\in[0,1]} |f_n(x) - f_m(x)| < \varepsilon$$

for $x \in [0,1]$ and for $n, m \geq N$. Then using the properties of limits we have,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| < \varepsilon$$

Therefore using the fact, we have

$$|f_n(x) - f(x)| < \varepsilon$$

For (3.), let $x_0 \in [0,1]$ then

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

Now, for $n \geq N$ we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(x_0) - f_n(x_0)| < \varepsilon/3$ from (2). and since $f_n \in C([0,1])$ we can pick a δ_1 for s.t whenever $0 < |x - x_0| < \delta_1$ we have $|f_n(x) - f_n(x_0)| < \varepsilon/3$.

Thus, if we choose $\delta = \delta_1$ then we have

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$

Thus $f \in C([0,1])$.

For (4.), since we know that $f \in C([0,1])$. Thus

$$\max_{x \in [0,1]} |f_n(x) - f(x)| < \varepsilon$$

Thus, $d(f_n, f) < \varepsilon$ for any $\varepsilon > 0$ so we have $f_n \to f$ in X.

8 Problem Set 8

Problem 8.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose both $f'(x)$ and $f''(x)$ are continuous on $\mathbb{R}$ and that $f(0) = 0$. Define the function

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

Show that $g'(x)$ exists for all $x \in \mathbb{R}$ and express $g'(x)$ in terms of $f(x)$ and its derivative.

Solution.
For $x \neq 0$, we have that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

For $x = 0$, we can use the definition of the derivative,

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - xf'(0)}{x^2}$$

Now using taylor's formula near 0, we have that

$$f(x) = f(0) + xf'(0) + x^2 f''(c_x) \cdot \frac{1}{2}$$

$$\implies \frac{f(x) - xf'(0)}{x^2} = \frac{1}{2} \cdot f''(c_x)$$

$$\implies \lim_{x \to 0} \frac{f(x) - xf'(0)}{x^2} = \lim_{x \to 0} \frac{f''(c_x)}{2} = \frac{f''(0)}{2}$$

Thus, $g'(0)$ also exists and

$$g'(x) = \begin{cases} \dfrac{xf'(x) - f(x)}{x^2} & x \neq 0 \\ \dfrac{f''(0)}{2} & x = 0 \end{cases}$$

Problem 8.2

Suppose there exist two functions $S : \mathbb{R} \to \mathbb{R}$ and $C : \mathbb{R} \to \mathbb{R}$ which satisfy the following properties:

- $\frac{d}{dx}S(x) = C(x), \quad \frac{d}{dx}C(x) = S(x).$
- $S(0) = 0, \quad C(0) = 1.$

(1) Let $S^{(n)}(x)$ be the nth derivative of $S(x)$. Show that for $k \in \mathbb{N} \cup \{0\}$,

$$S^{(2k)}(x) = S(x), \quad S^{(2k+1)}(x) = C(x).$$

(2) Show that for all $x \in \mathbb{R}$,

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Solution.
For (1.) just use induction.

For (2.), restrict S to $[0, x]$ for some fixed $x > 0$. Then,

$$S(x) = \sum_{n=0}^{k} \frac{S^n(0)x^n}{(n)!} + R_k(x) = \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{x^{2n+1}}{(2n+1)!} + R_k(x)$$

But since S and C are bounded and continuous on $[0, x]$ and we have $S^n(x) = S(x)$ or $C(x)$, thus $|S^n(x)| \le M$ for $n \ge 0$.

$$|R_k| = \left| \frac{S^{k+1}(c)x^{k+1}}{(k+1)!} \right| \le \left| \frac{Mx^{k+1}}{(k+1)!} \right| < \varepsilon$$

Thus,

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Problem 8.3

Let $a < b$ be two real numbers and let $f : [a, b] \to \mathbb{R}$ be a function. Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) > 0$ for all $x \in (a, b)$.

1. Show that $f(x)$ is strictly increasing on $[a, b]$. That is, $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in $[a, b]$.

2. Show that for all $y \in (f(a), f(b))$, there exists a unique $x \in (a, b)$ such that $f(x) = y$.

3. Let the function $g : (f(a), f(b)) \to (a, b)$ be the inverse function of $f(x)$. In other words, $g(y) = x$ if $f(x) = y$. Show that g is continuous on $(f(a), f(b))$.

4. Show that g is differentiable on $(f(a), f(b))$ and that

$$g'(y) = \frac{1}{f'(g(y))} \quad \text{for all } y \in (f(a), f(b)).$$

Solution.
For (1.), Let $x_1 < x_2$ be points in $[a, b]$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$, it follows that

$$f(x_2) - f(x_1) > 0,$$

and hence $f(x_1) < f(x_2)$. Therefore, f is strictly increasing on $[a, b]$.

For (2.), since f is continuous on $[a, b]$, the IVT implies that for every $y \in (f(a), f(b))$ there exists $x \in (a, b)$ such that $f(x) = y$. To prove uniqueness, suppose $x_1 < x_2$ and $f(x_1) = f(x_2)$. This is a contradiction as $f(x_1) < f(x_2)$.

For (3.), fix $y_0 \in (f(a), f(b))$ and let $x_0 = g(y_0)$. Let $\varepsilon > 0$ be given. Since f is strictly increasing, we have

$$f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon).$$

Define

$$\delta := \min\{f(x_0 + \varepsilon) - f(x_0),\ f(x_0) - f(x_0 - \varepsilon)\}.$$

Then $\delta > 0$.
Now suppose $y \in (f(a), f(b))$ satisfies $|y - y_0| < \delta$. Then

$$f(x_0 - \varepsilon) < y < f(x_0 + \varepsilon).$$

By the definition of the inverse function and since f is increasing , we have

$$x_0 - \varepsilon < g(y) < x_0 + \varepsilon.$$

Hence,

$$|g(y) - g(y_0)| < \varepsilon.$$

Thus g is continuous on $(f(a), f(b))$.

For (4.), fix $y_0 \in (f(a), f(b))$ and let $x_0 = g(y_0)$, so that $f(x_0) = y_0$. For $y \neq y_0$, write $y = f(x)$, so that $x = g(y)$. Then

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}}.$$

Taking the limit as $y \to y_0$ is equivalent to taking the limit as $x \to x_0$, since g is continuous. Therefore,

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}.$$

Thus g is differentiable at y_0 and

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Since y_0 was arbitrary, this holds for all $y \in (f(a), f(b))$.

Problem 8.4

Define the function $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in [-1, 0) \cup (0, 1], \\ 0, & x = 0. \end{cases}$$

Show that $f(x)$ is Riemann integrable on $[-1, 1]$ and that

$$\int_{-1}^{1} f(x)\, dx = 2.$$

Solution.

For any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[-1, 1]$, every sub interval $[x_{i-1}, x_i]$ contains points $x \neq 0$. Thus, the supremum on each subinterval is $M_i = 1$. The upper sum is:

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1 \cdot (1 - (-1)) = 2$$

Hence, the upper integral is $\overline{\int_{-1}^{1}} f(x)\, dx = \inf_P U(f, P) = 2.$

Let $\varepsilon > 0$. Choose a partition $P_\varepsilon = \{-1, -\frac{\varepsilon}{4}, \frac{\varepsilon}{4}, 1\}$. The infimums m_i for the three sub intervals are $m_1 = 1$, $m_2 = 0$ (since $0 \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$), and $m_3 = 1$. The lower sum is:

$$L(f, P_\varepsilon) = 1\left(1 - \frac{\varepsilon}{4}\right) + 0\left(\frac{\varepsilon}{2}\right) + 1\left(1 - \frac{\varepsilon}{4}\right) = 2 - \frac{\varepsilon}{2}$$

Since $L(f, P_\varepsilon) \leq \underline{\int_{-1}^{1}} f(x)\,dx \leq 2$, and we can make $L(f, P_\varepsilon)$ arbitrarily close to 2 by choosing a small ε, the lower integral is $\underline{\int_{-1}^{1}} f(x)\,dx = 2$.

Since the upper and lower integrals are equal:

$$\underline{\int_{-1}^{1}} f(x)\,dx = \overline{\int_{-1}^{1}} f(x)\,dx = 2$$

The function f is Riemann integrable on $[-1, 1]$ and $\int_{-1}^{1} f(x)\,dx = 2$.