Real Analysis

Basu Dev Karki

Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from MIT OCW and write the solutions to the problem set here.

1 Problem Set 1

Problem 1.1

Let $\mathbb{F}$ be a ordered field with $1 \neq 0$. Show that $1 > 0$.

Solution.
First let us prove that $(-1) \cdot (-1) = 1$. We know that for all $x \in \mathbb{F}$, there is an inverse element $-x$ such that,

$$x + (-x) = 0$$

Thus, $1 + (-1) = 0$ which means that

$$0 = (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1)$$

$$\implies 1 = (-1) \cdot (-1)$$

Since $\mathbb{F}$ is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that $1 \neq 0$ so the middle statement can't be true and if $1 < 0$ then $0 < (-1)$. But from **axiom of order and multiplication** $0 < (-1) \cdot (-1) = 1$. Thus a contradiction.

Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

Solution.
Suppose $\frac{n}{m} = \frac{n_1}{m_1}$ and $\frac{p}{q} = \frac{p_1}{q_1}$ then using the definition of when two rational numbers are equal, we get $nm_1 = n_1 m$ and $pq_1 = p_1 q$. Thus,

$$\begin{aligned} m_1 q_1 (nq + pm) &= m_1 q_1 nq + m_1 q_1 pm \\ &= n_1 m q_1 q + p_1 q m_1 m \\ &= mq(n_1 q_1 + m_1 p_1) \end{aligned}$$

Thus, $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$.

Problem 1.3

Find the $\sup E$ and $\inf E$ for the following set E.

1. $E = \left\{n \in \mathbb{Z} \mid n < \sqrt{12}\right\}$
2. $E = \left\{r \in \mathbb{Q} \mid r < \sqrt{12}\right\}$
3. $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
4. $E = \left\{\frac{n^2+n}{n+1} \mid n \in \mathbb{N}\right\}$

Solution.

1. $\sup E = 3$ but $\inf E$ doesn't exist.
2. $\sup E = \sqrt{12}$ but $\inf E$ doesn't exist.
3. $\sup E = \frac{1+\sqrt{5}}{2}$ and $\inf E = \frac{1-\sqrt{5}}{2}$.
4. $\inf E = 1$ but $\sup E$ doesn't exist.

Problem 1.4

Let $\mathbb{M}$ be the set of polynomials with integer coefficients i.e,

$$\mathbb{M} := \{f(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Z}\}$$

Define the relation $0 \prec f$ if $0 < f(x)$ for x large enough. More precisely, we say

$$0 \prec f \quad \text{if there exists } M > 0 \text{ such that } f(x) > 0 \text{ for all } x > M.$$

Then define

$$f \prec g \quad \text{if } 0 \prec (g - f).$$

Show that $(M, \prec)$ is an ordered set.

Solution.
We'll use the fact that for large enough x, $f(x) > 0$ for $a_n > 0$. Let $f, g \in \mathbb{M}$ such that $f \neq g$ and

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

Define $h(x) = g(x) - f(x) = (b_0 - a_0) + (b_1 - a_1)x + \cdots + (b_n - a_n)x^n$ and let $c_i = b_i - a_i$. Suppose c_k is the highest degree non zero coefficient, then if

1. $c_k > 0$ then $f \prec g$
2. $c_k < 0$ then $g \prec f$

Suppose $f \prec g$ and $g \prec h$. Then, $f(x) < g(x)$ and $g(x) < h(x)$ for large x and hence $f(x) < h(x)$ for large x. Thus, $0 \prec (h - f)$ which means $f \prec h$.

Problem 1.5

Prove that $(\mathbb{M}, \prec)$ **doesn't** satisfy the archimedean property.

Solution.
To show that it doesn't satisfy the archimedean property, we need to show that $\exists f, g \in \mathbb{M}$ such that

$$\forall n \in \mathbb{N}, g \not\prec nf$$

If we choose $g(x) = x^2$ and $f(x) = x$ and assume $g \prec nf$ then , $0 \prec x(n-x)$ which means that $0 < x(n-x)$ which we know is false for large x. Thus, $g \not\prec nf$.

Problem 1.6

Show that for any non-empty set $E \subset R$ which is bounded from below, E has the greatest lower bound.

Solution.
To show that E has greatest lower bound define

$$-E = \{-x \mid x \in E\}$$

If α is any lower bound of E then $x \geq \alpha \Rightarrow -x \leq -\alpha$. That means that $-E$ is bounded above by $-\alpha$. Since $-E$ is bounded above by $-\alpha$, it must have the least upper bound property. Let

$$\sup -E = \beta$$

Thus, $\beta \geq -x \Rightarrow x \geq -\beta$ and $\beta \leq -\alpha$ for any lower bound α of E. Thus, $\alpha \leq -\beta$. Hence, $-\beta = \inf E$.

Problem 1.7

Show that for any real number $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that $y^3 = x$.

Solution.
If $x = 0$ then $y = 0^3 = 0$. Let us define

$$A = \left\{a \mid a > 0 \text{ and } a^3 \leq x\right\}$$

for $x > 0$. Since A is non-empty and bounded, let $y = \sup A$. We will show that $y^3 = x$. Suppose $x > y^3$. Let $h = \min\{\frac{1}{2}, \frac{x-y^3}{3y^2+3y+1}\}$ then,

$$\begin{aligned}
(y+h)^3 &= y^3 + h^3 + 3yh^2 + 3y^2h \\
&< y^3 + h(1 + 3y + 3y^2) \\
&\leq y^3 + \frac{x-y^3}{3y^2+3y+1} \cdot (1 + 3y + 3y^2) \\
&\leq x
\end{aligned}$$

Thus, $y + h \in A$ but since y is the least upper bound of A we have, $y + h \leq y \Rightarrow h \leq 0$. This is a contradiction.

Suppose $x < y^3$. Let $h = \frac{y^3 - x}{3y^2}$ then

$$\begin{aligned}(y-h)^3 &= y^3 - h^3 - 3y^2h + 3yh^2 \\ &= y^3 - h^3 - 3y^2 \cdot \frac{(y^3 - x)}{3y^2} + 3yh^2 \\ &= x - h^3 + 3yh^2\end{aligned}$$

Since $3yh^2 - h^3 > 0$ we have $(y-h)^3 > x$. Let $q \in A$ then

$$q^3 \leq x < (y-h)^3$$

Thus, $(y-h)^3 - q^3 > 0 \Rightarrow (y-h-q)\left((y-h)^2 + (y-h)q + q^3\right) > 0$. Since, $(y-h)^2 + (y-h)q + q^3 > 0$ we have $y - h > q$. Thus $(y-h)$ is an upper bound for A. Thus $y \leq y - h \Rightarrow h \leq 0$. Contradiction!!

The proof also works for $x < 0$ as $-x > 0$. Thus,

$$\exists k \in \mathbb{R} \text{ s.t } k^3 = -x$$

Let $z := -k$ then $z^3 = (-k)^3 = -k^3 = -(-x) = x$.

2 Problem Set 2

Problem 2.1

Let a_n ad b_n be a sequence of real numbers.

1. Assume that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists. Show that $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} a_n\right)$ exists.
2. Give an example in which $\lim_{n\to\infty}(a_n b_n)$ exists but neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty} b_n$ exits.

Solutions.
To show that $\lim_{n\to\infty} a_n b_n = ab$ we need to show $\exists N \in \mathbb{N}$ such that for all $\varepsilon > 0$ we have

$$|a_n b_n - ab| < \varepsilon$$

for $n \geq N$. Now,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

Since $\{a_n\}$ is convergent, we know that it is bounded above. Suppose $|a_n| < C$ then

$$|a_n||b_n - b| + |b||a_n - a| < |C||b_n - b| + |b||a_n - a|$$

Also let $N_1 \in \mathbb{N}$ such that $|b_n - b| < \dfrac{\varepsilon}{2(|C|+1)}$ for all $n \geq N_1$ and let $N_2 \in \mathbb{N}$ such that $|a_n - a| < \dfrac{\varepsilon}{2(|b|+1)}$ for all $n \geq N_2$. Thus for $n \geq \max\{N_1, N_2\}$ we have,

$$\begin{aligned} 2|a_n||b_n - b| + |b||a_n - a| &< |C||b_n - b| + |b||a_n - a| \\ &< |C| \cdot \frac{\varepsilon}{2(|C|+1)} + |b| \cdot \frac{\varepsilon}{2(|b|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

For the second part just take $a_n = (-1)^n$ and $b_n = (-1)^n$. We know that $\lim_{n\to\infty} a_n b_n = 1$ but $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ doesn't exist.

Problem 2.2

Find the limit of the following sequence if it exists or show that the limit doesn't exists.

1. $a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$
2. $a_n = \frac{\sin(n)}{n}$
3. $a_n = \displaystyle\sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6 + i}}$

Solution.
For the first part we know that $a_n = \frac{1}{n+1} - \frac{1}{n} - 1$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n} - 1 &= \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 1 \\ &= 0 - 0 - 1 \\ &= -1\end{aligned}$$

The first equation is due to the algebraic property of of limit.

For the second part, we need to show that

$$\left|\frac{\sin(n)}{n}\right| < \varepsilon$$

We know that $0 \leq |\sin(n)| \leq 1 \Rightarrow 0 \leq \frac{|\sin(n)|}{|n|} \leq \frac{1}{|n|}$. Thus, for any $N > \frac{1}{\varepsilon}$ we have

$$\left|\frac{\sin(n)}{n}\right| \leq \frac{1}{|n|} < \frac{1}{N} < \varepsilon$$

For the third part, we will use **squeeze theorem** to show that the the limit is 1. Notice that,

$$\frac{n^2}{\sqrt{n^6+n}} < \frac{n^2}{\sqrt{n^6+i}} < \frac{n^2}{\sqrt{n^6}}$$

for $i = 1, 2, \ldots, n$. Thus,

$$\begin{aligned} n \cdot \frac{n^2}{\sqrt{n^6+n}} &\leq \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq n \cdot \frac{n^2}{\sqrt{n^6}} \\ \Longrightarrow \frac{1}{\sqrt{1+\frac{1}{n^5}}} &\leq \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq 1 \\ \Longrightarrow \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^5}}} &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq \lim_{n\to\infty} 1 \\ \Longrightarrow 1 &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} \leq 1 \end{aligned}$$

Thus, by **squeeze theorem** we have $\lim\limits_{n\to\infty} a_n = 1$.

Problem 2.3

Let a_n be a sequence of real numbers and L be a real number. Show that the following two statements are equivalent. One holds if and only if the other does.

1. There exists a subsequence a_{n_k} converging to L.
2. For any $\epsilon > 0$, there exist infinitely many a_n in $(L - \epsilon, L + \epsilon)$.

Solution.

Suppose (1.) holds then there exists a $N \in \mathbb{N}$ such that for all $k \geq N$ we have,

$$|a_{n_k} - L| < \varepsilon \Longrightarrow L - \varepsilon < a_{n_k} < L + \varepsilon$$

Thus, there are infinitely many a_i in the interval $(L - \varepsilon, L + \varepsilon)$.

Suppose (2.) holds then for any $\varepsilon > 0$ there are infinitely many a_n such that

$$L - \varepsilon < a_n < L + \varepsilon$$

Since there are infinitely many terms in $(L - 1, L + 1)$ choose n_1 such that

$$|a_{n_1} - L| < 1$$

Since there are infinitely many terms in $(L - \frac{1}{2}, L + \frac{1}{2})$ choose n_2 such that

$$n_2 > n_1 \text{ and } |a_{n_2} - L| < \frac{1}{2}$$

Continuing inductively we can produce a sequence such that $\{a_{n_i}\}$

$$|a_{n_k} - L| < \frac{1}{k}$$

Hence, $a_{n_k} \to L$.

Problem 2.4

Where possible, find a subsequence that is monotone and a subsequence that is convergent for the following sequences.

1. $a_n = \sin(n\pi/8)$
2. $a_n = (-1)^n n$

Solution.

Notice that

$$\sin\left(\frac{n\pi}{8}\right) = \begin{cases} 0 & \text{if } n = 8k \\ 1 & \text{if } n = 4(4k+1) \\ -1 & \text{if } n = 4(4k-1) \end{cases}$$

So a monotone subsequence would be $a_{n_k} = 1$. And the same example works for convergent subsequence.

For the second part, notice that

$$a_n = \begin{cases} 2k & \text{if } n = 2k \\ -2k-1 & \text{if } n = 2k+1 \end{cases}$$

So a monotone subsequence would be $a_{n_k} = 2k$. But there is no convergent subsequence, because we know that every subsequence would have a infinitely many odd numbers or infinitely many even numbers but we know that those aren't bounded.

3 Problem Set 3

Problem 3.1

Prove that for reals $x < y$, there exists a $r \in \mathbb{Q}$ such that $x < r < y$.

Solution. Using the archimedean property of $\mathbb{R}$

$$\exists n \in \mathbb{N}, \text{ s.t } \frac{1}{n} < y - x$$

$$\implies n(y-x) > 1$$

Since the difference is greater than 1, there exists a integer m such that

$$nx < m < ny \implies x < \frac{m}{n} < y$$

Hence we are done.

Problem 3.2

Let E be an non-empty subset of $\mathbb{R}$ which is bounded. Define

$$F := \left\{x^2 \mid x \in E\right\}$$

Show that $\sup F$ exists and that $\sup F = \max\{(\sup E)^2, (\inf E)^2\}$.

Solution.
Since E is bounded F is also bounded. Then since $\inf E \le x \le \sup E$ we have

$$x^2 \le \max\{(\sup E)^2, (\inf E)^2\}$$

If $\max\{(\sup E)^2, (\inf E)^2\} = (\sup E)^2$ then $\sup E \ge 0$ and $(\sup E)^2$ is an upper bound for F. Let C be any upper bound for F. Then $C \ge x^2 \Rightarrow \sqrt{C} \ge |x| \ge x$ for all $x \in E$. Hence $\sqrt{C}$ is an upper bound for E. Thus, $\sqrt{C} \ge \sup E \Rightarrow C \ge (\sup E)^2$. It follows that $(\sup E)^2 = \sup F$.

Similarly if $\max\{(\sup E)^2, (\inf E)^2\} = (\inf E)^2$ then $\sup F = (\inf E)^2$.

Problem 3.3

Let E be an non-empty subset of $\mathbb{R}$ which is bounded from above. Show that there exists a sequence $\{a_n\}$ such that $a_n \in E$ and $\lim\limits_{n\to\infty} a_n = \sup E$.

Solution.
We know that for each $\varepsilon > 0$ there exists at least one $a \in E$ such that $a > \sup E - \varepsilon$. Let $a_k \in E$ such that $a_k > \sup E - \dfrac{1}{k}$. Thus, $\dfrac{1}{k} > \sup E - a_k \ge 0$. Thus the sequence $\{a_n\}$ converges to $\sup E$ and $\lim\limits_{n\to\infty} a_n = \sup E$.

Problem 3.4

Let $a_1 = 4$ and define a_n inductively by

$$a_n = 4 - \frac{4}{a_{n-1}} \text{ for } n \geq 2$$

Show that $\lim_{n\to\infty} a_n = 2$.

Solution. Using induction one can prove that

$$a_n = 2 + \frac{2}{n}, \text{ for } n \geq 1$$

Thus, $\lim_{n\to\infty} a_n = 2$.

Problem 3.5

Let $T : \mathbb{R} \to \mathbb{R}$ be a contraction map and $x \in \mathbb{R}$ be a number. Define a sequence a_n by requiring $a_1 = x$ and $a_{n+1} = T(a_n)$.

1. Show that for any $m \in \mathbb{N}$, $|a_1 - a_m| \leq \frac{1}{1-\lambda}|a_1 - a_2|$
2. Show that a_n is a Cauchy sequence.

Solution.
Notice that,

$$\begin{aligned}
|a_n - a_{n+1}| &\leq \lambda |a_{n-1} - a_n| \\
&\leq \lambda^2 |a_{n-2} - a_{n-1}| \\
&\leq \qquad \vdots \\
&\leq \lambda^{n-1} |a_1 - a_2|
\end{aligned}$$

Therefore,

$$\begin{aligned}
|a_1 - a_m| &\leq |a_1 - a_{m-1}| + |a_{m-1} - a_m| \\
&\leq |a_1 - a_{m-2}| + |a_{m-2} - a_{m-1}| + |a_{m-1} - a_m| \\
&\leq \qquad \vdots \\
&\leq |a_1 - a_2| + \sum_{i=1}^{m-1} |a_i - a_{i+1}| \\
&\leq |a_1 - a_2| + \sum_{i=1}^{m-1} \lambda^i |a_1 - a_2| \\
&\leq |a_1 - a_2| \left(\sum_{i=0}^{m-1} \lambda^i \right) \\
&\leq |a_1 - a_2| \left(\frac{1 - \lambda^m}{1 - \lambda} \right) \leq \left(\frac{1}{1 - \lambda} \right) |a_1 - a_2|
\end{aligned}$$

For the second part, suppose $m > n$ then,

$$\begin{aligned}
|a_n - a_m| &\leq |a_n - a_{m-1}| + |a_{m-1} - a_m| \\
&\leq \qquad \vdots \\
&\leq \sum_{i=n}^{m-1} |a_i - a_{i+1}| \\
&\leq \sum_{i=n}^{m-1} \lambda^{i-1} |a_1 - a_2| \\
&\leq |a_1 - a_2| \cdot \left(\lambda^{n-1} \cdot \frac{1 - \lambda^{m-n}}{1 - \lambda} \right) \\
&\leq \frac{\lambda^{n-1}}{1 - \lambda} |a_1 - a_2|
\end{aligned}$$

So for any $\varepsilon > 0$ we can just choose a large n such that $\varepsilon > \dfrac{\lambda^{n-1}}{1 - \lambda} |a_1 - a_2|$.

4 Problem Set 4

Problem 4.1

Give an example of a sequence a_n that satisfies the following two conditions.

- a_n is divergent.
- For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_{n+1} - a_n| < \varepsilon \quad \text{for all } n \geq N.$$

Solution.
Take $a_n = \sum_{i=1}^{n} \frac{1}{i}$.

Problem 4.2

Let $p > 0$ be a positive number. Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

In the lecture we proved that the series diverges for $p = 1$ and converges for $p = 2$. Show that the series converges for $p > 1$ and diverges for $0 < p \leq 1$.

Solution.
For $0 < p \leq 1$ it is easy to see that $\frac{1}{n^p} \geq \frac{1}{n}$. From the comparison test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

For $p > 1$ we can use the same argument like we did for $p = 2$. We can group the terms and use the comparison test. I am skipping the details.

Problem 4.3

Let $r > 0$ be a positive number. Determine whether the series

$$\sum_{n=1}^{\infty} a_n$$

converges or diverges for the following cases.

1. $a_n = \sqrt{n+r} - \sqrt{n}$
2. $a_n = n^3 r^n$
3. $a_n = \frac{1}{n!} r^n$

The answer may depend on the value of r.

Solution.

For (1.), Let $r > 0$. Consider the series

$$\sum_{n=1}^{\infty} \left(\sqrt{n+r} - \sqrt{n}\right).$$

Rationalizing the terms, we have

$$\sqrt{n+r} - \sqrt{n} = \frac{r}{\sqrt{n+r} + \sqrt{n}}.$$

Since $\sqrt{n+r} \leq \sqrt{n} + \sqrt{r}$, it follows that

$$\sqrt{n+r} + \sqrt{n} \leq 2\sqrt{n} + \sqrt{r}.$$

Hence,

$$\sqrt{n+r} - \sqrt{n} = \frac{r}{\sqrt{n+r} + \sqrt{n}} \geq \frac{r}{2\sqrt{n} + \sqrt{r}}.$$

Choose N such that $2\sqrt{n} \geq \sqrt{r}$ for all $n \geq N$. Then for $n \geq N$,

$$2\sqrt{n} + \sqrt{r} \leq 3\sqrt{n},$$

and therefore

$$\sqrt{n+r} - \sqrt{n} \geq \frac{r}{3\sqrt{n}}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges, the comparison test implies that

$$\sum_{n=1}^{\infty} \left(\sqrt{n+r} - \sqrt{n}\right)$$

also diverges.

For (2.) we have and (3.) just do the ratio test.

Problem 4.4

Let b_n be a sequence of non-negative numbers which decreases to zero. That is,

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Let $a_n = (-1)^{n-1}b_n$. The purpose of this problem is to show that

$$\sum_{n=1}^{\infty} a_n$$

converges. This is called the *alternating series test*. Let

$$s_n = \sum_{k=1}^{n} a_k.$$

1. Show that s_{2k+1} is decreasing and that s_{2k} is increasing.
2. Show that s_{2k+1} is bounded from below and that s_{2k} is bounded from above.
3. Show that both $\lim_{k\to\infty} s_{2k+1}$ and $\lim_{k\to\infty} s_{2k}$ exist and are identical.
4. Show that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Solution.
For the first part notice that

$$a_{2k} + a_{2k+1} = b_{2k+1} - b_{2k} \leq 0$$

Thus $s_{2k+1} = s_{2k-1} + a_{2k} + a_{2k+1} \leq s_{2k-1}$.

Similarly,

$$a_{2k+1} + a_{2k+2} = b_{2k+1} - b_{2k+2} \geq 0$$

Thus, $s_{2k+2} = a_{2k+1} + a_{2k+2} + s_{2k} \geq s_{2k}$.

To show that $\{s_{2k+1}\}_{k=0}^{\infty}$ is bounded from below, we will show that $s_{2k+1} \geq 0$.

$$\begin{aligned} s_{2k+1} &= \underbrace{b_1 - b_2}_{\geq 0} + \underbrace{b_3 - b_4}_{\geq 0} + \cdots + \underbrace{b_{2k-1} - b_{2k}}_{\geq 0} + b_{2k+1} \\ &\geq b_{2k+1} \\ &\geq 0 \end{aligned}$$

Also,

$$\begin{aligned} s_{2k} &= b_1 - b_2 + b_3 - b_4 + \cdots + b_{2k-1} - b_{2k} \\ &\leq b_1 - b_2 + b_2 - b_3 + \cdots + b_{2k-2} - b_{2k-1} \\ &\leq b_1 - b_{2k-1} \\ &\leq b_1 \end{aligned}$$

Thus we're done with (2.).

For (3.), by **monotone convergence theorem**,

$$\lim_{n\to\infty} s_{2k+1} = \inf\{s_{2k+1}\} \quad \text{and} \quad \lim_{n\to\infty} s_{2k} = \sup\{s_{2k}\}$$

To show they are identical,

$$\begin{aligned} s_{2k+1} &= s_{2k} + a_{2k+1} = s_{2k} + b_{2k-1} \\ \implies \lim_{n\to\infty} s_{2k+1} &= \lim_{n\to\infty} s_{2k} \end{aligned}$$