Group Theory
Notes

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Content

1 Subgroups

First, we'll define some things which will come in handy later on.

1.1 Definitions and Examples

Definition 1.1. A subgroup H of G is subset of G such that every group axiom holds on H with the same group operation of G. If H is a subgroup of G we shall write $H \leq G$.

Proposition 1.1. A subset H of G is a subgroup of G if and only if

1. $H \neq \emptyset$ and
2. $\forall x, y \in H$, $xy^{-1} \in H$

Definition 1.2. A *kernel* of a function $\varphi : G \to H$ are groups, is the set

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e_H\}$$

where G and H are groups.

Proposition 1.2. A function $\varphi : G \to H$ is injective if and only if $\ker(\varphi) = \{0\}$.

Definition 1.3. A group action is a function $\mu : G \times A \to A$ such that

1. $\mu(g_1, \mu(g_2, a)) = \mu(g_1 \cdot g_2, a)$
2. $\mu(e, a) = a$

To make the notation simple enough, we abbreviate the notation of $\mu(g, a)$ to $g \cdot a$ or sometimes ga.

Remark. Instead of saying 'Group Action of G on A', we saying group G acts on the set A.

Definition 1.4. Let group G act on set A. A *stabilizer of* a, where $a \in A$, is a the set consisting of elements which fixes a. i.e

$$G_a = \{g \in G \mid g \cdot a = a\}$$

Proposition 1.3. The G_a is a subgroup of G for all $a \in A$.

Proposition 1.4. Let the group G act on a set A. The relation $\sim$ defined on A by

$$a \sim b \iff a = hb \quad \text{for some } h \in G$$

is a equivalence relation.

Definition 1.5. For each $a \in A$ the equivalence class under $\sim$ is called *orbit* of a under action of G. Thus,

$$\mathcal{O}(a) = \{x \in A \mid x \sim a\} = \{ha \mid h \in G\}$$

Definition 1.6. Let G be an abelian group. Define

$$t = \{g \in G \mid |g| < \infty\}$$

and call it the torsion subgroup of G.

You can check that the set is a subgroup of G.

Problems and Solutions

Problem : Find a non-abelian group G such that the set of all elements with finite order is not a subgroup of G.

Solution : $G = GL_2(\mathbb{Q})$. Take $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 2 \\ 1/2 & 0 \end{pmatrix}$.
Here, $|a| = |b| = 2$ but $|ab| = \infty$.

Problem : Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

Solution : The $(\Leftarrow)$ is pretty simple. For $(\Rightarrow)$, suppose neither of $H \subseteq K$ or $K \subseteq H$ is true. Then, there exists a element h, k s.t $h \in H$ and $h \notin K$ and $k \in K$ and $k \notin H$. But since $H \cup K$ is a subgroup, $h \cdot k$ must be either in H or K. If $h \cdot k \in H$ then $h^{-1} \cdot (h \cdot k) \in H$ and if $h \cdot k \in K$ then $(h \cdot k) \cdot k^{-1} \in K$

Problem : Let F be any field. Define

$$SL_n(F) = \{A \in GL_n(F) \mid \det(A) = 1\}$$

(called the *special linear group*). Prove that $SL_n(F) \leq GL_n(F)$.

Solution : If we use some basic properties of determinants we should be able to prove that this is a subgroup of the general linear group. We know that,

$$\det(AB) = \det(A) \cdot \det(B)$$

From this basic fact, we should be able to verify every subgroup axiom.

Problem : Prove that the intersection of arbitrary amount of non-empty collection of subgroups of G is also a subgroup of G.

Solution : Let us suppose

$$K = \bigcap G_i$$

where G_i are the subgroups.
Let us take $a \in K$. Since, $a \in K$ that implies that $a \in G_i$. Since, of them are subgroups $a^{-1} \in G_i$ thus $a^{-1} \in K$. It's easy to see that $e_G \in K$.Associativity is also pretty easy to check. If $a \in G$ and $b \in G$ then $ab \in G$ as $a, b \in G_i$ which means $ab \in G_i$.

Problem : Let A be an abelian group and fix some $n \in \mathbb{Z}$. Prove that the following subsets are subgroup of A,

1. $\{a^n \mid a \in A\}$
2. $\{a \in A \mid a^n = 1\}$

Solution : The problem can be easily solved if we know a facts about abelian group. Let $a, b \in A$ then

1. $(ab)^n = a^n b^n$
2. $(a^{-1})^n = (a^n)^{-1}$ *(This is true in general for all group A)*

Problem : Let H be a subgroup of additive group of rational numbers with the property that $1/x \in H$ for every non-zero element of $x \in H$. Prove that $H = 0$ or $H = \mathbb{Q}$.

Solution : If H has no non-zero element then $H = \{0\}$. If H has has a non-zero element x then $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. Since, $\frac{a}{b} \in H$ then $a \in H$ because of the additive nature of H. Since, $a \in H$ we have $\frac{1}{a} \in H$, thus $1 \in H$ as $a \cdot \frac{1}{a} = 1$. Since, $1 \in H$ we must have $-1 \in H$ as well. Thus, every integer a is in H. Since, $a \in H$ we have $\frac{1}{a} \in H$ thus, $\frac{b}{a} \in H$ for any $b \in \mathbb{Z}$. Thus, every rational number can be obtained by this method. Thus, $\mathbb{Q} = H$.

Problem : Show that $\{x \in D_{2n} \mid x^2 = 1\}$ is not a subgroup of D_{2n} $(n > 2)$.

Solution : We know that $(r^k s)^2 = 1$ for all $0 \leq k \leq n$. Thus, $(r^k s)(r^j s) = r^k(sr^j)s = r^{k-j}$. Thus $r^k = r^j \implies k = j$. But since $n > 2$ we can take different k, j. Thus, the set is not closed and thus it cannot be a subgroup of $D_2 n$.

Remark. Here, if you do not know about the Dihedral groups, then it might be little confusing but, essentially

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}$$

This Dihedral group is the set of symmetries of a regular n-gon. You can check that

$$sr^k = r^{-k}s$$

1.2 Centralizers and Normalizers, Stabilizer and Kernels

We'll look at some important families of subgroups. Let A be any non-empty subset of G.

Definition 1.7. Define $C_G(A) = \{g \in G \mid gag^{-1} = a \ \forall a \in A\}$. This subset is called *centralizers* of A in G. The set is the collection of elements in G which commutes with every element of A.

Proposition 1.5. $C_G(A) \leq G$

Proof. One can check that $1 \in C_G(A)$. Suppose $x \in C_G(A)$, then $xax^{-1} = a \implies a = x^{-1}ax$. Thus, $x^{-1} \in C_G(A)$. Let $x, y \in C_G(A)$ then

$$\begin{aligned}(xy)a(xy)^{-1} &= (xy)a(y^{-1}x^{-1}) \\ &= x(yay^{-1})x^{-1} \\ &= xax^{-1} \\ &= a\end{aligned}$$

Thus, $xy \in C_G(A)$. □

Definition 1.8. Define $Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\}$. This is the set of elements in G such that it commutes with every other element of G. This is called the *center* of G.

Remark. You may notice that $Z(G) = C_G(G)$. Thus, $Z(G) \leq G$.

Definition 1.9. Define $gAg^{-1} = \{gag^{-1} \in G \mid a \in A\}$. Define the *normalizers* of A in G to be the set

$$N_G(A) = \{g \in G \mid gAg^{-1} = A\}$$

Proposition 1.6. $N_G(A) \leq G$ and $C_G(A) \leq N_G(A)$.

Proof. Similar proof like **Proposition 1.4**. □

Examples

1. Let G be abelian group. Then $Z(G) = G$, also $N_G(A) = C_G(A) = G$.

2. Let $G = D_8$. And let $A = \{1, r, r^2, r^3\}$ be a subgroup of rotations in D_8. We show that $C_{D_8}(A) = A$. Since, powers of r commute with each other, we have $A \leq C_{D_8}(A)$. One can check $s \notin C_{D_8}(A)$ as $sr = r^{-1}s \neq rs$. Now, if $a \notin A$ and $a \in D_8$ then a must be of the form sr^i. If $a \in C_{D_8}(A)$ with $a = sr^i$ then $s = (sr^i)(r^{-i})$, thus a contradiction.

Stabilizer and Kernels of a Group Action

We have already seen what a stabilizer is, now lets look at what a kernel of a group action is.

Definition 1.10. A *kernel* of a group G acting on a set A is the set of elements in g such that it fixes every element in A. That is, if $\phi : G \times A \to A$ is a group action then

$$\ker(\phi) = \{g \in G \mid g \cdot s = s \ \forall s \in A\}$$

Proposition 1.7. Kernel of a group action is a subgroup of the group G.

Proof. If $g \in \ker(\phi)$ then $g \cdot s = s \implies g^{-1} \cdot (g \cdot s) = g^{-1} \cdot s \implies s = g^{-1} \cdot s$. Thus, $g^{-1} \in \ker(\phi)$. Similarly, you can verify other axioms. $\square$

We'll see that the centralizers, normalizers and kernels are some special case of facts that stabilizer and kernels of actions are subgroups. Let $S = P(G)$ be the collection of all the subsets of group G, and let G act on S by *conjugation* i.e

$$\phi : G \times S \to S \quad \text{where} \quad g \cdot A = gAg^{-1}$$

where gAg^{-1} is defined just like in **Definition 1.9.**

Under this action, the stabilizer of A is same as normalizer of A i.e $N_G(A) = G_A$. This is basically of the definition, $N_G(A) = \{g \in G \mid gAg^{-1} = A\} = \{g \in G \mid g \cdot A = A\} = G_A$. Thus, $N_G(A) \leq G$.

Next Let the group $N_G(A)$ act on $A \subseteq G$ by conjugation. One can check that the centralizer of A is the same as kernel of this action. Thus, $C_G(A) = \ker(\phi) \leq N_G(A)$ and from the above argument $C_G(A) \leq N_G(A) \leq G \implies C_G(A) \leq G$. One can also check that G acting on G by conjugation has kernel same as the center of the group i.e $Z(G)$ thus, $Z(G) \leq G$

Problems and Solutions

Problem : Prove that $C_G(Z(G)) = G$ and $N_G(Z(G) = G$.

Solution : We already know that $C_G(Z(G))$ and $N_G(Z(G))$ are the subgroups of G. Thus, if we prove every element of $C_G(Z(G))$ and $N_G(Z(G))$ is also an element of G then we're done. Let $a \in Z(G)$, then $ga = ag$ for any $g \in G$. Thus, $g \in C_G(Z(G))$. Since, $gZ(G)g^{-1} = \{gag^{-1} \mid a \in Z(G)\} = \{a \mid a \in Z(G)\} = Z(G)$. Thus, for any $g \in G$ we have $gZ(G)g^{-1} = Z(G)$ which means that $N_G(Z(G))$ collects all the $g \in G$. Thus, $N_G(Z(G)) = G$.

Problem : If A and B are the subsets of G such that $A \subseteq B$ then $C_G(B) \leq C_G(A)$.

Solution : Every element of $C_G(B)$ is in $C_G(A)$ as $xb = ba$ for all $b \in B$ so $xa = ax$ for all $a \in A$ thus, $x \in C_G(A)$. Thus, we are done.

Problem : Let H be a subgroup of G.

1. Show that $H \leq N_G(H)$.
2. Show that $H \leq C_G(H) \iff H$ is abelian.

Solution : For the first part, $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$. If we let $g \in H$ be an arbitrary element then $\{ghg^{-1} \mid h \in H\} = H$. This, can be proved by proving $\varphi_g : H \to H$ is a bijection for $g \in H$. Since, g is arbitrary $H \leq N_G(H)$.
For the second part, if H is abelian then $ga = ag$ for every $a, g \in H$ thus $H \leq C_G(H)$. If $H \leq C_G(H)$ then $ga = ag$ for all $a \in H$ and since H is a subgroup of $C_G(H)$ every element of H is in $C_G(H)$ that means $ga = ag$ for every $a, g \in H$. Thus H is abelian.

Problem : Let $n \in \mathbb{Z}$ and $n \geq 3$. Prove the following

1. $Z(D_{2n}) = \{1\}$ if n is odd
2. $Z(D_{2n}) = \{1, r^k\}$ if $n = 2k$

Solution : We know that only elements that commute with powers of r are powers of r. Thus, r^i be the element that commutes with every element of D_{2n}. Then, $r^i(sr^i) = (sr^i)r^i \implies r^i(r^{-i}s) = sr^{2i} \implies s = sr^{2i} \implies r^{2i} = 1 \implies n \mid i$ if n is odd. But $i < n$ so $i = 0$. If $n = 2k$ then $n \mid 2i \implies 2i = nk$ but $2i < 2n \implies 2 > k \implies k = 1$. Thus $i = n/2$.

Problem : Let $G = S_n$ and fix an $i \in \{1, 2, 3, \ldots, n\}$ and let $G_i = \{\sigma \in G \mid \sigma(i) = i\}$. Prove that G_i is a subgroup of G and find $|G_i|$.

Solution : The subgroup part of this is pretty easy. To find, $|G_i|$ we fix the map $i \to i$ and let the other maps vary. The number of ways to do this is $(n-1)!$ and this is the size of the group.

Problem : For any subgroup H of G and for any non-empty subset of A in G define $N_H(A) = \{h \in H \mid hAh^{-1} = A\}$. Show that $N_H(A) = N_G(A) \cap H$ and deduce that $N_H(A)$ is a subgroup of H.

Solution : $N_H(A)$ collects every $h \in H$ for which $hAh^{-1} = A$. $N_G(A) \cap H$ also collects $h \in H$ for which $hAh^{-1} = A$ thus $N_G(A) \cap H = N_H(A)$. To deduce $N_H(A)$ is a subgroup of H, you can easily check the axioms.

Problem : Let H be a subgroup of order 2 in G. Show that $N_G(H) = C_G(H)$. Deduce that if $N_G(H) = G$ then $H \leq Z(G)$.

Solution : Since, H has order 2 H must be $\{e, h\}$ where $h \neq e$ and $h^2 = e$. Now, if $gHg^{-1} = H$ then $\{ghg^{-1} \mid g \in G\} = \{e, ghg^{-1}\} = \{e, h\} \implies gh = hg$. Thus, $N_G(H)$ collects $g \in G$ which commutes with h which is exactly $C_G(H)$. For the second part, since $N_G(H) = G$ that means h commutes with every $g \in G$. Thus, $\{e, h\} \subseteq Z(G)$ and $H \leq Z(G)$.

Problem : Prove that $Z(G) \leq N_G(A)$ for any subset A of G.

Solution : Since $Z(G)$ collects every $g \in G$ such that it commutes with every other element of G, it must commute with every element of A. Thus, $gAg^{-1} = \{gag^{-1} \mid g \in Z(G)\} = \{a \mid g \in Z(G)\} = A$ which means every $g \in Z(G)$ is also an element of $N_G(A)$.

1.3 Cyclic and Cyclic Subgroups

Definition 1.11. A group G is called *cyclic* if if can be generated by a single element i.e there is some $x \in G$ such that $H = \{x^n \mid n \in \mathbb{Z}\}$. We write is as $G = \langle x \rangle$ and say G is generated by x.

Proposition 1.8. If $H = \langle x \rangle$ then $|H| = |x|$.

Proof. Suppose $|x| = n < \infty$ then $1, x, \ldots, x^{n-1}$ are all distinct. Thus $|H|$ is at least n. Now, using the division algorithm we can show that these are all of them.
Suppose now $|x| = \infty$ then that means there is no finite $n \in \mathbb{Z}$ s.t $x^n = 1$. If $x^b = x^c$ then $x^{b-c} = 1$ contradicting the fact that there is no n s.t $x^n = 1$. Thus, all of the powers of x are different and thus $|H| = \infty$. □

Proposition 1.9. Let G be a group and let $x \in G$. If $x^m = 1$ for some $m \in \mathbb{Z}$ then $|x|$ divides m.

Proposition 1.10. Any two cyclic group of same order are isomorphic. More specifically,

1. If $n \in \mathbb{Z}^+$ and $\langle x \rangle$ and $\langle y \rangle$ are both cyclic groups of order n, then the map

$$\varphi : \langle x \rangle \to \langle y \rangle$$

$$x^k \to y^k$$

is well defined and is an isomorphism.

2. If $\langle x \rangle$ is an infinite cyclic group then

$$\varphi : \mathbb{Z} \to \langle x \rangle$$

$$k \to x^k$$

is well defined and is an isomorphism.

Proposition 1.11. Let G be a group, let $x \in G$ and let $a \in \mathbb{Z} \setminus \{0\}$ then

1. If $|x| = \infty$ then $|x^a| = \infty$
2. If $|x| = n < \infty$ then $|x^a| = \frac{n}{(n,a)}$

Proof. 1. is pretty simple. Suppose $|x^a| = k$ then $x^{ak} = 1$ now $n \mid ak$. Write $(n, a) = d$ and $n = du$ and $a = dv$. Then, $du \mid dvk \implies u \mid k$. But

$$x^a = x^{dv} \implies (x^{du})^v = (x^n)^v = 1$$

$$\implies (x^a)^u = 1$$

$$\implies k \mid u$$

. Thus, $k = u \implies n = dk \implies k = \frac{n}{d} = \frac{n}{(n,a)}$. □

Proposition 1.12. Let $H = \langle x \rangle$.

1. Assume $|x| = \infty$. Then $H = \langle x^a \rangle \iff a = \pm 1$.

2. Assume $|x| = n < \infty$. Then $H = \langle x^a \rangle \iff (a, n) = 1$. The number of generators of H is $\varphi(n)$.

Proof. For 1. we know $x \in \langle x^a \rangle$ as $\langle x^a \rangle = \langle x \rangle$ thus

$$x = x^{ak} \implies ak = 1 \implies a = \pm 1$$

For 2. we know $H = \langle x^a \rangle \implies |H| = |x^a| \iff |x| = |x^a|$,

$$\iff \frac{n}{(n,a)} = n$$

$$\iff (n,a) = 1$$

Since, the number of positive integers less than n and co-prime to n are exactly $\varphi(n)$, thus the number of generators are exactly equal to $\varphi(n)$. □

Theorem 1.1. *Let $H = \langle x \rangle$ be a cyclic group.*

1. *Every subgroup of H is cyclic. More precisely, if $K \leq H$, then either $K = \{1\}$ or $K = \langle x^d \rangle$, where d is the smallest positive integer such that $x^d \in K$.*
2. *If $|H| = \infty$, then for any distinct nonnegative integers a and b, $\langle x^a \rangle \neq \langle x^b \rangle$.*
3. *If $|H| = n < \infty$, then for each positive integer a dividing n there is a unique subgroup of H of order a. This subgroup is the cyclic group $\langle x^d \rangle$, where $d = \frac{n}{a}$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{(n,m)} \rangle$, so that the subgroups of H correspond bijectively with the positive divisors of n.*

Proof. 1. and 2. are pretty easy. For 3. the cyclic group $\langle x^{n/a} \rangle$ has order a. To prove uniqueness, suppose K is any subgroup of H with order a, then $\langle x^b \rangle = K$ where b is the smallest positive integer b s.t $x^b \in K$(this is from 1.). Thus,

$$|\langle x^{n/a} \rangle| = |\langle x^b \rangle| \implies \frac{n}{d} = a = \frac{n}{(n,b)}$$

$$\implies d = (n,b) \implies d \mid b$$

Hence, $\langle x^b \rangle \leq \langle x^d \rangle$ and since they both have same order $\langle x^b \rangle = \langle x^d \rangle$.
For the assertion on 3., one can prove that $\langle x^m \rangle \leq \langle x^{(n,m)} \rangle$ as $(n,m) \mid n$, and since they have same order $\langle x^m \rangle = \langle x^{(n,m)} \rangle$. This means that the number of subgroups has a bijection with the divisors of n. □

Problems and Solutions

Problem : Find all subgroup of $\mathbf{Z}_{45} = \langle x \rangle$, giving a generator of each. Describe the containment between these subgroups.

Solution : There are exactly 6 different subgroups of $\mathbf{Z}_{45}$. Since there is a one to one correspondence between divisors of 45 and subgroups of $\mathbf{Z}_45$ we can list all of them,

$$\{1\}, \langle r \rangle, \langle r^3 \rangle, \langle r^5 \rangle, \langle r^9 \rangle, \langle r^15 \rangle$$

Problem : If x is an element of a finite group G and $|x| = |G|$. Prove that $G = \langle x \rangle$.

Solution : Let $|x| = n$. We know that $\{1, x, \dots, x^{n-1}\}$ is a subgroup of G. Since, it is a subgroup and has the same order as G thus $G = \{1, x, \dots, x^{n-1}\} = \langle x \rangle$.

Problem : Let $\mathbf{Z}_{48} = \langle x \rangle$ and use isomorphism $\mathbb{Z}/48\mathbb{Z} \cong \mathbf{Z}_{48}$ with $[1] \mapsto x$ to find all the subgroups of Z_{48}.

Solution : If $\langle [x] \rangle$ is a cyclic subgroup of $\mathbb{Z}/48\mathbb{Z}$ then $\langle \varphi([x]) \rangle$ is a subgroup of $\mathbf{Z}_{48}$ where φ is the isomorphic map.

Problem : Let $\mathbf{Z}_{48} = \langle x \rangle$. For which integer a does the map φ_a defined by $\varphi_a : [1] \mapsto x^a$ extends to an isomorphism from $\mathbb{Z}/48\mathbb{Z}$ to $\mathbf{Z}_{48}$.

Solution : We already know it is an homomorphism as

$$\varphi([u] + [v]) = (x^a)^{u+v} = (x^a)^u (x^a)^v = \varphi([u])\varphi([v])$$

But to be an isomorphism x^{na} needs to cover $\mathbf{Z}_{48}$ for all $n \in \mathbb{Z}$. Thus,

$$\mathbf{Z}_{48} = \langle x \rangle = \langle x^a \rangle$$

$$\implies (48, a) = 1$$

So, for all the a which are co-prime to 48 the map, φ_a is an isomorphism.

Problem : Let $\mathbf{Z}_{36} = \langle x \rangle$. For which integer a does the map $\psi_a : [1] \mapsto x^a$ extend to an well defined homomorphism from $\mathbb{Z}/48\mathbb{Z}$ onto $\mathbf{Z}_{36}$. Can ψ_a ever be surjective?

Solution : One can check that the map is a homomorphism. Now, we need to show that

$$[u] = [v] \implies \psi_a([u]) = \psi_a([v])$$

If $[u] = [v]$ then $u - v = 48m$

$$1 = \psi_a([0]) = \psi_a([u - v]) = x^{a(u-v)} = x^{48am}$$

$$\implies 36 \mid 48am$$

$$\implies 3 \mid am$$

Since $3 \mid am$ must hold for all integer m, if $3 \nmid a$ then $3 \mid m$ for all integer m which is clearly absurd thus $3 \mid a$. Thus, $x^{48 \cdot 3k \cdot m} = 1$ as $36 \mid 144km$. Thus,

$$x^{a(u-v)} = 1 \implies x^{au} = x^{av} \implies \psi_a([u]) = \psi_a([v])$$

Problem : Find a presentation for $\mathbf{Z}_n$ with one generator.

Solution : $\mathbf{Z}_n = \langle r \mid r^n = 1 \rangle$.

Problem : Show that if H is any group with $h^n = 1$ then there exists a unique homomorphism from $\mathbf{Z}_n = \langle x \rangle$ to H such that $x \mapsto h$.

Solution : Define $\psi : \mathbf{Z}_n \to H$ by $\psi(x^k) = h^k$. This is a homomorphism and is unique because the output is completely determined by h.

Problem : Show that if H is any group and h is an element of H, then there is a unique homomorphism from $\mathbb{Z}$ to H such that $1 \to h$.

Solution : Define $\psi : \mathbb{Z} \to H$ by $\psi(k) = h^k$. This is a homomorphism and is unique as the output is completely determined by h.

Problem : Let p be a prime and n be a positive integer. Show that if x is an element of the group G such that $x^{p^n} = 1$ then $|x| = p^m$ for some $m \leq n$.

Solution : We know that if $x^n = 1$ then $|x|$ must divide n. Thus, $|x|$ must divide p^n but the only divisors of p^n are powers of p. Thus, $|x| = p^m$ for some $m \leq n$.

Problem : Show that $(\mathbb{Z}/2^n\mathbb{Z})^\times$ is not cyclic.

Solution : Consider $\{1, -1\}$ and $1, 1 + 2^{n-1}$. They both are subgroups of order 2. But a cyclic group has exactly 1 subgroup of order d, where d is the divisor of order of the cyclic group G. But we found two distinct subgroups of the group with same order.

Problem : Let G be a finite group and let $x \in G$.

1. Prove that if $g \in N_G(\langle x \rangle)$ then $gxg^{-1} = x^a$ for some $a \in \mathbb{Z}$.
2. Prove conversely that if $gxg^{-1} = x^a$ for some $a \in \mathbb{Z}$ then $g \in N_G(\langle x \rangle)$. [Show first that $gx^kg^{-1} = (gxg^{-1})^k = x^{ak}$ for any integer k, so that $g\langle x \rangle g^{-1} \leq \langle x \rangle$. If x has order n, show the elements gx^ig^{-1}, $i = 0, 1, \ldots, n-1$, are distinct, so that $|g\langle x \rangle g^{-1}| = |\langle x \rangle| = n$ and conclude that $g\langle x \rangle g^{-1} = \langle x \rangle$.]

Problem : Let G be a cyclic group of order n and let k be an integer relatively prime to n. Prove that the map $x \mapsto x^k$ is surjective. Use Lagrange's Theorem (Exercise 19, Section 1.7) to prove the same is true for any finite group of order n. (For such k each element has a kth root in G. It follows from Cauchy's Theorem in Section 3.2 that if k is not relatively prime to the order of G then the map $x \mapsto x^k$ is not surjective.)

Problem : Let $\mathbf{Z}_n$ be a cyclic group of order n and for each integer a let

$$\sigma_a : \mathbf{Z}_n \to \mathbf{Z}_n \quad \text{by} \quad \sigma_a(x) = x^a \quad \text{for all } x \in Z_n.$$

1. Prove that σ_a is an automorphism of Z_n if and only if a and n are relatively prime.
2. Prove that $\sigma_a = \sigma_b$ if and only if $a \equiv b \pmod{n}$.

3. Prove that every automorphism of Z_n is equal to σ_a for some integer a.

4. Prove that $\sigma_a \circ \sigma_b = \sigma_{ab}$. Deduce that the map $a \mapsto \sigma_a$ is an isomorphism of $(\mathbb{Z}/n\mathbb{Z})^\times$ onto the automorphism group of Z_n (so $\mathrm{Aut}(Z_n)$ is an abelian group of order $\varphi(n)$).

1.4 Subgroups Generated by Subsets of a Group

Proposition 1.13. If $\mathcal{A}$ is any non empty collection of subsets of G then the intersection of all members of $\mathcal{A}$ is also a subgroup of G.

Proof. Trivial. $\square$

Definition 1.12. If A is any subset of group G define

$$\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H$$

This is called subgroup generated by A.

Definition 1.13. Let $A = \{a_1, \ldots, a_n\}$ then define

$$\bar{A} = \{a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_n^{\epsilon_n} \mid n \in \mathbb{Z}_{\geq 0}, a_i \in A, \epsilon_i = \pm 1\}$$

where $\bar{A} = \{1\}$ if $A = \emptyset$.

Remark. Here, a_i's need not to be distinct.

Proposition 1.14. $\bar{A} = \langle A \rangle$

Proof. First we prove that $\bar{A}$ is a subgroup. Note that $\bar{A} \neq \emptyset$. If $a, b \in \bar{A}$ then write $a = a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ and $b = b_1^{\delta_1} \cdots b_m^{\delta_m}$ then one can check that $ab^{-1} \in \bar{A}$. Thus, $\bar{A}$ is a subgroup of G.
Now, since $A \subseteq \bar{A}$ as $a = a^1$ for every $a \in A$, we can say that $\langle A \rangle \subseteq \bar{A}$. It is because $\langle A \rangle$ is the intersection of all the subgroups containing A. Now, since $\langle A \rangle$ contains A and is a group, it must contain every element of form $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ thus $\bar{A} \subseteq \langle A \rangle$. This completes the proposition. $\square$

Problems and Solutions

1. Prove that if H is a subgroup then $\langle H \rangle = H$.

Solution : From the definition,

$$\langle H \rangle = \bigcap_{\substack{H \subseteq K \\ K \leq G}} K$$

Since, $H \subseteq H$ and $H \leq G$ thus $\langle H \rangle \subseteq H$. But also $H \subseteq \langle H \rangle$.

2. Prove if A is a subset of B then $\langle A \rangle \leq \langle B \rangle$. Give an example of $A \subseteq B$ with $A \neq B$ but $\langle A \rangle = \langle B \rangle$.

Solution : From the definition we have,

$$\langle B \rangle = \bigcap_{\substack{B \subseteq K \\ K \leq G}} K$$

Since, $A \subseteq B$ we have $\langle A \rangle \leq \langle B \rangle$. For the example, take $G = D_{16}$ and $A = \{r\}$ and $B = \{r, r^3\}$.

3. Prove if H is an abelian subgroup of G then $\langle H, Z(G) \rangle$ is abelian. Give and explicit example of a abelian subgroup H such that $\langle H, C_G(H) \rangle$ is not abelian.

Solution : We know that, $\bar{A} = \langle A \rangle$ thus

$$\langle H, Z(G) \rangle = \{a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \mid e_i = \pm 1, n \in \mathbb{Z}_{\geq 0}, a_i \in H \cup Z(G)\}$$

So, if you take two elements from $\langle H, Z(G) \rangle$, they will commute thus $\langle H, Z(G) \rangle$ is an abelian group. For the example, choose $H = \{1, r^2\}$ and $G = D_8$.

3. Prove that H is a subgroup then H is generated by $H - \{1\}$.

Solution : Since, we know that $\bar{A} = \langle A \rangle$ thus

$$\langle H - \{1\} \rangle = \{a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \mid a_i \in H - \{1\}, n \in \mathbb{Z}_{\geq 0}, \epsilon_i = \pm 1\}$$

Thus, for $a \in H$ and $a \neq 1$, $a \in \langle H - \{1\} \rangle$ and also $1 = a^1 a^{-1} \in \langle H - \{1\} \rangle$. Also, $a \in \langle H - \{1\} \rangle$ is just some combination of elements in H thus $a \in H$. Thus, we have

$$\langle H - \{1\} \rangle = H$$

4. Prove that the multiplicative group of positive rational numbers is generated by the set $\{ \frac{1}{p} \mid p \text{ is a prime } \}$.

Solution : Since,

$$\left\langle \left\{ \frac{1}{p} \mid p \text{ is a prime} \right\} \right\rangle = \left\{ \frac{p_1^{a_1} \cdots p_n^{a_n}}{q_1^{b_1} \cdots q_m^{a_m}} \mid p_i, q_i \in \textbf{Primes} \right\} = \mathbb{Q}_{>0}$$

5. A group H is called *finitely generated* if there is a finite set A such that $H = \langle A \rangle$.

(a) Prove that every finite group is finitely generated.

(b) Prove that $\mathbb{Z}$ is finitely generated.

(c) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic. [If H is a finitely generated subgroup of $\mathbb{Q}$, show that $H \leq \langle \frac{1}{k} \rangle$, where k is the product of all the denominators which appear in a set of generators for H.]

(d) Prove that $\mathbb{Q}$ is not finitely generated.

6. Exhibit a proper subgroup of $\mathbb{Q}$ which is not cyclic.

7. A subgroup M of a group G is called a *maximal subgroup* if $M \neq G$ and the only subgroups of G which contain M are M and G.

(a) Prove that if H is a proper subgroup of the finite group G then there is a maximal subgroup of G containing H.

(b) Show that the subgroup of all rotations in a dihedral group is a maximal subgroup.

(c) Show that if $G = \langle x \rangle$ is a cyclic group of order $n \geq 1$ then a subgroup H is maximal if and only if $H = \langle x^p \rangle$ for some prime p dividing n.

8. This is an exercise involving Zorn's Lemma (see Appendix I) to prove that every nontrivial finitely generated group possesses maximal subgroups. Let G be a finitely generated group, say $G = \{g_1, g_2, \ldots, g_n\}$, and let $\mathcal{S}$ be the set of all proper subgroups of G. Then $\mathcal{S}$ is partially ordered by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{S}$.

(a) Prove that the union, H, of all the subgroups in $\mathcal{C}$ is a subgroup of G.

(b) Prove that H is a proper subgroup. [If not, each g_i must lie in H and so must lie in some element of the chain $\mathcal{C}$. Use the definition of a chain to arrive at a contradiction.]

(c) Use Zorn's Lemma to show that $\mathcal{S}$ has a maximal element (which is, by definition, a maximal subgroup).

9. Let p be a prime and let

$$Z = \{z \in \mathbb{C} \mid z^{p^m} = 1 \text{ for some } n \in \mathbb{Z}^+\}$$

(so Z is the multiplicative group of all p-power roots of unity in $\mathbb{C}$). For each $k \in \mathbb{Z}^+$ let

$$H_k = \{z \in Z \mid z^{p^k} = 1\}$$

(the group of p^kth roots of unity). Prove the following:

(a) $H_k \leq H_m$ if and only if $k \leq m$.

(b) H_k is cyclic for all k (assume that for any $n \in \mathbb{Z}^+$, $\{e^{2\pi i t/n} \mid t = 0, 1, \ldots, n-1\}$ is the set of all nth roots of 1 in $\mathbb{C}$).

(c) Every proper subgroup of Z equals H_k for some $k \in \mathbb{Z}^+$ (in particular, every proper subgroup of Z is finite and cyclic).

(d) Z is not finitely generated.

10. A nontrivial abelian group A (written multiplicatively) is called *divisible* if for each element $a \in A$ and each nonzero integer k there is an element $x \in A$ such that $x^k = a$, i.e., each element has a kth root in A (in additive notation, each element is the kth multiple of some element of A).

(a) Prove that the additive group of rational numbers, $\mathbb{Q}$, is divisible.

(b) Prove that no finite abelian group is divisible.

11. Prove that if A and B are nontrivial abelian groups, then $A \times B$ is divisible if and only if both A and B are divisible groups.

2 Quotients Groups and Homomorphism

2.1 Definition and Examples

Definition 2.1. Let $\varphi : G \to H$ be a homomorphism. A *fiber* over a, where $a \in \text{im}(\varphi)$, is the set of elements in G that gets mapped to a under φ i.e

$$X_a = \{g \in G \mid \varphi(g) = a\}$$

It is also denoted by $\varphi^{-1}(a)$.

Definition 2.2. We define the product of *fibers* as following

$$X_a \cdot X_b = \{g_1 g_2 \mid g_1 \in X_a, g_2 \in X_b\}$$

Remark. By definition of the product of fibers, we can see that

$$X_a \cdot X_b = X_{ab}$$

Proposition 2.1. The set of *fibers* over the elements of $\text{im}(\varphi)$ forms a group.

Proof. The identity element of the set is going to be X_{1_H}. The inverse of X_a is going to be $X_{a^{-1}}$. And one can check that the associativity the closure property holds. □

Definition 2.3. If φ is a homomorphism $\varphi : G \to H$, the *kernel* of φ is the set

$$\{g \in G \mid \varphi(g) = 1_H\}$$

and will be denoted by $\ker \varphi$.

Remark. The kernel of φ is the same as the fiber over 1_h i.e

$$\ker \varphi = X_{1_H}$$

Definition 2.4. Let $\varphi : G \to H$ be a homomorphism with kernel K. The *quotient group* or the *factor group*, G/K (read as G *mod* K), is the group whose elements are the *fibers* of φ.

Proposition 2.2. Let $\varphi : G \to H$ be a homomorphism of groups with kernel K. Let $X \in G/K$ be the fiber above a. i.e $X = \varphi^{-1}(a)$

1. For any $u \in X$, $X = \{uk \mid k \in K\} = uK$
2. For any $u \in X$, $X = \{ku \mid k \in K\} = Ku$

Proof. We'll prove 2. and leave 1. for the future me. Suppose $k \in K$ then

$$\begin{aligned}\varphi(ku) &= \varphi(k)\varphi(u) \\ &= 1 \cdot \varphi(u) \\ &= a\end{aligned}$$

Thus, $ku \in X \implies Ku \subseteq X$. Now, to show $X \subseteq Ku$, take any $g \in X$ then define $k := gu^{-1}$ thus

$$\begin{aligned}\varphi(k) = \varphi(gu^{-1}) &= aa^{-1} = 1 \\ &\implies k \in K\end{aligned}$$

Thus, $g = ku \in Ku \implies X \subseteq Ku$. □

Remark. Any coset(check below for the definition) is also an *fiber* for some element. It is because

$$uK = \varphi^{-1}(\varphi(u)) = uK$$

Definition 2.5. For a $N \leq G$ and any $g \in G$ let

$$gN = \{gn \mid n \in N\} \quad \text{and} \quad Ng = \{ng \mid n \in G\}$$

be the *left coset* and *right coset* of N in G. Any element of a coset is called a *representative* of the coset.

Remark. To verify a map is a well defined map, one can't just use the condition imposed on the map. For example, the proof of the theorem below, I have said the operation is indeed well-defined but didn't prove it. You can't go about doing the following.
Let $uK = u'K$ and $vK = v'K$ thus

$$\begin{aligned}(uv)K &= (uK)(vK) \\ &= (u'K)(v'K) \\ &= (u'v')K\end{aligned}$$

Here, you're assuming that $a = b \implies f(a) = f(b)$ which is true if f were to be a function. But to be a function, it needs to be well defined. Therefore you're assuming it's well defined to begin with.

Theorem 2.1. *Let G be a group and let K be the kernel of some homomorphism from G to another group. Then the set whose elements are left cosets of K in G with operation defined by*

$$(uK) \circ (vK) = (uv)K$$

forms a group, G/K.

Proof. One can check that the set whose elements are left cosets of K in G with operation defined above, does indeed form a group. Note that the operation is also well defined. Now, if X and Y are fibers then $Z = XY$ is also a fiber. Now, we can write each fiber as

$$X = uK, \quad Y = vK, \quad XY = jK$$

But we set $j = uv$ as $uv \in XY$. Thus, every element of G/K is in set $\{uK \mid u \in G\}$. But we also that every coset is also a fiber thus, every element of $\{uK \mid u \in G\}$ is in G/K. □

Proposition 2.3. Let N be any subgroup of the group G. The set of left cosets of N in G form a partition of G. Furthermore, for all $u, v \in G$, $uN = vN \iff v^{-1}u \in N$ and in particular, $uN = vN$ if and only if u and v are representative of the same coset.

Proof. Since, $g \in gN$ as $1 \in N$, we can say that

$$g = \bigcup_{g \in G} gN$$

Now, if $x \in uN \cap vN$ then

$$x = un = vm$$

Thus, $u = vmn^{-1} \implies ut = vmn^{-1}t \in vN$. Thus, $uN \subseteq vN$ as ut covers every element of uN.
Now, one can reverse the roles and prove $vN \subseteq uN$, which altogether implies $uN = vN$. Thus, if $uN \cap vN \neq \emptyset$ then $uN = vN$.
For the other part of the proposition, $uN = vN \iff u = vn \iff v^{-1}u = n \in N$.
If $uN = vN = K$ then $u, v \in K$. Thus they are the representative of the same coset. Also, if $u \in tN$ and $v \in tN$ then $uN = tN = vN$. □

Proposition 2.4. Let G be a group and let N be a subgroup of G.

1. The operation on the set of left cosets of N on G defined by
$$(uN) \cdot (vN) = (uv)N$$
is well defined if and only if $gng^{-1} \in N$ for all $g \in G$ and for all $n \in N$.

2. If the above operation is well-defined then the it makes the set of left coset into a group.

Proof. Assume the operation is well-defined i.e $u, u_1 \in uN,\ v, v_1 \in vN \implies uvN = u_1v_1N$. Let g be an arbitrary element of G and let n be an arbitrary element of N. Then, set $u = 1, u_1 = n$ and $v_1 = v = g^{-1}$ thus

$$1g^{-1}N = ng^{-1}N \implies g^{-1}N = ng^{-1}N$$

Thus, $ng^{-1} \in g^{-1}N \implies gng^{-1} = k \in N$.
Now, suppose $gng^{-1} \in N$ for all $g \in G$ and $n \in N$. Let $u, u_1 \in uN$ and $v, v_1 \in vN$. We need to show

$$(uv)N = (u_1v_1)N$$

Since, $u_1 \in uN$ and $v_1 \in vN$ we can write them as $u_1 = un_1$ and $v_1 = vm$ for some $n, m \in N$. Now, if we can prove $u_1v_1 \in (uv)N$ then we'd be done.

$$\begin{aligned} u_1v_1 &= (un)(vm) \\ &= u(vv^{-1})nvm \\ &= (uv)(v^{-1}nv)m = (un)(n_1m) \end{aligned}$$

where $n_1 = v^{-1}nv = v^{-1}n(v^{-1})^{-1} \in N$ as per the assumption. Thus, $u_1v_1 \in uvN \implies (u_1v_1)N = (uv)N$.
For the second part, just check the group axioms. □

Definition 2.6. The element gng^{-1} is called the *conjugate* of n by g. The set $gNg^{-1} = \{gng^{-1} \mid n \in N\}$ is called *conjugate* of N by g. The element g is said to *normalize* N if $gNg^{-1} = N$. A subgroup N is called normal if every element of G *normalizes* N i.e $gNg^{-1} = N$ for all $g \in G$. If N is a normal subgroup of G then we write it as $N \trianglelefteq G$.

Proposition 2.5. Let N be the subgroup of G. Then the following are equivalent

1. $N \trianglelefteq G$

2. $N_G(N) = G$

3. $gN = Ng$ for all $g \in G$

4. $gNg^{-1} \subseteq N$

Proof. Most of them easily follow from the definition and previous propositions. $\square$

Definition 2.7. We define $G/N = \{gN \mid g \in G\}$ for $N \leq G$.

Proposition 2.6. Let $N \leq G$. Then N is normal if and only if N is a kernel of some homomorphism.

Proof. Suppose N is a kernel of some homomorphism φ. Then $gN = Ng$ for all $g \in G$ and by previous proposition we can say $N \trianglelefteq G$. Now, suppose $N \trianglelefteq G$ then we define a map $\psi : G \to G/N$ such that $g \mapsto gN$. Then,

$$\begin{aligned} \psi(g_1 g_2) &= (g_1 g_2)N \\ &= g_1 N g_2 N \\ &= \psi(g_1)\psi(g_2) \end{aligned}$$

Thus, ψ is indeed a homomorphism. Now,

$$\begin{aligned} \ker \psi &= \{g \mid \psi(g) = 1N\} \\ &= \{g \mid gN = N\} \\ &= \{g \mid g \in N\} \\ &= N \end{aligned}$$

Thus, N is the kernel of ψ. $\square$

Definition 2.8. Let $N \trianglelefteq G$. The homomorphism $\psi : G \to G/N$ defined by $\psi(g) = gN$ is called the *natural projection* of G onto G/N. If $\bar{H} \leq G/N$ is a subgroup of G/N, the *complete preimage* of $\bar{H}$ in G is the preimage of $\bar{H}$ under the natural projection.

Problems and Solutions

Problem : Let $\varphi : G \to H$ be an homomorphism and Let E be a subgroup of H. Prove that $\varphi^{-1}(E) \leq G$, where $\varphi^{-1}(E) = \{x \mid \varphi(x) \in E\}$. If $E \trianglelefteq H$, prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

Solution : Since, $\varphi^{-1}(E) = \{x \mid \varphi(x) \in E\}$. This subset of G is clearly not empty and if $x, y \in \varphi^{-1}(E)$ then $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \in E$ as E is a subgroup and $\varphi(x), \varphi(y) \in E$. Thus, $xy^{-1} \in \varphi^{-1}(E)$ for all $x, y \in \varphi^{-1}(E)$ which implies that $\varphi^{-1}(E) \leq G$.
If $E \trianglelefteq H$ then take any arbitrary element g of G and take any arbitrary element of n of $\varphi^{-1}(E)$. Thus, $\varphi(gng^{-1}) = \varphi(g)\varphi(n)\varphi(g)^{-1} \in E$ as $\varphi(n) \in E$ and $\varphi(g) \in H$ and E is normal. Thus, $gng^{-1} \in \varphi^{-1}(E)$ which implies $g\varphi^{-1}(E)g \subseteq \varphi^{-1}(E) \implies \varphi^{-1}(E) \trianglelefteq G$.
For kernel part, take $E = \{1\} \trianglelefteq H$

Problem : Let $\varphi : G \to H$ be a homomorphism of groups with kernel K and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the the fiber above a and let Y be the fiber above b, i.e, $X = \varphi^{-1}(a)$ and $Y = \varphi^{-1}(b)$. Fix an element of X. Prove that if $XY = Z$ in the quotient group G/K and w is any member of Z, the there is some $v \in Y$ such that $uv = w$.

Solution : First part follows immediately from **Definition 1.2.** and for the second part look at $\varphi(u^{-1}w)$ where w is an arbitrary member of Z. Thus,

$$\begin{aligned} \varphi(u^{-1}w) &= a^{-1}(ab) \\ &= b \end{aligned}$$

Thus, $u^{-1}w \in Y \implies w = uv$ for some $v \in Y$.

Problem : Let A be an abelian group and let B be an subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.

Solution : Any subgroup of a abelian group is a normal subgroup. Notice that,

$$(uB) \circ (vB) = (uv)B = (vu)B = (uB) \circ (vB)$$

For the example part, take $G = S_3$ and $N = \{e, (123), (132)\}$.

Problem : Prove that in quotient group G/N, $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$.

Solution : If we let $(gN)^\alpha := \underbrace{(gN) \cdot (gN) \cdots (gN)}_{\alpha}$ for $\alpha \geq 0$ then

$$\begin{aligned} (gN) \cdot (gN) \cdots (gN) &= \{(gN) \cdot (gN)\} \cdots (gN) \\ &= (g^2N) \cdot (gN) \cdots (gN) \quad (\textbf{Proposition 2.4.}) \\ &= g^\alpha N \end{aligned}$$

Problem : Prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$(and gN has infinite order if no such n exists).Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.

Solution : Let n be the smallest positive integer such that $g^n \in N$. Then, $g^n N = N$. Now, if there exists a $s < n$ s.t $g^s \in N$, then it would contradict our assumption. Thus order of gN is n. Now, Suppose $g^n \notin N$ for any $n > 0$ then $g^n N \neq N$ for any $n > 0$. Thus the order is infinite if no such n exists.
An example of $g^s N = N$ and $s < |g|$ is $N = \{1, r, r^2\} \trianglelefteq D_6$, $g = r$.

Problem : Define $\varphi : \mathbb{R}^\times \to \{\pm 1\}$ by letting $\varphi(x)$ be x divided by the absolute value of x. Describe the fibers of φ and prove that φ is a homomorphism.

Solution : The fiber over -1 is $X_{-1} = \{x \mid \varphi(x) = -1\}$. Since $\varphi(x) = \frac{x}{|x|} = -1$, the only numbers that get mapped to -1 are $x < 0$. Similarly, the only number that get mapped to 1 are $x > 0$. Thus the fiber over 1 and -1 are the positive and negative reals.
Now, $\varphi(x \cdot y) = \frac{xy}{|xy|} = \frac{x}{|x|}\frac{y}{|y|} = \varphi(x)\varphi(y)$.

Problem : Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi(x, y) = x + y$. Prove that π is a surjective homomorphism and describe the kernel and fibers of π geometrically.

Solution : To prove its a homomorphism, take

$$\pi((x, y) + (a, b)) = \pi(x + a, y + b) = x + y + a + b = \pi(x, y) + \pi(a, b)$$

To prove its surjectivity , notice that for a real number x there are always two real numbers that add up to x.
The $\ker \pi$ is the set of solution to the equation $x + y = 0$ and the fiber over a of φ is the set of solution to the equation $x + y = a$.

Problem : Let $\varphi : \mathbb{R}^\times \to \mathbb{R}^\times$ be a map sending x to $|x|$. Prove it is a homomorphism and find the image of φ. Describe the kernel and the fibers of φ.

Solution : To show that it is a homomorphism,

$$\varphi(x \cdot y) = |x \cdot y| = |x| \cdot |y| = \varphi(x)\varphi(y)$$

The image of φ would be the positive reals. The $\ker \varphi = \{\pm 1\}$ and $X_a = \{\pm a\}$.

Problem : Define $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\varphi(a + ib) = a^2 + b^2$. Prove that the map is a homomorphism and find its image. Describe the kernel and the fibers of φ geometrically.

Solution : To prove its homomorphism,

$$\begin{aligned}\varphi((a + bi) \cdot (c + di)) &= \varphi(ac - bd + (ad + bc)i) \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= (a^2 + b^2)(c^2 + d^2)\end{aligned}$$

The image of φ is $\mathbb{R}_{>0}$ and the kernel is a set of solution to $x^2 + y^2 = 1$ which is a circle with radius 1.
The fiber of X_a is also the set of solution to the equation of circle with radius $\sqrt{a}$.

2.2 More on Cosets and Lagrange's Theorem

Theorem 2.2 (Lagrange's Theorem)**.** *If G is a finite group and H be its subgroup then order of H divides order of G and the number of left cosets of H in G is exactly $\frac{|G|}{|H|}$.*

Proof. Let $|H| = n$ and let k be the number of left cosets of H in G. By definition of a left coset the map

$$\Psi : H \to gH \quad \text{defined by} \quad h \to gh$$

is a surjection from H to left coset gH. Also, if $\Psi(h_1) = \Psi(h_2)$ then $gh_1 = gh_2 \implies h_1 = h_2$. Thus, the map Ψ is a bijection and

$$|H| = |gH|$$

Since, G is partitioned into k disjoint cosets of H which have the same order as H, we have

$$|G| = nk \implies \frac{|G|}{|H|} = k$$

□

Remark. We could add a similar proof for gH and conclude number of left cosets = number of right cosets for any finite group G.

Definition 2.9. If G is a group (possibly infinite) and $H \leq G$, then the number of left cosets of H in G is called the *index* of H in G and is denoted by $|G : H|$.

Proposition 2.7. If G is a finite group and $x \in G$, then order of x divides the order of G. In particular $x^{|G|} = 1$ for all $x \in G$.

Proof. Apply lagrange's theorem on $H = \langle x \rangle$. □

Proposition 2.8. If G is a group of prime order p, then G is cyclic, hence $G \cong \mathbf{Z}_p$.

Proof. Take any $x \in G$ such that $x \neq 1$ then $|\langle x \rangle|$ divides p which implies $\langle x \rangle = G$. □

Proposition 2.9. Let G be a group and H be a subgroup of G with $|G : H| = 2$. Then $H \trianglelefteq G$.

Proof. Let $g \in G$ be arbitrary. If $g \in H$, then clearly

$$gH = H = Hg.$$

If $g \notin H$, then since there are exactly two left cosets of H in G and $g \notin H$, these must be H and gH. Because left cosets are disjoint, we have

$$gH = G \setminus H.$$

Similarly, the right cosets of H in G are also disjoint, so the two right cosets must be H and Hg, and therefore

$$Hg = G \setminus H.$$

Thus,

$$Hg = G \setminus H = gH.$$

Hence $gH = Hg$ for all $g \in G$, and therefore $H \trianglelefteq G$. □

Theorem 2.3 (Cauchy's Theorem)**.** *If G is finite group and p is a prime dividing $|G|$ then G has an element of order p.*

Proof. Next Chapter $\square$

Theorem 2.4 (Sylow's Theorem)**.** *If G is a finite group and $|G| = p^\alpha m$, where $p \nmid m$ then G has a subgroup of order p^α.*

Proof. Next Chapter $\square$

Definition 2.10. Let G be a group and $H, K \leq G$ and define

$$HK = \{hk \mid h \in H, k \in K\}$$

Proposition 2.10. If H and K are finite subgroups of a group then

$$|HK| = \frac{|HK|}{|H \cap K|}$$

Proof. Notice that

$$HK = \bigcup_{h \in H} hK$$

Since, each coset has size $|K|$ it is enough to find the number of left cosets of form hK where $h \in H$. But $h_1K = h_2K$ where $h_1, h_2 \in H$ if and only if $h_2^{-1}h_1 \in K$. Thus,

$$h_1K = h_2K \iff h_2^{-1}h_1 \in (H \cap K) \iff h_1(H \cap K) = h_2(H \cap K)$$

Thus the number of distinct cosets of the form hK where $h \in H$ is the number of distinct left cosets of $H \cap K$ in H. Thus, by lagrange's theorem we've the number of distinct left cosets of $H \cap K$ in H equal to $\frac{|H|}{|H \cap K|}$. Since, there are $\frac{|H|}{|H \cap K|}$ distinct cosets and each coset has a size of $|K|$ we get our desired formula. $\square$

Proposition 2.11. If H and K are subgroups of a group, HK is a subgroup if and only if $HK = KH$.

Proof. Assume first that $HK = KH$ and let $a, b \in HK$. We prove $ab^{-1} \in HK$, so HK is a subgroup by the subgroup criterion. Let

$$a = h_1k_1 \quad \text{and} \quad b = h_2k_2,$$

for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $b^{-1} = k_2^{-1}h_2^{-1}$, so

$$ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}.$$

Let $k_3 = k_1k_2^{-1} \in K$ and $h_3 = h_2^{-1}$. Thus $ab^{-1} = h_1k_3h_3$. Since $HK = KH$,

$$k_3h_3 = h_4k_4 \quad \text{for some } h_4 \in H,\ k_4 \in K.$$

Therefore,

$$ab^{-1} = h_1(h_4k_4) = (h_1h_4)k_4,$$

and since $h_1h_4 \in H$, $k_4 \in K$, we obtain $ab^{-1} \in HK$, as desired.
Conversely, assume that HK is a subgroup of G. Since $K \leq HK$ and $H \leq HK$, by the closure property of subgroups, $KH \subseteq HK$. To show the reverse containment, let $hk \in HK$. Since HK is a subgroup, $hk = a^{-1}$ for some $a \in HK$. If $a = h_1k_1$, then

$$hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH,$$

so $HK \subseteq KH$. Hence $HK = KH$, completing the proof.

$\square$

Proposition 2.12. If H and K are subgroups of G and $H \leq N_G(K)$, then HK is a subgroup of G. In particular, if $K \trianglelefteq G$ then $HK \leq G$ for any $H \leq G$.

Proof. We prove $HK = KH$. Let $h \in H$, $k \in K$. By assumption, $hkh^{-1} \in K$, hence

$$hk = (hkh^{-1})h \in KH.$$

This proves $HK \subseteq KH$. Similarly, $kh = h(h^{-1}kh) \in HK$, proving the reverse containment.
For the second statement: if $K \trianglelefteq G$, then $N_G(K) = G$, so in particular $H \leq N_G(K)$ for any subgroup $H \leq G$. The result then follows from the first part. $\square$

Definition 2.11. If A is any subset of $N_G(K)$(or $C_G(K)$), we shall say A *normalizes* K(*centralizes* K, respectively).

Problems and Solutions

Will do later

2.3 The Isomorphism Theorems

Theorem 2.5. *(First Isomorphism Theorem) If $\varphi : G \to H$ is a group homomorphism, then $\ker\varphi \trianglelefteq G$ and $G/\ker\varphi \cong \varphi(G)$.*

Proof. we have already shown the first part of the theorem, which is $\ker\varphi \trianglelefteq G$. For the second part, define a map

$$\begin{aligned} \Phi : G/\ker\varphi &\to \varphi(G) \\ g\ker\varphi &\mapsto \varphi(g) \end{aligned}$$

This map is well-defined as

$$\begin{aligned} g\ker\varphi &= h\ker\varphi \\ \implies gh^{-1} &\in \ker\varphi \\ \implies \varphi(g) &= \varphi(h) \\ \implies \Phi(g\ker\varphi) &= \Phi(h\ker\varphi) \end{aligned}$$

This map is also a homomorphism as

$$\begin{aligned} \Phi((g\ker\varphi)\cdot(h\ker\varphi)) &= \Phi((gh)\ker\varphi) \\ &= \varphi(gh) \\ &= \varphi(g)\varphi(h) \\ &= \Phi(g\ker\varphi)\Phi(h\ker\varphi) \end{aligned}$$

This map is trivially surjective and for the injective part, suppose

$$\Phi(g\ker\varphi) = \Phi(h\ker\varphi)$$

$$\begin{aligned} \implies \varphi(g) &= \varphi(h) \\ \implies \varphi(gh^{-1}) &= e \\ \implies gh^{-1} &\in \ker\varphi \end{aligned}$$

Thus, $g \in (\ker\varphi)h = h(\ker\varphi)$, which implies $g\ker\varphi = h\ker\varphi$. □

Corollary 2.1. Let $\varphi : G \to H$ be a group homomorphism. Then,

1. φ is injective $\iff$ $\ker\varphi = \{e\}$.
2. $|G : \ker\varphi| = |\varphi(G)|$.

Theorem 2.6. *(Second Isomorphism Theorem) Let G be a group and let A and B be subgroups of G and assume $A \le N_G(B)$. Then AB is a subgroup, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.*

Proof. Since $A \le N_G(B)$, every element of A normalizes B; that is,

$$aBa^{-1} = B \qquad \text{for all } a \in A.$$

Also, $B \le N_G(B)$ trivially, because for any $b \in B$,

$$bBb^{-1} = B.$$

Now consider an arbitrary element of AB. Every element of AB can be written as ab for some $a \in A$ and $b \in B$. We compute the conjugate of B by such an element:

$$(ab)B(ab)^{-1} = a\big(bBb^{-1}\big)a^{-1} = aBa^{-1} = B,$$

since $bBb^{-1} = B$ and $aBa^{-1} = B$.
Thus every element of AB normalizes B, so

$$AB \le N_G(B).$$

Since $kBk^{-1} = B$ for any $k \in AB$, it follows that

$$B \trianglelefteq AB.$$

Since, B is normal in AB the quotient group AB/B is well defined. Define the map $\varphi : A \to AB/B$ by

$$\varphi(a) = aB$$

One can check that this map is a homomorphism. Also, it is clear that this map is surjective from the definition. The identity in AB/B is $1B$ so

$$\ker \varphi = \{a \in A \mid aB = 1B\} = \{a \in A \mid a \in B\} = A \cap B$$

By the first isomorphism theorem, $A \cap B \trianglelefteq A$ and $A/A \cap B \cong AB/B$. □

Theorem 2.7. *(The Third Isomorphism Theorem) Let G be a group and $H, K \trianglelefteq G$ with $H \le K$. Then, $K/H \trianglelefteq G/H$ and*

$$(G/H)/(K/H) \cong (G/K)$$

Proof. First of all notice that $H \trianglelefteq K$. Thus, K/H is well-defined. Now, let $gH \in G/H$ and $kH \in K/H$ be arbitrary element. Then, $(gH)(kH)(gH)^{-1} = (gkg^{-1})H$, since $gkg^{-1} \in K$ it implies that $(gkg^{-1})H \in K/H$. Thus, $K/H \trianglelefteq G/H$.
Now, define $\varphi : G/H \to G/K$ by $gH \mapsto gK$. One can check this is a well-defined map. Also,

$$\varphi((aH)(bH)) = (ab)K = (aK)(bK) = \varphi(aH)\varphi(bH)$$

Thus, it a homomorphism as well. Now,

$$\begin{aligned}
\ker \varphi &= \{gH \in G/H \mid \varphi(gH) = K\} \\
\ker \varphi &= \{gH \in G/H \mid gK = K\} \\
\ker \varphi &= \{gH \in G/H \mid g \in K\} \\
\ker \varphi &= K/H
\end{aligned}$$

□

Theorem 2.8. *(The Fourth or Lattice Isomorphism Theorem) Let G be a group and let N be a normal subgroup of G. Then there is a bijection from the set of subgroups A of G which contain N onto the set of subgroups $\overline{A} = A/N$ of G/N. In particular, every subgroup of $\overline{G}$ is of the form A/N for some subgroup A of G containing N (namely, its preimage in G under the natural projection homomorphism from G to G/N).*
This bijection has the following properties: for all $A, B \le G$ with $N \le A$ and $N \le B$,

1. *$A \leq B$ if and only if $\overline{A} \leq \overline{B}$,*

2. *if $A \leq B$, then $|B : A| = |\overline{B} : \overline{A}|$,*

3. *$\langle A, B \rangle = \langle \overline{A}, \overline{B} \rangle$,*

4. *$\overline{A \cap B} = \overline{A} \cap \overline{B}$, and*

5. *$A \trianglelefteq G$ if and only if $\overline{A} \trianglelefteq \overline{G}$.*

Problems and Solutions

Will do later

2.4 Composition Series and Hölder Program

Proposition 2.13. If G is a finite abelian group and p is a prime dividing $|G|$, then G contains and element of order p.

Proof. We'll use induction to prove the result.

Base case: $|G| = p$ If $|G| = p$ then x has order p by Lagrange's Theorem and we are done.

Inductive Hypothesis: Suppose for every abelian group H such that $p \mid |H|$ and $|H| < |G|$, there exists a element with order p.

Suppose $x \in G$ with $x \neq 1$ and p divides $|x|$ and write $|x| = pn$. By **Proposition 1.11**(2), $|x^n| = p$, and again we have an element of order p. We may therefore assume p does not divide $|x|$. Let $N = \langle x \rangle$. Since G is abelian, $N \trianglelefteq G$. By Lagrange's Theorem, $|G/N| = \frac{|G|}{|N|}$ and since $N \neq 1$, $|G/N| < |G|$. Since p does not divide $|N|$, we must have $p \mid |G/N|$. We can now apply the induction assumption to the smaller group G/N to conclude it contains an element, $\bar{y} = yN$, of order p. Thus, we get $y \notin N$ ($\bar{y} \neq \bar{1}$) but $y^p \in N$. Now, suppose $|y^p| = |y|$ then $(|y|, p) = 1$ but $|y^p| = |y|$ we have $\langle y^p \rangle = \langle y \rangle$, thus $y = y^{pz} \in N$ a contradiction.
Thus, $|y^p| \neq |y|$ and from **Proposition 1.11.**(2) we have $|y^p| = \frac{|y|}{(|y|,m)}$ which implies $(|y|, p) > 1$. Thus, $p \mid |y|$. We are now in the situation described in the preceding paragraph, so that argument again produces an element of order p. The induction is complete. □

Definition 2.12. A (finite or infinite) group G is called *simple* if $|G| > 1$ and the only normal subgroups of G are 1 and G.

Definition 2.13. In a group G a sequence of subgroups

$$1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{k-1} \leq N_k = G$$

is called *composition series* if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is a simple group for $0 \leq i \leq k-1$. If the above series is a composition series then the quotient groups N_{i+1}/N_i are called *composition factor*.

Remark. Normality is not **transitive**.

Theorem 2.9 (Jordan-Hölder). *Let G be a finite group with $G \neq 1$. Then,*

1. *G has a composition series and*
2. *The composition factors in a composition series are unique, namely if $1 = N_0 \leq \cdots \leq N_r = G$ and $1 = M_0 \leq \cdots \leq M_s = G$ are composition series for G then $r = s$ and there is some permutation, π, of $\{1, 2 \ldots, r\}$ such that*

$$M_{\pi(i)}/M_{\pi(i)-1} \cong N_i/N_{i+1}, \quad i \leq i \leq r$$