

# Real Analysis

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## Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from [MIT OCW](#) and write the solutions to the problem set here.

# 1 Problem Set 1

## Problem 1.1

Let  $\mathbb{F}$  be a ordered field with  $1 \neq 0$ . Show that  $1 > 0$ .

*Solution.*

First let us prove that  $(-1) \cdot (-1) = 1$ . We know that for all  $x \in \mathbb{F}$ , there is an inverse element  $-x$  such that,

$$x + (-x) = 0$$

Thus,  $1 + (-1) = 0$  which means that

$$\begin{aligned} 0 &= (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1) \\ &\implies 1 = (-1) \cdot (-1) \end{aligned}$$

Since  $\mathbb{F}$  is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that  $1 \neq 0$  so the middle statement can't be true and if  $1 < 0$  then  $0 < (-1)$ . But from **axiom of order and multiplication**  $0 < (-1) \cdot (-1) = 1$ . Thus a contradiction.

## Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

*Solution.*

Suppose  $\frac{n}{m} = \frac{n_1}{m_1}$  and  $\frac{p}{q} = \frac{p_1}{q_1}$  then using the definition of when two rational numbers are equal, we get  $nm_1 = n_1m$  and  $pq_1 = p_1q$ . Thus,

$$\begin{aligned} m_1q_1(nq + pm) &= m_1q_1nq + m_1q_1pm \\ &= n_1mq_1q + p_1qm_1m \\ &= mq(n_1q_1 + m_1p_1) \end{aligned}$$

Thus,  $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$ .

**Problem 1.3**

Find the  $\sup E$  and  $\inf E$  for the following set  $E$ .

1.  $E = \{n \in \mathbb{Z} \mid n < \sqrt{12}\}$
2.  $E = \{r \in \mathbb{Q} \mid r < \sqrt{12}\}$
3.  $E = \{x \in \mathbb{R} \mid x^2 - x - 1 < 0\}$
4.  $E = \left\{ \frac{n^2+n}{n+1} \mid n \in \mathbb{N} \right\}$

*Solution.*

1.  $\sup E = 3$  but  $\inf E$  doesn't exist.
2.  $\sup E = \sqrt{12}$  but  $\inf E$  doesn't exist.
3.  $\sup E = \frac{1+\sqrt{5}}{2}$  and  $\inf E = \frac{1-\sqrt{5}}{2}$ .
4.  $\inf E = 1$  but  $\sup E$  doesn't exist.

**Problem 1.4**

Let  $\mathbb{M}$  be the set of polynomials with integer coefficients i.e,

$$\mathbb{M} := \{f(x) = a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{Z}\}$$

Define the relation  $0 \prec f$  if  $0 < f(x)$  for  $x$  large enough. More precisely, we say

$$0 \prec f \quad \text{if there exists } M > 0 \text{ such that } f(x) > 0 \text{ for all } x > M.$$

Then define

$$f \prec g \quad \text{if } 0 \prec (g - f).$$

Show that  $(\mathbb{M}, \prec)$  is an ordered set.

*Solution.*

We'll use the fact that for large enough  $x$ ,  $f(x) > 0$  for  $a_n > 0$ . Let  $f, g \in \mathbb{M}$  such that  $f \neq g$  and

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \quad \text{and} \quad g(x) = b_0 + b_1x + \cdots + b_nx^n$$

Define  $h(x) = g(x) - f(x) = (b_0 - a_0) + (b_1 - a_1)x + \cdots + (b_n - a_n)x^n$  and let  $c_i = b_i - a_i$ . Suppose  $c_k$  is the highest degree non zero coefficient, then if

1.  $c_k > 0$  then  $f \prec g$
2.  $c_k < 0$  then  $g \prec f$

Suppose  $f \prec g$  and  $g \prec h$ . Then,  $f(x) < g(x)$  and  $g(x) < h(x)$  for large  $x$  and hence  $f(x) < h(x)$  for large  $x$ . Thus,  $0 \prec (h - f)$  which means  $f \prec h$ .

**Problem 1.5**

Prove that  $(\mathbb{M}, \prec)$  doesn't satisfy the archimedean property.

*Solution.*

To show that it doesn't satisfy the archimedean property, we need to show that  $\exists f, g \in \mathbb{M}$  such that

$$\forall n \in \mathbb{N}, g \not\prec nf$$

If we choose  $g(x) = x^2$  and  $f(x) = x$  and assume  $g \prec nf$  then,  $0 \prec x(n - x)$  which means that  $0 < x(n - x)$  which we know is false for large  $x$ . Thus,  $g \not\prec nf$ .

**Problem 1.6**

Show that for any non-empty set  $E \subset R$  which is bounded from below,  $E$  has the greatest lower bound.

*Solution.*

To show that  $E$  has greatest lower bound define

$$-E = \{-x \mid x \in E\}$$

If  $\alpha$  is any lower bound of  $E$  then  $x \geq \alpha \Rightarrow -x \leq -\alpha$ . That means that  $-E$  is bounded above by  $-\alpha$ . Since  $-E$  is bounded above by  $-\alpha$ , it must have the least upper bound property. Let

$$\sup -E = \beta$$

Thus,  $\beta \geq -x \Rightarrow x \geq -\beta$  and  $\beta \leq -\alpha$  for any lower bound  $\alpha$  of  $E$ . Thus,  $\alpha \leq -\beta$ . Hence,  $-\beta = \inf E$ .

**Problem 1.7**

Show that for any real number  $x \in \mathbb{R}$  there exists a real number  $y \in \mathbb{R}$  such that  $y^3 = x$ .

*Solution.*

If  $x = 0$  then  $y = 0^3 = 0$ . Let us define

$$A = \{a \mid a > 0 \text{ and } a^3 \leq x\}$$

for  $x > 0$ . Since  $A$  is non-empty and bounded, let  $y = \sup A$ . We will show that  $y^3 = x$ .

Suppose  $x > y^3$ . Let  $h = \min\{\frac{1}{2}, \frac{x-y^3}{3y^2+3y+1}\}$  then,

$$\begin{aligned} (y+h)^3 &= y^3 + h^3 + 3yh^2 + 3y^2h \\ &< y^3 + h(1 + 3y + 3y^2) \\ &\leq y^3 + \frac{x-y^3}{3y^2+3y+1} \cdot (1 + 3y + 3y^2) \\ &\leq x \end{aligned}$$

Thus,  $y + h \in A$  but since  $y$  is the least upper bound of  $A$  we have,  $y + h \leq y \Rightarrow h \leq 0$ . This is a contradiction.

Suppose  $x < y^3$ . Let  $h = \frac{y^3 - x}{3y^2}$  then

$$\begin{aligned}(y - h)^3 &= y^3 - h^3 - 3y^2h + 3yh^2 \\&= y^3 - h^3 - 3y^2 \cdot \frac{(y^3 - x)}{3y^2} + 3yh^2 \\&= x - h^3 + 3yh^2\end{aligned}$$

Since  $3yh^2 - h^3 > 0$  we have  $(y - h)^3 > x$ . Let  $q \in A$  then

$$q^3 \leq x < (y - h)^3$$

Thus,  $(y - h)^3 - q^3 > 0 \Rightarrow (y - h - q)((y - h)^2 + (y - h)q + q^3) > 0$ . Since,  $(y - h)^2 + (y - h)q + q^3 > 0$  we have  $y - h > q$ . Thus  $(y - h)$  is an upper bound for  $A$ . Thus  $y \leq y - h \Rightarrow h \leq 0$ . Contradiction!!

The proof also works for  $x < 0$  as  $-x > 0$ . Thus,

$$\exists k \in \mathbb{R} \text{ s.t } k^3 = -x$$

Let  $z := -k$  then  $z^3 = (-k)^3 = -k^3 = -(-x) = x$ .

## 2 Problem Set 2

### Problem 2.1

Let  $a_n$  ad  $b_n$  be a sequence of real numbers.

1. Assume that  $\lim_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} b_n$  exists. Show that  $\lim_{n \rightarrow \infty} (a_n b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) \left( \lim_{n \rightarrow \infty} b_n \right)$  exists.
2. Give an example in which  $\lim_{n \rightarrow \infty} (a_n b_n)$  exists but neither  $\lim_{n \rightarrow \infty} a_n$  nor  $\lim_{n \rightarrow \infty} b_n$  exits.

*Solutions.*

To show that  $\lim_{n \rightarrow \infty} a_n b_n = ab$  we need to show  $\exists N \in \mathbb{N}$  such that for all  $\varepsilon > 0$  we have

$$|a_n b_n - ab| < \varepsilon$$

for  $n \geq N$ . Now,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

Since  $\{a_n\}$  is convergent, we know that it is bounded above. Suppose  $|a_n| < C$  then

$$|a_n||b_n - b| + |b||a_n - a| < |C||b_n - b| + |b||a_n - a|$$

Also let  $N_1 \in \mathbb{N}$  such that  $|b_n - b| < \frac{\varepsilon}{2(|C| + 1)}$  for all  $n \geq N_1$  and let  $N_2 \in \mathbb{N}$  such that  $|a_n - a| < \frac{\varepsilon}{2(|b| + 1)}$  for all  $n \geq N_2$ . Thus for  $n \geq \max\{N_1, N_2\}$  we have,

$$\begin{aligned} 2|a_n||b_n - b| + |b||a_n - a| &< |C||b_n - b| + |b||a_n - a| \\ &< |C| \cdot \frac{\varepsilon}{2(|C| + 1)} + |b| \cdot \frac{\varepsilon}{2(|b| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

For the second part just take  $a_n = (-1)^n$  and  $b_n = (-1)^n$ . We know that  $\lim_{n \rightarrow \infty} a_n b_n = 1$  but  $\lim_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} b_n$  doesn't exist.

### Problem 2.2

Find the limit of the following sequence if it exists or show that the limit doesn't exists.

$$1. a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$$

$$2. a_n = \frac{\sin(n)}{n}$$

$$3. a_n = \sum_{i=1}^n \frac{n^2}{\sqrt{n^6 + i}}$$

*Solution.*

For the first part we know that  $a_n = \frac{1}{n+1} - \frac{1}{n} - 1$ . Thus,

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n+1} - \frac{1}{n} - 1 &= \lim_{n \rightarrow \infty} \frac{1}{n+1} - \lim_{n \rightarrow \infty} \frac{1}{n} - \lim_{n \rightarrow \infty} 1 \\ &= 0 - 0 - 1 \\ &= -1\end{aligned}$$

The first equation is due to the algebraic property of limit.

For the second part, we need to show that

$$\left| \frac{\sin(n)}{n} \right| < \varepsilon$$

We know that  $0 \leq |\sin(n)| \leq 1 \Rightarrow 0 \leq \frac{|\sin(n)|}{|n|} \leq \frac{1}{|n|}$ . Thus, for any  $N > \frac{1}{\varepsilon}$  we have

$$\left| \frac{\sin(n)}{n} \right| \leq \frac{1}{|n|} < \frac{1}{N} < \varepsilon$$

For the third part, we will use **squeeze theorem** to show that the limit is 1. Notice that,

$$\frac{n^2}{\sqrt{n^6 + n}} < \frac{n^2}{\sqrt{n^6 + i}} < \frac{n^2}{\sqrt{n^6}}$$

for  $i = 1, 2, \dots, n$ . Thus,

$$\begin{aligned}n \cdot \frac{n^2}{\sqrt{n^6 + n}} &\leq \sum_{i=1}^n \frac{n^2}{\sqrt{n^6 + i}} \leq n \cdot \frac{n^2}{\sqrt{n^6}} \\ \Rightarrow \frac{1}{\sqrt{1 + \frac{1}{n^5}}} &\leq \sum_{i=1}^n \frac{n^2}{\sqrt{n^6 + i}} \leq 1 \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n^5}}} &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n^2}{\sqrt{n^6 + i}} \leq \lim_{n \rightarrow \infty} 1 \\ \Rightarrow 1 &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n^2}{\sqrt{n^6 + i}} \leq 1\end{aligned}$$

Thus, by **squeeze theorem** we have  $\lim_{n \rightarrow \infty} a_n = 1$ .

### Problem 2.3

Let  $a_n$  be a sequence of real numbers and  $L$  be a real number. Show that the following two statements are equivalent. One holds if and only if the other does.

1. There exists a subsequence  $a_{n_k}$  converging to  $L$ .
2. For any  $\epsilon > 0$ , there exist infinitely many  $a_n$  in  $(L - \epsilon, L + \epsilon)$ .

*Solution.*

Suppose (1.) holds then there exists a  $N \in \mathbb{N}$  such that for all  $k \geq N$  we have,

$$|a_{n_k} - L| < \varepsilon \implies L - \varepsilon < a_{n_k} < L + \varepsilon$$

Thus, there are infinitely many  $a_i$  in the interval  $(L - \varepsilon, L + \varepsilon)$ .

Suppose (2.) holds then for any  $\varepsilon > 0$  there are infinitely many  $a_n$  such that

$$L - \varepsilon < a_n < L + \varepsilon$$

Since there are infinitely many terms in  $(L - 1, L + 1)$  choose  $n_1$  such that

$$|a_{n_1} - L| < 1$$

Since there are infinitely many terms in  $(L - \frac{1}{2}, L + \frac{1}{2})$  choose  $n_2$  such that

$$n_2 > n_1 \text{ and } |a_{n_2} - L| < \frac{1}{2}$$

Continuing inductively we can produce a sequence such that  $\{a_{n_i}\}$

$$|a_{n_k} - L| < \frac{1}{k}$$

Hence,  $a_{n_k} \rightarrow L$ .

#### Problem 2.4

Where possible, find a subsequence that is monotone and a subsequence that is convergent for the following sequences.

$$1. \quad a_n = \sin(n\pi/8)$$

$$2. \quad a_n = (-1)^n n$$

*Solution.*

Notice that

$$\sin\left(\frac{n\pi}{8}\right) = \begin{cases} 0 & \text{if } n = 8k \\ 1 & \text{if } n = 4(4k+1) \\ -1 & \text{if } n = 4(4k-1) \end{cases}$$

So a monotone subsequence would be  $a_{n_k} = 1$ . And the same example works for convergent subsequence.

For the second part, notice that

$$a_n = \begin{cases} 2k & \text{if } n = 2k \\ -2k - 1 & \text{if } n = 2k + 1 \end{cases}$$

So a monotone subsequence would be  $a_{n_k} = 2k$ . But there is no convergent subsequence, because we know that every subsequence would have a infinitely many odd numbers or infinitely many even numbers but we know that those aren't bounded.

### 3 Problem Set 3

#### Problem 3.1

Prove that for reals  $x < y$ , there exists a  $r \in \mathbb{Q}$  such that  $x < r < y$ .

*Solution.* Using the archimedean property of  $\mathbb{R}$

$$\exists n \in \mathbb{N}, \text{ s.t } \frac{1}{n} < y - x$$

$$\implies n(y - x) > 1$$

Since the difference is greater than 1, there exists a integer  $m$  such that

$$nx < m < ny \implies x < \frac{m}{n} < y$$

Hence we are done.

#### Problem 3.2

Let  $E$  be an non-empty subset of  $\mathbb{R}$  which is bounded. Define

$$F := \{x^2 \mid x \in E\}$$

Show that  $\sup F$  exists and that  $\sup F = \max\{(\sup E)^2, (\inf E)^2\}$ .

*Solution.*

Since  $E$  is bounded  $F$  is also bounded. Then since  $\inf E \leq x \leq \sup E$  we have

$$x^2 \leq \max\{(\sup E)^2, (\inf E)^2\}$$

If  $\max\{(\sup E)^2, (\inf E)^2\} = (\sup E)^2$  then  $\sup E \geq 0$  and  $(\sup E)^2$  is an upper bound for  $F$ . Let  $C$  be any upper bound for  $F$ . Then  $C \geq x^2 \Rightarrow \sqrt{C} \geq |x| \geq x$  for all  $x \in E$ . Hence  $\sqrt{C}$  is an upper bound for  $E$ . Thus,  $\sqrt{C} \geq \sup E \Rightarrow C \geq (\sup E)^2$ . It follows that  $(\sup E)^2 = \sup F$ .

Similarly if  $\max\{(\sup E)^2, (\inf E)^2\} = (\inf E)^2$  then  $\sup F = (\inf E)^2$ .

#### Problem 3.3

Let  $E$  be an non-empty subset of  $\mathbb{R}$  which is bounded from above. Show that there exists a sequence  $\{a_n\}$  such that  $a_n \in E$  and  $\lim_{n \rightarrow \infty} a_n = \sup E$ .

*Solution.*

We know that for each  $\varepsilon > 0$  there exists at least one  $a \in E$  such that  $a > \sup E - \varepsilon$ . Let  $a_k \in E$  such that  $a_k > \sup E - \frac{1}{k}$ . Thus,  $\frac{1}{k} > \sup E - a_k \geq 0$ . Thus the sequence  $\{a_n\}$  converges to  $\sup E$  and  $\lim_{n \rightarrow \infty} a_n = \sup E$ .

**Problem 3.4**

Let  $a_1 = 4$  and define  $a_n$  inductively by

$$a_n = 4 - \frac{4}{a_{n-1}} \text{ for } n \geq 2$$

Show that  $\lim_{n \rightarrow \infty} a_n = 2$ .

*Solution.* Using induction one can prove that

$$a_n = 2 + \frac{2}{n}, \text{ for } n \geq 1$$

Thus,  $\lim_{n \rightarrow \infty} a_n = 2$ .

**Problem 3.5**

Let  $T : \mathbb{R} \rightarrow \mathbb{R}$  be a contraction map and  $x \in \mathbb{R}$  be a number. Define a sequence  $a_n$  by requiring  $a_1 = x$  and  $a_{n+1} = T(a_n)$ .

1. Show that for any  $m \in \mathbb{N}$ ,  $|a_1 - a_m| \leq \frac{1}{1-\lambda} |a_1 - a_2|$
2. Show that  $a_n$  is a Cauchy sequence.

*Solution.*

Notice that,

$$\begin{aligned} |a_n - a_{n+1}| &\leq \lambda |a_{n-1} - a_n| \\ &\leq \lambda^2 |a_{n-2} - a_{n-1}| \\ &\leq \quad \vdots \\ &\leq \lambda^{n-1} |a_1 - a_2| \end{aligned}$$

Therefore,

$$\begin{aligned} |a_1 - a_m| &\leq |a_1 - a_{m-1}| + |a_{m-1} - a_m| \\ &\leq |a_1 - a_{m-2}| + |a_{m-2} - a_{m-1}| + |a_{m-1} - a_m| \\ &\leq \quad \vdots \\ &\leq |a_1 - a_2| + \sum_{i=1}^{m-1} |a_i - a_{i+1}| \\ &\leq |a_1 - a_2| + \sum_{i=1}^{m-1} \lambda^i |a_1 - a_2| \\ &\leq |a_1 - a_2| \left( \sum_{i=0}^{m-1} \lambda^i \right) \\ &\leq |a_1 - a_2| \left( \frac{1 - \lambda^m}{1 - \lambda} \right) \leq \left( \frac{1}{1 - \lambda} \right) |a_1 - a_2| \end{aligned}$$

For the second part, suppose  $m > n$  then,

$$\begin{aligned}
 |a_n - a_m| &\leq |a_n - a_{m-1}| + |a_{m-1} - a_m| \\
 &\leq \dots \\
 &\leq \sum_{i=n}^{m-1} |a_i - a_{i+1}| \\
 &\leq \sum_{i=n}^{m-1} \lambda^{i-1} |a_1 - a_2| \\
 &\leq |a_1 - a_2| \cdot \left( \lambda^{n-1} \cdot \frac{1 - \lambda^{m-n}}{1 - \lambda} \right) \\
 &\leq \frac{\lambda^{n-1}}{1 - \lambda} |a_1 - a_2|
 \end{aligned}$$

So for any  $\varepsilon > 0$  we can just choose a large  $n$  such that  $\varepsilon > \frac{\lambda^{n-1}}{1 - \lambda} |a_1 - a_2|$ .

## 4 Problem Set 4

### Problem 4.1

Give an example of a sequence  $a_n$  that satisfies the following two conditions.

- $a_n$  is divergent.
- For any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|a_{n+1} - a_n| < \varepsilon \quad \text{for all } n \geq N.$$

*Solution.*

Take  $a_n = \sum_{i=1}^n \frac{1}{i}$ .

### Problem 4.2

Let  $p > 0$  be a positive number. Consider the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

In the lecture we proved that the series diverges for  $p = 1$  and converges for  $p = 2$ . Show that the series converges for  $p > 1$  and diverges for  $0 < p \leq 1$ .

*Solution.*

For  $0 < p \leq 1$  it is easy to see that  $\frac{1}{n^p} \geq \frac{1}{n}$ . From the comparison test the series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  diverges.

For  $p > 1$  we can use the same argument like we did for  $p = 2$ . We can group the terms and use the comparison test. I am skipping the details.

### Problem 4.3

Let  $r > 0$  be a positive number. Determine whether the series

$$\sum_{n=1}^{\infty} a_n$$

converges or diverges for the following cases.

1.  $a_n = \sqrt{n+r} - \sqrt{n}$
2.  $a_n = n^3 r^n$
3.  $a_n = \frac{1}{n!} r^n$

The answer may depend on the value of  $r$ .

*Solution.*

For (1.), Let  $r > 0$ . Consider the series

$$\sum_{n=1}^{\infty} (\sqrt{n+r} - \sqrt{n}).$$

Rationalizing the terms, we have

$$\sqrt{n+r} - \sqrt{n} = \frac{r}{\sqrt{n+r} + \sqrt{n}}.$$

Since  $\sqrt{n+r} \leq \sqrt{n} + \sqrt{r}$ , it follows that

$$\sqrt{n+r} + \sqrt{n} \leq 2\sqrt{n} + \sqrt{r}.$$

Hence,

$$\sqrt{n+r} - \sqrt{n} = \frac{r}{\sqrt{n+r} + \sqrt{n}} \geq \frac{r}{2\sqrt{n} + \sqrt{r}}.$$

Choose  $N$  such that  $2\sqrt{n} \geq \sqrt{r}$  for all  $n \geq N$ . Then for  $n \geq N$ ,

$$2\sqrt{n} + \sqrt{r} \leq 3\sqrt{n},$$

and therefore

$$\sqrt{n+r} - \sqrt{n} \geq \frac{r}{3\sqrt{n}}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges, the comparison test implies that

$$\sum_{n=1}^{\infty} (\sqrt{n+r} - \sqrt{n})$$

also diverges.

For (2.) we have and (3.) just do the ratio test.

**Problem 4.4**

Let  $b_n$  be a sequence of non-negative numbers which decreases to zero. That is,

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = 0.$$

Let  $a_n = (-1)^{n-1}b_n$ . The purpose of this problem is to show that

$$\sum_{n=1}^{\infty} a_n$$

converges. This is called the *alternating series test*. Let

$$s_n = \sum_{k=1}^n a_k.$$

1. Show that  $s_{2k+1}$  is decreasing and that  $s_{2k}$  is increasing.
2. Show that  $s_{2k+1}$  is bounded from below and that  $s_{2k}$  is bounded from above.
3. Show that both  $\lim_{k \rightarrow \infty} s_{2k+1}$  and  $\lim_{k \rightarrow \infty} s_{2k}$  exist and are identical.
4. Show that  $\sum_{n=1}^{\infty} a_n$  converges.

*Solution.*

For the first part notice that

$$a_{2k} + a_{2k+1} = b_{2k+1} - b_{2k} \leq 0$$

Thus  $s_{2k+1} = s_{2k-1} + a_{2k} + a_{2k+1} \leq s_{2k-1}$ .

Similarly,

$$a_{2k+1} + a_{2k+2} = b_{2k+1} - b_{2k+2} \geq 0$$

Thus,  $s_{2k+2} = a_{2k+1} + a_{2k+2} + s_{2k} \geq s_{2k}$ .

To show that  $\{s_{2k+1}\}_{k=0}^{\infty}$  is bounded from below, we will show that  $s_{2k+1} \geq 0$ .

$$\begin{aligned} s_{2k+1} &= \underbrace{b_1 - b_2}_{\geq 0} + \underbrace{b_3 - b_4}_{\geq 0} + \cdots + \underbrace{b_{2k-1} - b_{2k}}_{\geq 0} + b_{2k+1} \\ &\geq b_{2k+1} \\ &\geq 0 \end{aligned}$$

Also,

$$\begin{aligned} s_{2k} &= b_1 - b_2 + b_3 - b_4 + \cdots + b_{2k-1} - b_{2k} \\ &\leq b_1 - b_2 + b_2 - b_3 + \cdots + b_{2k-2} - b_{2k-1} \\ &\leq b_1 - b_{2k-1} \\ &\leq b_1 \end{aligned}$$

Thus we're done with (2.).

For (3.), by **monotone convergence theorem**,

$$\lim_{n \rightarrow \infty} s_{2k+1} = \inf\{s_{2k+1}\} \quad \text{and} \quad \lim_{n \rightarrow \infty} s_{2k} = \sup\{s_{2k}\}$$

To show they are identical,

$$\begin{aligned} s_{2k+1} &= s_{2k} + a_{2k+1} = s_{2k} + b_{2k-1} \\ \implies \lim_{n \rightarrow \infty} s_{2k+1} &= \lim_{n \rightarrow \infty} s_{2k} = L \end{aligned}$$

For the (4.) part,

$$|s_n - L| < \varepsilon$$

for large enough even and odd  $n$ .

## 5 Problem Set 5

### Problem 5.1

Let  $a_n, b_n$  be two sequences of real numbers and  $x$  be a real number. The Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Show that if  $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$  converges, then so does the Fourier series.

*Solution.*

We can ignore the  $\frac{a_0}{2}$  term and just focus on the sum. Let

$$S_N(x) = \sum_{n=1}^N a_n \cos(nx) + b_n \sin(nx)$$

We will show that the sequence  $\{S_N(x)\}$  is Cauchy. Let  $N, M \in \mathbb{N}$  such that  $M > N$ .

$$|S_M(x) - S_N(x)| = \left| \sum_{n=N+1}^M a_n \cos(nx) + b_n \sin(nx) \right| \leq \sum_{n=N+1}^M |a_n| + |b_n|$$

As  $N \rightarrow \infty$  the  $\sum_{n=N+1}^M |a_n| + |b_n| \rightarrow 0$ . Thus the sequence is Cauchy and from the **Cauchy convergence theorem** we know the series converges.

### Problem 5.2

Give an example of a function  $f(x)$  defined on  $[-1, 1]$  with the following property:  $(f(x))^2$  is continuous on  $[-1, 1]$  but  $f(x)$  is not continuous on  $[-1, 1]$ .

*Solution.*

An example of such a function would be

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

### Problem 5.3

Determine at which points the function  $f(x) = \lfloor x \rfloor$  is continuous or discontinuous.

*Solution.*

Let  $n \in \mathbb{Z}$ . Suppose, for contradiction, that  $\lfloor x \rfloor$  is continuous at  $x = n$ . Then, by definition, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|x - n| < \delta \implies |\lfloor x \rfloor - \lfloor n \rfloor| < \varepsilon.$$

Take  $\varepsilon = \frac{1}{2}$ . Consider  $x = n - \frac{\delta}{2}$ . Clearly,  $|x - n| = \frac{\delta}{2} < \delta$ , so it should satisfy the continuity condition.

However,

$$\lfloor x \rfloor = \lfloor n - \frac{\delta}{2} \rfloor = n - 1$$

and

$$\lfloor n \rfloor = n.$$

Thus,

$$|\lfloor x \rfloor - \lfloor n \rfloor| = |(n - 1) - n| = 1 > \varepsilon = \frac{1}{2},$$

which is a contradiction. Hence,  $\lfloor x \rfloor$  is **discontinuous at every integer**.

Let  $x_0 \notin \mathbb{Z}$  and set  $n = \lfloor x_0 \rfloor$ . Then  $n < x_0 < n + 1$ . Define

$$\delta = \min\{x_0 - n, n + 1 - x_0\} > 0.$$

For any  $x$  with  $|x - x_0| < \delta$ , we have

$$n < x < n + 1 \implies \lfloor x \rfloor = n = \lfloor x_0 \rfloor.$$

Hence,

$$|\lfloor x \rfloor - \lfloor x_0 \rfloor| = 0,$$

proving that  $\lfloor x \rfloor$  is **continuous at every non-integer**.

#### Problem 5.4

Let  $f(x)$  and  $g(x)$  be two continuous functions defined on  $\mathbb{R}$  with  $f(x) = g(x)$  for all  $x \in \mathbb{Q}$ . Show that  $f(x) = g(x)$  for all  $x \in \mathbb{R}$ .

*Solution.*

Let  $x_n$  be a rational number such that

$$r - \frac{1}{n} < x_n < r + \frac{1}{n}$$

where  $r$  is any irrational number. From our definition of  $x_n$ , we have that  $x_n \rightarrow r$ . Thus,

$$\begin{aligned} g(r) &= g\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} g(x_n) \\ &= \lim_{n \rightarrow \infty} f(x_n) \\ &= f\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= f(r) \end{aligned}$$

Thus,  $f(r) = g(r)$  for any  $r \in \mathbb{R} \setminus \mathbb{Q}$  and therefore  $f(x) = g(x)$  for any  $x \in \mathbb{R}$ .

**Problem 5.5**

Recall that

$$E(x) := 1 + x + \frac{x^2}{2!} + \cdots + \sum_{n=0}^{\text{infty}} \frac{x^n}{n!}$$

1. For  $k \in \mathbb{N}$ , define

$$E_k(x) := 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$$

Show that  $E_k(x)$  is continuous on  $\mathbb{R}$  for any  $k \in \mathbb{N}$ .

2. Let  $M > 0$  be a fixed number. Show that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|E_k(x) - E(x)| < \varepsilon$$

for all  $k \geq N$  and for all  $x \in [-M, M]$ .

Remark: We require that a single number  $N$  that works for all  $x \in [-M, M]$ .

3. Show that  $E(x)$  is continuous on  $\mathbb{R}$ .

*Solution.*

For (1.) just use the fact that if  $f$  is continuous then  $c \cdot f$  is continuous for some fixed constant  $c$ .

For (2.), let  $M > 0$  and fix  $x \in [-M, M]$ . Then  $|x| \leq M$ . We have

$$E(x) - E_k(x) = \sum_{n=k+1}^{\infty} \frac{x^n}{n!}.$$

By the triangle inequality,

$$|E(x) - E_k(x)| \leq \sum_{n=k+1}^{\infty} \frac{|x|^n}{n!}.$$

Since  $|x| \leq M$ , we have

$$|E(x) - E_k(x)| \leq \sum_{n=k+1}^{\infty} \frac{M^n}{n!}$$

The series  $\sum_{n=0}^{\infty} \frac{M^n}{n!}$  converges, hence its tail  $\sum_{n=k+1}^{\infty} \frac{M^n}{n!} \rightarrow 0$  as  $k \rightarrow \infty$ . Therefore,

$$\sup_{x \in [-M, M]} |E(x) - E_k(x)| \leq \sum_{n=k+1}^{\infty} \frac{M^n}{n!} < \varepsilon$$

for some  $k \geq N$ . Therefore,

$$|E(x) - E_k(x)| < \varepsilon$$

for  $k \geq N$  and for all  $x \in [-M, M]$ .

For (3.), notice that

$$|E(x) - E(x_0)| \leq |E(x) - E_k(x)| + |E_k(x) - E_k(x_0)| + |E_k(x_0) - E(x_0)|$$

Now, choose a  $M > |x_0|$  such that  $x \in [-M, M]$ . Then, from (2.),

$$|E(x) - E_k(x)| < \frac{\varepsilon}{3} \quad \text{and} \quad |E(x_0) - E_k(x_0)| < \frac{\varepsilon}{3}$$

for large enough  $k$ . Also,

$$|E_k(x) - E_k(x_0)| < \frac{\varepsilon}{3}$$

by (1.) and thus,

$$\begin{aligned} |E(x) - E_k(x)| + |E_k(x) - E_k(x_0)| + |E_k(x_0) - E(x_0)| &< 3 \times \frac{\varepsilon}{3} \\ \implies |E(x) - E(x_0)| &< \varepsilon \end{aligned}$$

## 6 Problem Set 6

### Problem 6.1

Let  $X = C([0, 1])$  and  $d : X \times X \rightarrow \mathbb{R}$  be defined by

$$d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$$

Prove that  $d$  satisfies

$$d(f, g) \leq d(f, h) + d(h, g)$$

for all  $f, g, h \in X$ .

*Solution.* Notice that

$$\begin{aligned} d(f, h) + d(h, g) &= \max_{x \in [0, 1]} |f(x) - h(x)| + \max_{x \in [0, 1]} |h(x) - g(x)| \\ &\geq |f(x) - h(x)| + |h(x) - g(x)| \quad (\text{for all } x \in [0, 1]) \\ &\geq |f(x) - g(x)| \\ &= d(f, g) \end{aligned}$$

### Problem 6.2

Let  $X = \mathbb{N}_{>0}$  and  $d : X \times X \rightarrow \mathbb{R}$  defined by

$$d(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right|$$

1. Show that the sequence  $x_n = 5n$  is a cauchy sequence.
2. Show that the sequence  $x_n = 5n$  doesn't converge.

Thus the Metric space is not Cauchy complete.

*Solution.* Notice that,

$$\begin{aligned} d(x_n, x_m) &= \frac{1}{5} \left| \frac{1}{n} - \frac{1}{m} \right| \\ &\leq \left| \frac{1}{n} - \frac{1}{m} \right| \\ &\leq \left| \frac{1}{n} \right| + \left| \frac{1}{m} \right| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

the second last inequality is due to the archimedean property of  $\mathbb{R}$ .  
Let  $\varepsilon = \frac{1}{2} > 0$ . For any candidate  $x \in X$ , we have

$$d(x_n, x) = \left| \frac{1}{5n} - \frac{1}{x} \right|.$$

Since  $x \geq 1$ , we have  $\frac{1}{x} \geq 0$ , so for sufficiently large  $n$ ,

$$\frac{1}{5n} < \frac{1}{2x} \implies d(x_n, x) = \left| \frac{1}{5n} - \frac{1}{x} \right| > \frac{1}{2x} \geq \frac{1}{2} = \varepsilon.$$

Hence, for all  $N \in \mathbb{N}$ , we can find  $n \geq N$  such that  $d(x_n, x) \geq \varepsilon$ . Thus the sequence is not convergent.

### Problem 6.3

Let  $X = \mathbb{R}^2$  and consider the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1. Let  $(x_n, y_n)$  be a sequence in  $\mathbb{R}^2$ . Show that  $(x_n, y_n)$  converges if and only if both  $x_n$  and  $y_n$  converge as sequences in  $\mathbb{R}$ .
2. Let  $(x_n, y_n)$  be a sequence in  $\mathbb{R}^2$ . Show that  $(x_n, y_n)$  is a Cauchy sequence if and only if both  $x_n$  and  $y_n$  are Cauchy sequences in  $\mathbb{R}$ .
3. Show that  $\mathbb{R}^2$  is Cauchy Complete.

*Solution.*

For (1.), suppose the sequence  $(x_n, y_n) \rightarrow (x, y)$  in  $\mathbb{R}^2$  then

$$\begin{aligned} |y_n - y| &= \sqrt{(y_n - y)^2} \\ &\leq \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &= d((x_n, y_n), (x, y)) \\ &< \varepsilon \end{aligned}$$

Similarly for  $|x_n - x|$ . Now, suppose  $x_n \rightarrow x$  and  $y_n \rightarrow y$  then we claim that  $(x_n, y_n) \rightarrow (x, y)$ . Notice, that from the triangle inequality

$$\begin{aligned} d((x_n, y_n), (x, y)) &\leq d((x_n, y_n), (x_n, y)) + d((x_n, y), (x, y)) \\ &= \sqrt{(y_n - y)^2} + \sqrt{(x_n - x)^2} \\ &= |y_n - y| + |x_n - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Thus,  $(x_n, y_n) \rightarrow (x, y)$ .

For (2.), suppose  $(x_n, y_n)$  is a Cauchy sequence then

$$\begin{aligned} |y_n - y_m| &= \sqrt{(y_n - y_m)^2} \\ &\leq \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} \\ &= d((x_n, y_n), (x_m, y_m)) \\ &< \varepsilon \end{aligned}$$

Similarly for  $|x_n - x_m|$ . Now suppose  $x_n$  and  $y_n$  are cauchy sequence in  $\mathbb{R}$ . Then,

$$\begin{aligned} d((x_n, y_n), (x_m, y_m)) &\leq d((x_n, y_n), (x_n, y_m)) + d((x_n, y_m), (x_m, y_m)) \\ &= \sqrt{(y_n - y_m)^2} + \sqrt{(x_n - x_m)^2} \\ &= |y_n - y_m| + |x_n - x_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Thus,  $(x_n, y_n)$  is cauchy in  $\mathbb{R}^2$ .

For (3.), the sequence  $(x_n, y_n)$  is cauchy sequence if and only if  $x_n$  and  $y_n$  are cauchy sequence in  $\mathbb{R}$  from (2.), and  $x_n$  and  $y_n$  are cauchy if and only if  $x_n$  and  $y_n$  are convergent and  $x_n$  and  $y_n$  are convergent if and only if  $(x_n, y_n)$  convergent from (1.).

#### Problem 6.4

Let  $A_j$ ,  $j \in \mathbb{N}$  be open sets in metric space  $(X, d)$ .

1. Show that  $\bigcup_{j=1}^{\infty} A_j$  is open.
2. Show that  $\bigcap_{j=1}^{\infty} A_j$  maybe not be open.

*Solution.*

For (1.), take a arbitrary element  $x \in \bigcup_{j=1}^{\infty} A_j$ . That means  $x \in A_k$  for some  $k \in \mathbb{N}$ . Thus,

$\exists r > 0$  such that  $B_r(x) \subseteq A_k \subseteq \bigcup_{j=1}^{\infty} A_j$ .

For (2.) consider  $A_j = \left(0, 1 + \frac{1}{j}\right)$  the intersection is just  $(0, 1]$  and  $B_\varepsilon(1)$  covers element greater than 1 as  $1 - \varepsilon < y < 1 + \varepsilon$  for some  $\varepsilon > 0$ .

#### Problem 6.5

Let  $(X, d)$  be a metric space,  $K$  be a compact set in  $X$  and  $A_j$ ,  $j \in \mathbb{N}$  be closed set in  $X$ . Suppose for all  $k \in \mathbb{N}$

$$K \cap \left( \bigcap_{j=1}^k A_j \right) \text{ is non empty}$$

Show that

$$K \cap \left( \bigcap_{j=1}^{\infty} A_j \right) \text{ is non empty}$$

*Solution.*

Suppose

$$K \cap \left( \bigcap_{j=1}^{\infty} A_j \right) \text{ is empty}$$

Then,

$$K \subseteq \left( \bigcap_{j=1}^{\infty} A_j \right)^c = \bigcup_{j=1}^{\infty} A_j^c$$

Using the compactness of  $K$ , we would have

$$K \subseteq \bigcup_{j=1}^k A_j^c = \left( \bigcap_{j=1}^k A_j \right)^c$$

for some  $k \in \mathbb{N}$ . But that means,  $K \cap \left( \bigcap_{j=1}^k A_j \right) = \emptyset$ .

## 7 Problem Set 7

### Problem 7.1

Consider the set

$$X := \left\{ (a_1, a_2, a_3, \dots) \mid \sum_{j=1}^{\infty} a_j^2 \text{ converges} \right\}.$$

Define the function  $d : X \times X \rightarrow \mathbb{R}$  as follows. For

$$x = (a_1, a_2, a_3, \dots), \quad y = (b_1, b_2, b_3, \dots) \in X,$$

define

$$d(x, y) := \sqrt{\sum_{j=1}^{\infty} (a_j - b_j)^2}.$$

1. Show that the function  $d$  is well-defined. Equivalently, show that for

$$x = (a_1, a_2, a_3, \dots) \quad \text{and} \quad y = (b_1, b_2, b_3, \dots) \in X,$$

the series

$$\sum_{j=1}^{\infty} (a_j - b_j)^2$$

converges.

2. Show that the function  $d$  satisfies the triangle inequality. You may use the following triangle inequality in  $\mathbb{R}^n$  without proof. For all  $n \geq 1$ ,

$$\sqrt{\sum_{j=1}^n (a_j - c_j)^2} \leq \sqrt{\sum_{j=1}^n (a_j - b_j)^2} + \sqrt{\sum_{j=1}^n (b_j - c_j)^2}.$$

3. Consider a sequence in  $X$  defined as

$$x_1 = (1, 0, 0, 0, \dots), \quad x_2 = (0, 1, 0, 0, \dots), \quad \text{and so on.}$$

In general,

$$x_n = (0, \dots, 0, \underbrace{1}_{\text{nth position}}, 0, \dots).$$

Show that  $(x_n)$  has no convergent subsequence.

*Solution.*

For (1.), Let  $S_N^a = \sum_{j=1}^N a_j^2$  and  $S_N^b = \sum_{j=1}^N b_j^2$ . We know that  $\lim_{n \rightarrow \infty} S_n^a$  and  $\lim_{n \rightarrow \infty} S_n^b$  exists, therefore  $\lim_{n \rightarrow \infty} S_n^a + S_n^b$  exists. And since,

$$0 \leq (a_j - b_j)^2 = a_j^2 + b_j^2 - 2a_j b_j \leq a_j^2 + b_j^2 + 2|a_j b_j| \leq 2(a_j^2 + b_j^2)$$

The last inequality is due to AM-GM inequality. Therefore,

$$0 \leq (a_j - b_j)^2 \leq 2(a_j^2 + b_j^2)$$

and we have that  $\sum_{j=1}^{\infty} (a_j - b_j)^2$  converges from comparison test.

For (2.), just takes the limit as  $n \rightarrow \infty$ .

For (3.), suppose there is a convergent subsequence and let  $x_{n_k} \rightarrow x$

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, x) + d(x, x_{n_\ell})$$

Now, notice that for  $n_k \neq n_\ell$  we have  $d(x_{n_k}, x_{n_\ell}) = \sqrt{2}$ . Thus, for large  $n_k$  and  $m_\ell$  we have

$$\begin{aligned} d(x_{n_k}, x_{n_\ell}) &\leq d(x_{n_k}, x) + d(x, x_{n_\ell}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\implies \sqrt{2} < \varepsilon \end{aligned}$$

But for  $\varepsilon < \sqrt{2}$  that's false.

### Problem 7.2

Let  $(X, d)$  be a metric space and  $(x_n)$  a sequence in  $X$ . Denote

$$E = \{x_1, x_2, x_3, \dots\}.$$

Suppose  $(x_n)$  has no convergent subsequence. Show that for all  $k \in \mathbb{N}$ , there exists  $r_k > 0$  such that

$$B(x_k, r_k) \cap E = \{x_k\}.$$

You may use the following fact without proof. Fix  $x \in X$ . Suppose that for all  $r > 0$ , there are infinitely many elements in  $E \cap B(x, r)$ . Then  $(x_n)$  has a subsequence which converges to  $x$ .

*Solution.* Using the fact, since no sequence of  $(x_n)$  converges to  $x$  there exists a  $r > 0$  such that  $E \cap B(x, r)$  is finite. Since no subsequence converges to  $x_k$  for  $k \in \mathbb{N}$ , we have  $r > 0$  s.t  $E \cap B(x_k, r)$  is finite. If  $|E \cap B(x_k, r)| = 1$  then we have the desired condition but suppose  $|E \cap B(x_k, r)| > 1$  then

$$E \cap B(x_k, r) = \{x_k, x_{m_1}, \dots, x_{m_\ell}\}$$

Then, since  $d(x_k, x_{m_i}) > 0$  for all  $1 \leq i \leq \ell$  we have

$$E \cap B\left(x_k, \min_{1 \leq i \leq \ell} \{d(x_k, x_{m_i})\}\right) = \{x_k\}$$

### Problem 7.3

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Suppose that  $f(x)$  is differentiable on  $(-\infty, 0) \cup (0, \infty)$  and that the limit  $\lim_{x \rightarrow 0} f'(x)$  exists. Show that  $f(x)$  is differentiable at  $x = 0$  and that  $f'(0) = \lim_{x \rightarrow 0} f'(x)$ .

*Solution.*

Let  $h \neq 0$ . Since  $f$  is continuous on the closed interval with endpoints 0 and  $h$  and differentiable on the open interval with end points 0 and  $h$ , we can use MVT. Thus, there exists a point  $c_h$  between 0 and  $h$  such that

$$\frac{f(h) - f(0)}{h} = f'(c_h).$$

As  $h \rightarrow 0$ , the point  $c_h$  lies between 0 and  $h$ , and therefore  $c_h \rightarrow 0$ . Since  $\lim_{x \rightarrow 0} f'(x) = L$ , it follows that

$$f'(c_h) \rightarrow L \quad \text{as } h \rightarrow 0.$$

Hence,

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} f'(c_h) = L.$$

Therefore, the derivative of  $f$  at 0 exists and satisfies

$$f'(0) = L = \lim_{x \rightarrow 0} f'(x).$$

#### Problem 7.4

Use  $\frac{d}{dx} e^x = e^x$  to show that  $e^x \geq 1 + x$  for all  $x \in \mathbb{R}$ .

*Solution.*

Notice that  $f(x) = e^x$  is continuous everywhere and differentiable everywhere. For  $x = 0$  it is obvious  $e^0 = 1$ . Assume that  $x > 0$  then  $f$  is continuous on  $[0, x]$  and differentiable on  $(0, x)$  thus applying MVT we get,

$$\begin{aligned} \frac{f(x) - f(0)}{x} &= f(c) \\ \implies \frac{e^x - 1}{x} &= e^c \end{aligned}$$

some for  $0 < c < x$ , we have  $e^c \geq 1$  we have

$$\begin{aligned} e^x &= e^c \cdot x + 1 \\ &\geq x + 1 \end{aligned}$$

Now assume  $x < 0$ , we have  $x = -a$  thus using MVT again we get,

$$\frac{e^{-a} - 1}{-a} = e^c$$

Since  $0 > c > -a$  we get  $e^c \leq 1 \implies e^{-a} = e^c(-a) + 1 \geq -a + 1$ .

**Problem 7.5**

Let  $X = C([0, 1])$  and let  $d : X \times X \rightarrow \mathbb{R}$  be defined by

$$d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|.$$

Suppose  $(f_n) \subset X$  is a Cauchy sequence.

1. Fix an arbitrary  $x_0 \in [0, 1]$ . Show that  $\lim_{n \rightarrow \infty} f_n(x_0)$  exists.
2. Define the function  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(x) := \lim_{n \rightarrow \infty} f_n(x).$$

Show that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all  $x \in [0, 1]$  and for all  $n \geq N$ .

3. Show that  $f(x)$  is continuous on  $[0, 1]$ . Equivalently, show that  $f \in X$ .
4. Show that

$$\lim_{n \rightarrow \infty} f_n = f$$

as a sequence in  $X$ .

*Solution.*

For (1.), since the  $f_n$  is Cauchy that means  $f_n(x_0)$  is also Cauchy. But since its a real valued function and  $\mathbb{R}$  is complete we have that  $f_n(x_0)$  is convergent. Thus the  $\lim_{n \rightarrow \infty} f_n(x_0)$  exists.

For (2.) we will use the fact that,  $\lim_{n \rightarrow \infty} |a_n \pm b_n| = \left| \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n \right|$  if both the limits exist. Since  $f_n$  is cauchy, we have

$$|f_n(x) - f_m(x)| \leq \max_{x \in [0, 1]} |f_n(x) - f_m(x)| < \varepsilon$$

for  $x \in [0, 1]$  and for  $n, m \geq N$ . Then using the properties of limits we have,

$$\lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| < \varepsilon$$

Therefore using the fact, we have

$$|f_n(x) - f(x)| < \varepsilon$$

For (3.), let  $x_0 \in [0, 1]$  then

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

Now, for  $n \geq N$  we have  $|f(x) - f_n(x)| < \varepsilon/3$  and  $|f(x_0) - f_n(x_0)| < \varepsilon/3$  from (2). and since  $f_n \in C([0, 1])$  we can pick a  $\delta_1$  for s.t whenever  $0 < |x - x_0| < \delta_1$  we have  $|f_n(x) - f_n(x_0)| < \varepsilon/3$ .

Thus, if we choose  $\delta = \delta_1$  then we have

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$

Thus  $f \in C([0, 1])$ .

For (4.), since we know that  $f \in C([0, 1])$ . Thus

$$\max_{x \in [0, 1]} |f_n(x) - f(x)| < \varepsilon$$

Thus,  $d(f_n, f) < \varepsilon$  for any  $\varepsilon > 0$  so we have  $f_n \rightarrow f$  in  $X$ .

## 8 Problem Set 8

### Problem 8.1

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Suppose both  $f'(x)$  and  $f''(x)$  are continuous on  $\mathbb{R}$  and that  $f(0) = 0$ . Define the function

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

Show that  $g'(x)$  exists for all  $x \in \mathbb{R}$  and express  $g'(x)$  in terms of  $f(x)$  and its derivative.

*Solution.*

For  $x \neq 0$ , we have that

$$g'(x) = \frac{x f'(x) - f(x)}{x^2}$$

For  $x = 0$ , we can use the definition of the derivative,

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x) - x f'(0)}{x^2}$$

Now using taylor's formula near 0, we have that

$$\begin{aligned} f(x) &= f(0) + x f'(0) + x^2 f''(c_x) \cdot \frac{1}{2} \\ \implies \frac{f(x) - x f'(0)}{x^2} &= \frac{1}{2} \cdot f''(c_x) \\ \implies \lim_{x \rightarrow 0} \frac{f(x) - x f'(0)}{x^2} &= \lim_{x \rightarrow 0} \frac{f''(c_x)}{2} = \frac{f''(0)}{2} \end{aligned}$$

Thus,  $g'(0)$  also exists and

$$g'(x) = \begin{cases} \frac{x f'(x) - f(x)}{x^2} & x \neq 0 \\ \frac{f''(0)}{2} & x = 0 \end{cases}$$

**Problem 8.2**

Suppose there exist two functions  $S : \mathbb{R} \rightarrow \mathbb{R}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}$  which satisfy the following properties:

- $\frac{d}{dx}S(x) = C(x)$ ,  $\frac{d}{dx}C(x) = S(x)$ .
- $S(0) = 0$ ,  $C(0) = 1$ .

(1) Let  $S^{(n)}(x)$  be the  $n$ th derivative of  $S(x)$ . Show that for  $k \in \mathbb{N} \cup \{0\}$ ,

$$S^{(2k)}(x) = S(x), \quad S^{(2k+1)}(x) = C(x).$$

(2) Show that for all  $x \in \mathbb{R}$ ,

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

*Solution.*

For (1.) just use induction.

For (2.), restrict  $S$  to  $[0, x]$  for some fixed  $x > 0$ . Then,

$$S(x) = \sum_{n=0}^k \frac{S^n(0)x^n}{(n)!} + R_k(x) = \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{x^{2n+1}}{(2n+1)!} + R_k(x)$$

But since  $S$  and  $C$  are bounded and continuous on  $[0, x]$  and we have  $S^n(x) = S(x)$  or  $C(x)$ , thus  $|S^n(x)| \leq M$  for  $n \geq 0$ .

$$|R_k| = \left| \frac{S^{k+1}(c)x^{k+1}}{(k+1)!} \right| \leq \left| \frac{Mx^{k+1}}{(k+1)!} \right| < \varepsilon$$

Thus,

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

**Problem 8.3**

Let  $a < b$  be two real numbers and let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. Suppose  $f(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Suppose  $f'(x) > 0$  for all  $x \in (a, b)$ .

1. Show that  $f(x)$  is strictly increasing on  $[a, b]$ . That is,  $f(x_1) < f(x_2)$  for all  $x_1 < x_2$  in  $[a, b]$ .
2. Show that for all  $y \in (f(a), f(b))$ , there exists a unique  $x \in (a, b)$  such that  $f(x) = y$ .
3. Let the function  $g : (f(a), f(b)) \rightarrow (a, b)$  be the inverse function of  $f(x)$ . In other words,  $g(y) = x$  if  $f(x) = y$ . Show that  $g$  is continuous on  $(f(a), f(b))$ .
4. Show that  $g$  is differentiable on  $(f(a), f(b))$  and that

$$g'(y) = \frac{1}{f'(g(y))} \quad \text{for all } y \in (f(a), f(b)).$$

*Solution.*

For (1.), Let  $x_1 < x_2$  be points in  $[a, b]$ . By the Mean Value Theorem, there exists  $c \in (x_1, x_2)$  such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

Since  $f'(c) > 0$  and  $x_2 - x_1 > 0$ , it follows that

$$f(x_2) - f(x_1) > 0,$$

and hence  $f(x_1) < f(x_2)$ . Therefore,  $f$  is strictly increasing on  $[a, b]$ .

For (2.), since  $f$  is continuous on  $[a, b]$ , the IVT implies that for every  $y \in (f(a), f(b))$  there exists  $x \in (a, b)$  such that  $f(x) = y$ . To prove uniqueness, suppose  $x_1 < x_2$  and  $f(x_1) = f(x_2)$ . This is a contradiction as  $f(x_1) < f(x_2)$ .

For (3.), fix  $y_0 \in (f(a), f(b))$  and let  $x_0 = g(y_0)$ . Let  $\varepsilon > 0$  be given. Since  $f$  is strictly increasing, we have

$$f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon).$$

Define

$$\delta := \min\{f(x_0 + \varepsilon) - f(x_0), f(x_0) - f(x_0 - \varepsilon)\}.$$

Then  $\delta > 0$ .

Now suppose  $y \in (f(a), f(b))$  satisfies  $|y - y_0| < \delta$ . Then

$$f(x_0 - \varepsilon) < y < f(x_0 + \varepsilon).$$

By the definition of the inverse function and since  $f$  is increasing, we have

$$x_0 - \varepsilon < g(y) < x_0 + \varepsilon.$$

Hence,

$$|g(y) - g(y_0)| < \varepsilon.$$

Thus  $g$  is continuous on  $(f(a), f(b))$ .

For (4.), fix  $y_0 \in (f(a), f(b))$  and let  $x_0 = g(y_0)$ , so that  $f(x_0) = y_0$ . For  $y \neq y_0$ , write  $y = f(x)$ , so that  $x = g(y)$ . Then

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}.$$

Taking the limit as  $y \rightarrow y_0$  is equivalent to taking the limit as  $x \rightarrow x_0$ , since  $g$  is continuous. Therefore,

$$\lim_{y \rightarrow y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}.$$

Thus  $g$  is differentiable at  $y_0$  and

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Since  $y_0$  was arbitrary, this holds for all  $y \in (f(a), f(b))$ .

#### Problem 8.4

Define the function  $f : [-1, 1] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1, & x \in [-1, 0) \cup (0, 1], \\ 0, & x = 0. \end{cases}$$

Show that  $f(x)$  is Riemann integrable on  $[-1, 1]$  and that

$$\int_{-1}^1 f(x) dx = 2.$$