Real Analysis

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Abstract

I really don't feel like doing analysis the way I did group theory and linear algebra, where I type out my notes on a latex file. Instead, I'll do my **Analysis I** from MIT OCW and post the problems and the solutions here.

1 Problem Set 1

Problem 1.1

Let $\mathbb{F}$ be a ordered field with $1 \neq 0$. Show that $1 > 0$.

Solution. First let us prove that $(-1) \cdot (-1) = 1$. We know that for all $x \in \mathbb{F}$, there is an inverse element $-x$ such that,

$$x + (-x) = 0$$

Thus, $1 + (-1) = 0$ which means that

$$0 = (-1) \cdot 0 = (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) = (-1) + (-1) \cdot (-1)$$
$$\implies 1 = (-1) \cdot (-1)$$

Since $\mathbb{F}$ is a ordered field, one of the statement below must be true because of the **first axiom of order**.

$$1 < 0, \quad 1 = 0, \quad 0 < 1 \tag{1}$$

We assumed that $1 \neq 0$ so the middle statement can't be true and if $1 < 0$ then $0 < (-1)$. But from **axiom of order and multiplication** $0 < (-1) \cdot (-1) = 1$. Thus a contradiction.

Problem 1.2

Define the addition of two rational numbers by

$$\frac{n}{m} + \frac{p}{q} := \frac{nq + mp}{mq}.$$

Show that it is well-defined.

Solution. Suppose $\frac{n}{m} = \frac{n_1}{m_1}$ and $\frac{p}{q} = \frac{p_1}{q_1}$ then using the definition of when two rational numbers are equal, we get $nm_1 = n_1 m$ and $pq_1 = p_1 q$. Thus,

$$\begin{aligned} m_1 q_1 (nq + pm) &= m_1 q_1 nq + m_1 q_1 pm \\ &= n_1 m q_1 q + p_1 q m_1 m \\ &= mq(n_1 q_1 + m_1 p_1) \end{aligned}$$

Thus, $\frac{n}{m} + \frac{p}{q} = \frac{n_1}{m_1} + \frac{p_1}{q_1}$.

Problem 1.3

Find the $\sup E$ and $\inf E$ for the following set E.

1. $E = \left\{ n \in \mathbb{Z} \mid n < \sqrt{12} \right\}$
2. $E = \left\{ r \in \mathbb{Q} \mid r < \sqrt{12} \right\}$
3. $E = \{ x \in \mathbb{R} \mid x^2 - x - 1 < 0 \}$
4. $E = \left\{ \frac{n^2+n}{n+1} \mid n \in \mathbb{N} \right\}$

Solution.

1. $\sup E = 3$ but $infE$ doesn't exist.
2. $\sup E = \sqrt{12}$ but $infE$ doesn't exist.
3. $\sup E = \frac{1+\sqrt{5}}{2}$ and $\inf E = \frac{1-\sqrt{5}}{2}$.
4. $\inf E = 1$ but $\sup E$ doesn't exist.

Problem 1.4

Let $\mathbb{M}$ be the set of polynomials with integer coefficients i.e,

$$\mathbb{M} := \{f(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Z}\}$$

Define the relation $0 \prec f$ if $0 < f(x)$ for x large enough. More precisely, we say

$$0 \prec f \quad \text{if there exists } M > 0 \text{ such that } f(x) > 0 \text{ for all } x > M.$$

Then define

$$f \prec g \quad \text{if } 0 \prec (g - f).$$

Show that $(M, \prec)$ is an ordered set.

Solution.
We'll use the fact that for large enough x, $f(x) > 0$ for $a_n > 0$. Let $f, g \in \mathbb{M}$ such that $f \neq g$ and

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

Define $h(x) = g(x) - f(x) = (b_0 - a_0) + (b_1 - a_1)x + \cdots + (b_n - a_n)x^n$ and let $c_i = b_i - a_i$. Suppose c_k is the highest degree non zero coefficient, then if

1. $c_k > 0$ then $f \prec g$
2. $c_k < 0$ then $g \prec f$

Suppose $f \prec g$ and $g \prec h$. Then, $f(x) < g(x)$ and $g(x) < h(x)$ for large x and hence $f(x) < h(x)$ for large x. Thus, $0 \prec (h - f)$ which means $f \prec h$.

Problem 1.5

Prove that $(\mathbb{M}, \prec)$ **doesn't** satisfy the archimedean property.

Solution.
To show that it doesn't satisfy the archimedean property, we need to show that $\exists f, g \in \mathbb{M}$ such that

$$\forall n \in \mathbb{N}, g \not\prec nf$$

If we choose $g(x) = x^2$ and $f(x) = x$ and assume $g \prec nf$ then , $0 \prec x(n - x)$ which means that $0 < x(n - x)$ which we know is false for large x. Thus, $g \not\prec nf$.

Problem 1.6

Show that for any non-empty set $E \subset R$ which is bounded from below, E has the greatest lower bound.

Solution.
To show that E has greatest lower bound define

$$-E = \{-x \mid x \in E\}$$

If α is any lower bound of E then $x \geq \alpha \Rightarrow -x \leq -\alpha$. That means that $-E$ is bounded above by $-\alpha$. Since $-E$ is bounded above by $-\alpha$, it must have the least upper bound property. Let

$$\sup -E = \beta$$

Thus, $\beta \geq -x \Rightarrow x \geq -\beta$ and $\beta \leq -\alpha$ for any lower bound α of E. Thus, $\alpha \leq -\beta$.