Irrationality of π

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Introduction

The irrationality of π has been a subject of fascination for centuries due to the central role π plays in mathematics, particularly in geometry and analysis. While Lambert first proved the irrationality of π in the 18th century using continued fractions, several alternative proofs have since been discovered, some of which rely on elementary techniques. In this exposition, we present a proof inspired by the approach of Ivan Niven, which constructs a special auxiliary function involving π, analyzes its derivatives, and ultimately derives a contradiction under the assumption that π is rational.

The method is notable for being accessible yet elegant, requiring only tools from calculus, elementary number theory, and induction. We will proceed step-by-step to establish the necessary properties of the constructed function, leading to the desired conclusion.

Proof Outline

We proceed by contradiction, assuming that π is a rational number, say $\pi = \frac{a}{b}$ for some positive integers a and b. Based on this assumption, we define a function

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

for a fixed positive integer n. We then construct another function $G(x)$ as a particular linear combination of $f(x)$ and its higher derivatives.

The key steps of the proof are as follows:

1. We show that for each non-negative integer i, the values $f^{(i)}(0)$ and $f^{(i)}(\pi)$ are integers. This involves expanding $f(x)$, analyzing its coefficients, and proving an auxiliary lemma on the divisibility of certain factorial expressions.

2. Using the function $G(x)$, we derive that

   $$I = \int_0^\pi f(x)sin(x)dx$$

   is equal to $G(\pi) + G(0)$, which must therefore be an integer.

3. We obtain an upper bound for the integral I and show that as n becomes large, I approaches zero. Since I is both an integer and arbitrarily close to zero, this leads to a contradiction unless $I = 0$, which is impossible. Therefore, our initial assumption that π is rational must be false. This completes the proof that π is irrational.

Assume, for contradiction, that $\pi = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. We define:

- $f(x) = \frac{x^n(a-bx)^n}{n!}$,
- $G(x) = f(x) - f^{(2)}(x) + f^{(3)}(x) + \cdots + (-1)^n f^{(2n)}(x)$.

We first observe that $f^{(i)}(x)$ takes integer values at $x = 0$ and $x = \pi$. Indeed, we have:

$$f\left(\frac{a}{b} - x\right) = f(x)$$

Differentiating both sides:

$$\begin{aligned} f'\left(\frac{a}{b} - x\right) \cdot (-1) &= f'(x) \\ f''\left(\frac{a}{b} - x\right) &= f''(x) \\ &\vdots \\ f^{(k)}\left(\frac{a}{b} - x\right) &= (-1)^k f^{(k)}(x) \end{aligned}$$

Setting $x = 0$, it follows that whenever $f^{(i)}(0)$ is an integer, so is $f^{(i)}(\pi)$. This can be formalized via induction, but we will omit that for brevity. Our task now reduces to proving that $f^{(i)}(0)$ is an integer for all $i \in \mathbb{N}$.

We expand $f(x)$ as follows:

$$\begin{aligned} f(x) &= \frac{x^n(a-bx)^n}{n!} \\ &= \sum_{r=0}^{n} (-1)^r a^{n-r} b^r x^{r+n} \binom{n}{r} \cdot \frac{1}{n!} \end{aligned}$$

Define $Z_r = (-1)^r a^{n-r} b^r$ for convenience. Then:

$$f(x) = \sum_{r=0}^{n} \frac{Z_r \cdot \binom{n}{r} \cdot x^{n+r}}{n!}$$

For $i < n$, clearly $f^{(i)}(0) = 0$ since every term in $f(x)$ contains a factor of at least x^n.

Thus, it remains to prove that $f^{(i)}(0)$ is an integer for $i \geq n$. Consider:

$$f^{(n)}(x) = C_1 + C_2 x + C_3 x^2 + \cdots + C_{2n} x^n$$

If we can show that each C_i is an integer, then all higher derivatives evaluated at $x = 0$ will also be integers.
Now, consider the $(r+1)$-th term of $f(x)$:

$$f_{r+1}(x) = \frac{Z_r \cdot \binom{n}{r} \cdot x^{n+r}}{n!}$$

Taking the n-th derivative:

$$f_{r+1}^{(n)}(x) = \frac{Z_r \cdot \binom{n}{r} \cdot x^r}{n!} \cdot (n+r)(n+r-1)\cdots(r+1)$$

This gives:

$$C_{r+1} = \frac{Z_r \cdot \binom{n}{r} \cdot (n+r)(n+r-1)\cdots(r+1)}{n!} \in \mathbb{Z}$$

Since Z_r is an integer, it suffices to prove that $\frac{C_{r+1}}{Z_r}$ is an integer. For $r \geq 1$, we compute:

$$\begin{aligned}
\frac{C_{r+1}}{Z_r} &= \frac{\binom{n}{r} \cdot (n+r)(n+r-1)\cdots(r+1)}{n!} \\
&= \frac{\frac{n!}{r!(n-r)!} \cdot (n+r)(n+r-1)\cdots(r+1)}{n!} \\
&= \frac{(n+r)(n+r-1)\cdots(r+1)\cdot r!}{(r!)^2(n-r)!} \\
&= \frac{(n+r)!}{(r!)^2(n-r)!} \\
&= \frac{(n+r)(n+r-1)\cdots(n-r+1)}{(r!)^2}
\end{aligned}$$

We now invoke the following result:

Lemma 0.1. *For any $n, r \in \mathbb{N}$,*

$$\frac{(n+1)(n+2)\cdots(n+r)}{r!} \in \mathbb{Z}.$$

We prove this by induction. The base case $r = 2$ is clear.

Suppose the result holds for r. Then for $r+1$:

$$\frac{(n+1)(n+2)\cdots(n+r)}{r!} \cdot \frac{n+r+1}{r+1}$$

Since the numbers $n+1, n+2, \cdots, n+r, n+r+1$ form $r+1$ consecutive integers, one of them must be divisible by $r+1$. This completes the induction.
Returning, we write:

$$\begin{aligned}\frac{C_{r+1}}{Z_r} &= \frac{(n+r)(n+r-1)\cdots(n-r+1)}{(r!)^2} \\ &= \left(\frac{(n+r)\cdots(n+1)}{r!}\right)\left(\frac{n\cdots(n-r+1)}{r!}\right)\end{aligned}$$

By the lemma, both factors are integers. Thus, we have shown that $f^{(i)}(0)$ and $f^{(i)}(\pi)$ are integers for all i.

Next, consider:

$$\begin{aligned}\frac{d}{dx}\left(G'(x)\sin(x) - G(x)\cos(x)\right) &= G''(x)\sin(x) + G'(x)\cos(x) - G'(x)\cos(x) + G(x)\sin(x) \\ &= G''(x)\sin(x) + G(x)\sin(x) \\ &= f(x)\sin(x)\end{aligned}$$

Thus,

$$I = \int_0^\pi f(x)\sin(x)dx = [G'(x)\sin(x) - G(x)\cos(x)]_0^\pi = G(\pi) + G(0)$$

Since $G(\pi)$ and $G(0)$ are sums of $f^{(i)}(\pi)$ and $f^{(i)}(0)$ respectively, I is an integer.

However, for $0 < x < \pi$,

$$0 < f(x)\sin(x) \le f(x) = \frac{x^n(a-bx)^n}{n!} < \frac{\pi^n a^n}{n!}$$

since $x^n < \pi^n$ and $a - bx < a$ implies $(a-bx)^n < a^n$. Therefore,

$$0 < f(x)\sin(x) < \frac{\pi^n a^n}{n!}.$$

Integrating,

$$0 < \int_0^\pi f(x)\sin(x)dx < \frac{\pi^{n+1}a^n}{n!}.$$

But,

$$\lim_{n\to\infty} \frac{\pi^{n+1}a^n}{n!} = 0.$$

Thus, I is a positive integer arbitrarily close to zero, which is impossible. Therefore, π cannot be rational.

References

[1] Ivan Niven, *A simple proof that π is irrational*, Bulletin of the American Mathematical Society, **53** (1947), 509.
https://heuklyd.github.io/papers/pdf/Niven-1947.pdf