# Macroeconomics I: Problem Set 3

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## 1 Problem 1: Robots in an Overlapping Generations Model

### 1.1 Let $K(t) > 0$ denote the equilibrium level of capital in period $t$ (which is equal to $s(t-1)$). Show that the allocation of capital across technologies and equilibrium prices in period $t$ are as follows. If $K(t) < \frac{1}{4(A(t))^2}$, then $K^0(t) = K(t)$, $R(t) = \frac{1}{2}(K(t))^{-\frac{1}{2}}$ and $w(t) = \frac{1}{2}(K(t))^{1/2}$. If $K(t) \geq \frac{1}{4(A(t))^2}$, then $K^0(t) = \frac{1}{4(A(t))^2}$, $R(t) = A(t)$ and $w(t) = \frac{1}{4A(t)}$.

Let us begin with the situation in which $K(t) < \frac{1}{4(A(t))^2}$. In that case we have:

$$K(T) < \frac{1}{4(A(t))^2} \implies (A(t))^2 < \frac{1}{4K(t)} \implies A(t) < \frac{1}{2}(K(t))^{-\frac{1}{2}}.$$

This result implies that for every $K(t)$, we have that the marginal return of capital of the old firm is larger than the marginal return of capital for the robot technology. Therefore we indeed obtain $K^o(t) = K(t)$. Since in equilibrium the wage will be equal to the marginal return of labor, we can set

$$w(t) = \frac{\partial Y^o(t)}{\partial L(t)}\Big|_{L(t)=1,\ K^o(t)=K(t)} = \frac{1}{2}(K(t))^{\frac{1}{2}}(L(t))^{-\frac{1}{2}}\Big|_{L(t)=1} = \frac{1}{2}(K(t))^{\frac{1}{2}}.$$

We also know that in equilibrium $R(t)$ should be equal to the marginal return of capital. That is:

$$R(t) = \frac{\partial Y^o(t)}{\partial K^o(t)}\Big|_{L(t)=1,\ K^o(t)=K(t)} = \frac{1}{2}(K(t))^{-\frac{1}{2}}.$$

Using the results derived above we thus indeed have that if $K(t) < \frac{1}{4(A(t))^2}$, then $K^o(t) = K(t),\ R(t) = \frac{1}{2}(K(t))^{-\frac{1}{2}}$ and $w(t) = \frac{1}{2}(K(t))^{\frac{1}{2}}$.

Let us now consider the case where $K(t) \geq \frac{1}{4(A(t))^2}$. Then, $A(t) \geq \frac{1}{2}(K(t))^{-\frac{1}{2}}$. We can split this into $A(t) = \frac{1}{2}(K(t))^{-\frac{1}{2}}$ and $A(t) > \frac{1}{2}(K(t))^{-\frac{1}{2}}$. In the latter case, the demand of capital would be infinite, whereas there is only a finite amount of capital provided to the economy. Therefore, if $A(t) > \frac{1}{2}(K(t))^{-\frac{1}{2}}$, $K^o(t) = 0$ which results in a marginal return on capital of infinity (in the limit) for the firm with the old technology:

$$\lim_{x\to 0} \frac{\partial Y^o(t)}{\partial K^o(t)}\Big|_{L(t)=1,\ K^o(t)=x} = \lim_{x\to 0} \frac{1}{2(K^o(t))^{\frac{1}{2}}}\Big|_{K^o(t)=x} \to \infty.$$

Therefore, if all capital would be assigned to the robot technology, the marginal return of capital is extremely large for the old technology. Therefore, it is not possible that $A(t) > \frac{1}{2}(K^o(t))^{-\frac{1}{2}}$ in equilibrium and we thus require that $A(t) = \frac{1}{2}(K^o(t))^{-\frac{1}{2}} = R(t)$. Out of equilibrium, that is when $A(t) > \frac{1}{2}(K^o(t))^{-\frac{1}{2}}$ households will transfer capital to the robot technology until $A(t) = \frac{1}{2}(K^o(t))^{-\frac{1}{2}}$. On the other hand, if $A(t) < \frac{1}{2}(K^o(t))^{-\frac{1}{2}}$, households will move their capital from the robot technology to the old technology until $A(t) = \frac{1}{2}(K^o(t))^{-\frac{1}{2}}$. Some trivial rewriting of the equation indeed results in $K^0(t) = \frac{1}{4(A(t))^2}$.

Since $A(t) = R(t) = \frac{1}{2}(K^o(t))^{-\frac{1}{2}}$, we can plug in $A(t)$ in the marginal return for labor in for the old technology. This yields:

$$\begin{aligned}
w(t) &= \left.\frac{\partial Y^o(t)}{\partial L(t)}\right|_{L(t)=1,\ K^o(t)=\frac{1}{4(A(t))^2}} \\
&= \left.\frac{1}{2}(K^o(t))^{\frac{1}{2}}(L(t))^{-\frac{1}{2}}\right|_{L(t)=1,\ K^o(t)=\frac{1}{4(A(t))^2}} \\
&= \left.\frac{1}{2}(K^o(t))^{\frac{1}{2}}\right|_{K^o(t)=\frac{1}{4A(t)^2}} \\
&= \frac{1}{2}\left(\frac{1}{4(A(t))^2}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\frac{1}{2A(t)} \\
&= \frac{1}{4A(t)}.
\end{aligned}$$

The derivations above confirm that if $K(t) \geq \frac{1}{4(A(t))^2}$ we have that $w(t) = \frac{1}{4A(t)}$.

## 1.2 Show using induction that $K(t) > \frac{1}{4(A(t))^2}$ and $K(t+1) = \frac{1}{8A(t)}$ for all $t \geq 0$.

We have been provided with the information that $s(t-1) = K^o(t) + K^r(t)(= K(t))$ and $s(w(t), R(t+1)) = \frac{1}{2}w(t)$. For $K(0)$ we know that $K(0) > \frac{1}{4(A(0))^2}$. First we need to check whether $K(1) = \frac{1}{8(A(1))}$. From 1.1: if $K(t) \geq \frac{1}{4(A(t))^2}$ we have $w(t) = \frac{1}{4A(t)}$. We have $K(0) > \frac{1}{4(A(0))^2}$ so $w(0) = \frac{1}{4A(0)}$. Then:

$$K(1) = s\left(\frac{1}{4A(1)}, R(2)\right) = \frac{1}{2}\frac{1}{4A(1)} = \frac{1}{8A(1)}.$$

So the result indeed holds for $\tau = 0$. We will now show that it also holds for $\tau = t+1$, given that it holds for $\tau = t$. Concretely, we show that, given $K(t) > \frac{1}{4(A(t))^2}$ and $K(t+1) = \frac{1}{8A(t)}$, we have that $K(t+1) > \frac{1}{4(A(t+1))^2}$ and $K(t+2) = \frac{1}{8A(t+1)}$. We have that:

$$\begin{aligned}
K(t+1) = \frac{1}{8A(t)} &\overset{?}{>} \frac{1}{4(A(t+1))^2} \\
&\implies 4(A(t+1))^2 \overset{?}{>} 8A(t) \\
&\implies 4((1+g)^{2(t+1)}(A(0))^2 \overset{?}{>} 8(1+g)^t A(0) \\
&\implies (1+g)^{t+t+2}(A(0))^2 \overset{?}{>} \frac{1}{2}(1+g)^t A(0) \\
&\implies (1+g)^{t+2}A(0) \overset{?}{>} \frac{1}{2}.
\end{aligned}$$

Since we know that $(1+g)^{t+2} > 1$ and $A(0) > 2$, we indeed must have that $(1+g)^{t+2}A(0) > \frac{1}{2}$.

Since we have derived that $K(t+1) > \frac{1}{4(A(t+1))^2}$, we can show that indeed $K(t+2) = \frac{1}{8A(t+1)}$ since $K(t+2) = s(w(t+1), R(t+2)) = \frac{1}{2}w(t+1) = \frac{1}{2}\frac{1}{4A(t+1)} = \frac{1}{8A(t+1)}$.

### 1.3 Compute a formula that gives equilibrium aggregate output $Y(t) \equiv Y^0(t) + Y^r(t)$ as a function of $A(t)$ and $g$ for all $t \geq 1$. Is aggregate output increasing, constant, or decreasing over time? Explain the economic mechanism driving this evolution of output over time.

If $K(0) \geq \frac{1}{4(A(t))^2}$, then $K(t) = K^o(t) + K^r(t)$. Then, as shown in the previous questions, $K^o(t) = \frac{1}{4(A(t))^2}$ and $K(t) = \frac{1}{8A(t-1)}$. Then, $K^r(t)$ is the difference between them: $K^r(t) = \frac{1}{8A(t-1)} - \frac{1}{4(A(t))^2}$. Then, we substitute the quantities for $K^r(t)$ and $K^o(t)$ in their respective output formulas, and summing them gives, after simplification and realizing that $\frac{A(t)}{A(t-1)} = (1+g)$:

$$Y(t) = \frac{1}{4A(t)} + \frac{(1+g)}{8}$$

The output should be decreasing over time because $A(t)$ grows over time. Since the robots become more productive over time, capital accumulation shifts to the robot technology, making them more productive. To keep the marginal products between the two technologies equal, fewer and fewer capital is used in the old production technology. Aside from the shift in relative amounts of capital towards the robot technology, the households care about consumption. This implies that the amount of capital they will use will decrease over time, because the same amount of capital can sustain more and more consumption, or in other words, one can have the same consumption with less capital. This brings about a trend in which capital decreases from generation to generation, and consequently, output decreases, and more and more of that output will go to consumption.

### 1.4 Consider a policy intervention that prohibits using the robot technology. Would implementing this intervention yield a Pareto improvement? Explain your answer.

Since the output decreases over time following a robot technology introduction, it could be that households are better of with higher amounts of output, forcing them to use the old capital. However, generation -1 consumes only in the first period, consuming the rent plus the endowment: $R(0) = A(0)$. If the robots weren't there, we know the equilibrium rental rate is strictly lower: $R(0) = \frac{1}{2}(K(0))^{-\frac{1}{2}} < A(0)$ because the household has to rent more capital in comparison to when there were robots. Hence, generation -1 would be strictly worse off, whereas all other generations could be made better off.

## 2 Problem 2: Binding Non-Negative Investment

### 2.1 Set up the maximization problem on the right-hand side of equation (1) as a constrained maximization problem, letting $\lambda$ denote the Lagrange multiplier on the inequality constraint $K' \geq (1-\delta)K$ and ignoring the other inequality constraints mentioned earlier. Write down the first-order condition for $K'$. Provide an economic interpretation of this condition, explaining in particular the role of $\lambda$.

First, we set up the maximization problem as a constrained maximization problem with the $L$ as the Lagrangian,

$$L = u(F(K, A(z)(1-l)) + (1-\delta)K - K', l) + \beta \sum_{z' \in \mathcal{Z}} q[z'|z]V(K', z') - \lambda[(1-\delta)K - K'] \tag{1}$$

where the first part corresponds to the right-hand side of equation (1) and the last part corresponds to the inequality constraint.

We take the first order condition with respect to $K'$:

$$\frac{\partial L}{\partial K'} = -u_{K'}(F(K, A(z)(1-l)) + (1-\delta)K - K'l) + \beta \sum_{z' \in \mathcal{Z}} q[z'|z]V_{K'}(K', z') + \lambda = 0$$

Thus,

$$\beta \sum_{z' \in \mathcal{Z}} q[z'|z]V_{K'}(K', z') + \lambda = u_{K'}(F(K, A(z)(1-l)) + (1-\delta)K - K', l)$$

In this FOC, the right-hand side $u_{K'}(...)$ is the marginal costs (benefit) of increasing one unit of investment at the current time period. If $K'$ increases, investment is higher and thus consumption is lower. The left-hand side $\beta \sum(...)$ is the marginal cost of reducing one unit of investment at the current time period. In equilibrium, the marginal benefit of increasing one unit of investment at the current time period is equal to the marginal cost of reducing one unit of investment at the current time period subject to the capital constraint.

$\lambda$ is the Lagrange multiplier and reflects the capital constraint.. The marginal utility of increasing investment at the current time period and thus increasing capital at a later time period is the sum of the shadow value of capital at a later time period $\lambda$ and the change in value of additional capital, represented by $V_{K'}(K', z')$[1]. If the constraint weren't there, the agent would set marginal benefits equal to marginal costs, and $\lambda = 0$. The values are weighted with respect to their event probabilities and discounted to the current time period with discount factor $\beta$.

### 2.2 Taking as given that the value function is differentiable, use the envelope theorem to obtain an expression for $V_K(K, z)$. Provide an economic interpretation of this so-called envelope condition, explaining in particular the role of $\lambda$.

First, we differentiate the entire Lagrangian from part 1 with respect to $K$, and set it equal to $V_K(K, z)$:

[1] This reflects the change in the value of capital given a change in capital and is the derivative of the value function.

$$\frac{\partial L}{\partial K} = [F_K(K, A(z)(1-l)) + (1-\delta)]\, u_K(F(K, A(z)(1-l)) + (1-\delta) - K', l) - \lambda(1-\delta)$$
$$= V_K(K, z)$$

We rewrite, and get:

$$\lambda(1-\delta) = ((1-\delta) + F_K(...))u_c(...) + V_K(K, z)$$

In equilibrium, a change in capital has $V_K(K, z)$ marginal effect on the value of capital, and an effect on utility from a change in consumption. Note that the left-hand side of the equation is constant with $\lambda(1-\delta)$. This means that the effect on the value of capital and the effect utility from a change in consumption should cancel each other out (sum = 0). Thus, with the constraint $K' \geq (1-\delta)K$, the difference between the marginal benefits and the marginal costs of increased one unit of investment is the difference $\delta$.

The shadow value of capital at a later time period is reflected by both sides, and tells us that this depends on an indirect change of investment on the value function and a direct change in later time utility, due to consumption changes. Again, the values are weighted, with respect to changed productivity and depreciation.

### 2.3 Show that the functions $\pi(K, z), l(K, z), c(K, z)$ and $\lambda(K, z)$ must satisfy the equation

$$\begin{aligned}
1 = \beta \sum_{z' \in \mathcal{Z}} q[z'|z] & \frac{u_c(c(\pi(K, z), z', l(\pi(K, z), z'))}{u_c(c(K, z), l(K, z))} \\
& \cdot [F_K(\pi(K, z), A(z')(1 - l(\pi(K, z), z'))) + (1-\delta)] \\
& + \frac{\lambda(K, z) - \beta \sum_{z' \in \mathcal{Z}} q[z'|z](1-\delta)\lambda(\pi(K, z), z')}{u_c(c(K, z), l(K, z))}
\end{aligned}$$

From Question 2.2, we know that

$$V_K(K, z) = (F_K(K, A(z)(1-l)) + (1-\delta))u_K(F(K, A(z)(1-l)) + (1-\delta) - K', l) - \lambda(1-\delta)$$

We rewrite the envelope condition [2] $V_K(K, z)$ to get $V_K'(K', z')$:

$$V_K'(K', z) = (F_K'(K, A(z')(1-l')) + (1-\delta))u_K(F(K', A(z')(1-l)) + (1-\delta) - K', l') - \lambda'(1-\delta)$$

We rewrite using the conditions from the question, substituting in policy functions, and get:

$$\begin{aligned}
&\Leftrightarrow V_K(\pi(K, z'), z') = (F_K(\pi(K, z), A(z')(1 - l(K', z'))) + (1-\delta))u_c(c(\pi(K, z), A(z')(1 - l(K', z'))) \\
&+ (1-\delta) - \pi(K, z), l(K', z')) - \lambda(\pi(K, z), z')(1-\delta) \\
&\Leftrightarrow V_K(\pi(K, z), z') = (F_K(\pi(K, z), A(z')(1 - l(K', z'))) + \\
&(1-\delta))u_c(c(\pi(K, z), z', l(\pi(K, z), z')) - \lambda(\pi(K, z), z')(1-\delta)
\end{aligned}$$

Next, we plug in $V_K'(K', z')$ in the first order condition with respect to $K'$ that we got in question 2.1.

[2]Note that $\lambda' = \lambda(K', z') = \lambda(\pi(K, z), z')$.

$$
\begin{aligned}
&\beta \sum_{z' \in \mathcal{Z}} q[z'|z] V_{K'}(K', z') + \lambda = u_{K'}(F(K, A(z)(1-l)) + (1-\delta)K - K', l) \\
&\Leftrightarrow \beta \sum_{z' \in \mathcal{Z}} q[z'|z] V_{K'}(K', z') + \lambda(K, z) = u_c(c(\pi(K, z), l(K, z)) \\
&\Leftrightarrow \beta \sum_{z' \in \mathcal{Z}} q[z'|z][(F_K(\pi(K, z), A(z')(1 - l(\pi(K, z), z'))) + (1-\delta))u_c(c(\pi(K, z), z', l(\pi(K, z), z')) \\
&- \lambda(\pi(K, z), z')(1-\delta)] + \lambda(K, z) = u_c(c(\pi(K, z), l(K, z))
\end{aligned}
$$

We can divide both sides by $u_c(c(\pi(K, z), l(K, z))$ and rewrite to the equation given in the question:

$$
\begin{aligned}
1 = \beta &\sum_{z' \in \mathcal{Z}} q[z'|z] \frac{u_c(c(\pi(K, z), z', l(\pi(K, z), z'))}{u_c(c(K, z), l(K, z))} \\
&\cdot [F_K(\pi(K, z), A(z')(1 - l(\pi(K, z), z'))) + (1-\delta)] \\
&+ \frac{\lambda(K, z) - \beta \sum_{z' \in \mathcal{Z}} q[z'|z](1-\delta)\lambda(\pi(K, z), z')}{u_c(c(K, z), l(K, z))}
\end{aligned}
$$