Practice Problems for

grad div curl and all that

Brett Hopkins
bhopkins@unr.edu

Contents

1 Chapter I

Problem I-1:

Problem I-2:

Problem I-3:

A. Write a formula for a vector function in two dimensions which is in the positive radial direction and whose magnitude is 1.

Solution:

$$\boxed{\frac{\mathbf{i}y + \mathbf{j}x}{\sqrt{x^2 + y^2}}}$$

B. Write a formula for a vector function in two dimensions whose direction makes an angle of 45° with the x-axis and hwose magnitude at any point (x, y) is $(x + y)^2$.

Solution:

$$\boxed{(\mathbf{i} + \mathbf{j}) \cdot \frac{(x + y)^2}{\sqrt{2}}}$$

C. Write a formula for a vector function in two dimensions whose direction is tangential (in the sens of the example on page 5) and whose magnitude at any point (x,y) is equal to its distance from the origin.

Solution:

$$\boxed{-\mathbf{i}y + \mathbf{j}x}$$

D. Write a formula for a vector function in three dimensions which is in the positive radial direction and whose magnitude is 1.

Solution:

$$\boxed{\frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{\sqrt{x^2 + y^2 + z^2}}}$$

Problem I-4: An object moves in the xy-plane in such a way that its position vector $\mathbf{r}$ is given as a function of time t by

$$\mathbf{r} = \mathbf{i}a \cos \omega t + \mathbf{j}b \sin \omega t,$$

where a, b, and ω are constants.

A. How far is the object from the origin at any time t?

Solution:

$$\boxed{d = \sqrt{a^2 \cos^2 \omega t + b^2 \sin^2 \omega t}}$$

B. Find the object's velocity and acceleration as functions of time.

Solution:

$$\boxed{\frac{dr}{dt} = -\mathbf{i}\omega a \sin \omega t + \mathbf{j}\omega b \cos \omega t}$$

$$\boxed{\frac{d^2r}{dt^2} = -\mathbf{i}\omega^2 a \cos \omega t - \mathbf{j}\omega^2 b \sin \omega t}$$

C. Show that the object moves on the elliptical path

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Solution:

$$x(t) = a \cos \omega t.$$

$$y(t) = b \sin \omega t.$$

$$\left(\frac{a \cos \omega t}{a}\right)^2 + \left(\frac{b \sin \omega t}{b}\right)^2 = 1.$$

$$\cos^2 \omega t + \sin^2 \omega t = 1.$$

$$\boxed{1 = 1}$$

Problem I-5: A charge $+1$ is situated at the point $(1, 0, 0)$ and a charge -1 is situated at the point $(-1, 0, 0)$. Find the electric field of these two charges at an arbitrary point $(0, y, 0)$ on the y-axis.

Solution:

$$E(r) = \frac{1}{4\pi\epsilon_0} \sum_{l=1}^{N} \frac{qq_0}{|r - r_l|^2} \hat{\mathbf{u}}_l.$$

Contribution from $+1$ charge at $(1, 0, 0)$

$$\mathbf{r} - \mathbf{r_1} = (0, y, 0) - (1, 0, 0) = (-1, y, 0).$$

$$|\mathbf{r} - \mathbf{r_1}| = \sqrt{1 + y^2}.$$

$$\hat{\mathbf{r}} = \frac{(-1, y, 0)}{\sqrt{1 + y^2}}.$$

Inside the summation for $+1$ charge at $(1, 0, 0)$

$$\frac{1}{(\sqrt{1 + y^2})^2} \cdot \frac{(-1, y, 0)}{\sqrt{1 + y^2}}.$$

$$= \frac{(-1, y, 0)}{(1 + y^2)^{\frac{3}{2}}}.$$

Contribution from -1 charge at $(0, y, 0)$

$$-\frac{(1, y, 0)}{(1 + y^2)^{\frac{3}{2}}}.$$

Contribution from both charges

$$E(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{(-1,y,0)}{(1+y^2)^{\frac{3}{2}}} - \frac{(1,y,0)}{(1+y^2)^{\frac{3}{2}}}\right).$$

$$E(r) = \frac{\mathbf{i}}{4\pi\epsilon_0} \cdot \frac{2}{(1+y^2)^{\frac{3}{2}}}.$$

$$\boxed{E(r) = \frac{\mathbf{i}}{2\pi\epsilon_0} \cdot \frac{1}{(1+y^2)^{\frac{3}{2}}}}$$

Problem I-6: Instead of using arrows to represent vector functions, we sometimes use families of curves called *field lines*. A curve $y = y(x)$ is a field line of the vector function $\mathbf{F}(x, y)$ if at each point (x_0, y_0) on the curve, $\mathbf{F}(x_0, y_0)$ is tangent to the curve.

A. Show that the field lines $y = y(x)$ of a vector function

$$\mathbf{F}(x, y) = \mathbf{i}F_x(x, y) + \mathbf{j}F_y(x, y).$$

are solutions of the differential equation

$$\frac{dy}{dx} = \frac{F_y(x, y)}{F_x(x, y)}.$$

Solution:

Since the field line $y = y(x)$ has the vector function $\mathbf{F}(x, y)$ tangent to it at every point, the slope of the field line at any point must be equal to the slope of the vector at that point.

$$\text{slope of } \mathbf{F} = \frac{\text{vertical component}}{\text{horizontal component}} = \frac{F_y(x, y)}{F_x(x, y)}.$$

Since the vector function is tangential to the field line at every point, their slopes must be equal.

$$\frac{dy}{dx} = \frac{F_y(x, y)}{F_x(x, y)}.$$

Therefore, the field lines of the vector function are solutions of the differential equation. (This equation essentially states that the field line has the same slope at every point of the vector field, essentially the definition of a field line being tangent to a vector field.)

B. Determine the field lines of each of the funcitons of **Problem I-1**. Draw the field lines and compare with the arrow diagrams of **Problem I-1**.

A. $\mathbf{i}y + \mathbf{j}x$

Solution:

$$F_x = y,\ F_y = x.$$

$$\frac{dy}{dx} = \frac{x}{y}.$$

$$\int y\, dy = \int x\, dx.$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C.$$
$$y^2 = x^2 + C.$$
$$\boxed{y^2 - x^2 = C}$$

B. $(\mathbf{i}+\mathbf{j})/\sqrt{2}$

Solution:

$$F_x = \frac{1}{\sqrt{2}},\ F_y = \frac{1}{\sqrt{2}}.$$
$$\frac{dy}{dx} = 1.$$
$$\int dy = \int dx.$$
$$\boxed{y = x + C}$$

C. $\mathbf{i}x - \mathbf{j}y$

Solution:

$$F_x = x,\ F_y = -y.$$
$$\frac{dy}{dx} = -\frac{y}{x}.$$
$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx.$$
$$\ln y = -\ln x + C.$$
$$\ln x + \ln y = C.$$
$$\boxed{xy = C}$$

D. $\mathbf{i}y$

Solution:

$$F_x = y.$$
$$\frac{dy}{dx} = \frac{0}{y}.$$
$$\boxed{y = C}$$

E. $\mathbf{j}x$

Solution:

$$F_y = x.$$
$$\frac{dy}{dx} = \frac{x}{0}\,(undefined).$$
$$dx = 0.$$
$$\boxed{x = C}$$

F. $(\mathbf{i}y + \mathbf{j}x)/\sqrt{x^2+y^2}, (x,y) \neq (0,0)$

Solution:

$$F_x = \frac{y}{\sqrt{x^2+y^2}},\ F_y = \frac{x}{\sqrt{x^2+y^2}}.$$

$$\frac{dy}{dx} = \frac{x}{y}.$$

$$\int y\,dy = \int x\,dx.$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C.$$

$$\boxed{y^2 - x^2 = C}$$

G. $\mathbf{i}y + \mathbf{j}xy$

Solution:

$$F_x = y,\ F_y = xy.$$

$$\frac{dy}{dx} = \frac{xy}{y}.$$

$$\frac{dy}{dx} = x.$$

$$\int dy = \int x\,dx.$$

$$\boxed{y = \frac{x^2}{2} + C}$$

H. $\mathbf{i} + \mathbf{j}y$

Soltuion:

$$F_x = 1,\ F_y = y.$$

$$\frac{dy}{dx} = y.$$

$$\int \frac{1}{y}\,dy = \int dx.$$

$$\ln y = x + C.$$

$$\boxed{y = e^x + C}$$

2 Chapter II

Problem II-1:
Find a unit vector $\hat{\mathbf{n}}$ normal to each of the following surfaces.

In general,

$$\hat{\mathbf{n}}(x,y,z) = \frac{-\mathbf{i}\frac{\partial z}{\partial x} - \mathbf{j}\frac{\partial z}{\partial y} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}.$$

A. $z = 2 - x - y$

Solution:

$$\frac{\partial z}{\partial x} = -1.$$

$$\frac{\partial z}{\partial y} = -1.$$

$$\boxed{\hat{\mathbf{n}} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}}$$

B. $z = (x^2 + y^2)^{\frac{1}{2}}$

Soltuion:

$$\frac{\partial z}{\partial x} = 2x\frac{1}{\sqrt{x^2 + y^2}} = \frac{x}{z}.$$

$$\frac{\partial z}{\partial y} = 2y\frac{1}{\sqrt{x^2 + y^2}} = \frac{y}{z}.$$

$$\hat{\mathbf{n}} = \frac{-\mathbf{i}\frac{x}{z} - \mathbf{j}\frac{y}{z} + \mathbf{k}}{\sqrt{1 + \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2}}.$$

$$\boxed{\hat{\mathbf{n}} = \frac{-\mathbf{i}x - \mathbf{j}y + \mathbf{k}z}{\sqrt{2}}}$$

C. $z = \sqrt{1 - x^2}$

Solution:

$$\frac{\partial z}{\partial x} = \frac{1}{2} \cdot -2x \cdot \frac{1}{\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}}.$$

$$\frac{\partial z}{\partial y} = 0.$$

$$\frac{-\mathbf{i}\frac{-x}{\sqrt{1-x^2}} + \mathbf{k}}{\sqrt{1 + (\frac{-x}{\sqrt{1-x^2}})^2}}.$$

$$\frac{\mathbf{i}\frac{x}{z}+\mathbf{k}}{\sqrt{1+\frac{x^2}{1-x^2}}}.$$

$$\frac{\mathbf{i}\frac{x}{z}+\mathbf{k}}{\sqrt{\frac{1-x^2+x^2}{1-x^2}}}.$$

$$\frac{\mathbf{i}\frac{x}{z}+\mathbf{k}}{\sqrt{\frac{1}{z^2}}}.$$

$$\frac{\mathbf{i}\frac{x}{z}+\mathbf{k}}{\frac{1}{z}}.$$

$$\boxed{\mathbf{i}x+\mathbf{k}z}$$

D. $z = x^2 + y^2$

Solution:

$$\frac{\partial z}{\partial x} = 2x.$$

$$\frac{\partial z}{\partial y} = 2y.$$

$$\frac{-\mathbf{i}2x-\mathbf{j}2y+\mathbf{k}}{\sqrt{1+(2x)^2+(2y)^2}}.$$

$$\frac{-\mathbf{i}2x-\mathbf{j}2y+\mathbf{k}}{\sqrt{1+4x^2+4y^2}}$$

$$\boxed{\frac{-\mathbf{i}2x-\mathbf{j}2y+\mathbf{k}}{\sqrt{1+4z}}}$$

E. $z = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)^{\frac{1}{2}}$

Solution:

$$\frac{\partial z}{\partial x} = \frac{1}{2} - \frac{2x}{a^2} \cdot \frac{1}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}} = \frac{x}{a^2 z}$$

$$\frac{\partial z}{\partial y} = \frac{y}{a^2 z}$$

$$\hat{\mathbf{n}} = \frac{-\mathbf{i}\frac{x}{a^2z}-\mathbf{j}\frac{y}{a^2z}+\mathbf{k}}{\sqrt{1+(\frac{x}{a^2z})^2+(\frac{y}{a^2z})^2}}$$

$$\hat{\mathbf{n}} = \frac{-\mathbf{i}x-\mathbf{j}y+\mathbf{k}a^2z}{\sqrt{1+\frac{x^2}{a^4z^2}+\frac{y^2}{a^4z^2}}} = \frac{-\mathbf{i}x-\mathbf{j}y+\mathbf{k}a^2z}{\sqrt{1+\frac{x^2+y^2}{a^4z^2}}}$$

$$\boxed{\hat{\mathbf{n}} = \frac{-\mathbf{i}x-\mathbf{j}y+\mathbf{k}a^2z}{a\sqrt{1+(a^2-1)z^2}}}$$