DIV, GRAD, CURL, AND ALL THAT

AN INFORMAL TEXT ON VECTOR ANALYSIS

FOURTH EDITION

SOLUTION MANUAL

H. M. SCHEY
ROCHESTER INSTITUTE OF TECHNOLOGY
ROCHESTER, NEW YORK

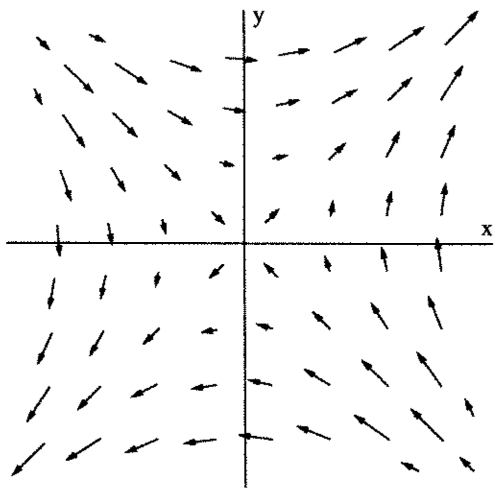
W • W • NORTON & COMPANY, INC.

The diagrams in this manual were made by Nathan D. Clinch using Mathwriter. We are grateful to Professor Rebecca E. Hill for her generous assistance in the use of Mathwriter.

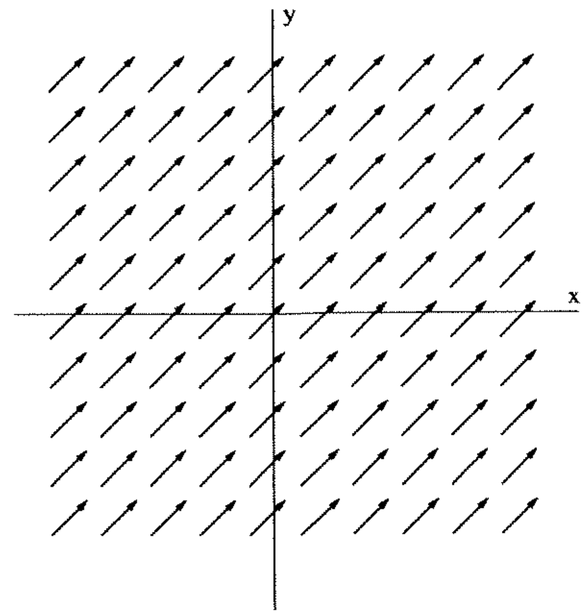
There is no harm in being sometimes wrong--especially if one is promptly found out.

--Maynard Keynes

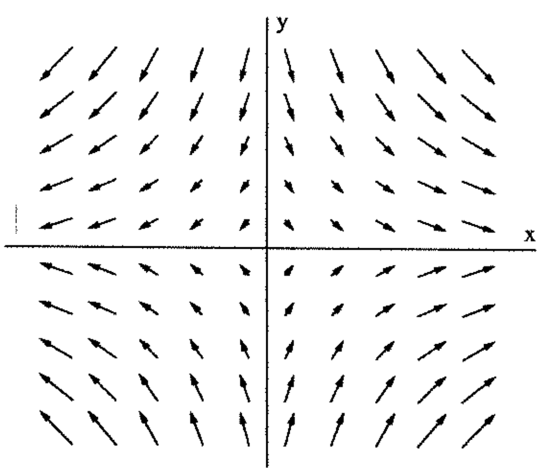
CHAPTER I



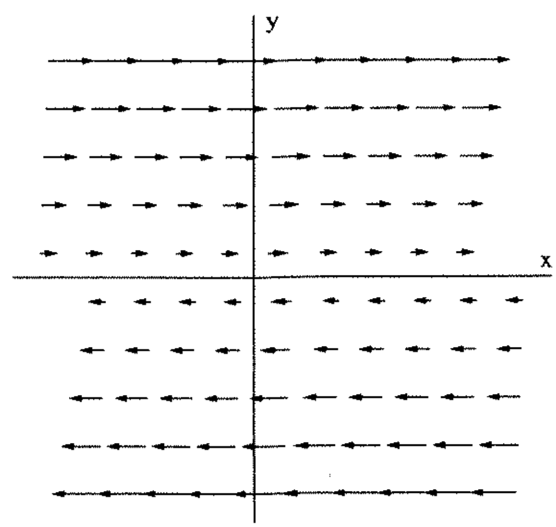
(a)



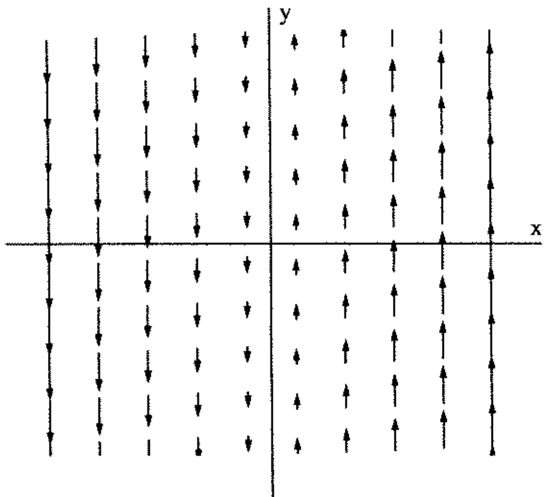
(b)



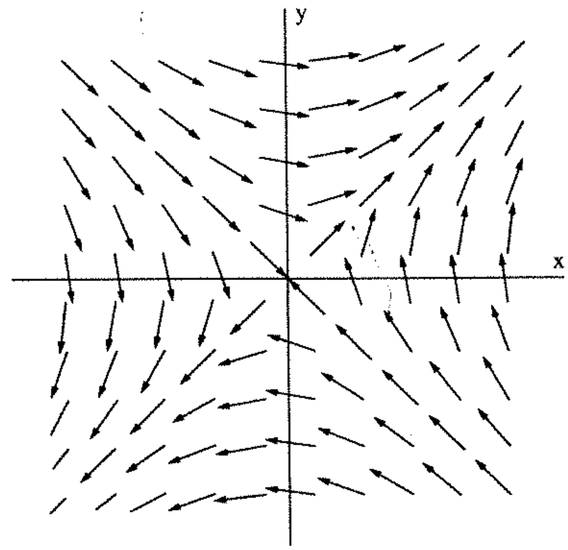
(c)



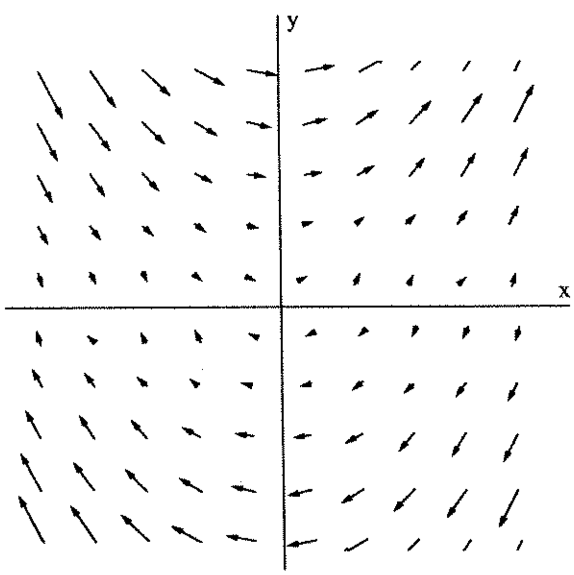
(d)



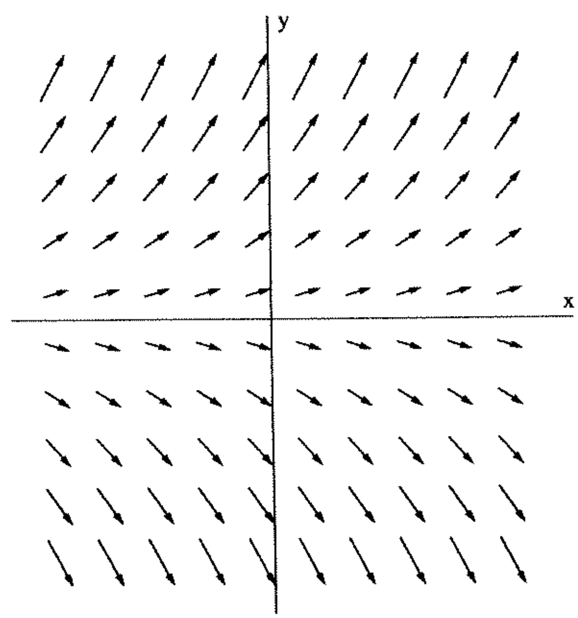
(e)



(f)

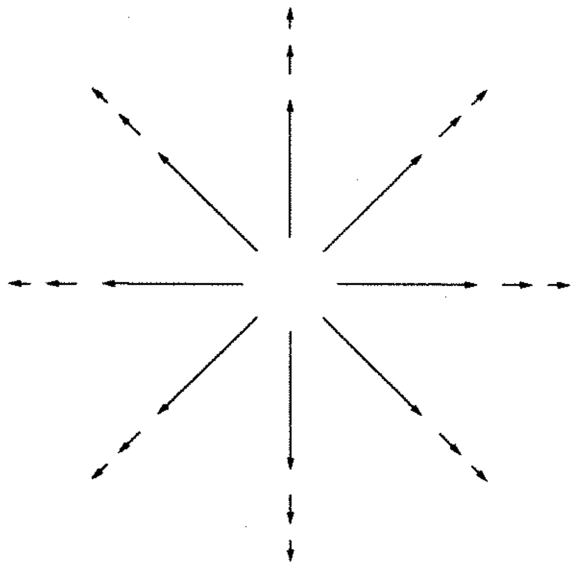


(g)



(h)

2.



3. a. $\dfrac{\mathbf{i}x + \mathbf{j}y}{\sqrt{x^2 + y^2}}$ c. $-\mathbf{i}y + \mathbf{j}x$

b. $(x^2 + y^2)\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$ d. $\dfrac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{\sqrt{x^2 + y^2 + z^2}}$

4. a. $|\mathbf{r}| = \sqrt{a^2\cos^2\omega t + b^2\sin^2\omega t}$

b. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t$

c. $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\mathbf{i}a\omega^2\cos\omega t - \mathbf{j}b\omega^2\sin\omega t = -\omega^2\mathbf{r}$

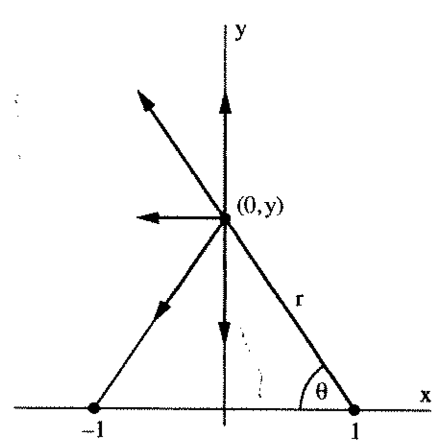
d. The x and y coordinates of the object at time t are given by $x = a\cos\omega t$ and $y = b\sin\omega t$. Hence $(x/a)^2 + (y/b)^2 = \cos^2\omega t + \sin^2\omega t = 1$.

5. It is clear from the figure that the y component of the field is 0. The x component of the field due to the charge at (1,0,0) is

$$E_x^{(1)} = -\frac{1}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}.$$

But $r = \sqrt{1 + y^2}$ and $\cos\theta = 1/r$. Hence

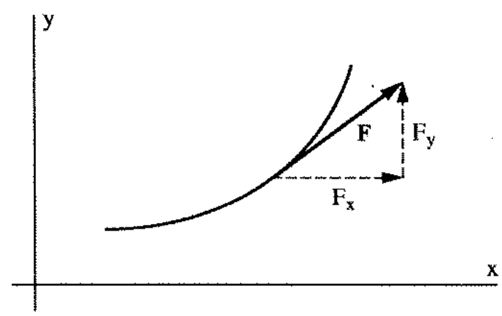
$$E_x^{(1)} = -\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3} = -\frac{1}{4\pi\varepsilon_0}\frac{1}{(1 + y^2)^{3/2}}.$$



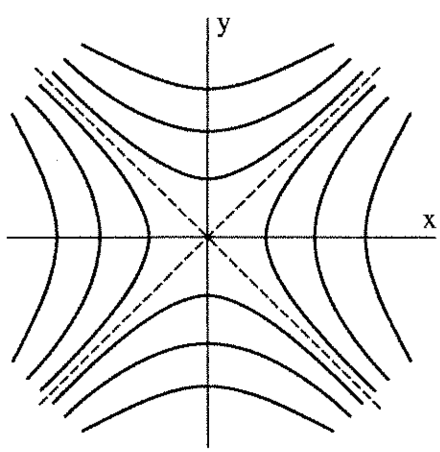
The x component of the field due to the charge at (−1,0,0), $E_x^{(-1)}$, is the same as this. Hence

$$\mathbf{E} = -\frac{1}{2\pi\varepsilon_0}\frac{\mathbf{i}}{(1 + y^2)^{3/2}}.$$

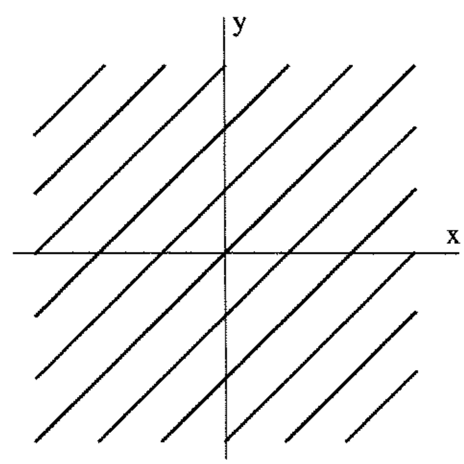
6. a. Since the function **F** is tangent to the field line, the slope of the field line at any point is F_y/F_x (see figure). But the slope is also given by dy/dx. Hence $dy/dx = F_y/F_x$.



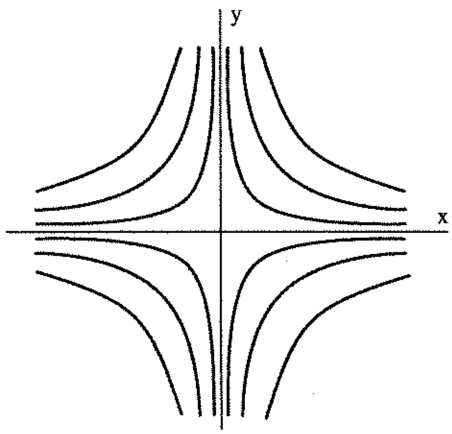
b. In the following, c represents an arbitrary constant.



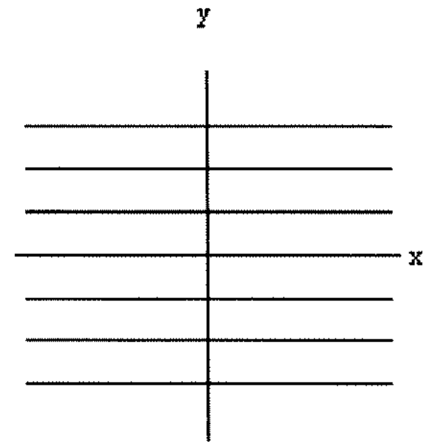
(i). $dy/dx = x/y$, $x^2 - y^2 = c$. A family of hyperbolas with asymptotes $y = \neq x$.



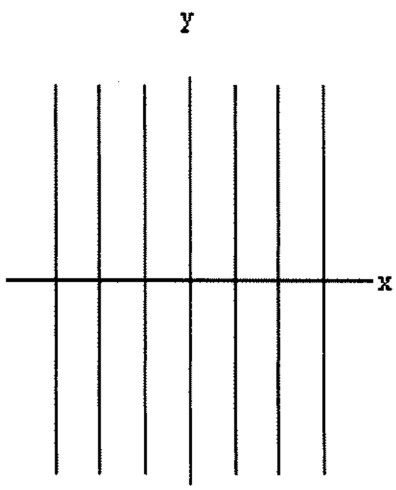
(ii). $dy/dx = 1$, $y = x + c$. A family of lines with slope 1.



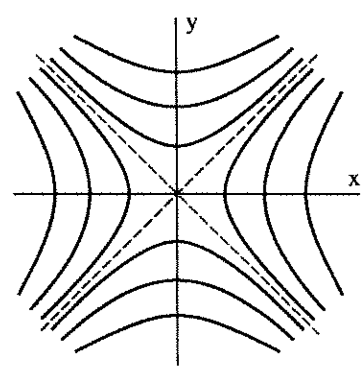
(iii). $dy/dx = -y/x$, $y = c/x$
A family of hyperbolas with asymptotes on the axes.



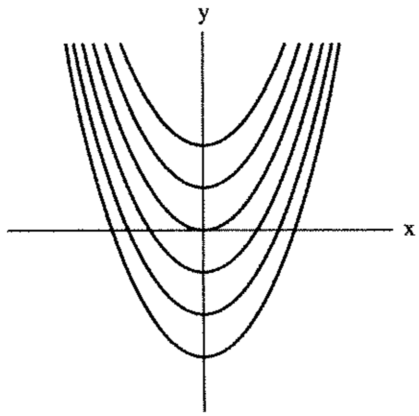
(iv). $dy/dx = 0$, $y = c$. A family of lines parallel to the x-axis.



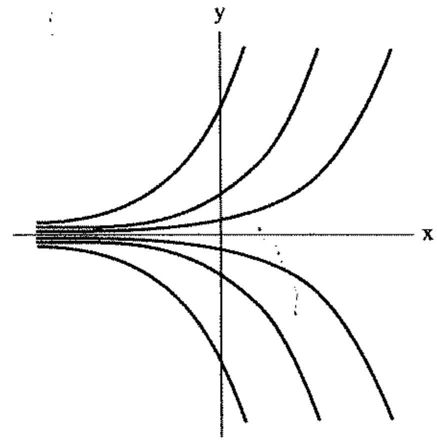
(v). dy/dx is not defined, but $dx/dy = 0$, $x = c$. A family of lines parallel to the y-axis.



(vi). $dy/dx = x/y$, $x^2 - y^2 = c$.
A family of hyperbolas with asymptotes $y = \pm x$.



(vii). $dy/dx = x$, $y = x^2/2 + c$. A family of parabolas.



(viii). $dy/dx = y$, $y = ce^x$. A family of exponentials.

CHAPTER II

1\. a. $\dfrac{\partial z}{\partial x} = -1,\ \dfrac{\partial z}{\partial y} = -1,\ \hat{\mathbf{n}} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$.

b. $\dfrac{\partial z}{\partial x} = \dfrac{x}{z},\ \dfrac{\partial z}{\partial y} = \dfrac{y}{z},\ \hat{\mathbf{n}} = \left(-\mathbf{i}\dfrac{x}{z}-\mathbf{j}\dfrac{y}{z}+\mathbf{k}\right)/\sqrt{(x/z)^2+(y/z)^2+1}$

$$= -\frac{\mathbf{i}x + \mathbf{j}y - \mathbf{k}z}{\sqrt{2}\,z}.$$

c. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z},\ \dfrac{\partial z}{\partial y} = 0,\ \hat{\mathbf{n}} = (\mathbf{i}x/z+\mathbf{k})/\sqrt{(x/z)^2+1} = (\mathbf{i}x+\mathbf{k}z)/\sqrt{x^2+z^2}$

$= \mathbf{i}x + \mathbf{k}z$ since $x^2 + z^2 = 1$.

d. $\dfrac{\partial z}{\partial x} = 2x,\ \dfrac{\partial z}{\partial y} = 2y,\ \hat{\mathbf{n}} = (-2\mathbf{i}x-2\mathbf{j}y+\mathbf{k})/\sqrt{1+4x^2+4y^2}$

$$= \frac{-2\mathbf{i}x - 2\mathbf{j}y + \mathbf{k}}{\sqrt{1 + 4z}}.$$

e. $\dfrac{\partial z}{\partial x} = -\dfrac{x/a^2}{z},\ \dfrac{\partial z}{\partial y} = -\dfrac{y/a^2}{z},\ \hat{\mathbf{n}} = \left(\mathbf{i}\dfrac{x}{a^2z}+\mathbf{j}\dfrac{y}{a^2z}+\mathbf{k}\right)/\sqrt{\dfrac{x^2}{a^4z^2}+\dfrac{y^2}{a^4z^2}+1}$

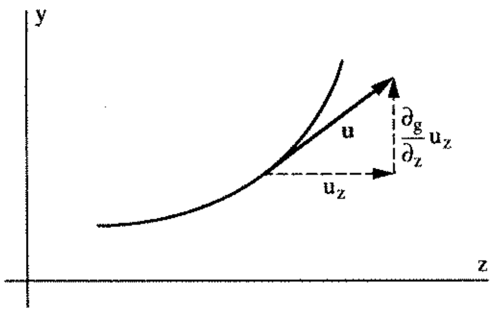
$$= \frac{\mathbf{i}x+\mathbf{j}y+\mathbf{k}a^2z}{a\sqrt{1-(a^2-1)z^2}}.$$

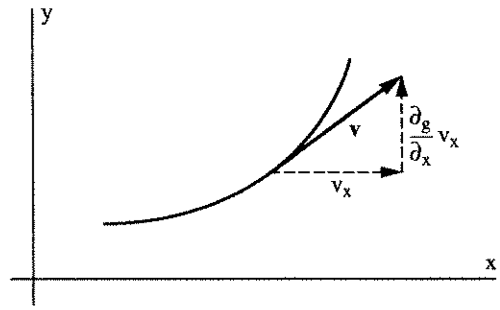
2\. a. $z = (d-ax-by)/c,\ \dfrac{\partial z}{\partial x} = -a/c,\ \dfrac{\partial z}{\partial y} = -b/c,\ \hat{\mathbf{n}} = \dfrac{\mathbf{i}a/c+\mathbf{j}b/c+\mathbf{k}}{\sqrt{a^2/c^2+b^2/c^2+1}}$

$$= \frac{\mathbf{i}a+\mathbf{j}b+\mathbf{k}c}{\sqrt{a^2+b^2+c^2}}.$$

b. As d varies with a, b, and c fixed, a family of *parallel* planes is generated. Because they are parallel they all have the same normal.

3.

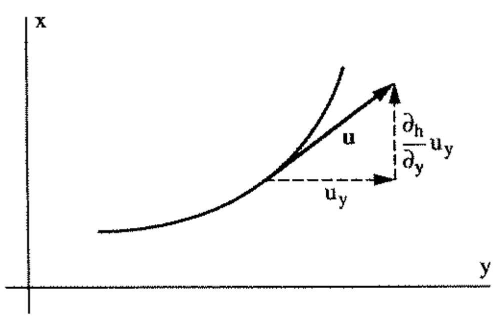


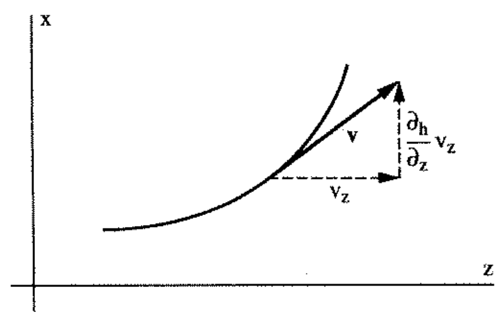


$$\mathbf{u} = u_z\left(\mathbf{k} + \mathbf{j}\frac{\partial g}{\partial z}\right) \qquad \mathbf{v} = v_x\left(\mathbf{i} + \mathbf{j}\frac{\partial g}{\partial x}\right)$$

$$\mathbf{u} \times \mathbf{v} = u_z v_x\left(\mathbf{j} - \mathbf{i}\frac{\partial g}{\partial x} - \mathbf{k}\frac{\partial g}{\partial z}\right)$$

$$\therefore\ \hat{\mathbf{n}} = \mathbf{u} \times \mathbf{v}/|\mathbf{u} \times \mathbf{v}| = \left(\mathbf{j} - \mathbf{i}\frac{\partial g}{\partial x} - \mathbf{k}\frac{\partial g}{\partial z}\right)\Big/\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}\ .$$





$$\mathbf{u} = u_y\left(\mathbf{j} + \mathbf{i}\frac{\partial h}{\partial y}\right) \qquad \mathbf{v} = v_z\left(\mathbf{k} + \mathbf{i}\frac{\partial h}{\partial z}\right)$$

$$\mathbf{u} \times \mathbf{v} = u_y v_z\left(\mathbf{i} - \mathbf{k}\frac{\partial h}{\partial z} - \mathbf{j}\frac{\partial h}{\partial y}\right)$$

$$\therefore\ \hat{\mathbf{n}} = \mathbf{u} \times \mathbf{v}/|\mathbf{u} \times \mathbf{v}| = \left(\mathbf{i} - \mathbf{j}\frac{\partial h}{\partial y} - \mathbf{k}\frac{\partial h}{\partial z}\right)\Big/\sqrt{1 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}$$

4. a. $z = f(x,y) = 1 - x - y$ so $\partial f/\partial x = \partial f/\partial y = -1$. Thus

$$\iint_S G dS = \iint_R z\sqrt{3}\ dxdy = \sqrt{3}\iint_R (1 - x - y)\ dxdy$$

where R is the triangle in the xy-plane bounded by the coordinate axes and the line $x + y = 1$. Hence the integral is

$$\sqrt{3}\int_0^1\int_0^{1-x} dxdy - \sqrt{3}\int_0^1\int_0^{1-x} xdydx - \sqrt{3}\int_0^1\int_0^{1-x} ydydx =$$

$$\frac{\sqrt{3}}{2} - \sqrt{3}\int_0^1 x(1 - x)dx - \frac{\sqrt{3}}{2}\int_0^1 (1 - x)^2 dx = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{6}.$$

b. $z = f(x,y) = x^2 + y^2$ so $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$. Thus

$$\iint_S G dS = \iint_R \frac{1}{1 + 4(x^2 + y^2)}\sqrt{1 + 4x^2 + 4y^2}\, dxdy = \iint_R \frac{dxdy}{\sqrt{1 + 4(x^2 + y^2)}}$$

where R is the circle of radius 1 lying in the xy-plane with its center at the origin. Transforming to polar coordinates we get

$$\int_0^{2\pi}\int_0^1 \frac{rdrd\theta}{\sqrt{1 + 4r^2}} = 2\pi\int_0^1 \frac{rdr}{\sqrt{1 + 4r^2}} = 2\pi\left(\frac{1}{4}\right)(1 + 4r^2)^{1/2}\Big|_0^1 = \frac{\pi}{2}\left(\sqrt{5} - 1\right).$$

c. $z = f(x,y) = (1 - x^2 - y^2)^{1/2}$ so $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$. Hence

$$\iint_S G dS = \iint_R (1 - x^2 - y^2)^{3/2}\sqrt{1 + \frac{x^2 + y^2}{z^2}}\ dxdy =$$

$$\iint_R (1 - x^2 - y^2)^{3/2}\frac{1}{(1 - x^2 - y^2)^{1/2}}\ dxdy = \iint_R (1 - x^2 - y^2)\ dxdy,$$

where R is the circle of radius 1 lying in the xy-plane with its center at the origin. Transforming to polar coordinates we get

$$\int_0^{2\pi}\int_0^1 (1 - r^2)rdrd\theta = 2\pi\int_0^1 (1 - r^2)rdr = 2\pi\int_0^1 (r - r^3)\ dr = \frac{\pi}{2}.$$

5. a. $z = f(x,y) = 1 - x/2 - y/2$ so $\partial f/\partial x = \partial f/\partial y = -1/2$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_R \left[-\left(-\frac{1}{2}\right)x - z\right] dxdy = \iint_R \left[\frac{x}{2} - \left(1 - \frac{x}{2} - \frac{y}{2}\right)\right] dxdy =$$

$$= \iint_R \left(x + \frac{y}{2} - 1\right) dxdy,$$

where R is the region in the xy-plane bounded by the coordinate axes and the line $x + y = 2$. Thus the integral is

$$\int_0^2 \int_0^{2-x} xdydx + \frac{1}{2}\int_0^2 \int_0^{2-x} ydydx - 2 = \int_0^2 x(2 - x)dx + \frac{1}{4}\int_0^2 (2 - x)^2\ dx - 2$$

$$= 4/3 + 2/3 - 2 = 0.$$

b. $z = f(x,y) = \sqrt{a^2 - x^2 - y^2}$ so $\partial f/\partial x = -x/z$ and $\partial f \partial y = -\ y/z$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_R \left[-x\left(-\frac{x}{z}\right) - y\left(-\frac{y}{z}\right) + z\right] dxdy = \iint_R \frac{x^2 + y^2 + z^2}{z} dxdy =$$

$$a^2 \iint_R \frac{dxdy}{\sqrt{a^2 - x^2 - y^2}},$$

where R is the circle of radius a lying in the xy-plane with its center at the origin. Transforming to polar coordinates we get

$$a^2 \int_0^{2\pi} \int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}} = 2\pi a^2 \int_0^a \frac{rdr}{\sqrt{a^2 - r^2}} = 2\pi a^2[-(a^2 - r^2)^{1/2}]\ \Big|_0^a = 2\pi a^3.$$

c. $z = f(x,y) = 1 - x^2 - y^2$ so $\partial f/\partial x = -2x$ and $\partial f/\partial y = -2y$. Thus

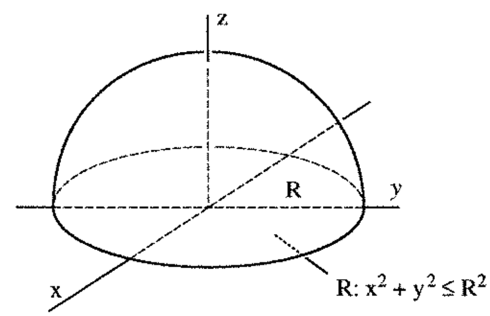
$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_R [-y(-2y) + 1]dxdy = \iint_R (1 + 2y^2)\ dxdy,$$

where R is the circle of radius 1 lying in the xy-plane with its center at the origin. Therefore the integral is

$$\iint_R dxdy + 2\iint_R y^2 dxdy = \pi + 2\int_0^{2\pi} \int_0^1 r^2 \sin^2\theta\ rdrd\theta = \pi + 2\pi \int_0^1 r^3 dr$$

$$= \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

6. $m = \iint_S \sigma(x,y,z)\, dS = \frac{\sigma_0}{R^2}\iint_S (x^2 + y^2)\, dS$ where S is the surface $z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$. Hence $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$, so



$$m = \frac{\sigma_0}{R^2}\iint_{\mathbf{R}} (x^2 + y^2)\sqrt{1 + (x/z)^2 + (y/z)^2}\, dxdy$$

where $\mathbf{R}$ is the disc $x^2 + y^2 \le R^2$. Thus,

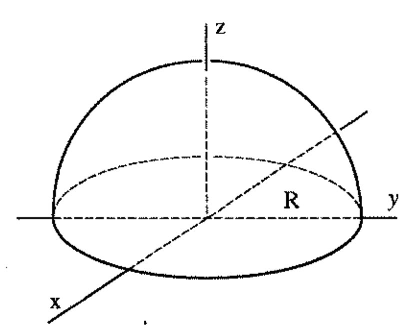
$$m = \frac{\sigma_0}{R^2}\iint_{\mathbf{R}} (x^2 + y^2)\sqrt{\frac{x^2 + y^2 + z^2}{z^2}}\, dxdy = \iint_{\mathbf{R}} \frac{x^2 + y^2}{\sqrt{R^2 - x^2 - y^2}}\, dxdy$$

$$= \frac{\sigma_0}{R}\int_0^{2\pi}\int_0^R \frac{r^3\, drd\theta}{\sqrt{R^2 - r^2}} = \frac{2\pi\sigma_0}{R}\int_0^R \frac{r^3 dr}{\sqrt{R^2 - r^2}} = 2\pi\sigma_0 R^2\int_0^1 \frac{w^3 dw}{\sqrt{1 - w^2}}.$$

This integral can be done by elementary methods; its value is 2/3. Hence we have, finally, $m = 4\pi\sigma_0 R^2/3$.

7. $I = \iint_S \sigma(x,y,z)(x^2 + y^2)\, dS$. This integral differs from the one in Problem 6 by an extra factor $x^2 + y^2 = r^2$ in the integrand. Hence $I = \frac{2\pi\sigma_0}{R}\int_0^R \frac{r^5\, dr}{\sqrt{R^2 - r^2}} = \pi\sigma_0 R^4\int_0^1 \frac{w^5\, dw}{\sqrt{1 - w^2}}$. This integral can be done by elementary methods; its value is 8/15, and so $I = 16\pi\sigma_0 R^4/15$.

8. To find the total charge q we use Gauss' law, $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = q/\varepsilon_0$. The surface S consists of the circular base $x^2 + y^2 \le R^2$ and the hemispherical shell $z = \sqrt{R^2 - x^2 - y^2}$. On the base, $\hat{\mathbf{n}} = -\mathbf{k}$ so that $\mathbf{E}\cdot\hat{\mathbf{n}} = -\lambda xy$. Hence



$$\iint_{\text{Base}} \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = -\lambda\iint_{\text{Base}} xy\, dxdy =$$

$-\lambda\int_0^{2\pi}\int_0^R r^3\sin\theta\cos\theta\ dr d\theta$. But the θ integration yields 0. (This result can also be obtained using symmetry arguments.) On the hemispherical shell, $\partial z/\partial x = -x/z$ and $\partial z\partial y = -y/z$. Also, since the projection of the shell onto the xy-plane is the circular base,

$$\iint_{\text{Shell}} \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = \iint_{\text{Base}}\left[-\lambda yx\left(-\frac{x}{z}\right) - \lambda xz\left(-\frac{y}{z}\right) + \lambda xy\right] dxdy = -3\lambda \iint_{\text{Base}} xy\ dxdy$$

$= 0$ as before. Thus $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = q/\varepsilon_0 = 0 \Rightarrow q = 0$.

9. To find the total charge q we use Gauss' law, $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = q/\varepsilon_0$. On the rectangular base in the xy-plane, $\hat{\mathbf{n}} = -\mathbf{k}$ and so $\mathbf{E}\cdot\hat{\mathbf{n}} = 0$. Thus the base makes no contribution to the surface integral. On the semi-circular end at $x = h/2$ we have $\hat{\mathbf{n}} = \mathbf{i}$ so $\mathbf{E}\cdot\hat{\mathbf{n}} = \lambda x = \lambda h/2$. Hence $\iint_{\text{End}} \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = (\lambda h/2)(\pi r^2/2) = \lambda h\pi r^2/4$. On the semi-circular end at $x = -h/2$, $\hat{\mathbf{n}} = -\mathbf{i}$ so $\mathbf{E}\cdot\hat{\mathbf{n}} = -\lambda x = \lambda h/2$. Thus this end also contributes $\lambda h\pi r^2/4$ to the surface integral. On the cylindrical top $z = \sqrt{r^2 - y^2}$ so $\partial z/\partial x = 0$ and $\partial z\partial y = -y/z$. Hence, noting that the projection of the cylindrical top onto the xy-plane is the rectangular base R, we have

$$\iint_{\text{Top}} \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = \iint_R [-\lambda x\cdot 0 - \lambda y(-y/z) + 0]\ dxdy = \lambda\iint_R \frac{y^2}{z}\ dxdy =$$

$$\lambda\int_{-h/2}^{h/2}\int_{-r}^{r}\frac{y^2}{\sqrt{r^2 - y^2}}\ dydx = \lambda h\int_{-r}^{r}\frac{y^2}{\sqrt{r^2 - y^2}}\ dydx = \lambda\pi r^2 h/2.$$

Thus

$$\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = \frac{\lambda\pi r^2 h}{4} + \frac{\lambda\pi r^2 h}{4} + \frac{\lambda\pi r^2 h}{2} = \lambda\pi r^2 h = q/\varepsilon_0 \Rightarrow q = \lambda\pi r^2 h\ \varepsilon_0.$$

10. a. On the face in the yz-plane, $\hat{\mathbf{n}} = \pm\mathbf{i}$, so $\mathbf{F}\cdot\hat{\mathbf{n}} = \pm x = 0$ (because $x = 0$ in the yz-plane). The other two faces can be handled in the same way. Hence $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = 0$.

b. On the circular top and bottom, $\hat{\mathbf{n}} = \pm\mathbf{k}$ and $\mathbf{F}\cdot\hat{\mathbf{n}} = 0$. On the curved surface $\hat{\mathbf{n}} = (\mathbf{i}x + \mathbf{j}y)/R$ and $x^2 + y^2 = R^2$. Hence

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \frac{x^2 + y^2}{R}\ \ln(x^2 + y^2) = R\ \ln R^2 = 2R\ \ln R.$$

Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = (2R \ln R)(2\pi Rh) = 4\pi R^2 h \ln R.$

c. On the spherical surface, $\hat{\mathbf{n}} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)/R$ and $x^2 + y^2 + z^2 = R^2$. Hence

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \frac{x^2 + y^2 + z^2}{R} e^{-(x^2 + y^2 + z^2)} = \frac{R^2 e^{-R^2}}{R} = Re^{-R^2}.$$

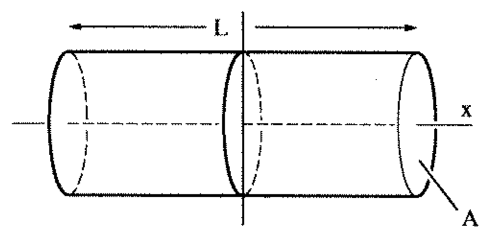
Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = Re^{-R^2}(4\pi R^2) = 4\pi R^3 e^{-R^2}.$

d. The only surfaces to contribute to the surface integral are the one at $x = 0$ and the one at $x = b$. At $x = 0$, $\hat{\mathbf{n}} = -\mathbf{i}$ and so $\mathbf{F}\cdot\hat{\mathbf{n}} = -E(x) = -E(0)$. At $x = b$, $\hat{\mathbf{n}} = \mathbf{i}$ and $\mathbf{F}\cdot\hat{\mathbf{n}} = E(x) = E(b)$. Thus

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = [E(b) - E(0)]b^2.$$

11. a. By symmetry

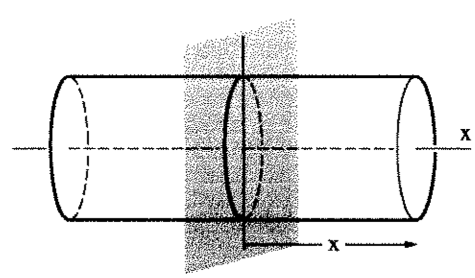
$$\mathbf{E} = \mathbf{i}E(x), \quad x > 0$$
$$\quad = -\mathbf{i}E(x), \quad x < 0,$$



and $E(x) = E(-x)$. For the surface S choose the cylinder of cross-sectional area A with its axis coinciding with the x-axis, and extending from $x = -L/2$ to $x = L/2$. Now $\mathbf{E}\cdot\hat{\mathbf{n}} = 0$ on the curved surface of the cylinder. On the flat end at $x = L/2$, $\mathbf{E}\cdot\hat{\mathbf{n}} = E(L/2)$, and so $\iint_{End} \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = E(L/2)A$. Similarly over the flat surface at $x = -L/2$ the surface integral is $E(-L/2)A$. Using Gauss' law and the fact that $E(-L/2) = E(L/2)$, we have $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = E(L/2)A + E(-L/2)A = 2E(L/2)A = \sigma A/\varepsilon_0$, and so, $E = \sigma/(2\varepsilon_0)$. Thus $\mathbf{E} = \mathbf{i}\sigma/(2\varepsilon_0)$, $x > 0$, and $\mathbf{E} = -\mathbf{i}\sigma/(2\varepsilon_0)$, $x < 0$.

b Symmetry considerations are the same here as in (a). Hence

$$\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = 2E(x)A.$$



We consider two cases. In the first, $-b \le x \le b$. Then the charge enclosed is $2xA\rho_0$, and Gauss' law gives

$$2E(x)A = 2xA\rho_0/\varepsilon_0$$

or $E(x) = \rho_0 x/\varepsilon_0$, $-b \le x \le b$. In the second case, $x > b$. The enclosed charge is $2bA\rho_0$ and so $2E(x)A = 2bA\rho_0/\varepsilon_0$, or $E(x) = b\rho_0/\varepsilon_0$. Summarizing,

$$\begin{aligned} \mathbf{E} &= \mathbf{i}\rho_0 x/\varepsilon_0, & -b \le x \le b \\ &= \mathbf{i}\rho_0 b/\varepsilon_0, & x > b \\ &= -\mathbf{i}\rho_0 b/\varepsilon_0, & x < -b \end{aligned}$$

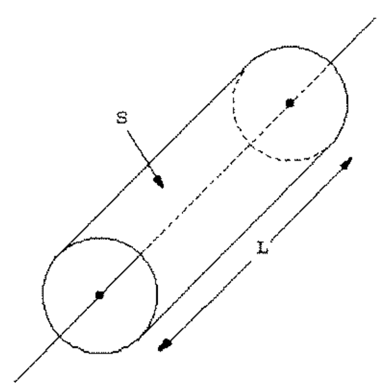
c. Symmetry considerations are the same here as in (a). so $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = 2E(x)A$. Now, however, the charge enclosed by the surface is

$$q = \iiint_V \rho\, dV = A\rho_0 \int_{-x}^{x} e^{-|x'/b|} dx' = 2A\rho_0 \int_{0}^{x} e^{-x'/b} dx = 2A\rho_0 b(1 - e^{-x/b}).$$

Thus $2AE(x) = \dfrac{2A\rho_0 b}{\varepsilon_0}(1 - e^{-x/b})$, or $E(x) = \dfrac{\rho_0 b}{\varepsilon_0}(1 - e^{-x/b})$. Hence

$$\begin{aligned} \mathbf{E} &= \mathbf{i}\frac{\rho_0 b}{\varepsilon_0}(1 - e^{-x/b}), & x > 0 \\ &= -\mathbf{i}\frac{\rho_0 b}{\varepsilon_0}(1 - e^{x/b}), & x < 0 \end{aligned}$$

12. a. By symmetry the field is radially outward and depends only upon r: $\mathbf{E} = E(r)\hat{\mathbf{e}}_r$. Take as the surface S the right circular cylinder of length L whose axis coincides with the line of charge as shown in the figure. The flux through the flat circular ends of S is 0 because $\mathbf{E}\cdot\hat{\mathbf{n}} = 0$ on the ends. Only the curved surface (CS) of the cylinder contributes to the flux which is given by



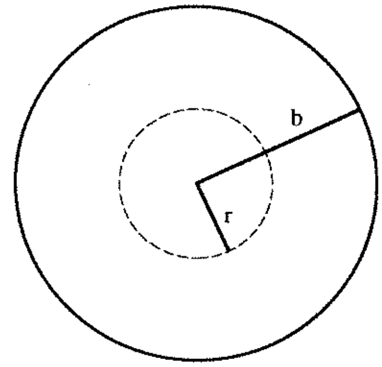
$$\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, dS = \iint_{CS} E(r)\, dS = E(r)\iint_{CS} dS = E(r)(2\pi rL).$$

The total charge enclosed by S is λL and so $2\pi rLE(r) = \lambda L/\varepsilon_0$ and $E(r) = \lambda/(2\pi\varepsilon_0 r)$. Hence

$$\mathbf{E}(r) = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\mathbf{e}}_r}{r}.$$

Note that if the line of charge coincides with the z-axis then $\hat{\mathbf{e}}_r = \dfrac{\mathbf{i}x + \mathbf{j}y}{\sqrt{x^2 + y^2}}$.

b. The symmetry is the same as in (a). We consider two cases. In the first the radius of the cylinder $r < b$. The flux is $2\pi rLE(r)$, as it was in (a), but the charge enclosed is $\pi r^2 L\rho_0$. Thus $2\pi rLE(r) = \pi r^2 L\rho_0/\varepsilon_0$ or $E(r) = r\rho_0/(2\varepsilon_0)$. When $r > b$ the enclosed charge is $\pi b^2 L\rho_0$ so Gauss' law reads $2\pi rLE(r) = \pi b^2 L\rho_0/\varepsilon_0$ so $E(r) = b^2\rho_0/(2\varepsilon_0 r)$. Hence



$$\begin{aligned} \mathbf{E} &= \frac{\rho_0 r}{2\varepsilon_0}\hat{\mathbf{e}}_r, \qquad r < b \\ &= \frac{\rho_0 b^2}{2\varepsilon_0 r}\hat{\mathbf{e}}_r, \qquad r \geq b \end{aligned}$$

c. The symmetry is the same as in (a) but now the enclosed charge is

$$\rho_0 L\int_0^{2\pi}\int_0^{r} e^{-r'/b}\, r'dr'd\theta = 2\pi\rho_0 L\int_0^{r} e^{-r'/b}\, r'dr'$$

$$= 2\pi\rho_0 Lb^2\left[1 - \left(1 + \frac{r}{b}\right)\right]e^{-r/b}.$$

Hence Gauss' law reads $2\pi rLE(r) = \dfrac{2\pi\rho_0 Lb^2}{\varepsilon_0}\left[1 - \left(1 + \dfrac{r}{b}\right)\right]e^{-r/b}$, or

$$\mathbf{E}(r) = \frac{\rho_0 b^2}{\varepsilon_0 r}\left[1 - \left(1 + \frac{r}{b}\right)\right]e^{-r/b}\ \hat{\mathbf{e}}_r.$$

13. a. Here the field is in the radial direction and depends only upon r: $\mathbf{E} = E(r)\ \hat{\mathbf{e}}_r$ where $\hat{\mathbf{e}}_r = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)/\sqrt{x^2 + y^2 + z^2}$. The surface S is the sphere of radius r with its center at the center of charge, which is also the origin of the coordinate system. The flux of $\mathbf{E}$ is then

$$\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = E(r)(4\pi r^2).$$

We consider two cases. In the first $r < b$. The enclosed charge is $4\pi r^3\rho_0/3$ so $4\pi r^2 E(r) = 4\pi r^3\rho_0/(3\varepsilon_0)$ and $E(r) = \rho_0 r/(3\varepsilon_0)$. In the second case, $r > b$ so that the enclosed charge is $4\pi b^3\rho_0/3$ and Gauss'law is $4\pi r^2 E(r) = 4\pi b^3\rho_0/(3\varepsilon_0)$, and we get $E(r) = b^3\rho_0/(3\varepsilon_0 r^2)$. Summarizing,

$$\mathbf{E}(r) = \frac{\rho_0 r}{3\varepsilon_0}\,\hat{\mathbf{e}}_r, \qquad r < b$$
$$= \frac{\rho_0 b^3}{3\varepsilon_0 r^2}\,\hat{\mathbf{e}}_r, \qquad r \geq b$$

b. The symmetry is the same as in (a) but now the charge enclosed by S is

$$q = \iiint_V \rho_0 e^{-r'/b} dV = 4\pi\rho_0 \int_0^r e^{-r'/b}(r')^2\, dr'$$

$$= 4\pi\rho_0 b^3\left[2 - e^{-r/b}\left(\frac{r^2}{b^2} + \frac{2r}{b} + 2\right)\right].$$

Therefore Gauss' law gives

$$4\pi r^2 E(r) = \frac{4\pi\rho_0 b^3}{\varepsilon_0}\left[2 - e^{-r/b}\left(\frac{r^2}{b^2} + \frac{2r}{b} + 2\right)\right]$$

so

$$\mathbf{E}(r) = \frac{\rho_0 b^3}{\varepsilon_0 r^2}\left[2 - e^{-r/b}\left(\frac{r^2}{b^2} + \frac{2r}{b} + 2\right)\right]\hat{\mathbf{e}}_r.$$

c. The symmetry is the same as in (a). To determine the total charge enclosed we must consider three cases. In the first case, $r < b$ and $q = 4\pi r^3\rho_0/3$. Hence $4\pi r^2 E(r) = \dfrac{4\pi r^3}{3\varepsilon_0}\rho_0$, and $E(r) = \dfrac{\rho_0 r}{3\varepsilon_0}$.

In the second case, $b < r < 2b$ and $q = \dfrac{4\pi b^3\rho_0}{3} + \dfrac{4\pi(r^3 - b^3)\rho_1}{3}$, so from Gauss' law we have $4\pi r^2 E(r) = \dfrac{4\pi}{3\varepsilon_0}[b^3\rho_0 + (r^3 - b^3)\rho_1]$, and $E(r) = \dfrac{1}{3\varepsilon_0 r^2}[b^3\rho_0 + (r^3 - b^3)\rho_1]$.

Finally, in the third case we have $r > 2b$ whence the total enclosed charge is given by

$$q = \frac{4\pi}{3}\rho_0 b^3 + \frac{4\pi}{3}\rho_1(8b^3 - b^3) = \frac{4\pi b^3}{3}(\rho_0 + 7\rho_1),$$

and so from Gauss' law we get $4\pi r^2 E(r) = \dfrac{4\pi b^3}{3\varepsilon_0}(\rho_0 + 7\rho_1)$, so $E(r) =$

$\frac{b^3}{3\varepsilon_0 r^2}(\rho_0 + 7\rho_1)$. Summarizing we have

$$\begin{aligned}
\mathbf{E}(r) &= \frac{\rho_0}{3\varepsilon_0}\frac{\hat{\mathbf{e}}_r}{r^2}, \qquad r < b \\
&= \frac{1}{3\varepsilon_0 r^2}[b^3\rho_0 + (r^3 - b^3)\rho_1]\hat{\mathbf{e}}_r, \qquad b \le r \le 2b \\
&= \frac{b^3}{3\varepsilon_0 r^2}(\rho_0 + 7\rho_1)\hat{\mathbf{e}}_r, \qquad r > 2b.
\end{aligned}$$

To have $\mathbf{E} = 0$ when $r > 2b$, we must have $\rho_0 + 7\rho_1 = 0$, or $\rho_0 = -7\rho_1$. The total charge is then

$$-\frac{4\pi b^3}{3}(7\rho_1) + \frac{4\pi}{3}(8b^3 - b^3)\rho_1 = 0.$$

14. (a). $\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}z^2 = 2(x + y + z)$.

(b). $\frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}xz + \frac{\partial}{\partial z}xy = 0$.

(c). $\frac{\partial}{\partial x}e^{-x} + \frac{\partial}{\partial y}e^{-y} + \frac{\partial}{\partial z}e^{-z} = -(e^{-x} + e^{-y} + e^{-z})$.

(d). $\frac{\partial}{\partial x}1 + \frac{\partial}{\partial y}(-3) + \frac{\partial}{\partial z}z^2 = 2z$.

(e). $\frac{\partial}{\partial x}\left[-\frac{xy}{x^2 + y^2}\right] + \frac{\partial}{\partial y}\left[\frac{xy}{x^2 + y^2}\right] = -\frac{y}{x^2 + y^2}$.

(f). $\frac{\partial}{\partial z}\sqrt{x^2 + y^2} = 0$.

(g). $\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$.

(h). $\frac{\partial}{\partial x}\left[-\frac{y}{\sqrt{x^2 + y^2}}\right] + \frac{\partial}{\partial y}\left[\frac{x}{\sqrt{x^2 + y^2}}\right] =$

$$xy(x^2 + y^2)^{-3/2} - xy(x^2 + y^2)^{-3/2} = 0.$$

15. (a). In the following we evaluate the function at the center of the relevant face of the cube.

On S_1 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{i} \cong (x_0 + s/2)^2$

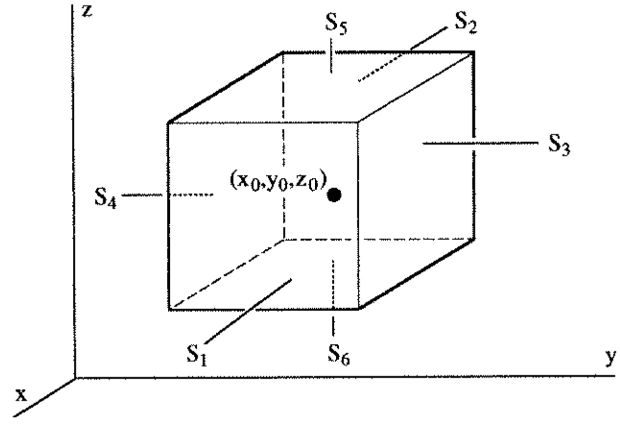
On S_2 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{i} \cong -(x_0 - s/2)^2$

On S_3 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{j} \cong (y_0 + s/2)^2$

On S_4 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{j} \cong -(y_0 - s/2)^2$

On S_5 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{k} \cong (z_0 + s/2)^2$

On S_6 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{k} \cong -(z_0 - s/2)^2$



Hence $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS + \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS \cong s^2[(x_0 + s/2)^2 - (x_0 - s/2)^2] = 2x_0s^3,$ with analogous results for the othe two pairs of faces. Hence

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS \cong 2s^3(x_0 + y_0 + z_0).$$

(b). The volume of the cube is $V = s^3$ so

$$(1/V)\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \frac{2s^3(x_0 + y_0 + z_0)}{s^3} = 2(x_0 + y_0 + z_0).$$

By definition this is $\nabla\cdot\mathbf{F}$ at (x_0,y_0,z_0) and it agrees with Prob. II-14(a). [Note that there is no need to calculate the limit of this expression as $s \to 0$ since the result is independent of s.]

(c). For $\mathbf{F} = \mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy$ (evaluating $\mathbf{F}\cdot\hat{\mathbf{n}}$ at the center of the face),

$$\text{On } S_1\ \mathbf{F}\cdot\hat{\mathbf{n}} = yz \text{ so } \iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_{S_1} yz\ dS \cong y_0z_0s^2$$

$$\text{On } S_2\ \mathbf{F}\cdot\hat{\mathbf{n}} = -yz \text{ so } \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = -\iint_{S_2} yz\ dS \cong -y_0z_0s^2.$$

Note that these two results cancel. Calculations analogous to this one show that the other two pairs of faces also give cancelling results. Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = 0$ and so $\nabla\cdot\mathbf{F} = 0$, which is the result obtained in Prob II-14(b).

For $\mathbf{F} = \mathbf{i}e^{-x} + \mathbf{j}e^{-y} + \mathbf{k}e^{-z}$ (evaluating $\mathbf{F}\cdot\hat{\mathbf{n}}$ at the center of the face), we find

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_{S_1} e^{-x}\ dS \cong e^{-(x_0 + s/2)}s^2$$

and

$$\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_{S_2} e^{-x}\ dS \cong -e^{-(x_0 - s/2)}s^2.$$

Hence

$$\iint_{S_1 + S_2} e^{-x}\ dS \cong s^2[e^{-(x_0 + s/2)} - e^{-(x_0 - s/2)}]$$

Dividing this by the volume, s^3, gives

$$\frac{e^{-(x_0 + s/2)} - e^{-(x_0 - s/2)}}{s} \to e^{-x_0}$$

as $s \to 0$. The other two pairs of faces are treated in the same way and yield e^{-y_0} and e^{-z_0}. The sum of the three contributions is thus $e^{-x_0} + e^{-y_0} + e^{-z_0}$, which is the result of ProbII-14(c), evaluated at (x_0, y_0, z_0).

16. a. Let $f'(u) = df/du$. Then $\nabla\cdot\mathbf{F} = f'(x) + f'(y) + f'(-2z)(-2)$. With $(x,y,z) = (c,c,-c/2)$ we get $\nabla\cdot\mathbf{F} = f'(c) + f'(c) - 2f'(c) = 0$.

b. $\nabla\cdot\mathbf{G} = \frac{\partial}{\partial x}f(y,z) + \frac{\partial}{\partial y}g(x,z) + \frac{\partial}{\partial z}h(x,y) = 0.$

17. For the triangle at the front, $\mathbf{F}\cdot\hat{\mathbf{n}} \cong F_x\left(x_0 + \frac{\Delta x_0}{2}, y_0, z_0\right)$ (see Figure a). Hence

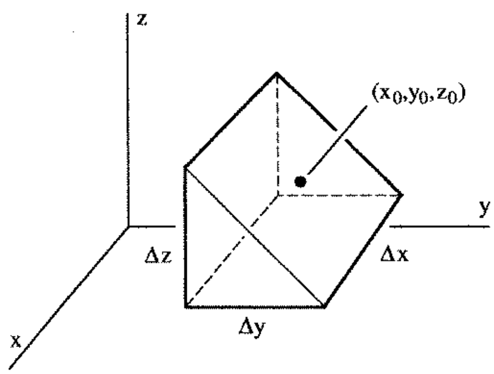


Figure a

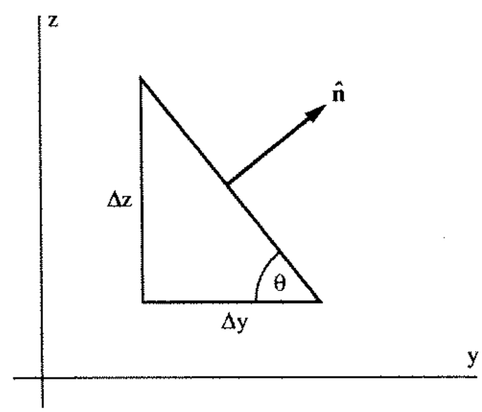


Figure b

$\iint_{\text{Front}} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \cong F_x\left(x_0 + \frac{\Delta x_0}{2}, y_0, z_0\right)\Delta y\Delta z/2$. Similarly on the back triangle, $\iint_{\text{Back}} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \cong -F\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\Delta y\Delta z/2$. For the rectangular side parallel to the xz-plane, $\iint_{\text{Rect side}} \mathbf{F}\cdot\hat{\mathbf{n}}dS \cong -F_y\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right)\Delta x\Delta z$. On the rectangular base, $\iint_{\text{Rect Base}} \mathbf{F}\cdot\hat{\mathbf{n}}dS \cong -F_z\left(x_0, y_0, z_0 - \frac{\Delta z}{2}\right)\Delta x\Delta z$. Next we must find $\hat{\mathbf{n}}$ for the slanted surface. From Figure b we see that

$$\hat{\mathbf{n}} = \mathbf{j}\sin\theta + \mathbf{k}\cos\theta = \frac{\mathbf{j}\Delta z + \mathbf{k}\Delta y}{\sqrt{(\Delta y)^2 + (\Delta z)^2}}.$$

Therefore

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \frac{F_y\Delta z + F_z\Delta y}{\sqrt{(\Delta y)^2 + (\Delta z)^2}}\,.$$

Hence

$$\iint_{\text{Slant. surf.}} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \cong \frac{F_y(x_0,y_0,z_0)\Delta z + F_z(x_0,y_0,z_0)\Delta y}{\sqrt{(\Delta y)^2 + (\Delta z)^2}}\Delta x\sqrt{(\Delta y)^2 + (\Delta z)^2}$$

$$\cong F_y(x_0,y_0,z_0)\Delta x\Delta z + F_z(x_0,y_0,z_0)\Delta x\Delta y.$$

Adding all these results and dividing by $\Delta V = \frac{1}{2}\Delta x\Delta y\Delta z$ we get

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \cong \frac{F_x\left(x_0+\frac{\Delta x}{2},y_0,z_0\right) - F_x\left(x_0-\frac{\Delta x}{2},y_0,z_0\right)}{\Delta x}$$
$$+ \frac{F_y(x_0,y_0,z_0) - F_y\left(x_0,y_0-\frac{\Delta y}{2},z_0\right)}{\Delta y/2}$$
$$+ \frac{F_z(x_0,y_0,z_0) - F_z\left(x_0,y_0,z_0-\frac{\Delta z}{2}\right)}{\Delta z/2}$$

$$\to \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)_{(x_0,y_0,z_0)} \quad \text{as } \Delta x,\ \Delta y, \text{ and } \Delta z \to 0.$$

18.

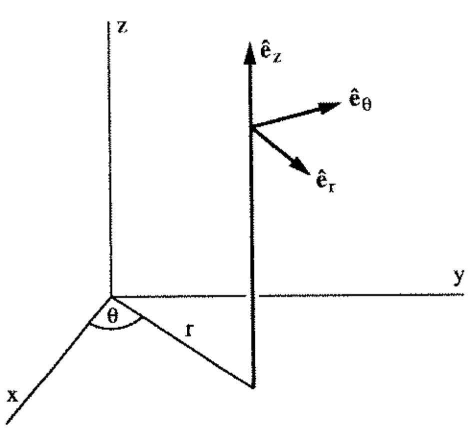


Figure a

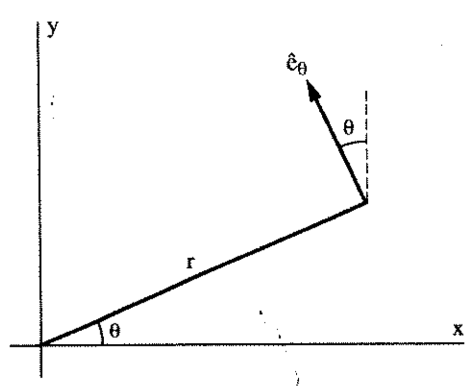


Figure b

a. Using Figures (a) and (b) we find

$$\hat{\mathbf{e}}_r = \frac{\mathbf{i}x + \mathbf{j}y}{r} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$
$$\hat{\mathbf{e}}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$
$$\hat{\mathbf{e}}_z = \mathbf{k}$$

These are three equations in the three unkonwns $\mathbf{i},\mathbf{j},\mathbf{k}$. Solving, we find $\mathbf{i} = \hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta$, $\mathbf{j} = \hat{\mathbf{e}}_r\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta$, and $\mathbf{k} = \hat{\mathbf{e}}_z$.

b. $$\mathbf{F} = \frac{-\mathbf{i}xy + \mathbf{j}y^2}{x^2 + y^2} = -\frac{r^2\sin\theta\cos\theta}{r^2}(\hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta) + \frac{r^2\cos^2\theta}{r^2}(\hat{\mathbf{e}}_r\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta)$$

$$= \hat{\mathbf{e}}_r(-\sin\theta\cos^2\theta + \sin\theta\cos^2\theta) + \hat{\mathbf{e}}_\theta(\sin^2\theta\cos\theta + \cos^3\theta)$$
$$= \hat{\mathbf{e}}_\theta\cos\theta.$$

But $\nabla\cdot\mathbf{F} = \dfrac{1}{r}\dfrac{\partial}{\partial\theta}(\cos\theta) = -\dfrac{\sin\theta}{r} = -\dfrac{y}{x^2 + y^2}$, which agrees with Prob. II–14(e).

c. $\mathbf{F} = \mathbf{k}r = \hat{\mathbf{e}}_z r$. Then $\nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial z}(r) = 0$, which agrees with Prob. II–14(f).

19.

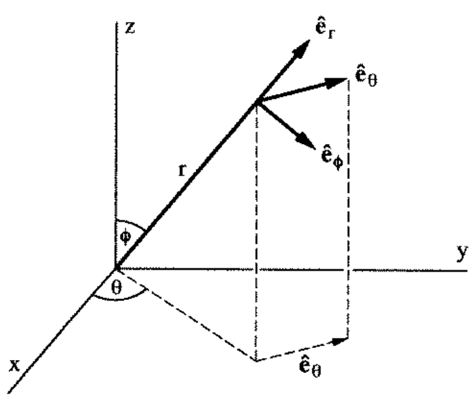


Figure a

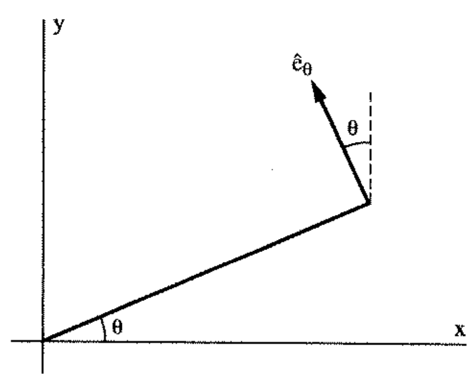


Figure b

a. Using Figures a and b we have

$$\hat{\mathbf{e}}_r = \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{r} = \mathbf{i}\sin\phi\cos\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\phi$$

$$\hat{\mathbf{e}}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

Then

$$\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{vmatrix}$$

$$= \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\sin\theta\cos\phi - \mathbf{k}\sin\phi.$$

These expressions for $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$, and $\hat{\mathbf{e}}_\theta$ are three equations in the three unknowns $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Solving these three equations yields the three expressions for $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ given in the statement of the problem.

b. $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. Using the results given in (a) and the standard expressions for x, y, and z in terms of r, θ, and ϕ, we find that $\mathbf{F} = \hat{\mathbf{e}}_r r$. Then $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r) = 3$, in agreement with the result for Prob II14-(g).

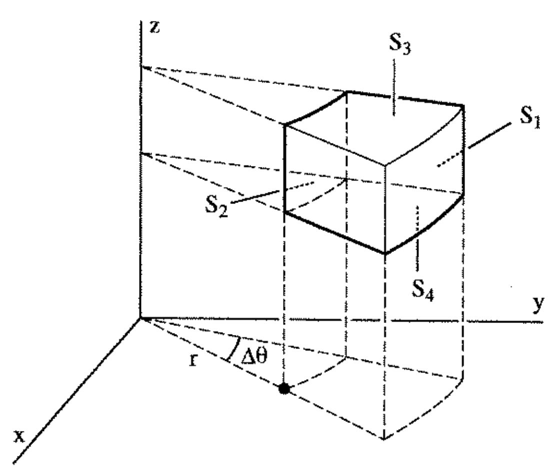
c. $\mathbf{F} = \dfrac{-\mathbf{i}y + \mathbf{j}x}{\sqrt{x^2 + y^2}}$. Using the results given in (a) and the standard expressions for x, y, and z in terms of r, θ, and ϕ, we find that $\mathbf{F} = \hat{\mathbf{e}}_\theta$. Then $\nabla\cdot\mathbf{F} = \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}(1) = 0$, in agreement with the results of ProbII14-(h).

20. For the side S_1 we have

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS \cong F_\theta\left(r, \theta + \frac{\Delta\theta}{2}, z\right)\Delta r\Delta z,$$

and for S_2,

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS \cong -F_\theta\left(r, \theta - \frac{\Delta\theta}{2}, z\right)\Delta r\Delta z.$$



Thus

$$\frac{1}{\Delta V}\iint_{S_1 + S_2} \mathbf{F}\cdot\mathbf{n}\,dS \cong \frac{1}{r\Delta\theta\Delta r\Delta z}\left[F_\theta\left(r, \theta + \frac{\Delta\theta}{2}, z\right) - F_\theta\left(r, \theta - \frac{\Delta\theta}{2}, z\right)\right]$$

$$\cong \frac{1}{r}\frac{F_\theta\left(r, \theta + \frac{\Delta\theta}{2}, z\right) - F_\theta\left(r, \theta - \frac{\Delta\theta}{2}, z\right)}{\Delta\theta} \rightarrow \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} \text{ as } \Delta\theta\rightarrow 0.$$

For S_3,

$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}dS \cong F_z\left(r,\theta,z + \frac{\Delta z}{2}\right)r\Delta\theta\Delta r,$$

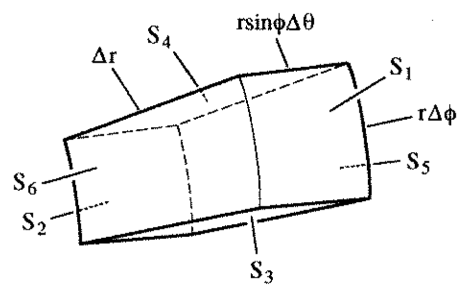
and for S_4,

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}dS \cong -F_z\left(r,\theta,z - \frac{\Delta z}{2}\right)r\Delta\theta\Delta r.$$

Hence

$$\frac{1}{\Delta V}\iint_{S_3 + S_4} \mathbf{F}\cdot\mathbf{n}dS \cong \frac{F_z\left(r,\theta,z + \frac{\Delta z}{2}\right) - F_z\left(r,\theta,z - \frac{\Delta z}{2}\right)}{\Delta z} \to \frac{\partial F_z}{\partial z} \text{ as } \Delta z \to 0.$$

21. For sides S_1 and S_2 we have



$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS \cong F_r\left(r + \frac{\Delta r}{2},\phi,\theta\right)r^2\sin\phi\Delta\phi\Delta\theta$$

and

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS \cong -F_r\left(r - \frac{\Delta r}{2},\phi,\theta\right)r^2\sin\phi\Delta\phi\Delta\theta.$$

Hence

$$\frac{1}{\Delta V}\iint_{S_1 + S_2} \mathbf{F}\cdot\mathbf{n}dS \cong \frac{F_r\left(r + \frac{\Delta r}{2},\phi,\theta\right) - F_r\left(r - \frac{\Delta r}{2},\phi,\theta\right)}{\Delta r} \to \frac{\partial F}{\partial r} \text{ as } \Delta r \to 0.$$

For sides S_3 and S_4 we have

$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}dS \cong F_\phi\left(r,\phi + \frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi + \frac{\Delta\phi}{2}\right)\Delta r\Delta\theta,$$

and

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}dS \cong -F_\phi\left(r,\phi - \frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi - \frac{\Delta\phi}{2}\right)\Delta r\Delta\theta.$$

Thus

$$\frac{1}{\Delta V}\iint_{S_3 + S_4} \mathbf{F}\cdot\mathbf{n}dS \cong \frac{F_\phi\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)\sin\left(\phi+\frac{\Delta\phi}{2}\right) - F_\phi\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)\sin\left(\phi-\frac{\Delta\phi}{2}\right)}{r\sin\theta\Delta\theta}$$

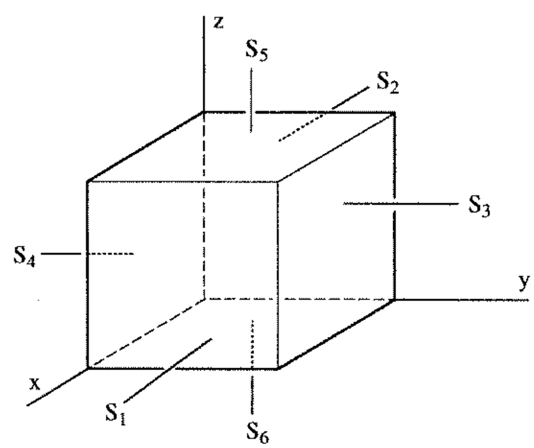
$$\to \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(F_\phi \sin\phi) \text{ as } \Delta\phi \to 0.$$

For sides S_5 and S_6 we get $\iint_{S_5} \mathbf{F}\cdot\mathbf{n}dS \cong F_\theta\left(r,\phi,\theta + \frac{\Delta\theta}{2}\right)r\Delta\phi\Delta r$ and $\iint_{S_6} \mathbf{F}\cdot\mathbf{n}dS \cong -F_\theta\left(r,\phi,\theta - \frac{\Delta\theta}{2}\right)r\Delta\phi\Delta r$. Thus $\frac{1}{\Delta V}\iint_{S_5+S_6} \mathbf{F}\cdot\mathbf{n}dS \cong$

$$\frac{F_\theta\left(r,\phi,\theta + \frac{\Delta\theta}{2}\right) - F_\theta\left(r,\phi,\theta - \frac{\Delta\theta}{2}\right)}{r\sin\phi\Delta\theta} \to \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta} \quad \text{as } \Delta\theta\to 0.$$

22. $\nabla\cdot[\hat{\mathbf{e}}_r f(r)] = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 f(r)] = 0$ if $r^2 f(r) = \text{const}$. Thus $f(r) = \frac{\text{const}}{r^2}$.

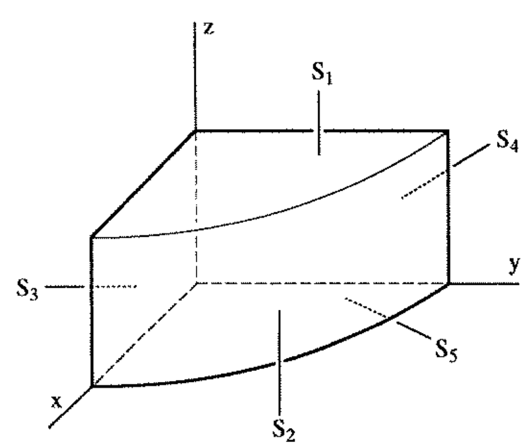
23. a. For faces S_1 and S_2 we have



$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} x dydx = b\iint_{S_1} dydx = b^3$, and $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS = -\iint_{S_2} x dydx = 0$ because $x = 0$ on S_2. In exactly the same way S_3 and S_5 each yield b^3 and S_4 and S_6 both give 0. Hence $\iint_S \mathbf{F}\cdot\mathbf{n}dS = 3b^3$. But

$$\nabla\cdot\mathbf{F} = \nabla\cdot(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = 3. \text{ Thus } \iiint_V \nabla\cdot\mathbf{F}\, dV = 3\iiint_V dV = 3b^3.$$

b. On S_1 we have



$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} z dS = h\iint_{S_1} dS = \pi R^2 h/4$, because $z = h$ on S_1. On S_2, $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS = -\iint_{S_2} z\, dS = 0$, because $z = 0$ on S_2. Next, $\iint_{S_3} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_4} \mathbf{F}\cdot\mathbf{n}dS = 0$ because $\hat{\mathbf{n}} = \pm\hat{\mathbf{e}}_\theta$ on S_3 and S_4 and $\mathbf{F}\cdot\hat{\mathbf{n}} = F_\theta = 0$. Finally,

$\iint_{S_5} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_5} \mathbf{F}\cdot\hat{\mathbf{e}}_r\, dS = \iint_{S_5} r\, dS = R\iint_{S_5} dS = R\left[\frac{2\pi Rh}{4}\right] = \frac{\pi R^2 h}{2}$. Adding the non-zero contributions from S_1 and S_5, we get $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \frac{3\pi R^2 h}{4}$.

Next we have $\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot r) + \frac{1}{r}\frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial z}(z) = 3$. Thus the volume integral is

$$\iiint_V \nabla\cdot\mathbf{F}\ dV = 3\iiint_V dV = 3\left(\frac{\pi r^2 h}{4}\right) = \frac{3\pi R^2 h}{4}.$$

c. $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_S r^2\hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r dS = R^2\iint_S dS = 4\pi R^2$. But $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^2)$

$= 4r$. Therefore $\iiint_V \nabla\cdot\mathbf{F}\ dV = \iiint_V 4r dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 4r\cdot r^2\sin\theta\ dr\ d\theta\ d\phi =$ $4\pi R^2$.

24. a. Using the divergence theorem, $\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = \iiint_V \nabla\cdot\mathbf{B}\ dV = 0$ because $\nabla\cdot\mathbf{B} = 0$.

b. From (a) $\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = \iint_{CS} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS + \iint_{Base} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = 0$, where CS

means the curved surface of the cone. It follows then that $\iint_{CS} \mathbf{B}\cdot\hat{\mathbf{n}}dS$

$= -\iint_{Base} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS$. But on the base $\mathbf{B}\cdot\hat{\mathbf{n}} = -B$, because the normal points outward from the volume. Hence

$$\iint_{CS} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = B\iint_{Base} dS = \pi R^2 B.$$

25. In the divergence theorem let $\mathbf{F} = \mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector. Then $\nabla\cdot\mathbf{F} = \nabla\cdot\mathbf{c} = 0$ and so

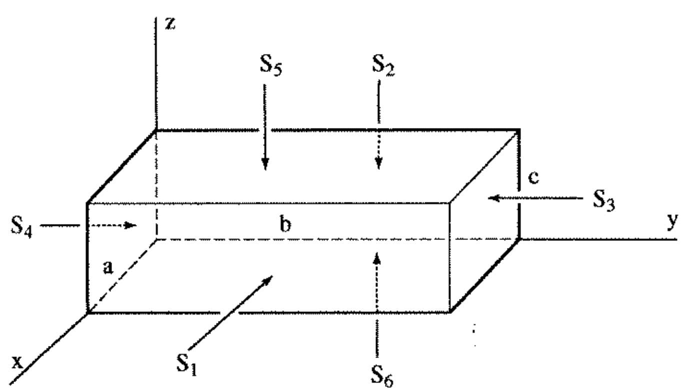
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_S \mathbf{c}\cdot\hat{\mathbf{n}}\ dS = \mathbf{c}\cdot\iint_S \hat{\mathbf{n}}\ dS = 0.$$

But because $\mathbf{c}$ is an arbitrary vector, this last equality implies that $\iint_S \hat{\mathbf{n}}\ dS = 0$.

26. a. In the divergence theorem set $\mathbf{F} = \mathbf{r}$. Recalling that $\nabla\cdot\mathbf{r} = 3$, we have $\iint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS = \iiint_V 3dV = 3V$, or $V = \frac{1}{3}\iint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS$.

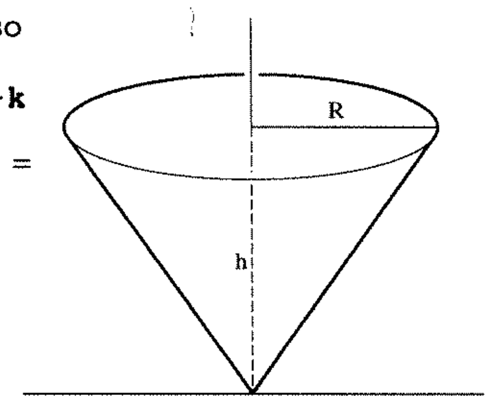
b. (i) We have the following (see figure below):

On S_1, $\mathbf{r}\cdot\hat{\mathbf{n}} = x = a$ On S_2, $\mathbf{r}\cdot\hat{\mathbf{n}} = -x = 0$
On S_3, $\mathbf{r}\cdot\hat{\mathbf{n}} = y = b$ On S_4, $\mathbf{r}\cdot\hat{\mathbf{n}} = -y = 0$
On S_5, $\mathbf{r}\cdot\hat{\mathbf{n}} = z = c$ On S_6, $\mathbf{r}\cdot\hat{\mathbf{n}} = -z = 0$



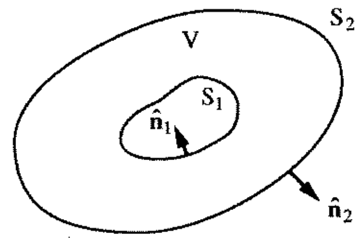
$\therefore \iint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS = abc + bac + cab = 3abc$.

(ii). On the curved surface, $\mathbf{r} \perp \hat{\mathbf{n}}$ so $\mathbf{r}\cdot\hat{\mathbf{n}} = 0$. On the top (see figure), $\mathbf{r}\cdot\hat{\mathbf{n}} = \mathbf{r}\cdot\mathbf{k} = z = h$. Hence $\iint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS = h\pi R^2$ and so $V = \frac{\pi R^2 h}{3}$.



(iii). $\iint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS = \iint_S \mathbf{r}\cdot\hat{\mathbf{e}}_r\,dS = R\iint_S dS = R(4\pi R^2) = 4\pi R^3$, and so $V = 4\pi R^3/3$.

27. a. S_1 and S_2 are the two parts of the surface that encloses V. With $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ oriented as shown in the figure, we have

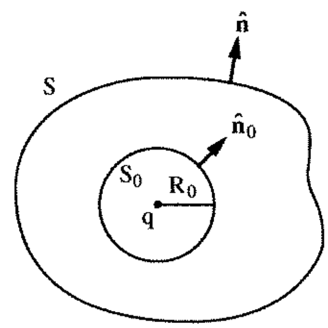


$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS + \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iiint_V \nabla\cdot\mathbf{F}\ dV = 0,$$

because $\nabla\cdot\mathbf{F} = 0$ in V. Thus $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = -\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS$. If we now replace $\hat{\mathbf{n}}_1$ by $-\hat{\mathbf{n}}_1$ we get $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS$. Note that this choice of $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ corresponds to the normals pictured in the text.

b. Using the divergence in spherical coordinates, $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{c}{r^2}\right) = 0$ (c is a constant). Note that this holds only if $r \neq 0$.

c. Let S_0 be a sphere of radius R_0 centered at the charge and lying entirely within S (see figure). The field of a point charge q at the origin is $\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{e}}_r}{r^2}$. Hence $\iint_{S_0} \mathbf{E}\cdot\hat{\mathbf{n}}dS = \frac{q}{4\pi\varepsilon_0 R_0^2}(4\pi R_0^2) = \frac{q}{\varepsilon_0}$. But from (a), $\iint_{S_0} \mathbf{E}\cdot\hat{\mathbf{n}}dS = \iint_S \mathbf{E}\cdot\hat{\mathbf{n}}dS$ with the normals as shown in the figure. Hence $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}dS = \frac{q}{\varepsilon_0}$.



d. We have established in (c) that $\iint_S \mathbf{E}_i\cdot\hat{\mathbf{n}}\ dS = \frac{q_i}{\varepsilon_0}$, where $\mathbf{E}_i$ is the field due to the charge q_i. Thus, using superposition,

$$\sum_i \iint_S \mathbf{E}_i\cdot\hat{\mathbf{n}}\ dS = \iint_S \sum_i \mathbf{E}_i\cdot\hat{\mathbf{n}}\ dS = \frac{1}{\varepsilon_0}\sum_i q_i,$$ or $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}dS = \frac{q}{\varepsilon_0}$, where $\mathbf{E} = \sum_i \mathbf{E}_i$ is the field due to the entire collection of charges, and $q = \sum_i q_i$ is the total charge.

28. a. $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_S \frac{\hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r}{r^2} dS = \left(\frac{1}{R^2}\right)(4\pi R^2) = 4\pi$. But $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$ and so $\iiint_V \nabla\cdot\mathbf{F}dV = 0 \neq 4\pi$. The divergence theorem is apparently violated. However, for the theorem to hold, $\mathbf{F}$ must be well-behaved everywhere in V (and on S). But V includes the point $r = 0$ where $\mathbf{F}$ is not defined.

b. $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS + \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = -\iint_{S_1} \frac{\hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r}{R_1^2} dS + \iint_{S_2} \frac{\hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r}{R_2^2} dS =$

$$-\frac{4\pi R_1^2}{R_1^2} + \frac{4\pi R_2^2}{R_2^2} = 0 = \iiint_V \nabla\cdot\mathbf{F}dV,$$

which is the desired result because $\nabla\cdot\mathbf{F} = 0$ in the region V between the two spheres.

c. The region enclosed by S must not include the point $r = 0$.

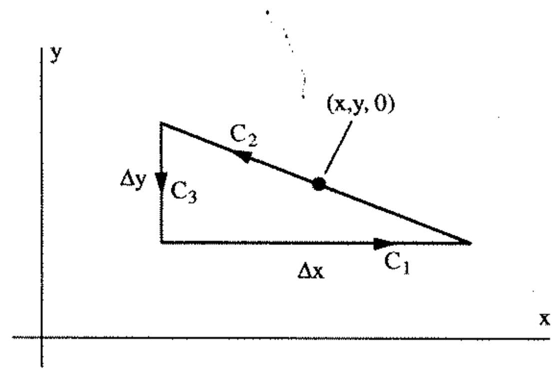
1. $\mathbf{F} = f(r)\hat{\mathbf{u}} = f(r)\hat{\mathbf{e}}_r = f(r)\dfrac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{r}$. Hence

$$\mathbf{F}\cdot\hat{\mathbf{t}}ds = \frac{1}{r}f(r)(xdx + ydy + zdz) = \frac{f(r)}{r}rdr = f(r)dr$$

$$\therefore \quad \int_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = \int_{r_1}^{r_2} f(r)dr = F(r_2) - F(r_1), \text{ where } F' = f.$$

This result is independent of C.

2. Defining the sides of the triangle as shown in the figure, we get the following:



On C_1 $\hat{\mathbf{t}} = \mathbf{i}$ so $\displaystyle\int_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}ds = \int_{C_1}F_x dx \cong$

$F_x\left(x, y - \dfrac{\Delta y}{2}, z\right)\Delta x,$

On C_3 $\hat{\mathbf{t}} = -\mathbf{j}$ so $\displaystyle\int_{C_3}\mathbf{F}\cdot\hat{\mathbf{t}}ds = -\int_{C_3}F_y dy \cong -F_y\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y.$

On C_2 we have (see figure) $\hat{\mathbf{t}} = \dfrac{-\mathbf{i}\Delta x + \mathbf{j}\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$ and so

$$\mathbf{F}\cdot\hat{\mathbf{t}} = \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}[-F_x(x,y,z)\Delta x + F_y(x,y,z)\Delta y].$$

Hence

$$\int_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}ds \cong \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}[-F_x(x,y,z)\Delta x + F_y(x,y,z)\Delta y]\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\cong -F_x(x,y,z)\Delta x + F_y(x,y,z)\Delta y.$$

The area of the triangle is $\Delta S = \Delta x\Delta y/2$ so we get

$$\frac{1}{\Delta S}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds \cong \frac{2}{\Delta x\Delta y}\Big[F_x\left(x, y - \frac{\Delta y}{2}, z\right)\Delta x - F_y\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y -$$

$$F_x(x,y,z)\Delta x + F_y(x,y,z)\Delta y]$$

$$= \frac{F_y(x,y,z) - F_y\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x/2} - \frac{F_x(x,y,z) - F_x\left(x, y - \frac{\Delta y}{2}, z\right)}{\Delta y/2}$$

$$\to \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \text{ as } \Delta x \text{ and } \Delta y \to 0.$$

3\. a. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & x^2 & -y^2 \end{vmatrix} = 2(-\mathbf{i}y + \mathbf{j}z + \mathbf{k}x).$

b. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xz & 0 & -x^2 \end{vmatrix} = 5\mathbf{j}x.$

c. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-y} & e^{-z} & e^{-x} \end{vmatrix} = \mathbf{i}e^{-z} + \mathbf{j}e^{-x} + \mathbf{k}e^{-y}.$

d. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \mathbf{i}(x - x) + \mathbf{j}(y - y) + \mathbf{k}(z - z) = 0.$

e. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -yz & xz & 0 \end{vmatrix} = -\mathbf{i}x - \mathbf{j}y + 2\mathbf{k}z.$

f. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & x^2+y^2 \end{vmatrix} = 2(\mathbf{i}y - \mathbf{j}x).$

g. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 & yz \end{vmatrix} = \mathbf{i}z - \mathbf{k}x.$

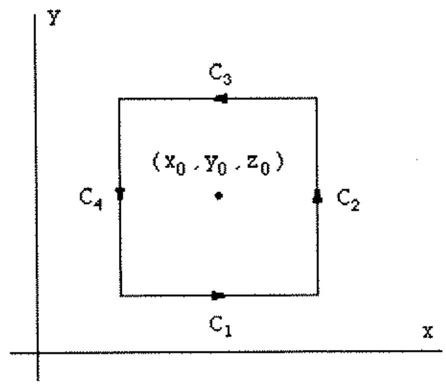
h. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/D & y/D & z/D \end{vmatrix}$ where $D = (x^2 + y^2 + z^2)^{3/2}$.

The x component of this is

$$\frac{\partial}{\partial y}\frac{z}{D} - \frac{\partial}{\partial z}\frac{y}{D} = -\frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} = 0.$$

The other two components yield 0 in the same way.

4. a. $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_1} z^2 ds = 0$ (because $z = 0$ on C_1),



$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_2} x^2 ds = (x_0 + s/2)^2 s,$$

$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_3} z^2 ds = 0$ (because $z = 0$ on C_3), and

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_4} x^2 ds = -(x_0 - s/2)^2 s.$$

Combining these results we find

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = [(x_0 + s/2)^2 - (x_0 - s/2)^2]s = 2x_0 s^2.$$

b. Since the area of the square is s^2 we get

$\frac{1}{s^2}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = 2x_0 = (\mathbf{k}\cdot\nabla\times\mathbf{F})_{(x_0,y_0,z_0)}$. (Note that there is no need to take the limit as $s \to 0$ because our result, $2x_0$, is independent of s.) This result agrees with Prob III3-(a).

c. For $\mathbf{F} = 3\mathbf{i}xz - \mathbf{k}x^2$

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = 3\int_{C_1} xz ds = 0 \quad \text{(because } z = 0 \text{ on } C_1)$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_2} F_y\ ds = 0 \quad \text{(because } F_y = 0)$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -3\int_{C_3} xz\ ds = 0 \quad \text{(because } z = 0 \text{ on } C_3)$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_4} F_y\ ds = 0 \quad \text{(because } F_y = 0)$$

Combining these results we get $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = 0$. This implies that $\nabla \times \mathbf{F} = 0$, which agrees with Prob III3-(b).

For $\mathbf{F} = \mathbf{i}e^{-y} + \mathbf{j}e^{-z} + \mathbf{k}e^{-x}$,

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_1} e^{-y} ds \cong e^{-(y_0 - s/2)}s$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_3} e^{-z} ds \cong s$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_3} e^{-y}\ ds \cong -e^{-(y_0 + s/2)}s$$

$$\int_{C_4} \mathbf{F}\cdot\mathbf{t}\ ds = -\int_{C_4} e^{-z}\ ds \cong -s$$

Combining these results and dividing by the area of the square, s^2, we get

$$\frac{1}{s^2}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \frac{e^{-(y_0 - s/2)} - e^{-(y_0 + s/2)}}{s}$$

The limit of this expression as $s \to 0$ is by definition $-\frac{d}{dy}e^{-y}$ evaluated at y_0, that is e^{-y_0}. This agrees with Prob III4-(d).

5. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \oint_C (y + y^2)dz$, and the curve C consists of the three

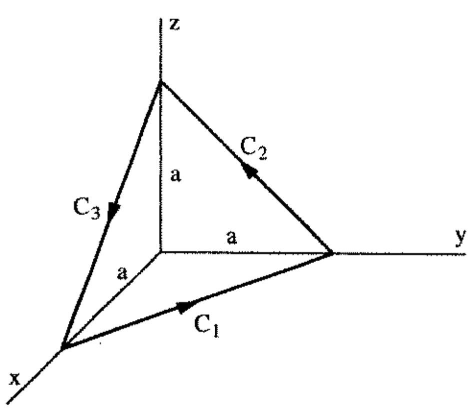


Figure a

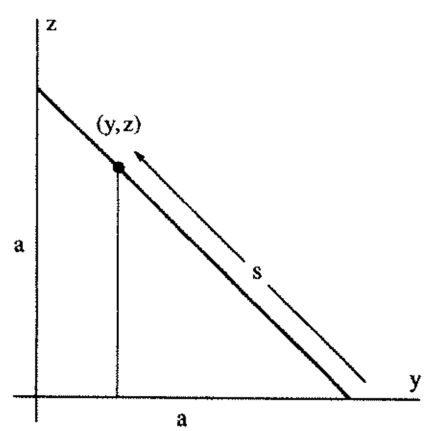


Figure b

sides of the triangle as shown in Figure (a). However, on C_1 $y = 0$ and on C_2 $z = 0$, so the only contribution to the integral comes from C_3. From Figure (b) we get $(a - y)/s = 1/\sqrt{2}$ or $y = a - s/\sqrt{2}$ and $z/s = 1/\sqrt{2}$ or $z = s/\sqrt{2}$. Here s the length along C_3 measured from the point (0,a,0) as shown in Figure (b). Hence

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_3} (y + y^2)dz = \int_0^{\sqrt{2}\,a}\left[\left(a - \frac{s}{\sqrt{2}}\right) + \left(a - \frac{s}{\sqrt{2}}\right)^2\right]\frac{ds}{\sqrt{2}}$$

$$= a\int_0^1 [a(1 - w) + a^2(1 - w)^2]\ dw = \frac{a^2}{2} + \frac{a^3}{3}. \qquad \left(w = \frac{s}{\sqrt{2}\,a}\right)$$

The triangle is equilateral with side $\sqrt{2}\,a$ and its area $A = \sqrt{3}\,a^2/2$. Hence

$$\frac{1}{A}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \frac{2}{\sqrt{3}\,a^2}\left(\frac{a^2}{2} + \frac{a^3}{3}\right) = \frac{1}{\sqrt{3}} + \frac{2a}{3\sqrt{3}} \rightarrow \frac{1}{\sqrt{3}} \quad \text{as} \quad a \rightarrow 0.$$

But $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & y + y^2 \end{vmatrix} = (1 + 2y)\mathbf{i}$, and $\hat{\mathbf{n}} = \dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$.

Hence $\mathbf{n}\cdot\nabla \times \mathbf{F} = \dfrac{1 + 2y}{\sqrt{3}} \rightarrow \dfrac{1}{\sqrt{3}}$ as $y \rightarrow 0$, in agreement with the limit calculated above.

6. $\mathbf{A} \times \mathbf{r} = \mathbf{i}(A_y z - A_z y) + \mathbf{j}(A_z x - A_x z) + \mathbf{k}(A_x z - A_z x)$. Hence

$$\nabla \times (\mathbf{A} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_y z - A_z y & A_z x - A_x z & A_x y - A_z x \end{vmatrix}$$

$$= \mathbf{i}(A_x + A_x) + \mathbf{j}(A_y + A_y) + \mathbf{k}(A_z + A_z) = 2\mathbf{A}.$$

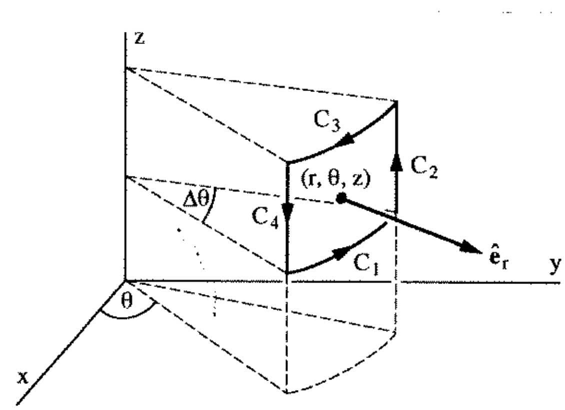
Hence $\nabla \times \frac{1}{2}(\mathbf{A} \times \mathbf{r}) = \mathbf{A}$.

7. $\nabla \times \mathbf{F} = \mathbf{i}\left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)$. Thus

$$\nabla\cdot(\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0$$

because terms cancel in pairs: (1 & 4), (2 & 5), and (3 & 6).

8. To get the radial component of the curl we proceed as follows.



$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_\theta\left(r,\theta,z - \frac{\Delta z}{2}\right) r\Delta\theta,$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_z\left(r,\theta + \frac{\Delta\theta}{2},z\right)\Delta z,$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_\theta\left(r,\theta,z + \frac{\Delta z}{2}\right) r\Delta\theta,$$

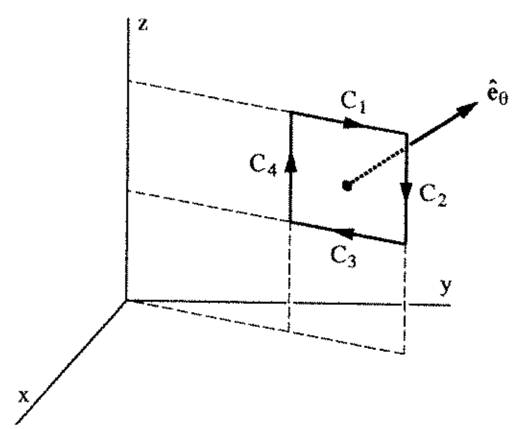
$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_z\left(r,\theta - \frac{\Delta\theta}{2},z\right)\Delta z.$$

Adding these four integrals and dividing by $\Delta A = r\Delta\theta\Delta z$ we get

$$\frac{1}{\Delta A}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_z\left(r,\theta + \frac{\Delta\theta}{2},z\right) - F_z\left(r,\theta - \frac{\Delta\theta}{2},z\right)}{r\Delta\theta} - \frac{F_\theta\left(r,\theta,z + \frac{\Delta z}{2}\right) - F_\theta\left(r,\theta,z - \frac{\Delta z}{2}\right)}{\Delta z}$$

$$\to \frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} = (\nabla \times \mathbf{F})_r \quad \text{as } \Delta\theta \text{ and } \Delta z \to 0.$$

To get the θ component of the curl we proceed as follows.



$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_r\left(r,\theta,z + \frac{\Delta z}{2}\right)\Delta r$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_z\left(r + \frac{\Delta r}{2},\theta,z\right)\Delta z$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_r\left(r,\theta,z - \frac{\Delta z}{2}\right)\Delta r$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_z\left(r - \frac{\Delta r}{2},\theta,z\right)\Delta z$$

Adding these four integrals and dividing by $\Delta A = \Delta r\Delta z$ we get

$$\frac{1}{\Delta A}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_r\left(r,\theta,z + \frac{\Delta z}{2}\right) - F_r\left(r,\theta,z - \frac{\Delta z}{2}\right)}{\Delta z} - \frac{F_z\left(r + \frac{\Delta r}{2},\theta,z\right) - F_z\left(r - \frac{\Delta r}{2},\theta,z\right)}{\Delta r}$$

$$\rightarrow \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} = (\nabla \times \mathbf{F})_\theta \quad \text{as } \Delta z \text{ and } \Delta r \rightarrow 0.$$

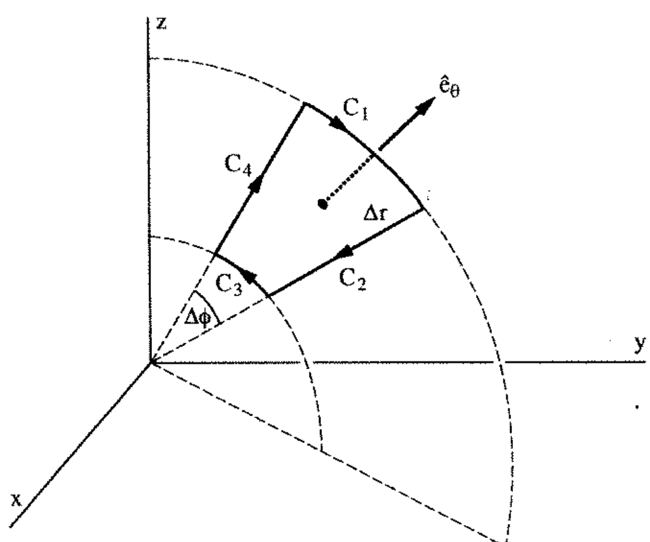
9. To get the θ component of the curl we proceed as follows (see figure below):

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_1} F_\phi ds \cong F_\phi\left(r + \frac{\Delta r}{2}, \phi, \theta\right)\left(r + \frac{\Delta r}{2}\right)\Delta\phi$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_2} F_r\, ds \cong -F_r\left(r, \phi + \frac{\Delta\phi}{2}, \theta\right)\Delta r$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_3} F_\phi\, ds \cong -\,F_\phi\left(r - \frac{\Delta r}{2}, \phi, \theta\right)\left(r - \frac{\Delta r}{2}\right)\Delta\phi$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_4} F_r\, ds \cong F_r\left(r, \phi - \frac{\Delta\phi}{2},\right)\Delta r$$



Adding the contributions from C_1 and C_3 and dividing by $\Delta A = r\Delta\theta\Delta r$, we get

$$\frac{1}{\Delta A}\int_{C_1+C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_\phi\left(r + \frac{\Delta r}{2}, \phi, \theta\right)\left(r + \frac{\Delta r}{2}\right) - F_\phi\left(r - \frac{\Delta r}{2}, \phi, \theta\right)\left(r - \frac{\Delta r}{2}\right)}{r\Delta r}$$

$$\rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) \quad \text{as } \Delta r \rightarrow 0$$

Adding the contributions of C_2 and C_4 and dividing by ΔA we get

$$\frac{1}{\Delta A}\int_{C_2+C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong -\frac{F_r\left(r,\phi+\frac{\Delta\phi}{2},\theta\right) - F_r\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)}{r\Delta\theta}$$

$$\to -\frac{1}{r}\frac{\partial F_r}{\partial\phi} \quad \text{as } \Delta r \to 0.$$

Combining these two results we get

$$(\nabla \times \mathbf{F})_\theta = \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) - \frac{1}{r}\frac{\partial F_r}{\partial\phi}$$

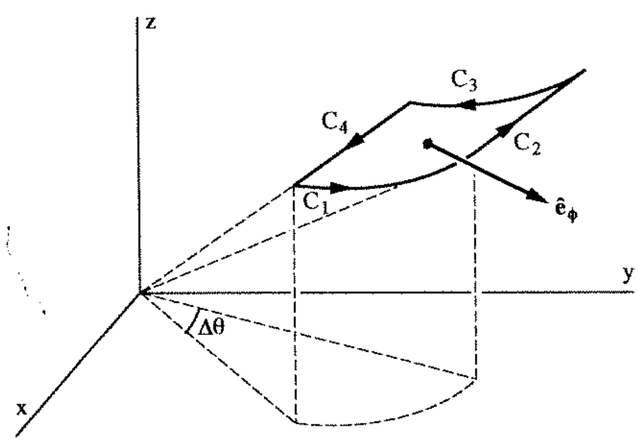
We next obtain $(\nabla \times \mathbf{F})_\phi$ (see figure below):

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_1} F_\theta ds \cong \theta\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)\sin\phi\Delta\theta$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_2} F_r ds \cong F_r\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)\Delta r$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_1} F_\theta ds \cong -F_\theta\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right)\sin\phi\Delta\theta$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_2} F_r ds \cong -F_r\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)\Delta r$$



Adding the contributions from C_1 and C_3 and dividing by $\Delta A = r\sin\phi\Delta\theta\Delta r$, we get

$$\frac{1}{\Delta A}\int_{C_1+C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong -\frac{1}{r}\,\frac{F_\theta\left(r + \frac{\Delta r}{2},\phi,\theta\right)\left(r + \frac{\Delta r}{2}\right) - F_\theta\left(r - \frac{\Delta r}{2},\phi,\theta\right)\left(r - \frac{\Delta r}{2}\right)}{\Delta r}$$

$$\to -\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) \quad \text{as } \Delta r \to 0.$$

Adding the contributions from C_2 and C_4 and dividing by ΔA we get

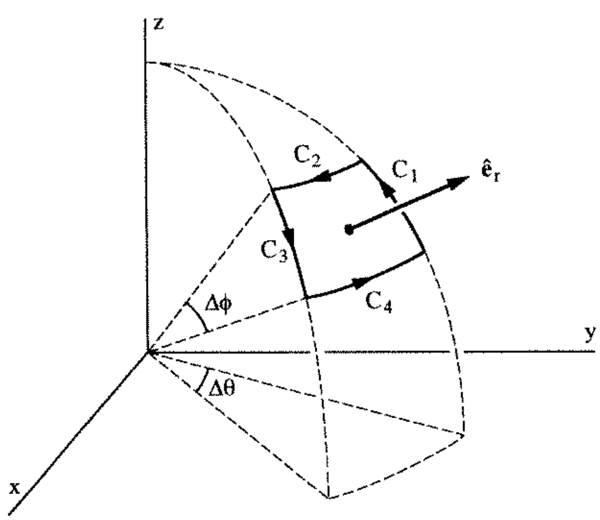
$$\frac{1}{\Delta A}\int_{C_2+C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong \frac{F_r\left(r,\phi,\theta + \frac{\Delta\theta}{2}\right) - F_r\left(r,\phi,\theta - \frac{\Delta\theta}{2}\right)}{r\sin\phi\Delta\theta}$$

$$\to \frac{1}{r\sin\phi}\frac{\partial F_r}{\partial\theta} \quad \text{as } \Delta\theta \to 0.$$

Combining these two results we find

$$(\nabla \times \mathbf{F})_\phi = \frac{1}{r\sin\phi}\frac{\partial F_r}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\theta).$$

Finally we obtain $(\nabla\times \mathbf{F})_r$ (see figure below):



$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_1} F_\phi ds \cong -F_\phi\left(r,\phi,\theta + \frac{\Delta\theta}{2}\right)r\Delta\phi$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_2} F_\phi ds \cong -F_\theta\left(r,\phi - \frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi - \frac{\Delta\phi}{2}\right)\Delta\phi$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_3} F_\phi ds \cong F_\phi\left(r,\phi,\theta - \frac{\Delta\theta}{2}\right)r\Delta\phi$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_4} F_\theta ds \cong F_\theta\left(r,\phi + \frac{\Delta\phi}{2},\theta\right) r\sin\left(\phi + \frac{\Delta\phi}{2}\right)\Delta\theta$$

Adding the integrals over C_1 and C_3 and dividing by $\Delta A = r^2\sin\phi\Delta\phi\Delta\theta$ we get

$$\frac{1}{\Delta A}\int_{C_1+C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong -\frac{F_\phi\left(r,\phi,\theta + \frac{\Delta\theta}{2}\right) - F_\phi\left(r,\phi,\theta - \frac{\Delta\theta}{2}\right)}{r\sin\phi\Delta\theta}$$

$$\to -\frac{1}{r\sin\phi}\frac{\partial F_\phi}{\partial\theta} \quad \text{as } \Delta\theta \to 0.$$

Adding the integrals over C_2 and C_4 and dividing by ΔA we get

$$\frac{1}{\Delta A}\int_{C_2+C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong$$

$$\frac{F_\theta\left(r,\phi + \frac{\Delta\phi}{2},\theta\right)\sin\left(\phi + \frac{\Delta\phi}{2}\right) - F_\theta\left(r,\phi - \frac{\Delta\phi}{2},\theta\right)\sin\left(\phi - \frac{\Delta\phi}{2}\right)}{r\mathrm{si}\phi\Delta\phi}$$

$$\to \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\ F_\theta) \quad \text{as } \Delta\phi \to 0.$$

Thus

$$(\nabla \times \mathbf{F})_r = \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\ F_\theta) - \frac{1}{r\sin\phi}\frac{\partial F_\phi}{\partial\theta}.$$

10. a. $\mathbf{F} = -\mathbf{i}yz + \mathbf{j}xz$

$$= -(\hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta)zr\sin\theta + (\hat{\mathbf{e}}_r\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta)zr\cos\theta$$

$$= rz\hat{\mathbf{e}}_\theta$$

Thus

$$(\nabla \times \mathbf{F})_r = -\frac{\partial}{\partial z}(rz) = -r$$

$$(\nabla \times \mathbf{F})_\theta = 0$$

$$(\nabla \times \mathbf{F})_z = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot rz) = 2z$$

and so

$$\nabla \times \mathbf{F} = -r\hat{\mathbf{e}}_r + 2z\hat{\mathbf{e}}_z = -r(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) + 2\mathbf{k}z$$

$$= -\mathbf{i}x - \mathbf{j}y + 2\mathbf{k}z.$$

b. $\mathbf{F} = \mathbf{i}x + \mathbf{j}y \quad \mathbf{k}(x^2 + y^2)$

$$= (\hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta)r\cos\theta + (\hat{\mathbf{e}}_r\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta)r\sin\theta + \hat{\mathbf{e}}_z r^2$$

$$= \hat{\mathbf{e}}_r r + \hat{\mathbf{e}}_z r^2.$$

Thus

$$(\nabla \times \mathbf{F})_r = \frac{1}{r}\frac{\partial}{\partial\theta}(r^2) = 0$$

$$(\nabla \times \mathbf{F})_\theta = \frac{\partial r}{\partial z} - \frac{\partial}{\partial r}(r^2) = -2r$$

$$(\nabla \times \mathbf{F})_z = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot 0) - \frac{\partial}{\partial\theta}(r) = 0$$

and so

$$\nabla \times \mathbf{F} = -2r\hat{\mathbf{e}}_\theta = -2\mathbf{j}x + 2\mathbf{i}y = 2(\mathbf{i}y - \mathbf{j}x).$$

11. a. $\mathbf{F} = \mathbf{i}xy + \mathbf{j}y^2 + \mathbf{k}yz$

$$= (\hat{\mathbf{e}}_r\sin\phi\cos\theta + \hat{\mathbf{e}}_\phi\cos\phi\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta)r^2\sin^2\phi\cos\theta\sin\theta$$

$$+ (\hat{\mathbf{e}}_r\sin\phi\sin\theta + \hat{\mathbf{e}}_\phi\cos\phi\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta)r^2\sin^2\phi\sin^2\theta$$

$$+ (\hat{\mathbf{e}}_r\cos\phi - \hat{\mathbf{e}}_\phi\sin\phi)r^2\sin\phi\sin\theta\cos\phi$$

$$= \hat{\mathbf{e}}_r r^2\sin\phi\sin\theta$$

Thus

$$(\nabla \times \mathbf{F})_r = 0$$

$$(\nabla \times \mathbf{F})_\phi = \frac{1}{r}\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}(r^2\sin\phi\sin\theta) = -r\cos\theta$$

$$(\nabla \times \mathbf{F})_\theta = -\frac{1}{r}\frac{\partial}{\partial\phi}(r^2\sin\phi\sin\theta) = -r\cos\phi\sin\theta$$

and so

$$\nabla \times \mathbf{F} = \hat{\mathbf{e}}_\phi r\cos\theta - \hat{\mathbf{e}}_\theta r\cos\phi\sin\theta$$

$$= \mathbf{i}z - \mathbf{k}x.$$

b. $$\mathbf{F} = \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{(x^2 + y^2 + z^2)^{3/2}} = \frac{r\hat{\mathbf{e}}_r}{r^3} = \frac{\hat{\mathbf{e}}_r}{r^2}.$$

Thus $\mathbf{F}$ has only a radial component and depends only upon r. Using the form of the curl in spherical coordinates, the only terms involving F_r are one in the ϕ component which is proportional to $\frac{\partial F_r}{\partial\theta}$, and one in the θ component which is proportional to $\frac{\partial F_r}{\partial\phi}$. Both these derivatives are zero, so $\nabla \times \mathbf{F} = 0$.

12. The argument given in Prob III-11 shows that $\nabla \times \mathbf{F} = 0$.

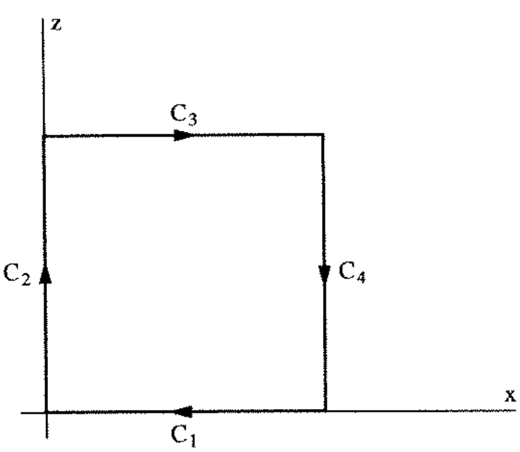
13. The ones with zero curl, (d) and (h).

14. In Stokes' theorem put $\mathbf{F} = \mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector. Then Stokes' theorem reads

$$\oint_C \mathbf{c}\cdot\hat{\mathbf{t}}\ ds = \iint_S \mathbf{n}\cdot\nabla \times \mathbf{c}\ dS = 0$$

because $\nabla \times \mathbf{c} = 0$. Hence $\mathbf{c}\cdot\oint_C \hat{\mathbf{t}}\ ds = 0$. But because $\mathbf{c}$ is an arbitrary vector this implies that $\oint_C \hat{\mathbf{t}}\ ds = 0$.

15. a. $\mathbf{F} = \mathbf{i}z^2 - \mathbf{j}y^2$. Thus $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \oint_C z^2dx - y^2dy = \oint_C z^2dx$ because $y = 0$ on C. Now



$\int_{C_1} z^2dx = 0$, because $z = 0$ on C_1

$\int_{C_2} z^2dx = 0$, because $dx = 0$ on C_2

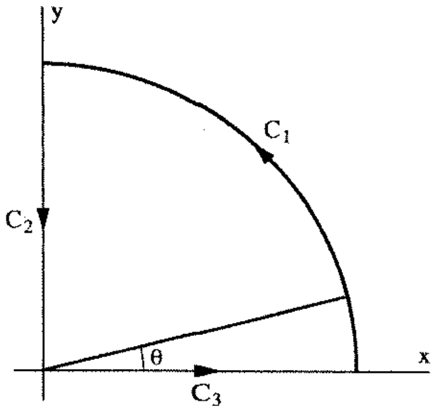
$\int_{C_3} z^2dx = \int_0^1 dx = 1$, because $z = 1$ on C_3

$\int_{C_4} z^2dx = 0$, because $dx = 0$ on C_4

$\therefore \quad \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = 1.$

It is easy to show that $\nabla \times \mathbf{F} = 2\mathbf{j}z$ which implies that $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = 0$ on all surfaces except S_5. But on S_5 $\hat{\mathbf{n}} = \mathbf{j}$ so $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = 2z$. Thus $\iint_{S_5} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\ dS = 2\int_0^1\int_0^1 z\ dxdz = 2\int_0^1 z\ dz = 1$, in agreement with the line integral given above.

b. $\mathbf{F} = \mathbf{i}y + \mathbf{j}z + \mathbf{k}x$. Hence $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_1} y dx$ (because $z = 0$ on C_1). Letting $x = \cos\theta$, $y = \sin\theta$, this integral becomes $-\int_0^{\pi/2} \sin^2\theta d\theta = -\pi/4$. The integrals over C_2 and C_3 are treated in exactly the same way and both yield $-\pi/4$. Hence $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -3\pi/4$.



A straightforward calculation gives $\nabla \times \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, while the equation of the surface is $z = \sqrt{1 - x^2 - y^2}$. Hence $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$. We therefore have

$$\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS = \iint_R \left(-\frac{x}{z} - \frac{y}{z} - 1\right)dxdy$$

where R is the quarter circle of radius 1 lying in the xy-plane and centered at the origin. The integral can be written

$$-\iint_R \frac{x}{\sqrt{1 - x^2 - y^2}}dxdy - \iint_R \frac{y}{\sqrt{1 - x^2 - y^2}}dxdy - \iint_R dxdy.$$

But

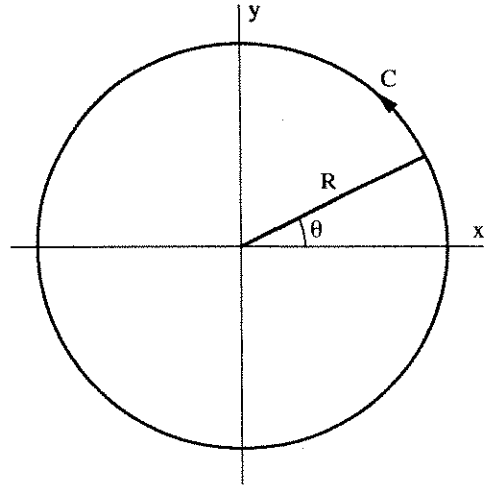
$$\iint_R \frac{x}{\sqrt{1 - x^2 - y^2}}dxdy = \int_0^{\pi/2}\int_0^1 \frac{r^2\cos\theta\ drd\theta}{\sqrt{1 - r^2}} = \int_0^1 \frac{r^2\ dr}{\sqrt{1 - r^2}} = \frac{\pi}{4}.$$

The second integral above can be treated in exactly the same way and also yields $\pi/4$. The third integral is just the area of the quarter-circle and thus also equals $\pi/4$. It follows then that

$$\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\ dS = -\frac{3\pi}{4},$$

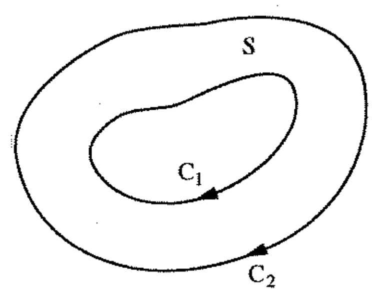
in agreement with the line integral calculated above.

c. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C ydx - xdy + zdz =$ $\oint_C ydx - xdy$, because $z = 0$ on C. But $\oint_C ydx = -R^2\int_0^{2\pi}\sin^2\theta d\theta = -\pi R^2$ where we have put $x = R\cos\theta$ and $y = R\sin\theta$, as shown in the figure. Using the same transformation we find $\oint_C xdy =$ $-R^2\int_0^{2\pi}\cos^2\theta d\theta = -\pi R^2$. Adding these two



results we get $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -2\pi R^2$. A straightforward calculation gives $\nabla \times \mathbf{F} = -2\hat{\mathbf{k}}$. On the curved surface of the cylinder $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = -2\hat{\mathbf{n}}\cdot\mathbf{k} = 0$. On the top of the cylinder $\hat{\mathbf{n}} = \mathbf{k}$ so $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = \mathbf{k}\cdot(-2\mathbf{k}) = -2$. Therefore we have $\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\ dS = -2\iint_S dS = -2(\pi R^2) = -2\pi R^2$, in agreement with the line integral calculated above.

16. a. C_1 and C_2 together constitute a curve (in two parts) which encloses a surface S (see figure; note that the orientation of C_1 is the oppposite of that in the figure in the text). Applying Stokes'theorem, $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds =$



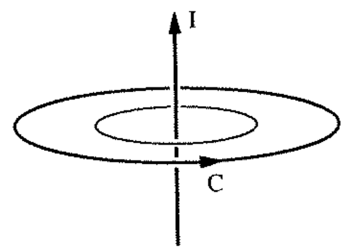
$$\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds + \oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \iint_S \mathbf{n}\cdot\nabla \times \mathbf{F}\ dS = 0,$$

because $\nabla \times \mathbf{F} = 0$ on C and S. Thus $\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds$. Changing the direction of C_1 to conform to the diagram in the text we get

$$\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds.$$

b. Using cylindrical coordinates, $\nabla \times \left[\frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta\right] = \frac{\mu_0 I}{2\pi}\nabla \times \frac{\hat{\mathbf{e}}_\theta}{r} = \frac{\mu_0 I}{2\pi}\hat{\mathbf{e}}_z\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{1}{r}\right) = 0$. Note that this result does not hold at $r = 0$ where $\mathbf{B}$ is undefined.

c. $\oint_{\text{Circle}} \mathbf{B}\cdot\hat{\mathbf{t}}\ ds = \oint_{\text{Circle}} \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{t}}\ ds$. But on the circle $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\theta$ so $\oint_{\text{Circle}} \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{t}}\ ds = \frac{\mu_0 I}{2\pi R}\oint_{\text{Circle}} ds = \frac{\mu_0 I}{2\pi R}(2\pi R) = \mu_0 I$. Now consider any closed curve C enclosing the line of current.



Construct a circular path lying entirely within C (see figure). Since $\nabla \times \mathbf{B} = 0 \quad (r \neq 0)$ the result from (a) applies and gives $\oint_C \mathbf{B}\cdot\hat{\mathbf{t}}\ ds = \oint_{\text{Circle}} \mathbf{B}\cdot\hat{\mathbf{t}}\ ds = \mu_0 I$.

17. a. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \oint_C \frac{\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{t}}}{r}ds$. But $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\theta$ and $r = R$ on C so $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \frac{1}{R}(2\pi R) = 2\pi$. On the other hand, $\nabla \times \frac{\hat{\mathbf{e}}_\theta}{r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{1}{r}\right) = 0$ so that $\iint_S \hat{\mathbf{n}}\cdot\nabla \times \frac{\hat{\mathbf{e}}_\theta}{r}\ dS = 0$. Thus Stokes' theorem fails. The reason is that Stokes' theorem requires $\mathbf{F}$ to be smooth on C and S, but $\mathbf{F} = \frac{\hat{\mathbf{e}}_\theta}{r}$ is not defined at $r = 0$.

b. $\mathbf{F}$ is smooth in D and Stokes' theorem holds. D is not simply connected.

18. Since $\mathcal{E} = -\frac{d\Phi}{dt}$ we have $\oint_C \mathbf{E}\cdot\hat{\mathbf{t}}\ ds = -\frac{d}{dt}\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = -\iint_S \hat{\mathbf{n}}\cdot\frac{\partial \mathbf{B}}{\partial t}\ dS$. Applying Stokes' theorem to the line integral, we find $\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{E}dS = -\iint_S \hat{\mathbf{n}}\cdot\frac{\partial \mathbf{B}}{\partial t}\ dS$. Because this result holds for any capping surface S, this implies that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

19. A simple calculation shows that $\nabla \times \mathbf{F} = 0$ so that the line integral is independent of path. We may therefore replace the complicated path given in the statement of the problem by a simple one. We choose $x = w$, $y = w$, $z = w$ $(0 \le w \le 1)$. Then $\int_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_0^1 (e^{-w} - we^{-w})dw + \int_0^1 (e^{-w} - we^{-w})dw + \int_0^1 (e^{-w} - we^{-w})dw = 3\int_0^1 (e^{-w} - we^{-w})dw = 3/e$.

20. Take the divergence of the fourth equation recalling that $\nabla\cdot\nabla \times \mathbf{F} = 0$. Then $\nabla\cdot\nabla \times \mathbf{B} = 0 = \varepsilon_0\mu_0\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} + \mu_0\nabla\cdot\mathbf{J}$. But using the first equation we get $\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = \frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right)$. Hence $\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right) + \nabla\cdot\mathbf{J} = 0$, or $\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$. This equation asserts that electric charge is conserved.

21. From the second of Maxwell's equations we have $\mathbf{B}\cdot\nabla \times \mathbf{E} = -\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right)$, and from the fourth, $\mathbf{E}\cdot\nabla \times \mathbf{B} = \varepsilon_0\mu_0\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J}\cdot\mathbf{E} = \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{E^2}{2}\right) + \mu_0\mathbf{J}\cdot\mathbf{E}$. Subtracting these two equations we get

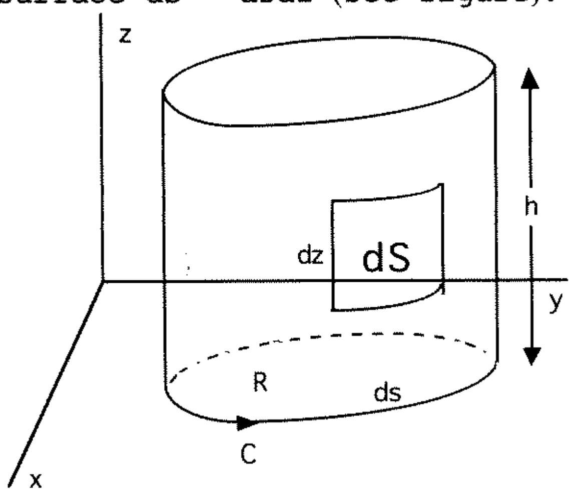
$$\mathbf{E}\cdot(\nabla \times \mathbf{B}) - \mathbf{B}\cdot(\nabla \times \mathbf{E}) = \frac{\partial}{\partial t}\left(\frac{\varepsilon_0\mu_0E^2 + B^2}{2}\right) + \mu_0\mathbf{J}\cdot\mathbf{E}$$

Using the fourth identity on the inside front cover of text we find that the left hand side of this last equation can be written $\nabla\cdot(\mathbf{B} \times \mathbf{E})$. Thus

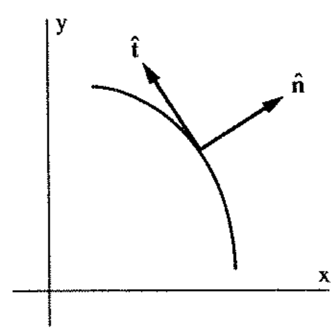
$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0E^2 + B^2/\mu_0}{2}\right) + \nabla\cdot\left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right) = -\mathbf{J}\cdot\mathbf{E}.$$

We interpret $\rho_E = \dfrac{\varepsilon_0E^2 + B^2/\mu_0}{2}$ as the electromagnetic energy density, and $\mathbf{J}_E = \dfrac{\mathbf{E} \times \mathbf{B}}{\mu_0}$ as the electromagnetic energy current density. Thus our last equation reads $\frac{\partial\rho_E}{\partial t} + \nabla\cdot\mathbf{J}_E = -\mathbf{J}\cdot\mathbf{E}$. If the right-hand side of this equation were 0, the equation would assert that electromagnetic energy is conserved. However, the term $\mathbf{J}\cdot\mathbf{E}$ is the rate at which the electric field does work in moving electric charges. Thus the electromagnetic energy is not conserved: it decreases when the field does work on the charges, and it increases when the charges do work on the field.

22\. a. Let CS stand for curved surface and T&B for top and bottom. Then $\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = \iint_{CS} \mathbf{G}\cdot\hat{\mathbf{n}}\ dS + \iint_{T\&B} \mathbf{G}\cdot\hat{\mathbf{n}}\ dS$. However, the integral over the top and bottom is zero because $\hat{\mathbf{n}} = \pm\mathbf{k}$ and $\mathbf{G}\cdot\mathbf{k} = 0$. Now we note that on the curved surface $dS = ds dz$ (see figure). Hence



$\iint_{CS} \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = \iint_{CS} \mathbf{G}\cdot\hat{\mathbf{n}}\ dsdz = h\oint_C \mathbf{G}\cdot\hat{\mathbf{n}}\ ds$, where h is the height of the cylinder. Now $\hat{\mathbf{n}}\cdot\hat{\mathbf{t}} = 0$ (see Figure b). Thus $n_x t_x + n_y t_y = 0$ so that $n_x = -n_y t_y/t_x$. Hence $\hat{\mathbf{n}} = (-n_y t_y/n_x)\mathbf{i} + n_y\mathbf{j} = (-n_y/t_x)(t_y\mathbf{i} - t_x\mathbf{j})$. To make this a unit vector we take $\hat{\mathbf{n}} = t_y\mathbf{i} - t_x\mathbf{j}$. Thus $\mathbf{G}\cdot\hat{\mathbf{n}} = G_x t_y - G_y t_x$, and so $\oint_C \mathbf{G}\cdot\hat{\mathbf{n}} ds = \oint_C (G_x t_y - G_y t_x)\ ds = \oint_C (G_x dy - G_y dx)$.



Thus we have

$$\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = h\oint_C (G_x dy - G_y dx) = \iiint_V \nabla\cdot\mathbf{G}\ dV = \iiint_V \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dxdydz$$

$$= h\iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dxdy.$$

$\therefore \oint_C (G_x dy - G_y dx) = \iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dxdy.$

b. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C F_x dx + F_y dy = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS$. But $\hat{\mathbf{n}} = \mathbf{k}$ so $\oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy.$

c. In the equation derived in (b) put $F_x = -G_y$ and $F_y = G_x$. Then we get $\oint_C G_x dy - G_y dx = \iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dxdy$, which is the equation derived in (a).

23. a. Using the result of Prob III22-(b), $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS$ $= \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy = A$ provided $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1.$

b. $\mathbf{F} = \mathbf{j}x$ or $\mathbf{F} = -\mathbf{i}y$.

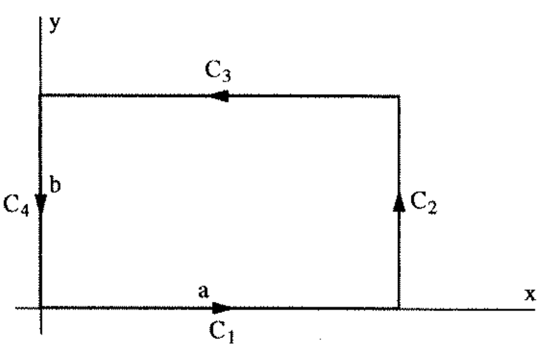
c. We use $\mathbf{F} = \mathbf{j}x$ so that $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C x dy.$

i. $\int_{C_1} x dy = \int_{C_3} x dy = 0$ because $dy = 0$

$\int_{C_2} x dy = a\int_0^b dy = ab$

$\int_{C4} x dy = 0$ because $x = 0$

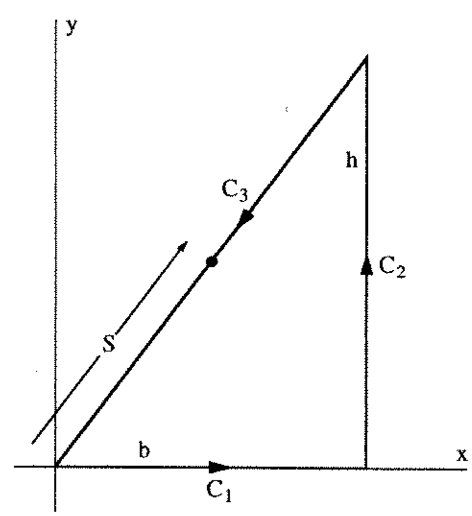
$\therefore A = ab$



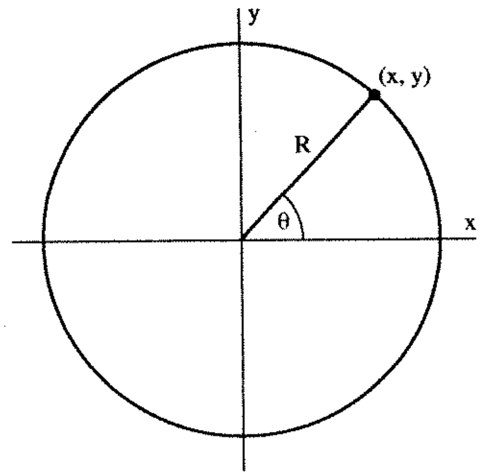
ii. $\int_{C_1} x dy = 0$ because $dy = 0$

$\int_{C_2} x dy = h\int_0^b dy = hb$

On C_3 $x/s = b/\sqrt{h^2 + b^2}$ so $x = bs/\sqrt{h^2 + b^2}$ and $y/s = h/\sqrt{h^2 + b^2}$ so $y = hs/\sqrt{h^2 + b^2}$. Thus $\int_{C_3} x dy = -\frac{hb}{b^2 + h^2}\int_0^{\sqrt{b^2 + h^2}} s ds = -\frac{hb}{2}$. Hence we have $A = hb - \frac{hb}{2} = \frac{hb}{2}.$



iii. $A = \oint_C x\,dy = r^2\int_0^{2\pi}\cos^2\theta\, d\theta = \pi r^2.$



d. $\mathbf{r} \times \hat{\mathbf{t}} = (\mathbf{i}x + \mathbf{j}y) \times (\mathbf{i}dx + \mathbf{j}dy)$ $= \mathbf{k}(xdy - ydx)$. Thus $\oint_C(-ydx + xdy) =$ $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C F_x dx + F_y dy$ when $F_x = -y$ and $F_y = x$. Hence using III22-(b), $\oint_C(-ydx + xdy)$

$$= \iint_R\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]dxdy = 2\iint_R dxdy = 2A.$$

Thus $A = \frac{1}{2}\oint_C(-ydx + xdy) = \left|\frac{1}{2}\oint_C \mathbf{r} \times \hat{\mathbf{t}}\, ds\right|.$

24. a. $(\nabla \times \mathbf{H})_x = \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} =$

$$= -\int_{x_0}^{x}\frac{\partial}{\partial y}G_y(x',y,z)dx' + G_x(x_0,y,z) - \int_{x_0}^{x}\frac{\partial}{\partial z}G_z(x',y,z)dx'$$

$$= G_x(x_0,y,z) - \int_{x_0}^{x}\left[\frac{\partial}{\partial y}G_y(x',y,z) + \frac{\partial}{\partial z}G_z(x',y,z)\right]dx'$$

But $\nabla\cdot\mathbf{G}(x',y,z) = 0 = \dfrac{\partial}{\partial x'}G_x(x',y,z) + \dfrac{\partial}{\partial y}G_y(x',y,z) + \dfrac{\partial}{\partial z}G_z(x',y,z)$.

Hence

$$(\nabla \times \mathbf{H})_x = G_x(x_0,y,z) + \int_{x_0}^{x}\frac{\partial}{\partial x'}G_x(x',y,z)dx'$$

$$= G_x(x_0,y,z) + G_x(x,y,z) - G_x(x_0,y,z) = G_x(x,y,z).$$

We also have $(\nabla \times \mathbf{H})_y = -\dfrac{\partial H_z}{\partial x} = G_y(x,y,z)$ and $(\nabla \times \mathbf{H})_z = \dfrac{\partial H_y}{\partial x} =$ $G_z(x,y,z)$. Thus $\nabla \times \mathbf{H} = \mathbf{i}G_x(x,y,z) + \mathbf{j}G_y(x,y,z) + \mathbf{k}G_z(x,y,z)$.

b. We can add to $\mathbf{H}$ any vector function whose curl is 0. Thus if $\nabla \times \mathbf{K} = 0$, then $\mathbf{G} = \nabla \times (\mathbf{H} + \mathbf{K}) = \nabla \times \mathbf{H}$.

25. a. $\nabla\cdot\mathbf{G} = 0$ as is easily verified. This implies that an $\mathbf{H}$ exists such that $\mathbf{G} = \nabla \times \mathbf{H}$. Using the equations given in Prob III24 we have $H_x = 0$, $H_y = \int_{x_0}^{x} x'dx' = \frac{1}{2}(x^2 - x_0^2)$, and $H_z = -\int_{x_0}^{x} zdx' + \int_{y_0}^{y} y'dy' = -xz + x_0z + \frac{1}{2}(y^2 - y_0)^2$. Thus

$$\mathbf{H} = \frac{1}{2}(x^2 - x_0^2)\mathbf{j} + \left[-xz + x_0z + \frac{1}{2}(y^2 - y_0)^2\right]\mathbf{k}.$$

b. $\nabla\cdot\mathbf{G} = 0$ as is easily verified. Then $H_x = 0$, $H_y = \int_{x_0}^{x} B_0dx' = B_0(x - x_0)$, and $H_z = 0$. Thus $\mathbf{H} = B_0(x - x_0)\mathbf{k}$.

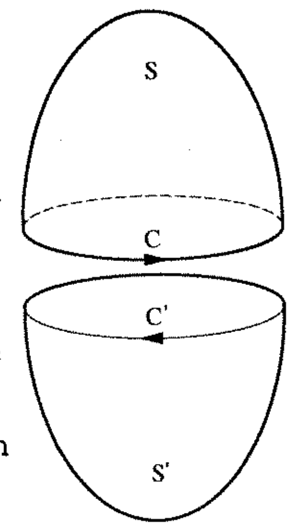
c. Here it can easily be verified that $\nabla\cdot\mathbf{G} \neq 0$, so there is no $\mathbf{H}$ in this case.

d. $\nabla\cdot\mathbf{G} = 0$ as is easily verified. Then $H_x = 0$, $H_y = -\int_{x_0}^{x} zdx' = -z(x - x_0)$, and $H_z = \int_{x_0}^{x} ydx' + 2\int_{y_0}^{y} x_0dy' = y(x - x_0) + 2x_0(y - y_0)$. Hence $\mathbf{H} = -z(x - x_0)\mathbf{j} + (xy + x_0y - 2x_0y_0)\mathbf{k}$.

e. Here it can easily be verified that $\nabla\cdot\mathbf{G} \neq 0$, so there is no $\mathbf{H}$ in this case.

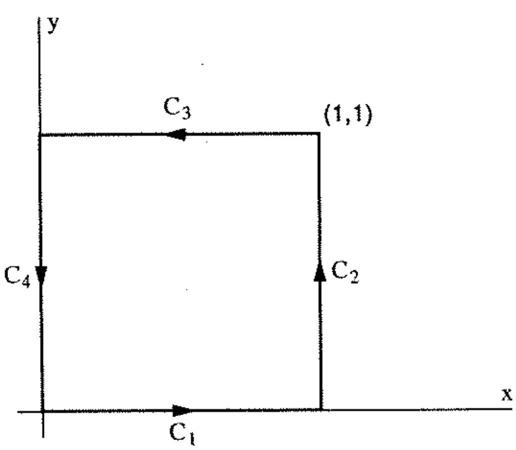
26. Using Stokes' theorem, $\oint_C \mathbf{A}\cdot\hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{A}\, dS = \iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\, dS$.

27. $\oint_C \mathbf{H}\cdot\hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{H}\, dS$ and $\oint_{C'} \mathbf{H}\cdot\hat{\mathbf{t}}\, ds = \iint_{S'} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{H}\, dS$. Adding these, the line integrals cancel because C and C′ are oppositely directed. Hence we get $\iint_{S+S'} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{H}\, dS = 0$. But by the divergence theorem, $\iint_{S+S'} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{H}\, dS = \iiint_V \nabla\cdot(\nabla \times \mathbf{H})\, dV$. Now let $\mathbf{G} = \nabla \times \mathbf{H}$. Then $\iiint_V \nabla\cdot\mathbf{G}\, dV = 0$. But since V is arbitrary this implies $\nabla\cdot\mathbf{G} = 0$.



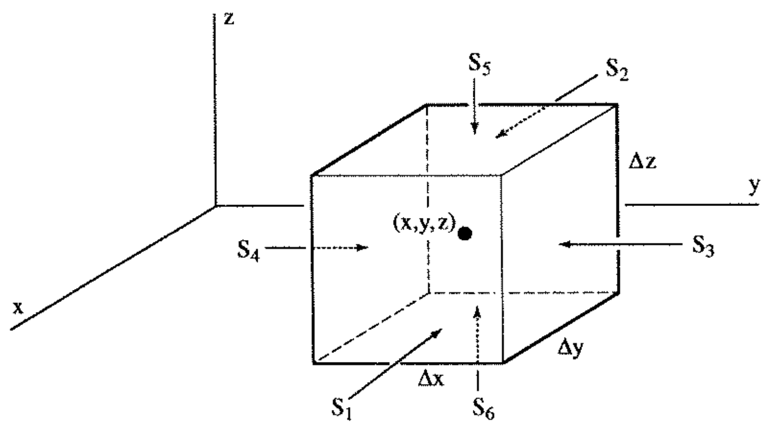
28. a. By Stokes' theorem $\oint_C \mathbf{H}\cdot\hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{H}\, dS = \iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\, dS$. Thus the integral form of the equation is $\oint_C \mathbf{H}\cdot\hat{\mathbf{t}}\, ds = \iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\, dS$.

b. Using Problem III25-(a), $\mathbf{G} = \mathbf{i}y + \mathbf{j}z + \mathbf{k}x$ for which $\nabla\cdot\mathbf{G} = 0$. Then from Problem III24 we find $H_x = 0$, $H_y = \int_{x_0}^{x} x'dx' = \frac{1}{2}(x^2 - x_0^2)$, and $H_z = -\int_{x_0}^{x} zdx' + \int_{y_0}^{y} y'dy' = -z(x - x_0) + \frac{1}{2}(y^2 - y_0^2)$. Let us select the path shown in the diagram and put $x_0 = y_0 = 0$. Note that $y = 0$ and $dy = 0$ on C. Thus $\oint_C \mathbf{H}\cdot\hat{\mathbf{t}}\, ds = -\oint_C zxdz$. Now $\int_{C_1} zxdz = \int_{C_3} zxdz = 0$ because $dz = 0$ on C_1 and C_3. Also $\int_{C_4} zxdz = 0$ because $x = 0$ on C_4. But $\int_{C_2} (-zx)dz = -\int_0^1 zdz = -\frac{1}{2}$. Hence $\oint_C \mathbf{H}\cdot\hat{\mathbf{t}}\, ds = -\frac{1}{2}$. For S take the square in the xz-plane enclosed by C. Now



$\mathbf{G}\cdot\hat{\mathbf{n}} = \mathbf{G}\cdot(-\mathbf{j}) = -z$ and so $\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = -\int_0^1\int_0^1 z\,dx\,dz = -\int_0^1 z\,dz = -\frac{1}{2}$.

29\. a. With the sides labelled as shown in the diagram and using the notation $F_q(S_k)$ to mean the value of F_q ($q = x,y,z$) at the center of the face S_k we have



on S_1 $\hat{\mathbf{n}} = \mathbf{i}$ and $\mathbf{i}\times\mathbf{F} = \mathbf{k}F_y(S_1) - \mathbf{j}F_z(S_1)$

on S_2 $\hat{\mathbf{n}} = -\mathbf{i}$ and $-\mathbf{i}\times\mathbf{F} = -\mathbf{k}F_y(S_2) + \mathbf{j}F_z(S_2)$

on S_3 $\hat{\mathbf{n}} = \mathbf{j}$ and $\mathbf{j}\times\mathbf{F} = -\mathbf{k}F_x(S_3) + \mathbf{i}F_z(S_3)$

on S_4 $\hat{\mathbf{n}} = -\mathbf{j}$ and $-\mathbf{j}\times\mathbf{F} = \mathbf{k}F_x(S_4) - \mathbf{i}F_z(S_4)$

on S_5 $\hat{\mathbf{n}} = \mathbf{k}$ and $\mathbf{k}\times\mathbf{F} = \mathbf{j}F_x(S_5) - \mathbf{i}F_y(S_5)$

on S_6 $\hat{\mathbf{n}} = -\mathbf{k}$ and $-\mathbf{k}\times\mathbf{F} = -\mathbf{j}F_x(S_6) + \mathbf{i}F_y(S_6)$

We must now integrate each of these over the appropriate face of the cuboid and add the results. Examining only those terms which are multiplied by $\mathbf{i}$ we get

$$\iint_{S_3} F_z\ dS - \iint_{S_4} F_z\ dS - \iint_{S_5} F_y\ dS + \iint_{S_6} F_y\ dS$$

$$\cong \left[F_z\left(x,y + \frac{\Delta y}{2},z\right) - F_z\left(x,y - \frac{\Delta y}{2},z\right)\right]\Delta x\Delta y$$

$$- \left[F_y\left(x,y,z + \frac{\Delta z}{2}\right) - F_y\left(x,y,z - \frac{\Delta z}{2}\right)\right]\Delta x\Delta y.$$

Dividing this expression by $\Delta V = \Delta x \Delta y \Delta z$ we get

$$\frac{F_z\left(x, y + \frac{\Delta y}{2}, z\right) - F_z\left(x, y - \frac{\Delta y}{2}, z\right)}{\Delta y} - \frac{F_y\left(x, y, z + \frac{\Delta z}{2}\right) - F_y\left(x, y, z - \frac{\Delta z}{2}\right)}{\Delta z}$$

Taking the limit of this expression as Δy and $\Delta z \to 0$ this becomes $\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = (\nabla \times \mathbf{F})_x$. The other two components of $\nabla \times \mathbf{F}$ can be handled in the same way.

b. Divide the volume V enclosed by S into subvolumes. We then assert that $\iint_S \hat{\mathbf{n}} \times \mathbf{F}\, dS = \sum_l \iint_{S_l} \hat{\mathbf{n}} \times \mathbf{F}\, dS$, where S_l is the surface of subvolume l. To show this consider two adjacent subvolumes; they have a face in common, S_0, and following the argument given in the text we can show that S_0 makes no contribution to the sum given above. Thus only the contributions from the faces which combine to give the original surface S are non-zero. Hence we have

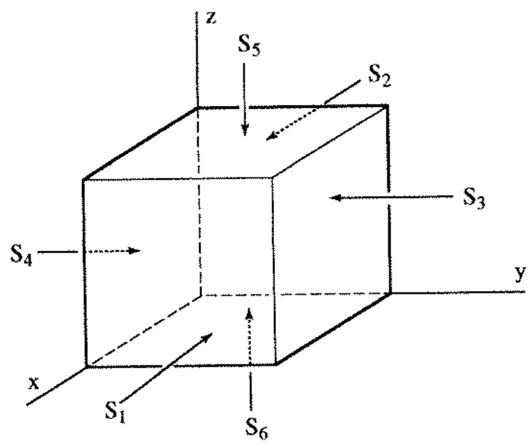
$\iint_S \hat{\mathbf{n}} \times \mathbf{F}\, dS = \sum_l \iint_{S_l} \hat{\mathbf{n}} \times \mathbf{F}\, dS = \sum_l \left[\frac{1}{\Delta V_l} \iint_{S_l} \hat{\mathbf{n}} \times \mathbf{F}\, dS\right] \Delta V_l$. In the limit as

$\Delta V_l \to 0$ the expression in the square brackets in this last equation is $(\nabla \times \mathbf{F})_l$ and so $\sum_l \left[\frac{1}{\Delta V_l} \iint_{S_l} \hat{\mathbf{n}} \times \mathbf{F}\, dS\right] \Delta V_l \to \iiint_V \nabla \times \mathbf{F}\, dV$,

and the result is established.

c. From the divergence theorem we get $\iint_S \hat{\mathbf{n}} \cdot (\mathbf{e} \times \mathbf{F})\, dS = \iiint_V \nabla \cdot (\mathbf{e} \times \mathbf{F})\, dV$. But $\nabla \cdot (\mathbf{e} \times \mathbf{F}) = \mathbf{F} \cdot (\nabla \times \mathbf{e}) - \mathbf{e} \cdot \nabla \times \mathbf{F} = -\mathbf{e} \cdot \nabla \times \mathbf{F}$ since the curl of a constant vector is 0. Using a familiar vector identity we have $\hat{\mathbf{n}} \cdot (\mathbf{e} \times \mathbf{F}) = \mathbf{e} \cdot (\mathbf{F} \times \hat{\mathbf{n}}) = -\mathbf{e} \cdot (\hat{\mathbf{n}} \times \mathbf{F})$. Hence we find $-\mathbf{e} \cdot \iint_S \hat{\mathbf{n}} \times \mathbf{F}\, dS = -\mathbf{e} \cdot \iiint_V \nabla \times \mathbf{F}\, dV$. Because $\mathbf{e}$ is an arbitrary vector this implies that $\iint_S \hat{\mathbf{n}} \times \mathbf{F}\, dS = \iiint_V \nabla \times \mathbf{F}\, dV$.

d. With $\mathbf{F} = \mathbf{i}y - \mathbf{j}z + \mathbf{k}x$ we get $\nabla \times \mathbf{F} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Therefore $\iiint_V \nabla \times \mathbf{F}\, dV = (\mathbf{i} - \mathbf{j} - \mathbf{k})\iiint_V dV = \mathbf{i} - \mathbf{j} - \mathbf{k}$, because the volume of the cube is 1. Next we note that



$$\hat{\mathbf{n}} \times \mathbf{F} = -\mathbf{k}z - \mathbf{j} \text{ on } S_1 \qquad \hat{\mathbf{n}} \times \mathbf{F} = -\mathbf{i}/2 \text{ on } S_4$$

$$= \mathbf{k}z \text{ on } S_2 \qquad = \mathbf{j}/2 + \mathbf{i} \text{ on } S_5$$

$$= -\mathbf{k} + \mathbf{i}x \text{ on } S_3 \qquad = -\mathbf{j}/2 \text{ on } S_6$$

From these we get

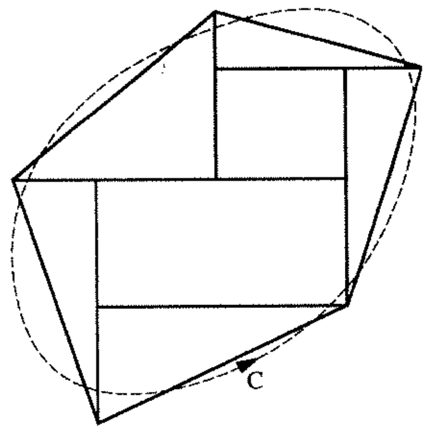
$$\iint_{S_1} \hat{\mathbf{n}} \times \mathbf{F}\, dS = -\mathbf{j} - \frac{\mathbf{k}}{2}\int_0^1\int_0^1 z\,dx\,dz = -\mathbf{j} - \frac{\mathbf{k}}{2} \qquad \iint_{S_4} \hat{\mathbf{n}} \times \mathbf{F}\, dS = -\frac{\mathbf{i}}{2}$$

$$\iint_{S_2} \hat{\mathbf{n}} \times \mathbf{F}\, dS = \frac{\mathbf{k}}{2} \qquad \iint_{S_5} \hat{\mathbf{n}} \times \mathbf{F}\, dS = \mathbf{i} + \frac{\mathbf{j}}{2}$$

$$\iint_{S_3} \hat{\mathbf{n}} \times \mathbf{F}\, dS = -\mathbf{k} + \mathbf{i}\int_0^1\int_0^1 x\,dx\,dz = \frac{\mathbf{i}}{2} - \mathbf{k} \qquad \iint_{S_5} \hat{\mathbf{n}} \times \mathbf{F}\, dS = -\frac{\mathbf{j}}{2}$$

Adding these six results gives $\iint_S \hat{\mathbf{n}} \times \mathbf{F}\, dS = \mathbf{i} - \mathbf{j} - \mathbf{k}$, which agrees with the volume integral computed above.

30. a.



b. $\oint_P \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \sum_l \iint_{C_l} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \sum_l \iint_{S_l} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\ dS$, using Stokes' theorem. If we use the result of (a) and divide the region into rectangles and right triangles, then

$$\sum_l \iint_{S_l} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\ dS = \sum_l \iint_{S_l} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy = \sum_l \iint_{S_l} C(x,y)dxdy$$

$$= \sum_l \iint_{S_l} \left[C(x_0,y_0) + \left(\frac{\partial C}{\partial x}\right)_{x_0,y_0}(x - x_0) + \left(\frac{\partial C}{\partial y}\right)_{x_0,y_0}(y - y_0) + \cdots \right] dxdy$$

$$= C(x_0,y_0)\Delta A + \left(\frac{\partial C}{\partial x}\right)_{x_0,y_0}\left[\sum_l \iint_{S_l}(x - x_0)dxdy\right] +$$

$$\left(\frac{\partial C}{\partial y}\right)_{x_0,y_0}\left[\sum_l \iint_{S_l}(y - y_0)dxdy\right] + \cdots$$

where ΔA is the area enclosed by the approximating polygon.

Now let (x_l,y_l) be an arbitrary point in ΔS_l. Then $\iint_{S_l}(x - x_0)dxdy$ $\cong (x_l - x_0)\Delta S_l$ and $\iint_{S_l}(y - y_0)dxdy \cong (y_l - y_0)\Delta S_l$. Hence

$$\oint_P \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong C(x_0,y_0)\Delta A + \left(\frac{\partial C}{\partial x}\right)_{x_0,y_0}\sum_l (x_l - x_0)\Delta S_l + \left(\frac{\partial C}{\partial y}\right)_{x_0,y_0}\sum_l (y_l - y_0)\Delta S_l$$
$$+ \cdots$$

In the limit as $N \to \infty$ and each $\Delta S_l \to 0$, we have $\Delta A \to \Delta S$, and $\oint_P \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \to \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds$, as well as $\sum_l (x_l - x_0)\Delta S_l \to \iint_R (x - x_0)dxdy$ and $\sum_l (y_l - y_0)\Delta S_l \to \iint_R (y - y_0)dxdy$. Furthermore, $\iint_R xdxdy = \bar{x}\Delta S$ and $\iint_R ydxdy = \bar{y}\Delta S$. Hence

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = C(x_0,y_0)\Delta S + (\bar{x} - x_0)\left(\frac{\partial C}{\partial x}\right)_{x_0,y_0}\Delta S + (\bar{y} - y_0)\left(\frac{\partial C}{\partial y}\right)_{x_0,y_0}\Delta S + \cdots$$

d. From the equation just derived we get

$$\frac{1}{\Delta S}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = C(x_0,y_0) + (\bar{x} - x_0)\left(\frac{\partial C}{\partial x}\right)_{x_0,y_0} + (\bar{y} - y_0)\left(\frac{\partial C}{\partial y}\right)_{x_0,y_0} + \cdots$$

If we now take the limit as $\Delta S \to 0$ about the point (x_0,y_0), then $\bar{x} \to x_0$ and $\bar{y} \to y_0$ so

$$\lim_{\Delta S\to 0} \frac{1}{\Delta S}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \equiv (\nabla \times \mathbf{F})_z = C(x_0,y_0) = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)_{x_0,y_0}$$

CHAPTER IV

1. a. (i). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(xyz) + \mathbf{j}\frac{\partial}{\partial y}(xyz) + \mathbf{k}\frac{\partial}{\partial z}(xyz) = \mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy.$

(ii). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \mathbf{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \mathbf{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2) =$

$= 2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z).$

(iii). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(xy + yz + xz) + \mathbf{j}\frac{\partial}{\partial y}(xy + yz + xz) + \mathbf{k}\frac{\partial}{\partial z}(xy + yz + xz)$

$= \mathbf{i}(y + z) + \mathbf{j}(x + z) + \mathbf{k}(y + x).$

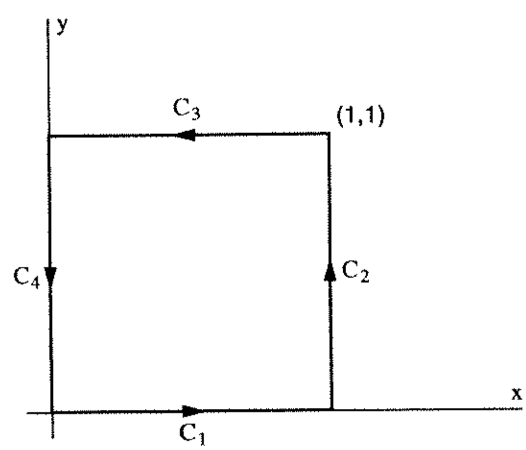
(iv). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(3x^2 - 4z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2 - 4z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2 - 4z^2)$

$= 6\mathbf{i}x - 8\mathbf{k}z.$

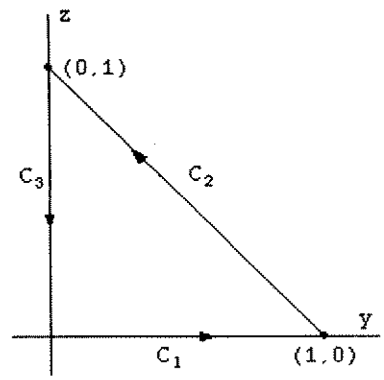
(v). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(e^{-x}\sin y) + \mathbf{j}\frac{\partial}{\partial y}(e^{-x}\sin y) + \mathbf{k}\frac{\partial}{\partial z}(e^{-x}\sin y)$

$= -\mathbf{i}e^{-x}\sin y + \mathbf{j}e^{-x}\cos y$

b. Using $f = x^2 + y^2 + z^2$, we have $\mathbf{F} = 2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$. For the square path shown in the figure, $z = 0$ so $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\oint_C (xdx + ydy)$. On C_1 $y = 0$ so $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_0^1 xdx = 1$. On C_2 $dx = 0$ so $\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_0^1 ydy = 1$. On C_3 $dy = 0$ so $\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_1^0 xdx = -1$. On C_4 $\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_1^0 ydy = -1$. Hence $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 1 + 1 - 1 - 1 = 0.$



For the triangular path shown in the figure $x = 0$ so we have $\int_C \mathbf{F}\cdot\mathbf{t}\,ds = 2\int_C (ydy + zdz)$. On C_1 $z = 0$ so $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_0^1 ydy = 1$. On C_3 $y = 0$ so $\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_1^0 zdz = -1$. For

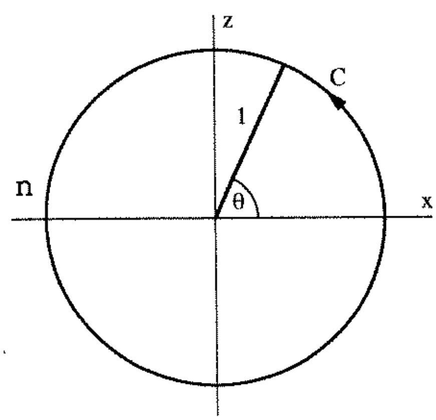


the path C_2 we have $x = 1 - \frac{s}{\sqrt{2}}$ and $y = \frac{s}{\sqrt{2}}$. Thus $\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}ds =$

$$2\int_0^{\sqrt{2}} \left(1 - \frac{s}{\sqrt{2}}\right)\left(-\frac{ds}{\sqrt{2}}\right) + 2\int_0^{\sqrt{2}} \frac{s}{\sqrt{2}}\frac{ds}{\sqrt{2}} = 2\int_0^{\sqrt{2}} \left(\sqrt{2}s - 1\right)ds = 0.$$ Hence

$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 0.$

For the circular path shown in the figure $y = 0$ so $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\oint_C (xdx + zdz)$. Set $x = \cos\theta$ and $z = \sin\theta$. Then



$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\oint_C (xdx + zdz) = 2\int_0^{2\pi} \cos\theta(-\sin\theta)d\theta$$

$$+ 2\int_0^{2\pi} \sin\theta\cos\theta d\theta = 0.$$

2. a. $\nabla(fg) = \mathbf{i}\frac{\partial}{\partial x}(fg) + \mathbf{j}\frac{\partial}{\partial y}(fg) + \mathbf{k}\frac{\partial}{\partial z}(fg)$

$$= \mathbf{i}(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}) + \mathbf{j}(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}) + \mathbf{k}(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z})$$

$$= f(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}) + g(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z})$$

$$= f\nabla g + g\nabla f.$$

b. The x-component of the left hand side of the identity is

$$\frac{\partial}{\partial x}(\mathbf{F}\cdot\mathbf{G}) = F_x\frac{\partial G_x}{\partial x} + G_x\frac{\partial F_x}{\partial x} + F_y\frac{\partial G_y}{\partial x} + G_y\frac{\partial F_y}{\partial x} + F_z\frac{\partial G_z}{\partial x} + G_z\frac{\partial F_z}{\partial x}.$$

The x-component of the right hand side is

$$G_x\frac{\partial F_x}{\partial x} + G_y\frac{\partial F_x}{\partial y} + G_z\frac{\partial F_x}{\partial z} + F_x\frac{\partial G_x}{\partial x} + F_y\frac{\partial G_x}{\partial y} + F_z\frac{\partial G_x}{\partial z} + F_y\left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)$$

$$- F_z\left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right) + G_y\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) - G_z\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)$$

$$= F_x\frac{\partial G_x}{\partial x} + G_x\frac{\partial F_x}{\partial x} + F_y\frac{\partial G_y}{\partial x} + G_y\frac{\partial F_y}{\partial x} + F_z\frac{\partial G_z}{\partial x} + G_z\frac{\partial F_z}{\partial x}.$$

Analogous procedures will establish the equality of the y- and z-components.

c. $$\nabla\cdot(f\mathbf{F}) = \frac{\partial}{\partial x}(fF_x) + \frac{\partial}{\partial y}(fF_y) + \frac{\partial}{\partial z}(fF_z)$$

$$= f\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + F_x\frac{\partial f}{\partial x} + F_y\frac{\partial f}{\partial y} + F_z\frac{\partial f}{\partial z}$$

$$= f\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla f.$$

d. $$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \frac{\partial}{\partial x}(F_yG_z - F_zG_y) + \frac{\partial}{\partial y}(F_zG_x - F_xG_z) + \frac{\partial}{\partial z}(F_xG_y - F_yG_x)$$

$$= G_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + G_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + G_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$+ F_x\left(\frac{\partial G_y}{\partial z} - \frac{\partial G_z}{\partial y}\right) + F_y\left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z}\right)\partial z + F_z\left(\frac{\partial G_x}{\partial y} - \frac{\partial G_y}{\partial x}\right)$$

$$= \mathbf{G}\cdot\nabla\times\mathbf{F} - \mathbf{F}\cdot\nabla\times\mathbf{G}.$$

e. $$[\nabla\times(f\mathbf{F})]_x = \frac{\partial}{\partial y}(fF_z) - \frac{\partial}{\partial z}(fF_y)$$

$$= f\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + F_z\frac{\partial f}{\partial y} - F_y\frac{\partial f}{\partial z}$$

$$= f(\nabla\times\mathbf{F})_x + [(\nabla f)\times\mathbf{F}]_x.$$

The other two components are handled in the same way.

f. $[\nabla \times (\nabla \times \mathbf{F})]_x = \frac{\partial}{\partial y}(F_xG_y - F_yG_x) - \frac{\partial}{\partial z}(F_zG_x - F_xG_z)$

$$= F_x\frac{\partial G_y}{\partial y} + G_y\frac{\partial F_x}{\partial y} - F_y\frac{\partial G_x}{\partial y} - G_x\frac{\partial F_y}{\partial y} - F_z\frac{\partial G_x}{\partial z} - G_x\frac{\partial F_z}{\partial z} + F_x\frac{\partial G_z}{\partial z} + G_z\frac{\partial F_x}{\partial y}$$

$$= (\mathbf{G}\cdot\nabla)F_x - G_x\frac{\partial F_x}{\partial x} - \mathbf{F}\cdot\nabla G_x + F_x\frac{\partial G_x}{\partial x} + F_x\nabla\cdot\mathbf{G} - F_x\frac{\partial G_x}{\partial x} - G_x(\nabla\cdot\mathbf{F}) + G_x\frac{\partial F_x}{\partial x}$$

$$= (\mathbf{G}\cdot\nabla)F_x - \mathbf{F}\cdot\nabla G_x + F_x\nabla\cdot\mathbf{G} - G_x(\nabla\cdot\mathbf{F})$$

$$= [(\mathbf{G}\cdot\nabla)\mathbf{F}]_x - [(\mathbf{F}\cdot\nabla)\mathbf{G}] + [\mathbf{F}(\nabla\cdot\mathbf{G})]_x - [\mathbf{G}(\nabla\cdot\mathbf{F})]_x.$$

The other two components are handled in the same way.

g. $[\nabla \times (\nabla \times \mathbf{F})]_x = \frac{\partial}{\partial y}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - \left(\frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}\right)$$

$$= \frac{\partial}{\partial x}(\nabla\cdot\mathbf{F}) - \nabla^2 F_x = [\nabla\cdot(\nabla\mathbf{F}) - \nabla^2\mathbf{F}]_x$$

The other two components are handled in the same way.

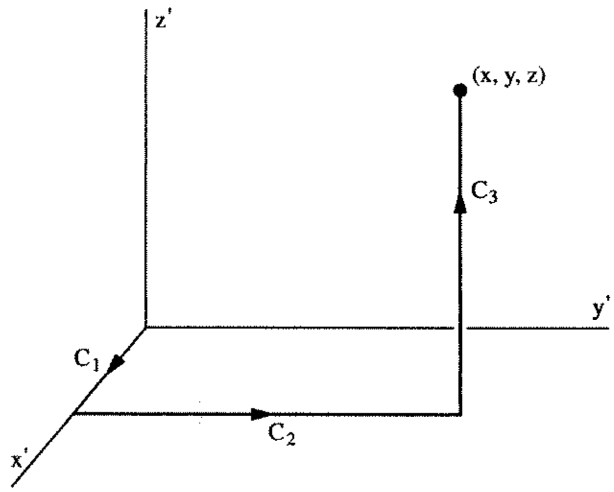
3. $[\nabla \times \nabla f]_x = \frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y} = 0$. The other two components are handled in the same way.

4. a. (i). $\nabla \times \mathbf{F} = -\mathbf{k} \neq 0 \Rightarrow$ not path independent.

(ii). $\nabla \times \mathbf{F} = 0 \Rightarrow \psi(x,y,z) = \int_{x_0,y_0,z_0}^{x,y,z} C\,dz' = C(z - z_0)$ or $Cz + \text{const}$

(iii). $\nabla \times \mathbf{F} = 0 \Rightarrow \psi(x,y,z) = \int_{x_0,y_0,z_0}^{x,y,z} y'z'dx' + x'z'dy' + x'y'dz'$.

Choose the path shown in the figure. On C_1 $y' = z' = 0$, so there is



no contribution to the integral. On C_2 $x' = x$ and $z' = 0$, so again there is no contribution to the integral. On C_3 $x' = x$, $y' = y$ and so the integral is $\int_0^z xydz' = xyz$ to which a constant may be added to give $xyz + \text{const}$.

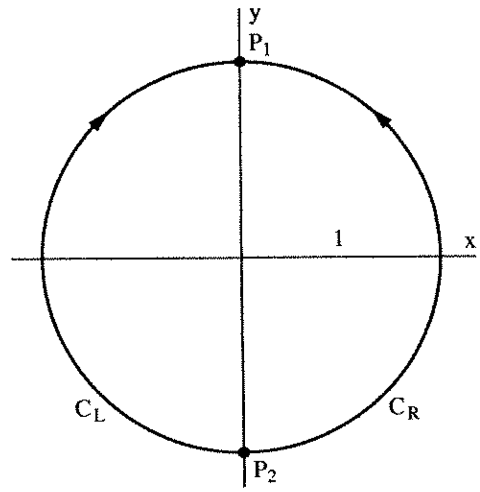
(iv). $\nabla \times \mathbf{F} = 0 \Rightarrow \psi(x,y,z) = \int_{x_0,y_0,z_0}^{x,y,z} x'dx' + y'dy' + z'dz'$. Using the same path as in (iii), $\psi(x,y,z) = \int_0^x x'dx' + \int_0^y y'dy' + \int_0^z z'dz' = \frac{1}{2}(x^2 + y^2 + z^2)$ or $\frac{1}{2}(x^2 + y^2 + z^2) + \text{const}$.

(v). $\nabla \times \mathbf{F} = -\mathbf{k}e^{-z}\cos y \neq 0 \Rightarrow$ not path independent.

b. (i). $\psi = \int \frac{xdx + ydy}{r^2} = \int \frac{rdr}{r^2} = \int \frac{dr}{r} = \ln r + \text{const}$ provided $r \neq 0$.

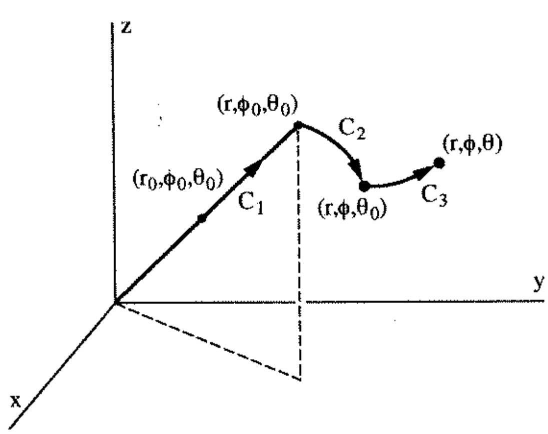
(ii). $\psi = \int \frac{xdx + ydy + zdz}{r^{1/2}} = \int \frac{rdr}{r^{1/2}} = \int r^{1/2}dr = \frac{2}{3}r^{3/2} + \text{const}$

5. $F(r,\theta,z) = \frac{\hat{\mathbf{e}}_\theta}{r}$. Hence on C_R, $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\theta$ and $\int_{C_R} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_R} \frac{\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_\theta}{r} ds = \int_{C_R} ds = \pi$, since $r = 1$ on C_R. On C_L $\hat{\mathbf{t}} = -\hat{\mathbf{e}}_r$ and $\int_{C_L} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_L} \frac{\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_\theta}{r} ds = -\int_{C_L} ds = -\pi$.



If we try to define $\psi = \int_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds$ we see that the value assigned to ψ at a point depends on the path used to reach that point; thus ψ is multiple valued and therefore not a function.

6. a. $\int_C \mathbf{E}\cdot\hat{\mathbf{t}}ds = \frac{p}{4\pi\varepsilon_0}\int_C\left(\frac{2\cos\phi}{r^3}dr + \frac{\sin\phi}{r^3}rd\phi\right)$. On C_1 this integral becomes $\frac{p}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{\cos\phi_0}{r^3}dr = \frac{p}{4\pi\varepsilon_0}\cos\phi_0\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)$. On C_2 we get



$\frac{p}{4\pi\varepsilon_0}\frac{1}{r^2}\int_{\phi_0}^{\phi}\sin\phi d\phi = \frac{p}{4\pi\varepsilon_0}\frac{1}{r^2}(\cos\phi_0 - \cos\phi)$. On C_3 we get no contribution because $\mathbf{E}$ has no θ component. Adding our results we find

$$\int_{C_1+C_2+C_3}\mathbf{E}\cdot\hat{\mathbf{t}}ds = \frac{p}{4\pi\varepsilon_0}\left(\frac{\cos\phi_0}{r_0^2} - \frac{\cos\phi}{r^2}\right).$$

By definition Φ is the negative of this. Thus, dropping the additive constant, $\Phi(r,\phi,\theta) = \frac{p}{4\pi\varepsilon_0}\frac{\cos\phi}{r^2}$.

b. Over a sphere centered at the origin, $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$ so $\mathbf{E}\cdot\hat{\mathbf{n}} = \frac{p}{2\pi\varepsilon_0}\frac{\cos\phi}{r^3}$ and we get $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = \frac{p}{2\pi\varepsilon_0 R^3}\int_0^{\pi}\int_0^{2\pi}\cos\phi d\phi d\theta = \frac{p}{\varepsilon_0 R^3}\int_0^{\pi}\cos\phi d\phi = 0$.

c. Gauss' law tells us that $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = q/\varepsilon_0$, where S is any closed surface, and q is the charge enclosed by S. We have shown that the flux of $\mathbf{E}$ is zero over *any* sphere centered at the origin. Thus the net charge contained in any such sphere is 0. Hence the flux through any closed surface not passing though the origin is 0.

7. The divergence theorem says $\begin{Bmatrix}\text{surface}\\ \text{integral}\end{Bmatrix} = \begin{Bmatrix}\text{volume}\\ \text{integral}\end{Bmatrix}$ and Stokes' theorem says $\begin{Bmatrix}\text{line}\\ \text{integral}\end{Bmatrix} = \begin{Bmatrix}\text{surface}\\ \text{integral}\end{Bmatrix}$. In this problem we have attempted to eliminate the surface integral from these two equations to get $\begin{Bmatrix}\text{line}\\ \text{integral}\end{Bmatrix} = \begin{Bmatrix}\text{volume}\\ \text{integral}\end{Bmatrix}$. This is not valid because the surface integral in the divergence theorem is over a *closed* surface, while the surface integal in Stokes' theorem is over a capping surface which is *open*. The surface integral in the divergence theorem can never be equal to the surface integral in Stokes' theorem.

8. $\mathbf{J} = \rho\mathbf{v} = -k\nabla\rho$. Thus $\mathbf{v} = -\frac{k}{\rho}\nabla\rho$. Using the identity given in Prob IV2-(e) we find $\nabla \times \mathbf{v} = -k\nabla \times \left(\frac{\nabla\rho}{\rho}\right) = -\frac{k}{\rho}\nabla \times \nabla\rho - k\left(\nabla\frac{1}{\rho}\right) \times \nabla\rho$ $= - k\left(\nabla\frac{1}{\rho}\right) \times \nabla\rho$ because $\nabla \times \nabla\rho = 0$ (see Prob IV-3). But $\nabla\frac{1}{\rho} = -\frac{1}{\rho^2}\nabla\rho$. Thus $\nabla \times \mathbf{v} = \frac{k}{\rho^2}\nabla\rho \times \nabla\rho = 0$, because the cross product of a vector function with itself is 0.

9. a. Because the diffusing matter is conserved, the continuity equation is valid: $\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$. But $\mathbf{J} = -k\nabla\rho$. Combining these two equations yields $\frac{\partial\rho}{\partial t} + \nabla\cdot(-k\nabla\rho) = 0$ or $\frac{\partial\rho}{\partial t} = k\nabla^2\rho$.

b. The rate of change of the number of bacteria in any volume V is equal to the rate at which they flow through the surface S of V plus the rate at which they reproduce. Hence

$$\frac{d}{dt}\iiint_V \rho dV = -\iint_S \mathbf{J}\cdot\hat{\mathbf{n}}\ dS + \lambda\iiint_V \rho dV.$$

Applying the divergence theorem to the first integral on the right hand side of this equation we get

$$\iiint_V \frac{\partial\rho}{\partial t} dV = -\iiint_V \nabla\cdot\mathbf{J}\ dV + \lambda\iiint_V \rho dV$$

or, because the volume V is arbitrary, $\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J} + \lambda\rho$. Using Fick's law we can rewrite this $\frac{\partial\rho}{\partial t} = = k\nabla^2\rho + \lambda\rho$.

10. a. Since the fluid is incompressible, $\frac{\partial\rho}{\partial t} = 0$. Using this in the continuity equation gives $\nabla\cdot\mathbf{J} = 0$. But since $\mathbf{J} = \rho\mathbf{v}$, this means that $\nabla\cdot(\rho\mathbf{v}) = 0$. Making use of the fact that ρ is constant,

this last equation reads $\rho\nabla\cdot\mathbf{v} = 0$. However, $\rho \neq 0$, so we have $\nabla\cdot\mathbf{v} = 0$.

b. Since $\nabla \times \mathbf{v} = 0$ there is a scalar function, ϕ, such that $\mathbf{v} = \nabla\phi$. But $\nabla\cdot\mathbf{v} = 0 = \nabla\cdot\nabla\phi$, or $\nabla^2\phi = 0$.

11. $Q = c\iiint_V T\rho dV$, so $\dfrac{dQ}{dt} = c\iiint_V \rho\dfrac{\partial T}{\partial t}dV = k\iint_S \hat{\mathbf{n}}\cdot\nabla T\ dS = k\iiint_V \nabla^2 T\ dV$,

where we have applied the divergence theorem. Since V is arbitrary, this implies that $c\rho\dfrac{\partial T}{\partial \rho} = k\nabla^2 T$ or $\nabla^2 T = \dfrac{c\rho}{k}\dfrac{\partial T}{\partial t} = \alpha\dfrac{\partial T}{\partial t}$.

12. a. Schrödinger equation and its complex conjugate are

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \quad \text{and} \quad -i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*$$

where $\hbar$ is Planck's constant divided by 2π. Multiplying the first of these equations by ψ^* and the second by ψ and then subtracting we get

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)$$

or

$$\frac{\partial}{\partial t}\psi^*\psi + \nabla\cdot\left[\frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right] = 0.$$

Let $\rho = \psi^*\psi$ and $\mathbf{J} = \dfrac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$. Then we get $\dfrac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$.

b. ρ is interpreted as the probability density and $\mathbf{J}$ as the probability current density. The continuity equation derived in (a) then implies that probability is conserved.

13. a. We know that $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$. Hence $\rho = \varepsilon_0\nabla\cdot\mathbf{E} = 3\varepsilon_0 g$, using the given field.

b. Because $\mathbf{E} = -\nabla\Phi$ we have $\dfrac{\partial\Phi}{\partial x} = -gx$, $\dfrac{\partial\Phi}{\partial y} = -gy$, $\dfrac{\partial\Phi}{\partial z} = -gz$

whence $\Phi = -\frac{1}{2}(x^2 + y^2 + z^2) + \text{const}$. Alternatively, we can integrate over an arbitrary path. Using the one shown in the

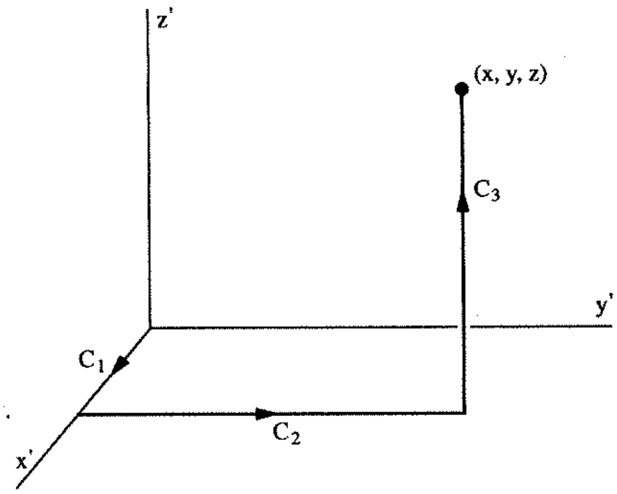


figure we have $\Phi = -\int_C \mathbf{E}\cdot\hat{\mathbf{t}}\, ds = -g\int_C x'dx' + y'dy' + z'dz' =$ $-\frac{1}{2}(x^2 + y^2 + z^2)$ (to which an arbitrary constant may be added).

c. $\nabla^2\Phi = -\frac{g}{2}(2 + 2 + 2) = -3g = -\frac{\rho}{\varepsilon_0}$.

14. a. In the divergence theorem, $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \iiint_V \nabla\cdot\mathbf{F}\, dV$, put $\mathbf{F} = u\nabla v$. Then we get $\iint_S \hat{\mathbf{n}}\cdot(u\nabla v)\, dS = \iiint_V \nabla\cdot(u\nabla v)\, dV$. But $\nabla\cdot(u\nabla v) = u\nabla^2 v + (\nabla u)\cdot(\nabla v)$, and so we have $\iint_S \hat{\mathbf{n}}\cdot(u\nabla v)\, dS = \iiint_V [u\nabla^2 v + (\nabla u)\cdot(\nabla v)]\, dV$.

b. If $\nabla^2 u = 0$, the the expression found in (a) becomes

$$\iint_S \hat{\mathbf{n}}\cdot(u\nabla v)\, dS = \iiint_V (\nabla u)\cdot(\nabla v)\, dV.$$

If we now set v = u in this expresion we get

$$\iint_S \hat{\mathbf{n}}\cdot(u\nabla u)\, dS = \iiint_V |\nabla u|^2\, dV.$$

c. In the expression found in (a) interchange u and v to get

$$\iint_S \hat{\mathbf{n}}\cdot(v\nabla u)\ dS = \iiint_V [v\nabla^2 u + (\nabla v)\cdot(\nabla u)]\ dV.$$

If we now subtract this from the result found in (a) we find

$$\iint_S \hat{\mathbf{n}}\cdot(u\nabla v - v\nabla u)\ dS = \iiint_V (u\nabla^2 v - v\nabla^2 u)\ dV.$$

15. Maxwell's equations in the absence of charges and currents read

$$\nabla\cdot\mathbf{E} = 0 \quad (1) \qquad \nabla\cdot\mathbf{B} = 0 \quad (3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2) \qquad \nabla \times \mathbf{B} = \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

Taking the time derivative of (4) we get $\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \varepsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2}$. But $\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\nabla \times \mathbf{E})$, where we have used (2). Now from the identity in Prob IV2-(g) we get $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$, using (1). Hence $\nabla^2\mathbf{E} = \varepsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$ where $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = \frac{1}{\sqrt{8.854 \times 10^{-12} \times 1.257 \times 10^{-6}}} = 2.997 \times 10^8$ m/sec. which we recognize as the velocity of light. The wave equation for $\mathbf{B}$ can be derived by an analogous procedure.

16. a. We add V_0 to the potential to get

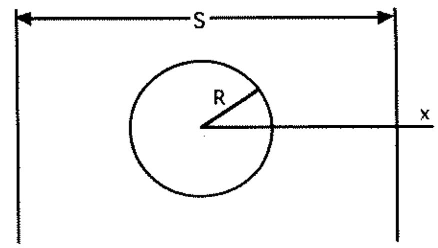
$$\Phi(r,\theta) = V_0 - E_0 r\left(1 - \frac{R^2}{r^2}\right)\cos\theta.$$

Since this expression satisfies Laplace's equation and the boundary conditions, by uniqueness it must be *the* solution. Note that because $\mathbf{E} = -\nabla\Phi$, the added constant does not alter the field.

b. Surround the cylinder by another of radius $a > R$ concentric with the first. The charge enclosed by a segment of length L of the cylinder is, by Gauss' law,

$$\varepsilon_0\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\ dS = \varepsilon_0 E_0 a\iint_S\left[1 + \left(\frac{R}{r}\right)^2\right]\cos\theta\, d\theta\, dz = \varepsilon_0 E_0 a L\left[1 + \left(\frac{R}{r}\right)^2\right]\int_0^{2\pi}\cos\theta\ d\theta = 0.$$

17. a. The potential function $\Phi(r,\theta,\phi)$ satisfies Laplace's equation and the boundary conditons $\Phi(R,\theta,\phi) = 0$ and $\Phi(r,\theta,\phi) \to -E_0x = -E_0 r\sin\phi\cos\theta$ as $r \to \infty$. We try a solution of the form $\Phi(r,\theta,\phi) = f(r)\sin\phi\cos\theta$. Substituting this into



LaPlace's equation (in spherical coordinates) yields $\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 2f = 0$. Trying $f(r) = r^\lambda$ yields $\lambda^2 + \lambda - 2 = 0$ which has roots 1, and -2. Hence we have $f(r) = Ar + \frac{B}{r^2}$ where A and B are arbitrary constants.and our solution is

$$\Phi(r,\theta,\phi) = \left(Ar + \frac{B}{r^2}\right)\sin\phi\cos\theta.$$

As r gets large, the solution approaches $Ar\sin\phi\cos\theta$. so $A = -E_0$. The condition at $r = R$ will be satisfied if we take

$$\Phi(R,\theta,\phi) = \left(-E_0R + \frac{B}{R^2}\right)\sin\phi\cos\theta = 0.$$

Hence we must have $B = E_0R^3$. Our solution is thus

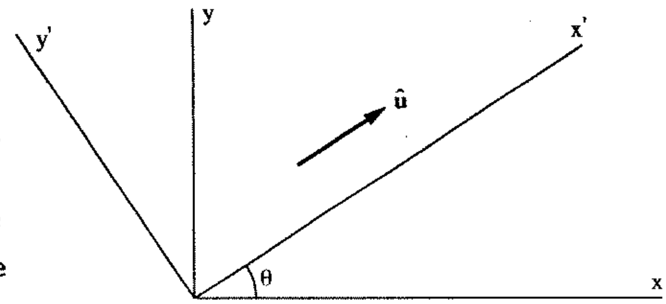
$$\Phi(r,\theta,\phi) = -E_0 r\left(1 - \frac{R^3}{r^3}\right)\sin\phi\cos\theta.$$

b. Place a sphere S_0 of radius $a > R$ around, and concentric with, the given sphere. Then the charge on the given sphere is proportional to $\iint_{S_0} \mathbf{E}\cdot\hat{\mathbf{n}}\, dS$, by Gauss' law. But $\mathbf{n} = \hat{\mathbf{e}}_r$ so $\mathbf{E}\cdot\hat{\mathbf{n}} = E_r = -\frac{\partial\Phi}{\partial r} = E_0\left(1 + \frac{2R^3}{a}\right)\sin\phi\cos\theta$. Thus the charge is proportional to

$$E_0\left(1 + \frac{2R^3}{a}\right)a^2\int_0^{\pi}\int_0^{2\pi} \sin^2\phi\, \cos\theta\, d\phi d\theta = 0.$$

c. Add V_0 to the potential found in (a).

18. We have $\frac{df}{ds} = \hat{\mathbf{u}}\cdot\nabla f = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ and $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$ (see figure). Using the chain rule we obtain



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial f}{\partial x'}\cos\theta - \frac{\partial f}{\partial y'}\sin\theta$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y} = \frac{\partial f}{\partial x'}\sin\theta + \frac{\partial f}{\partial y'}\cos\theta$$

$$\therefore \quad \frac{df}{ds} = \cos\theta\left(\frac{\partial f}{\partial x'}\cos\theta - \frac{\partial f}{\partial y'}\sin\theta\right) + \sin\theta\left(\frac{\partial f}{\partial x'}\sin\theta + \frac{\partial f}{\partial y'}\cos\theta\right) = \frac{\partial f}{\partial x'}$$

19. a. The rate of change of $z = f(x,y)$ in the direction $\hat{\mathbf{u}} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ is $\frac{df}{ds} = \hat{\mathbf{u}}\cdot\nabla f$. But $\frac{\partial f}{\partial x} = -\frac{x}{z}$ and $\frac{\partial f}{\partial y} = -\frac{y}{z}$. Hence

$$\frac{df}{ds} = \frac{-x\cos\theta - y\sin\theta}{\sqrt{r^2 - a^2 - b^2}} = -\frac{a\cos\theta + b\sin\theta}{c} \quad \text{at the point } (a,b,c).$$

To make this 0 we must have $a\cos\theta + b\sin\theta = 0$, or $\cos\theta = -\frac{b}{a}\sin\theta$.

Therefore $\hat{\mathbf{u}} = \mathbf{i}\left(-\frac{b}{a}\right)\sin\theta + \mathbf{j}\sin\theta = \frac{\sin\theta}{a}(-\mathbf{i}b + \mathbf{j}a)$. To make this a unit vector we write $\hat{\mathbf{u}} = \frac{\mathbf{i}b - \mathbf{j}a}{\sqrt{a^2 + b^2}}$.

b. The rate of increase is greatest in the direction of the gradient. But $\nabla f = -\frac{x\mathbf{i}}{z} - \frac{y\mathbf{j}}{z} = -\frac{a\mathbf{i} + b\mathbf{j}}{c}$ at the point (a,b,c).

c. The greatest rate of decrease is in the direction opposite to the gradient, that is, $\frac{a\mathbf{i} + b\mathbf{j}}{c}$.

20. Let $F(x,y,z) = z - f(x,y)$. Then $z = f(x,y)$ and $F(x,y,z) = 0$

describe the same surface. Now $\nabla F = -\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}$, and $\frac{\nabla F}{|\nabla F|}$ is clearly the expression for $\hat{\mathbf{n}}$ given in the text.

21. $\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$ and $\nabla f = \hat{\mathbf{e}}_r\frac{\partial f}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\mathbf{e}}_z\frac{\partial f}{\partial z}$.

Let $F = \nabla f$. Then

$$\nabla\cdot(\nabla f) = \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

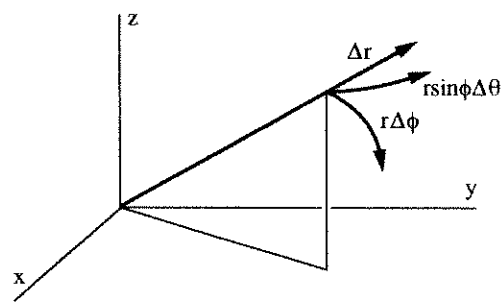
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

22. $\Delta\mathbf{s} = \hat{\mathbf{e}}_r\Delta r + \hat{\mathbf{e}}_\phi r\Delta\phi + \hat{\mathbf{e}}_\theta r\sin\phi\Delta\theta$ and

$$\Delta\psi = \frac{\partial\psi}{\partial r}\Delta r + \frac{\partial\psi}{\partial\phi}\Delta\phi + \frac{\partial\psi}{\partial\theta}\Delta\theta$$

$$= \frac{\partial\psi}{\partial r}\Delta r + \frac{1}{r}\frac{\partial\psi}{\partial\phi}r\Delta\phi + \frac{1}{r\sin\phi}\frac{\partial\psi}{\partial\theta}r\sin\phi\Delta\theta.$$

Hence $\nabla\psi = \hat{\mathbf{e}}_r\frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{e}}_\theta\frac{1}{r\sin\phi}\frac{\partial\psi}{\partial\theta}$.



23. $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\ F_\phi) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}$ and $\nabla\psi = \hat{\mathbf{e}}_r\frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{e}}_\theta\frac{1}{r\sin\phi}\frac{\partial\psi}{\partial\theta}$. Let $\mathbf{F} = \nabla\psi$. Then $\nabla\cdot(\nabla\psi) = \nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\ \frac{1}{r}\frac{\partial\psi}{\partial\phi}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\phi}\frac{\partial\psi}{\partial\theta}\right)$. Hence

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\ \frac{\partial\psi}{\partial\phi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}.$$

24. a. $\nabla^2 u = 0$ and $\nabla^2 v = 0$. Hence if $w = u - v$ then $\nabla^2 w = \nabla^2 u - \nabla^2 v = 0$.

b. On S_i, $u = \Phi_i$ and $v = \Phi_i$ so $w = \Phi_i - \Phi_i = 0$ on each S_i.

c. $\iint_S \hat{\mathbf{n}}\cdot(w\nabla w)\ dS = \iiint_V \nabla\cdot(w\nabla w)\ dV$. But $\nabla\cdot(w\nabla w) = \nabla w\cdot\nabla w + w\nabla^2 w = \nabla w\cdot\nabla w = |\nabla w|^2$ because $\nabla^2 w = 0$, as shown in (a). Therefore we have $\iint_S \hat{\mathbf{n}}\cdot(w\nabla w)\ dS = \iiint_V |\nabla w|^2\ dV$. But $\iint_S \hat{\mathbf{n}}\cdot(w\nabla w)\ dS = \sum_i \iint_{S_i} \hat{\mathbf{n}}\cdot(w\nabla w)\ dS = 0$

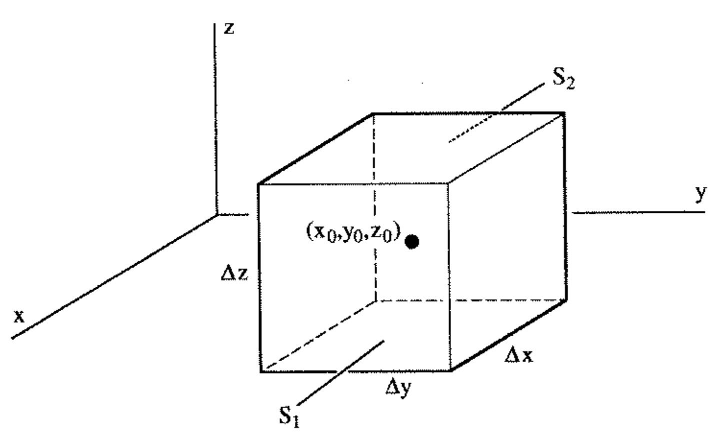
because $w = 0$ on each S_i. Thus $\iiint_V |\nabla w|^2\ dV = 0$.

d. From (c) we have $\iiint_V |\nabla w|^2\ dV = 0$. But $|\nabla w|^2$ is never negative. Hence the only way this integral can be 0 is by having $\nabla w = 0$. But this implies that w is a constant.

e. Since w is a constant, and is 0 on each S_i we must have $w = 0$ everywhere in V and on S. But this means $u = v$. Hence the solution is unique.

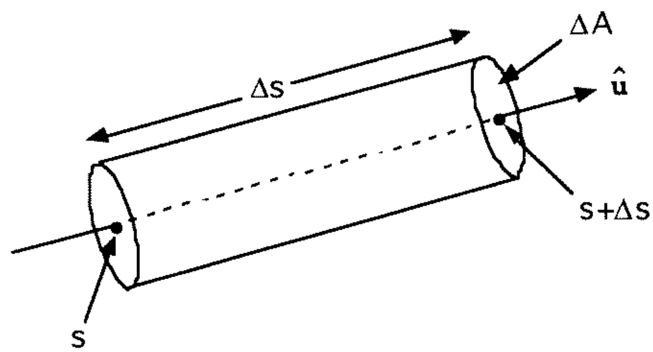
f. Note that $\iiint_V |\nabla w|^2\ dV = = \iint_S w\hat{\mathbf{n}}\cdot\nabla w dS$. We see that this is 0 if, instead of specifying Φ on all surfaces, we specify $\hat{\mathbf{n}}\cdot\nabla\Phi$ instead. Then $\hat{\mathbf{n}}\cdot\nabla w = 0$ on all surfaces so $\nabla w = 0$ as before. Hence we can say that w is a constant, but we cannot claim it is 0 as in (e). Our conclusion in this case is that two solutions, u and v, can differ at most by a constant.

25. a. We have $\iint_{S_1} \hat{\mathbf{n}} f\ dS \cong \mathbf{i} f\left(x + \frac{\Delta x}{2}, y, z\right)\Delta y \Delta z$ and $\iint_{S_2} \hat{\mathbf{n}} f\ dS \cong -\mathbf{i} f\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y \Delta z$. Adding these two results and dividing by ΔV



$= \Delta x \Delta y \Delta z$ we get $\mathbf{i}\dfrac{f\left(x + \dfrac{\Delta x}{2}, y, z\right) - f\left(x - \dfrac{\Delta x}{2}, y, z\right)}{\Delta x} \rightarrow \mathbf{i}\dfrac{\partial f}{\partial x}$ as $\Delta x \rightarrow 0$. The other two terms of the gradient come from the other two pairs of faces, and can be derived in an analogous way

b. $\hat{\mathbf{u}}\cdot\nabla f = \lim_{\Delta V\to 0}\dfrac{1}{\Delta V}\iint_S \hat{\mathbf{u}}\cdot\hat{\mathbf{n}} f dS$. But $\hat{\mathbf{u}}\cdot\mathbf{n} = 0$ on the curved surface



of the cylinder. On the ends of the cylinder we have $\iint_{\text{Ends}} \hat{\mathbf{u}}\cdot\hat{\mathbf{n}} f\ dS \cong [f(s_0 + \Delta s) - f(s_0)]\Delta A$. Dividing this by $\Delta v = \Delta A \Delta s$, we get $\dfrac{1}{\Delta V}\iint_S \hat{\mathbf{u}}\cdot\hat{\mathbf{n}} f\ dS \cong \dfrac{f(s_0 + \Delta s) - f(s_0)}{\Delta s} \rightarrow \dfrac{df}{ds}$ as $\Delta s \rightarrow 0$. Thus $\hat{\mathbf{u}}\cdot\nabla f = \dfrac{df}{ds}$, which is the rate of change of f in the direction specified by $\hat{\mathbf{u}}$.

c. Divide the region into subvolumes. Then

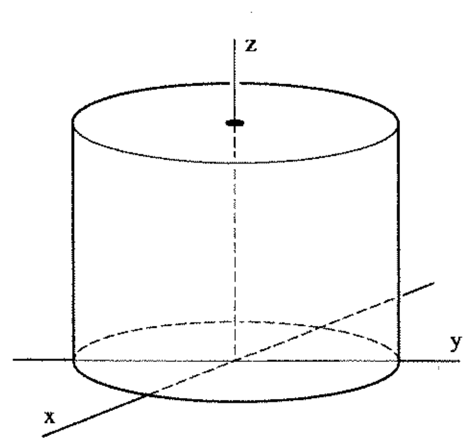
$$\iint_S \hat{\mathbf{n}} f\ dS = \sum_l \iint_{S_l} \hat{\mathbf{n}} f\ dS$$

where S_l is the surface of the lth subvolume. This is justified by the same reasoning used in the text in the derivation of the divergence thorem, namely, that subvolume faces common to two adjacent subvolumes make no net contribution to the sum. Thus

$$\iint_S \hat{\mathbf{n}} f\ dS = \sum_l \left[\frac{1}{\Delta V_l}\iint_{S_l} \hat{\mathbf{n}} f\ dS\right]\Delta V_l \to \iiint_V \nabla f\ dV \quad \text{as } \Delta V \to 0.$$

d. In the divergence theorem, $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iiint_V \nabla\cdot\mathbf{F}\ dV$, let $\mathbf{F} = \mathbf{c}f$ where $\mathbf{c}$ is an arbitrary constant vector. Then $\iint_S \mathbf{c}\cdot\hat{\mathbf{n}} f\ dS = \iiint_V \nabla\cdot(\mathbf{c}f)\ dV$, or $\mathbf{c}\cdot\iint_S \hat{\mathbf{n}} f\ dS = \mathbf{c}\cdot\iiint_V \nabla f\ dV$. Because $\mathbf{c}$ is arbitrary this implies that $\iint_S \hat{\mathbf{n}} f\ dS = \iiint_V \nabla f\ dV$.

e. On the base of the cylinder $\hat{\mathbf{n}} = -\mathbf{k}$ so $\iint_{Base} f\hat{\mathbf{n}}\ dS = -\mathbf{k}\iint_{Base}(x^2+y^2+z^2)dS = -\mathbf{k}\iint_{Base}(x^2+y^2)\ dS = -\mathbf{k}\int_0^{2\pi}\int_0^1 r^2\cdot r dr d\theta = -\frac{\pi}{2}\mathbf{k}$. On the top of the cylinder, $\hat{\mathbf{n}} = \mathbf{k}$ and we have



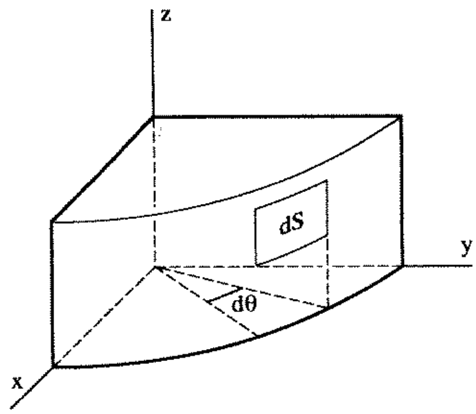
$$\iint_{Top} f\hat{\mathbf{n}}\ dS = \mathbf{k}\iint_{Top}(x^2 + y^2 + z^2)dxdy = \mathbf{k}\iint_{Top}(x^2 + y^2 + 1)dxdy = \frac{3\pi}{2}\mathbf{k}.$$

On the curved surface of the cylinder $\hat{\mathbf{n}} = \mathbf{i}x + \mathbf{j}y$ so we have

$$\iint_{CS} f\cdot\mathbf{n}\ dS = \iint_{CS}(x^2 + y^2 + z^2)(\mathbf{i}x + \mathbf{j}y)\ dS = \iint_{CS}(z^2 + 1)(\mathbf{i}x + \mathbf{j}y)\ dS,$$

where CS stands for curved surface. But $dS = r d\theta dz = d\theta dz$ because $r = 1$ on the curved surface. Thus the integral becomes

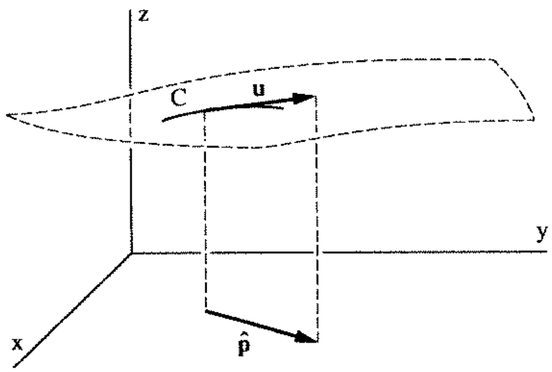
$\int_0^1\int_0^{2\pi}(z^2 + 1)(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)\ d\theta\ dz = 0$ because both the sine and the

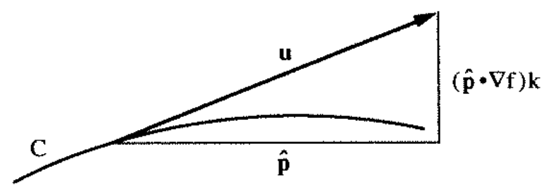


cosine integrate to 0. Adding together all the contributions we see that $\iint_{CS} f\hat{\mathbf{n}}\, dS = -\frac{\pi}{2}\mathbf{k} + \frac{3\pi}{2}\mathbf{k} + 0 = \pi\mathbf{k}$. Next we have $\nabla f = 2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$ and so $\iiint_V \nabla f dV = 2\int_0^1 \int_0^{2\pi} \int_0^1 (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) r dr d\theta dz$. But $\int_0^{2\pi} x d\theta = \int_0^{2\pi} \cos\theta d\theta = 0$ and $\int_0^{2\pi} y d\theta = \int_0^{2\pi} \sin\theta d\theta = 0$. Hence $\iiint_V \nabla f dV = 2\mathbf{k}\int_0^1 \int_0^{2\pi} \int_0^1 z r dr d\theta dz = \pi\mathbf{k}$ in agreement with the surface integral computed above.

26. a. We know that $\hat{\mathbf{p}}\cdot\nabla f$ is the rate of change of f in the direction of $\hat{\mathbf{p}}$. We see from the diagrams then that $\mathbf{u} = \hat{\mathbf{p}} + (\hat{\mathbf{p}}\cdot\nabla f)\mathbf{k}$. In the same way we have $\mathbf{v} = \hat{\mathbf{q}} + (\hat{\mathbf{q}}\cdot\nabla f)\mathbf{k}$. Hence

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \hat{\mathbf{p}} \times \hat{\mathbf{q}} + (\hat{\mathbf{p}} \times \mathbf{k})(\hat{\mathbf{q}}\cdot\nabla f) + (\mathbf{k} \times \hat{\mathbf{q}})(\hat{\mathbf{p}}\cdot\nabla f)\\ &= \mathbf{k}(p_x q_y - p_y q_x) + (-\mathbf{j}p_x + \mathbf{i}p_y)\left(q_x\frac{\partial f}{\partial x} + q_y\frac{\partial f}{\partial y}\right)\\ &\qquad + (\mathbf{j}q_x - \mathbf{i}q_y)\left(p_x\frac{\partial f}{\partial x} + p_y\frac{\partial f}{\partial y}\right)\end{aligned}$$





$$= [\mathbf{k}\cdot(\hat{\mathbf{p}} \times \hat{\mathbf{q}})]\mathbf{k} + \mathbf{i}(p_y q_x - q_y p_x)\frac{\partial f}{\partial x} + \mathbf{j}(-p_x q_y + p_y q_x)\frac{\partial f}{\partial y}$$

$$= [\mathbf{k}\cdot(\hat{\mathbf{p}} \times \hat{\mathbf{q}})]\mathbf{k} - (p_y q_x - p_x q_y)\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}\right)$$

$$= [\mathbf{k}\cdot(\hat{\mathbf{p}} \times \hat{\mathbf{q}})]\mathbf{k} - \mathbf{k}\cdot(\hat{\mathbf{p}} \times \hat{\mathbf{q}})\nabla f = \mathbf{k}\cdot(\hat{\mathbf{p}} \times \hat{\mathbf{q}})(\mathbf{k} - \nabla f).$$

But $\hat{\mathbf{n}} = \dfrac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \dfrac{\mathbf{k} - \nabla f}{|\mathbf{k} - \nabla f|} = \dfrac{-\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}.$

27. a. Maxwell's equations are

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \quad (1) \qquad \nabla\cdot\mathbf{B} = 0 \quad (3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2) \qquad \nabla \times \mathbf{B} = \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

In addition we have two other relations:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (5) \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad (6)$$

Eq (3) is satisfied because $\nabla\cdot(\nabla \times \mathbf{A}) = 0$ since the divergence of the curl is 0 (see Prob III7). Eq(2) reads

$$\nabla \times (-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \quad (7)$$

However, the left hand side of this equation is

$$-\nabla \times \nabla\Phi - \nabla \times \frac{\partial \mathbf{A}}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$$

because $\nabla \times \nabla\Phi = 0$ (see Prob IV3). Assuming $\mathbf{A}$ is well-behaved we can write $\nabla \times \dfrac{\partial \mathbf{A}}{\partial t} = \dfrac{\partial}{\partial t}(\nabla \times \mathbf{A})$ and so (7) is satisfied.

Eq (1) now reads $\nabla\cdot\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}\right) = \rho/\varepsilon_0$ or $\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\rho/\varepsilon_0$, and eq (4) reads $\nabla \times (\nabla \times \mathbf{A}) = \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}\right) + \mu_0\mathbf{J}$. However, $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$, and so

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = -\varepsilon_0\mu_0\left(\nabla\frac{\partial\Phi}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) + \mu_0\mathbf{J}$$

which, after some algebra becomes

$$\nabla^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \nabla\left(\nabla\cdot\mathbf{A} + \varepsilon_0\mu_0\frac{\partial\Phi}{\partial t}\right).$$

b. With $\mathbf{A}' = \mathbf{A} + \nabla\chi$ and $\Phi' = \Phi - \frac{\partial\psi}{\partial t}$ we have $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\mathbf{A}' - \nabla\chi) = \nabla \times \mathbf{A}'$ because the curl of the divergence is 0 (see Prob IV3). Next we have

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} = -\nabla\left(\Phi' + \frac{\partial\chi}{\partial t}\right) - \frac{\partial}{\partial t}(\mathbf{A}' - \nabla\chi)$$

$$= -\nabla\Phi' - \frac{\partial\mathbf{A}'}{\partial t} - \nabla\frac{\partial\chi}{\partial t} - \frac{\partial}{\partial t}\nabla\chi = -\nabla\Phi' - \frac{\partial\mathbf{A}'}{\partial t}.$$

c. We have $\nabla\chi = \mathbf{A}' - \mathbf{A}$ whence $\nabla^2\chi = \nabla\cdot\mathbf{A}' - \nabla\cdot\mathbf{A}$, and $\frac{\partial\chi}{\partial t} = \Phi - \Phi'$ whence $\frac{\partial^2\chi}{\partial t^2} = \frac{\partial\Phi}{\partial t} - \frac{\partial\Phi'}{\partial t}$. Thus

$$\nabla^2\chi - \varepsilon_0\mu_0\frac{\partial^2\chi}{\partial t^2} = \nabla\cdot\mathbf{A}' - \nabla\cdot\mathbf{A} - \varepsilon_0\mu_0\left(\frac{\partial\Phi}{\partial t} - \frac{\partial\Phi'}{\partial t}\right).$$

$$= \nabla\cdot\mathbf{A}' + \varepsilon_0\mu_0\frac{\partial\Phi'}{\partial t} - \left(\nabla\cdot\mathbf{A} + \varepsilon_0\mu_0\frac{\partial\Phi}{\partial t}\right).$$

Using the equation given in (c) we see that this result implies that $\nabla\cdot\mathbf{A}' + \varepsilon_0\mu_0\frac{\partial\Phi'}{\partial t} = 0$.

d. Eq.(1) can be written $\nabla\cdot\left(-\nabla\Phi' - \frac{\partial\mathbf{A}'}{\partial t}\right) = \rho/\varepsilon_0$ or $\nabla^2\Phi' + \nabla\cdot\frac{\partial\mathbf{A}'}{\partial t} = -\rho/\varepsilon_0$. But $\nabla\cdot\frac{\partial\mathbf{A}'}{\partial t} = \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = -\varepsilon_0\mu_0\frac{\partial^2\Phi'}{\partial t^2}$, using the equation derived in (c). Thus we have $\nabla^2\Phi' - \varepsilon_0\mu_0\dfrac{\partial^2\Phi'}{\partial t^2} = -\rho/\varepsilon_0$.

Eq (4) may be written $\nabla \times (\nabla \times \mathbf{A}') = \varepsilon_0\mu_0\dfrac{\partial}{\partial t}\left(-\nabla\Phi' - \dfrac{\partial \mathbf{A}'}{\partial t}\right) + \mu_0\mathbf{J}$. Using the identity $\nabla \times (\nabla \times \mathbf{A}') = \nabla(\nabla\cdot\mathbf{A}') - \nabla^2\mathbf{A}'$ we can rewrite this equation $\nabla(\nabla\cdot\mathbf{A}') - \nabla^2\mathbf{A}' = \varepsilon_0\mu_0\dfrac{\partial}{\partial t}\nabla\Phi' - \varepsilon_0\mu_0\dfrac{\partial^2\mathbf{A}'}{\partial t^2} + \mu_0\mathbf{J}$.

Rearranging this gives $\nabla^2\mathbf{A}' - \varepsilon_0\mu_0\dfrac{\partial^2\mathbf{A}'}{\partial t^2} = -\mu_0\mathbf{J} + \nabla\left(\nabla\cdot\mathbf{A}' + \varepsilon_0\mu_0\dfrac{\partial\Phi'}{\partial t}\right)$.

But in (a) we showed that the expression in parentheses on the right hand side of this last equation is 0. Hence

$$\nabla^2\mathbf{A}' - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}'}{\partial t^2} = -\mu_0\mathbf{J}.$$

28\. a. We have $\iint_S \hat{\mathbf{n}}p\,dS = \iiint_V \nabla p\,dV$. Thus the equation of motion can be written

$$\iiint_V \rho\mathbf{f}_{ext}\,dV - \iiint_V \nabla p\,dV = \iiint_V \rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right]dV.$$

But because V is arbitrary, this expression implies

$$\mathbf{f}_{ext} - \frac{1}{\rho}\nabla p = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}.$$

b. If $\mathbf{v} = 0$ the equation found in (a) becomes $\mathbf{f}_{ext} = \dfrac{1}{\rho}\nabla p$.

c. The external force is $\mathbf{f}_{ext} = -\mathbf{k}g$. Thus the equation found in (b) can be written $-g\mathbf{k} = \dfrac{1}{\rho}\left(\mathbf{i}\dfrac{\partial p}{\partial x} + \mathbf{j}\dfrac{\partial p}{\partial y} + \mathbf{k}\dfrac{\partial p}{\partial z}\right)$. However, because p depends only upon z, $\dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial y} = 0$ and $\dfrac{\partial p}{\partial z} = \dfrac{dp}{dz}$. Hence our equation becomes $\dfrac{dp}{dz} = -\rho g$ which has solution $p = p_0 - \rho g z$.