

Book Notes on

div grad curl and all that

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1 Chapter I: Introduction, Vector Functions, and Electrostatics

2 Chapter II: Surface Integrals and the Divergence

2.1 Gauss' Law

2.2 The Unit Normal Vector

2.3 Definition of Surface Integrals

2.4 Evaluating Surface Integrals

2.5 Flux

An integral of the type

$$\iint_S F(x, y, z) \bullet \hat{n} dS \quad (1)$$

is sometimes referred to as the "flux of F ." Gauss' law states that the flux of the electromagnetic field over some closed surface is the enclosed charge divided by ϵ_0 .

To properly understand what the Flux (Latin for "flow") represents, it is good to get a geometric feeling. Imagine we have a fluid of density ρ moving with velocity \mathbf{v} . We want to find the total mass of the fluid that crosses an area ΔS perpendicular to the direction of flow in a time Δt . The fluid in the cylinder of length $v\Delta t$ with the patch ΔS will cross ΔS in the interval Δt .

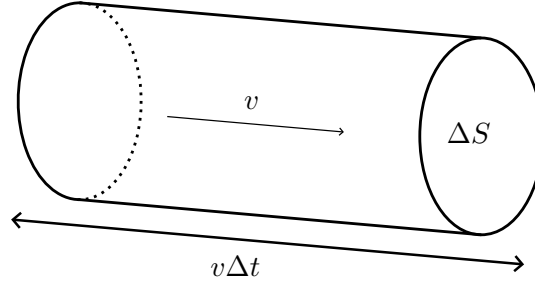


Figure 1: Cylinder

The volume of this cylinder is $v\Delta t\Delta S$ and contains a total mass $\rho v\Delta t\Delta S$. Dividing out the Δt will give the rate of flow. Thus, we can see that the rate of flow is

$$(\text{Rate of flow through } \Delta S) = \rho v \Delta S \quad (2)$$

Considering a more complicated case in which ΔS is not perpendicular to the direction of flow, we can apply earlier used techniques to see that the total amount of flow is equal to the previous rate of flow multiplied by $\cos(\theta)$, where θ is the angle between the velocity vector \mathbf{v} and the unit vector $\hat{\mathbf{n}}$.

$$(\text{Rate of flow through } \Delta S) = \rho \mathbf{v} \bullet \hat{\mathbf{n}} \Delta S \quad (3)$$

Now, if we consider an arbitrary surface S in some region containing flowing matter,

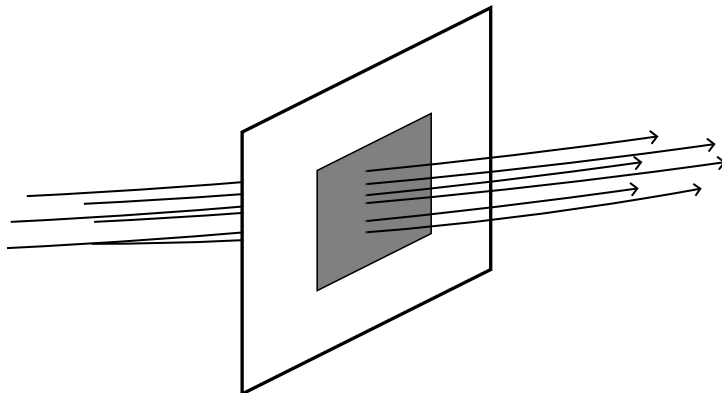


Figure 2: Flowing Matter

we can approximate the surface by a polyhedron, and, using previous arguments, the rate at which matter flows through the l th face of this polyhedron is approximately

$$\rho(x_l, y_l, z_l) \mathbf{v}(x_l, y_l, z_l) \bullet \hat{\mathbf{n}}_l \Delta S_l \quad (4)$$

Knowing that we can sum over all of the faces while taking the limit to produce an integral, we see that

$$(\text{Rate of flow through } \Delta S) = \iint_S \rho(x, y, z) \mathbf{v}(x, y, z) \bullet \hat{\mathbf{n}} dS \quad (5)$$

If this surface S is a closed surface and there is a net rate flow out of the volume it encloses, the integral will be positive. If there is a net rate of flow in, the integral will be negative.

If in this last equation we put

$$\mathbf{F}(x, y, z) = \rho(x, y, z) \mathbf{v}(x, y, z) \quad (6)$$

the integral is identical to that of the flux equation. For this reason, any integral of (1) is called "the flux of \mathbf{F} over the surface S ," even when the function is not the product of a density and a velocity.

In the terms of Gauss' law, the electromagnetic field flows in or out of a surface enclosing charge, whose amount of "flow" is proportional to the net charge enclosed.

2.6 Using Gauss' Law to Find the Field

Gauss' Law is the only good general method for calculating the electric field, although, it is not an explicit expression for \mathbf{E} . The equation does not say " \mathbf{E} equals something," but rather "the flux of \mathbf{E} (the surface integral of the normal component of \mathbf{E}) equals something."

Thus, in order to utilize Gauss' law, we must "distangle" \mathbf{E} from its surroundings. Still, there are situations in which Gauss' law can be used to find the field.

Consider a point charge q placed at the origin of a coordinate system. From what we know about a point charge, the electric field must be in the radial direction and must have the same magnitude at all points on the surface of a sphere centered at the origin. Symbolically, we have $\mathbf{E} = \hat{\mathbf{e}}_r E(r)$ where $\hat{\mathbf{e}}_r = \mathbf{r}/r$ is a unit vector in the radial direction. Thus, Gauss' law becomes

$$\iint_S E(r) \hat{\mathbf{e}}_r \bullet \hat{\mathbf{n}} dS = q/\epsilon_0 \quad (7)$$

If, for the surface S , we choose a spherical shell of radius r centered at the origin,

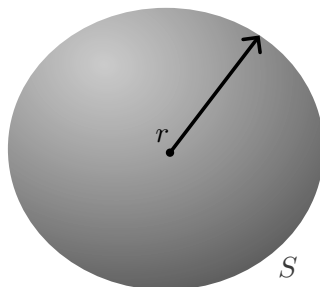


Figure 3: Sphere

we can see that $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$ (both pointing outwards from the origin) and that $\hat{\mathbf{n}} \bullet \hat{\mathbf{e}}_r = 1$. Plugging this into the integral, we get

$$\iint_S E(r) dS = q/\epsilon_0 \quad (8)$$

If we realize that r is a constant over the spherical surface S , we see that $E(r)$ is also a constant of S and we get

$$\iint_S E(r) dS = E(r) \iint_S dS = 4\pi r^2 E(r) = q/\epsilon_0 \quad (9)$$

whence

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (10)$$

and

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{e}}_r E(r) = \frac{\hat{\mathbf{e}}_r}{4\pi\epsilon_0} \frac{q}{r^2} \quad (11)$$

We can see how dependent this example is on the symmetry when using Gauss' law. To use Gauss' law in the form of (1) there are a total of three situations to yield the electric field. A spherically symmetric distribution of charge, an infinitely long cylindrically symmetric distribution, and an infinite slab of charge. The real value of (1) is that it can be changed into a more useful form.

This form makes it so difficult to find \mathbf{E} as it requires knowing the electric field at all infinitesimally small points within a surface S .

$$\iint_S \mathbf{E} \bullet \hat{\mathbf{n}} dS = q/\epsilon_0 \quad (12)$$

This could be simplified by taking the sum of lets say 100 points,

$$\sum_{l=1}^{100} \mathbf{E}_l \cdot \hat{\mathbf{n}}_l \Delta S_l \simeq q/\epsilon_0 \quad (13)$$

but even this requires a large amount of calculations of \mathbf{E} to get an accurate value.

The next step is to consider dealing with the "flux at a single point" (whatever that may mean) rather than the flux through a surface. This will allow Gauss' law to yeild something usable. Let us consider at point P surrounded by a set of concentric spherical shells S_1, S_2, S_3 , and so on, and calculate the flux Φ_1, Φ_2, Φ_3 , and so on, through each shell. We can then attempt to define the "flux at point P " as the limiting value approached by the sequence of fluxes approaches infinitely small shells centered at point P .

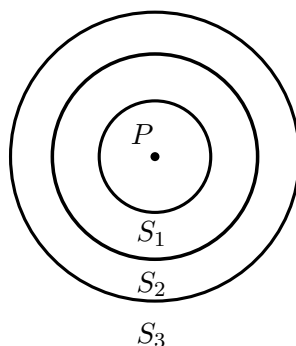


Figure 4: P and Spherical Shells S_n

This seemed to be a good solution, as it would employ the symmetry we used earlier and would simplify the calculation. But, it does not work because the flux approaches zero for any point P . This is because the flux describes the amount of matter flowing through an area. When that area approaches zero, as in point P , the flux approaches zero. Proving this mathematically will suggest a way to fix the issue. If $\bar{\rho}_{\Delta V}$ denotes the average density of electric charge in some region of volume ΔV , then the total charge in ΔV is $\bar{\rho}_{\Delta V} \Delta V$. Thus, Gauss' law equation may be written as

$$\iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \bar{\rho}_{\Delta V} \Delta V / \epsilon_0 \quad (14)$$

Here, we can see the validity of our assertion: As $S \rightarrow 0$, the enclosed volume ΔV must, of course, also approach zero. Thus, the flux also tends to zero. Now, with this proof, we can now isolate a quantity that does not vanish as $S \rightarrow 0$. Dividing by ΔV , we get

$$\frac{1}{\Delta V} \iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \bar{\rho}_{\Delta V} / \epsilon_0 \quad (15)$$

Now if we take the limit as S shrinks to zero about some point in ΔV whose coordinates are (x, y, z) , then we see that the average density approaches $\rho(x, y, z)$, the density at (x, y, z) , and we get

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \bar{\rho}_{\Delta V} / \epsilon_0 \quad (16)$$

This expression is hideous and can only have practical use if we can pound the left-hand side into a form that looks and acts half civilized.

2.7 Divergence

Let us consider the surface integral of some arbitrary vector function $\mathbf{F}(x, y, z)$

$$\iint_S \mathbf{F} \bullet \hat{\mathbf{n}} dS \quad (17)$$

We are interested in the ratio of this integral to the volume enclosed by the surface S as the volume shrinks to zero about some point, for that is exactly the type of quantity that appears in equation (16). This limit is important enough to warrant a special name and notation. It is called the divergence of \mathbf{F} and is designated $\text{div } \mathbf{F}$.

$$\text{div } \mathbf{F} = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \bullet \hat{\mathbf{n}} dS \quad (18)$$

This quantity is a scalar, and in general, will have different values at different points (x, y, z) . Thus, the divergence of a vector function is a scalar function of position.

So, equation (17) can be written as

$$\text{div } \mathbf{E} = \rho/\epsilon_0 \quad (19)$$

If we take our new terminology literally, we can interpret equation (19) to mean that the field "diverges" from a point, and how much it diverges is representative to the quantity of charge there is at that point as represented by the density there.

The next thing to do is simplify the expression for the divergence of a vector function. Consider a small rectangular cuboid with edges of length Δx , Δy , Δz parallel to the coordinate axes. Let the point at the center of the cuboid have coordinates (x, y, z) . We calculate the surface integral of \mathbf{F} over the surface of the cuboid by regarding the integral as a sum of six terms, one for each cuboid face.

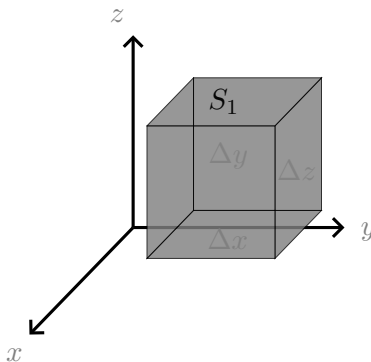


Figure 5: Cube Faces

We begin by considering the face marked S_1 in the figure. We want

$$\iint_{S_1} \mathbf{F} \bullet \hat{\mathbf{n}} dS \quad (20)$$

In this case it is obvious that the unit vector normal to this face and pointing outward from the enclosed volume is \mathbf{i} , thus, $\mathbf{F} \bullet \mathbf{i} = F_x$. The preceding integral is

$$\iint_{S_1} F_x(x, y, z) dS \quad (21)$$

By then assuming the cuboid is small, we can calculate this integral approximately as F_x at the center of the face S_1 , multiplied by the area of the face (assuming that the cuboid face is so small that the value of the vector function is approximately equal at all points of the face). The coordinates of this center are $(x + \Delta x/2, y, z)$. Thus,

$$\iint_{S_1} F_x(x, y, z) dS \simeq F_x\left(x + \frac{\Delta x}{2}, y, z\right) \Delta y \Delta z \quad (22)$$

This same reasoning can be applied to the opposite face S_x whose outward normal vector is $-\mathbf{i}$ and whose center is at $(x - \Delta x/2, y, z)$. Solving this integral leads to

$$-F_x\left(x - \frac{\Delta x}{2}, y, z\right) \Delta y \Delta z \quad (23)$$

Adding together the contributions of these two faces, we get:

$$\begin{aligned} & \iint_{S_1+S_2} \mathbf{F} \cdot \hat{\mathbf{n}} dS \\ & \simeq \left[F_x\left(x + \frac{\Delta x}{2}, y, z\right) - F_x\left(x - \frac{\Delta x}{2}, y, z\right) \right] \Delta y \Delta z \\ & = \frac{F_x\left(x + \frac{\Delta x}{2}, y, z\right) - F_x\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x} \Delta y \Delta z \Delta x \end{aligned} \quad (24)$$

Recognizing that $\Delta x \Delta y \Delta z = \Delta V$ is the volume of the cuboid, we can simplify the integral to

$$\begin{aligned} & \frac{1}{\Delta V} \iint_{S_1+S_2} \mathbf{F} \cdot \hat{\mathbf{n}} dS \\ & = \frac{F_x\left(x + \frac{\Delta x}{2}, y, z\right) - F_x\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x} \end{aligned} \quad (25)$$

Recognizing that as the limit as ΔV approaches zero the sides of the cuboid approach zero allow us to write $\lim_{\Delta x \rightarrow 0}$ on the right hand side in place of $\lim_{\Delta V \rightarrow 0}$ and we find

$$\begin{aligned} & \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_{S_1+S_2} \mathbf{F} \cdot \hat{\mathbf{n}} dS \\ & = \lim_{\Delta x \rightarrow 0} \frac{F_x\left(x + \frac{\Delta x}{2}, y, z\right) - F_x\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x} \end{aligned} \quad (26)$$

$$= \frac{\partial F_x}{\partial x} \quad (27)$$

evaluated at (x, y, z) . This last equation is the definition of the partial derivative. Now, when considering the contributions of the other faces of the cube,

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (28)$$

The limit on the left hand side of this equation is the divergence of \mathbf{F} . Thus, we have just demonstrated that

$$\operatorname{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (29)$$

This equation provides a much more straitforward method to determining the divergence. If we consider the function

$$\mathbf{F} = \mathbf{i}x^2 + \mathbf{j}xy + \mathbf{k}yz \quad (30)$$

We then calculate

$$\frac{\partial F_x}{\partial x} = 2x, \quad \frac{\partial F_y}{\partial y} = x, \quad \text{and} \quad \frac{\partial F_z}{\partial z} = y \quad (31)$$

Thus, the divergence of \mathbf{F} is

$$\operatorname{div} \mathbf{F} = 2x + x + y = 3x + y \quad (32)$$

Returning to the electrostatic field, we combine the equations to find

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho/\epsilon_0 \quad (33)$$