2067(3rd batch)

Group A(10X2=20)

1. **Define a relation and a function from a set into another set. Give suitable examples.**

   Any subset of Cartesian product A X B is called relation from A to B.

   For eg,

   Relation= {(x,y):x+y$\geq$ 5}

   ={(1,4),(2,3),(2,4),(3,3),(3,4)}

   A function from set A to the set B is a relation in which each element of A associates with unique element of B is denoted by $f: A \to B$

   For example

   F(x)=2x+5

2. **Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by using integral test.**

   Solution

   We determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by comparing it with $\int_1^{\infty} \frac{1}{x^2} dx$. To carry out the comparison, we think of the terms of the series as values of the function $f(x) = \frac{1}{x^2}$ and interpret these values as the areas of rectangle under the curve $y = \frac{1}{x^2}$

   Therefore,

$$S_n = \frac{1}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \cdots . + \frac{1}{n^2}$$
$$= f(1) + f(2) + f(3) + \cdots .. + f(n)$$

$$< f(1) + \int_1^n \frac{1}{x^2} dx$$
$$< 1 + \int_1^\infty \frac{1}{x^2} dx$$
$$< 1 + 1 = 2$$

Thus the partial sum of given problem are bounded above (by 2) and the series converges.

3. **Investigate the convergence of the series $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$**

Solution

For the series $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}$$
$$= \frac{1}{3} \cdot \frac{2^{n+1}+5}{2^n+5}$$
$$= \frac{1}{3} \cdot \frac{2+5.2^{-n}}{1+5.2^{-n}}$$
$$\to \frac{1}{3} \cdot \frac{2}{1}$$
$$= \frac{2}{3}$$

4. **Find the foci, vertices, center of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$**

Solution

From the given equation we get,

$$a^2 = 16 \text{ and } b^2 = 9$$
$$\therefore a = 4 \text{ and } b = 3$$

Now,

Semimajor axis: $a = 4$

Semiminor axis: $b = 3$

Center to focus distance: $c = \sqrt{16-9} = \sqrt{7}$

Foci: $(\pm c, 0) = (\pm\sqrt{7}, 0)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Center: (0,0)

5. **Find the equation for the plane through (-3,0,7) perpendicular to $\vec{n} = \overrightarrow{5i} + \overrightarrow{2j} - \vec{k}$**

Solution

The component of equation is

$$5\big(x-(-3)\big) + 2(y-0) + (-1)(z-7) = 0$$

Simplifying we obtain

$$5x + 15 + 2y - z + 7 = 0$$
$$5x + 2y - z = -22$$

Which is the required equation

6. **Define cylindrical co-oridinate (r,∅,z). Find an equation for the circular cylinder $4x^2 + 4y^2 = 9$ in cylindrical co-ordinate**

Solution

Cylindrical coordiantes represent a point P in space by ordered triples (r, θ, z) in which

- r and θ are polar coordinates for the vertical projection of P on the xy-plane
- z is the rectangular vertical coordinate.

Given equation is $\boldsymbol{4x^2 + 4y^2 = 9}$

Now,

$$4(\rho cos\emptyset)^2 + 4(\rho sin\emptyset)^2 = 9$$
$$4\rho^2 \cos^2 \emptyset + 4\rho^2 \sin^2 \emptyset = 9$$
$$4\rho^2(\cos^2 \emptyset + \sin^2 \emptyset) = 9$$
$$4\rho^2 = 9$$
$$\rho = \pm\frac{3}{2}$$

Which is the required equation

7. **Calculate $\iint f(x,y)da$ for f(x,y)=$1 - 6x^2y$, $R: 0 \le x \le 2, -1 \le y \le 1$**
8. **Define Jacobian Determinant for x=g(u,v,w), y=h(u,v,w), z=k(u,v,w)**

Solution

Jacobian determinant for X=g(u,v,w), y=h(u,v,w), z=k(u,v,w) is

$$J(u,v,w) = \frac{\partial(x,y,z}{\partial(u,v,w)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} \end{vmatrix}$$

Q.N.9)What do you mean by local extreme point of f(x,y)? Illustrate the concept by graphs.

Solution:

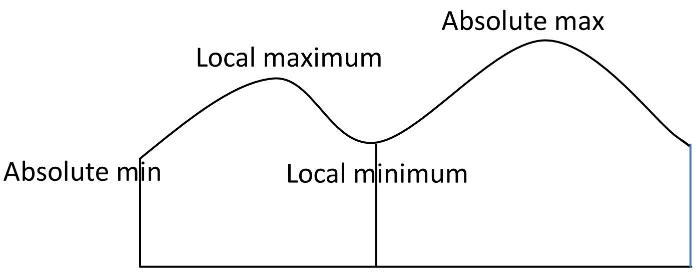
Definition of Local Maximum and Local Minimum

A function f has a local maximum value at an interior point **c** of its domain if

$f(x) \le f(c)$ for all x in some open interval containing c.

A function f has a local maximum value at an interior point **c** of its domain if

$f(x) \geq f(c)$ for all x in some open interval containing c.



Q.N.10)Define partial differential equation of the first index with suitable examples.

Partial differential equation

If a dependent variable is a function of two or more than two independent variable then on equation involving with partial differential coefficient it is known as partial differential equation. This is the relation of dependent variable independent variable and partial differential coefficient.

For eg:-

$\frac{dz}{dx} - \frac{2dz}{dy} = 0$ is a first order partial differential equation

$\frac{d^2z}{dx^2} + \frac{d^2}{dxdy} + \frac{2d^2}{dy^2} = 0$ is a second order partial differential equation

$\frac{d^3z}{dx^3} + \frac{d^3z}{dx^2dy} + \frac{d^3z}{dxd\,y^2} + \frac{d^3z}{dy^3} = 0$ is a third order partial differential equation

Group B(5X4=20)

Q.N.11)State the mean value theorem for a differentiable function and verify it for the function $f(x) = \sqrt{1 - x^2}$ on the interval of $[-1, 1]$.

Solution

Mean value theorem for a differentiable function states that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Proof

If we divide both sides of the Max-Min Inequality by (b-a) we obtain

$\min f \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \max f$

Since f is continuous, the Intermediate value theorem for Continuous Function say that f must assume every value between min f and max f. It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)\, dx$ at same point c in [a,b].

Solution

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$
$$= \frac{1}{2}\int_{-1}^{1}\sqrt{(1 - x^2)}\ dx$$
$$= \frac{1}{2}(\sin^{-1}(1) - \sin^{-1}(-1))$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{3\pi}{2}\right)$$
$$= \frac{1}{2}\left(-\frac{2\pi}{2}\right)$$
$$= -\frac{\pi}{2}$$

The average value of $f(x) = \sqrt{1-x^2}$ over [-1,1] is $-\frac{\pi}{2}$. The function is assume this value when $\sqrt{1-x^2} = -\frac{\pi}{2}$

Q.N.12)What do you mean by Taylor's Polynomial of order n?Obtain taylor's polynomial and Taylor's series generated by the funcion f(x)=cosx at x=0.

Solution:

The functions & its derivatives are

$f(x) = cosx \qquad f(0) = 1$

$f^{'}(x) = -sinx \qquad f^{'}(0) = 0$

$f^{''}(x) = -cosx \qquad f^{''}(0) = -1$

$f^{'''}(x) = sinx \qquad f^{'''}(0) = 0$

$f^{2k}(0\,) = (-1^k)$

The series has only even power term for n=2k, taylor's representation is given as:

$$cosx = f(0) + f^{'}(0)(x-0) + \frac{f^{''}(0)}{2!}(x-0)^2 + \frac{f^{'''}(0)}{3!}(x-0)^3 + \frac{f^4(0)}{4!}(x-0)^4 + \cdots + \frac{f^{2k}(0)(x-0)^{2k}}{(2k)!} + R_{2k}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \ldots \ldots \ldots \ldots .. + \frac{(-1)^k(x)^{2k}}{(2k)!} + R_{2k}(x)$$

[Note: even term are 0 since every sinx=sin(0)=0]

Because the derivative of the cosine have absolute values less than or equal to 1, the remainder estimation theorem with M=1 gives

$$|R_{2k}(x)| \le 1.\frac{|x|^{2k+1}}{(2k+1)!}$$

For every value of x, $R_{2k} \to 0 \ as \ k \to \infty$. Therefore, the series converges to $cosx$ for every value of x. Thus,

$$cosx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \ldots \ldots \ldots \ldots \ldots ..$$

Q.N.13)Find the length of the cardoid $\boldsymbol{r = 1 - \cos\theta}$

Given equation is $r = 1 - cos\theta$

$$\frac{dr}{d\theta} = sin\theta$$

We have the formula length of cardioid as follows

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - cos\theta)^2 + (sin\theta)^2$$
$$= 1 - 2cos\theta + \cos^2\theta + \sin^2\theta$$
$$= 2 - 2cos\theta$$

And$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$= \int_\pi^{2\pi} \sqrt{2 - 2cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 4 + 4$$
$$= \mathbf{8}$$

Q.N.14)Define the partial derivative of $f(x,y)$ at a point (x_0, y_0) with respect to all variable. Find the derivative of $f(x,y) = xe^y + \cos(x,y)$ at the point (2,0) in the direction of $A = 3\vec{i} - 4\vec{j}$

Solution

The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is

$\left|\frac{df}{dx}\right|_{x_0,y_0} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ provide limit exists

Here

$$f(x,y) = xe^y + \cos(x,y)\ at\ (2,0)$$

$$f_x = e^y - \sin(x,y)$$

$$fy = xe^y - \sin(x,y)$$

Here

$$A = 3\vec{i} - 4\vec{j}$$

$$U = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$

$$\nabla f = f_x p_0\vec{i} + f_y p_0\vec{j}$$

$$= e^y - ysinxy\vec{i} + xe^y - xsinxy$$

$$= (e^0 - 0.\sin(2\times 0))\vec{i} + 2\times e\text{^}0\ + \cos(2\times 0).2)\vec{j})$$
$$= \vec{i} + 2\vec{j}$$

Q.N.15)Find the general solution of the differential equation $\frac{x^2dz}{dx}+\frac{y^2dz}{dx}=(x+y)z$

Solution:

$$\frac{x^2dz}{dx}+\frac{y^2dz}{dx}=(x+y)z$$

Pp+Qq=R

$p=x^2\ ,q=y^2,R=x+y$

$$\frac{dx}{p}=\frac{dy}{y^2}=\frac{dz}{R}$$

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{z}{x+y}$$

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

$-\frac{1}{x}=-\frac{1}{y}+C_1$ Note: $\left\{\frac{x^{-2+1}}{-2+1}=\frac{x^{-1}}{-1}=-\frac{1}{x}\right\}$

$$C_1=\frac{1}{y}-\frac{1}{x}$$

$$Now,\frac{dx}{p}-\frac{dy}{q}=\frac{dz}{R}$$

$$or;\frac{dx}{x^2}-\frac{dy}{y^2}=\frac{z}{x+y}$$

$$or;\frac{d(x-y)}{x^2-y^2}=\frac{dz}{x+yz}$$

$$or;\frac{d(x-y)}{(x-y)(x+y)}=\frac{dz}{(x+y)z}$$

$$or;\frac{d(x-y)}{x-y}=\frac{dz}{1}$$

$or; 1 = dz$ integrating, we get

$logz + C_2 = 1$

$C_2 = 1 - logz$

$\therefore (C_1, C_2) = (\frac{1}{g} - \frac{1}{x}, 1 - logz)$

Q.N.16)Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line y=x-2

Here

The given curve is $y = \sqrt{x} - - - - - - - - - - - (i)$

And the lines are on x-axis

$$\therefore y = 0 - - - - - - (ii)$$
$$y = x - 2 - - - - - (iii)$$

Here upper curve is $y = \sqrt{x}$

But the lower boundary changes from $y = 0$ for $0 \le x \le 2$ to y=x-2

Here,

$$f(x) = \sqrt{x}$$

$$g(x) = 0$$

Solving (i) and (ii)

$$x = 0$$

Solving (ii) and (iii)

$$x - 2 = 0$$

$$x = 2$$

Therefore limit of integration are

$$a = 0, b = 2$$

Now

$$Area(A) = \int_0^2 [f(x) - g(x)]dx$$
$$= \int_0^2 (\sqrt{x})dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2$$
$$= \frac{3}{2}\left[2^{\frac{3}{2}} - 0\right]$$
$$= \frac{4\sqrt{2}}{3}$$

OR

Q.N.16)Investigate the convergence of the integrals

- $\int_0^1 \frac{1}{1-x} dx$

  The integral $\int_0^1 \frac{1}{1-x} dx$ is continuous on [0, 1] but becomes infinite at x=1 so we evaluate the integral as $b \to 1^-$

$$\therefore I = \lim_{b \to 1^-} \int_a^b \frac{1}{1-x} dx$$
$$= \lim_{b \to 1^-} \left[\frac{\log(1-x)}{1}\right]_0^b$$
$$= \lim_{b \to 1} - \log(1-b) + \log(1)$$
$$= \lim_{b \to 1} \log\left(\frac{1}{1-b}\right)$$

$= \lim_{b \to 1} \log\left(\frac{1}{1-1}\right) = log\infty$ which is infinite so the given integral diverges

- $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

Solution

The integral $f(x) = \frac{1}{(x-1)^{\frac{2}{3}}}$ becomes infinite at x=1 but continuous on [0, 1] and [1, 3]. The converges of the integral are [0, 3]. The convergence of the integral are [0, 3] depends on the integrals from 0 to 1 and 1 to 3. Now, on [0, 1] we have

$$I_1 = \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \lim_{b\to 1^-} \int_a^b (x-1)^{-\frac{2}{3}} dx$$

$$= \lim_{b\to 1^-} \left[\frac{3(x-1)^{\frac{1}{3}}}{1}\right]_0^b = 3\left[0-(-1)^{\frac{1}{3}}\right]$$

$$= 3[-1-1] = 3$$

Now, on [1, 3] we have

$$I_2 = \int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx$$

$$= \lim_{b\to 1+} \left[\frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_1^3$$

$$= 3 \times 2^{\frac{1}{3}}$$

Now,

$I_1 + I_2 = 3 + 3.2^{\frac{1}{3}} = 3(1+\sqrt[3]{2})$

Which is finite, the given integral convergence

Q.N.17)**Calculate the curvature and torsion of the for the helix,**

$$\boldsymbol{r(t) = (acost)i + (asint)j + btk, a, b \geq 0, a^2 + b^2 \neq 0}$$

$\boldsymbol{r(t) = (acost)i + (asin\ t)j + btk(a, b \geq 0, a^2 + b^2 \neq 0)}$

$r(t) = (a\cos t\ )i + (a\sin t)j + btk$

$\frac{dr}{dt} = \vec{v} = -a\sin t\ \vec{\imath} + acost\ \vec{\jmath} + \overrightarrow{bk}$..................(i)

$|\vec{v}|$=$\sqrt{(-asint)^2 + (acost)^2 + b^2}$..................(ii)

$= \sqrt{a^2 + b^2}$

$\therefore T = \frac{\vec{v}}{|\vec{v}|}$

$$\frac{dT}{dt} = \frac{-asint\vec{\imath} + acost\vec{\jmath} + b\vec{k}}{(\sqrt{a^2 + b^2})}$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\left(\frac{acost}{\sqrt{a^2 + b^2}}\right)^2 + \left(-\frac{asint}{\sqrt{a^2 + b^2}}\right)}$$

Diff equation (i) w.r.t 't' we get,

$$\vec{a} = \frac{dv}{dt} = -acost\vec{\imath} - asint\vec{\jmath}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -asint & acost & b \\ -acost & -asint & 0 \end{vmatrix}$$

$= (0 + absint)\vec{\imath} - (0 + abcost)\vec{\jmath} + (\mathrm{a}^2\sin^2\mathrm{t} + \mathrm{a}^2\cos^2\mathrm{t})\ \vec{k}$

$= absint\vec{\imath} - abcost\vec{\jmath} + a^2\vec{k}$

$|\vec{v} \times \vec{a}| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4}$

$= \sqrt{a^2b^2 + a^4}$

$= a\sqrt{a^2 + b^2}$

Now,

$(k) = \frac{|\vec{v} + \vec{a}|}{|\vec{v}|^3}$

$= \frac{a}{a^2 + b^2}$

$$curvature(\rho) = \frac{1}{k} = \frac{a^2 + b^2}{a}$$

Now, to evaluate the torsion, we find the entries in the determinant by differentiating r with respect to t. we have v and a,

$$\dot{a} = \frac{da}{dt} = (asint)i - (acost)j$$

Hence,

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|vXa|^2}$$

$$\frac{\begin{vmatrix} -asint & acost & b \\ -acost & -asint & 0 \\ asint & -acost & 0 \end{vmatrix}}{a\sqrt{a^2+b^2)}}$$

$$= \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2 + b^2)}$$

$$\therefore \tau = \frac{b}{a^2 + b^2}$$

Q.N.18)Find the volume of the region D enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

Solution:

Solution:

$z = x^2 + 3y^2 \dots \dots \dots \dots \dots \dots (i)$

$z = 8 - x^2 - y^2 \dots \dots \dots \dots . . (ii)$

From(i)&(ii)

$x^2 + 3y^2 = 8 - x^2 - y^2$

$or; 2x^2 + 4y^2 - 8$

$or; 2y^2 = 4 - x^2$

$or; x^2 + 2y^2 - 4 \ldots\ldots\ldots\ldots\ldots (iii)$

$$or; y = \pm\sqrt{\frac{(4-x^2)}{2}}$$

$i.e; -\sqrt{\frac{4-x^2}{2}} \leq \sqrt{\frac{4-x^2}{2}}$

Now , to find the points for x-axis.

$y = 0$

$$\sqrt{\frac{4-x^2}{2}} = 0$$

$\therefore x = \pm 2. \quad i.e;\ -2 \leq x \leq 2.$

The points of x-axis are(2,0,0) &(-2,0,0)

Then x^2 plane will be (2,0,4) &(-2,0,4)

$$voume(v) = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} 8 - 2x^2 - 4y^2 dydx$$

$$\int_{-2}^{2}\left[8 - 2x^2 - y - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$\left[\int_{-2}^{2} 8\sqrt{\frac{(4-x^2)}{2}} - 2x^2\sqrt{\frac{(4-x^2)}{2}} - 4\left(\sqrt{\frac{(4-x^2)}{2}}\right)^3 + 8\sqrt{\frac{(4-x^2)}{2}} - 2x^2\sqrt{\frac{(4-x^2)}{2}} - \frac{4}{3}\left(\sqrt{\frac{(4-x^2)}{2}}\right)^3\right]dx$$

$$= \left[\int_{-2}^{2} [16\sqrt{\frac{(4-x^2)}{2}} - 4x^2\sqrt{\frac{(4-x^2)}{2}} - \frac{8}{3}\left(\sqrt{\frac{(4-x^2)}{2}}\right)^3\right]dx$$

$$= \int_{0}^{2} \frac{8}{3\sqrt{2}}(4-x^2)^{\frac{3}{2}}\,dx$$

Put $x = 2sin\theta$

$dx = 2cos\theta d\theta$

$$= \frac{16}{3\sqrt{2}}\int_{0}^{\frac{\pi}{2}} (4 - 4\sin^2\theta)^{\frac{3}{2}}\, 2cos\theta d\theta$$

$$\frac{32}{3\sqrt{2}}\int_{0}^{\frac{\pi}{2}} (4\cos^2\theta)^{\frac{3}{2}}\, cos\theta d\theta$$

$$= \frac{32}{3\sqrt{2}}\int_{0}^{\frac{\pi}{2}} 8\cos^4\theta d\theta$$

$$= \frac{32}{3\sqrt{2}}\int_{0}^{\frac{\pi}{2}} 2(2\cos^2\theta)^2\, d\theta$$

$$\frac{64}{3\sqrt{2}}\int_{0}^{\frac{\pi}{2}} (1+\cos\theta)^2\, d\theta$$

Therefore volume= $\boldsymbol{8\sqrt{2\pi}}$

Q.N.19)Find the absolute maximum and minimum values of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular plate in the first quadrant bounded by lines x=0, y=0 and x+y=9

Solution: Since f is differentiable, the only places where can assume these values are points inside the triangle where $f_x=f_y=0$ and points on the boundary .

a) **Interior points .** For these we have

$f_x=2-2x=0, f_y=2-2y=0,$

yielding the single point(x,y)=(1,1). The valuable of f there is f(1,1)=4

b) **Boundary points**. We take the triangle one side at a time.
i)On the segment OA,y=0. The function
$f(x,y)=f(x,0)=2+2x-x^2$
may now be regarded as function of x defined on the closed interval
$0 \le x \le 9$.Its extreme values may occur at the endpoints
x=0 where f(0,0)=2
x=9 where f(9,0)=2+18-81= -61
and at the interior points where $f'(x,0)=2-2x=0$. The only interior point where $f'(x,0)=0$ is x=1, where
f(x,0)=f(1,0)=3
ii)On the segment OB,x=0 and
$f(x,y0=f(0,y)=2+2y-y^2$
we know from the symmetry of f in x and y and from the analysis we just carried out that the candidates on this segment are
f(0,0)=2, f(0,9)= -61, f(0,1)=3.
iii) we have already acounted for the values of f at the endpoints of AB, so we need only
$f(x,y)=2+2x+2(9-x)-x^2-(9-x)^2= -61 + 18x - 2x^2$
setting $f'(x,9-x)=18- 4x =0$ gives

$x = \frac{18}{4} = \frac{9}{2}$.

At this value of x,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \text{ and } f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}.$$

Summary, We list all the candidates: 4,2,-61,3,-(41/2). The maximum is 4, which f assumes at(1,1). The minimum is -61 , which f assumes at (0,9) and (9,0).

OR

Q.N.19)Find the points on the curve $xy^2 = 54$ nearest to the origin. How the lagrange multipliers defined?

Solution

Since we want the points nearest the origin, let f be the square of the distance from the origin to a point on the curve.

$f = x^2 + y^2 \qquad g = xy^2 - 54 = 0$

$\nabla f = 2x\vec{i} + 2y\vec{j} \qquad \nabla g = y^2\vec{i} + 2xy\vec{j}$

$\nabla f = \lambda \nabla g \rightarrow 2x = y^2\lambda \rightarrow 2y = 2xy\lambda$

$2y = 2xy\lambda \rightarrow 2y = y^3\lambda^2 \rightarrow 2y - y^3\lambda^2 = 0$

$$y(2 - y^2\lambda^2) = 0 \rightarrow y = 0 \text{ or } y^2 = \frac{2}{\lambda^2}$$

CASE 1: $y = 0 \rightarrow x = 0$, but (0, 0) does not satisfy the constraint.

Therefore, $y \neq 0$.

CASE2:

$$y \neq 0 \text{ and } y^2 = \frac{2}{\lambda^2}$$

Since $y \neq 0 \rightarrow 2 = 2x\lambda \rightarrow x = \frac{1}{\lambda}$

$$xy^2 = 54 \rightarrow \frac{1}{\lambda} \cdot \frac{2}{\lambda^2} = 54 \rightarrow \frac{2}{54} = \lambda^3 \rightarrow \lambda = \frac{1}{3}$$

$$x = \frac{1}{\frac{1}{3}} = 3 \; and \; y = \pm 3\sqrt{2}$$

So the points nearest the origin are $(3, \pm 3\sqrt{2})$

Method of lagrange multipliers

Suppose that f(x,y,z) and g(x,y,z) are differentiable. To find the local maximum and minimum values of f subject to the constraints g(x,y,z)=0, find the values of x,y,z and λ that simultaneously satisfy the equations. $\nabla f = \lambda \nabla g \; and \; g(x,y,z) = 0$

Q.N.20)Derive D'Alembert's solution satisfying the initials conditions of the one-dimensional wave equation.

Solution:

The wave equation $\frac{d^2u}{dt^2} = \frac{C^2 d^2u}{dx^2}$

$$where \; C^2 = \frac{T}{\rho}$$

The solution of equation(i) can be obtained by introducing thwo independent variables v and z defined by $v = x + ct \; and \; z = x - ct$

Differentiating v w.r.t x

$$\frac{dv}{dx} = \frac{du}{dv}.\frac{dv}{dx} + \frac{du}{dz}.\frac{dz}{dx}$$

$$= \frac{du}{dv} + \frac{du}{dz}$$

Again differentiate w.r. to x

$$\frac{d^2}{dv}\left(\frac{du}{dv}.\frac{du}{dz}\right)\frac{dv}{dx} + \frac{d}{dz}\left(\frac{du}{dv} + \frac{du}{dz}\right)\frac{dz}{dx}$$

$$= \frac{d^2u}{dvz} + \frac{d^{2u}}{dv^2z} + \frac{d^{2u}}{dzdv} + \frac{d^2u}{dz^2}$$

$$\frac{d^{2u}}{dvz}+\frac{zd^2u}{dvdz}+\frac{d^2u}{dz^2}$$

Again diff u w.r.to t.

$$\frac{du}{dt}=\frac{du}{dv}.\frac{dv}{dt}+\frac{du}{dz}.\frac{dz}{dt}$$

$$=\frac{du}{dv}.c+\frac{du}{dz}(-c)$$

$$=\frac{cdu}{dv}-\frac{cdu}{dz}$$

$$\frac{d^2u}{dt^2}=c\left[\frac{du}{dv}\left(\frac{du}{dv}-\frac{du}{dz}\right)\frac{dv}{dt}+\frac{du}{dz}\left(\frac{du}{dv}-\frac{du}{dz}\right)\frac{dz}{dt}\ \right]$$

$$c^2\left[\frac{d^2u}{dv^2}-\frac{d^2u}{dv^2z}-\frac{d^2u}{dzdv}+\frac{d^2u}{dz^2}\right]$$

$$=C^2\left[\frac{d^2u}{dv^2}-\frac{2d^2u}{dvdz}+\frac{d^2u}{dz^2}\right]$$

Inserting these value in eqn (1)

We get,

$$C^2\left[\frac{d^2u}{dv^2}-\frac{2d^2u}{dvdz}+\frac{d^2u}{dz^2}\right]=C^2\left(\frac{d^2u}{dv^2}+\frac{2d^2u}{dvdz}+\frac{d^2u}{dz^2}\right)$$

$$\frac{4d^2u}{dvdz}=0$$

$$or;\frac{d^2u}{dvdz}=0\ldots\ldots..(2)$$

To get solution integrating partially w.r. to z .

$$\frac{du}{dv}=\gamma(v)$$

Integrating w.r to v

$$u = \int \gamma(v)dv + \varphi(2)$$

$$u(x,t) = \emptyset(v) + \varphi(2)$$

$$= \boldsymbol{\emptyset(x + ct)}$$
$$+ \boldsymbol{\varphi(x}$$
$$\boldsymbol{- ct)} \dots \dots (\mathbf{3}) \boldsymbol{is\ known\ as\ D'Alembart's soltion\ of\ wave\ equation}.$$

D'Alembert's Solution satisfies initial condition

$$u(x,a) = f(x) \dots \dots \dots \dots \dots (4)$$

$$u_t(x,c) = g(x) \dots \dots \dots \dots . . (5)$$

Differentiating equation (iii) with respect to t

$$u_t(x,t) = c\emptyset'(x + ct) - c\varphi(x - ct) - - - - - (6)$$

$$Now,$$
$$u(x,0) = \emptyset(x) + \varphi(x) = f(x) \dots . . (7)$$
$$u_t(x,0) = c\emptyset^{'(x)} - c\varphi'(x) = g(x) \dots \dots \dots \dots . (8)$$

Equation (8) can be written as

$$\emptyset(x) - \varphi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)ds + k(x_0) \dots \dots \dots \dots (9)$$

Integrating (9)

$\emptyset(x) - \varphi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)ds + k(x_0)$ where $k(x_0) = \emptyset(x_0) - \varphi(x_0) \dots \dots \dots (10)$

Adding (7)and (10), we get

$$2\emptyset(x) = f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)ds + k(x_0)$$

$$\emptyset(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + \frac{1}{2}k(x_0) \dots \dots \dots \dots . . (11)$$

Substituting (7) and (11)

$$\varphi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - \frac{1}{2}k(x_0) \ldots\ldots\ldots\ldots\ldots\ldots..(12)$$

Replacing x by(x+ct) in (ii) and x by (x-ct) in equation (12)

$$\emptyset(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds - \frac{1}{2}(kx_0)$$

$$\varphi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)ds - \frac{1}{2}k(x_0)$$

Placing the values in wave equation

$$u(x,t) = \emptyset(x+ct) + \varphi(x-ct)$$

$$\frac{1}{2}f(x+ct) + \frac{1}{2}c\int_{x_0}^{x+ct} g(s)ds + \frac{1}{2}k(x_0) + \frac{1}{2}f(x-ct)$$
$$- \frac{1}{2c}\int_{x_o}^{x-ct} g(s)ds - \left(\frac{1}{2}\right)k(x_0)$$

$$= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2}c\int_{x_0}^{x+ct} g(s)ds - \frac{1}{2}c\int_{x_0}^{x-ct} g(s)ds$$

$$\left[\because \int_b^a f(x)dx = -\int_a^b f(x)dx\right]$$

$$\left[\because \int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx\right]$$

$$= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2}c\int_{x_0}^{x-ct} g(s) + \frac{1}{2}\int_{x-ct}^{x_0} g(s)ds$$

$$= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2}\int_{x-ct}^{x_0} g(s)ds + \frac{1}{2}c\int_{x_o}^{x-ct} g(s)$$

$$u(x.t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

When initial velocity is zero. The above solution will reduce to

$$\boldsymbol{u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)}$$