Csc. MTH 104-2069
Tribhuvan University
Institute of Science and Technology
2069

Bachelor Level / First Year / First Semester / Science — Full Marks: 80
Computer Science and Information Technology – Mth.104 — Pass Marks: 32
(Calculus and Analytical Geometry) — Time: 3 hours.

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.
Attempt all questions.

Group A (10X2=20)

1. Verify the mean value theorem for the function $f(x) = \sqrt{x(x-1)}$ in the interval [0,1].

   Solution

   The function $f(x) = \sqrt{x(x-1)}$ is continuous for [0, 1] and differentiable for (-1, 1).
   Now,
   $f(0) = 0$ and $f(1) = 0$
   We have,
   $f'(x) = \frac{d(\sqrt{x(x-1)}}{dx} = \frac{d\sqrt{x(x-1)}}{d(x(x-1))} \times \frac{d(x(x-1))}{dx} = \frac{1}{2\sqrt{x(x-1)}} \times (2x-1)$
   $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{0-0} = 0$

2. Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$ for $0 \le x \le 1$.

   Here, the given curve is
   $$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$
   So,
   $\frac{dy}{dx} = \frac{3}{2} \times \frac{4\sqrt{2}}{3}x^{1/2} = 2\sqrt{2}x^{1/2}$
   And,
   $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(2\sqrt{2}x^{1/2}\right)^2} = \sqrt{1+8x}$
   Now, the length of the curve for $0 \le x \le 1$ is
   $L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
   $= \int_0^1 \sqrt{1+8x}\, dx$
   $= \int_0^1 (1+8x)^{1/2}\, dx$

$= \left[\frac{(1+8x)^{\frac{1}{2}+1}}{8(\frac{1}{2}+1)}\right]_0^1$
$= \frac{2}{3} \times \frac{1}{8}\left[(1+8x)^{3/2}\right]_0^1$
$= \frac{1}{12}\left[(1+8\times 1)^{3/2} - (1+8\times 0)^{3/2}\right]$
$= \frac{1}{12}[27-1]$
$= \frac{13}{6}$units

3. Test the convergence of the given series $\sum_{n=1}^{\infty}\frac{1}{n!}$ by comparison test.

The series $\sum_{n=1}^{\infty}\frac{1}{n!} = \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ converges because all of its terms are positive and less than or equal to the corresponding terms of $\sum_{n=0}^{\infty}\frac{1}{2^{\mathrm{n}}} = 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots$

The geometric series on the left converges and we have $\sum_{n=0}^{\infty}\frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$

4. Obtain the semi-major axis, semi-minor axis, foci, vertices $\frac{x^2}{25}+\frac{y^2}{16} = 1$.

Here, the given equation is,

$$\frac{x^2}{25}+\frac{y^2}{16} = 1\ldots\text{(i)}$$

Comparing (i) with the standard equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, we get,
$a^2 = 25$ and $b^2 = 16$
$\therefore a = 5 \qquad b = 4$
Distance from focus to centre, $c = \sqrt{a^2-b^2} = \sqrt{5^2-4^2} = \sqrt{9} = 3$
Foci: $(\pm c, 0) = (\pm 3, 0)$
Vertices: $(\pm a, 0) = (\pm 5, 0)$
Centre: (0,0)

5. Find the angle between the vectors $2i+j+k$ and $-4i+3j+k$.

Let, $\vec{u} = 2i+j+k$ and $\vec{v} = -4i+3j+k$
Now,

$$|\vec{\mathrm{u}}| = \sqrt{\mathrm{u}_1{}^2+\mathrm{u}_2{}^2+\mathrm{u}_3{}^2} = \sqrt{2^2+1^2+1^2} = \sqrt{6}$$
$$|\vec{\mathrm{v}}| = \sqrt{\mathrm{v}_1{}^2+\mathrm{v}_2{}^2+\mathrm{v}_3{}^2} = \sqrt{(-4)^2+3^2+1^2} = \sqrt{26}$$

The angle between the vectors is given by

$$cos\theta = \frac{\mathrm{u}_1\mathrm{v}_1+\mathrm{u}_2\mathrm{v}_2+\mathrm{u}_3\mathrm{v}_3}{|\vec{\mathrm{u}}||\vec{\mathrm{v}}|} = \frac{-8+3+1}{\sqrt{6}\times\sqrt{26}} = \frac{-4}{\sqrt{156}} = -\frac{2}{\sqrt{39}}$$
$$\therefore \theta = \cos^{-1} - \frac{2}{\sqrt{39}} = 108.68°$$

6. Obtain the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Here, the given equations are:
$y = x$ and $y = x^2$

Now, from the figure, the area of the region R bounded by $y = x$ and $y = x^2$ is
$A = \int_0^1 \int_{x^2}^{x} dydx$
$= \int_0^1 [y]_{x^2}^{x} dx$
$= \int_0^1 (x - x2)dx$
$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$
$= \left[\frac{1^2}{2} - \frac{1^3}{3}\right] - \left[\frac{0^2}{2} - \frac{0^3}{3}\right]$
$= \frac{1}{2} - \frac{1}{3}$
$= \frac{1}{6}$

7. Show that the function $f(x, y) = f(x) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$ is continuous at every point in the plane except the origin.

Solution

The function f is continuous at any point $(x, y) \neq (0, 0)$ because its values are then given by a rational function of x and y.

At (0, 0) the value of f is defined, we claim, but f has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results as we see.

For any value of m, the function f has a constant value on the punctured line $y = mx, x \neq 0$, because

$f(x, y)|_{y=mx} = \frac{2xy}{x^2+y^2}\Big|_{y=mx} = \frac{2x.mx}{x^2+(mx)^2} = \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}$

Therefore, f has thus number as its limit as (x, y) approaches $(0, 0)$ along the line

$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} f(x, y)|_{y=mx} = \frac{2m}{1+m^2}$
along $y = mx$.

This limit changes with m. there is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous.

8. Using partial derivatives, find $\frac{dy}{dx}$ if $2xy + \tan y - 4y^2 = 0$.

Let $F(x, y) = 2xy + \tan y - 4y^2$

Now,

$$F_x = \frac{\partial f}{\partial x} = \frac{\partial(2xy+\tan y-4y^2)}{\partial x} = 2y$$

$$F_y = \frac{\partial f}{\partial y} = \frac{\partial(2xy+\tan y-4y^2)}{\partial y} = 2x + sec^2y - 8y$$

We know,

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2y}{2x+sec^2y-8y}$$

9. Verify that the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$ is satisfied by $z = \frac{1}{x}\emptyset(y-x) + \emptyset'(y-x)$.

Solution: We have,

$$\frac{\partial z}{\partial x} = -\frac{1}{x}\emptyset'(y-x) - \frac{1}{x^2}\emptyset(y-x) - \emptyset''(y-x)$$

$$= -\frac{1}{x}\left[\frac{1}{x}\emptyset(y-x) + \emptyset''(y-x)\right] - \emptyset''(y-x)$$

$$= -\frac{z}{x} - \emptyset''(y-x)$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{x}\frac{\partial z}{\partial x} + \frac{z}{x^2} + \emptyset'''(y-x)$$

$$= -\frac{1}{x}\left[-\frac{z}{x} - \emptyset''(y-x)\right] + \frac{z}{x^2} + \emptyset'''(y-x)$$

$$= \frac{z}{x^2} + \frac{1}{x}\emptyset''(y-x) + \frac{z}{x^2} + \emptyset'''(y-x)$$

$$= \frac{2z}{x^2} + \frac{1}{x}\emptyset''(y-x) + \emptyset'''(y-x)$$

$$\frac{\partial z}{\partial y} = \frac{1}{x}\emptyset'(y-x) + \emptyset''(y-x)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{x}\emptyset'(y-x) + \emptyset''(y-x)$$

So,

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$$

Hence, $z = \frac{1}{x}\emptyset(y-x) + \emptyset'(y-x)$ is a solution of the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$.

10. Find the general solution of the equation $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$.

Solution

The integral surfaces of this PDE is given by the equation

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

Taking the first two ratios,

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$or, x^{-2}dx = y^{-2}dy$$

Integrating both sides,

$$or, -x^{-1} = -y^{-1} + c_1$$

$$or, \frac{1}{y} - \frac{1}{x} = c_1 \dots (i)$$

The ratios are equal to

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$
$$or, \frac{dx - dy}{x - y} = \frac{dz}{z}$$
$$or, \log(x-y) = \log z + \log c_2$$
$$or, x - y = z.c_2$$
$$or, \frac{x-y}{z} = c_2 \dots (ii)$$

Then, the general solution is

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0$$

Group B (5X4=20)

11. State and prove mean value theorem for definite integral.

Mean value theorem for definite integrals(Statement): If f is continuous on $[a, b]$, then at some point c on $[a, b]$, $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

Proof:

We know,

$$\min f(b-a) \le \int_a^b f(x)dx \le \max f(b-a)$$

Dividing all sides by $(b-a)$,

$$\min f \le \frac{1}{b-a}\int_a^b f(x)dx \le \max f$$

Since f is continuous, the intermediate value theorem for continuous function says that f must assume every value between $\min f$ and $\max f$. It must therefore, assume the value $\frac{1}{b-a}\int_a^b f(x)dx$ at some point c in $[a, b]$, i.e., $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ at some point c in $[a, b]$.

12. Find the area of the region that lies in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Here, the given equation is,

$$r = 2(1 + \cos\theta)$$

Now, we know that area of the region enclosed by the cardioids is

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta$$
$$= \int_0^{2\pi} \frac{1}{2} (2(1 + \cos\theta))^2 d\theta$$

$$= 2\int_0^{2\pi}(1+\cos\theta)^2\, d\theta$$
$$= 2\int_0^{2\pi}(1+2\cos\theta+cos^{\,2}\theta)\, d\theta$$
$$= 2\int_0^{2\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$
$$= 2\int_0^{2\pi}\left(\frac{2+4\cos\theta+1+\cos 2\theta}{2}\right) d\theta$$
$$= 2\int_0^{2\pi}\left(\frac{3+4\cos\theta+\cos 2\theta}{2}\right) d\theta$$
$$= \int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\, d\theta$$
$$= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$
$$= \left[3\times 2\pi+4\sin 2\pi+\frac{\sin 2\times 2\pi}{2}\right]-\left[3\times 0+4\sin 0+\frac{\sin 2\times 0}{2}\right]$$
$$= 6\pi+0+0-(0+0+0)$$
$$= 6\pi$$

13. What do you mean by principal unit normal vector? Find unit tangent vector and principal unit normal vector for the circular motion $\vec{r}(t) = (\cos 2t)i + (\sin 2t)\, j$.

At a point where $\kappa \neq 0$, the principal unit normal vectors for a smooth curve in the plane is

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

where $\kappa = \frac{1}{\vec{v}}\left|\frac{d\vec{T}}{dt}\right|$ and $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is the unit tangent vector.

Here,

$$\vec{r}(t) = (\cos 2t)i + (\sin 2t)\, j$$
$$\vec{v} = \frac{d\vec{r}(t)}{dt} = \frac{d(\cos 2t)i+(\sin 2t)j}{dt} = \frac{-\sin 2t}{2}\vec{\imath} + \frac{\cos 2t}{2}\vec{\jmath}$$

and,

$$|\vec{v}| = \sqrt{\left(\frac{-\sin 2t}{2}\right)^2 + \left(\frac{\cos 2t}{2}\right)^2} = \sqrt{\left(\frac{sin^2 2t}{4}\right) + \left(\frac{cos^2 2t}{4}\right)} = \frac{1}{2}$$

$$\therefore \vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\frac{-\sin 2t}{2}\vec{\imath}+\frac{\cos 2t}{2}\vec{\jmath}}{\frac{1}{2}} = -\sin 2t\vec{\imath} + \cos 2t\vec{\jmath}$$

Now,

$$\frac{d\vec{T}}{dt} = \frac{d(-\sin 2t\vec{\imath}+\cos 2t\vec{\jmath})}{dt} = \left(\frac{-\cos 2t}{2}\right)\vec{\imath} + \left(\frac{-\sin 2t}{2}\right)\vec{\jmath}$$

$$\therefore \left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(\frac{-\cos 2t}{2}\right)^2 + \left(\frac{-\sin 2t}{2}\right)^2} = \sqrt{\frac{cos^2 2t}{4} + \frac{sin^2 2t}{4}} = \frac{1}{2}$$

Hence,

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \frac{\left(\frac{-\cos 2t}{2}\right)\vec{\imath}+\left(\frac{-\sin 2t}{2}\right)\vec{\jmath}}{\frac{1}{2}} = -\cos 2t\vec{\imath} - \sin 2t\vec{\jmath}$$

14. Define partial derivative of a function $f(x,y)$ with respect to x at the point (x_0, y_0). State Euler's theorem, verify it for the function $(x,y) = x^2 + 5xy + \sin x + 7e^x$ $x = \left({}^{y}/_{2}\right) + 1$.

The partial derivative of $f(x,y)$ w.r.t. x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

provided the limit exists.

Euler's Theorem (Statement):

Here,

The given function is,

$$F(x,y) = x^2 + 5xy + \sin x + 7e^x$$

We know,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\left(x^2+5xy+\sin x+7e^x\right)}{\partial y}\right) = \frac{\partial(5x)}{\partial x} = 5$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\left(x^2+5xy+\sin x+7e^x\right)}{\partial x}\right) = \frac{\partial(2x+5y+\cos x+7e^x)}{\partial y} = 5$$

$$\therefore \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 5$$

Hence, Euler's theorem is verified.

15. Find a particular integral of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 2y - x^2$.

Here, the given equation is,

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 2y - x^2$$

$$or, (D^2 - D')z = 2y - x^2$$

The auxiliary equation is,

$$m^2 - 1 = 0$$

$$\therefore m = \pm 1$$

Then the complementary equation is,

$$z = \phi_1(y+x) + \phi_2(y-x)$$

Now, the partial integration (P.I.)$= \frac{1}{D^2 - D'}(2y - x^2)$

$$= \frac{1}{D^2\left(1-\frac{D'}{D^2}\right)}(2y-x^2)$$

$$= \frac{1}{D^2}\left(1-\frac{D'}{D^2}\right)^{-1}(2y-x^2)$$

$$= \frac{1}{D^2}\left(1+\frac{D'}{D^2}+\frac{D'^2}{D^4}+\cdots\right)(2y-x^2)$$

$$= \frac{1}{D^2}\left[(2y-x^2)+\frac{1}{D^2}D'(2y-x^2)\right]$$

$$= \frac{1}{D^2}\left[(2y-x^2)+\frac{1}{D^2}\times\frac{\partial\left(2y-x^2\right)}{\partial y}\right]$$

$$= \frac{1}{D^2}(2y-x^2)+\frac{1}{D^4}\times 2$$

$$= 2y\times\frac{x^2}{2}-\frac{x^4}{4}+2\times\frac{x^3}{2\times 3}$$

$$= x^2y-\frac{x^4}{4}+\frac{x^3}{3}$$

<u>Group C</u> (5X8=40)

16. Graph the function $y = x^{5/3} - 5x^{2/3}$.

Here,

$$y = x^{5/3} - 5x^{2/3}$$

$\therefore y' = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}\left(\frac{x-2}{x^{1/3}}\right)$

and,

$y'' = \frac{5}{3} \times \frac{2}{3}x^{-1/3} - \frac{10}{3} \times \left(\frac{-1}{3}\right)x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1)$

Since the first derivative of f is zero at $x = 2$ and undefined at $x = 0$, so the critical points are $x = 0$ and $x = 2$. These critical points divide the X-axis into intervals $(-\infty, 0)$, $(0,2)$ and $(2, \infty)$. By using the first derivative test for monotonic functions, we can assume that the function is increasing on $(-\infty, 0)$, decreasing on $(0,2)$, and increasing on $(2, \infty)$ because $f'(x) > 0$ at each point $x \in (-\infty, 0)$ *and* $x \in (2, \infty)$ and $f'(x) < 0$ at each point $x \in (0,2)$.

f has a local maximum at $x = 0$(f' changes from positive to negative) and that f has a local minimum at $x = 2$(f' changes from negative to positive).

The local minimum value is

$$f(2) = 2^{2/3}(2-5) = -3(2^{2/3})$$

And the local maximum value is

$$f(0) = 0$$

Since $f''(x) = \frac{10}{9}x^{-4/3}(x+1)$ is zero at $x = -1$ and undefined at $x = 0$, so the possible inflection points are $x = -1$ and $x = 0$.

By using the second derivative test for concavity, we see that f is concave down on $(-\infty, -1)$ and concave up on the intervals $(0, \infty)$ and $(-1,0)$ because $f''(x) < 0$ at each points $x \in (-\infty, -1)$ and $f''(x) > 0$ at each points $x \in (0, \infty)$ and $x \in (-1,0)$.

Hence, f is increasing concave down on $(-\infty, -1)$, increasing concave up on $(-1,0)$, decreasing concave up on $(0,2)$ and increasing concave up on $(2, \infty)$.

We see that there is an inflection point at $x = -1$, but not at $x = 0$ because the concavity does not change at $x = 0$.

However,

- The function $y = x^{5/3} - 5x^{2/3}$ is continuous
- $y' \to \infty$ as $x \to 0^-$ and $y' \to -\infty$ as $x \to 0^+$
- The concavity does not change at $x = 0$ tells us that the graph has a cusp at $x = 0$.

Graph: $y = x^{5/3} - 5x^{2/3}$

x	-1	0	2	5
y	-6	0	$-3(2^{2/3})$	0

17. What is meant by Maclaurin series? Obtain the Maclaurin series for the function $f(x) = e^{-x}$.

Let f be a function with derivatives of all orders throughout some interval containing 0 as an interior point. Then the Maclaurin series generated by f at $x = 0$ is

$$\sum_{k=0}^{\infty}\frac{f^k(0)x^k}{k!} = f(0) + f'(o)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n$$

We have,

$$f(x) = e^{-x} \Rightarrow f(0) = e^{-0} = 1$$
$$f'(x) = e^{-x} \Rightarrow f'(0) = e^{-0} = 1$$
$$f''(x) = e^{-x} \Rightarrow f''(0) = e^{-0} = 1$$

Similarly,

$$f^n(x) = e^{-x} \Rightarrow f^n(0) = e^{-0} = 1$$

Then the Maclaurin series is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n$$
$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

18. Evaluate the double integral $\int_0^4 \int_{x=y/2}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$ by applying the transformation $u = \frac{2x-y}{2}, v = \frac{y}{2}$ and integrating over an appropriate region in the uv-plane.

<u>Solution</u>

Here,

$$u = \frac{2x-y}{2} \text{ and } v = \frac{y}{2}$$
$$\Rightarrow u + v = \frac{2x-y}{2} + \frac{y}{2} = \frac{2x}{2} = x$$
$$\therefore u + v = x$$

and,

$$y = 2v$$

We find the boundaries of G by substituting these expressions into the equations for the boundaries of R.

xy boundaries for the boundary of R	Corresponding uv - equations for the boundary of G	Simplified uv-equation
$x = \frac{y}{2}$	$u + v = \frac{2v}{2}$	$u = 0$
$x = \frac{y}{2} + 1$	$u + v = \frac{2v}{2} + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 0$	$2v = 4$	$v = 2$

The Jacobian of the transformation is,

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial(u+v)}{\partial u} & \frac{\partial(u+v)}{\partial v} \\ \frac{\partial(2v)}{\partial u} & \frac{\partial(2v)}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Now,

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} dxdy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u|J(u,v)|\, dudv$$

$$= \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \times 2dudv$$
$$= 2\int_{v=0}^{v=2} \int_{u=0}^{u=1} u\, dudv$$
$$= 2\int_{v=0}^{v=2} \left[\frac{u^2}{2}\right]_0^1 dv$$
$$= \int_{v=0}^{v=2} 1^2 dv$$
$$= \int_{v=0}^{v=2} dv$$
$$= [v]_0^2$$
$$= 2$$

19. Define maximum and minimum of a function at a point. Find the local maximum and local minimum of the function $f(x,y) = 2xy - 5x^2 + 2y^2 + 4x + 4y - 4$.

<u>Solution</u>

Here, the given function is,

$$f(x,y) = 2xy - 5x^2 + 2y^2 + 4x + 4y - 4$$

We know,

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial\left(2xy - 5x^2 + 2y^2 + 4x + 4y - 4\right)}{\partial x} = 2y - 10x + 4$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial\left(2xy - 5x^2 + 2y^2 + 4x + 4y - 4\right)}{\partial y} = 2y + 4y + 4$$

For local extreme values,

$$f_x = 0$$

$or, 2y - 10x + 4 = 0$

$or, \quad -10x + 2y = 4$ …(i)

and,

$$f_y = 0$$

$or, \quad 2x - 4y + 4 = 0$

$or, \qquad 2x - 4y = -4$ …(ii)

Solving (i) and (ii),

$$x = \frac{2}{11} \; and \; y = -\frac{48}{3}$$

$\therefore f$ takes its extreme value at $\left(\frac{2}{11}, \frac{-48}{3}\right)$.

Now,

$$f_{xx} = -10 < 0$$
$$f_{xy} = 2$$
$$f_{yy} = -4 < 0$$

So,

$$f_{xx}f_{yy} - {f_{xy}}^2 = (-20) \times (-4) - 2^2 = 72 > 0$$

Since, $f_{xx} < 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$, there exists a maximum value at $\left(\frac{2}{11}, \frac{-48}{3}\right)$.

Now, the maximum value is,

$$f\left(\frac{2}{11}, \frac{-48}{3}\right) = 2xy - 5x^2 + 2y^2 + 4x + 4y - 4 = 438$$

OR

Find the volume of the region D enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution

The volume is given by,

$$V = \iiint\limits_D dzdydx$$

The integral of $F(x, y, z) = 1$ over D. to find the limits of integration for evaluating the integral, we first sketch the region the surfaces intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2\ or, x^2 + 2y^2 = 4, z > 0$, the boundary of the region P, the projection of D out of the xy-plane is an ellipse with the same equation $x^2 + 2y^2 = 4$.

The upper boundary of R is the curve $y = \sqrt{\frac{4-x^2}{2}}$.

The lower boundary of R is the curve $y = -\sqrt{\frac{4-x^2}{2}}$.

$$V = \iiint_D dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} [z]_{x^2+3y^2}^{8-x^2-y^2} dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} [(8 - x^2 - y^2) - (x^2 + 3y^2)]\, dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} [8 - 2x^2 - 4y^2]\, dydx$$

$$= \int_{-2}^{2} \left[8y - 2x^2y - \frac{4}{3}y^3\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[2(4 - x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)\right]^{3/2} dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)\right]^{3/2} dx$$

$$= \int_{-2}^{2} \frac{16}{3}\left(\frac{1}{\sqrt{2}}\right)^3 (4 - x^2)^{3/2} dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2} dx$$

$= 8\pi\sqrt{2}$ [After integrating with substitution $x = 2\sin x$]

20. Find the solution of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$.

Solution

The given equation is,

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

$$or, (D^2 - D')z = x - y$$

The auxiliary equation is,

$$m^2 - 1 = 0$$

$\therefore m = \pm 1$

Then, the C.F. is,

$$C.F. = \phi_1(y - x) + \phi_2(y + x)$$

Now, the particular integral is,

$$P.I. = \frac{1}{D^2 - D'^2}(x - y)$$

$$= \frac{1}{D^2}\left(1 - \frac{D'^2}{D^2}\right)^{-1}(x - y)$$

$$= \frac{1}{D^2}\left(1 + \frac{D'^2}{D^2} + \cdots\right)(x - y)$$

$$= \frac{1}{D^2}(x - y)$$

$$= \frac{1}{D^2}x - \frac{1}{D^2}y$$

$$= \frac{x^3}{2\times 3} - y \times \frac{x^2}{2}$$

$$\therefore P.I. = \frac{x^3}{6} - \frac{x^2 y}{2}$$

Hence, the general solution is

$$z = C.F. + P.I. = \phi_1(y - x) + \phi_2(y + x) + \frac{x^3}{6} - \frac{x^2 y}{2}$$

OR

Find the particular integral of the equation $(D^2 - D')z = 2y - x^2$ where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$.

Here, the given equation is,

$$(D^2 - D')z = 2y - x^2$$

The auxiliary equation is,

$$m^2 - 1 = 0$$

$$\therefore m = \pm 1$$

Then the complementary equation is,

$$z = \phi_1(y + x) + \phi_2(y - x)$$

Now, the partial integration (P.I.)= $\frac{1}{D^2 - D\prime}(2y - x^2)$

$$= \frac{1}{D^2\left(1 - \frac{D\prime}{D^2}\right)}(2y - x^2)$$

$$= \frac{1}{D^2}\left(1 - \frac{D'}{D^2}\right)^{-1}(2y - x^2)$$
$$= \frac{1}{D^2}\left(1 + \frac{D'}{D^2} + \frac{D'^2}{D^4} + \cdots\right)(2y - x^2)$$
$$= \frac{1}{D^2}\left[(2y - x^2) + \frac{1}{D^2}D'(2y - x^2)\right]$$
$$= \frac{1}{D^2}\left[(2y - x^2) + \frac{1}{D^2} \times \frac{\partial(2y - x^2)}{\partial y}\right]$$
$$= \frac{1}{D^2}(2y - x^2) + \frac{1}{D^4} \times 2$$
$$= 2y \times \frac{x^2}{2} - \frac{x^4}{4} + 2 \times \frac{x^3}{2 \times 3}$$
$$= x^2 y - \frac{x^4}{4} + \frac{x^3}{3}$$

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