Randomness and Computation

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We have $\mathbb{E}(X|Y) = z$?
if $\mathbb{E}(|x|) < \infty$

1. $\mathbb{F}_z \subset \mathbb{F}_y$
2. $\forall A \in \mathbb{F}_Y, \mathbb{E}(\mathrm{II}_A z) = \mathbb{E}(\mathrm{II}_A X)$

Note, that

$$\mathbb{E}(X|A) = \frac{\mathbb{E}(x\mathrm{II}_A)}{\mathbb{P}(A)}$$

Compatible with the previous definition, when y is discrete. Where $\mathbb{E}(|X|) < \infty$ then, $(w) = \mathbb{E}(X|\{y = y_i\})$ and $y_i : y(w) = y_i$

- $\mathbb{E}(X + Y|Z) = \mathbb{E}(X|Z) + \mathbb{E}(Y|Z)$
- $\mathbb{E}(aX|Y) = a\mathbb{E}(X|Y)$
- $\mathbb{E}(P(y)X|Y) = f((y)\mathbb{E}(X|Y))$
- $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{P}(y = y_1)\mathbb{E}(X|Y = Y_1) \\
&= +\mathbb{P}(y = y_2)\mathbb{E}(X|Y = Y_2) \\
&= +\mathbb{P}(y = y_3)\mathbb{E}(X|Y = Y_3) \\
&= +...
\end{aligned}$$

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x If x8y are jointly continuous.
$f_y(y \neq 0)$ and if $\mathbb{E}(|X|) < \infty$, then $\mathbb{E}(X|Y) = G(y)$

$$G(y) = \int_{-\infty}^{\infty} x\frac{f_{xy}(x, y)}{f_y(y)}dx$$

which we arrived at from

$$\begin{aligned}
\mathbb{P}(Y \in A) &= \mathbb{P}\big((x, y) \in R \times A\big) \\
&= \int_{-\infty}^{\infty}\int_A f_{xy}(x, y)dydx \\
f_y(y) &= \int_{-\infty}^{\infty} f_{xy}(x, y)dx
\end{aligned}$$

<u>*Exercise :*</u>

$$f_{xy}(x, y) = \begin{cases} k(x + y) & : x, y \in [0, 1] \\ 0 & : otherwise \end{cases}$$

<u>*Answer :*</u> From the first property, we use the theorem:

Th: if $F_z \subset F_y$
then $z = f(y)$ for some measurable function f.

If the questions that z generates are contained in $f(y)$, then points mapped previously to the arrival space may be mapped to the same point.

1. This implies $\mathbb{E}(X|Y) = G(y)$
2. if true for $\forall a \in F_y$ then let $A = \{y \in [0,a]\} \in F_y$ then $\mathbb{E}(\mathrm{II}_A z) = \mathbb{E}(\mathrm{II}_A x)$ must be true and equal to

$$\mathbb{E}(\mathrm{II}_{Y\in[0,x]} G(y)) = \mathbb{E}(\mathrm{II}_{Y\in[0,x]} X)$$

Now we can start doing some calculations. Lets calculate f_y.

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)dx = k(\frac{1}{2} + y), y \in [0,1]$$

$$\int_{-\infty}^{\infty} \mathrm{II}_y \in [a,a] \cdot G(y) f_y(y) dy = \int_0^x G(y)k(\frac{1}{2} + y)dy$$

This is the same an important equivalence:

$$\mathbb{E}(\mathrm{II}_A y \in [0,x]G(y)) = \mathbb{E}(\mathrm{II}_{y\in[0,a]} x)$$

is equivalent to:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{II}_{y\in[0,a]} x f_{xy}(x,y)dxdy = \int_{0a}^{a}\int_0^1 xk(x+y)dxdy$$

Thus:

$$Eq\mathrm{I} = Eq\mathrm{II}\forall a \implies \frac{d}{da}\mathrm{I} = \frac{d}{da}\mathrm{II}$$

$$\begin{aligned} G(y)k(\frac{1'}{2} + y) &= \int_0^1 k(x+y)dx \\ &= k(\frac{1}{3} + \frac{1}{2}y) \end{aligned}$$

From which we can conclude:

$$\begin{aligned} G(y) &= \frac{\frac{1}{3} + \frac{1}{2}y}{\frac{1}{2} + y} \\ \mathbb{E}(X|Y) &= \frac{\frac{1}{3} + \frac{1}{2}y}{\frac{1}{2} + y} \end{aligned}$$

If we use the formula we have, we can check the results we have.

Formula:

$$G(y) = \int_{-\infty}^{\infty} \frac{f_{xy}(x,y)}{f_y(y)} dx$$

Confirm results:

$$\begin{aligned} G(y) &= \int_{-\infty}^{\infty} x \frac{f_{xy}(x,y)}{f_y(y)} dx \\ &= \frac{\int_0 k(x+y)xdx}{\int_{-\infty}^{\infty} kx(x+y)dx} \\ &= \frac{\frac{1}{3}+\frac{1}{2}y}{\frac{1}{2}+y} \end{aligned}$$