UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2011:25

Modeling and simulation of highway traffic using a cellular automaton approach

Ding Ding

Examensarbete i matematik, 30 hp
Handledare och examinator: Ingemar Kaj
December 2011

Department of Mathematics
Uppsala University

Modeling and simulation of highway traffic using a cellular automaton approach

Ding Ding

Abstract

The purpose of this paper is to discover how Cellular Automata (CA) can be applied to traffic flow simulations. First, we introduce the three types of traffic model: microscopic traffic model, macroscopic traffic model and mesoscopic traffic model. Second, to evaluate dynamic traffic flow, we developed a traffic flow simulator that uses cellular automata model. We extend the existing CA models to describe the influence of a car accident in single-lane and double-lane traffic flow model. We also add the lane changing rules to simulate the reality traffic condition. By simulation, we analyze all possible situations. The simulation was implemented in Matlab programming language.

1. Main Features of Traffic Stream

Traffic phenomena are an important question in modern society. Investigating on regular pattern of traffic flow has significant meaning. Analysis and simulation of traffic flow can be wildly applied in Transportation Planning, Traffic Control and Traffic Engineering. In the early 90S, New York City government decided to construct the tunnel to New Jersey. After analyzed and modeled the traffic flow, they adjusted traffic management strategy which increased the capacity of current existing facilities. So the tunnel construction was avoided.

Traffic stream is complex and nonlinear and defined as multi-dimensional traffic lanes with flow of vehicles over time. Traffic phenomena are complex and nonlinear. Vehicles followed each other on each lane and they can choose different lane when the former position is empty. There are three main characteristics to visualize a traffic stream: speed, density, and flow.

1.1. Speed, Density and Flow

Speed (*V*) is defined as travel distance per unit time in traffic flow. The precise speed of each car is difficult to measure. In practice, we calculate the average speed of the sample vehicles. In a time space diagram, time is measured along the horizontal axis and distance is measured along the vertical axis. The velocity of the traffic stream equals to the slope of the traffic trajectory (*v=dx/dt*).The figure below shows the nonlinear traffic stream.

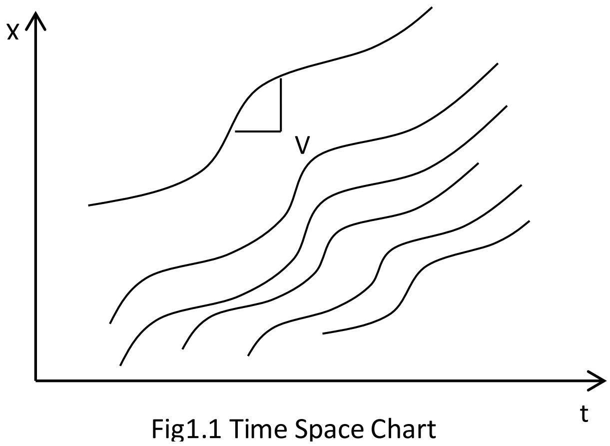


Fig1.1 Time Space Chart

The common method speed is to calculate the time mean speed. Time mean speed is measured by the average speed of a traffic stream passing a fixed point along a roadway over a fixed period of time. Time mean speed can be sampled by loop detectors and other fixed-location speed detection equipment. The time-mean speed can be calculated as:

$$v_t = \frac{1}{m}\sum_{i=1}^{m} v_i \,,$$

Where *m* is the number of vehicles passing the fix point, v_i is the speed of the passing vehicles.

The density (*K*) of the traffic flow is defined as the numbers of vehicles per unit road. Inverse of the density is spacing, which corresponds to the distance between two vehicles $k = 1/s$, where *k* represents density, s represents spacing.

There are two major densities in the traffic stream: critical density and jam density. The critical density the maximum density for unlimited flow and the jam density is the density under congestion. In a roadway which with the length *L*, the density of the traffic equals to the numbers of the vehicles at time *t* divides the roadway length. The density is also the inverse of the spacing of the vehicles.

Flow (*Q*) of the traffic stream is defined as the numbers of vehicles per unit of time. In practice, it usually counts as hour. Flow can be calculated as the inverse of the interval of time between continuous vehicles:

$$q = \frac{1}{h_{i+1} - h_i},$$

where *q* represents flow, *h* represents the *i-th* vehicle pass the settle point

There exists inverse relationship between the density and flow. If the traffic has high density then the flow will be low. The relationship can be shown as below

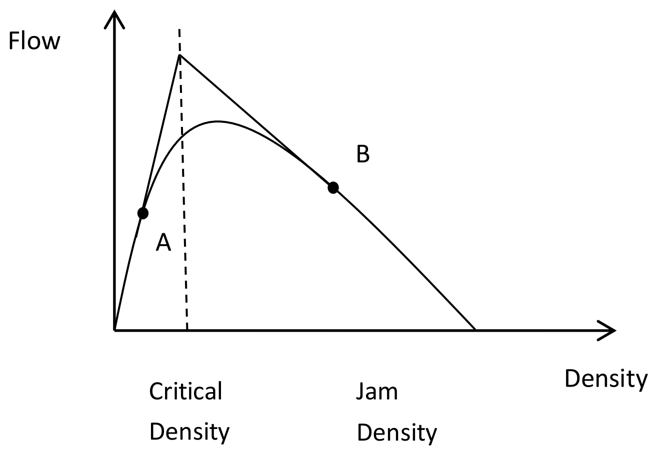


Fig 1.2 Density and Flow Relationship Chart

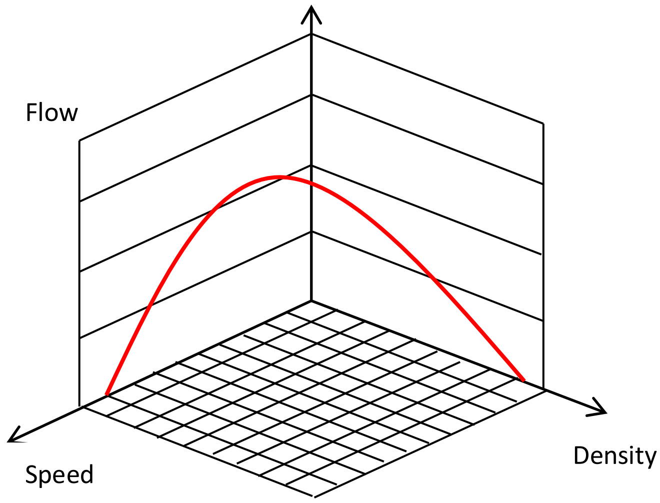


Fig 1.3 Flow, Density and Density Relationship Chart

The relation between density and flow is not as apparent. Under an uninterrupted situation, the relationship between speed, density and flow can be presented as

$$Q = V \cdot K$$

Where *Q*=Flow (vehicles/hour), *V*=Speed (miles/hour, kilometers/hour), *K*=Density (vehicles/mile, vehicles/kilometer).

The traffic flow is depended on the speed and density. The following diagram shows the relationship between speed, density and flow.

1.2. Traffic Congestion

Traffic congestion is a condition on roadway which can present as slower speeds, longer travel times, and increased vehicular queuing. It has caused a lot of inconvenience to people's life and work. When traffic demand is great enough that the interaction between vehicles slows the speed of the traffic stream, congestion is incurred. Time-space diagrams can illustrate the congestion phenomenon. Traffic congestion will move up stream. Congestion waves will vary in propagation length, depending upon the upstream traffic flow and density. The figure 1.4 shows how the congestion move.

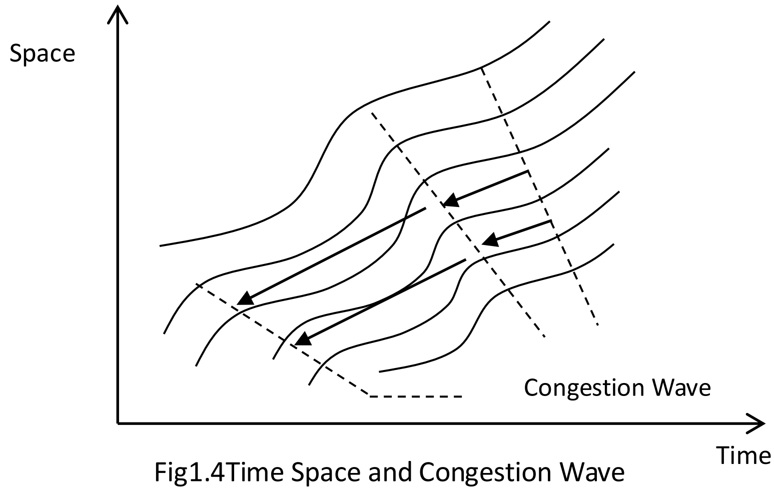


Fig1.4Time Space and Congestion Wave

2. Three Types of Traffic Model

The former research on traffic modeling can be classified as three parts: Microscopic modeling, mesoscopic modeling and macroscopic modeling.

Microscopic traffic flow models simulate single vehicle-driver units, based on driver's behavior. The dynamic variables of the models represent microscopic properties like the position and velocity of the vehicles. There are two modeling approach are known as Car-following model and Cellular automaton model. Richards (1956) establish the Car-following models which are defined by ordinary differential equations describing the vehicles' positions and velocities. Newell (1961) set up an optimal velocity base on a distance dependent velocity. Cellular automaton models describe the dynamical properties of the system in a discrete setting. It consists of a regular grid of cells. For traffic model, the road is divided into a constant length Δx and the time is divided in to steps of Δt. Each grid of cells can either be occupied by a vehicle or empty.

Macroscopic traffic flow model study the characteristics of traffic flow like average velocity, density, flow and mean speed of a traffic stream. The first major step in macroscopic modeling of traffic was taken by Lighthill and Whitham (1955). They establish the L-W model which indexed the comparability of 'traffic flow on long crowded roads' with 'flood movements in long rivers'. Richards (1956) complemented the model by introducing of 'shock-waves on the highway' into the model as an identical approach known as the LWR model. Payne (1971) changes the microscopic variables to macroscopic scale. Helbing (1996) proposed a third order macroscopic traffic model with the traffic density, velocity and variance on the velocity.

Mesoscopic models combine the properties of both microscopic and macroscopic models. Mesoscopic models simulate individual vehicles separately, but use the macroscopic view to express their activities and interactions. The classic model is the Gas-Kinetic based model.

1.1. Microscopic traffic model: Car-Following Model

The Car-following model describes the dynamics between the vehicles' positions and the velocities. The purpose of the model is to determine how cars follow another in the road. The basic assumption of this model is that the vehicle will maintain a minimum time and length between each other. If the front car changes its speed then the following cars will also change speed.

The speed of the vehicle n is denoted as $\frac{dx_n(t)}{dt} = \dot{x}_n(t)$

Acceleration of the vehicle n is denoted as $\frac{d\dot{x}_n(t)}{dt} = \frac{d^2x_n(t)}{dt} = \ddot{x}_n(t)$

Chandler et al (1958) first developed the linear car- following model. The model can be express as

$$\ddot{x}_{n+1}(t+T) = \alpha[\dot{x}_n(t) - \dot{x}_{n+1}(t)]$$

Where

α : Sensitivity Coefficient

$\ddot{x}_{n+1}(t+T)$: Acceleration of the (n+1) th car at time $(t+T)$

$\dot{x}_n(t)$: The speed of the (n) th car at time t

$\dot{x}_{n+1}(t)$: The speed of the (n+1) th car at time t

Chandler et al (1959) discussed the stability of the linear model and define two kinds of the stability: Local Stability and Asymptotic Stability. Local Stability refers the stability of the following car distance. Asymptotic Stability refers the velocity fluctuation of the following car. The expressions is

$$C = \alpha T$$

C represents the characteristics of distance between the two cars. If C becomes smaller, the distance between the cars is smaller and traffic stream is more stable. For Local Stability, when $C \leq 1/2$ the traffic stream is almost stable. If $C > 1/2$, the traffic stream is turbulent.

Gazis et al. (1961) developed the non-linear car-following model, known as the General Motors Nonlinear Model. The model is given by

$$\ddot{x}_{n+1}(t+T) = \alpha \frac{x_{n+1}^m(t+T)}{[x_n(t) - x_{n+1}(t)]^l}[\dot{x}_n(t) - \dot{x}_{n+1}(t)]$$

l : Sensitivity Coefficient of the distance $\left(x_n(t) - x_{n+1}(t)\right)$

m : Sensitivity Coefficient of the speed $\left(\dot{x}_n(t) - \dot{x}_{n+1}(t)\right)$

1.2. Macroscopic traffic model: LWR model

The LWR model (lighthill and Whitham, 1995; Richard, 1956) describes the traffic flow by using fluid dynamic differential equation. The law of the conservation of the vehicles in traffic can be shown as

$$\frac{n(x)\partial\, C(x,t)}{\partial t}+\frac{\partial\, q(x,t)}{\partial x}=0$$

$C(x,t)$: Traffic density in vehicles per lane per kilometer at location x and at time t

$n(x)$: The numbers of lanes at position x.

$q(x,t)$: The traffic flow (traffic intensity) in vehicles per hour at location x and at time t.

The aggregated variable $C(x,t)$ and $q(x,t)$ are continuous functions of space and time.The equation expresses the physical principle of the traffic flow. The traffic flow can also be expressed in terms of the traffic density and the traffic speed.

$$q(x,t)=C(x,t)\cdot v(x,t)\cdot n(x)$$

Lighthill, Whitham and Richard observed that

$$v(x,t)=F(C(x,t))$$

The LWR model is continuous in both time and space. The analytical solution of the model can be hard to calculate. The practical traffic flow is discrete both in time and space. The discretization of the LWR model with time step Δt is

$$C_j(k+1)=C_j(k)+\frac{\Delta t}{l_j n_j}[q_{in,j}(k)-q_{out,j}(k)]$$

$C_j(k)$: The average traffic density in space section j and in time period k.

Δt : The time step.

l_j : The length of the section j.

n_j : The number of lanes.

$q_{in,j}(k)$: The inflow in section j and in period k.

$q_{out,j}(k)$: The outflow in section j and in period k.

There are two major drawbacks of the LWR model. First, the model first assumed there is equilibrium in the traffic flow. In practice, the traffic flow is more complex. It might be impossible to prove the existence of the equilibrium. Second, the LMR model does not consider the external conditions such like road condition and the micro-condition such like driving behavior.

1.3. Mesoscopic traffic model: Gas-Kinetic Traffic Flow Model

A gas-kinetic traffic flow model describes the heterogeneous traffic flow operations. The former study has shown that the expression reflecting vehicle interactions in traditional models is only valid for dilute traffic. The kinetic theory treats the vehicles as gas particles. The unconstrained and constrained traffic are governed by continuum and non-continuum processes. The simulation by using gas-kinetic dynamics can be well fit in mesoscopic traffic flow. The continuum process reflects the smooth changes such like acceleration. The non-continuum process reflects the violent fluctuations such like deceleration. There are various versions of the Gas-Kinetic models have been developed to extend the adaptability. Prigogine and Herman (1971) have proposed the Boltzmann equation for traffic flow.

$$\frac{\partial\, f(x,v,t)}{\partial\, t} + v\frac{\partial\, f(x,v,t)}{\partial\, x} = -\frac{f(x,v,t) - \rho(x,t)F_{des}(v)}{\tau_{rel}} + \left(\frac{\partial f(x,v,t)}{\partial\, t}\right)_{int}$$

$$\left(\frac{\partial f(x,v,t)}{\partial\, t}\right)_{int} = \int_{w>v} dw[1-\hat{p}(\rho)]|w-v|f(x,w,t)f(x,v,t) - \int_{w<v} dw[1-\hat{p}(\rho)]|v-w|f(x,v,t)f(x,w,t)$$

$f(x,v,t)$: Velocity distribution function

τ_{rel} : Relaxation time

$F_{des}(v)$: Desired velocity distribution

Paver-Fontana (1975) improved the model by taking into account the different personalities of the drivers and proposed generalized gas-kinetic traffic flow model.

$$\frac{\partial\, g(x,v,v_{des},t)}{\partial\, t} + v\frac{\partial\, g(x,v,v_{des},t)}{\partial\, x} + \frac{\partial}{\partial\, v}\left[\left(\frac{v_{des}-v}{\tau}\right) g(x,v,v_{des},t)\right] = +\left(\frac{g(x,v,v_{des},t)}{\partial\, t}\right)_{int}$$

$$\left(\frac{g(x,v,v_{des},t)}{\partial\, t}\right)_{int} = f(x,v,t)\int_v^\infty dv'(1-p_{pass})(v'-v)g(x,v,v_{des},t) - g(x,v,v_{des},t)\int_0^v\int_v^\infty dv'(1-p_{pass})(v-v')f(x,v,t)$$

Where $f(x,v,t) = \int_0^\infty dv_{des}\, g(x,v,v_{des},t)$

p_{pass} : The probability of passing.

$g(x,v,v_{des},t)$: Velocity distribution function.

3. Cellular Automaton model

Cellular automaton model is one of the microscopic traffic models. In this model, a roadway is made up of cells like the points in a lattice or like the checkerboard and time is also discredited. Vesicles move from on cell to another. The first research using Cellular Automaton model for traffics simulation was conducted by Nagel and Schreckenberg (1992). They simulate the single-lane highway traffic flow by a stochastic CA model. The basic rule of the traffic flow is that each vehicle move v sites at each time. The velocity v will add 1 if there is no cars v space ahead and slow down to $i-1$ if there is another vehicle i spaces ahead. The velocity will slow down randomly with the probability p. There are some CA models have been quiet used, like Nagel-Schreckenberg model (1992) and BJH model (Benjamin, Johnson and Hui 1996).

In the CA model, the street is divided into cells at a typical space which is the space occupied by vehicles in a dense jam. The space is depended by car length and distance to the preceding car. Each cell can be occupied at most one car or empty. There exist a maximum speed v_{max} and the velocity of each car can take the value between $v = 0,1,2,\dots,v_{max}$.

The simplest traffic CA model is developed by Wolfram (1986, 1994) and Biham et al (1992). The model is described as the asymmetric simple exclusion model on one dimensional roadway. The formula is as following

$$x_i(t+1) = x_i(t) + \; min(1, x_{i+1}(t) - x_i(t) - 1)$$

In this model, the vehicle moves to forward cell if the cell ahead is not occupied. Then the velocity of all vehicles adds 1 simultaneously. The velocity is either one or zero. Fukui and Ishibashi (1996) proposed extension of this model.

$$x_i(t+1) = x_i(t) + \; min(v_{max}, x_{i+1}(t) - x_i(t) - 1)$$

The model makes the assumption there exists the maximum speedv_{max}.

There are four steps movement in the simplest rule set, which leads to a realistic behavior, has been introduced in 1992 by Nagel und Schreckenberg.

Step 1.All the vehicles whose velocity has not reached the maximum v_{max} will accelerate by one unit.

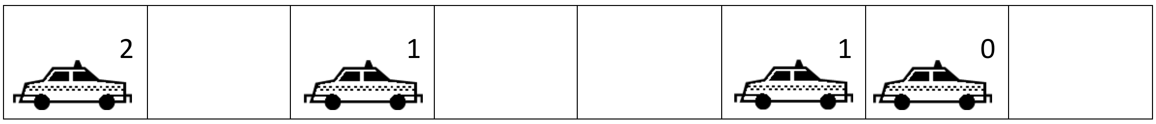
Step 2.Assume a car has m empty cells in front of it. If the velocity of the car (v) is bigger than m, then the velocity becomes tom. If the velocity of the car (v) is smaller than m, then the velocity changes to v. $(v \to min[v, m])$

Step 3.The velocity of the car may reduce by one unit with the probability p.

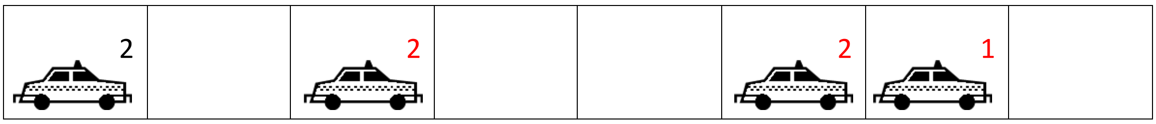
Step 4. After 3 steps, the new position of the vehicle can be determined by the current velocity and current position. $(x_n{}' \to x_n + v_n)$

The following figure shows the four steps movements.

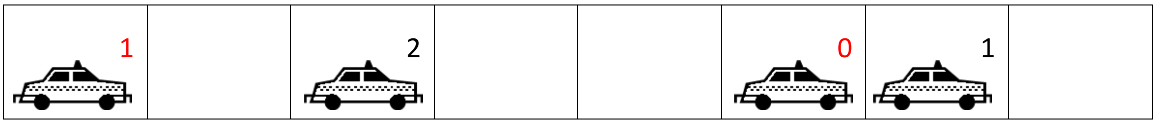
Configuration at time t:



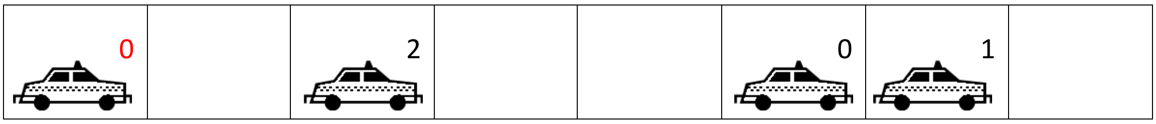
Step 1 Acceleration ($v_{max} = 2$)



Step 2 Safety distance



Step 3 Randomization



Step 4 Driving

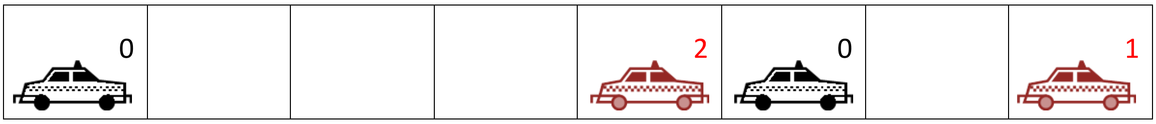


Fig 3.1 Movement of CA Model

The mathematical formula can be shown as

$$v_{i+1} = max\{0, min(v_{max}, d_i - 1, v_i + 1) - \xi_i{}^{(t)}\}$$

$$x_i(t+1) = x_i(t) + \ max\{0, min(v_{max}, x_{i+1}(t) - x_i(t) - 1, x_i(t) - x_i(t-1) - 1 + 1) - \xi_i{}^{(t)}\}$$

$\xi_i{}^{(t)}$: the Boolean random variable. $\xi_i{}^{(t)} = 1$ with the probability *p*, $\xi_i{}^{(t)} = 0$ with the probability *1-p*.

4. Simulation of Traditional Cellular Automaton Model

The one-lane highway traffic model is based on the former Cellular Automaton model. There exists one highway which is a close boundary system. The highway is divided in equal size cells. Each cell can either occupy one vehicle or is empty. Each vehicle can be described by position and velocity. x_i is the position of i th vehicle and v_i is the velocity of i th vehicle. Before each movement, we first define the gap between successive vehicles. gap_i is the gap space between i thvehicle and i- 1 th vehicle. There are four steps in the model.

4.1 Rules and Algorithm

Acceleration: If the speed of i thvehicle$\mathrm{v_i}$ is lower than the maximum speed v_{max}, then the speed will increase by a. But the speed will remains smaller than v_{max}. *a* is the acceleration rate. The rule is given as:

$$v_i \rightarrow min(v_i + a, v_{max})$$

Deceleration: The vehicles reduce speed reduce its speed if the front gap is not enough for current speed. The speed will reduce to $gap_i - 1$. The rule is given as:

$$v_i \rightarrow min(v_i, gap_i - 1)$$

Where $gap_i = x_i - x_{i-1}$

Randomization: In the model, driver will decrease the speed randomized. If the $v_i \geq 0$, then the speed of i th will reduce the speed one unit with the probability *p*. According to

D.Chowdhury, L.Santen and A. Schadchneider (2000), realistic data shows city traffic has a higher value of random probability than the number in highway traffic. For city traffic, we choose the probability of randomization $p = 0.5$. For highway traffic, we choose the probability of randomization $p = 0.3$. The rule is given as:

$$v_i \rightarrow max(v_i - 1,0)\ \ with\,probabilty\,p$$

Move: After 4 steps, the new position of the vehicle can be determined by the current velocity and current position

$$x_i \rightarrow x_i + v_i$$

Parameters are defined as follow:

a is the acceleration rate

$gap_i{}^f = x_i - x_{i-1}$ the front gap

$x_i{}^t$ is the position of $i\ th$ vehicle at time t.

x_{ip} is the incident place.

T1 is the time when accident happens.

T2 is the time when accident end.

v_i is the velocity of $i\ th$ vehicle.

v_{max} is the maximum speed.

p is the probability of decrease speed, here we choose 0.25.

q is the probability to switch lane.

L is the length of road.

Input
The length of the highway, The length of the cell, Maximum velocity, Initial density, Incident details and Driver behavior probability p

Initialization
Generate initial vehicles

Begin
Calculate gap

$$\mathrm{gap}_i{}^f = x_i - x_{i-1}$$

Acceleration

$$v_i \rightarrow min(v_i + 1, v_{max})$$

Deceleration

$$v_i \rightarrow min(v_i, gap_i - 1)$$

Randomization

$$v_i \rightarrow max(v_i - 1,0)\ \ with\,probabilty\,p$$

Vehicle position update

$$x_i \rightarrow x_i + v_i$$

Vehicle generation

Fig 4.1 Algorithm for one-lane model

To simulate the one-lane traffic model, we first need to input parameters: the length of the highway, the length of the cell, numbers of vehicles, maximum velocity, the initial density, incident details and driver behavior probability p. According to D.Chowdhury, L.Santen and A. Schadchneider (2000), realistic data shows city traffic has a higher value of random probability than the number in highway traffic. For city traffic, we choose the probability of randomization $p = 0.5$. For highway traffic, we choose the probability of randomization $p = 0.3$. We set the number of cells is 2000 which is also the length of road. Each cell can be either occupied or empty. We randomly generate the position and speed of cars. And the number of cars is defined by the density. Each car will follow the 4 rules (Acceleration, Deceleration, Randomization, and Move) and make the movement. We use two different methods to compare the models: Flow-Density diagram and Time-Space diagram.

4.2 Simulation

We use Matlab to simulate the one-lane traffic flow. First, we define there are 500 cells in the roadway. Second, we randomly generate the position and speed for each car. We set the parameter *p=0*, which means there is no chance car driver will slow down the speed. Without the slowdown step, the Cellular Automaton can avoid the noise. We model 500 steps movement and choose the last 100 movements. We use Time-Space Diagrams to show the movements of vehicle. The Time-Space diagram shows as following:

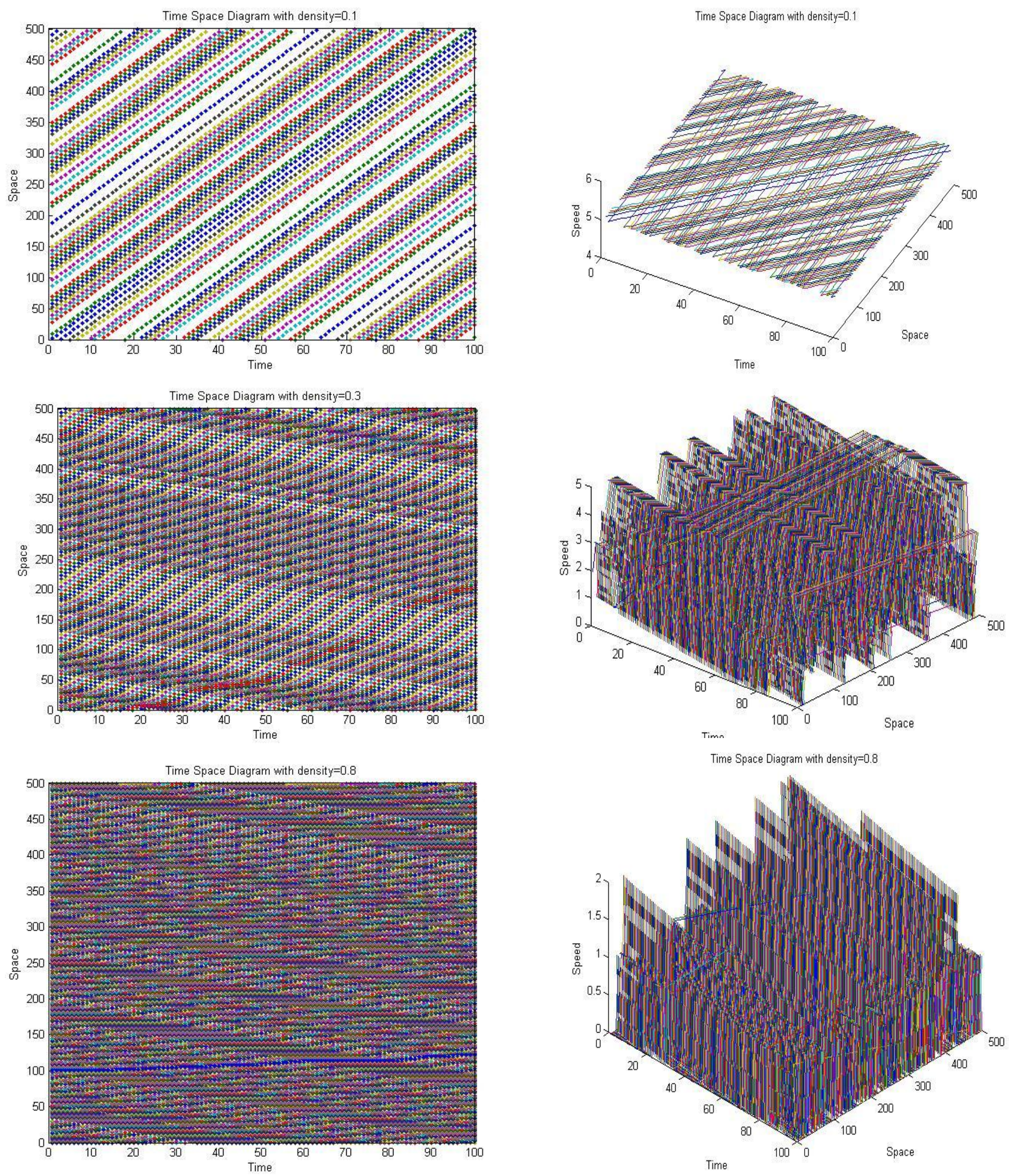


Fig 4.2 Time Space Speed Diagram with density=0.1, 0.3, 0.8

Figure 4.2 are Time Space Diagram for one-lane model with maximum speed=5, total cells number=500, density=0.1, 0.3, 0.8 and probability slow down=0.3. We choose the last 100 steps of 500 steps. Left diagrams are the Time-Space diagram and right diagrams are the Time-Space-Speed diagram. From the figure we can see the traffic flow movements and congestion. Congestion happened in the place where the thick lines cluster. Traffic congestion tend to move up stream. At the low density, there exists little congestion. At high level, there exists lot congestion and the speeds of the vehicles are at lower level.

To see the movement of a single vehicle, we choose the one third position of the whole traffic flow. The following picture shows the vehicle movement at different density.

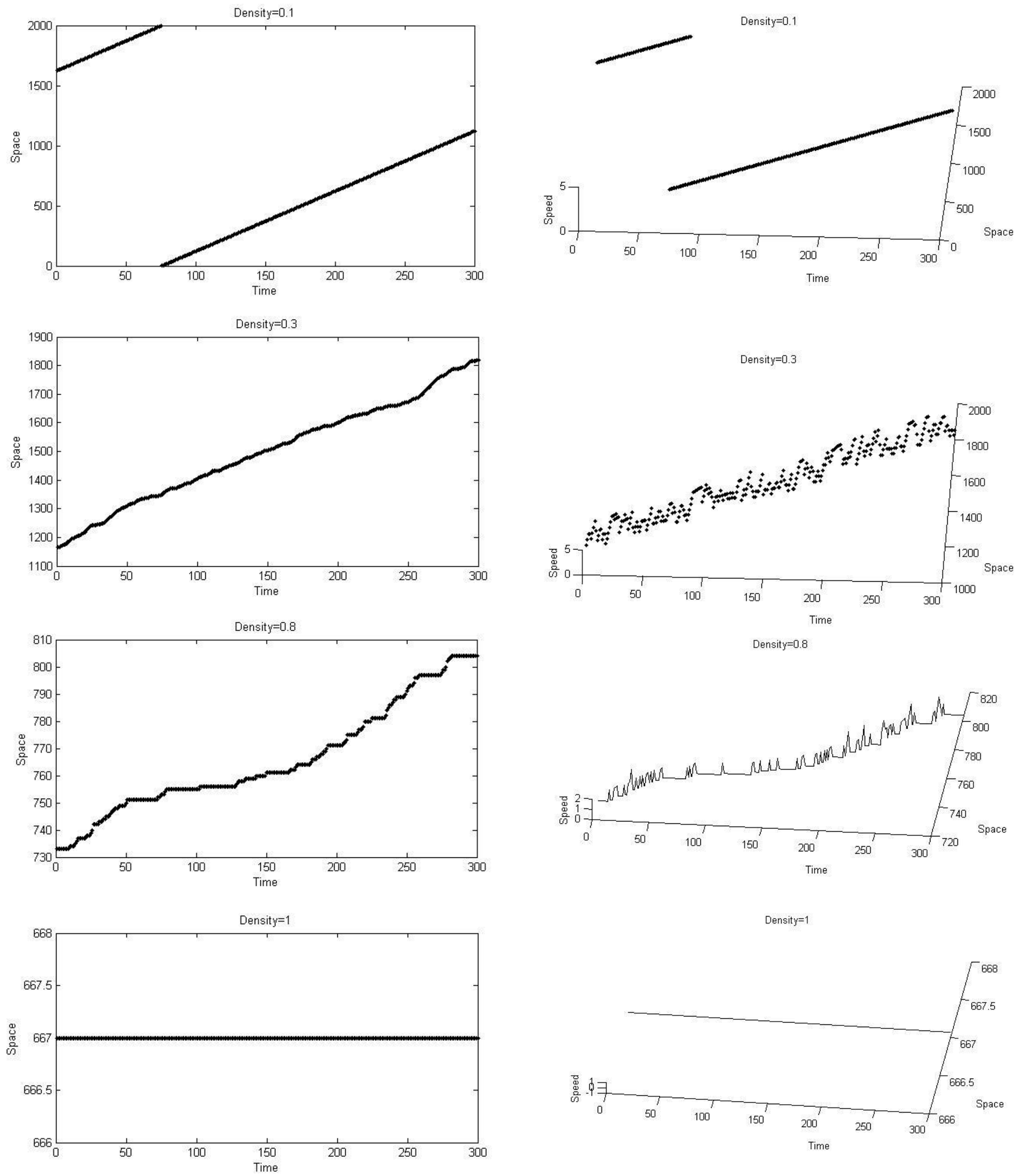


Fig 4.3 Time Space and Time Space Speed Diagram with density=0.1, 0.3, 0.8, 1

Figure 4.3 shows the movement at the one third position of the whole traffic flow. At density 0.1, the target vehicle is unobstructed and keeps the max speed 5. At density 0.3, there is some congestion in the roadway and the target vehicle move slower than before. The vehicle keeps a high speed for the most time. At density 0.8, the congestion is serious. The target vehicle move slow and its speed are under 2. At density equal to 1, all vehicles cannot move.

We define the flow at a given time step as $flow = \frac{\sum K_i V_i}{L}$, where *L* is the total number of cars, K_i is the density and V_i is the speed of $i\text{-}th$ car. To study the relationship between density and flow, we calculate the flow under different density and max speed. The result can be shown in following picture.

Figure 4.4 shows the relationship between flow and density with different maximum speed. We can tell from the figure, the higher maximum speed, kurtosis of curve is higher. The critical density is 0.5, 0.33, 0.26, 0.2, and 0.17 for max speed 1 to 5. Critical density is the density when the flow is biggest.

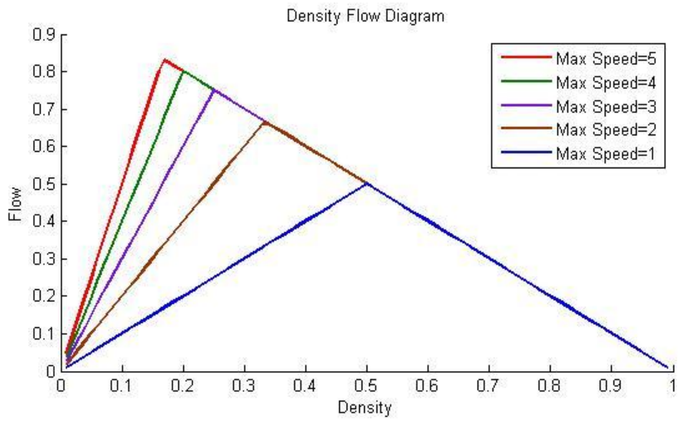


Fig 4.4 Density-Flow Chart

5. Incident Simulation

2.1. Incident Occurrence Rule

If any accident happened in the road, the vehicles in the upstream at current time and downstream at next time are blocked. The vehicles start queue from the incident place. We define $x_{incident}$ is the incident place. *T1* is the time when accident happens. *T2* is the time when accident end. The rule is given as:

$$v_{incident} = 0 \;\; for\, T1 \leq t \leq T2$$

2.2. Simulation

To simulate the accident in one lane, we assume that the accident happen between *T1* and *T2. T1* is the time when accident happens. *T2* is the time when accident end. The vehicles in the upstream at current time and downstream at next time are blocked. In the last 100 steps, we assume that the accident happens at time 30 and end at time 70. The incident place is in the middle of the roadway. After the accident happened, the downstream traffic flow blocked from the incident place. The speed of the downstream traffic reduces to zero. Serious congestion begins. After time 70, accident is excluded and traffic flow begins to start again. At simulation we assume the incident happened in the middle of the traffic steam.

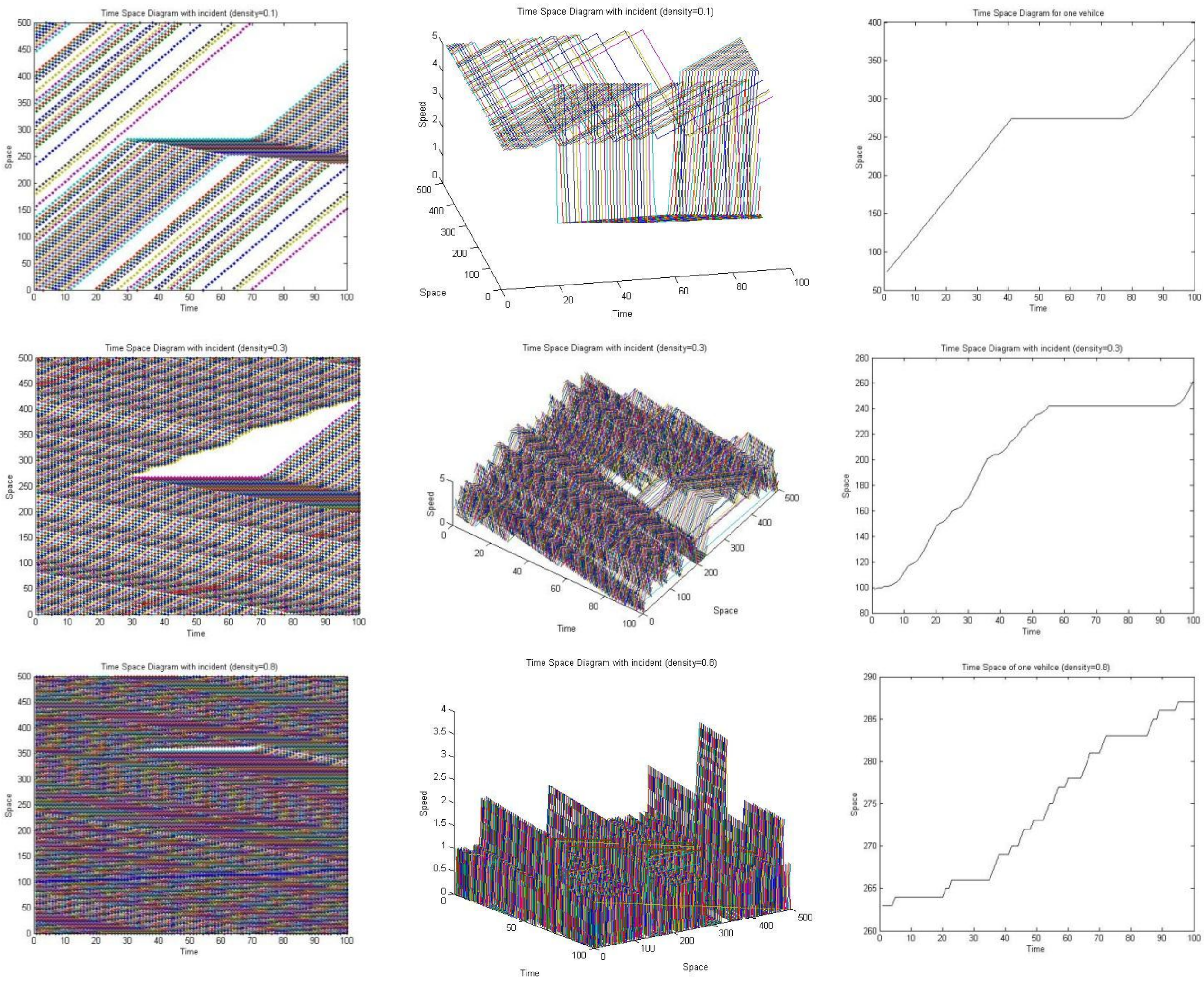


Fig 5.1 Time-Space Diagrams with accident One-lane, V_max=5, L=500, K=0.1,0.3,0.8, P=0, Steps=100

Figure 5.1 shows the Time-Space Diagram when accident happens with maximum speed=5, total cells number=500, density=0.1, 0.3 and 0.8.Left chart are the time space diagram for whole traffic stream. Middle picture is the 3D time space diagram for whole traffic stream. Right chart is the trajectory of one vehicle in a third of the traffic stream. From the figure, we can tell from time step 30-70 there exists serious congestion. The entire vehicles stop during the accident.

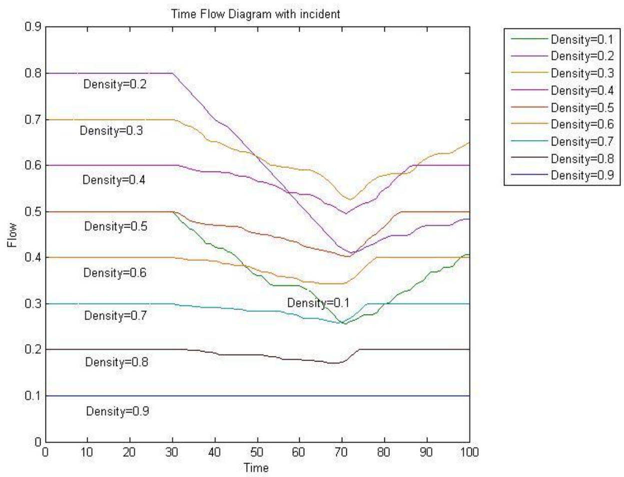


Fig 5.2 Time-Flow Diagrams with accident One-lane, V_max=5, L=500, P=0, Steps=100

Figure 5.2 shows the relationship with time and flow under different density. We can tell that the flow reduce when the accident happens at time 30. During the accident period (time steps 30-70), the capacity of the traffic flow is zero. When the accident excludes at time 70, we can see that traffic flow begins to restore.

6. Application of Single lane model

In the previous study, we assume an ideal situation. The roadway is separated into several cells. Each cell can be occupied by one vehicle or not. The max speed is defined from 1 to 5. Typical values to model highway traffic are $v_{max} = 5,\ p = 0.25$. In the deterministic limit ($p = 0$), the Nagel-Schreckenberg model shows a sharp transition between the free flow stage and congested flow stage at a critical density $K_c = 1/(v_{max} + 1)$. In the free flow stage, there is almost no traffic jam in the flow. The speed vehicles in the free flow stage are close to the maximum speed. In the congested flow stage, there exists some congestion. The congestion condition will get worse with the density increase.

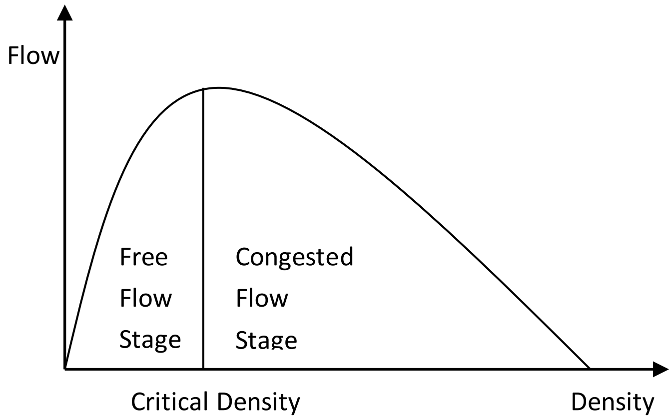


Fig 6.1 Free Flow Stage and Congested Flow Stage

Under the steady stage in which $p = 0$. There is no random slow down. The rule for speed can be written as

$$v_{i+1} = min(v_{max}, d_i, v_i + 1). \tag{1}$$

For density is small than the critical density ($K < K_c$), the speed vehicles are close to the maximum speed. Hence the speed is satisfied by

$$v_{i+1} = v_{max},\ \ K < K_c. \tag{2}$$

For density is large than the critical density ($K_c < K$), vehicles' speed is determined by the gaps. Hence the speed is satisfied by

$$v_{i+1} = d_i,\ K_c < K. \tag{3}$$

The average gap in the congestion flow is equal to the numbers of the empty cells ($L - KL$) divided into cells for vehicles (KL). The Hence the gap is satisfied by

$$d_{average} = \frac{L-KL}{K} = \frac{1-K}{K}. \tag{4}$$

We can calculate the critical density when the $v_{max} = \frac{1-K}{K}$. We get the critical density is $K_c = 1/(v_{max} + 1)$. There is an analytic relation between velocity and density as follows:

$$V = \begin{cases} V_{max} & K \leq \frac{1}{v_{max}+1} \\ \frac{1-K}{K} & K > 1/(v_{max}+1) \end{cases} . \quad (5)$$

The flow of the traffic can be calculate by the average speed multiply the density $Q(K) = K \cdot V(K)$. Combining the equation (2), (3) and (4), the flow can be express as

$$Q(K) = min\left[v_{max} \cdot K, \frac{1-K}{K} \cdot K\right] = min[v_{max} \cdot K, 1 - K]$$

We compare the critical density by formula and simulation as following table:

Max Speed (p=0)	Critical density of the Block (Simulation)	Critical density of the Block (Formula)
1	0.5	0.5
2	0.33	0.33
3	0.26	0.25
4	0.2	0.2
5	0.17	0.167

Table 6.2 Critical density of the Block by Simulation and Formula

7. Modeling and simulation of single-lane highway traffic with open boundary and queuing system

The traditional CA model has a close boundary for each time step the cars leaving the system will entry the road immediately. The initial cars will forever stay in the road. This flaw does not meet the reality. In the new model, we set the following rules:

- For each time step, a car will come to the road with a probability λ.
- If the cars can not entry into the road, they will line up in the entrance.
- The length of queuing is l.
- The cars reach the end of the road will leave.

Fig 7.1 Movement of improved CA Model

The above can shows the situation of the road.

There is a fixed density which defines the initial number of cars in the roadway. After the initialization, the cars will move follow the rules for each time step. The new speed of each vehicle will be decided by the gap, forward speed and maximum speed. The cars reach the end of the road will leave and never come back. For each time step, a new car will come with the probability λ. If there are no empty space in the first grid of the road, the car will wait outside the road. The length of the queue will be l. The left side shows the algorithm for the improved one lane model.

Input

The length of highway, The length of the cell, Maximum velocity, Initial density, Incident details and probability p

Initialization

Generate initial vehicles

Begin

Calculate gap

$$gap_i{}^f = x_i - x_{i-1}$$

Acceleration

$$v_i \rightarrow min(v_i + 1, v_{max})$$

Deceleration

$$v_i \rightarrow min(v_i, gap_i - 1)$$

Randomization

$v_i \rightarrow max(v_i - 1,0)$ $with\, probabilty\; p$

A new car come with probaility λ come into the queue.

Vehicle position update

$$x_i \rightarrow x_i + v_i$$

Vehicle generation

The car which reach the end will leave the road.

Fig 7.2 Algorithm for improved one-lane model

Figure 7.3 are Time Space Diagram for one-lane model with maximum speed=5, total cells number=500, density=0.1, 0.3, 0.8 and the entry probability is 0.8. We choose 100 time steps. Right diagrams are the Time-Space diagram. From the figure we can see the traffic flow movements and congestion. Congestion happened in the place where the thick lines cluster. Traffic congestion tend to move up stream. At the low density, there exists little congestion. At high level, there exists lot congestion and the speeds of the vehicles are at lower level.

From the diagram we can tell that with the cars come and leave, the density of the roadway change all the time. The car leaves with the probability 1 and come with the probability 0.8. The total number of cars decreases with time. In the last 50 time steps, we can tell from the diagram there are less congestion the before.

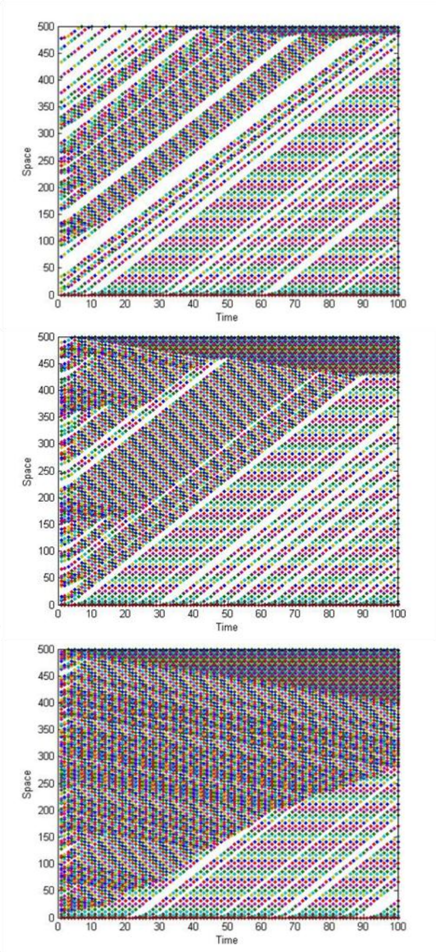


Fig 7.3 Time Space Diagram with density=0.1, 0.3, 0.8

In the single-lane highway traffic model with open boundary and queuing system, there exists one critical entry probability λ_c. Then the entry probability $\lambda<\lambda_c$, the length of queue l is around zero. When the entry probability $\lambda>\lambda_c$, the length of queue l will increase with time which means the capacity of the roadway in beyond the limit. The critical entry probability λ_c can be an index to describe the capacity of the roadway.

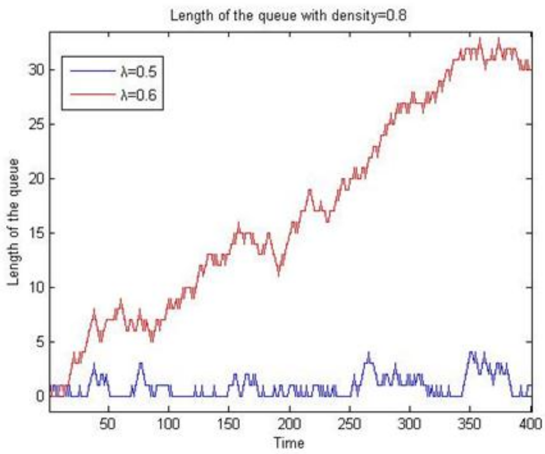


Fig 7.4 The length of the queue under density 0.6

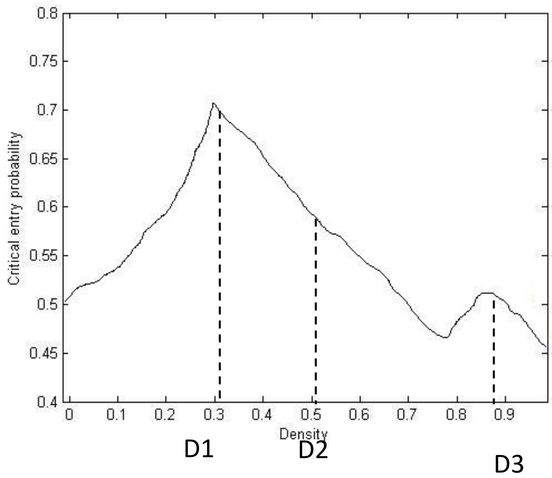


Fig 7.5 The critical entry probability under different density

From the diagram left, we can tell that when the entry probability beyond the critical entry probability the length will be increase with time. The diagram right shows the critical entry probability change with different density. The density is about D1, the road have the maximum capacity. When the density>D3, the roadway is in periodic oscillation state.

8. Modeling and simulation of Double-lane highway traffic with open boundary and queuing system

To simulate two-lane highway traffic, we started with the Nagel-Schreckenberg model. In the former model, we assume that the traffic flow system is a close boundary system. The number of vehicles in the traffic flow of model is constant which means there is no incoming or outgoing car in the system. The vehicles circular in the traffic system like a circle. Now we improve the model with and open boundary and queuing system. We simplify the model as a two-lane highway. Each lane is allowed to have its maximum velocity. Here we define, v_{max}^1 is the maximum speed of lane 1 and v_{max}^2 is the maximum speed of lane 2. From the former study, we know for the cars in one lane there are four steps: accelerate, keep safety distance, decrease speed randomized and move. Two-lane system is similar to the one-lane model. We introduce a parameter q, which describe the probability of a car change lane if that is allowed. A car is allowed to change lane if there are no cars right now in the sections that it will move through in this step or next step. The movements of the two-lane traffic model are as followed:

8.1. Rules

Acceleration:

All the vehicles whose velocity has not reached the maximum v_{max}^{1} or v_{max}^{2} will accelerate by one unit.

$$v_i \rightarrow min(v_i + a, v_{max})$$

Deceleration:

The vehicles reduce speed reduce its speed if the front gap is not enough for current speed. The speed will reduce to $gap_i - 1$. The rule is given as:

$$v_i \rightarrow min(v_i, gap_i - 1)$$

Where $\mathrm{gap_i} = \mathrm{x_i} - \mathrm{x_{i-1}}$

Randomization:

In the model, driver will decrease the speed randomized. If the $v_i \geq 0$, then the speed of i th will reduce the speed one unit with the probability p. According to D.Chowdhury, L.Santen and A. Schadchneider (2000), realistic data shows city traffic has a higher value of random probability than the number in highway traffic. For city traffic, we choose the probability of randomization $p = 0.5$. For highway traffic, we choose the probability of randomization $p = 0.3$. The rule is given as:

$$v_i \rightarrow max(v_i - 1,0) \;\; with\, probabilty\, p$$

Switch lane:

A car will change its lane for its own benefit. We conclude the criteria for lane-changing.

1. The distance ahead in current lane is smaller than the car speed.

2. The distance ahead in another lane is larger than in the current lane.

3. There exists an empty cell right in another lane.

4. The distance ahead of the following vehicle in another lane is larger than the speed of the following vehicle.

Criteria 1 and 2 are known as the trigger criteria (incentive criteria). Incentive criteria describe motivation that drivers are likely to drive fast in the target lane. Criteria 3 and 4 are the safety rules which assure the lane changing will not cause the bump of following vehicle. The vehicle which are meet the criteria will allow to switch lane with the probably q. If the accident happened, the car drivers attempt to switch lane more often for the purpose of minimization the travel time. Nagel (1998) developed a two-lane model to describe the lane changing behavior. Because of the fluctuations of the vehicle, the vehicles will not keep a constant speed in the roadway. The rule is given as:

Incentive criteria

$$gap_i < v_i$$

$$gap_{pred} > gap_i$$

Safety criteria

$$gap_{succ} > gap_{safe}$$

$gap_i = x_i - x_{i-1}$ is the gap between $i\ th$ vehicle and $(i-1)\ th$ vehicle.

gap_{pred} and gap_{succ} are the gaps between $i\text{-}th$ vehicle and preceding vehicle and the succeeding vehicle in the target lane.

gap_{safe} is the maximum possible speed of the succeeding vehicle in the target lane. For simplicity we choose the safety distance is the speed of the following vehicle.

New car entry

For each time step, a car will come to the road with a probability λ. If the cars can not entey into the road, they will line up in the entrance. The length of queuing is *l*.

Move:

After 5 steps, the new position of the vehicle can be determined by the current velocity, current position and the changes the lanes.

$$x_i \rightarrow x_i + v_i$$

Parameters are defined as follow:

a is the acceleration rate

${gap_i}^f = x_i - x_{i-1}$ the front gap

${x_i}^t$ is the position of i thvehicle at time t

x_{ip} is the incident place

T1 is the time when accident happen

T2 is the time when accident end

v_i is the velocity of i thvehicle

v_{max}^1 is the maximum speed of lane 1

v_{max}^2 is the maximum speed of lane 2

p is the probability of decrease speed.

q is the probability to switch lane.

L is the length of road.

Q is defined as flow of the model. $Q = \frac{\sum_{i=1}^{L} K_i V_i}{L}$

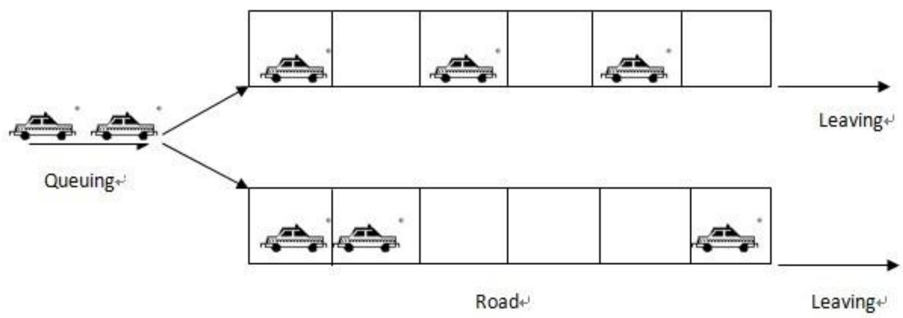


Fig 8.1 Movement of improved CA Model of two lanes

The above can shows the situation of the road.

8.2. Algorithm

Input

The length of the highway, The length of the cell, Maximum velocity of each lanes, Initial density, Incident details, Driver behavior probability p and the probability to change lanes q

Initialization

Generate initial vehicles

Begin

- Calculate gap
  - Acceleration
  - Deceleration
  - Randomization
  - Vehicle position update
  - A new car coming
  - Vehicle generation
  - The car leave the road
- Lane changing
- t=t+1

End

Fig 8.2 Algorithm for two-lane model

To simulate the two-lane traffic model, we first need to input parameters: the length of the highway, the length of the cell, numbers of vehicles, maximum velocity of each lanes, the initial density, incident details, driver behavior probability p and probability q to change lanes. According to D.Chowdhury, L.Santen and A. Schadchneider (2000), realistic data shows city traffic has a higher value of random probability than the number in highway traffic. For city traffic, we choose the probability of randomization $p = 0.5$. For highway traffic, we choose the probability of randomization $p = 0.25$.

8.3. Two-lane traffic Simulation accident

We also use Matlab to apply Monte Carlo method. We first generate the position and speed for each car in two lanes. Then use the five rules: Acceleration, Deceleration, Randomization and Lane change. We opted the parameter *p=0.3*, which means there is 30% probability the car driver will slow down the speed. We model 400 steps movement.

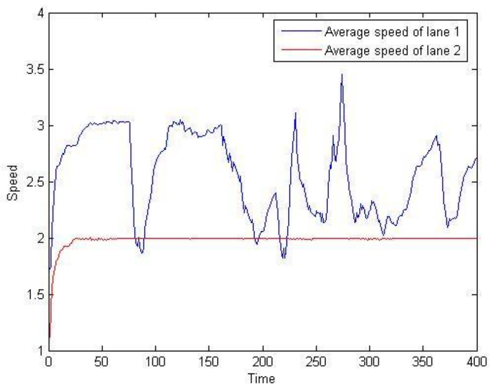


Fig 8.3 The average speed of lane 1 and lane 2.

The left chart shows the average speed of the different lanes. We settle the initial density lane one is 0.3 and lane 2 is 0.5. when the density of lane 1 is smaller than the lane 3, the average speed will always be lower. The moving car will choose lane 1 to speed up. With the density of lane 1 increase, the difference of speed will decrease.

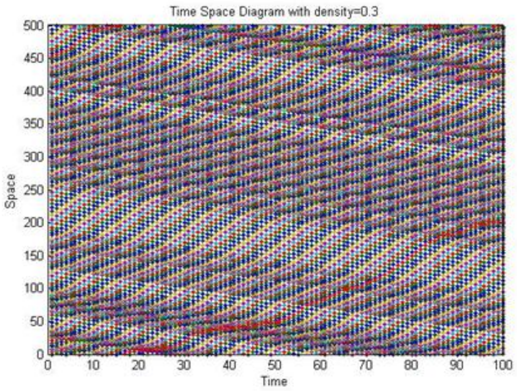


Fig 8.4 Time-Flow Diagrams of the first lane
Two-lane, V_max=5, L=1000, K=0.3, P=0.3, Steps=300

Figure 8.4 are Time Space Diagram of first lane for two-lane model with maximum speed=5, total cells number=1000, density=0.3 and probability slow down=0.3. We choose the last 500 steps of 1000 steps. From the figure we can see the traffic flow movements and congestion in the two pictures are similar. Congestion happened in the place where the thick lines cluster. The lane 2 has the same pattern.

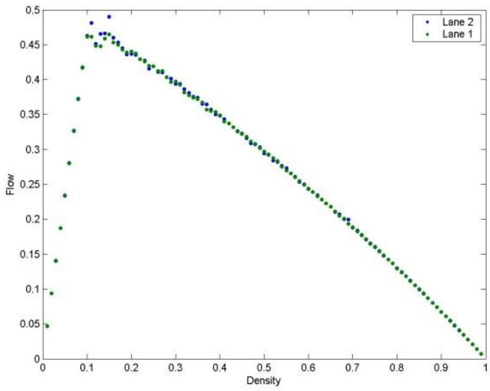


Fig 8.5 Density-Flow Diagram
Two-lane, V_max=5, L=1000, P=0.25, Steps=300

Figure 8.5 shows the Density-Flow Diagram for two-lane model with maximum speed=5, total cells number=2000 and probability slow down=0.3. From the diagram, we can tell that with the flow increase from zero to 0.5 with density increase from zero to one. Both for first lane and second lane, the flow reaches its highest value when density is around 0.13. Two lanes show no big difference.

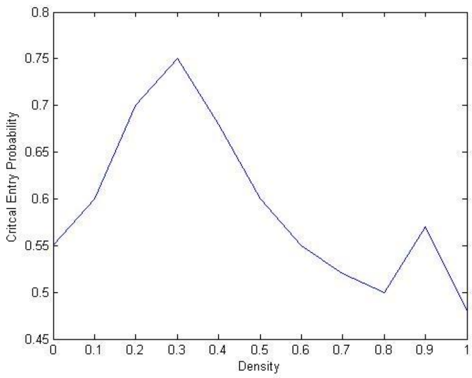


Fig 8.6 The critical entry probability under different density

Figure 8.6 shows the critical entry probability change with different density. The critical entry probability can describe the capacity of the roadway; the critical entry probability will reach highest point when density is about 0.3. It will decrease after that. In periodic oscillation state, it will grow again.

8.4. Modeling and simulation of incident in double lane highway traffic with open boundary

- Incident Occurrence Rule

If any accident happened in the road, the vehicles in the upstream at current time and downstream at next time are blocked. The vehicles start queue from the incident place. We define x_{ip} is the incident place. ${x_i}^{t-1}$ is the position of $i\ th$ vehicle at time t-1. ${x_i}^{t}$ is the position of i thvehicle at time t. T1 is the time when accident happens. T2 is the time when accident end. The rule is given as:

$$if\left({x_i}^{t-1} \le x_{ip}\ and\ {x_i}^{t} \ge x_{ip}\right)\ then \rightarrow\ x_i = 0\ \ for\ \mathrm{T1}\ \le \mathrm{t} \le \mathrm{T2}$$

- Double-lane traffic Simulation with accident

To simulate the accident in two lane, we assume that the accident happen between T1 and T2. T1 is the time when accident happens. T2 is the time when accident end. The accident happens in the first lane and the vehicles in the upstream at current time and downstream at next time are blocked. The block vehicle can switch lane into another lane. In the last 300 steps, we assume that the accident happens at time 100 and in the 1000th cell which is in the middle of the roadway. After the accident happened, the downstream traffic flow blocked from the incident place. The speed of the downstream traffic reduces to zero. Serious congestion begins. After time 150, accident is excluded and traffic flow begins to start again.

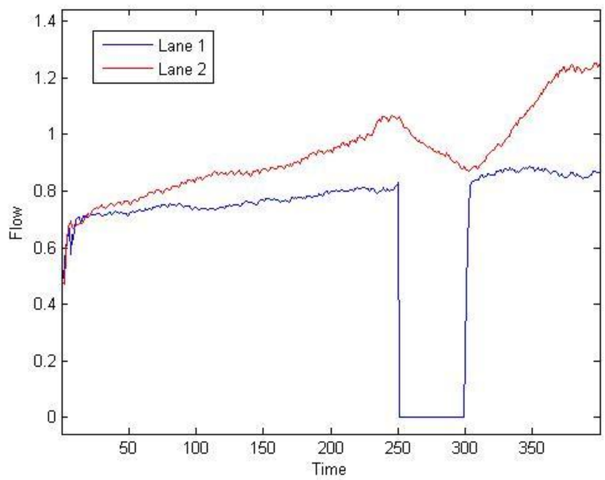


Fig 8.7 Time-Flow Diagrams with accident
Two-lane, V_max=5, L=500, K=0.25, P=0.25, Steps=400

Figure 8.7 shows the Flow-Time Diagram when accident happens. We can tell that the flow of the first lane reduce to 0 when the accident happens at time 250. The flow of second lane decreases also because most cars are move to this lane. When the accident excludes at time 300, we can see that traffic flow begins to restore.

9. Application of the CA model

In this paper, first we establish the one lane CA model with closed boundary. In the closed boundary system, the traffic flow has regular pattern without uncertain disturbing. From the time and space diagram, we can tell the traffic flow movements and congestion. Congestion happened in the place where the thick lines cluster. Traffic congestion tends to move up stream. We choose the index flow and critical density to measure the capacity of the roadway.

We assume an ideal situation. The roadway is separated into several cells. Each cell can be occupied by one vehicle or not. The max speed is defined from 1 to 5. In reality, the vehicle can be separated as two kinds: car and truck. The average length of cars is about 7.5m, and the length of trucks id about 11.5m. When max speed is 5, in reality the speed limit is about 135km/h.

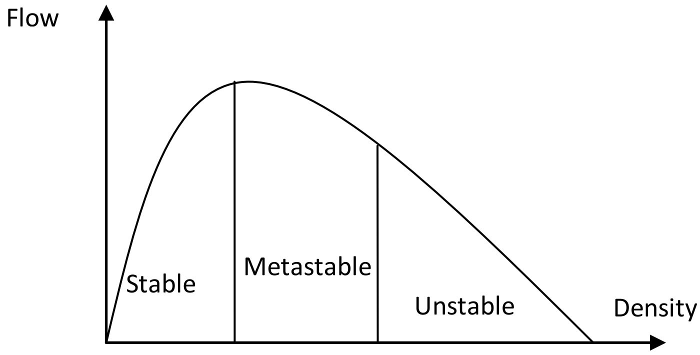


Fig 9.1 Three-phase traffic theory

According to the Three-phase traffic theory, the traffic flow can be divided into three stages: stable stage, meta stable stage and unstable stage. In stable stage, the traffic flow is a free flow. There is almost no traffic jam in the flow. The speed vehicles in the stable stage are close to the maximum speed. In the meta stable stage, there exists some congestion. The traffic flow has small disturbances, but traffic state remains stable. In the unstable stage, the traffic is congested at most time. The vehicles move slowly.

When the traffic flow has a lower density, the lag of each vehicle is large. Based on our simulation, the speed vehicles in the stable stage are close to the maximum speed. Since the vehicles will decrease their speed with the probability p. So, the average speed of free flow can be express as following:

$$v_f = (1-p)v_{max} + p(v_{max} - 1)$$

The flow of the traffic stream can be express as following:

$$Q = (v_{max} - p) \cdot K$$

For the high density traffic flow, the lag of each vehicle is small. We assume that the flow can be divided into congested flow and free flow. The vehicles in the free flow have the average speed v_f. The vehicles in the jam have the speed $v_j = 0$. The density of the free flow K_f can be determined by two factors. First factor is the time of the first vehicle waiting in the congested flow T_w. The second factor is the average free flow speed v_f. According to the Barlovic, Santen and Schadshneider (1998) T_w can be describe as

$$T_w = \frac{1}{1-p}$$

Because neglecting interactions between cars, the average distance of two consecutive cars is given by

$$gap = \ T_w \ \ v_f + 1$$

Using the normalization, the total length can be express as:

$$L = K_j L + K_f \ L \ gap$$

Which can be simply as

$$K_j + K_f \ gap = 1$$

K_j is the density of the jam

K_f is the dentisy of the free flow

And $\mathrm{K_j} + \mathrm{K_f} = \mathrm{K}$

The average flow under the high density is

$$Q(K) = K_j \ \ v_j + K_f \ \ v_f = K_f \ \ v_f$$

From the formulas above, we can have the high density flow:

$$Q(K) = (1-p)(1-K)$$

More detail can be found in Barlovic, Santen and Schadshneider (1998).

We choose the $(v_{max} - p) \cdot K = (1-p)(1-K)$, to get the critical density.

The critical density of the Block distribution can be described as following:

$$\text{Critical density of the Block: } K = \frac{1-P}{V_{max} + 1 - 2p} \ ,$$

Where p is the probability of slow down.

In our case, we choose slow down probability to be 0.3. From the formula above on the condition the roadway is one lane, when the maximum speed is 5, critical density of the block is 0.1296. When the maximum speed is 4, critical density of the block is 0.159. When the maximum speed is 3, critical density of the block is 0.212. When the maximum speed is 2, critical density of the block is 0.318. When the maximum speed is 1, critical density of the

Max Speed (p=0.3)	Critical density of the Block (Simulation)	Critical density of the Block (Formula)
1	0.5	0.5
2	0.3	0.318
3	0.2	0.212
4	0.15	0.159
5	0.12	0.1296

Table 9.2 Critical density of the Block by Simulation and Formula

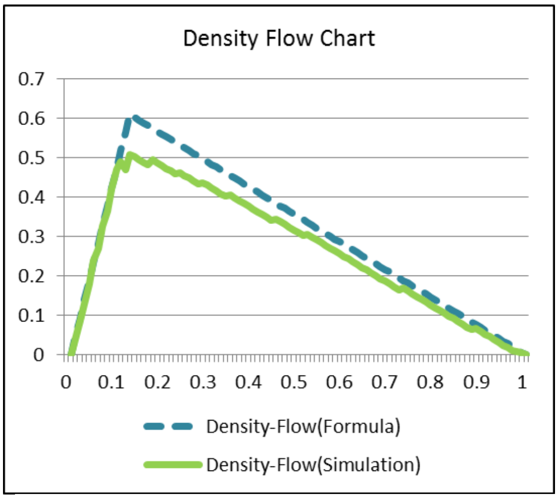


Table 9.3 Density Flow Chart by Simulation and Formula

block is 0.5.

Combine the low density flow formula and high density flow formula:

$$Q(K) = \begin{cases} (v_{max} - p) \cdot K & K < \dfrac{1-P}{V_{max} + 1 - 2p} \\ (1-p)(1-K) & K \geq \dfrac{1-P}{V_{max} + 1 - 2p} \end{cases}$$

In the former model, we assume that the traffic flow system is a close boundary system. The number of vehicles in the traffic flow of model is constant which means there is no incoming or outgoing car in the system. The vehicles circular in the traffic system like a circle. Now we improve the model into two lanes with and open boundary and queuing system which makes the simulation more realistic. The vehicles can transfer in both lanes. And the cars reach the end of the road will leave. The entry probability λ represents the probability a new car will enter the road at each time. There exists one critical entry probability λ_c. When the entry probability $\lambda<\lambda_c$, the length of queue l is around zero. When the entry probability $\lambda>\lambda_c$, the length of queue l will increase with time which means the capacity of the roadway in beyond the limit. The critical entry probability λ_c can be an index to describe the capacity of the roadway.

To analyze the effect of incident to the roadway, we run the simulation under different length and place of the incident. We choose the length of the incident (10 grids, 400 grids and 3000 grids) and position of the incident, simultaneously the length for roadway is settled to be 6000. In the following graph, L1 is the length between the incident and the entrance of the road.

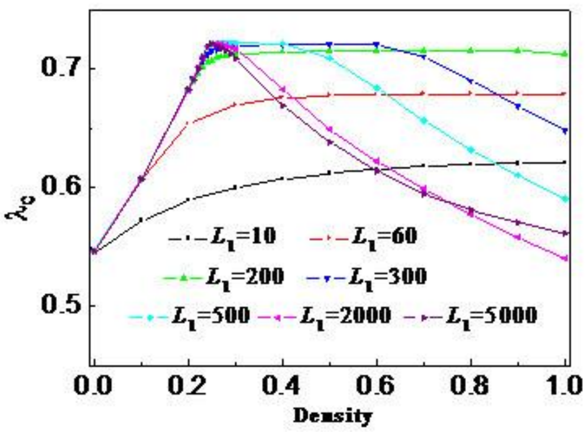


Fig 9.4 Relationship between critical entry probability and density (incident is 10 grids)

The left chart shows the impact of λ_c for a short bottleneck (10 grids) in different positions.

When the length of the bottleneck is relatively small, the impact to the roadway capacity of the roadway becomes quite complicated. For small L1, which means the incident is closed to the entrance, the figure of the branches increases into flat line. For bigger L1, the critical entry probability first increases and then reduces with density.

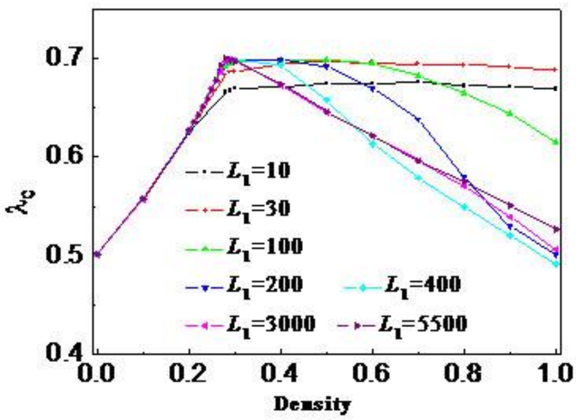


Fig 9.5 Relationship between critical entry probability and density (incident is 400 grids)

The left chart shows the impact of λ_c for a medium bottleneck (400 grids) in different positions.

For medium size incident, large L1 will lead to a first increased and then decreased diagram. For small L1, the first half curve will goes up and then becomes smooth.

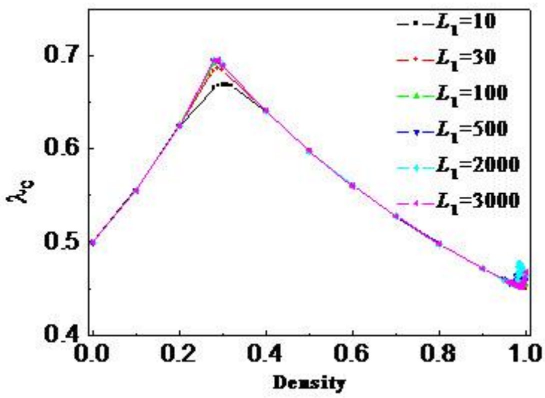


Fig 9.6 Relationship between critical entry probability and density (incident is 3000 grids)

The left chart shows the impact of λ_c for a long bottleneck (3000 grids) in different positions.

When the incident is not closed to the entrance, it will not make great impact on the shape of the critical entry probability. The critical entry probability is lower compare with short and medium bottleneck. If the bottleneck is near from the entrance (L1 is very small), the peak will be slightly lower.

From the simulation, we can have the relationship between the incident and capacity of the roadway. According to the actual location and length of the bottleneck we can apply the conclusion to the actual traffic guide to optimize the traffic situation.

10. Conclusion

In 1975, Treiterer and Myers found the phantom traffic jam phenomenon (ghost traffic jam) through the satellite map, which shows in the fig 9.1. Phantoms jam show a traffic pattern, where high volume cars move in the roadway. Congestion and disturbances will occur when vehicle cluster and become amplified. Until 1992 Nagel and Schreckenbergd used CA model to simulate and explain the phantom traffic jam phenomenon.

From the simulation we had, the phantom jam can be explained. With the increase in the number of vehicles on the road, vehicle density increases. The smaller the spacing between vehicles, the higher interaction between vehicles is. When density maintains in a low level, the vehicles' movement are free. When the vehicle is moving forward, the relationship between position and time is linear and vehicle keeps constant speed. When the density increase, the degree of free movement reduces and traffic blocking is generated in roadway. The relationship between position and time is non-linear. Some regional has intensive vehicles and others have few vehicles. Traffic movement and congestion appear alternately, similar to the peaks and troughs of the wave propagation.

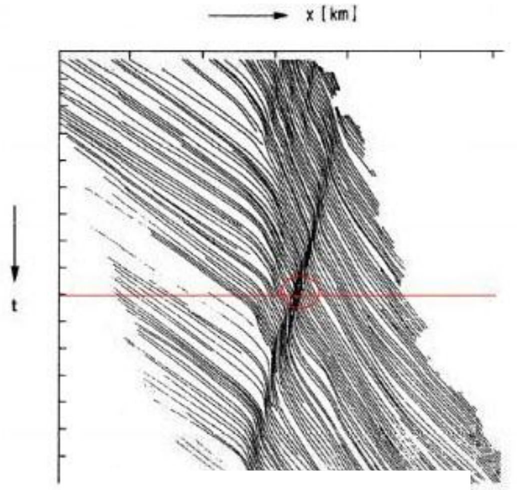


Fig 10.1 Phantom jams

Cellular automaton approach has the following advantages:

1. CA model is easy to understand and to implement in computer.
2. CA model is able to reproduce the complex traffic phenomenon and reflect the characteristics of traffic flow. The simulation shows the change of cellular in every time steps. Observer can not only get each vehicle's speed at any time but also microscopic description such as displacement and distance of each cars and macroscopic description such as average speed, density and flow of traffic flow.
3. CA model is able to simulate both one-lane roadway and multi-lane roadway, distinguish small vehicles and big vehicles by setting.

Reference

Greenshields, B.D. (1933). The Photographic Method of studying Traffic Behaviour, Proceedings of the 13th Annual Meeting of the Highway Research Board.

Greenshields, B.D. (1935). A study of highway capacity, Proceedings Highway Research, Record, Washington Volume 14, pp. 448-477.

Lighthill, M.H., Whitham, G.B.,(1955). On kinematic waves II: A theory of traffic flow on long, crowded roads. Proceedings of The Royal Society of London Ser. A 229, 317-345.

Richards, P.J., (1956), Shock waves on the highway, Operations Research 4, 42–51.

Chandler, R., R. Herman, and E. Montroll (1958).Traffic dynamics; studies in car-following.Operations Research 6, 165+.

Herman R, Montroll E W, Potts R B, et al (1959). Traffic dynamic: Analysis of Stability in car-following [J]. Operations Research7, 86-106.

Gazis, D., R. Herman, and R. Rothery (1961). Nonlinear follow-the-leader models of traffic flow, Operations Research 9, 545+.

Prigogine I and Herman R (1971). Kinetic Theory of Vehicular Traffic, New York: Elsevier.

Wolfram S (1984), Cellular Automata and Complexity, Reading, MA: Addison-Wesley.

Wolfram S (1986), Theory and Applications of Cellular Automata, Singapore: World Scientific.

Bank J (1991). Two-capacity phenomenon at freeway bottlenecks: a basis for ramp metering [J]. Transportation Research Record, 1320: 83-90.

Biham O (1992), Middleton AA and Levine D, Phys. Rev. A 46 R6124.

K. Nagel, M. Schreckenberg (1992). A cellular automaton model for freeway traffic, Journal de Physique I 2(12), 2221-2229.

Fukui M and Ishibashi Y (1996), J. Phys. Soc. Japan 65 1868.

S. C. Benjamin, N.F. Johnson, P. M. Hui (1996). Cellular automata models of traffic flow along a highway containing a junction, Journal of Physics A: Mathematical and General 29(12) 3119-3127.

P. Wagner, K. Nagel, D.E. Wolf (1997). Realistic multi-lane traffic rules for cellular antomata, PhysicaA 234, 687-698.

R. Barlovic, L. Santen, A. Schadschneider and M. Schreckenberg (1998), Metastable states in cellular automata for traffic flow, Eur. Phys. J. B 5, 793.

D. Chowdhury, L. Santen and A. Schadschneider (2000). Statistical physics of vehicular traffic and some related systems, Physics Reports 329, 199.

Transportation Research Board (2000), Highway Capacity Manual [M], Washington DC.National Research Council.

Persaud B N (1998). Exploration of the Breakdown phenomenon in freeway traffic [J]. Transportation Research Record, 1634: 64-69.

R. Barlovic, L.Santen, A.Schadschneider, M.Schreckenberg, (1998), Metastable states in cellular automata for traffic flow[J]. Eur. Phys.J.B, 5:793-800

Xue Y, Dong L Y, Dai S Q. (2005), Effects of changing order in the update rules on traffic flow. Phys. Rev. E. 2005, 71:026123 1-6.