CENG 384 - Signals and Systems for Computer Engineers
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Homework 4

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1. (a)

$$y''(t) = 4x'(t) - 5y'(t) + x(t) - 6y(t)$$

(b) To find frequency reponse, we need to put $x(t) = e^{jwt}$ to the system, and we will get $y(t) = H(jw) \times e^{jwt}$. Therefore, when we put these values to the above system equation:

$$(jw)^2 H(jw)e^{jwt} = 4jwe^{jwt} - 5jwH(jw)e^{jwt} + e^{jwt} - 6H(jw)e^{jwt}$$

$$(jw)^2 H(jw)e^{jwt} + 5jwH(jw)e^{jwt} + 6H(jw)e^{jwt} = 4jwe^{jwt} + e^{jwt}$$

$$H(jw)e^{jwt}(j^2w^2 + 5jw + 6) = e^{jwt}(4jw + 1)$$

$$H(jw) = \frac{4jw + 1}{j^2w^2 + 5jw + 6}$$

(c) In LTI system, we know that Fourier transform of impulse response gives us frequency response. From part (b), we already know frequency response. Therefore,

$$H(jw) = \frac{4jw + 1}{j^2w^2 + 5jw + 6}$$

$$= \frac{4jw + 1}{(jw + 3)(jw + 2)}$$

$$= \frac{4jw + 1}{(jw + 3)(jw + 2)} = \frac{A}{jw + 3} + \frac{B}{jw + 2}$$

$$4jw + 1 = Ajw + 2A + Bjw + 3B$$

$$4 = A + B$$

$$1 = 2A + 3B$$

Here, we find $A = 11$ and $B = -7$. Therefore, our frequency response can be written as:

$$H(jw) = \frac{11}{jw + 3} - \frac{7}{jw + 2}$$

We know that when $x(t) = e^{-at}u(t)$, $X(jw) = \frac{1}{a+jw}$ (eqn 1).
By using this, our impulse response will be:

$$h(t) = (11e^{-3t} - 7e^{-2t})u(t)$$

(d) We know that $Y(jw) = H(jw) \times X(jw)$. We have already found $H(jw)$ in part (b). Therefore, we need to find $X(jw)$ for $x(t) = \frac{1}{4}e^{-t/4}u(t)$.

$$X(jw) = \frac{1}{4} \times \frac{1}{\frac{1}{4} + jw} \ \textit{(by using eqn 1 again)}$$

$$X(jw) = \frac{1}{1 + 4jw}$$

Now, we can find $Y(jw) = H(jw) \times X(jw)$

$$Y(jw) = H(jw) \times X(jw) = \frac{4jw + 1}{(jw + 3)(jw + 2)} \times \frac{1}{1 + 4jw}$$

$$Y(jw) = \frac{1}{(jw+3)(jw+2)}$$
$$\frac{1}{(jw+3)(jw+2)} = \frac{A}{jw+3} + \frac{B}{jw+2}$$
$$1 = Ajw + 2A + Bjw + 3B$$
$$A + B = 0$$
$$2A + 3B = 1$$

Here, we find $A = -1$ and $B = 1$. Therefore, $Y(jw)$ will be:

$$Y(jw) = -\frac{1}{jw+3} + \frac{1}{jw+2}$$

By using eqn 1, we can reach $y(t)$ as:

$$y(t) = (-e^{-3t} + e^{-2t})u(t)$$

2. (a)

$$H(jw) = \frac{jw+4}{-w^2 + 5jw + 6}$$

We know that $Y(jw) = X(jw) \times H(jw)$. Therefore,

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{jw+4}{-w^2+5jw+6}$$

$$(-w^2)Y(jw) + 5jwY(jw) + 6Y(jw) = jwX(jw) + 4X(jw)$$

Since $j^2 = -1$, we can write $(-w^2) = (jw)^2$. Therefore, above equation will be:

$$(jw)^2Y(jw) + 5jwY(jw) + 6Y(jw) = jwX(jw) + 4X(jw)$$

By looking the Fourier transform table we can find inverse Fourier transform of this equation. And, this gives us a differential equation which represents the system.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

(b)

$$H(jw) = \frac{jw+4}{-w^2+5jw+6} = \frac{jw+4}{(jw+2)(jw+3)}$$
$$= \frac{jw+4}{(jw+2)(jw+3)} = \frac{A}{jw+2} + \frac{B}{jw+3}$$
$$Ajw + 3A + Bjw + 2B = jw + 4$$
$$A + B = 1$$
$$3A + 2B = 4$$

Here, we find $A = 2$ and $B = -1$. Therefore,

$$H(jw) = \frac{2}{jw+2} - \frac{1}{jw+3}$$

By using Fourier transform of $e^{-at}u(t)$ is $\frac{1}{a+jw}$, impulse response will be:

$$h(t) = (2e^{-2t} - e^{-3t})u(t)$$

(c) We know that $Y(jw) = X(jw) \times H(jw)$. We have already known $H(jw)$. Thus, we need to find $X(jw)$ for given input $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$
By using Fourier transform of $e^{-at}u(t)$ is $\frac{1}{a+jw}$ and by using Fourier transform of $te^{-at}u(t)$ is $\frac{1}{(a+jw)^2}$ from the transformation tables of the textbook, we can find:

$$X(jw) = \frac{1}{4+jw} - \frac{1}{(4+jw)^2}$$

$$X(jw) = \frac{3+jw}{(4+jw)^2}$$

Since $Y(jw) = X(jw) \times H(jw)$

$$Y(jw) = X(jw) \times H(jw) = \frac{3+jw}{(4+jw)^2} \times \frac{jw+4}{(jw+2)(jw+3)}$$

$$Y(jw) = \frac{1}{(jw+4)(jw+2)}$$

(d)

$$Y(jw) = \frac{1}{(jw+4)(jw+2)} = \frac{A}{jw+4} + \frac{B}{jw+2}$$
$$Ajw + 2A + Bjw + 4B = 1$$
$$A + B = 0$$
$$2A + 4B = 1$$

Here, we find $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Therefore, $Y(jw)$ will be

$$Y(jw) = \frac{-\frac{1}{2}}{jw+4} + \frac{\frac{1}{2}}{jw+2}$$

By using Fourier transform of $e^{-at}u(t)$ is $\frac{1}{a+jw}$

$$y(t) = (-\frac{1}{2}e^{-4t} + \frac{1}{2}e^{-2t})u(t)$$

3. (a) By using analysis equation:

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

When we put $x(t) = e^{-|t|}$

$$X(jw) = \int_{-\infty}^{\infty} e^{-|t|}e^{-jwt}\,dt$$
$$= \int_{-\infty}^{0} e^{t}e^{-jwt}\,dt + \int_{0}^{\infty} e^{-t}e^{-jwt}\,dt$$
$$= \int_{-\infty}^{0} e^{(1-jw)t}\,dt + \int_{0}^{\infty} e^{(-1-jw)t}\,dt$$
$$= \left(|_{-\infty}^{0} \quad \frac{e^{(1-jw)t}}{1-jw}\right) \quad + \quad \left(|_{0}^{\infty} \quad \frac{e^{(-1-jw)t}}{-1-jw}\right)$$
$$= \frac{1}{1-jw} + \frac{1}{1+jw}$$
$$X(jw) = \frac{2}{1+w^2} \quad [RESULT]$$

(b) We can use "Differentiation in Frequency" property of Table 4.1 of our Textbook for this question. In other words,

Fourier transform of $tx(t)$ will be $j\frac{d}{dw}X(jw)$.

Therefore, Fourier transform of $te^{-|t|}$ will be:

$$= j\frac{d}{dw}X(jw) = j\frac{d}{dw}\left(\frac{2}{1+w^2}\right)$$
$$= j\left(\frac{-(2w)\times 2}{(1+w^2)^2}\right)$$
$$= \frac{-4jw}{(1+w^2)^2} \quad [RESULT]$$

(c) By using synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}\,dw$$

And, by using duality property of Fourier transform with the result of part (b):

$$x(t) = te^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4jw}{(1+w^2)^2}e^{jwt}\,dw$$

Multiplying both side by 2π and replacing t by $-t$, we get:

$$2\pi(-t)e^{-|-t|} = \int_{-\infty}^{\infty} \frac{-4jw}{(1+w^2)^2}e^{jw(-t)}\,dw$$
$$-2\pi te^{-|t|} = \int_{-\infty}^{\infty} \frac{-4jw}{(1+w^2)^2}e^{-jwt}\,dw$$

Then, we can interchange the name of the variables t and w:

$$-2\pi w e^{-|w|} = \int_{-\infty}^{\infty} \frac{-4jt}{(1+t^2)^2} e^{-jwt}\, dt$$

Multiply right hand side's both numerator and denominator with j:

$$-2\pi w e^{-|w|} = \int_{-\infty}^{\infty} \frac{-4(j^2)t}{(1+t^2)^2 \times (j)} e^{-jwt}\, dt$$

$$-2\pi w e^{-|w|} = \frac{1}{j}\int_{-\infty}^{\infty} \frac{4t}{(1+t^2)^2} e^{-jwt}\, dt$$

$$-j2\pi w e^{-|w|} = \int_{-\infty}^{\infty} \frac{4t}{(1+t^2)^2} e^{-jwt}\, dt$$

By considering analysis equation $X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\, dt$, it is clear that:
Fourier transform of $\frac{4t}{(1+t^2)^2}$ is $-j2\pi w e^{-|w|}$ [RESULT]

4. (a) $\frac{3}{4}y[n-1] + 2x[n] - \frac{1}{8}y[n-2] = y[n]$

(b) By applying Fourier transform to the above equation:

$$\frac{3}{4}e^{-jw}Y(e^{jw}) + 2X(e^{jw}) - \frac{1}{8}e^{-2jw}Y(e^{jw}) = Y(e^{jw})$$

$$2X(e^{jw}) = Y(e^{jw}) - \frac{3}{4}e^{-jw}Y(e^{jw}) + \frac{1}{8}e^{-2jw}Y(e^{jw})$$

$$2X(e^{jw}) = Y(e^{jw})[1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}]$$

$$\frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}} = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$\frac{16}{8 - 6e^{-jw} + e^{-2jw}} = H(e^{jw}) \quad [RESULT]$$

(c)

$$H(e^{jw}) = \frac{16}{e^{-2jw} - 6e^{-jw} + 8} = \frac{16}{(e^{-jw}-4)(e^{-jw}-2)}$$

$$\frac{16}{(e^{-jw}-4)(e^{-jw}-2)} = \frac{A}{e^{-jw}-4} + \frac{B}{e^{-jw}-2}$$

$$16 = Ae^{-jw} - 2A + Be^{-jw} - 4B$$

$$A + B = 0$$

$$-2A - 4B = 16$$

Here, we get $A = 8$ and $B = -8$. Therefore,

$$H(e^{jw}) = \frac{8}{e^{-jw}-4} - \frac{8}{e^{-jw}-2}$$

We know that Fourier transform of $a^n u[n]$ is $\frac{1}{1-ae^{-jw}}$. Therefore, our frequency response can be:

$$H(e^{jw}) = \frac{-2}{1 - \frac{1}{4}e^{-jw}} + \frac{4}{1 - \frac{1}{2}e^{-jw}}$$

Finally, impulse response will be:

$$h[n] = \left(-2(\frac{1}{4})^n + 4(\frac{1}{2})^n\right) u[n]$$

(d) This is LTI system. Since $y[n] = x[n] * h[n]$, we can say that $Y(e^{jw}) = X(e^{jw}) \times H(e^{jw})$. We have already found frequency response at part (b). Therefore, now, we need to find $X(e^{jw})$ for given input $x[n] = (\frac{1}{4})^n u[n]$.
By using (Fourier transform of $a^n u[n]$ is $\frac{1}{1-ae^{-jw}}$)

$$X(e^{jw}) = \frac{1}{1 - \frac{1}{4}e^{-jw}}$$

Now, we can find $Y(e^{jw}) = X(e^{jw}) \times H(e^{jw})$

$$Y(e^{jw}) = X(e^{jw}) \times H(e^{jw}) = \frac{1}{1 - \frac{1}{4}e^{-jw}} \times (\frac{8}{e^{-jw}-4} - \frac{8}{e^{-jw}-2})$$

$$= \frac{4}{4-e^{-jw}} \times (\frac{8}{e^{-jw}-4} - \frac{8}{e^{-jw}-2})$$
$$= \frac{4}{e^{-jw}-4} \times (-\frac{8}{e^{-jw}-4}) + \frac{8}{e^{-jw}-2})$$
$$= -\frac{32}{(e^{-jw}-4)^2} + (\frac{32}{(e^{-jw}-4)(e^{-jw}-2)})$$
$$= -\frac{32}{(e^{-jw}-4)^2} + \frac{16}{e^{-jw}-4} - \frac{16}{e^{-jw}-2}$$
$$= -\frac{32}{(e^{-jw}-4)^2} + (\frac{-4}{1-\frac{1}{4}e^{-jw}}) - (\frac{-8}{1-\frac{1}{2}e^{-jw}})$$
$$Y(e^{jw}) = -\frac{2}{(1-\frac{1}{4}e^{-jw})^2} + (\frac{-4}{1-\frac{1}{4}e^{-jw}}) - (\frac{-8}{1-\frac{1}{2}e^{-jw}})$$

By using (Fourier transform of $a^n u[n]$ is $\frac{1}{1-ae^{-jw}}$) and by using Fourier transform of $(n+1)a^n u[n]$ is $\frac{1}{(1-ae^{-jw})^2}$

$$y[n] = \left(-2(n+1)(\frac{1}{4})^n - 4(\frac{1}{4})^n + 8(\frac{1}{2})^n\right) u[n]$$

5.

$$x[n] * h_1[n] + x[n] * h_2[n] = y[n]$$
$$x[n] * (h_1[n] + h_2[n]) = y[n]$$

Therefore, the overall system's impulse response is $h[n] = h_1[n] + h_2[n]$.
Therefore, frequency response of the overall system is $H(e^{jw}) = H_1(e^{jw}) + H_2(e^{jw})$

For given $h_1 n = (\frac{1}{3})^n u[n]$, frequency response is:

$$H_1(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}}$$

Now, substract $H_1(e^{jw})$ from frequency response of overall system $H(e^{jw})$

$$H(e^{jw}) - H_1(e^{jw}) = H_2(e^{jw})$$
$$\frac{5e^{-jw}-12}{e^{-2jw}-7e^{-jw}+12} - \frac{1}{1-\frac{1}{3}e^{-jw}} = H_2(e^{jw})$$
$$\frac{5e^{-jw}-12}{(e^{-jw}-3)(e^{-jw}-4)} - \frac{3}{3-e^{-jw}} = H_2(e^{jw})$$
$$\frac{5e^{-jw}-12}{(e^{-jw}-3)(e^{-jw}-4)} + \frac{3}{e^{-jw}-3} = H_2(e^{jw})$$
$$\frac{5e^{-jw}-12}{(e^{-jw}-3)(e^{-jw}-4)} + \frac{3e^{-jw}-12}{(e^{-jw}-3)(e^{-jw}-4)} = H_2(e^{jw})$$
$$\frac{8(e^{-jw}-3)}{(e^{-jw}-3)(e^{-jw}-4)} = H_2(e^{jw})$$
$$H_2(e^{jw}) = \frac{8}{e^{-jw}-4}$$
$$H_2(e^{jw}) = \frac{-2}{1-\frac{1}{4}e^{-jw}}$$

By using (Fourier transform of $a^n u[n]$ is $\frac{1}{1-ae^{-jw}}$)

$$h_2[n] = -2(\frac{1}{4})^n u[n]$$

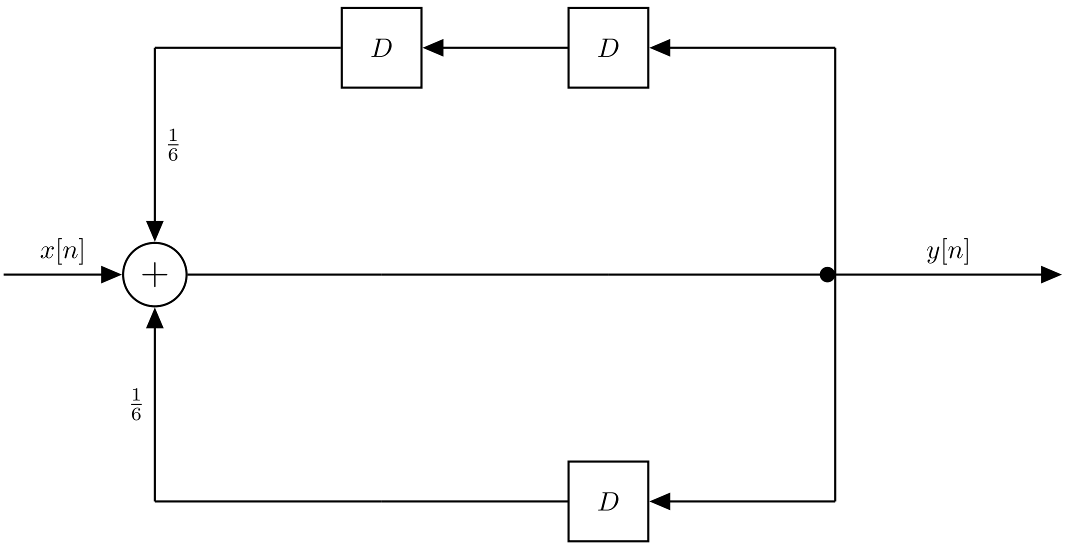
6. (a) Since this is LTI system, we know that $H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$. Therefore,

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1-\frac{1}{6}e^{-jw}-\frac{1}{6}e^{-2jw}}$$
$$Y(e^{jw}) - \frac{1}{6}e^{-jw}Y(e^{jw}) - \frac{1}{6}e^{-2jw}Y(e^{jw}) = X(e^{jw})$$

By using Fourier transform of $y[n-n_0]$ is $e^{-jwn_0}Y(e^{jw})$ (i.e. time-shifting property), we get:

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$
$$y[n] = x[n] + \frac{1}{6}y[n-1] + \frac{1}{6}y[n-2] \quad [RESULT]$$

(b) The block diagram is:



(c)

$$H(e^{jw}) = \frac{1}{1 - \frac{1}{6}e^{-jw} - \frac{1}{6}e^{-2jw}} = \frac{-6}{e^{-2jw} + e^{-jw} - 6}$$

$$= \frac{-6}{(e^{-jw}+3)(e^{-jw}-2)}$$

$$= \frac{-6}{(e^{-jw}+3)(e^{-jw}-2)} = \frac{A}{e^{-jw}+3} + \frac{B}{e^{-jw}-2}$$

$$-6 = Ae^{-jw} - 2A + Be^{-jw} + 3B$$

$$A + B = 0$$

$$-2A + 3B = -6$$

Here, we get $A = \frac{6}{5}$ and $B = -\frac{6}{5}$. Therefore,

$$H(e^{jw}) = \frac{\frac{6}{5}}{e^{-jw}+3} + \frac{(-\frac{6}{5})}{e^{-jw}-2}$$

And it can be written as:

$$H(e^{jw}) = \frac{\frac{2}{5}}{1 + \frac{1}{3}e^{-jw}} + \frac{\frac{3}{5}}{1 - \frac{1}{2}e^{-jw}}$$

We know that Fourier transform of $a^n u[n]$ is $\frac{1}{1-ae^{-jw}}$. Therefore, impulse response will be:

$$h[n] = \left(\frac{2}{5}(-\frac{1}{3})^n + \frac{3}{5}(\frac{1}{2})^n\right) u[n]$$