CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 3

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1. (a)

$$x(t) = \frac{1}{2} + \cos(w_o t)$$
$$= \frac{1}{2} + \frac{1}{2}(e^{jw_o t} + e^{-jw_o t})$$
$$= \frac{1}{2} + \frac{1}{2}e^{jw_o t} + \frac{1}{2}e^{-jw_o t}$$

Therefore, $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$ and $a_{-1} = \frac{1}{2}$. Magnitude graph is:

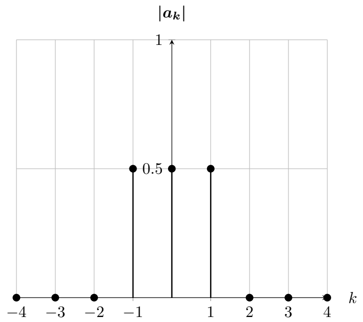

Figure 1: k vs. $|a_k|$

All phases are 0. Therefore, phase graph is not drawn.

(b)

$$y(t) = \frac{3}{2} + 2\sin(w_o t)$$
$$= \frac{3}{2} + 2[\frac{1}{2j}(e^{jw_o t} - e^{-jw_o t})]$$
$$= \frac{3}{2} + \frac{1}{j}e^{jw_o t} - \frac{1}{j}e^{-jw_o t}$$

Therefore, $b_0 = \frac{3}{2}$, $b_1 = \frac{1}{j}$ and $b_{-1} = -\frac{1}{j}$. Magnitude graph is:

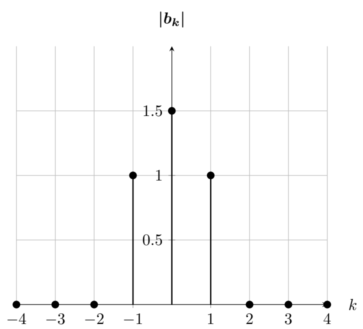

Figure 2: k vs. $|b_k|$

Phase graph is:

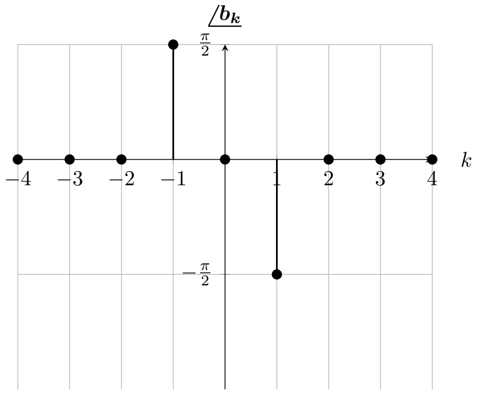

Figure 3: k vs. $\angle b_k$

(c)

$$z(t) = x(t) + y(t) = \frac{1}{2} + \cos(w_o t) + \frac{3}{2} + 2\sin(w_o t) + \cos(2w_o t + \frac{\pi}{4})$$

$$= \frac{1}{2} + \frac{1}{2}(e^{jw_o t} + e^{-jw_o t}) + \frac{3}{2} + 2[\frac{1}{2j}(e^{jw_o t} - e^{-jw_o t})] + \frac{1}{2}(e^{j(2w_o t + \frac{\pi}{4})} + e^{-j(2w_o t + \frac{\pi}{4})})$$

$$= \frac{1}{2} + (\frac{1}{2} + \frac{1}{j})e^{jw_o t} + \frac{3}{2} + (\frac{1}{2} - \frac{1}{j})e^{-jw_o t} + (\frac{e^{j\pi/4}}{2})e^{j2w_o t} + (\frac{e^{-j\pi/4}}{2})e^{-j2w_o t}$$

$$= 2 + (\frac{j+2}{2j})e^{jw_o t} + (\frac{j-2}{2j})e^{-jw_o t} + (\frac{e^{j\pi/4}}{2})e^{j2w_o t} + (\frac{e^{-j\pi/4}}{2})e^{-j2w_o t}$$

Therefore, Fourier coefficients are:

$$c_0 = 2$$

$$c_1 = \frac{j+2}{2j} = \frac{1}{2} - j$$

$$c_{-1} = \frac{j-2}{2j} = \frac{1}{2} + j$$

$$c_2 = \frac{e^{j\pi/4}}{2} = \frac{\sqrt{2}}{4}(1+j)$$

$$c_{-2} = \frac{e^{-j\pi/4}}{2} = \frac{\sqrt{2}}{4}(1-j)$$

Magnitude graph is:

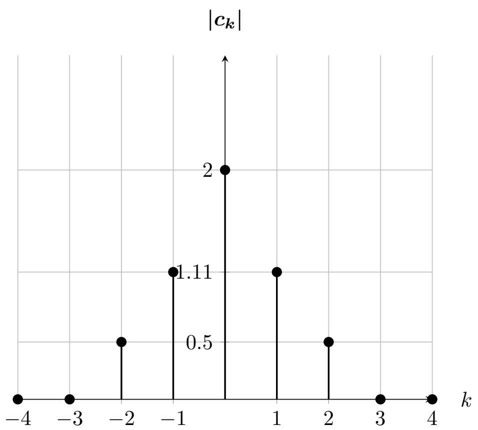

Figure 4: k vs. $|c_k|$

Phase graph is:

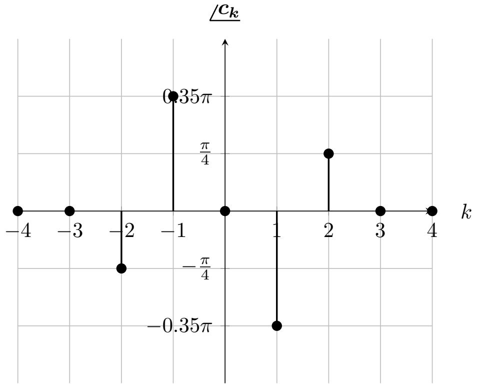

Figure 5: k vs. $\angle c_k$

2.

$$x(t) = \begin{cases} M & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

$$a_0 = \begin{cases} \frac{1}{T} * \int_0^{T_1} x(t)\,dt & t < T_1 \\ \frac{1}{T} * \int_0^{T_1} x(t)\,dt & T_1 \leq t < T \end{cases}$$

$$a_0 = \begin{cases} \frac{1}{T} * \int_0^{T_1} M\,dt & t < T_1 \\ \frac{1}{T} * \int_0^{T_1} 0\,dt & T_1 \leq t < T \end{cases}$$

$$a_0 = \begin{cases} \frac{1}{T} * (M * T_1 - 0) & t < T_1 \\ \frac{1}{T} * 0 & T_1 \leq t < T \end{cases}$$

$$a_0 = \begin{cases} \frac{M * T_1}{T} & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

$A_0 = 2 * a_0$

$$A_0 = \begin{cases} 2 * \frac{M * T_1}{T} & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

$b_0 = \frac{2}{T} * \int_0^{T_1} x(t) * cos(kW_0 t)\,dt$

$$b_0 = \begin{cases} \frac{2}{T} * \int_0^{T_1} M * cos(kW_0 t)\,dt & t < T_1 \\ \frac{2}{T} * \int_0^{T_1} 0 * cos(kW_0 t)\,dt & T_1 \leq t < T \end{cases}$$

$$b_0 = \begin{cases} \frac{2 * M}{T * k * W_0} * (sin(kW_0 T1) - 0) & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

$A_k = b_0$

$$A_k = \begin{cases} \frac{2 * M}{T * k * W_0} * (sin(kW_0 T1) - 0) & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

$c_0 = \frac{2}{T} * \int_0^{T_1} x(t) * sin(kW_0 t)\,dt$

$$c_0 = \begin{cases} \frac{2}{T} * \int_0^{T_1} M * sin(kW_0 t)\,dt & t < T_1 \\ \frac{2}{T} * \int_0^{T_1} 0 * sin(kW_0 t)\,dt & T_1 \leq t < T \end{cases}$$

$$c_0 = \begin{cases} \frac{-2 * M}{T * k * W_0} * (cos(kW_0 T1) - 1) & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

$_k = -c_0$

$$B_0 = \begin{cases} \frac{2*M}{T*k*W_0} * (cos(kW_0T1) - 1) & t < T_1 \\ 0 & T_1 \leq t < T \end{cases}$$

Magnitude graph is:

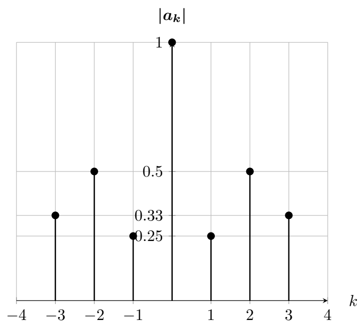

Figure 6: k vs. $|a_k|$

3. (a) x(t) $= 1+ \frac{1}{2} * cos(2\pi t) + cos(4\pi t) + \frac{2}{3} * cos(6\pi t)$

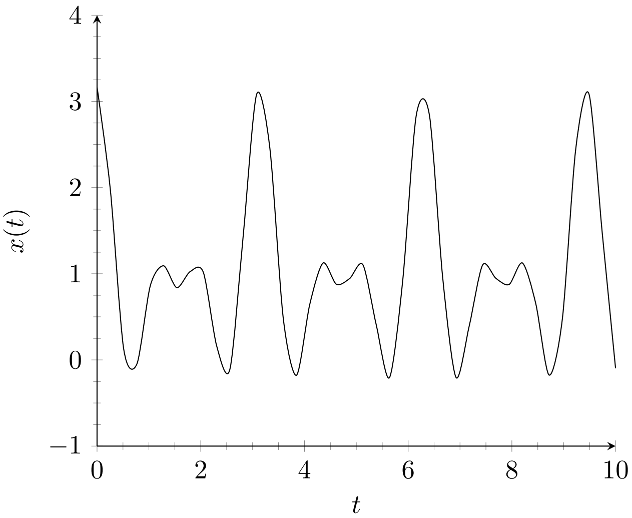

Figure 7: $x(t)$ vs t.

(b) x(t) $= 1+ \frac{1}{2} * cos(2\pi t) + cos(4\pi t) + \frac{2}{3} * cos(6\pi t)$
$x(t) = \frac{1}{2} * (\frac{1}{2} * (e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2} * (e^{2j2\pi t} + e^{-2j2\pi t}) + (\frac{2}{3} * (\frac{1}{2} * (e^{3j2\pi t} + e^{-3j2\pi t}))$
$x(t) = \frac{1}{4} * (e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2} * (e^{2j2\pi t} + e^{-2j2\pi t}) + \frac{1}{3} * (e^{3j2\pi t} + e^{-3j2\pi t})$

$a_0 = 1$
$a_1 = a_{-1=\frac{1}{4}}$
$a_2 = a_{-2} = \frac{1}{2}$
$a_3 = a_{-3} = \frac{1}{3}$

(c) In a_k graph we use frequency domain which is how much of signal exist within a given frequency. In x_t graph we see signal behavior in given time domain.Signal properties are same but we observe them different way.

(d) $b_k = a_k * H(jkW_0)$
$H(jW_0) = \int_{-\infty}^{\infty} h(t)e^{-jw_0t}\, dt$
$H(jW_0) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-jw_0t}\, dt$
$H(jW_0) = \int_{0}^{\infty} e^{-2t}e^{-jw_0t}\, dt$
$H(jW_0) = \frac{1}{2+jW_0}$
$H(jkW_0) = \frac{1}{2+jkW_0}$

Fundamental period T is 1. Fundamental frequency W_0 is $\frac{2\pi}{T} = 2\pi$
$b_k = a_k * \frac{1}{2+jk2\pi}$

$b_0 = 1 * \frac{1}{2-0} = \frac{1}{2}$
$b_1 = \frac{1}{4} * \frac{1}{2+j2\pi} = \frac{1}{8*(1+\pi j)}$
$b_{-1} = \frac{1}{4} * \frac{1}{2-j2\pi} = \frac{1}{8*(1-\pi j)}$
$b_2 = \frac{1}{2} * \frac{1}{2+j4\pi} = \frac{1}{4*(1+2\pi j)}$
$b_{-2} = \frac{1}{2} * \frac{1}{2-j4\pi} = \frac{1}{4*(1-2\pi j)}$
$b_3 = \frac{1}{3} * \frac{1}{2+j6\pi} = \frac{1}{6*(1+3\pi j)}$
$b_{-3} = \frac{1}{3} * \frac{1}{2-j6\pi} = \frac{1}{6*(1-3\pi j)}$

4. (a) Basic periodic signal is x(t) = e^{jW_0t}
$a_k = \frac{1}{4}\int_T x(t)e^{-jkw_0t}\,dt \quad x(t) => a_k$
$x(t-3) => b_k$
Because of time shifting; $b_k = e^{-jkW_03} * a_k$
$x(-t) => b_k$
Because of time reversal; $b_k = a_{-k}$
Because of linearity ; $\frac{e^{-jkW_03}*a_k}{3} - \frac{2}{7} * a_k$

   (b) $f'(x) = c_k$
Because of differentiation property; $c_k = jW_0k * a_k$
$(f'(x))^3 = d_x$
$d_x = j^3(W_0)^3 a_k$

5. (a) $x[n] = \sin(\frac{\pi}{2}n)$ it is periodic with fundamental period: $N = \frac{2\pi}{\frac{\pi}{2}} = 4$
$x[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$
Therefore, Fourier coefficients of $x[n]$ are:
$a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}$

   (b) $y[n] = 1 + \cos(\frac{\pi}{2}n)$ it is periodic with fundamental period: $N = \frac{2\pi}{\frac{\pi}{2}} = 4$
$y[n] = 1 + \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})$
Therefore, Fourier coefficients of $y[n]$ are:
$b_0 = 1$, $b_1 = \frac{1}{2}$ and $b_{-1} = \frac{1}{2}$

   (c) According to multiplication property, since our signals are periodic with same period $T = 4$, fourier series coefficients of their multiplication $x[n] \times y[n]$ will be :

$$h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

We can use convolution for the right hand side of this equation like $h_k = a_k * b_k$. Therefore:
$a_k = \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]$ from part a
$b_k = \delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$ from part a

$$h_k = (\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]) * (\delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1])$$

$$= \frac{1}{2j}\delta[k-1] + \frac{1}{4j}\delta[k-2] + \frac{1}{4j}\delta[k] - \frac{1}{2j}\delta[k+1] - \frac{1}{4j}\delta[k] - \frac{1}{4j}\delta[k+2]$$

$$h_k = \frac{1}{2j}\delta[k-1] + \frac{1}{4j}\delta[k-2] - \frac{1}{2j}\delta[k+1] - \frac{1}{4j}\delta[k+2]$$

Therefore: $h_1 = \frac{1}{2j}$, $h_2 = \frac{1}{4j}$, $h_{-1} = -\frac{1}{2j}$, $h_{-2} = -\frac{1}{4j}$

   (d)

$$x[n] \times y[n] = \sin(\frac{\pi}{2}n) \times (1 + \cos(\frac{\pi}{2}n))$$

$$= \sin(\frac{\pi}{2}n) + (\sin(\frac{\pi}{2}n) \times \cos(\frac{\pi}{2}n))$$

$$= \sin(\frac{\pi}{2}n) + \frac{1}{2}(\sin(\frac{\pi}{2}n + \frac{\pi}{2}n) + \sin(\frac{\pi}{2}n - \frac{\pi}{2}n))$$

$$= \sin(\frac{\pi}{2}n) + \frac{1}{2}(\sin(\pi n) + 0)$$

$$= \sin(\frac{\pi}{2}n) + \frac{1}{2}\sin(\pi n)$$

$$= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n} + \frac{1}{2}\sin(\pi n)$$

$$= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n} + \frac{1}{2}(\frac{1}{2j}e^{j\pi n} - \frac{1}{2j}e^{-j\pi n})$$
$$= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n} + \frac{1}{4j}e^{2j\frac{\pi}{2}n} - \frac{1}{4j}e^{-2j\frac{\pi}{2}n}$$

Therefore, $h_1 = \frac{1}{2j}$, $h_2 = \frac{1}{4j}$, $h_{-1} = -\frac{1}{2j}$, $h_{-2} = -\frac{1}{4j}$
We got the same result that we have already found in part c.

6.

$$a_k = \cos(\frac{k\pi}{6}) + \sin(\frac{5k\pi}{6})$$
$$a_k = \frac{1}{2}(e^{jk\frac{\pi}{6}} + e^{-jk\frac{\pi}{6}}) + \frac{1}{2j}(e^{j5k\frac{\pi}{6}} - e^{-j5k\frac{\pi}{6}})$$
$$a_k = \frac{1}{2}e^{j\frac{\pi}{6}k} + \frac{1}{2}e^{-j\frac{\pi}{6}k} + \frac{1}{2j}e^{j\frac{5\pi}{6}k} - \frac{1}{2j}e^{-j\frac{5\pi}{6}k}$$

We know $N = 12$. By considering analysis equation $a_k = \frac{1}{N}\sum_{n=N} x[n]e^{-jk(2\pi/N)n}$, we can create below similarity easily:

$$a_k = \frac{1}{2}e^{j\frac{2\pi}{12}k1} + \frac{1}{2}e^{-j\frac{2\pi}{12}k1} + \frac{1}{2j}e^{j\frac{2\pi}{12}k5} - \frac{1}{2j}e^{-j\frac{2\pi}{12}k5}$$

However, we need to e^{-j} at each term to create a full similarity to the analysis equation. For this reason, we can use $e^{-j2\pi k}$ which is actually equals to 1 according to Euler equation:
$e^{-j2\pi k} = \cos(-2\pi k) + j\sin(-2\pi k) = \cos(-2\pi k) = 1$
By multiplying the terms that do not contain e^{-j} in our equation with $e^{-j2\pi k}$, we get:

$$a_k = \frac{1}{2}(e^{-j2\pi k})e^{j\frac{2\pi}{12}k1} + \frac{1}{2}e^{-j\frac{2\pi}{12}k1} + \frac{1}{2j}(e^{-j2\pi k})e^{j\frac{2\pi}{12}k5} - \frac{1}{2j}e^{-j\frac{2\pi}{12}k5}$$
$$a_k = \frac{1}{2}e^{(-j2\pi k)(1-\frac{1}{12})} + \frac{1}{2}e^{-j\frac{2\pi}{12}k1} + \frac{1}{2j}e^{(-j2\pi k)(1-\frac{5}{12})} - \frac{1}{2j}e^{-j\frac{2\pi}{12}k5}$$
$$a_k = \frac{1}{2}e^{(-j2\pi k)(\frac{11}{12})} + \frac{1}{2}e^{-j\frac{2\pi}{12}k1} + \frac{1}{2j}e^{(-j2\pi k)(\frac{7}{12})} - \frac{1}{2j}e^{-j\frac{2\pi}{12}k5}$$
$$a_k = \frac{1}{2}e^{-j\frac{2\pi}{12}k\cdot 11} + \frac{1}{2}e^{-j\frac{2\pi}{12}k\cdot 1} + \frac{1}{2j}e^{-j\frac{2\pi}{12}k\cdot 7} - \frac{1}{2j}e^{-j\frac{2\pi}{12}k\cdot 5} \ (eq.\ 1)$$

Now, we can use analysis equation without any problem. We also already know $N = 12$. Therefore:

$$a_k = \frac{1}{N}\sum_{n=N} x[n]e^{-jk(2\pi/N)n}$$
$$a_k = \frac{1}{12}\sum_{n=0}^{11} x[n]e^{-jk(2\pi/12)n}$$
$$a_k = \frac{1}{12}(x[0]e^{-jk(2\pi/12)\cdot 0} + x[1]e^{-jk(2\pi/12)\cdot 1} + x[2]e^{-jk(2\pi/12)\cdot 2} + \ \ldots \ + x[11]e^{-jk(2\pi/12)\cdot 11})$$

Here, we can easily say that we have nonzero Fourier coefficients for only $n = 1, 5, 7, 11$ by checking our derived equation above (eq .1).

$$for\ n = 1: \ \frac{1}{12}x[1] = \frac{1}{2},\ so\ x[1] = 6$$
$$for\ n = 5: \ \frac{1}{12}x[5] = -\frac{1}{2j},\ so\ x[5] = 6j$$
$$for\ n = 7: \ \frac{1}{12}x[7] = \frac{1}{2j},\ so\ x[7] = -6j$$
$$for\ n = 11: \ \frac{1}{12}x[11] = \frac{1}{2},\ so\ x[11] = 6$$

Therefore, according to these values, our signal $x[n]$ will be:

$$x[n] = 6\delta[n-1] + 6j\delta[n-5] - 6j\delta[n-7] + 6\delta[n-11]$$

7. (a) $N = 4$

We can use analysis equation. It is:

$$a_k = \frac{1}{N}\sum_{n=N} x[n]e^{-jk(2\pi/N)n}$$
$$= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n}$$
$$= \frac{1}{4}(x[0] + x[1]e^{-jk(2\pi/4)} + x[2]e^{-jk(2\pi/4)2} + x[3]e^{-jk(2\pi/4)3})$$
$$= \frac{1}{4}(0 + e^{-jk(2\pi/4)} + 2e^{-jk(2\pi/4)2} + e^{-jk(2\pi/4)3})$$
$$= \frac{1}{4}(e^{-jk\pi/2} + 2e^{-jk\pi} + e^{-jk3\pi/2})$$
$$= \frac{1}{4}((\cos(-\pi/2) + j\sin(-\pi/2))^k + 2(\cos(-\pi) + j\sin(-\pi))^k + (\cos(-3\pi/2) + j\sin(-3\pi/2))^k$$
$$a_k = \frac{1}{4}((-j)^k + 2(-1)^k + (j)^k)$$

We know that $a_k = a_k + N$ since the signal is periodic with $N = 4$. Therefore,

$$a_0 = \frac{1}{4}((-j)^0 + 2(-1)^0 + (j)^0) = 1$$
$$a_1 = \frac{1}{4}((-j)^1 + 2(-1)^1 + (j)^1) = \frac{1}{4}(-j - 2 + j) = -\frac{1}{2}$$
$$a_2 = \frac{1}{4}((-j)^2 + 2(-1)^2 + (j)^2) = \frac{1}{4}(-1 + 2 - 1) = 0$$
$$a_3 = \frac{1}{4}((-j)^3 + 2(-1)^3 + (j)^3) = \frac{1}{4}(j - 2 - j) = -\frac{1}{2}$$

We do not need to find other coefficients, we can just use the period ($N = 4$) for other coefficients according to above fact ($a_k = a_k + 4$). Therefore, magnitude graph of a_k's is:

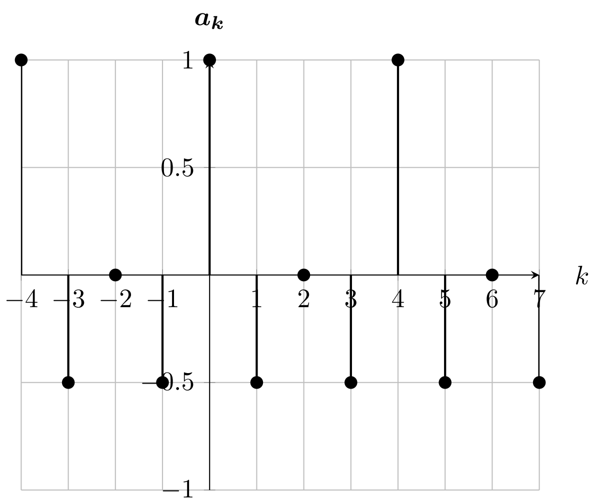

Figure 8: k vs. a_k (continues towards left and right with period N=4)

Also, it is easily seen that spectrum graph consists of all 0. Therefore, it is not drawn.

(b) **i.** It is easily seen that $y[n]$ is 0 for $n = ..., -5, -1, 3, 7, ...$ However, $x[n]$ is 1 for same n values. This is the only difference between two graphs. Therefore, we can substract 1 from $x[n]$ for these n values to reach $y[n]$ signal. We can use unit impulse for this substraction.

$$y[n] = x[n] - \sum_{m=-\infty}^{\infty} \delta[n - 4m + 1]$$

ii.

$N = 4$ again.

We can use analysis equation. It is:

$$b_k = \frac{1}{N}\sum_{n=N} y[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-jk(2\pi/4)n}$$

$$= \frac{1}{4}(y[0] + y[1]e^{-jk(2\pi/4)} + y[2]e^{-jk(2\pi/4)2} + y[3]e^{-jk(2\pi/4)3})$$

$$= \frac{1}{4}(0 + e^{-jk(2\pi/4)} + 2e^{-jk(2\pi/4)2}$$

$$= \frac{1}{4}(e^{-jk\pi/2} + 2e^{-jk\pi})$$

$$= \frac{1}{4}((\cos(-\pi/2) + j\sin(-\pi/2))^k + 2(\cos(-\pi) + j\sin(-\pi))^k$$

$$b_k = \frac{1}{4}((-j)^k + 2(-1)^k)$$

We know that $b_k = b_k + N$ since the signal is periodic with $N = 4$. Therefore,

$$b_0 = \frac{1}{4}((-j)^0 + 2(-1)^0) = \frac{3}{4}$$

$$b_1 = \frac{1}{4}((-j)^1 + 2(-1)^1) = \frac{-j-2}{4}$$

$$b_2 = \frac{1}{4}((-j)^2 + 2(-1)^2) = \frac{1}{4}(-1+2) = \frac{1}{4}$$

$$b_3 = \frac{1}{4}((-j)^3 + 2(-1)^3) = \frac{1}{4}(j-2) = \frac{j-2}{4}$$

We do not need to find other coefficients, we can just use the period ($N = 4$) for other coefficients according to above fact ($b_k = b_k + 4$). Magnitudes are:

$$|b_0| = \sqrt{(\frac{3}{4})^2 + 0^2} = \frac{3}{4}$$

$$|b_1| = \sqrt{(\frac{-2}{4})^2 + (\frac{-1}{4})^2} = \sqrt{\frac{5}{16}}$$

$$|b_2| = \sqrt{(\frac{1}{4})^2 + 0^2} = \frac{1}{4}$$

$$|b_3| = \sqrt{(\frac{-2}{4})^2 + (\frac{1}{4})^2} = \sqrt{\frac{5}{16}}$$

Therefore, magnitude graph of b_k's is:

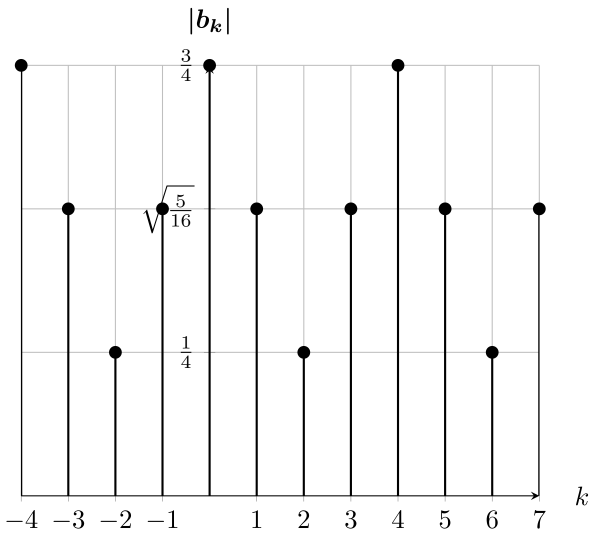

Figure 9: k vs. $|b_k|$ (continues towards left and right with period N=4)

Phases are:

$$\angle b_0 = \arctan(0) = 0$$

$$\underline{/b_1} = \arctan(\frac{-\frac{1}{4}}{-\frac{2}{4}}) - \pi(since\ it\ is\ in\ third\ region\ in\ unit\ circle) = -0.85\pi$$

$$\underline{/b_2} = \arctan(0) = 0$$

$$\underline{/b_3} = \arctan(\frac{\frac{1}{4}}{-\frac{2}{4}}) + \pi(since\ it\ is\ in\ second\ region\ in\ unit\ circle) = 0.85\pi$$

The phase graph of b_k's is:

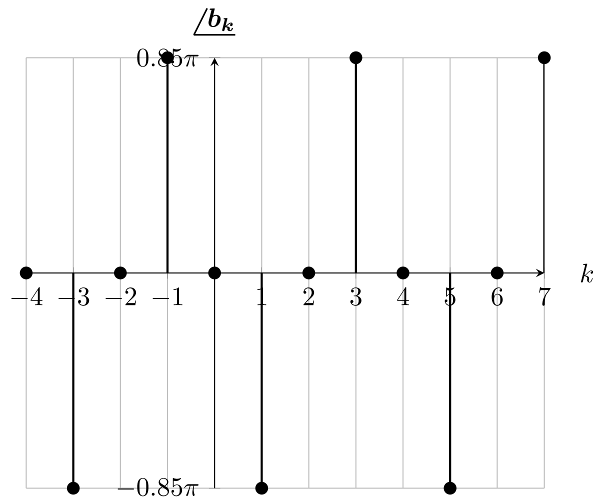

Figure 10: k vs. $\underline{/b_k}$ (continues towards left and right with period N=4)