CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 1

TONKAL, Özlem
e1881531@ceng.metu.edu.tr

BAŞŞİMŞEK, Orçun
e2098804@ceng.metu.edu.tr

July 23, 2021

---

1. In the lectures, we have seen that:

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

If we expand this summation:
$e^t = \frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + ...$

$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + ...$

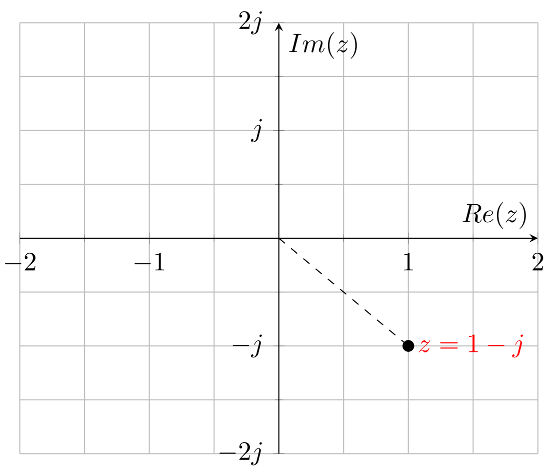
If we take derivative of both side:
$\frac{de^t}{dt} = \frac{d}{dt}(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + ...)$

$\frac{de^t}{dt} = 0 + \frac{1}{1!} + \frac{2t}{2!} + \frac{3t^2}{3!} + \frac{4t^3}{4!} + \frac{5t^4}{5!} + \frac{6t^5}{6!} + ...$

$\frac{de^t}{dt} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + ... = e^t$
As you see, after simplifying the right hand side, we reached the same e^t according to above definition since it goes to infinity.
Therefore, $\frac{de^t}{dt} = e^t$

2. (a) If $z = x + jy$ and $z - 3 = j - 2\bar{z}$ and, we also know that conjugate of z is: $\bar{z} = x - jy$. Therefore,
$x + jy - 3 = j - 2(x - jy)$
$x - 3 + jy = -2x + (2y + 1)j$
Then,
$x - 3 = -2x$ and $y = 2y + 1$
$x = 1$ and $y = -1$
Therefore,
$z = 1 - j$

For $z = x + jy$, we know that $|z| = \sqrt{x^2 + y^2}$. Therefore,
$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
$|z|^2 = 2$ (answer for first part).
<u>For second part:</u> We have already found z as $z = 1 - j$ above, and its plot is:



(b) Given $z = re^{j\theta}$ and $z^4 = -81$

$z^4 = r^4 e^{4j\theta} = -81$
$r^4(\cos(4\theta) + j\sin(4\theta)) = -81$

Here, we have only real -81 at the right hand side of the equation. For this reason, left hand side should also be real.
Therefore, $j\sin(4\theta) = 0$.
Therefore,
$4\theta = 0$, i.e. $\theta = 0$ or
$4\theta = \pi$, i.e. $\theta = \frac{\pi}{4}$

First case: If we put $\theta = 0$ to our equation:
$r^4(\cos(0) + j\sin(0)) = -81$
$r^4(1 + 0) = -81$
$r^4 = -81$ Since r(magnitude) must be a real value, there can not be a real number such that its fourth power is equal to negative number. Therefore, we could not reach the solution.

Second case: If we put $\theta = \frac{\pi}{4}$ to our equation:
$r^4(\cos(4\frac{\pi}{4}) + j\sin(4\frac{\pi}{4})) = -81$
$r^4(-1 + 0) = -81$
$-r^4 = -81$
$r^4 = 81$
$r = 3$ Now, we get the real value for r(magnitude). Therefore our complex number z is:
$z = 3e^{j\frac{\pi}{4}}$ [RESULT]

(c) $z = \frac{(\frac{1}{2}+\frac{1}{2}j)(1-j)}{1-\sqrt{3}j} = \frac{\frac{1}{2}+\frac{-j}{2}+\frac{j}{2}-\frac{1}{2}j^2}{1-\sqrt{3}j} = \frac{1}{1-\sqrt{3}j}$

if we multiply both numerator and denominator of z with conjugate of $1-\sqrt{3}j$ (which is equal to $1+\sqrt{3}j$):

$z = \frac{1+\sqrt{3}j}{(1+\sqrt{3}j)(1-\sqrt{3}j)} = \frac{1+\sqrt{3}j}{1-\sqrt{3}j+\sqrt{3}j-3j^2} = \frac{1+\sqrt{3}j}{4}$

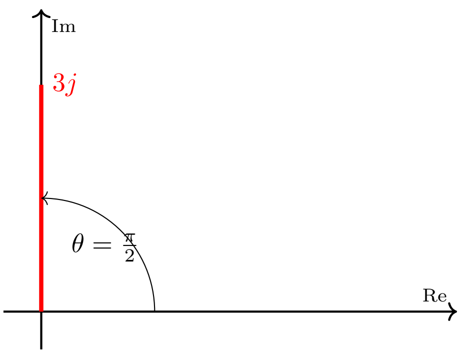
Hence, $z = \frac{1+\sqrt{3}j}{4}$

For magnitude: $|z| = \sqrt{(\frac{1}{4})^2 + (\frac{\sqrt{3}}{4})^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

For angle: We know that $\theta = \tan^{-1}(\frac{b}{a})$ for $z = a + bj$
Therefore, $\theta = \tan^{-1}(\frac{\frac{\sqrt{3}}{4}}{\frac{1}{4}}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
Therefore, magnitude is: $|z| = \frac{1}{2}$ and angle is: $\theta = \frac{\pi}{3}$ [RESULT]

(d) $z = -\frac{3}{j}e^{j\frac{\pi}{2}}$
Let's say $z = z_1 z_2$, $z_1 = -\frac{3}{j}$ and $z_2 = e^{j\frac{\pi}{2}}$
First, we need to convert $z_1 = -\frac{3}{j}$ to polar form.
$z_1 = \frac{-3}{j}$, multiply numerator and denominator with conjugate of denominator which is $-j$.
$\frac{(-3)(-j)}{(j)(-j)} = \frac{3j}{-j^2} = 3j$

When we plot $z_1 = 3j$ on the complex plane:



We see that magnitude of z_1 is equal to $r = 3$ and angle of z_1 is $\theta = \frac{\pi}{2}$.
Therefore, polar form of z_1 is equal to:
$z_1 = 3e^{j\frac{\pi}{2}}$

Our $z_2 = e^{j\frac{\pi}{2}}$ was already in polar form.
Therefore, when we put these values on given equation:
$z = z_1 z_2$
$z = 3e^{j\frac{\pi}{2}} e^{j\frac{\pi}{2}}$
$z = 3e^{j\pi}$ [RESULT]

3. $y(t) = 2x(\frac{1}{2}t + 3)$

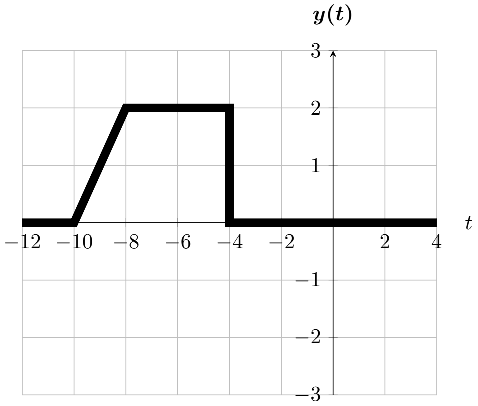


Figure 1: t vs. $y(t)$.

4. (a) Firstly, we need to find $x[-n]$:

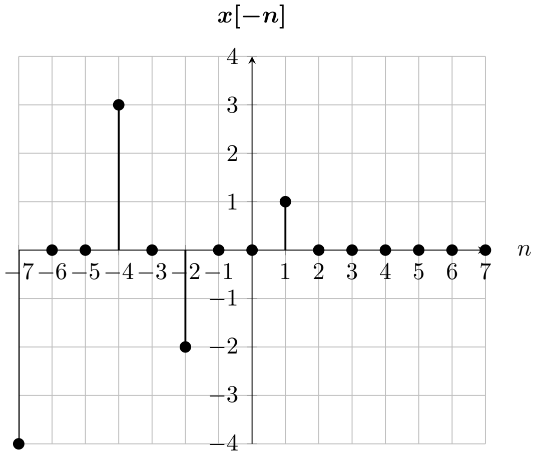


Figure 2: n vs. $x[-n]$.

Secondly, we need to find $x[2n+1]$:

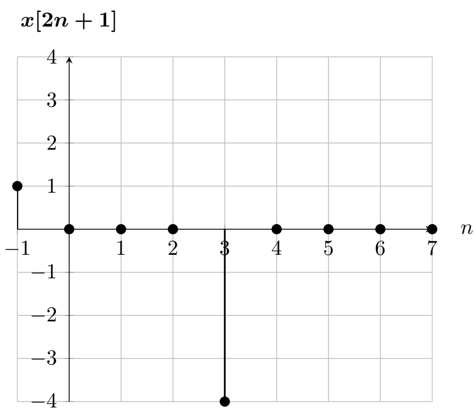


Figure 3: n vs. $x[2n+1]$.

RESULT is $x[-n]+x[2n+1]$:

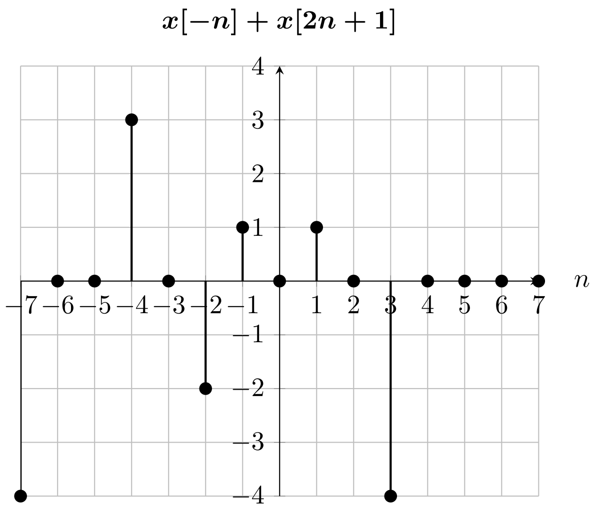


Figure 4: n vs. $x[-n]+x[2n+1]$.

(b) $x[-n]+x[2n+1] = -4\delta[n+7]+3\delta[n+4]-2\delta[n+2]+\delta[n+1]+\delta[n-1]-4\delta[n-3]$

5. (a) We know that $\cos(t)$ is a periodic function with fundamental period $T_0 = 2\pi$.
For given signal $x(t) = 3\cos(7\pi t - \frac{4\pi}{5})$:

- Coefficient 3 just determines amplitude, it does not affect period.
- $-\frac{4\pi}{5}$ is just time-shift and it also does not affect period of cosine.
- 7π which is coefficient of t is time-scale and it affects period of cosine.

Thus, given signal is periodic with fundamental period $T_0 = \frac{2\pi}{7\pi} = \frac{2}{7}$

(b) We know that discrete time signal $\sin[wn]$ is periodic with fundamental period $N_0 = \frac{2\pi}{w}k$ for possible minimum $k \in \mathbb{Z}^+$ which makes $N_0 \in \mathbb{Z}^+$ also.
Therefore, for given discrete time signal $\sin[4n - \frac{\pi}{2}]$:

$N_0 = \frac{2\pi}{4}k$
Here, since π is irrational number, there is no any $k \in \mathbb{Z}^+$ which makes N_0 integer.
Therefore, given discrete time signal is **not periodic**.

(c) For $2\cos[\frac{7\pi}{5}n]$, fundamental period $N_0 = \frac{2\pi}{\frac{7\pi}{5}}k$ such that k should be minimum possible positive integer that makes $N_0 \in \mathbb{Z}^+$ also.

Therefore, $N_0 = \frac{2\pi}{\frac{7\pi}{5}}k = 2\pi\frac{5}{7\pi}k = \frac{10}{7}k$
When $k = 7$, $N_0 = 10$

For $7\sin[\frac{5\pi}{2}n - \frac{\pi}{3}]$, again according to above definition:

$N_0 = \frac{2\pi}{\frac{5\pi}{2}}k = 2\pi\frac{2}{5\pi}k = \frac{4}{5}k$
When $k = 5$, $N_0 = 4$

As a result, for given signal in the question $x[n] = 2\cos[\frac{7\pi}{5}n] + 7\sin[\frac{5\pi}{2}n - \frac{\pi}{3}]$:
We need to find least common multiple of periods of these two signals which is equal to:
LCM(10,4) = 20
Thus, given signal is periodic with period $N = 20$.

6. (a) We know that the signal is even if $x(t) = x(-t)$. When we look at the given figure:
For $-2 < t < -1$, outputs are: $0 < x(t) < 1$
However, for $1 < -t < 2$, outputs are: $x(-t) = 0$. This contradicts to above statement.
In other words, for this region of inputs, the signal is not symmetric with respect to y axis. Therefore, this is a contradiction and given signal is not even.

We know that the signal is odd if $x(t) = -x(-t)$. When we look at the given figure:
$x(1) = 1$, and $-x(-1) = -1$. They are not equal to each other. This is a contradiction to above statement.
Thus, this signal is not odd.

(b) We know that $Even\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$ and $Odd\{x(t)\} = \frac{1}{2}(x(t) - x(-t))$

$$x(t) = \begin{cases} 0 & t < -2 \\ t+2 & -2 \le t < -1 \\ 1 & -1 \le t \le 1 \\ 0 & 1 < t \end{cases}$$

$$x(-t) = \begin{cases} 0 & t < -1 \\ 1 & -1 \le t \le 1 \\ -t+2 & 1 < t \le 2 \\ 0 & 2 < t \end{cases}$$

According to above definitions:

$$Even\{x(t)\} = \frac{1}{2}(x(t) + x(-t)) = \begin{cases} 0 & t < -2 \\ \frac{t+2}{2} & -2 \le t < -1 \\ 1 & -1 \le t \le 1 \\ \frac{-t+2}{2} & 1 < t \le 2 \\ 0 & 2 < t \end{cases}$$

$$Odd\{x(t)\} = \frac{1}{2}(x(t) - x(-t)) = \begin{cases} 0 & t < -2 \\ \frac{t+2}{2} & -2 \le t < -1 \\ 0 & -1 \le t \le 1 \\ \frac{t-2}{2} & 1 < t \le 2 \\ 0 & 2 < t \end{cases}$$

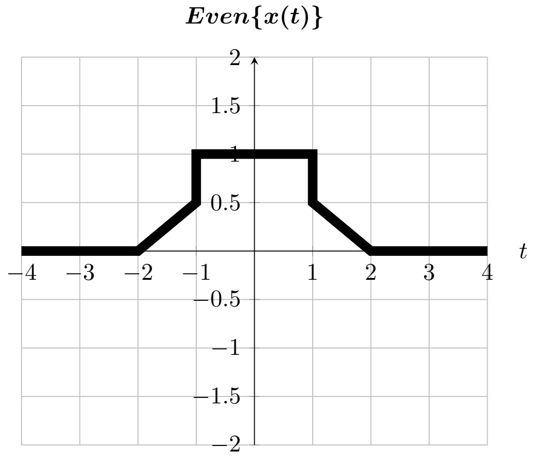


Figure 5: t vs. $Even\{x(t)\}$.

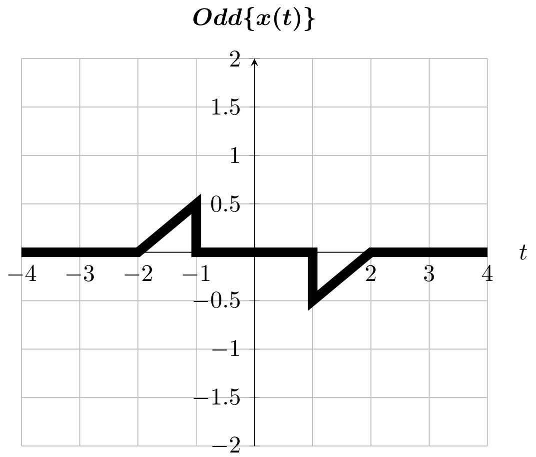


Figure 6: t vs. $Odd\{x(t)\}$.

7. (a) $x(t) = -3u(t-2) + 5u(t-3) - 3u(t-5)$

(b) $\frac{dx(t)}{dt} = -3\frac{du(t-2)}{dt} + 5\frac{du(t-3)}{dt} - 3\frac{du(t-5)}{dt}$

$\frac{dx(t)}{dt} = -3\delta(t-2) + 5\delta(t-3) - 3\delta(t-5)$

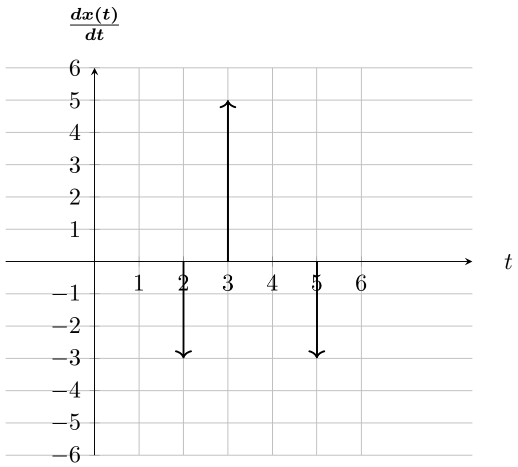


Figure 7: t vs. $\frac{dx(t)}{dt}$.

8. (a)
- The system has memory because current output depends on past value of input. For example, for $n = 1$:
  $y[1] = x[3 \times 1 - 5] = x[-2]$ (Output at $n = 1$ depends on past input at $n = -2$)
- The system is stable because when we choose input signal as bounded signal such as unit step function, outputs will also be bounded.
- The system is **not** causal because for some outputs, output depends on future value of input. For example, for $n = 4$:
  $y[4] = x[3 \times 4 - 5] = x[7]$ (Output at $n = 4$ depends on future input at $n = 7$, and this contradicts to causality principle)
- The system is not time invariant.
  Definition of time-invariance is: $y[n - n_0] = h[x[n - n_0]]$
  $=> x_1[n] => x[n - n_0]$
  $=> y_1[n] = x_1[3n - 5]$
  $=> y_1[n] = x[3n - n_0 - 5]$
  $=> y_1'[n] = y[n - n_0]$
  $=> y_1'[n] = x[3(n - n_0) - 5]$
  $=> y_1'[n] = x[3n - 3n_0 - 5]$
  $y_1' \neq y_1$
  Thus, this system is not time-invariant.
- The system is invertible,we can find inverse of h
  $y[n] = h[x[3n - 5]]$
  $x[n] = h^{-1}[y[(n + 5)/3]]$
- The system is linear if superposition property hold
  $y[n] = x[3n - 5]$
  $y_1[n] = x_1[3n - 5]$
  $y_2[n] = x_2[3n - 5]$
  $x_3 = a_1 \times x_1 + a_2 \times x_2$
  $y_3 = x_3[3n - 5]$
  $y_3 = a_1 \times x_1[3n - 5] + a_2 \times x_2[3n - 5]$
  $y_3' = a_1 \times y_1 + a_2 \times y_2$
  $y_3 = y_3'$ This system is linear

(b)
- The system has memory because current output depends on past value of input. For example, for $n = 1$:
  $y(1) = x(3 \times 1 - 5) = x(-2)$ (Output at $n = 1$ depends on past input at $n = -2$)
- The system is stable because when we choose input signal as bounded signal, outputs will also be bounded.
- The system is **not** causal because for some outputs, output depends on future value of input. For example, for $n = 4$:
  $y(4) = (3 \times 4 - 5) = x(7)$ (Output at $n = 4$ depends on future input at $n = 7$, and this contradicts to causality principle)
- The system is not time invariant.
  Definition of time-invariance is: $y[t - t_0] = h[x[t - t_0]]$

$=> x_1(t) => x(t-t_0)$
$=> y_1(t) = x_1(3t-5)$
$=> y_1(t) = x(3t-t_0-5)$
$=> y_1'(t) = y(t-t_0)$
$=> y_1'(t) = x(3(t-t_0)-5)$
$=> y_1'(t) = x(3t-3t_0-5)$
$y_1' \neq y_1$
Thus, this system is not time-invariant.

- The system is invertible,we can try find inverse of h
  $y(t) = h(x(3t-5))$
  $x(t) = h^{-1}(y((t+5)/3))$
  The system is invertible.
- The system is linear if superposition property hold
  $y(n) = x(3n-5)$
  $y_1(n) = x_1(3n-5)$
  $y_2(n) = x_2(3n-5)$
  $x_3 = a_1 \times x_1 + a_2 \times x_2$
  $y_3 = x_3(3n-5)$
  $y_3 = a_1 \times x_1(3n-5) + a_2 \times x_2(3n-5)$
  $y_3' = a_1 \times y_1 + a_2 \times y_2$
  $y_3 = y_3'$ This system is linear

(c)
- The system has memory because
  $y(t) = tx(t-1)$. For $t = 1$
  $y(t) = 1x(0)$ we need past value of x
- The system is causal, we do not need any future value of x.
- $y(t) = h(tx(t-1))$
  $x(t) = h^{-1}(y(t+1)/(t+1))$
  we can find invertible of h.The system is invertible.
- The system is not time invariant
  $y(t) = tx(t-1)$
  $y'(t) = y(t-t0) = (t-t0) \times x(t-t0-1)$
  $x_1(t) = x(t-t0-1)$
  $y(t) = t \times x_1(t) = t \times (t-t0-1)$
  $y(t) \neq y'(t)$
  Thus, the system is not time-invariant.
- The system is linear if superposition property hold
  $y(t) = x(t-1) \times t$
  $y_1(t) => t \times x_1(t-1)$
  $y_2(t) => t \times x_2(t-1)$
  $x_3 = a_1 \times x_1 + a_2 \times x_2$
  $y_3 = x_3 \times t$
  $y_3 = t \times (a_1 \times x_1 + a_2 \times x_2) = a_1 \times y_1 + a_2 \times y_2$
  $t \times (a_1 \times x_1(t-1) + a_2 \times x_2(t-1)) = a_1 \times t \times x_1(t-1) + a_2 \times t \times x_2(t-1)$
  The system is linear
- The system is not stable
  $y(t) = tx(t-1)$
  $y(t)/t = x(t-1)$ If t=0
  $y(0)/0 = x(-1)$
  Output is ∞ for $x(-1)$. Thus, output signal is unbounded and system is not stable.

(d)
- The system has memory.
  $k = 1 => x[n-1]$
  $k = 2 => x[n-2]$. It continues like this in the summation. Thus, we need past values of inputs.
- The system is casual because we do not need any future inputs. The system depends only the past inputs.
- The system is not stable. Even if all inputs are bounded and their summation can also be bounded,the output goes towards infinity since k's upper limit is also infinity, so output is not bounded.
- The system is invertible.
  $y[n+1] = x[n] + x[n-1] + x[n-2] + ...$
  $y[n] = x[n-1] + x[n-2] + ...$
  $y[n+1] - y[n] = x[n]$
  As you see, we can reach $x[n]$ again. Thus, the system is invertible.
- The system is time invariant
  $x_1[n] = x[n-n_0]$
  $y[n] = x_1[n]$

$y[n] = \sum_{k=1}^{\infty} x[n - n_0 - k]$
$y'[n] = y[n - n_0]$
$y'[n] = \sum_{k=1}^{\infty} x[n - n_0 - k]$
$y[n] = y'[n]$

- As we explained in previous items, the sytsem is linear if superposition property hold.
  $y_1[n] = \sum_{k=1}^{\infty} x_1[n - k]$
  $y_2[n] = \sum_{k=1}^{\infty} x_2[n - k]$
  $y_3[n] = a_1 y_1[n] + a_2 y_2[n]$
  $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$
  $y'_3[n] = \sum_{k=1}^{\infty} x_3[n - k]$
  $y'_3[n] = a_1 \sum_{k=1}^{\infty} x_1[n - k] + a_2 \sum_{k=1}^{\infty} x_2[n - k]$
  $y_3[n] = y'_3[n]$
  Thus, the system is linear.