CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 2

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1. (a) $-6y(t) - 5y'(t) + x'(t) = y''(t)$

   (b) We need to form general solution for $y(t)$ by finding homogeneous solution $y_h(t)$ and $y_p(t)$ for given particular input. After that, we need to sum according to superposition property.

   $y(t) = y_h(t) + y_p(t)$

   For homogeneous solution, we assume that the system has no input (i.e. $x(t) = 0$) and we can benefit from exponential function like $y_h(t) = Ce^{\alpha t}$.
   Since $x(t) = 0$, $x'(t) = 0$ also. Therefore our equation for homogeneous solution will be:

   $y''(t) + 5y'(t) + 6y(t) = 0$

   Now, we need to put $y(t) = Ce^{\alpha t}$ into above equation by finding first and second derivative of it.
   $y'(t) = C\alpha e^{\alpha t}$
   $y''(t) = C\alpha^2 e^{\alpha t}$

   $C\alpha^2 e^{\alpha t} + 5C\alpha e^{\alpha t} + 6Ce^{\alpha t} = 0$
   $Ce^{\alpha t}(\alpha^2 + 5\alpha + 6) = 0$
   When we equate $(\alpha^2 + 5\alpha + 6)$ to zero, there are two α values. These are:
   $\alpha_1 = -2$ and $\alpha_2 = -3$. Therefore, our homogeneous solution will be:

   $y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$ [equation no. 1]

   Now, we also need to find particular solution $y_p(t)$ for given particular input. Here, we can use the definition of linearity. Since the given system is LTI, any input-output pair of this system should be proportional since superposition property is hold. In other words, since given particular input is $x(t) = (e^{-t} + e^{-4t})u(t)$, we can say that:

   $y_p(t) = Ae^{-t} + Be^{-4t}$

   Now, similar to homogeneous solution case, we need to find derivatives of $y_p(t)$ and put them on the given equation again.

   $y_p'(t) = -Ae^{-t} - 4Be^{-4t}$
   $y_p''(t) = Ae^{-t} + 16Be^{-4t}$
   Also this time, for this particular solution, we need $x'(t) = (-e^{-t} - 4e^{-4t})$
   Now, put them on the main equation from part (a):

   $-6y(t) - 5y'(t) + x'(t) = y''(t)$
   $x'(t) = y''(t) + 5y'(t) + 6y(t)$

   $(-e^{-t} - 4e^{-4t}) = Ae^{-t} + 16Be^{-4t} + 5(-Ae^{-t} - 4Be^{-4t}) + 6(Ae^{-t} + Be^{-4t})$
   $-e^{-t} - 4e^{-4t} = Ae^{-t} + 16Be^{-4t} - 5Ae^{-t} - 20Be^{-4t} + 6Ae^{-t} + 6Be^{-4t}$
   $-e^{-t} - 4e^{-4t} = 2Ae^{-t} + 2Be^{-4t}$
   Hence,
   $2A = -1$, $A = -\frac{1}{2}$
   $2B = -4$, $B = -2$

By substituting A and B on $y_p(t) = Ae^{-t} + Be^{-4t}$
We get,
$y_p(t) = -\frac{1}{2}e^{-t} - 2e^{-4t}$ [equation no. 2]

Now, we need to form general solution by adding homogeneous and particular solutions (from equation no. 1 and equation no. 2).
$y(t) = y_h(t) + y_p(t)$

$y(t) = C_1e^{-2t} + C_2e^{-3t} - \frac{1}{2}e^{-t} - 2e^{-4t}$ [equation no. 3]

Since the system is initially at rest, we know that $y(0) = y'(0) = 0$. Therefore, we can use these initial conditions to find constants C_1 and C_2.

$y(0) = C_1 + C_2 - \frac{1}{2} - 2 = 0$
$C_1 + C_2 = \frac{5}{2}$ [equation no. 4]

$y'(t) = -2C_1e^{-2t} - 3C_2e^{-3t} + \frac{1}{2}e^{-t} + 8e^{-4t}$
$y'(0) = -2C_1 - 3C_2 + \frac{1}{2} + 8 = 0$
$2C_1 + 3C_2 = \frac{17}{2}$ [equation no. 5]

By solving [equation no. 4] and [equation no. 5] together, we get:
$C_1 = -1$ and $C_2 = \frac{7}{2}$
When we put these values into [equation no. 3], we finally reach the result as:

$y(t) = [-e^{-2t} + \frac{7}{2}e^{-3t} - \frac{1}{2}e^{-t} - 2e^{-4t}]u(t)$

2. (a) We know that the system is LTI. When we analyze graph of $x[n]$, it is easily seen that input $x_1[n]$ is the summation of $x[n]$ and its 2 unit shifted to the left and then multiplicated by 1 version. In other words:

$$x_1[n] = x[n] - x[n-2]$$

Here, since the system is time invariance, time shift at the input reflects same time shift at the output. In other words:

$$x[n-2] \;\; gives \;\; y[n-2]$$

Also, we already know that:

$$x[n] \;\; gives \;\; y[n]$$

Therefore, to create input $x_1[n] = x[n] - x[n-2]$, we can use superposition property since the system is linear.

$$x_1[n] = x[n] - x[n-2] \;\; gives \;\; y_1[n] = y[n] - y[n-2]$$

As a result, according to given figure of $y[n]$, we can reach $y_1[n]$ as follows:

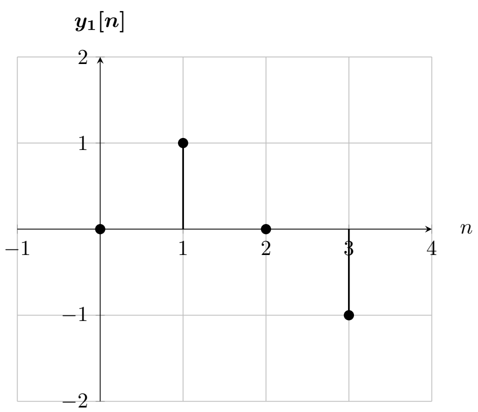


Figure 1: n vs. $y_1[n]$.

(b) We know that for discrete LTI systems, convolution summation is:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Here, we can simply use given input-output pair's values to reach impulse response $h[n]$.
Firstly, when we put $n = 0$:

$$y[0] = 0 = \sum_{k=-\infty}^{\infty} x[k]h[-k] = x[0]h[0] + x[1]h[-1]$$

$$= h[0] + h[-1]$$

Since the system is initially at rest, for any response of this system: $y[n] = 0$ when $n < 0$. Thus, for also impulse response, this condition is still valid. Therefore, $h[-1] = 0$.
Since we found $h[0] + h[-1] = 0$ at above. $h[0] = 0$ also.
When we continue to write the convolution summation terms for other n values:

$$when\ n = 1: \ y[1] = 1 = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = x[0]h[1] + x[1]h[0] \rightarrow h[1] = 1$$

$$when\ n = 2: \ y[2] = 0 = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = x[0]h[2] + x[1]h[1] \rightarrow h[2] + h[1] = 0 \rightarrow h[2] = -1$$

$$when\ n = 3: \ y[3] = 0 = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = x[0]h[3] + x[1]h[2] \rightarrow h[3] + h[2] = 0 \rightarrow h[3] = 1$$

$$when\ n = 4: \ y[4] = 0 = \sum_{k=-\infty}^{\infty} x[k]h[4-k] = x[0]h[4] + x[1]h[3] \rightarrow h[4] + h[3] = 0 \rightarrow h[4] = -1$$

Actually, we reached the pattern here for impulse response $h[n]$. It is $h[n] = 1$ when n is odd and $h[n] = -1$ when n is even (except $h[0] = 0$). And, it continues forever with this pattern. Thus, it is infinite impulse response. We already know the unit step function whose value is always 1 when $n \geq 0$. Therefore, we can write $h[n]$ in terms of it as:

$$h[n] = (-1)^{n-1}u[n-1]$$

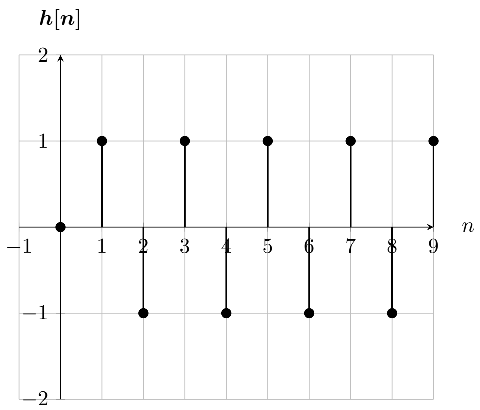


Figure 2: n vs. $h[n]$. (n continues towards infinity with this pattern)

(c) From part (b), we have already found $h[n] = (-1)^{n-1}u[n-1]$. We can also write it in terms of unit impulse functions as follows:

$$h[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4] + \delta[n-5] - \delta[n-6]...$$

Hence, with $y[n] = x[n] * h[n]$, we can easily say that:

$$y[n] = x[n-1] - x[n-2] + x[n-3] - x[n-4] + x[n-5] - x[n-6]...$$

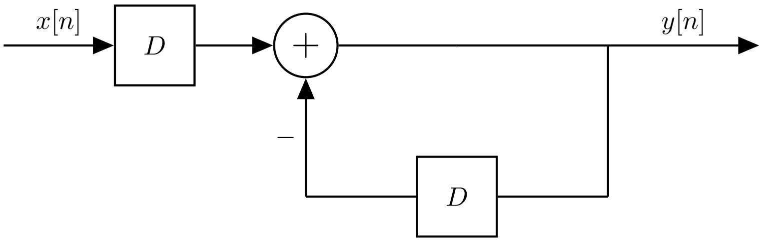
If we also evaluate $y[n-1]$ and add it to $y[n]$, we can reach more compact equation as follows:

$$y[n-1] = x[n-2] - x[n-3] + x[n-4] - x[n-5] + x[n-6] - x[n-7]...$$

$$y[n] = x[n-1] - x[n-2] + x[n-3] - x[n-4] + x[n-5] - x[n-6]...$$

$$y[n] + y[n-1] = x[n-1] \quad (RESULT\ of\ 2.c)$$

(d)



3. (a) $y[n] = x[n] * h[n]$

By using distributive property:
$y[n] = x[n] * \delta[n-1] + x[n] * 3\delta[n+2]$

By using sampling property:
$y[n] = x[n-1] + 3x[n+2]$

Hence,
$y[n] = \delta[n-4] + 3\delta[n-1] + 2\delta[n] + 6\delta[n+3]$

Graphic of $y[n]$ is as follows:

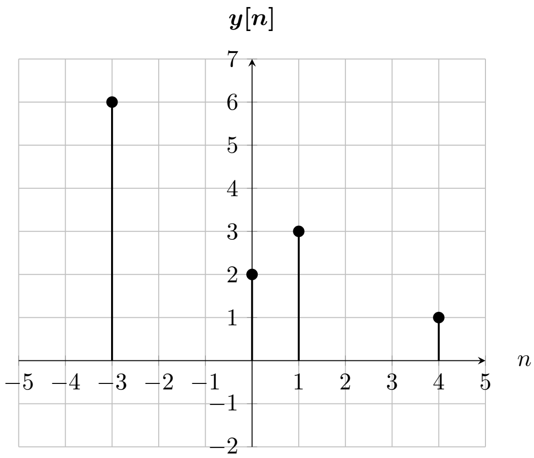


Figure 3: n vs. $y[n]$.

(b)

$$x[n] = u[n+3] - u[n]$$
$$x[n] = \sum_{k=0}^{\infty} \delta[n+3-k] - \sum_{k=0}^{\infty} \delta[n-k]$$
$$x[n] = \delta[n+3] + \delta[n+2] + \delta[n+1]$$

On the other hand, for $h[n]$:

$$h[n] = u[n-1] - u[n-3]$$
$$h[n] = \sum_{k=0}^{\infty} \delta[n-1-k] - \sum_{k=0}^{\infty} \delta[n-3-k]$$
$$h[n] = \delta[n-1] + \delta[n-2]$$

Now, for convolution:
$y[n] = x[n] * h[n]$

By using distributive property:
$y[n] = x[n] * \delta[n-1] + x[n] * \delta[n-2]$

By using sampling property:

$y[n] = x[n-1] + x[n-2]$

Hence,
$y[n] = \delta[n+2] + 2\delta[n+1] + 2\delta[n] + \delta[n-1]$

Graphic of $y[n]$ is as follows:

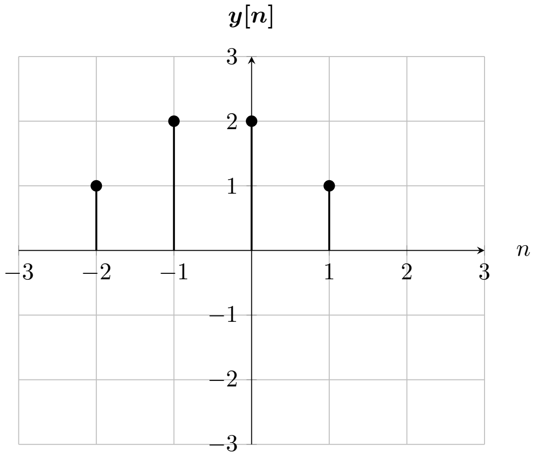


Figure 4: n vs. $y[n]$.

4. (a)

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_0^t e^{-2\tau}e^{-3(t-\tau)}\,d\tau \\ &= e^{-3t}\int_0^t e^{\tau}\,d\tau \\ &= e^{-3t}(e^t - 1) \end{aligned}$$

Thus, $y(t) = (e^{-2t} - e^{-3t})u(t)$

(b) Here, $x(t)$ is simply finite pulse between $t = 0$ and $t = 2$. In other words, $x(t)$ is:

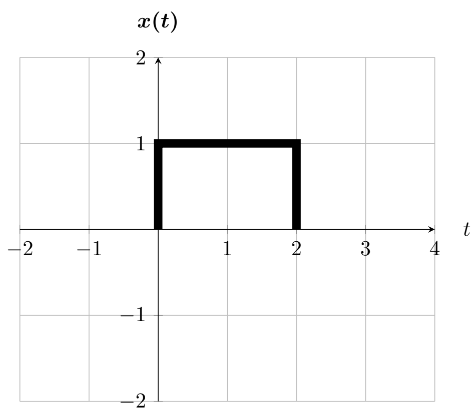


Figure 5: t vs. $x(t)$.

We need to find $x(t-\tau)$ for convolution operation. We first change the variable t to τ, then do time-reverse, and then shift by t. The resulting graph is as follows:

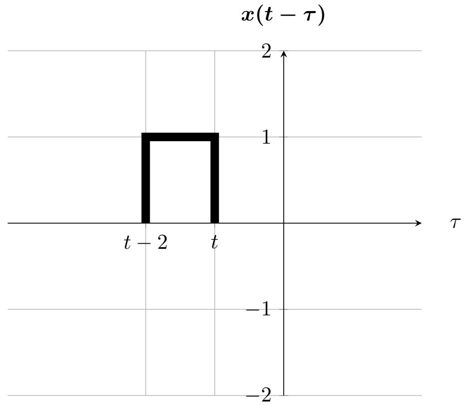


Figure 6: τ vs. $x(t-\tau)$.

We also need to $h(\tau) = e^{2\tau}u(\tau)$. Its graph is also as follows:

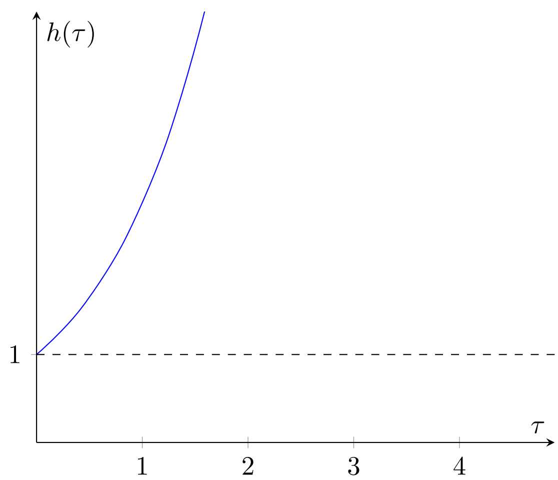


Figure 7: τ vs. $h(\tau)$.

Here, when we analyze graphs, there are 3 different cases according to values of t.
First case: When $t \leq 0$, there is no overlap between functions $x(t-\tau)$ and $h(\tau)$ as seen on the figures. Therefore, $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau = 0$

Second case: When $t > 0$ and $t-2 \leq 0$, overlap between functions $x(t-\tau)$ and $h(\tau)$ starts to occur. Therefore,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \\ &= \int_{0}^{t} x(t-\tau)h(\tau)\,d\tau \\ &= \int_{0}^{t} 1 \times e^{2\tau}\,d\tau \\ &= |_0^t \ \frac{e^{2\tau}}{2} \\ &= \frac{e^{2t}-1}{2} \end{aligned}$$

Third case: When $t-2 > 0$, overlap between functions $x(t-\tau)$ and $h(\tau)$ still continues. Therefore,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \\ &= \int_{t-2}^{t} x(t-\tau)h(\tau)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_{t-2}^{t} 1 \times e^{2\tau}\, d\tau \\
&= |_{t-2}^{t} \; \frac{e^{2\tau}}{2} \\
&= \frac{e^{2t} - e^{2t-4}}{2} \\
&= \frac{e^{2t}(1 - e^{-4})}{2}
\end{aligned}
$$

When we combine these three cases, we get $y(t)$ as:

$$
y(t) = \begin{cases} 0 & t \leq 0 \\ \frac{e^{2t}-1}{2} & 0 < t \leq 2 \\ \frac{e^{2t}(1-e^{-4})}{2} & t > 2 \end{cases}
$$

5. (a)

$$
\begin{aligned}
h[n] &= s[n] - s[n-1] \\
&= nu[n] - (n-1)u[n-1] \\
&= nu[n] - nu[n-1] + u[n-1] \\
&= n(u[n] - u[n-1]) + u[n-1] \\
&= n\delta[n] + u[n-1]
\end{aligned}
$$

Since $n\delta[n]$ term is 0, impulse response is:

$$
h[n] = u[n-1]
$$

(b) We can use convolution summation to find given output $y[n] = \delta[n] - \delta[n-1]$:

$$
y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]
$$

We have already found impulse response, so put it on this equation:

$$
y[n] = \delta[n] - \delta[n-1] = \sum_{k=-\infty}^{\infty} x[k]u[n-k-1]
$$

Since left-hand side of the equation contains unit impulse functions, on the right-hand side we can write $u[n-k-1]$ in terms of unit impulse functions as well.

$$
y[n] = \delta[n] - \delta[n-1] = \sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k-1] + \delta[n-k-2] + \delta[n-k-3] + \delta[n-k-4] + ...\}
$$

For right-hand side of this equation, when we write the terms of summation a little bit:

$$
\begin{aligned}
&when\ k = -1: \ x[-1]\{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + ...\} \\
&when\ k = 0: \ x[0]\{\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + ...\} \\
&when\ k = 1: \ x[1]\{\delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + ...\}
\end{aligned}
$$

Since we had $y[n] = \delta[n] - \delta[n-1]$ on left-hand side, we need to have $x[-1] = 1$, $x[0] = -2$, $x[1] = 1$ to obtain same result on right-hand side. Hence, the graph of $x[n]$ is as follows:

$x[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$ (RESULT of 5.b)

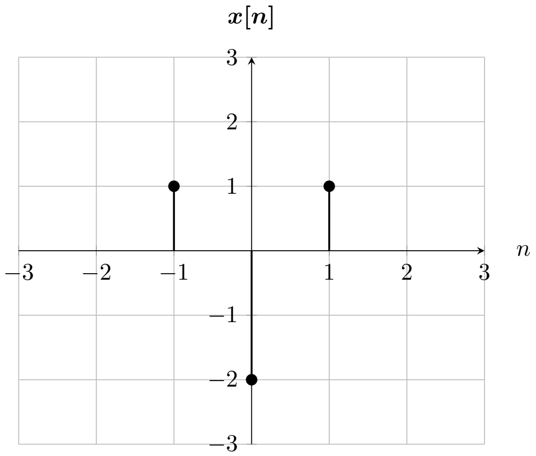


Figure 8: n vs. $x[n]$.

(c) In part (a), we found $h[n] = u[n-1]$ which is also equal to:

$$h[n] = \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + ...$$

Hence, with $y[n] = x[n] * h[n]$, we can easily say that:

$$y[n] = x[n-1] + x[n-2] + x[n-3] + x[n-4] + ...$$

If we also evaluate $y[n+1]$ and subtract $y[n]$ from it, we can reach more compact equation as follows:

$$-y[n] = -x[n-1] - x[n-2] - x[n-3] - x[n-4] - ...$$
$$y[n+1] = x[n] + x[n-1] + x[n-2] + x[n-3] + ...$$
$$y[n+1] - y[n] = x[n] \quad (RESULT\ of\ 5.c)$$

6. We know that $h(t) = \frac{ds(t)}{dt}$. Therefore,

$\frac{ds(t)}{dt} = tu(t)$

$h(t) = tu(t)$
Now, we can use this impulse response for convolution operation to find $y(t)$.
As an input, we already have $x(t) = e^{-t}u(t)$
Also, since both $h(t)$ and $x(t)$ are multiplied with unit step function, there is no overlap between them when $t < 0$. Therefore, when $t < 0$, $y(t) = 0$.
When $t \geq 0$:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\
&= \int_0^t e^{-\tau}(t-\tau)\,d\tau \\
&= \int_0^t te^{-\tau}\,d\tau - \int_0^t \tau e^{-\tau}\,d\tau \\
&= t\int_0^t e^{-\tau}\,d\tau - \int_0^t \tau e^{-\tau}\,d\tau \\
&= t(|_0^t - e^{-\tau}) - \int_0^t \tau e^{-\tau}\,d\tau \\
&= t(1-e^{-t}) - \int_0^t \tau e^{-\tau}\,d\tau \\
&= (t - te^{-t}) - \int_0^t \tau e^{-\tau}\,d\tau\ [eq.1]
\end{aligned}$$

Now, for this integral $\int_0^t \tau e^{-\tau}\,d\tau$, we need to use integration by parts method.

$\tau = u$ implies that $d\tau = du$
$\int e^{-\tau}\, d\tau = dv$ implies that $-e^{-\tau} = v$

$$\begin{aligned}\int_0^t \tau e^{-\tau}\, d\tau &= |_0^t uv - \int_0^t vdu \\ &= |_0^t \tau(-e^{-\tau}) - \int_0^t -e^{-\tau}\, d\tau \\ &= (-te^{-t}) + \int_0^t e^{-\tau}\, d\tau \\ &= -te^{-t} + |_0^t(-e^{-\tau}) \\ &= -te^{-t} + (1 - e^{-t})\end{aligned}$$

When we put this result to its place above on [eq.1]:

$$y(t) = (t - te^{-t}) - (-te^{-t} + 1 - e^{-t})$$
$$y(t) = e^{-t} + t - 1$$

7. (a) $x(t) * u(t) * \delta(t-3) - x(t) * u(t) * \delta(t-5) = y(t)$

When we put $\delta(t)$ as an input in this system, we will get impulse response $h(t)$:

$\delta(t) * u(t) * \delta(t-3) - \delta(t) * u(t) * \delta(t-5) = h(t)$

Here, we know that $\delta(t) * u(t) = u(t)$ according to sampling property. Therefore,

$u(t) * \delta(t-3) - u(t) * \delta(t-5) = h(t)$

Again, $u(t) * \delta(t-3) = u(t-3)$ and $u(t) * \delta(t-5) = u(t-5)$ according to sampling property. Therefore, impulse response of this system is:

$h(t) = u(t-3) - u(t-5)$

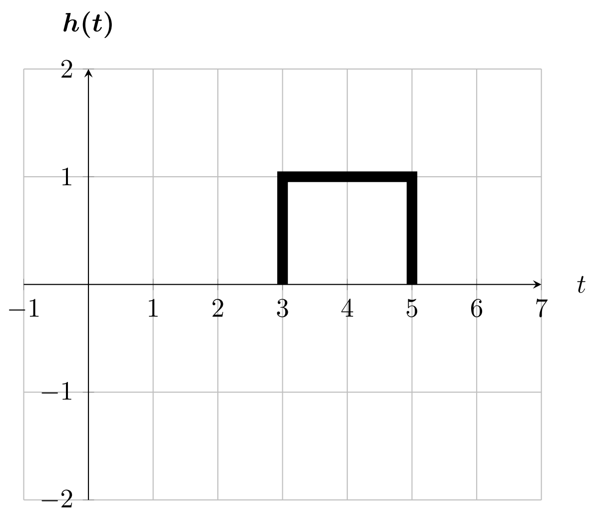


Figure 9: t vs. $h(t)$.

(b)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Here, since we already know $h(t)$ from part (a), firstly, we can plot the graph of $h(t-\tau)$ easily.

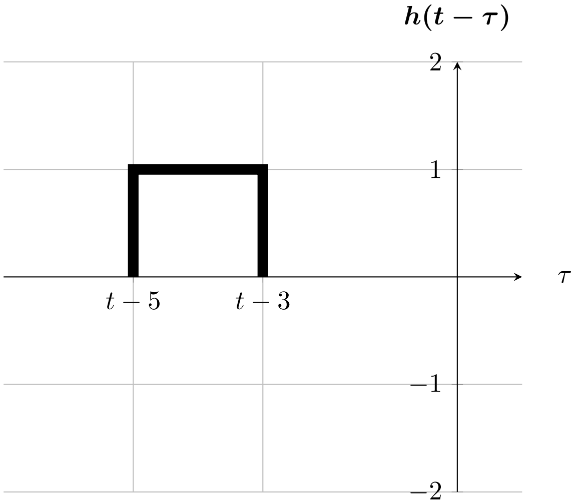


Figure 10: τ vs. $h(t-\tau)$.

We also need to $x(\tau) = e^{-3\tau}u(\tau)$. Its graph is also as follows:

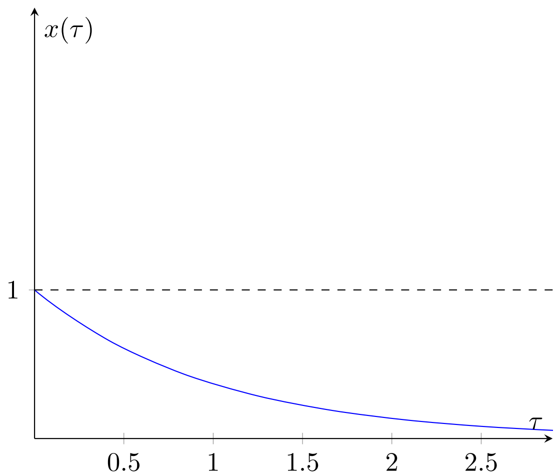


Figure 11: τ vs. $x(\tau)$.

Here, when we analyze graphs, there are 3 different cases according to values of t.
First case: When $t - 3 \leq 0$, there is no overlap between functions $x(t-\tau)$ and $h(\tau)$ as seen on the figures. Therefore,
$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = 0$

Second case: When $t - 3 > 0$ and $t - 5 \leq 0$, overlap between functions $x(t-\tau)$ and $h(\tau)$ starts to occur. Therefore,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{0}^{t-3} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{0}^{t-3} e^{-3\tau} \times 1\,d\tau \\ &= |_0^{t-3} \quad \frac{e^{-3\tau}}{-3} \\ &= \frac{1 - e^{-3t+9}}{3} \end{aligned}$$

Third case: When $t-5>0$, overlap between functions $x(t-\tau)$ and $h(\tau)$ still continues. Therefore,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\
&= \int_{t-5}^{t-3} x(\tau)h(t-\tau)\,d\tau \\
&= \int_{t-5}^{t-3} e^{-3\tau} \times 1\,d\tau \\
&= |_{t-5}^{t-3} \ \frac{e^{-3\tau}}{-3} \\
&= \frac{e^{-3t+15} - e^{-3t+9}}{3} \\
&= \frac{e^{-3t+9}(e^{6}-1)}{3}
\end{aligned}$$

When we combine these three cases, we get $y(t)$ as:

$$y(t) = \begin{cases} 0 & t \le 3 \\ \frac{1-e^{-3t+9}}{3} & 3 < t \le 5 \\ \frac{e^{-3t+9}(e^{6}-1)}{3} & t > 5 \end{cases}$$

(c) We have already found $h(t) = u(t-3) - u(t-5)$. Therefore, $\frac{dh(t)}{dt} = \delta(t-3) - \delta(t-5)$

$$\begin{aligned}
(\frac{dh(t)}{dt}) * x(t) &= (\delta(t-3) - \delta(t-5)) * x(t) \\
&= \delta(t-3) * x(t) - \delta(t-5) * x(t) \\
&= x(t-3) - x(t-5)
\end{aligned}$$