Beyond Graph Signal Processing

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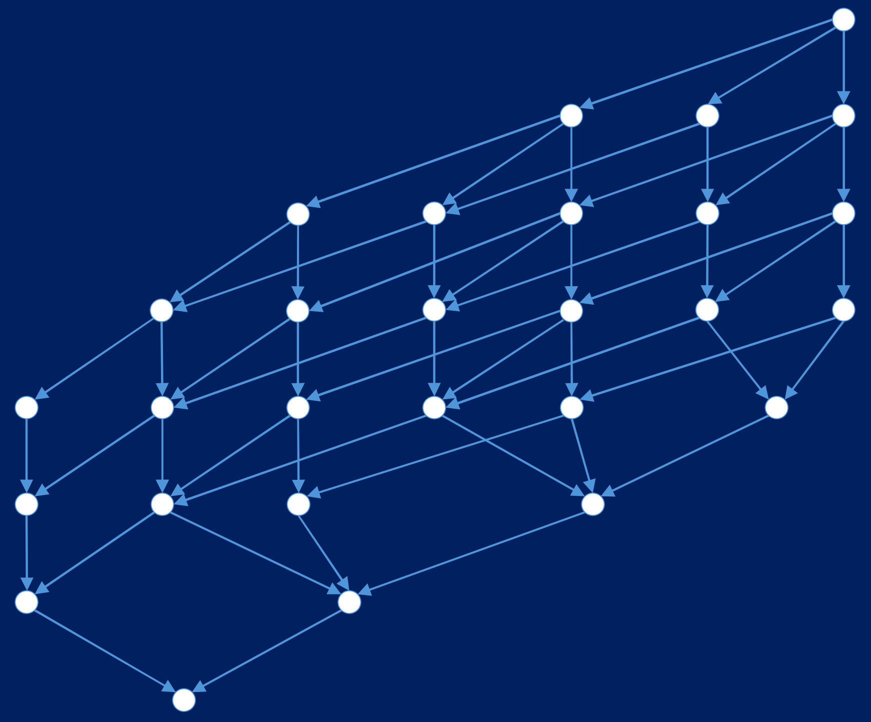
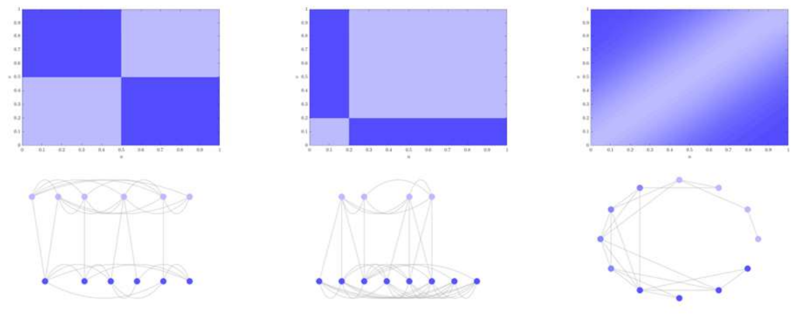
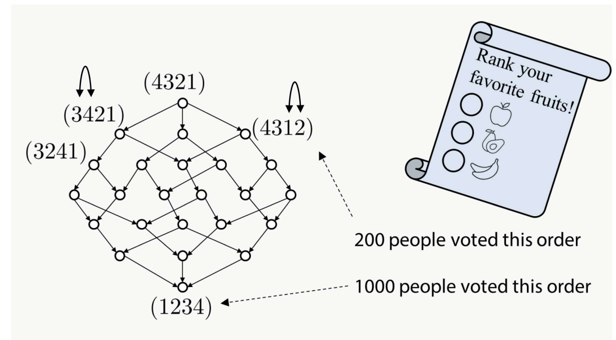
Sommersemester 2025



There is more than graphs



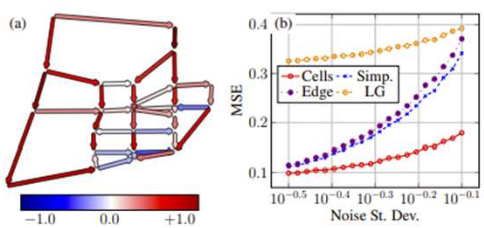
(a) SBM with balanced communities (b) SBM with imbalanced communities (c) Exponential

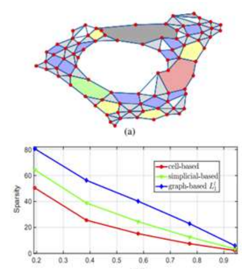


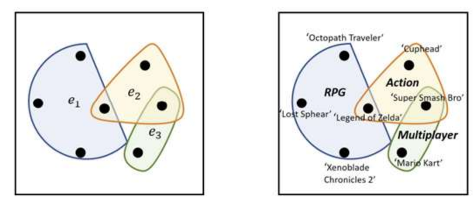
Graphons: Continuous limits of graphs

Direction, Order, and Causality

Graphon Signal Processing, Ruiz et al







Edge flows and higher-order structures

Signal Processing on Cell Complexes, Roddenberry et al. Topological Signal Processing over Cell Complexes, Sardellitti et al. ntroducing Hypergraph Signal Processing, Zhang et al.

Ruiz, Luana, Luiz FO Chamon, and Alejandro Ribeiro. "Graphon signal processing." *IEEE Transactions on Signal Processing* 69 (2021): 4961-4976.

Seifert, Bastian, Chris Wendler, and Markus Püschel. "Causal fourier analysis on directed acyclic graphs and posets." *IEEE Transactions on Signal Processing* 71 (2023): 3805-3820.

Misiakos, Panagiotis, and Markus Püschel. "SpinSVAR: Estimating Structural Vector Autoregression Assuming Sparse Input." *The 41st Conference on Uncertainty in Artificial Intelligence*.

Rey, Samuel, Hamed Ajorlou, and Gonzalo Mateos. "Directed Acyclic Graph Convolutional Networks." *arXiv preprint arXiv:2506.12218* (2025).

Zhang, Songyang, Zhi Ding, and Shuguang Cui. "Introducing hypergraph signal processing: Theoretical foundation and practical applications." *IEEE Internet of Things Journal* 7.1 (2019): 639-660.

Roddenberry, T. Mitchell, Michael T. Schaub, and Mustafa Hajij. "Signal processing on cell complexes." *ICASSP 2022-2022 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2022.

Schaub, Michael T., et al. "Signal processing on simplicial complexes." *Higher-Order Systems*. Cham: Springer International Publishing, 2022. 301-328.

Graphons: Limit of Dense Graphs

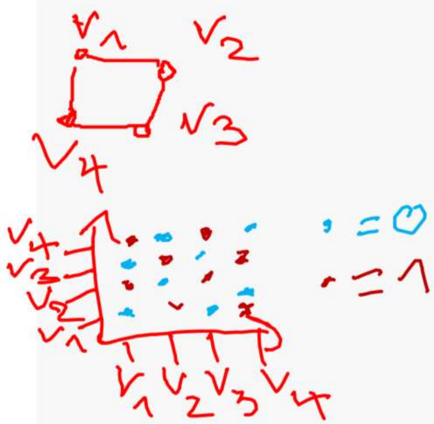
Graphons generalize large dense graphs as measurable functions, allowing a continuous signal domain.

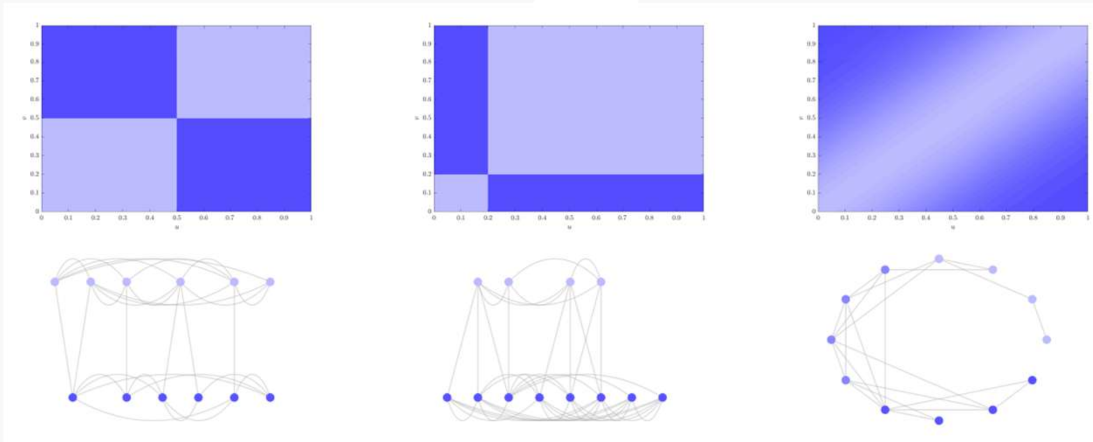
$$W : [0,1]^2 \to [0,1]$$

Graphon Signals

Signals on graphons are functions on [0,1], analogous to node signals on finite graphs.

$$f : [0,1] \to \mathbb{R}$$





(a) SBM with balanced communities (b) SBM with imbalanced communities (c) Exponential

Graphons provide a continuous analog of graph signals on dense networks

Graphon Operator and Fourier Basis

The graphon operator generalizes the graph Laplacian, defining a spectral basis for analysis.

$$(\mathbb{W}f)(x) = \int_0^1 W(x,y) f(y)\, dy$$

Graphon Fourier Transform (GFT)

The GFT uses eigenfunctions of the graphon operator to analyze and process signals.

$$f(x) = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i(x)$$

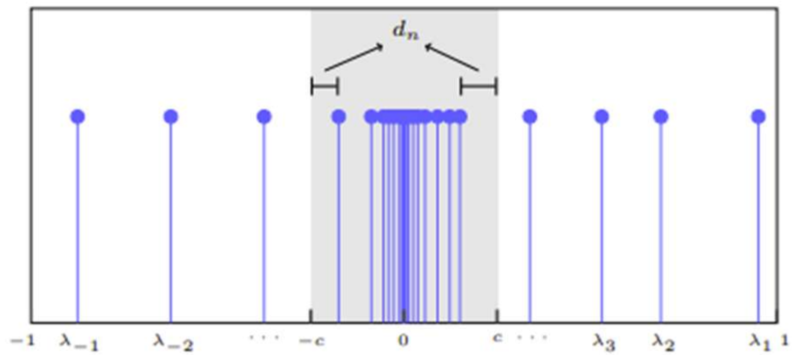


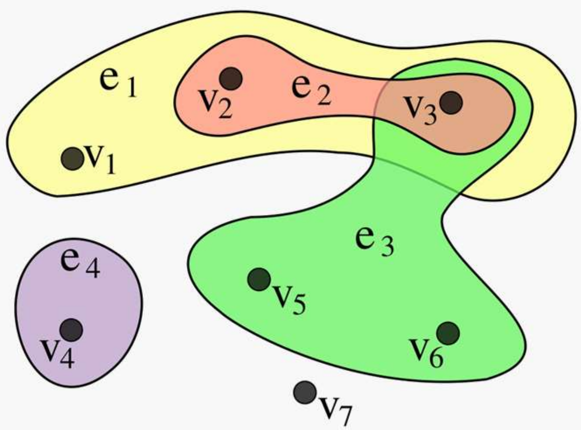
Figure 3. Graphon eigenvalues. A graphon has an infinite number of eigenvalues λ_i but for any fixed constant c the number of eigenvalues $|\lambda_i| \geq c$ is finite. Thus, eigenvalues accumulate at 0 and this is the only accumulation point for graphon eigenvalues. The quantity d_n is approximately equal to the minimum distance between c (or $-c$) and the eigenvalues in the set $\{\lambda_i \mid \lambda_i < c\}$.

Graphon Fourier analysis extends GSP to a functional and infinite-dimensional setting

Hypergraphs

A hypergraph extends graphs by allowing hyperedges to connect more than two nodes simultaneously.

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \quad e \in \mathcal{E} \subseteq 2^{\mathcal{V}}, \ |e| \geq 2$$



Hypergraph Signal

A hypergraph signal assigns a value to each node, just like in GSP, but processed over hyperedges.

$$s : \mathcal{V} \to \mathbb{R}, \quad s \in \mathbb{R}^N$$

Shifts on hypergraphs are represented via high-order adjacency using tensors instead of matrices.

$$\mathcal{A}_M \in \mathbb{R}^{N \times N \times \cdots \times N}$$

Thus the signal is extended as tensor to higher-dimensions modeling the combined signal on higher relations

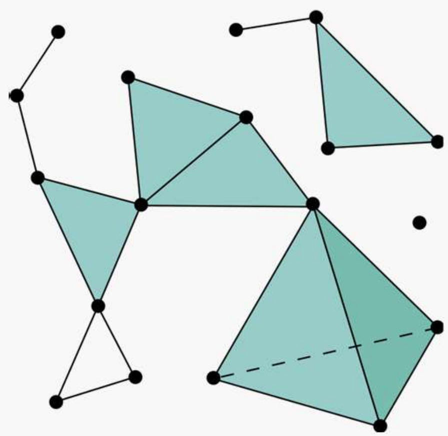
$$s^{[M-1]} = s \circ \ldots \circ s$$

Hypergraphs model multi-node relations; signals live on nodes as in GSP

Simplicial Complexes

Simplicial complexes are built from simplices (nodes, edges, triangles...) and allow the definition of discrete calculus and topology-aware signal processing.

$$\mathrm{SC} = \{\sigma \subset V \mid \text{ all subsets of } \sigma \text{ also in SC}\}$$



Cell Complexes

Cell complexes generalize graphs using cells of varying dimensions (nodes, edges, polygons, etc.), enabling modeling of higher-order interactions.

$$\text{Cell complex } X = \bigcup_{k=0}^{K} C_k \text{ with } C_k \text{ set of } k\text{-cells}$$

Fig. 1. Compared to simplicial complexes, cell complexes can describe the same domain structure with fewer building blocks. For instance, the topology of a torus can be described much more succinctly by a cell complex (right) than a simplicial complex (left).

Higher-order structures model multi-way relations beyond pairwise graph links

Simplicial Signal Processing

Signale für alle n-Simplizes

Defines signals and Laplacians on simplices to generalize Fourier analysis to higher-order domains.

$$L_k = B_k^\top B_k + B_{k+1} B_{k+1}^\top$$

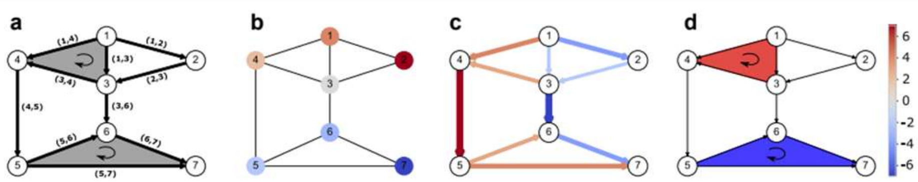


Fig. 2: **Signals on simplicial complexes of different order. a**: Structure of the simplicial complexes used as a running example in the text. Arrows represent the chosen reference orientation. Shaded areas correspond to the 2-simplices $\{1,3,4\}$ and $\{5,6,7\}$. **b**: Signal on 0-simplices (nodes). **c**: Signal on 1-simplices (edges). **d**: Signal on 2-simplices (triangles). Reproduced from [46].

Cell Complex Neural Networks

CCNNs learn on cell complexes by propagating signals across cells of all dimensions via boundary and coboundary maps.

$$h_{t+1}^{(k)} = \sigma\left(B^\top h_t^{(k-1)} + B h_t^{(k+1)} + W h_t^{(k)}\right)$$

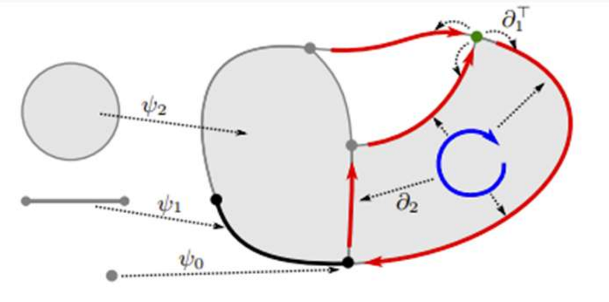


Fig. 2. Illustration of a regular cell complex. Left: homeomorphisms ψ_0, ψ_1, ψ_2 to closed balls in Euclidean space. Right: cochains, chains, boundary maps, and coboundary maps.

Learning and signal processing extend to cells and simplices with structure-aware tools

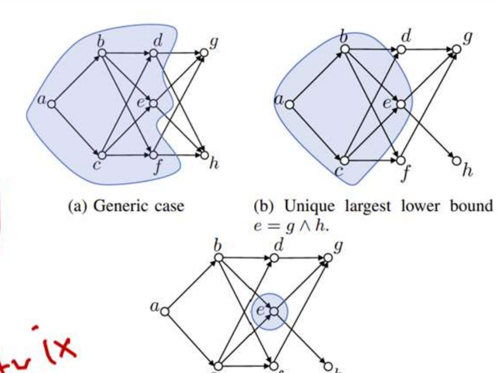
Causal Fourier Analysis

Extends graph Fourier transform to signals on weighted DAGs / posets by defining shift and convolution via edge-causal structure

A =

DAG = Dreiecksmatrix

alle EW 0



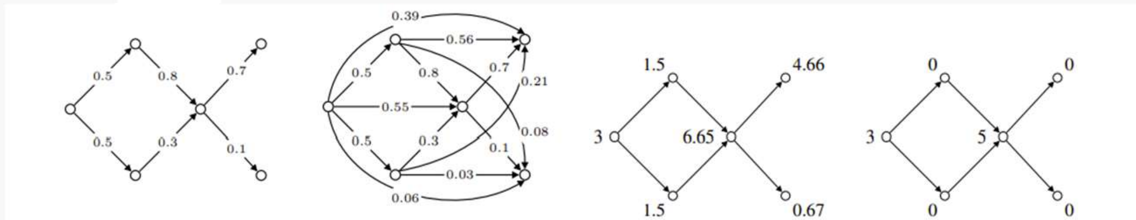
(a) Generic case (b) Unique largest lower bound $e = g \wedge h$. (c) Unique largest lower bound $e = g \wedge h$ and Boolean weights.

Fig. 6: $\mathbf{s}$ shifted with h at node g (i.e., $(T_h\mathbf{s})_g$). (a) Generic case: linear combination of common predecessors of g and h such that common causes are removed. (b) Special case in which g and h have a unique largest lower bound in the partial order: linear combination of predecessors of e. (c) As in (b) but in addition with Boolean edge weights: $(T_h\mathbf{s})_g = s_e$.

Structural SEM Interpretation

Assumes linear structural equation model: signal = filter over root-cause inputs via DAG, enabling spectral analysis of causality

$$X = XA + N$$



(a) Original DAG (b) Transitive closure (c) Measurements vector (d) Root causes

Fourier basis on DAGs enables causally-aware spectral filtering.

Application: SpinSVAR

Method for estimating structural vector autoregression from time-series data under the assumption of sparse input

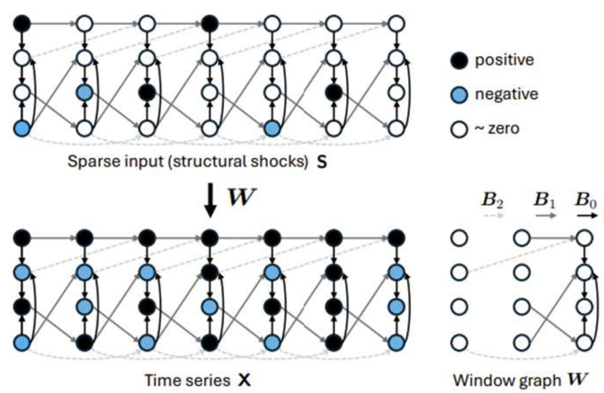
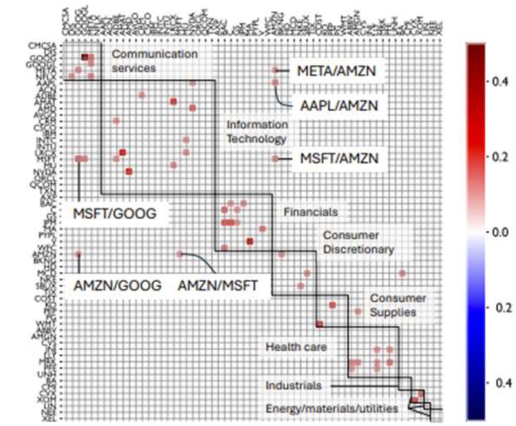
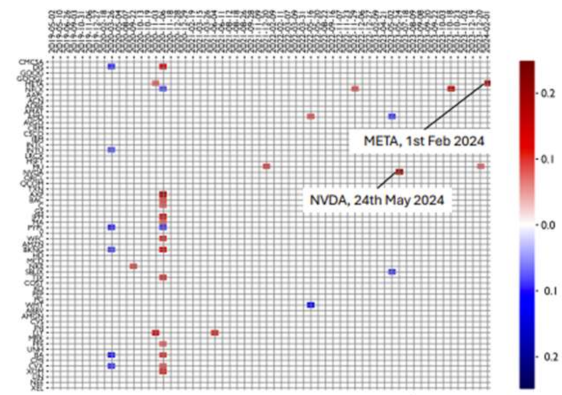


Figure 1: Visualizing an SVAR (3) with sparse input **S**. Out of 28 structural shocks in **S** only seven are significant (positive or negative) and the rest are approximately zero. The window graph $\boldsymbol{W}$, composed of $\boldsymbol{B}_0, \boldsymbol{B}_1, \boldsymbol{B}_2$, generates the observed dense time series **X** (bottom) via (3).



(a) SpinSVAR estimate for $\widehat{\boldsymbol{B}}_0$



(b) SpinSVAR estimate for $\widehat{\boldsymbol{S}}$

Figure 3: Real experiment on the S&P 500 stock market index. (a) Instantaneous relations $\widehat{\boldsymbol{B}}_0$ between the 45 highest weighted stocks within S&P 500, grouped by sectors (squares), and (b) the discovered structural shocks $\widehat{\boldsymbol{S}}$ for 60 days. In (a) the direction of influence is from row to column.

DAG Network (DCN)

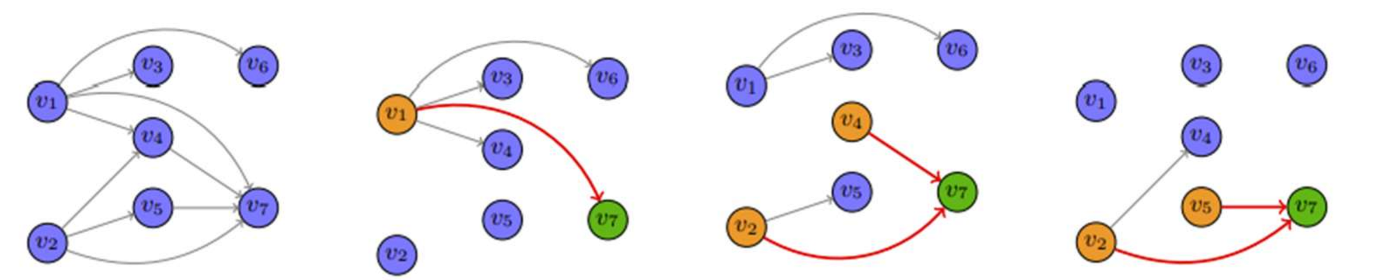
A GNN tailored for signals on DAGs, respecting directionality and acyclicity via causal receptive fields.

$$h_v^{(l+1)} = \sigma\Big(W_1 h_v^{(l)} + W_2 \sum_{u \in \mathrm{pred}(v)} h_u^{(l)}\Big)$$

Causal Convolution

Signal propagation follows topological order, ensuring no cycles and causal consistency in feature aggregation.

$$H^{(l+1)} = \sigma\big(W_1 H^{(l)} + W_2 A_{\mathrm{DAG}} H^{(l)}\big)$$



(a) Reachability DAG $\bar{\mathcal{D}}$. (b) GSO $\mathbf{S}_1$. (c) GSO $\mathbf{S}_4$. (d) GSO $\mathbf{S}_5$.

Fig. 2: An example of how information is propagated by causal shifts $\mathbf{S}_k$ towards downstream nodes. For the GSOs $\mathbf{S}_1$, $\mathbf{S}_4$, and $\mathbf{S}_5$, we indicate the nodes whose signal values contribute to the respective shifted signals at node v_7. Self-loops in the directed graphs induced by the GSOs are excluded to avoid clutter.

DCN extends spectral/GNN methods to DAG-signals using causal directed convolutions