Graph Signal Processing Exercise Set 5

Graph Signal Sampling

June 11, 2025

Learning Goals

By the end of this exercise, you should be able to...

- Compute sampling and interpolation operators for bandlimited graph signals
- Verify perfect reconstruction from sampled values
- Analyze blue noise sampling using the graph Fourier transform

Exercise 1: Sampling and Interpolation Operators

Consider a path graph with 3 nodes, adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = D - A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The eigenvectors of L are:

$$U = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

a) Let the signal be 2-bandlimited: $x = U_{(:,1:2)}\alpha$. Choose nodes 1 and 3 for sampling and define

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Compute SU_2 and its pseudo-inverse. Then reconstruct α from $y = Sx$.

c) Show that reconstruction $\hat{x} = U_2\hat{\alpha}$ recovers x exactly.

Exercise 2: Ill-posed Sampling

a) Choose nodes 1 and 2 as sampling nodes. Define S accordingly.

b) Compute SU_2, check whether it has full column rank.

c) Explain why this sampling scheme leads to unstable reconstruction for certain frequencies.

Exercise 3: Blue Noise Sampling

a) Consider the sampling pattern $s = [0, 1, 0]^T$.

b) Compute the GFT of s: $\hat{s} = U^\top s$.

c) Plot or describe the squared GFT magnitude $|\hat{s}|^2$.

d) Explain why this constitutes blue noise sampling using spectral terms.