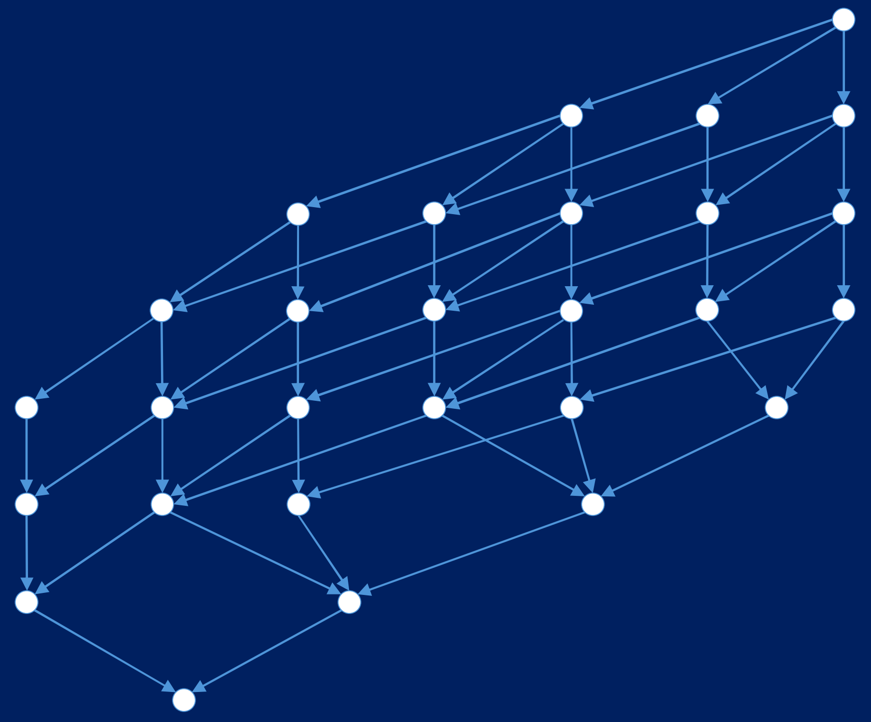
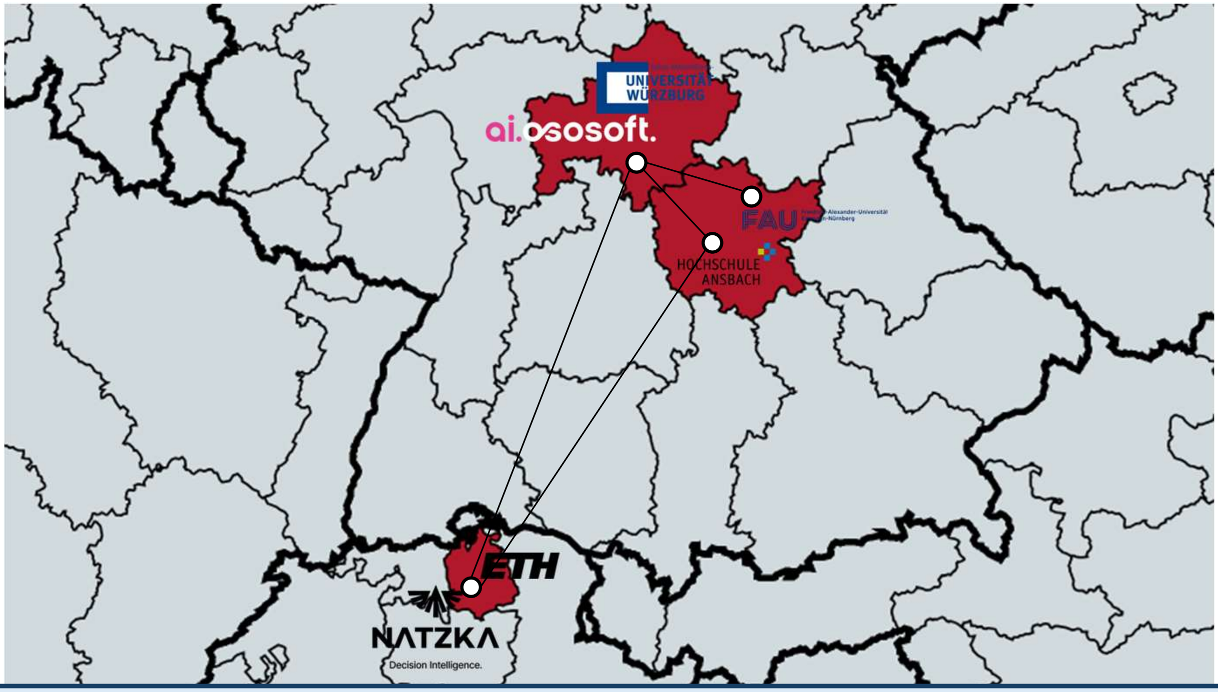
Graph Signal Processing

Bastian Seifert

Sommersemester 2025



Lecturer: Dr. Bastian Seifert



Q: What is your background?

Organizational matters

Graded student research project (Studienarbeit):

- Topics: Your own or one from list (will be distributed by mail)
- Use AI (!), but the right one (e.g., https://notebooklm.google.com/ to make sure you have no hallucinated sources etc.)

Deliverables:

- ~4 pages, IEEE conference format (will distribute template)
- Code used to implement solutions

Exercises:

- Used to deepen knowledge
- You solve between lectures – we talk about possible solutions

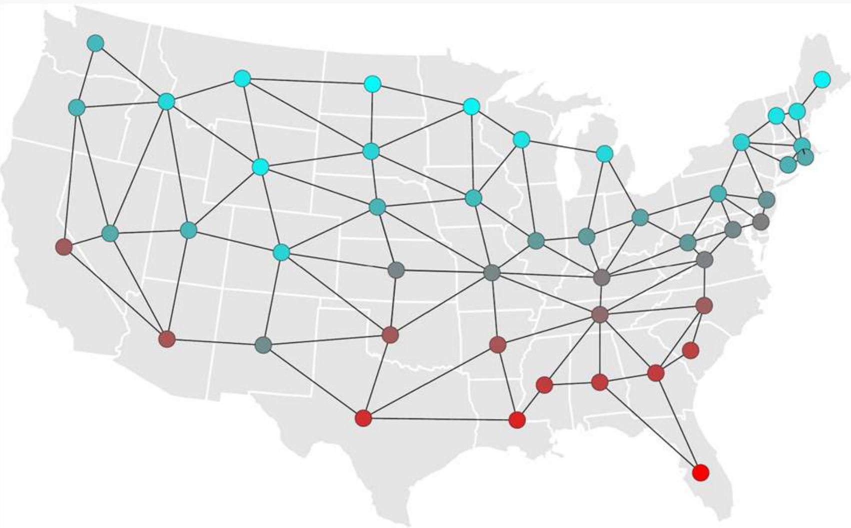
Lecture dates (Wednesday, 13:15-14:45, exercise 15:00-16:30):

19.3 (no exercise), 2.4, 16.4, 30.4, 14.5, 28.5 (no exercise), 11.6, 25.6

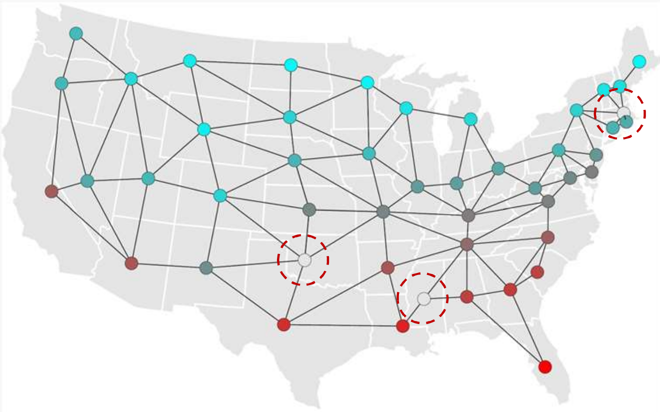
Github repository:

https://github.com/bastian-seifert/gsp-lecture

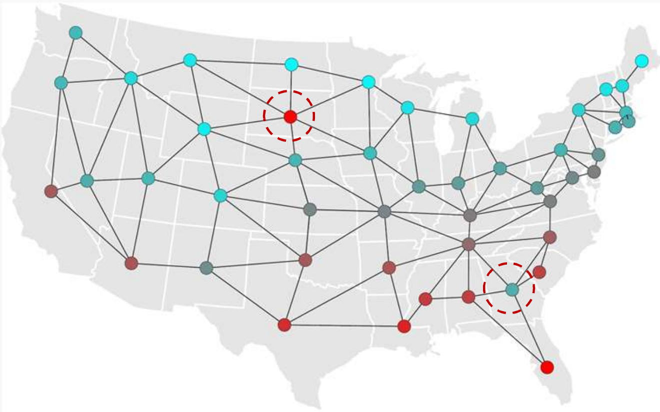
What we want to investigate



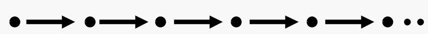
Fill in missing data (signal imputation)



Detect outliers

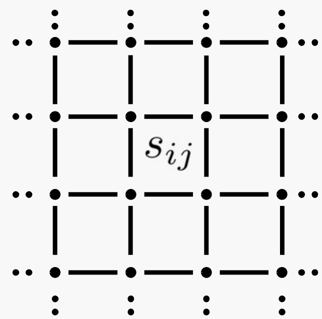


Classical Signal Process.



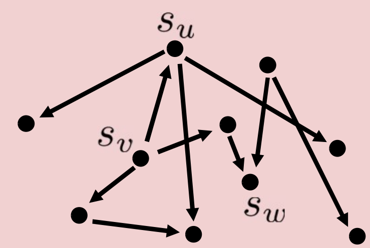
Signals indexed by time (discrete)

Image Processing



Signals indexed by space (discrete)

Graph Signal Processing



Signals indexed by graph

Shumann 2012
Sandryhaila 2013

Goal: Analyze, process, learn with data supported on graph

Graph

$\mathrm{G} = \{\mathrm{V},\ \mathrm{E}\}$

Nodes (vertices): a finite set

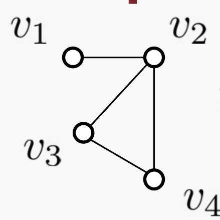
$$V = \{v_1, \ldots, v_n\}$$

Edges (links):

$$E = \{\{v_i, v_k\}, \ldots, \{v_j, v_h\}\}$$

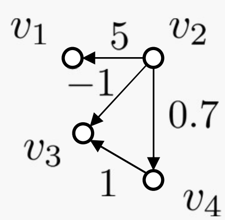
Note that if a graph is disconnected, we consider it in this lecture as two graphs.

Example



$$V = \{v_1, v_2, v_3, v_4\}$$

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$$



More structure

Directed graph

edges have direction

$$E = \{(v_i, v_k), \ldots, (v_h, v_j)\}$$

note: there are now pairs instead of a set, i.e., $(v_i, v_k) \neq (v_k, v_i)$

Weighted graph

edges (directed or undirected) have weights

$$W = \{w_{v_i, v_k} \mid (v_i, v_k) \in E\}$$

Q: Can you think of some real-world examples?

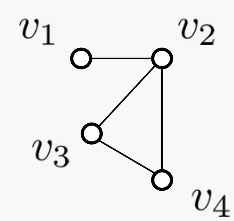
Adjacency matrix

Represents a graph as matrix

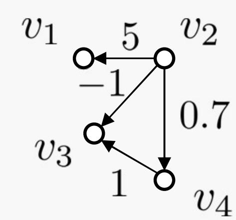
$$A = (A_{v_i,v_j}) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

it has 1 if the edge between nodes at index i and j exists, and 0 otherwise.

Examples



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -1 & 0.7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sparse matrices

It's very inefficient (and sometimes impossible) to store all these zeros. Hence when working with graphs, we use **sparse matrices**, special data structures for matrices which avoid storing the zeros.
In this lecture you will probably have contact with **Coordinate list (COO)** for creating and **Compressed sparse column (CSC)** for applications formats (lookup *sparse* for Scipy/Numpy or Matlab).

Laplacian matrix

- The **degree** of a node, is the number of incoming edges
- Collect all degrees in diagonal matrix

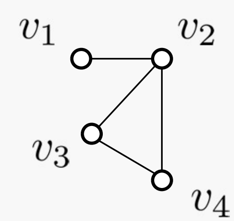
$$D = \mathrm{diag}(\deg(v_1), \ldots, \deg(v_n))$$

- The Laplacian matrix is

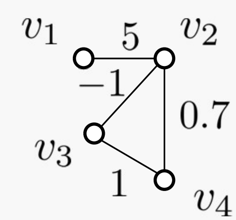
$$L = D - A$$

- Many choices for directed graphs, often one just uses the undirected Laplacian

Examples

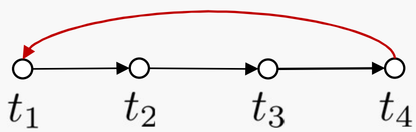


$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

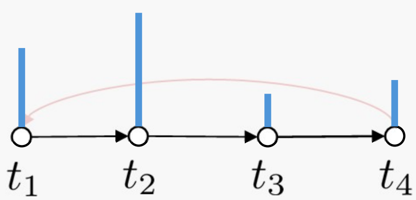


$$L = \begin{bmatrix} 1 & -5 & 0 & 0 \\ -5 & 3 & 1 & -0.7 \\ 0 & 1 & 2 & -1 \\ 0 & -0.7 & -1 & 2 \end{bmatrix}$$

Time signals (as graphs)



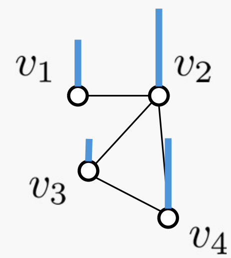
Time graph, representing the flow of time.
Red edge is there, as one considers periodic signals (or approximates with periodic signals).



Numbers associated to each time step are the time signal

$$\mathbf{s} = \begin{bmatrix} s_{t_1} \\ s_{t_2} \\ s_{t_3} \\ s_{t_4} \end{bmatrix} = (s_t)_{t \in T}$$

Graph signals

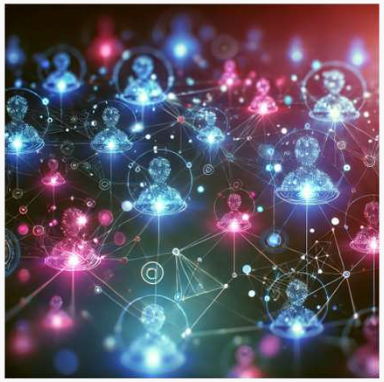


Numbers associated to each *node* are the **graph signal**

$$\mathbf{s} = \begin{bmatrix} s_{v_1} \\ s_{v_2} \\ s_{v_3} \\ s_{v_4} \end{bmatrix} = (s_v)_{v \in V}$$

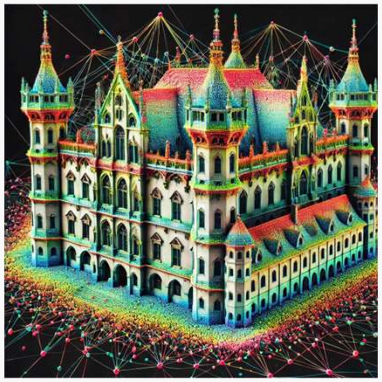
Social Network

- Nodes = Users
- Edges = Friendship
- Signal = Number of interactions



3D point cloud

- Nodes = Location in 3D space
- Edges = Nearest neighborhood
- Signal = Color of voxel



Sensor Network

- Nodes = Sensors
- Edges = Connection
- Signal = Sensor measurements

Q: Can you think of more examples?

Time shift

$$(T\mathbf{s})_i = s_{(i-k) \mod 4}$$

Consider z-transform

$$\mathbf{s} \mapsto s_0 z^0 + s_1 z^{-1} + s_2 z^{-2} + s_3 z^{-3}$$

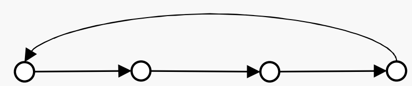
shifting of signals is done by circular delay

$$\begin{aligned} z^{-1} \cdot \mathbf{s} &= s_0 z^{-1} + s_1 z^{-2} + s_2 z^{-3} + s_3 z^0 \\ &= s_3 z^0 + s_0 z^{-1} + s_1 z^{-2} + s_2 z^{-3} \end{aligned}$$

Matrix representation

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} s_3 \\ s_0 \\ s_1 \\ s_2 \end{bmatrix}$$

is adjacency matrix of graph



The time shift/delay is the central building block of discrete time signal processing

Space shift

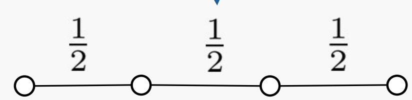
$$(T\mathbf{s})_i = \frac{1}{2}(s_{i-1} + s_{i+1})$$

We will see in later lectures how variations of this shift are connected to discrete cosine/sine transformations

Matrix representation

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s_1 \\ \frac{1}{2}(s_0 + s_2) \\ \frac{1}{2}(s_1 + s_3) \\ \frac{1}{2}s_2 \end{bmatrix}$$

is adjacency matrix of graph



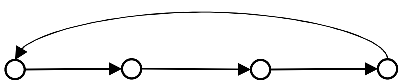
The space shift can be used to define image processing concepts.

Algebraic Signal Processing Theory

Püschel & Moura, 2008, IEEE Trans. Signal Proc.

Shift → Convolution → Fourier Transformation

Definition of shift → Linear shift-equivariant mapping → Eigendecomposition of convolution



$$(T\mathbf{s})_i = s_{(i-k) \mod 4}$$

$$(H\mathbf{s})_i = \sum_{k=0}^{3} h_k s_{(i-k) \mod 4}$$

$$\mathrm{DFT}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}$$

Q: How to define shift(s) on Graphs?

Shift = adjacency matrix

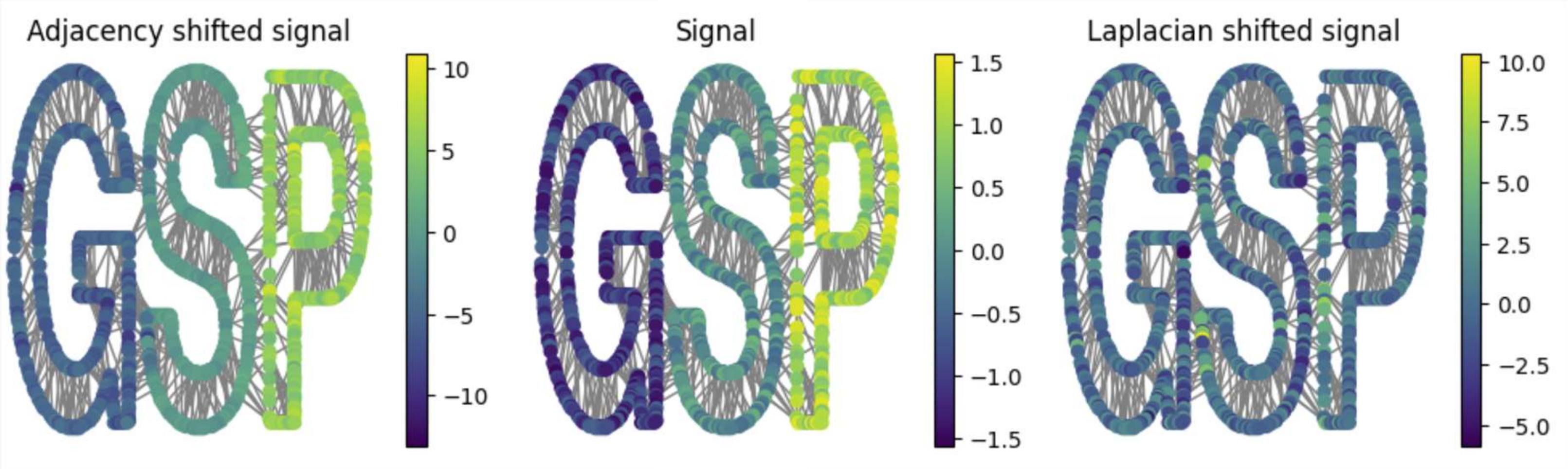
Sandryhaila & Moura 2013, IEEE Trans. Sign. Proc.

- Linear combination of one-hop neighbours
- Generalization of delay

Shift = graph Laplacian

Shuman & Ricaud & Vandergheynst, IEEE SSP Workshop 2012

- Discrete difference operator
- Generalization of diffusion
- Hard to generalize to directed graphs



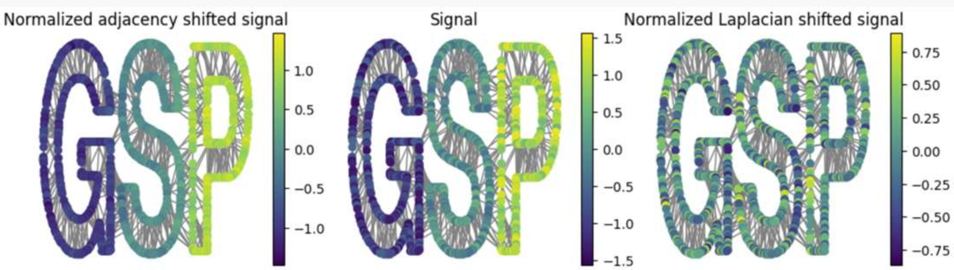
You have to test which shift is better suited for your application.

Shifts, normalized

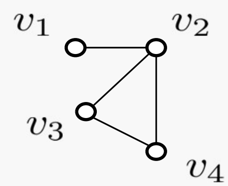
- Normalization is done via

$$A_{\text{norm}} = D^{-1}A$$
$$L_{\text{norm}} = D^{-1}L$$

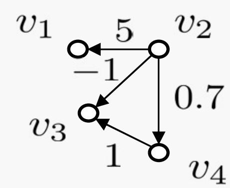
- For Laplacian other normalizations are possible (see exercises)



Examples



$$A_{\text{norm}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$



$$A_{\text{norm-inflow}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{5}{3} & 0 & -\frac{1}{3} & 0.7 \cdot \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_{\text{norm-outflow}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -\frac{1}{2} & 0.7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

For directed graphs it matters if you normalize to the left or right!

Low-pass filter

First-order low-pass filter

$$y_n = b_0 s_n + b_1 s_{n-1}$$

is a polynomial in the delay/shift

$$\mathbf{y} = b_0 T^0 \mathbf{s} + b_1 T \mathbf{s}$$

using the unit impulse

$$\mathbf{u} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$$

the impulse response is

$$\mathbf{h} = \begin{bmatrix} b_0 & b_1 & 0 \dots & 0 \end{bmatrix}^T$$

Moving average

Moving average filter of length M is defined as

$$y_n = \frac{1}{M} \sum_{k=0}^{M-1} s_{n-k}$$

can be rewritten as polynomial in the delay/shift

$$\mathbf{y} = \frac{1}{M} \sum_{k=0}^{M-1} T^k \mathbf{s}$$

the impulse response is

$$\mathbf{h} = \begin{bmatrix} \frac{1}{M} & \frac{1}{M} & \dots & \frac{1}{M} & 0 \dots & 0 \end{bmatrix}^T$$

All finite impulse response (FIR) filters can be expressed as polynomials

Graph convolution/filter

Graph filters produce output for any signal as input, i.e.,

$$H \in \mathbb{C}^{N \times N}$$

$$H \cdot \mathbf{s}$$

Properties:

- graph filters are *linear* systems

$$H(\alpha \mathbf{s}_1 + \beta \mathbf{s}_2)) = \alpha H \mathbf{s}_1 + \beta H \mathbf{s}_2$$

- a graph filter is *shift-invariant* if

$$A(H\mathbf{s}) = H(A\mathbf{s})$$

Theorem: All linear, shift-invariant graph filters are polynomials in the shift.
Proof: Exercise.

Graph moving average

The graph moving average filter of length M isft

$$H = \frac{1}{M} \sum_{k=0}^{M-1} A^k$$

the impulse response is

$$\mathbf{h} = \begin{bmatrix} \frac{1}{M} & \frac{1}{M} & \dots & \frac{1}{M} & 0 \dots & 0 \end{bmatrix}^T$$

Label propagation

Binary classification via graph filter

$$s_n^{\text{pred}} > 0 \mapsto \text{Class 1}$$
$$s_n^{\text{pred}} < 0 \mapsto \text{Class 2}$$

where the prediction is done by constructing a graph filter so that

$$\mathbf{s}^{\text{pred}} = H \cdot \mathbf{s}^{\text{known}}$$

where the unknown signal values are set to 0. Use subset of known values as filter training

$$V^{\text{train}} \subset V^{\text{known}}$$

then the filter is good if

$$\text{diag}(\mathbf{s}^{\text{known}}) H \mathbf{s}^{\text{train}} \geq 0$$

we want to find filter of max order L

$$H = \sum_{k=0}^{L} h_k A^k$$

The coefficients can be found by least-squares minimization

$$\text{argmin}_H ||DH\mathbf{s}^{\text{train}} - 1_N||_2$$
$$= \text{argmin}_{\mathbf{h}} ||(DA^0\mathbf{s}^{\text{train}} \ldots DA^L\mathbf{s}^{\text{train}})\mathbf{h} - 1_N||_2$$

where we used

$$D = \text{diag}(\mathbf{s}^{\text{known}})$$
$$\mathbf{h} = [h_0 \ldots h_L]^T$$

then prediction is

$$\mathbf{s}^{\text{pred}} = H \cdot \mathbf{s}^{\text{known}} > 0$$

Customer churn prediction

In Jupyter notebook we predict customer churn from customer data.