Graph Signal Processing Exercise Set 1

Solution: Graphs - Signals - Shifts

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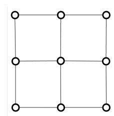
Image signals

Explain how (greyscale) images can be interpreted as signals on a rectangular graph. That is,

(a) define a rectangular graph (e.g., do it on a small example and write down the adjacency matrix),

(b) explain how the greyscale defines a signal value on each node.

Solution: As graph we use the grid graph The greyscale is a num-



ber associated to each pixel. We associate each pixel with a node of the graph and use the greyscale number as signal on the respective node.

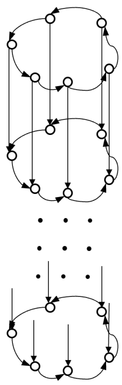
The adjacency matrix can be represented as $A \otimes \mathbf{1} + \mathbf{1} \otimes A$ (the cartesian product of graphs, more on this in the next exercise), with A the adjacency matrix of the path graph on three nodes.

Transportation network

Consider a city's public transportation system with 6 bus stops. The daily passenger count boarding at each stop is recorded for a week. Model this situation as a graph and graph signal. Draw one simple example graph with 6 nodes and assign realistic passenger count values. Then calculate both, normalized adjacency and Laplacian shift, and apply them to your signal. What could you learn from the results? What additional data could you collect to make this application even more useful?

Solution: Take for example a circle graph with 6 stops, copied for each day as depicted in the figure below and some two-digit values each day. Applying the shift operators leads to averaging respective smoothing of the numbers.

It would be very valuable to also collect t he number of passengers which exit the bus at each station, not only the ones boarding.



Graph shift

Consider the graph C_4 with signal $\mathbf{s} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$. Calculate the adjacency A and Laplacian L matrix and apply both to the signal. Discuss the differences between the results.

Solution: We have

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Applying them to the signal yields

$$A \cdot \mathbf{s} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \; L \cdot \mathbf{s} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix}.$$

More shift normalizations

For the adjacency matrix from the lecture

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Graph Signal Processing Exercise Set 1

compare the normalization by the degree matrix D from the left $D^{-1} \cdot A$ (normalization used in consensus network) and normalization from the right $A \cdot D^{-1}$ (normalization used in physical distribution network), for example by applying it to a signal.

The Laplacian can also be normalized from the right, but more interesting is normalizing it symmetrically from both sides with the square root of the degrees, i.e.,

$$\mathcal{L} = D^{-1/2} L D^{-1/2}.$$

Calculate this symmetric normalized Laplacian for above graph and study it's behavior.

Solution: We have

$$D^{-1/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\mathcal{L} = \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 1 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 1 & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{3} & 1 \end{bmatrix}$$

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