Graph Signal Processing

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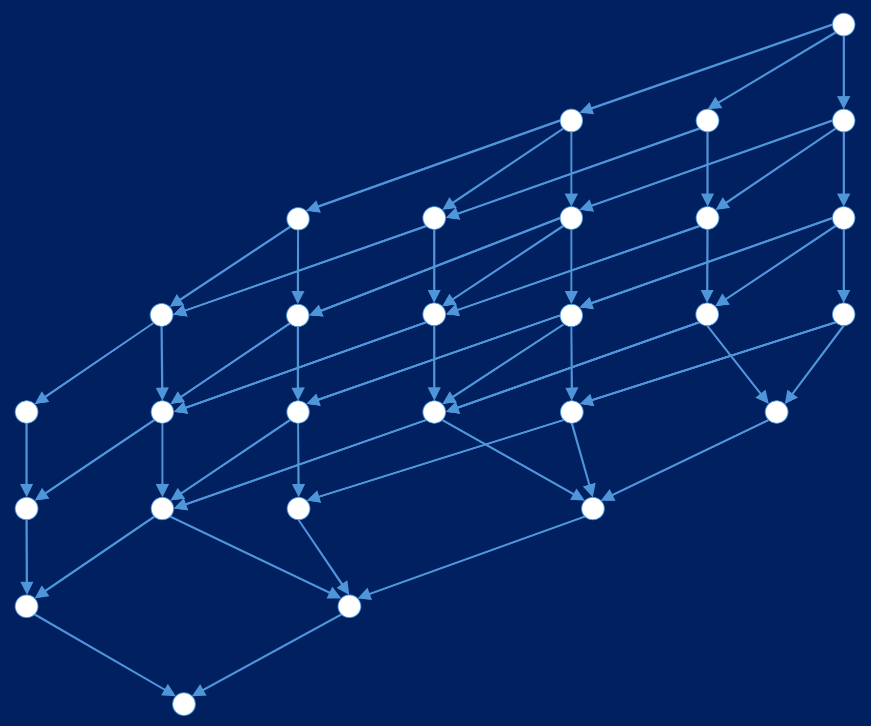
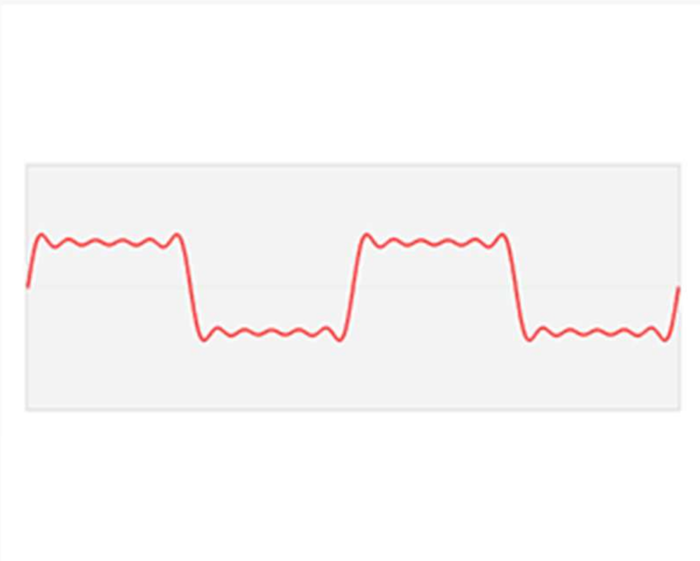
Sommersemester 2025



Fourier Analysis



Animation: Lucas V. Barbosa - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=24830373

Integrate against eigenfunctions of Laplace operator

$$\mathcal{L} = -\frac{d^2}{dx^2}$$

which are

$$\mathrm{e}^{\mathrm{i}nx} = \cos(nx) + \mathrm{i}\sin(nx)$$

Hearing the shape of a drum

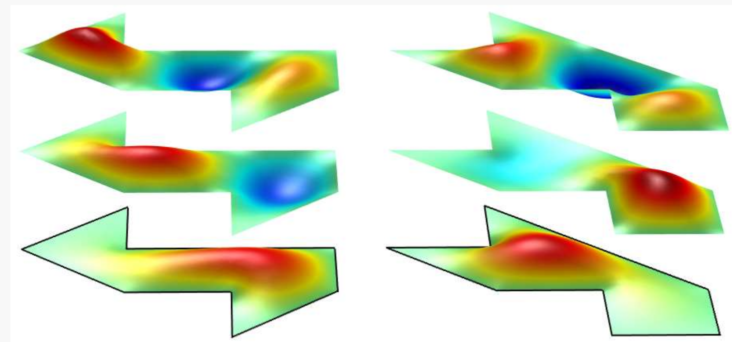


Figure: Chien Liu https://www.comsol.com/blogs/can-we-hear-the-shape-of-a-drum

Associate eigenvalues of Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

to properties of the shape

Eigenvectors and Eigenvalues

A vector $\mathbf{u}$ and a scalar λ are an eigenvector and and eigenvalue of the shift if

$$T\mathbf{u} = \lambda\mathbf{u}$$

Independent eigenvectors are eigenvectors which can not be written as linear combination of the others.
A matrix is diagonalizable if it has as many independent eigenvectors as its size. Then

$$T = U\Lambda U^{-1}$$

where Λ is a diagonal matrix.

Examples

$$\begin{bmatrix} 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1&1&0&-2\\ 1&1&-2&0\\ -1&1&0&2\\ 1&1&2&0 \end{bmatrix}\begin{bmatrix} -2&&&\\ &2&&\\ &&0&\\ &&&0 \end{bmatrix}\begin{bmatrix} -1&1&-1&1\\ 1&1&1&1\\ 0&-1&0&1\\ -1&0&1&0 \end{bmatrix}$$

$U^{-1} = U^T$
orthogonal
(unitär)

$$\begin{bmatrix} 0&1\\ 1&0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1&1\\ -1&1 \end{bmatrix}\begin{bmatrix} -1&0\\ 0&1 \end{bmatrix}\begin{bmatrix} 1&-1\\ 1&1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 0&1\\ 1&0 \end{bmatrix}\cdot\begin{bmatrix} x\\ y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 1&1\\ -1&1 \end{bmatrix}\begin{bmatrix} -1&0\\ 0&1 \end{bmatrix}\cdot\begin{bmatrix} x-y\\ x+y \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1&1\\ -1&1 \end{bmatrix}\cdot\begin{bmatrix} -x+y\\ x+y \end{bmatrix} \\ &= \begin{bmatrix} y\\ x \end{bmatrix} \end{aligned}$$

Shifts of undirected graphs are diagonalizable

Diagonalizable matrices

Real symmetric matrices, i.e., ones with

$$A = A^T$$

are always diagonalizable and have only real eigenvalues.

Not every matrix is diagonalizable.

Non-symmetric diagonalizable matrices typically have pairs of complex eigenvalues.

Simultaneous diagonalization

Matrices are simulatenous diagonalizable if they can be diagonalized by the same matrix, i.e.,

$$T = U\Lambda_1 U^{-1}, H = U\Lambda_2 U^{-1}$$

If matrices commute, they preserve eigenvectors,

$$T(H\mathbf{u}) = H(T\mathbf{u}) = H(\lambda\mathbf{u}) = \lambda H\mathbf{u}$$

so they can be simultaneously diagonalized

$T^k = U\Lambda^k U^{-1} \Leftrightarrow U^{-1}T^kU = \Lambda^k$

H filter $\Rightarrow p(T) \overset{(*)}{=} \sum_{k=0}^{N} a_k T^k = H$

$U\Lambda U^{-1} = T$

$U^{-1}HU \overset{(*)}{=} U^{-1}\left(\sum_{k=0}^{N} a_k T^k\right)U = \sum_k a_k \left(U^{-1}T^kU\right) = \sum_k a_k \Lambda^k = p(\Lambda)$

$H = U p(\Lambda) U^{-1}$

Filters and shift can be diagonalized by the same matrix

Graph Fourier transform

Eigenvalues of shift = graph frequencies, spectrum

Eigenvectors = frequency component

Graph Fourier transform of signal is

$$\hat{\mathbf{s}} = \mathcal{F}\mathbf{s}$$

with

$$\mathcal{F} = U^{-1}$$

for

$$A = U\Lambda U^{-1}$$

Frequency response

The frequency response describes the effect of a filter to the frequency content of a signal

$$\tilde{\mathbf{s}} = H(A)\mathbf{s} = \mathcal{F}^{-1}H(\Lambda)\mathcal{F}\mathbf{s}$$

$$\Leftrightarrow$$

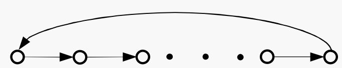
$$\mathcal{F}\tilde{\mathbf{s}} = H(\Lambda)\hat{\mathbf{s}}$$

(convolution theorem) and is defined as

$$\widehat{H(A)} = H(\Lambda)$$

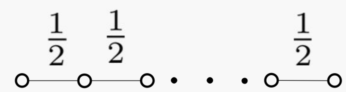
Filtering a signal on a graph is equivalent to multiplying the signal spectrum with the frequency response of the filter

Cycle graph



$$A = \frac{1}{N}\mathrm{DFT}_N^{-1}\begin{bmatrix} e^{-i\cdot\frac{2\pi\cdot 0}{N}} & & \\ & \ddots & \\ & & e^{-i\cdot\frac{2\pi\cdot(N-1)}{N}} \end{bmatrix}\mathrm{DFT}_N$$

Weighted path graph

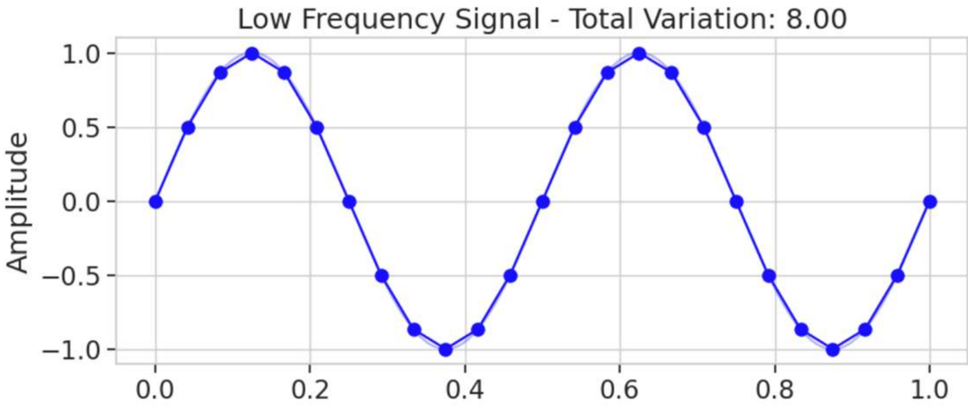


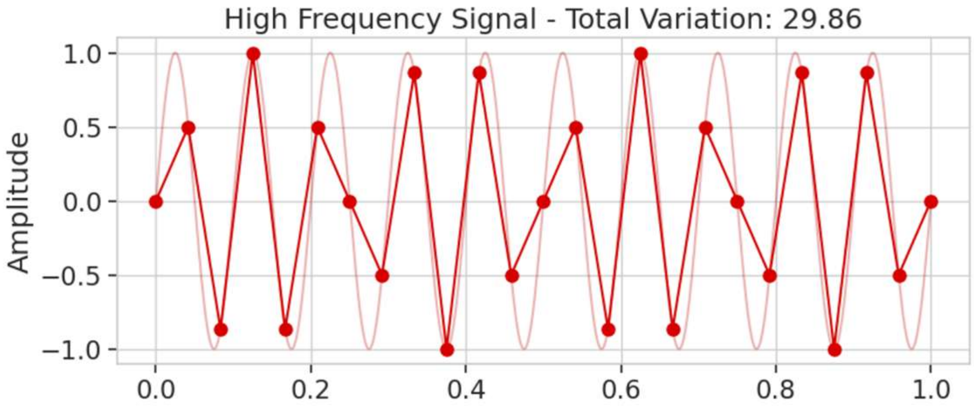
$$A = \frac{1}{N}\mathrm{DCT}_N^{-1}\begin{bmatrix} \cos(\frac{\pi\cdot 0}{N}) & & \\ & \ddots & \\ & & \cos(\frac{\pi\cdot(N-1)}{N}) \end{bmatrix}\mathrm{DCT}_N$$

The graph Fourier transform generalizes classical signal transforms

Total variation of a discrete time signal is defined as

$$\mathrm{TV}(\mathbf{s}) = \sum_{t \in T} |s_t - s_{t-1}|$$





Higher total variation = higher frequency

Graph total variation

The total variation of a graph signal is defined as

$$\mathrm{TV}_G(\mathbf{s}) = ||\mathbf{s} - A^{\mathrm{norm}}\mathbf{s}||_1$$

It's important to take the normalized shift here, since otherwise don't have the proper scalation for comparision of the shifted with the original signal.

$$||s - A^{norm} s||_1 = \sum_{v \in V} |s_v - (A s)_v|$$

$$A = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \qquad A s = \begin{pmatrix} s_3 \\ s_1 \\ s_2 \end{pmatrix}$$

$$||s - A s||_1 = |s_1 - s_3| + |s_2 - s_1| + |s_3 - s_2|$$

$$= \sum |s_t - s_{\underbrace{t+1 \bmod I}_{\text{Periodisierung}}}|$$

Graph frequency ordering

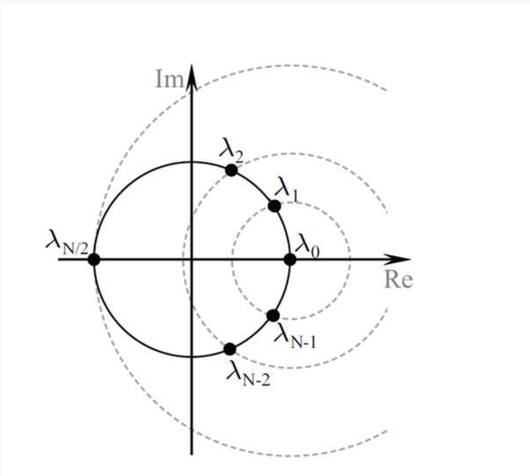
For real eigenvalues $\lambda_1 < \lambda_2$ one has

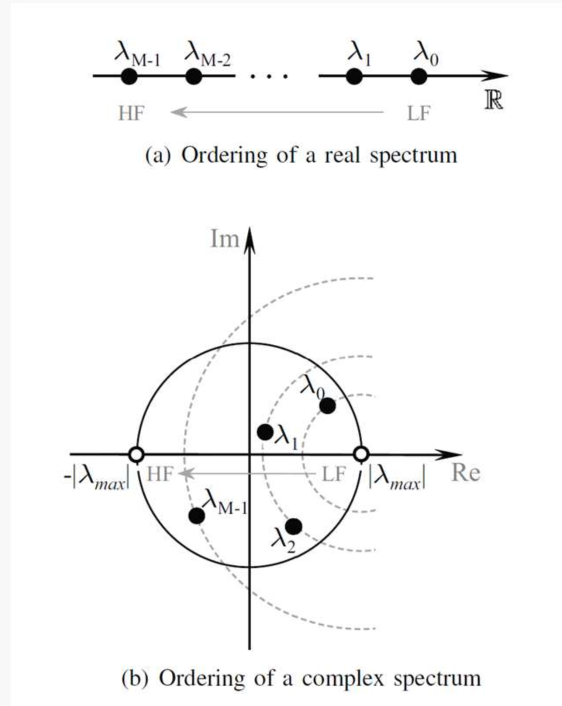
$$\mathrm{TV}_G(u_1) > \mathrm{TV}_G(u_2)$$

while for complex eigenvalues this is the case if if λ_1 is located nearer than λ_2 to $|\lambda_{\max}|$

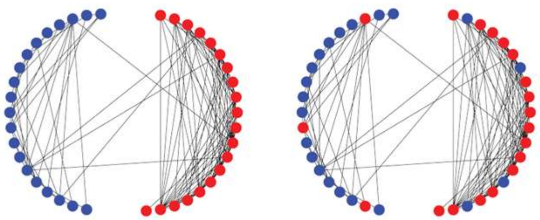
Graph frequency hence are ordered by their total variations

Visualization of graph frequency ordering





Figures: Sandryhaila & Moura, IEEE Trans. Signal Proc. 62(12), 2014



(a) True labels (b) Synthetic labels

Fig. 11. A subgraph of 40 blogs labels: blue corresponds to "liberal" blogs and red corresponds to "conservative" ones. Labels in (a) form a smoother graph signal than labels in (b).

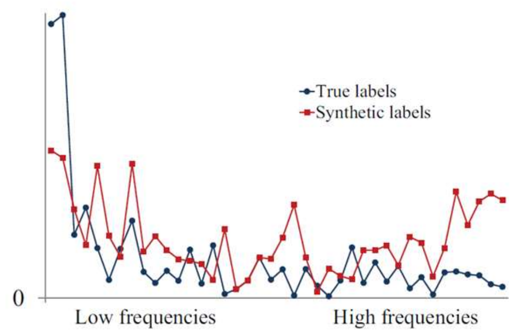


Fig. 12. Magnitudes of the spectral coefficients for graph signals formed by true and synthetic labels in Fig. 11.

Outliers typically show by higher high frequency content

Figures: Sandryhaila & Moura, IEEE Trans. Signal Proc. 62(12), 2014