Evaluation of flow rate for a one-dimensional lava flow with power-law rheology

Antonello Piombo[1,2] and Michele Dragoni[1]

Received 17 September 2009; revised 14 October 2009; accepted 16 October 2009; published 25 November 2009.

[1] During the emplacement lava behaves as a non-Newtonian, pseudoplastic fluid. Laboratory experiments on lava samples suggest that a power-law constitutive equation may be appropriate. We consider a horizontally unbounded, isothermal layer of lava, flowing down a slope driven by the gravity force. We consider a constitutive equation where shear stress is proportional to strain rate raised to a power n, ranging from 0 to 1, and we take into account the temperature dependence of the rheological parameters. Formulae are obtained relating the lava rheology to the geometric and dynamic parameters of a lava flow. Under the model assumptions, if we know the temperature dependence of the rheological parameters, these formulae allow to evaluate the flow rate from the field measurements of the temperature, and the thickness or the surface velocity of a lava flow. **Citation:** Piombo, A., and M. Dragoni (2009), Evaluation of flow rate for a one-dimensional lava flow with power-law rheology, *Geophys. Res. Lett.*, *36*, L22306, doi:10.1029/2009GL041024.

1. Introduction

[2] Once the lava leaves the eruption vent and flows downhill, it begins to cool. Lava flows lose heat by conduction to the substrate, by radiation to the surroundings, and by convection and conduction to the atmosphere. The temperature variation along the flow is important, because cooling is the main factor that limits the downslope flow of lava; but the temperature gradient along the flow is very small. In addition, if the lava cooling is not too advanced, the thermal boundary layers have a limited effect on the flow dynamics: this is true close to the source of the lava flow. Therefore isothermal models of lava flows are a reasonable approximation in describing a limited segment of flow not too far from the vent, where temperature can be considered uniform.

[3] During the emplacement lava behaves as a non-Newtonian, pseudoplastic fluid. The nonlinearity is generally ascribed to the increasing crystal content of lava, due to progressive cooling [e.g., *Griffiths*, 2000]; when the crystal concentration is large enough, lava may gain a yield strength [e.g., *Kerr and Lister*, 1991; *Pinkerton and Stevenson*, 1992; *Saar et al.*, 2001]. Power-law and Bingham rheologies have been suggested for basaltic lavas, since they allow to reproduce some of known shear-dependent behaviors of basaltic melts [e.g., *Shaw et al.*, 1968; *Pinkerton and Sparks*, 1978; *Hardee and Dunn*, 1981; *Pinkerton and Norton*, 1995; *Sakimoto et al.*, 1997; *Lavalle et al.*, 2007]. Non-Newtonian rheologies need a minimum of two rheological parameters such as yield strength and plastic viscosity for Bingham flow or flow consistency and power-law exponent for power-law flow. In the Bingham rheology the shear strain is zero until a yield value is reached and is a linear function of stress above [e.g., *Hulme*, 1974; *Dragoni et al.*, 1986]. Laboratory experiments on lava samples suggest that a power-law constitutive equation may be more appropriate [*Spera et al.*, 1988; *Hardee and Dunn*, 1981; *Pinkerton and Norton*, 1995; *Sonder et al.*, 2006]. In the constitutive equation of the power-law rheology, the shear stress is proportional to strain rate raised to an exponent. Power-law rheologies, such as the Herschel-Bulkley fluid, have been often employed to model mud flows [e.g., *Ng and Mei*, 1994; *Huang and García*, 1998].

[4] The effusion rate of lava from a volcanic vent is an important quantity controlling the length of the ensuing lava flow. The ability to evaluate the effusion rate is therefore of importance to estimate the hazard connected with the flow. *Tallarico and Dragoni* [1999, 2000] considered 2-D models for near-vent channeled lava flows, assuming the lava as an isothermal Newtonian or Bingham liquid flowing in a rectangular channel down a constant slope. For these models the authors showed how to calculate the effusion rate, given the lava flow width, the topographic slope, the lava density, the surface flow velocity, and either the lava viscosity or flow thickness [*Tallarico et al.*, 2006]. In recent years it has become common to evaluate the effusion rate from the radiant energy emitted from a lava flow [e.g., *Harris et al.*, 1997]. However this method is far from providing data with a good accuracy [*Dragoni and Tallarico*, 2009].

[5] In the present paper we consider a power-law rheology to model an isothermal and steady-state lava flow. We consider a horizontally unbounded layer of lava, flowing down a slope driven by the gravity force. Under these conditions, our model aims to give an estimate of flow rate on the basis of measurable quantities as surface velocity, thickness, and temperature of lava.

2. Constitutive Equations

[6] After the lava has been effused, cooling induces a progressive, partial crystallization of liquid lava. Crystallization is believed to be responsible for the non-Newtonian, pseudoplastic behavior of lava. In pseudoplastic rheology, the relationship between strain rate and stress is nonlinear and stress must increase to a significant value before an appreciable strain rate takes place. This relationship can be

[1]Dipartimento di Fisica, Alma Mater Studiorum Università di, Bologne, Italy.
[2]CIRSA, Alma Mater Studiorum Università di, Bologne, Italy.

0094-8276/09/2009GL041024

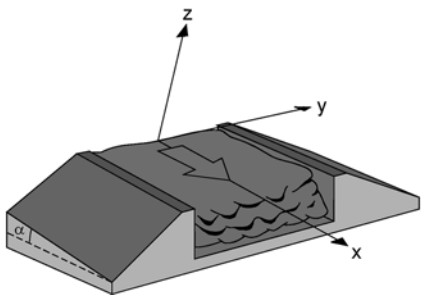

Figure 1. The coordinate system.

expressed by a power law where the strain rate is raised to an exponent $n < 1$.

[7] An alternative rheological model which is often employed in describing lava flows is the Bingham body (or Bingham plastic), which is characterized by a yield strength. According to what has been said above, a close relationship exists between a pseudoplastic fluid and a Bingham body. In fact the Bingham body is often used as an approximation of a pseudoplastic fluid with smaller values of n. There is an obvious advantage in doing that: the Bingham body is such that it behaves as a Newtonian fluid after the yield stress is exceeded, so that the simpler Navier-Stokes equation can be still used in describing its motion.

[8] It is not obvious that lava has a definite yield strength, such as is introduced by the Bingham rheology. A clear indication of a yield stress is the presence of a plug, which is however difficult to observe in the field and to distinguish from the solid crust. As is implied by its name, the pseudoplastic rheology can mimic the dynamical effects typical of a plastic (i.e., Bingham) body without the need of a definite yield strength, but with an appropriate choice of the value of n. Moreover it is consistent with laboratory experiments indicating a nonlinear dependence of stress on strain rate over the whole stress range. Therefore it should be considered as a more general constitutive equation for lava.

[9] We consider a constitutive equation where shear stress σ is proportional to strain rate $\dot{e}$ raised to a power n:

$$\sigma = 2K\dot{e}^n \tag{1}$$

where the consistency K and the exponent n are positive constants, and n ranges from 0 to 1. Equation (1) reduces to the Newtonian model if n is unity; then K is the viscosity. Viscometric studies of magma suggest that a power-law constitutive equation may be appropriate [*Spera et al.*, 1988; *Hardee and Dunn*, 1981; *Pinkerton and Norton*, 1995; *Sonder et al.*, 2006]; in these laboratory experiments $0.5 \leq n \leq 1$, while K can vary over several orders of magnitude.

[10] In general, the lava behaves as a Newtonian fluid close to liquidus temperature; at lower temperatures lava behaves as a pseudoplastic fluid. For rhyolitic magma, *Spera et al.* [1988] suggest for K and n and for $1370 \text{ K} \leq T \leq 1620$ K the following functions of temperature:

$$n(T) = n_0 + aT \tag{2}$$

$$K(T) = K_{01}\, e^{\frac{b}{T}} \tag{3}$$

where T is the absolute temperature, K_{01}, n_0, a, and b are constants. Equation (2) is a rough approximation of the measurements of *Pinkerton and Norton* [1995], which refer to basaltic lava of Mt. Etna for $1360 \text{ K} \leq T \leq 1393$ K.

[11] Other exponential forms for the temperature dependence of K (or of viscosity) are used for lavas. K can depend on temperature by the Arrhenius law

$$K(T) = K_{02}\, e^{\frac{E_a}{RT}} \tag{4}$$

where K_{02} is a positive constant, E_a is the activation energy, and R is the universal gas constant. Equation (4) is similar to (3). *Dragoni et al.* [1986] adopted for the viscosity a temperature dependence which can be expressed by the law

$$K(T) = K_{03}\, e^{-c(T-T_L)} \tag{5}$$

where K_{03}, c are positive constant, and T_L is the liquidus. To give an example of application of our model, we assume in the following a choice of values of the constants in (2) and (3) which are consistent with the measurements of *Pinkerton and Norton* [1995], which we extrapolate to a larger temperature range, due to the scarcity of experimental data on the dependence of rheological parameters on temperature in the melting range of silicates.

3. Model

[12] We consider a horizontally unbounded layer of lava, flowing down a slope driven by the gravity force. We assume that the lava is isothermal. The solution was calculated for the flow in a conduit by *Turcotte and Schubert* [1982]. Similar solutions were calculated by *Ng and Mei* [1994] and *Huang and García* [1998] for a mud flow.

[13] We introduce a coordinate system (x, y, z) (Figure 1). No quantities depend on the coordinates x and y, so that the problem is one-dimensional. The Cauchy equation for steady-state flow is

$$v_j v_{i,j} = -p_{,i} + \sigma_{ij,j} + \rho g_i \tag{6}$$

where v_i is the flow velocity, p is the pressure, σ_{ij} is the viscous stress, ρ is the density of lava, and g_i is the gravity acceleration. Since the flow occurs in the x direction, the only nonvanishing component of stress is σ_{xz}. The x component of equation (6) is then

$$\sigma_{xz,z} + \rho g \sin\alpha = 0 \tag{7}$$

where α is the slope angle (Figure 1). As to the constitutive equation, we assume the power law

$$\sigma_{xz} = 2K\dot{e}^n_{xz} \tag{8}$$

where $\dot{e}_{xz}$ is the strain rate, and $0 < n \leq 1$. Equation (8) can also be written as

$$\sigma_{xz} = \frac{K}{2^{n-1}}\left(\frac{dv_x}{dz}\right)^n \tag{9}$$

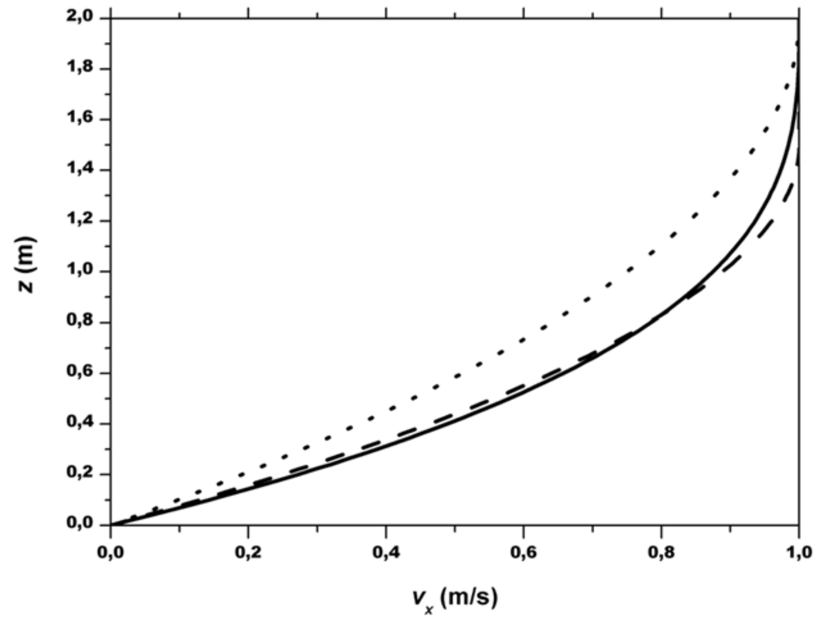

Figure 2. A comparison among the velocity profiles through the thickness of lava flow for three rheological models and for the values of parameters given in Table 1: power-law (solid line), Bingham (dashed), and Newtonian (dotted) rheology.

Introduction of (9) into (7) yields

$$\frac{d}{dz}\left(\frac{dv_x}{dz}\right)^n + \frac{2^{n-1}\rho g \sin\alpha}{K} = 0 \tag{10}$$

A first integration can be made with the free-surface boundary condition at $z = h$, where h is the thickness of lava flow:

$$\sigma_{xz}(h) = 0 \tag{11}$$

This gives

$$\dot{e}_{xz}(z) = \left(\frac{\rho g \sin\alpha}{2K}\right)^{\frac{1}{n}} (h - z)^{\frac{1}{n}} \tag{12}$$

and

$$\sigma_{xz}(z) = \rho g \sin\alpha\,(h - z) \tag{13}$$

A second integration can be made on (13) with the boundary condition of vanishing velocity at $z = 0$:

$$v_x(0) = 0 \tag{14}$$

whence

$$v_x(z) = \frac{2n}{n+1}\left(\frac{\rho g \sin\alpha}{2K}\right)^{\frac{1}{n}}\left[h^{\frac{n+1}{n}} - (h - z)^{\frac{n+1}{n}}\right] \tag{15}$$

The maximum value V of velocity is at the top of the lava flow:

$$V = \frac{2n}{n+1}\left(\frac{\rho g \sin\alpha}{2K}\right)^{\frac{1}{n}} h^{\frac{n+1}{n}} \tag{16}$$

and the average velocity is

$$\bar{v}_x = \frac{1}{h}\int_0^h v_x(z)dz = \frac{2n}{2n+1}\left(\frac{\rho g \sin\alpha}{2K}\right)^{\frac{1}{n}} h^{\frac{n+1}{n}} \tag{17}$$

The volume flow rate (per unit width in the y direction) is then

$$q = h\bar{v}_x = \frac{2n}{2n+1}\left(\frac{\rho g \sin\alpha}{2K}\right)^{\frac{1}{n}} h^{\frac{2n+1}{n}} \tag{18}$$

Table 1. Three Possible Rheologies for a Lava Flow[a]

Parameter	Value
Geometrical and Dynamical Parameters	
ρ	2800 kg/m^3
g	9.8 m/s^2
α	0.2 rad
h	2 m
V	1 m/s
q	1.3 m^3/s
Rheological Parameters	
Power-Law Rheology	
n	0.5
K	5 10^3 Pa s^n
Bingham Rheology	
η_B	6 10^3 Pa s
h_p	0.5 m
τ	3 10^3 Pa
Newton Rheology	
η_N	10^4 Pa s

[a]Here η, τ, and h_p are the viscosity, the yield stress, and the plug thickness, respectively.

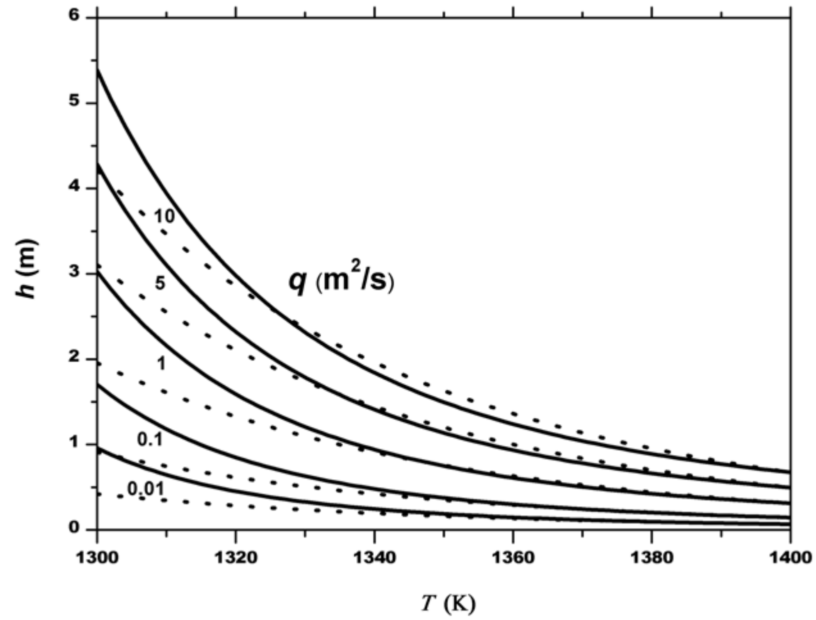

Figure 3. Flow thickness h as a function of temperature for different values of q and ρ = 2800 kg/m^3, α = 0.2 rad, g = 9.8 m/s^2, a = 5 10^{-3} K^{-1}, $n_0 = -6$, K_{01} = 8 10^{-30} Pa s^n, and $b = 10^5$ K; the dotted line refers to the Newtonian case (n = 1).

From (16) and (18) q can be expressed as a function of the exponent n, the surface velocity V, and the flow thickness h:

$$q = \beta h V \tag{19}$$

where

$$\beta \equiv \frac{n+1}{2n+1} \tag{20}$$

We note that, in the expression of volume flow rate q, the lava rheology is represented by β, while q is independent of consistency K. The value of β can range from $\frac{2}{3}$ to 1 (for a Newtonian rheology $\beta = \frac{2}{3}$); as a consequence, for given h and V, q can vary only in a narrow range, according to (19).

[14] From (16) we can express h as a function of V for a given choice of ρ, α, K, and n:

$$h = \left(\frac{n+1}{2n}\right)^{\frac{n}{n+1}} \left(\frac{2K}{\rho g \sin\alpha}\right)^{\frac{1}{n+1}} V^{\frac{n}{n+1}} \tag{21}$$

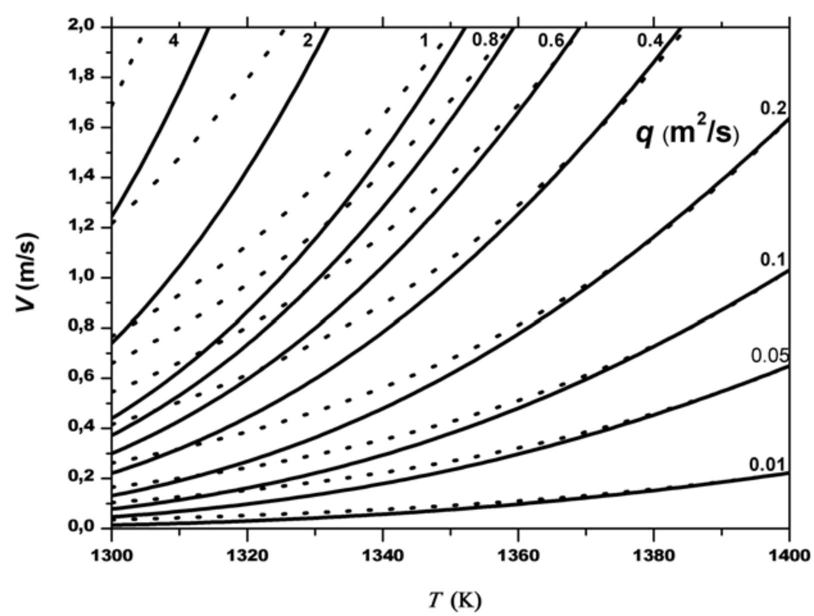

Figure 4. Surface velocity V as a function of temperature for different values of q and ρ = 2800 kg/m^3, α = 0.2 rad, g = 9.8 m/s^2, a = 5 10^{-3} K^{-1}, $n_0 = -6$, K_{01} = 8 10^{-30} Pa s^n, and $b = 10^5$ K; the dotted line refers to the Newtonian case (n = 1).

Table 2. Measured and Calculated Values of Parameters of a Lava Flow[a]

Symbol	Value
Measured Values	
ρ	2800 kg/m^3
g	9.8 m/s^2
α	0.2 rad
T	1350 K
V	0.3 m/s
Calculated Values	
n	0.75
β	0.70
K	1183 Pa s^n
h	3.4 m
q	0.7 m^2/s

[a]Here $a = 5\ 10^{-3}$ K^{-1}, $n_0 = -6$, $K_{01} = 8\ 10^{-30}$ Pa s^n, and $b = 10^5$ K.

As an alternative, from (18) and (21), we can express q as a function of V for a given choice of ρ, α, K, and n:

$$q = \frac{n+1}{2n+1}\left(\frac{n+1}{2n}\right)^{\frac{n}{n+1}}\left(\frac{2K}{\rho g \sin\alpha}\right)^{\frac{1}{n+1}} V^{\frac{2n+1}{n+1}}. \qquad (22)$$

4. Discussion and Conclusions

[15] Figure 2 shows a comparison among different rheological models: three possible velocity profiles shown for a lava flow, characterized by given values of surface velocity, thickness, and flow rate (Table 1); the values are chosen in order to be consistent with a Newtonian flow. We note that a power-law constitutive equation for $n < 1$ can describe the presence of a nearly undeformed part of the flow, similar to the plug of Bingham rheology, but with a smaller thickness. In general, the thickness of this layer increases with decreasing n for a power-law rheology.

[16] If the rheology is non-Newtonian, the evaluation of flow rate would require the values of rheological parameters in addition to thickness and average velocity. However, if we know the temperature dependence of the lava rheological parameters, the flow rate can be evaluated from the field measurements of the temperature and the thickness (Figure 3) or the surface velocity (Figure 4). If we adopt for K and n the temperature dependences given in (2) and (3), we can obtain the flow rate as a function of T, h and V by equations (18) and (22). Figures 3 and 4 show that flow thicknesses increase exponentially and flow velocities decrease exponentially as temperature decreases, for a given value of q. This strong temperature dependence is mainly due to the effect of the increase in K, as temperature decreases. We also note that, for any temperature value, there is a value of flow rate at which the thickness and the surface velocity of the power-law flow are equal to the corresponding quantities of a Newtonian flow. In these cases, an additional measurement of surface velocity (or thickness) is required to solve the ambiguity. Hence, Figures 3 and 4 show that, if we know the temperature dependence of n and K, and we can measure T, ρ and α and either h or V, then we can evaluate the flow rate q by (18) or (22), respectively.

[17] Table 2 shows the values of n (and β), K, h, and q, for a choice of values for T, V, α, and ρ, according to the temperature dependencies (2) and (3).

[18] If we suppose that the effects of lateral boundaries on the flow can be neglected, and w is the width of the lava flow, from (19) we can write the volume flow rate as

$$Q = \beta h w V \qquad (23)$$

In summary, formulae relating the lava rheology to geometric and dynamic parameters of a lava flow have been obtained for a power-law rheology. If we know the temperature dependence of the lava rheological parameters, the flow rate can be evaluated from the field measurements of the temperature, and the thickness or the surface velocity of the flow. A limit of this method is the inaccuracy of the empirical laws giving the dependence of K and n on temperature. Further experimental investigations are needed in order to improve the knowledge of the temperature dependence on rheological parameters, also as a function of the chemical composition of lavas.

[19] **Acknowledgments.** This work was funded by MIUR and INGV. Many thanks to the editor F. Florindo and two anonymous reviewers for useful comments on the first version of the manuscript.

References

Dragoni, M., and A. Tallarico (2009), Assumptions in the evaluation of lava effusion rates from heat radiation, *Geophys. Res. Lett.*, *36*, L08302, doi:10.1029/2009GL037411.
Dragoni, M., M. Bonafede, and E. Boschi (1986), Downslope flow model of a Bingham liquid: Implications for lava flows, *J. Volcanol. Geotherm. Res.*, *30*, 305–325.
Griffiths, R. W. (2000), The dynamics of lava flows, *Annu. Rev. Fluid Mech.*, *32*, 477–518.
Hardee, H. C., and J. C. Dunn (1981), Convective heat transfer in magmas near the liquidus, *J. Volcanol. Geotherm. Res.*, *10*, 195–207.
Harris, A. J. L., S. Blake, D. A. Rothery, and N. F. Stevens (1997), A chronology of the 1991 to 1993 Mount Etna eruption using advanced very high resolution radiometer data: Implications for real-time thermal volcano monitoring, *Geophys. Res. Lett.*, *102*, 7985–8003.
Huang, X., and M. H. García (1998), A Herchel-Bulkley model for mud flow down a slope, *J. Fluid Mech.*, *374*, 305–333.
Hulme, G. (1974), The interpretation of lava flow morphology, *Geophys. J. R. Astron. Soc.*, *39*, 361–383.
Kerr, R. C., and J. R. Lister (1991), The effects of shape on crystal settling and on the rheology of magmas, *Geology*, *99*, 457–467.
Lavalle, Y., K. U. Hess, B. Cordonnier, and D. B. Dingwell (2007), Non-Newtonian rheological law for highly crystalline dome lavas, *Geology*, *35*, 843–846.
Ng, C. O., and C. C. Mei (1994), Roll waves on a shallow layer of mud modelled as a power-law fluid, *J. Fluid Mech.*, *263*, 151–183.
Pinkerton, H., and G. Norton (1995), Rheological properties of basaltic lavas at sub-liquidus temperatures: Laboratory and field measurements on lavas from Mount Etna, *J. Volcanol. Geotherm. Res.*, *68*, 307–323.
Pinkerton, H., and R. S. J. Sparks (1978), Field measurements of the rheology of lava, *Nature*, *276*, 383–385.
Pinkerton, H., and R. J. Stevenson (1992), Methods of determining the rheological of magmas at sub-liquidus temperatures, *J. Volcanol. Geotherm. Res.*, *53*, 47–66.
Saar, M. O., M. Manga, K. V. Cashman, and S. Fermouw (2001), Numerical models of the onset of yield strength in crystal-melt suspensions, *Earth Planet. Sci. Lett.*, *187*, 367–379.
Sakimoto, S. E. H., J. Crisp, and S. M. Baloga (1997), Eruption constraints on tube-fed planetary lava flows, *J. Geophys. Res.*, *102*, 6597–6613.
Shaw, H. R., T. L. Wright, D. L. Peck, and R. Okamura (1968), The viscosity of basaltic magma: An analysis of field measurements in Makaopuhi lava lake, Hawaii, *Am. J. Sci.*, *266*, 255–264.
Sonder, I., B. Zimanowski, and R. Büttner (2006), Non-Newtonian viscosity of basaltic magma, *Geophys. Res. Lett.*, *33*, L02303, doi:10.1029/2005GL024240.
Spera, F. J., A. Borgia, J. Strimple, and M. Feigenson (1988), Rheology of melts and magmatic suspensions: 1. Design and calibration of a concentric cylinder viscometer for application to rhyolitic magma, *J. Geophys. Res.*, *93*, 10,273–10,294.

Tallarico, A., and M. Dragoni (1999), Viscous Newtonian laminar flow in a rectangular channel: Application to Etna lava flows, *Bull. Volcanol.*, *61*, 40–47.
Tallarico, A., and M. Dragoni (2000), A three-dimensional Bingham model for channeled lava flows, *J. Geophys. Res.*, *105*, 25,969–25,980.
Tallarico, A., M. Dragoni, and G. Zito (2006), Evaluation of lava effusion rate and viscosity from other flow parameters, *J. Geophys. Res.*, *111*, B11205, doi:10.1029/2005JB003762.
Turcotte, D. L., and G. Schubert (1982), *Geodynamics: Application of Continuum Physics to Geological Problems*, John Wiley, New York.

M. Dragoni and A. Piombo, Dipartimento di Fisica, Alma Mater Studiorum Università di Bologna, I-40127 Bologna, Italy. (michele.dragoni@unibo.it; antonello.piombo@unibo.it)