Exercise set 1

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March 8, 2022

Deadline: 4 March 2022

1 In-class exercises

Exercise 1. Case 1:

1. The reward is 500,000 with certainty.
2. The reward is 2,500,000 with probability 0.10. It is 500,000 with probability 0.89, and 0 with probability 0.01.

Case 2:

1. The reward is 500,000 with probability 0.11, or 0 with probability 0.89.
2. The reward is 2,500,000 with probability 0.1, or 0 with probability 0.9.

Show that under the expected utility hypothesis, if gamble 1 is preferred in Case 1, gamble 1 must also be preferred in the Case 2, for any increasing utility function.

Exercise 2 (15). Consider the St. Petersburg paradox, where a coin is tossed until it comes heads. We have estabilshed that if your utility for money is $U(x) = x$, and the coin was fair, i.e. the probability of heads was 1/2 then you should be willing to pay any amount to play the game. In this exercise, let us consider two alternative hypotheses:

- Assume that the coin is fair, but your utility for money is $U(x) = ln x$. How much would you now be willing to pay to play the game? Hint: Calculate the expected utility of playing.
- Assume that the coin is not fair, but only comes head with probability 0.4, and your utility is linear: U(x) = x. How much would you be willing to play now?
- Perform an experiment to validate your claims using the code in `src/paradox.ipynb`

2 Homework

Exercise 3. Assuming that U is increasing and absolutely continuous, consider the following experiment:

1. You specify an amount a, then observe random value Y.
2. If $Y \geq a$, you receive Y currency units.
3. If $Y < a$, you receive a random amount X with known distribution (independent of Y).

Show that we should choose a such that $U(a) = \mathbb{E}[U(X)]$.

EXERCISE 4 (usefulness of probability and utility).
1. Would it be useful to separate randomness from uncertainty? What would be desirable properties of an alternative concept to probability?
2. Give an example of how the expected utility assumption might be violated.

Exercise 3 solution. Let $\mu = \mathbb{E}[U(X)]$. From the problem definition, the expected utility of choosing a such that $U(a) = \mu$ is:

$$\mathbb{E}(U \mid a) = \int_a^\infty U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^a \mathrm{d}P_Y(y).$$

Let some $\epsilon > 0$. Now select $a - \epsilon$:

$$\begin{aligned}
\mathbb{E}(U \mid a - \epsilon) &= \int_{a-\epsilon}^\infty U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^{a-\epsilon} \mathrm{d}P_Y(y) \\
&= \int_a^\infty U(y) \, \mathrm{d}P_Y(y) + \int_{a-\epsilon}^a U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^{a-\epsilon} \mathrm{d}P_Y(y)
\end{aligned}$$

Since $U(a) = \mu$ and U is increasing, $U(y) \leq \mu$ for all $y < a$. Consequently:

$$\begin{aligned}
\mathbb{E}(U \mid a - \epsilon) &\leq \int_a^\infty U(y) \, \mathrm{d}P_Y(y) + \int_{a-\epsilon}^a U(a) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^{a-\epsilon} \mathrm{d}P_Y(y) \\
&= \int_a^\infty U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^a \mathrm{d}P_Y(y) = \mathbb{E}(U \mid a).
\end{aligned}$$

Similarly, if we select $a + \epsilon$, we have:

$$\begin{aligned}
\mathbb{E}(U \mid a + \epsilon) &= \int_{a+\epsilon}^\infty U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^{a+\epsilon} \mathrm{d}P_Y(y) \\
&= \int_{a+\epsilon}^\infty U(y) \, \mathrm{d}P_Y(y) + \int_a^{a+\epsilon} U(a) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^a \mathrm{d}P_Y(y) \\
&\leq \int_{a+\epsilon}^\infty U(y) \, \mathrm{d}P_Y(y) + \int_a^{a+\epsilon} U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^a \mathrm{d}P_Y(y) \\
&= \int_a^\infty U(y) \, \mathrm{d}P_Y(y) + \mu \int_{-\infty}^a \mathrm{d}P_Y(y) = \mathbb{E}(U \mid a).
\end{aligned}$$

An alternative proof may be obtained by applying the fundamental theorem of calculus:

$$\mathrm{d}/\mathrm{d}x \left(\int_a^x f(t) \, \mathrm{d}t \right) = f(x)$$

to show that the derivative of $\mathbb{E}(U \mid a)$ with respect to a is zero at $U(a) = \mu$:

$$\begin{aligned}
\mathbb{E}[U \mid a] &= \mathbb{E}[U \mid Y \geq a]\, \mathbb{P}(Y \geq a) + \mathbb{E}[U \mid Y < a]\, \mathbb{P}(Y < a) \\
&= \int_a^\infty U(y) p_Y(y) \, \mathrm{d}y + \mu \int_{-\infty}^a p_Y(y) \, \mathrm{d}y
\end{aligned}$$

Taking derivatives wrt a:

$$\begin{aligned}
\mathrm{d}\, \mathbb{E}[U \mid a] / \, \mathrm{d}a &= \frac{\mathrm{d}}{\mathrm{d}a} \int_a^\infty U(y) p_Y(y) \, \mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}a} \mu \int_{-\infty}^a p_Y(y) \, \mathrm{d}y \\
&= -U(a) p_Y(a) + \mu p_Y(a).
\end{aligned}$$

Solving for $\mathrm{d}\, \mathbb{E}[U \mid a] / \, \mathrm{d}a = 0$ gives $U(a) = \mu$. □

EXERCISE 5 (usefulness of probability and utility).
1. Would it be useful to separate randomness from uncertainty? What would be desirable properties of an alternative concept to probability?
2. Give an example of how the expected utility assumption might be violated.

Exercise 2 solution. I am not sure if it is useful, but it is certainly possible. We would like some function $\xi(\omega)$ telling us how sure we are that some ω is true. What values could this take? It could be in a bounded range, like probability. It could have multiple values. For example, it might return two values, an upper and lower bound on the likelihood of an event.

What about properties? I think one fundamental property is conditioning on information. If we start with some initial belief $\xi(\cdot)$, we would like to be able to calculate $\xi(\cdot | D)$ given some data D in a consistent manner.

A real-world example is not possible: it is easy to simply have an arbitrary utility function, that can explain arbitrary individual behaviours. However, in practice we can assume that people have "similar" utility functions, and so statistically infer that their aggregate behaviour may not correspond to maximising expected utility.

Mathematically, consider the first exercise. If somebody says they prefer A to B in the first case but B to A in the second case, then the expected utility hypothesis is indeed violated. □