

Exp3

Seminar Advanced Topics in Reinforcement Learning and Decision Making

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Overview

1. Adversarial Bandits
2. Importance weighted estimators
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Proof outline
4. Adversarial linear bandits
5. Exp3 for adversarial linear bandits

Adversarial Bandits

- Reward sequence $(x_t)_{t=1}^n$, with $x_{t,i}$ reward of arm i at time t , **fixed by an adversary**.
- Agent's action is drawn from a distribution $P_t \in \mathcal{P}_{k-1}$, where \mathcal{P}_d is a probability simplex over $d+1$ actions (i.e. $\mathcal{P}_d = \{p \in [0, 1]^{d+1} : \|p\|_1 = 1\} = \{p \in [0, 1]^{d+1} : \sum_i p_i = 1\}$).
- The policy π maps from histories onto the distribution over action

$$\pi : ([k] \times [0, 1])^* \rightarrow \mathcal{P}_{k-1}.$$

- At each timestep t :
 1. Agent chooses distribution P_t
 2. Action is drawn $A_t \sim P_t$
 3. Reward $X_t = x_{tA_t}$ is observed.

Adversarial Bandits regret

Regret in case of an adversarial bandit can be summarized as:

$$R_n(\pi, x) = \max_{i \in [k]} \sum_{t=1}^n x_{ti} - \mathbb{E} \left[\sum_{t=1}^n x_{tA_t} \right] \quad (1)$$

While Worst case regret for some policy π is:

$$R_n^*(\pi) = \sup_{x \in [0,1]^{n \times k}} R_n(\pi, x) \quad (2)$$

Importance weighted estimators

Before we go into the Exp3 algorithm, we have to introduce an unbiased estimator for reward \hat{X}_{ti} at time t for arm i . It is defined as:

$$\hat{X}_{ti} = \frac{\mathbb{I}\{A_t = i\}X_t}{P_{ti}} \quad (3)$$

where P_{ti} is the probability of selecting arm i at time t . Why is \hat{X}_{ti} unbiased?

$$\mathbb{E}[\hat{X}_{ti}] = \mathbb{E}\left[\frac{\mathbb{I}\{A_t = i\}X_t}{P_{ti}}\right] = \sum_j P_{tj} \hat{X}_{ti} = P_{ti} \hat{X}_{ti} + \sum_{j \neq i} P_{tj} \cdot 0 = P_{ti} \frac{X_t}{P_{ti}} = x_{ti} \quad (4)$$

Loss-based importance weighted estimator

Another unbiased estimator would be:

$$\hat{X}_{ti} = 1 - \frac{\mathbb{I}\{A_t = i\}(1 - X_t)}{P_{ti}} \quad (5)$$

where we can prove it is unbiased by substituting $Y_t = 1 - X_t$. The advantage of loss based estimator is, that while in the case of estimator from previous slide its variance is:

$$\mathbb{V} \left[\frac{\mathbb{I}\{A_t = i\} X_t}{P_{ti}} \right] = x_{ti}^2 \frac{1 - P_{ti}}{P_{ti}} \quad (6)$$

while loss based estimator's variance is proportional to $(1 - x_{ti})^2$:

$$\mathbb{V} \left[1 - \frac{\mathbb{I}\{A_t = i\}(1 - X_t)}{P_{ti}} \right] = (1 - x_{ti})^2 \frac{1 - P_{ti}}{P_{ti}} \quad (7)$$

Exp3

Exponential-weight algorithm for exploration and exploitation (hence **Exp3**) is based on changing the probabilities for each actions based on term $\hat{S}_t i = \sum_{s=1}^t \hat{X}_s i$, which is the sum of estimated rewards up until current round t .

Probability for each arm at each timestep is calculated using the following, softmax-like, formula:

$$P_{ti} = \frac{\exp(\eta \hat{S}_{t-1,i})}{\sum_j \exp(\eta \hat{S}_{t-1,j})} \quad (8)$$

Then, $A_t \sim P_t$ is played, reward X_t observed and estimated rewards adjusted:

$$\hat{S}_{ti} = \hat{S}_{t-1,i} + 1 - \frac{\mathbb{I}\{A_t = i\}(1 - X_t)}{P_{ti}} \quad (9)$$

Exp3 expected regret

Regret for Exp3 is bounded from above by:

$$R_n \leq \frac{\log(k)}{\eta} + \eta nk \quad (10)$$

when we set learning rate $\eta = \sqrt{\log(k)/nk}$, then the regret bound would be optimized as follows:

$$R_n \leq 2\sqrt{nk\log(k)} \quad (11)$$

Define

$$\hat{S}_n = \sum_{t,i} P_{ti} \hat{X}_{t,i}, \quad W_t = \sum_{j=1}^k \exp \left(\eta \hat{S}_{t,j} \right)$$

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We now bound the exponential term

$$\exp(\eta \hat{S}_{ni}) \leq \sum_{j=1}^k$$

Adversarial linear bandits

In adversarial linear settings, actions from action set $\mathcal{A} \subset \mathbb{R}^d$ are d -dimensional vectors, just as reward x_t at time t . Reward in this setting is given by inner product $\langle A_t, x_t \rangle$. Without loss of generality, we can switch to losses $y_t = 1 - x_t$. Therefore, if observed loss would be defined as $Y_t = \langle A_t, y_t \rangle$, then regret after n steps is defined as:

$$R_n = \mathbb{E} \left[\sum_{t=1}^n Y_t \right] - \min_{a \in \mathcal{A}} \sum_{t=1}^n \langle a, y_t \rangle \quad (12)$$

Exp3 for finite exponential weights

Probability distribution $P_t(a)$ is given by mixture distribution:

$$P_t(a) = (1 - \gamma)\tilde{P}_t(a) + \gamma\pi(a) \quad (13)$$

where $\pi(a)$ is an exploration distribution mapping simplex $\mathcal{A} \rightarrow [0, 1]$; $\sum_{a \in \mathcal{A}} \pi(a) = 1$, while $\tilde{P}_t(a)$ is a probability mass function:

$$\tilde{P}_t(a) \propto \exp\left(-\eta \sum_{s=1}^{t-1} \hat{Y}_s(a)\right) \quad (14)$$

Finally, loss estimate is estimated by $\hat{Y}_t = Q_t^{-1} A_t Y_t$, where Q_t is given by:

$$Q_t = \sum_{a \in \mathcal{A}} P_t(a) a a^\top \quad (15)$$

Exp3 for finite exponential weights

Distribution is calculated at each step by:

$$P_t(a) = \gamma\pi(a) + (1 - \gamma) \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \hat{Y}_s(a)\right)}{\sum_{a' \in \mathcal{A}} \exp\left(-\eta \sum_{s=1}^{t-1} \hat{Y}_s(a')\right)} \quad (16)$$

Action $A_t \sim P_t$ is sampled, loss $Y_t = \langle A_t, y_t \rangle$ is observed and loss estimate is updated using:

$$\hat{Y}_t = Q_t^{-1} A_t Y_t \quad (17)$$

Exp3 regret for adversarial linear bandits

Exp3 regret is bounded from above by:

$$R_n \leq 2\sqrt{(2g(\pi) + d)n \log(k)} \quad (18)$$

where d is the dimension of \mathcal{A} , k is the number of arms and $g(\pi)$ equals:

$$g(\pi) = \max_{a \in \mathcal{A}} \|a\|_{Q^{-1}(\pi)}^2 \quad (19)$$

Exp3 for continuous exponential weights

If the number of arms is big, or if it goes to ∞ , then this algorithm becomes intractable. Instead of computing P_t for every arm, we can switch to continuous exponential weights. Assuming that \mathcal{A} is convex, distribution is calculated by:

$$P_t(B) = \gamma\pi(B) + (1 - \gamma) \frac{\int_B \exp\left(-\eta \sum_{s=1}^{t-1} \hat{Y}_s(a)\right) da}{\int_{\mathcal{A}} \exp\left(-\eta \sum_{s=1}^{t-1} \hat{Y}_s(a)\right) da} \quad (20)$$

Rest of the algorithm is analogous to the Exp3 for finite action sets.