1 2.14

We will show that $x_k = 1 + (0.5)^{2^k}$ converges Q-quadratically to 1. So, by definition, we need to find a constant $M > 0$ such that

$$\frac{|x_{k+1} - 1|}{|x_k - 1|^2} \leq M$$

for all k sufficiently large. Because of the equalities

$$\frac{|x_{k+1} - 1|}{|x_k - 1|^2} = \frac{(0.5)^{2^{k+1}}}{(0.5)^{2^{k+1}}} = 1$$

that holds for all k, we see that the claim follows with $M = 1$.

2 2.15

We will show that $x_k = \frac{1}{k!}$ converges Q-superlinearly to 0 but not Q-quadratically. In order to prove the first claim, we need to show that

$$\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = 0$$

holds. This follows easily after simplfying the expression.

$$\frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \xrightarrow{k\to\infty} 0$$

To show the latter claim, note that

$$\frac{|x_{k+1}|}{|x_k|^2} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!\cdot k!}} \frac{k!}{k+1} \xrightarrow{k\to\infty} \infty$$

which implies that there is no constant M fulfilling the definition of Q-quadratic convergence.

3 2.16

We will show that the sequence

$$x_k = \begin{cases} \left(\frac{1}{4}\right)^{2^k}, & k \text{ even} \\ \frac{x_{k-1}}{k}, & k \text{ odd} \end{cases}$$

converges both Q-superlinearly and R-quadratically to zero but not Q-quadratically. By

$$\frac{|x_{k+1}|}{|x_k|} = \begin{cases} \frac{x_k}{k x_k} \\ \frac{(1/4)^{2^{k+1}} k}{(1/4)^{2^{k-1}}} \end{cases} = \begin{cases} \frac{1}{k}, & k \text{ even} \\ k(1/4)^{3\cdot 2^{k-1}}, & k \text{ odd} \end{cases}$$

As both of these two sequences covnerge to zero, the claim follows.

To show that the sequence converges R-quadratically, we consider the sequence $v_k = (1/4)^{2^{k-1}}$. Since $(1/4)^{2^k} \leq v_k$ and $(1/4)^{2^{k-1}}/k \leq v_k$, we get $x_k \leq v_k$. Also, v_k converges Q-quadratically, where the proof is essentially the same as the argument given in exercise 2.14. Hence, x_k converges R-quadratically.

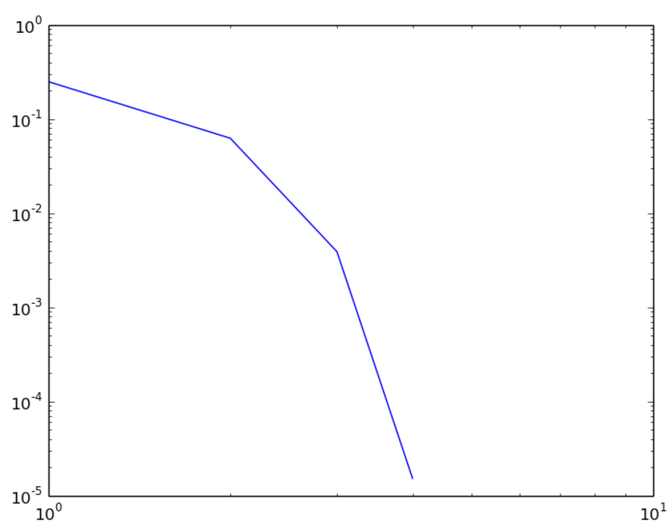
Lastly, we will show that x_k does not converge Q-quadratically. For k even, we see that

$$\frac{|x_{k+1}|}{|x_k|^2} = \frac{(1/4)^{2^k}}{(k+1) \cdot \left((1/4)^{2^k}\right)^2} = \frac{1}{(k+1)(1/4)^{2^k}},$$

which converges to infinity and is therefore unbounded. This implies that the sequence does not converge Q-quadratically.

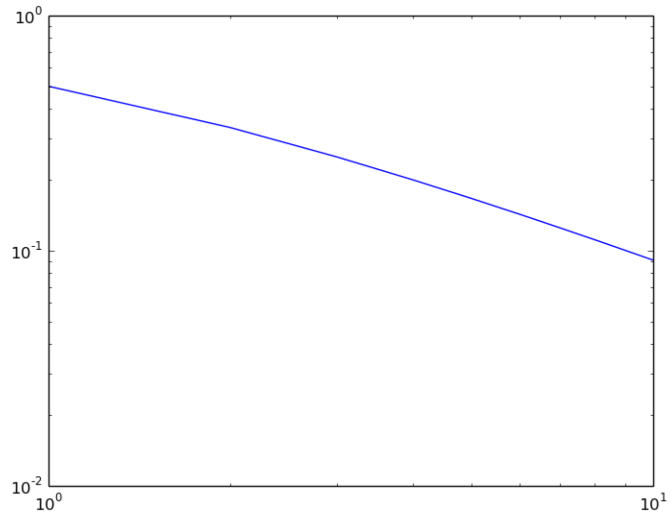
4 Pictures Exercise 1

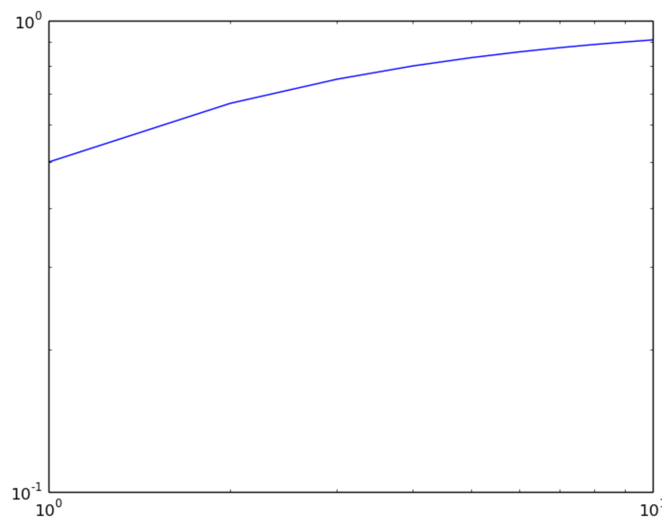
The following pictures show the convergence properties. The first picture displays the $q = 1$-quotient of $1 + \left(\frac{1}{2}\right)^{2^k}$. The second picture displays the $q = 1$-



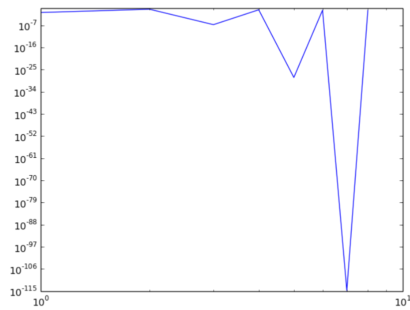
quotient of $\frac{1}{k!}$ The next two pictures show the quotient for $q = 1$ and $q = 2$ for the sequence from exercise 2.16.

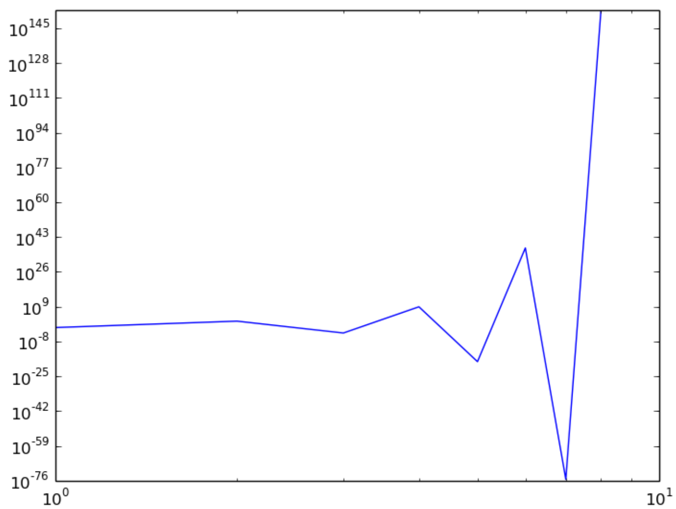
The last picture displays the $q = 1$ quotient for $\frac{1}{k}$ which converges q-linearly





but not superlinearly.





5 Exercise 2

5.1 2.1

We will discretize the optimization problem

$$\min_x \int_0^1 \left[\left(\frac{\mathrm{d}\, x_1(t)}{\mathrm{d}\, t} \right)^2 + \left(\frac{\mathrm{d}\, x_2(t)}{\mathrm{d}\, t} \right)^2 \right] \rho(x(t))\, \mathrm{d}\, t$$

,subject to the condtion $x(0) = a$ and $x(1) = b$, by discretizing x on n points using finite differences and the midpoint rule. Let $t_i = \frac{i}{n}$ for $0 \leq i \leq n$.

$$
\begin{aligned}
&\int_0^1 \left[\left(\frac{\mathrm{d}\, x_1(t)}{\mathrm{d}\, t} \right)^2 + \left(\frac{\mathrm{d}\, x_2(t)}{\mathrm{d}\, t} \right)^2 \right] \rho\left(x\left(t\right)\right) \mathrm{d}\, t \\
=& \sum_{i=o}^{n-1} \int_{t_i}^{t_{i+1}} \left[\left(\frac{\mathrm{d}\, x_1(t)}{\mathrm{d}\, t} \right)^2 + \left(\frac{\mathrm{d}\, x_2(t)}{\mathrm{d}\, t} \right)^2 \right] \rho\left(x\left(t\right)\right) \mathrm{d}\, t \\
\approx& \sum_{i=0}^{n-1} \Delta x \left[\left(\frac{\mathrm{d}\, x_1(\frac{t_i+t_{i+1}}{2})}{\mathrm{d}\, t} \right)^2 + \left(\frac{\mathrm{d}\, x_2(\frac{t_i+t_{i+1}}{2})}{\mathrm{d}\, t} \right)^2 \right] \rho\left(x\left(\frac{t_i + t_{i+1}}{2}\right)\right) \mathrm{d}\, t \\
\approx& \sum_{i=0}^{n-1} \Delta x \left[\left(\frac{x_{1,i+1} - x_{1,i}}{\Delta x} \right)^2 + \left(\frac{x_{2,i+1} - x_{2,i}}{\Delta x} \right)^2 \right] \frac{\rho(x_{i+1}) + \rho(x_i)}{2} \\
=& \sum_{i=0}^{n-1} \frac{1}{2\Delta x} \left[(x_{1,i+1} - x_{1,i})^2 + (x_{2,i+1} - x_{2,i})^2 \right] \left(\rho(x_{i+1}) + \rho(x_i)\right)
\end{aligned}
$$

We will rewrite this now using matrix notation to facilitate differentiation. For this, we will set

$$
\begin{aligned}
x &= (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, x_{2,1}, \ldots, x_{2,n-1})^T && \in \mathbb{R}^{2(n-1)} \\
v &= (-a_1, 0, \ldots, 0, b_1, -a_2, 0, \ldots, 0, b_2)^T && \in \mathbb{R}^{2n} \\
\rho(x) &= (\rho(x_1), \ldots, \rho(x_{n-1}))^T && \in \mathbb{R}^{n-1} \\
w &= (\rho(a), 0, \ldots, 0, \rho(b), \rho(a), \ldots, 0, \rho(b))^T && \in \mathbb{R}^{2n} \\
A' &= \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 0 \end{pmatrix} && \in \mathbb{R}^{n \times (n-1)} \\
A &= \begin{pmatrix} A' & 0 \\ 0 & A' \end{pmatrix} && \in \mathbb{R}^{(2n) \times (2(n-1))} \\
M' &= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} && \in \mathbb{R}^{n \times (n-1)} \\
M &= \begin{pmatrix} M' \\ M' \end{pmatrix} && \in \mathbb{R}^{(2n) \times (n-1)}
\end{aligned}
$$

which will allow us to rewrite our discretization and define our objective function.

$$\begin{aligned} f :&\mathbb{R}^{2(n-1)} \to \mathbb{R} \\ x \mapsto& \frac{1}{2\Delta x} \left((Ax+v) \odot (Ax+v)\right)^T (M\rho(x) + w)) \end{aligned}$$

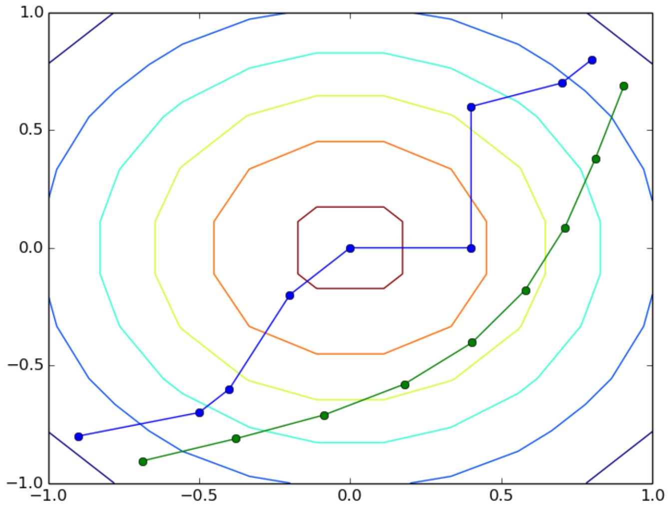
By use of the differentiation rules discussed in class, we get that

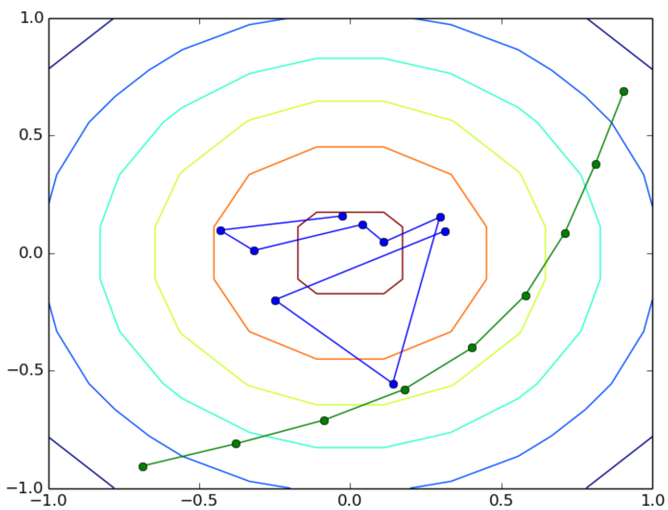
$$\nabla f(x) = \frac{1}{2 * \Delta x} \left[\nabla\rho(x) M^T (Ax+v) \odot (Ax+v) + 2A^T \operatorname{diag}(Ax+v)(M\rho(x)+w)\right]$$

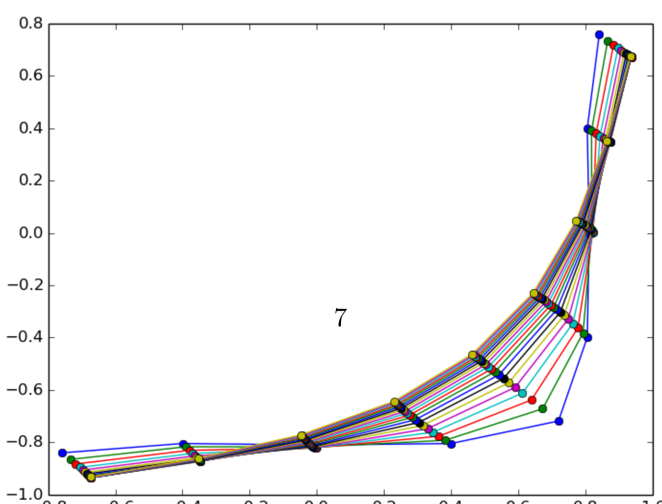
up to transpose ;-).

6 Pictures Exercise 2

The following pictures show my favorite examples and one of them shows the first 20 iterations because I thought that that looks cool.


1.0
0.5
0.0
−0.5
−1.0
−1.0 −0.5 0.0 0.5 1.0


1.0
0.5
0.0
−0.5
−1.0
−1.0 −0.5 0.0 0.5 1.0


0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−0.8 −0.6 −0.4 −0.2 0.0 0.2 0.4 0.6 0.8 1.0